Longest Palindromic Substring

We'll cover the following

- Problem Statement
- Basic Solution
- Top-down Dynamic Programming with Memoization
- Bottom-up Dynamic Programming
- Manacher's Algorithm

Problem Statement

Given a string, find the length of its Longest Palindromic Substring (LPS). In a palindromic string, elements read the same backward and forward.

Example 1:

```
Input: "abdbca"
Output: 3
Explanation: LPS is "bdb".
```

Example 2:

```
Input: = "cddpd"
Output: 3
Explanation: LPS is "dpd".
```

Example 3:

```
Input: = "pqr"
Output: 1
Explanation: LPS could be "p", "q" or "r".
```

Basic Solution

This problem follows the Longest Palindromic Subsequence (https://www.educative.io/collection/page/5668639101419520/5633779737559040/5748119283171328/) pattern. The only difference is that in a palindromic subsequence characters can be non-adjacent, whereas in a substring all characters should form a palindrome. We will follow a similar approach though.

The brute-force solution will be to try all the substrings of the given string. We can start processing from the beginning and the end of the string. So at any step, we will have two options::

1. If the element at the beginning and the end are the same, we make a recursive call to check if the remaining substring is also a palindrome. If so, the substring is a palindrome from beginning to the end.
2. We will skip either the element from the beginning or the end to make two recursive calls for the remaining substring. The length of LPS would be the maximum of these two recursive calls.

Here is the code:

Java | JS | Python3 | C++

```python
def find_LPS_length(st):
  return find_LPS_length_recursive(st, 0, len(st) - 1)


def find_LPS_length_recursive(st, startIndex, endIndex):
  if startIndex > endIndex:
    return 0

  # every string with one character is a palindrome
  if startIndex == endIndex:
    return 1

  # case 1: elements at the beginning and the end are the same
  if st[startIndex] == st[endIndex]:
    remainingLength = endIndex - startIndex - 1
    # check if the remaining string is also a palindrome
    if remainingLength == find_LPS_length_recursive(st, startIndex + 1, endIndex - 1):
      return remainingLength + 2

  # case 2: skip one character either from the beginning or the end
  c1 = find_LPS_length_recursive(st, startIndex + 1, endIndex)
  c2 = find_LPS_length_recursive(st, startIndex, endIndex - 1)
  return max(c1, c2)


def main():
  print(find_LPS_length("abdbca"))
  print(find_LPS_length("cddpd"))
  print(find_LPS_length("pqr"))


main()

```

Due to the three recursive calls, the time complexity of the above algorithm is exponential $O(3^n)$, where 'n' is the length of the input string. The space complexity is $O(n)$ which is used to store the recursion stack.

Top-down Dynamic Programming with Memoization

We can use an array to store the already solved subproblems.

The two changing values to our recursive function are the two indexes, startIndex and endIndex. Therefore, we can store the results of all the subproblems in a two-dimensional array. (Another alternative could be to use a hash-table whose key would be a string (startIndex + "|" + endIndex))

Here is the code for this:

Java | JS | Python3 | C++

```python
def find_LPS_length(st):
  n = len(st)
  dp = [[-1 for _ in range(n)] for _ in range(n)]
  return find_LPS_length_recursive(dp, st, 0, n - 1)


def find_LPS_length_recursive(dp, st, startIndex, endIndex):
  if startIndex > endIndex:
    return 0

  # every string with one character is a palindrome
  if startIndex == endIndex:
    return 1

  if dp[startIndex][endIndex] == -1:
    # case 1: elements at the beginning and the end are the same
    if st[startIndex] == st[endIndex]:
      remainingLength = endIndex - startIndex - 1
      # if the remaining string is a palindrome too
      if remainingLength == find_LPS_length_recursive(dp, st, startIndex + 1, endIndex - 1
        dp[startIndex][endIndex] = remainingLength + 2
        return dp[startIndex][endIndex]

    # case 2: skip one character either from the beginning or the end
    c1 = find_LPS_length_recursive(dp, st, startIndex + 1, endIndex)
    c2 = find_LPS_length_recursive(dp, st, startIndex, endIndex - 1)
    dp[startIndex][endIndex] = max(c1, c2)

  return dp[startIndex][endIndex]


def main():
  print(find_LPS_length("abdbca"))
  print(find_LPS_length("cddpd"))
  print(find_LPS_length("pqr"))


main()
```

The above algorithm has a time and space complexity of $O(n^2)$ because we will not have more than $n * n$ subproblems.

Bottom-up Dynamic Programming

Since we want to try all the substrings of the given string, we can use a two-dimensional array to store the subproblems' results. So `dp[i][j]` will be 'true' if the substring from index 'i' to index 'j' is a palindrome.

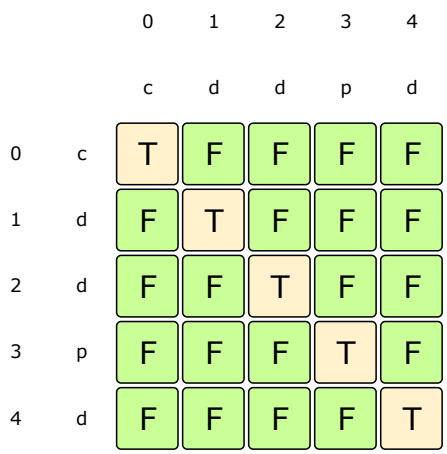
We can start from the beginning of the string and keep adding one element at a time. At every step, we will try all of its substrings. So for every `endIndex` and `startIndex` in the given string, we need to check the following thing:

If the element at the `startIndex` matches the element at the `endIndex`, we will further check if the remaining substring (from `startIndex+1` to `endIndex-1`) is a substring too.

So our recursive formula will look like:

```
1  if st[startIndex] == st[endIndex], and
2          if the remaing string is of zero length or dp[startIndex+1][endIndex-1] is a palin
3     dp[startIndex][endIndex] = true
```

Let's draw this visually for "cddpd", starting with a substring of length '1'. As we know, every substring with one element is a palindrome:



Every string with one element is a palindrome

1 of 8

		0	1	2	3	4
		c	d	d	p	d
0	c	T	F	F	F	F
1	d	F	T	F	F	F
2	d	F	F	T	F	F
3	p	F	F	F	T	F
4	d	F	F	F	F	T

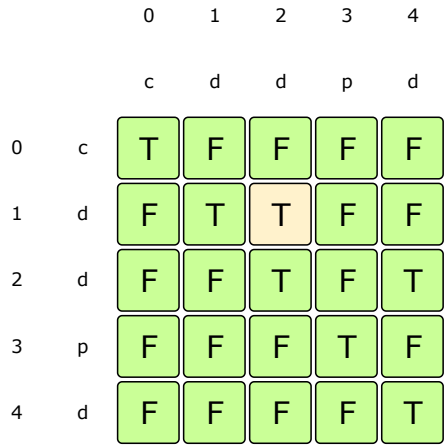
startIndex:3, endIndex:4 => since st[startIndex] != st[endIndex] => false

		0	1	2	3	4
		c	d	d	p	d
0	c	T	F	F	F	F
1	d	F	T	F	F	F
2	d	F	F	T	F	F
3	p	F	F	F	T	F
4	d	F	F	F	F	T

startIndex:2, endIndex:3 => since st[startIndex] != st[endIndex] => false

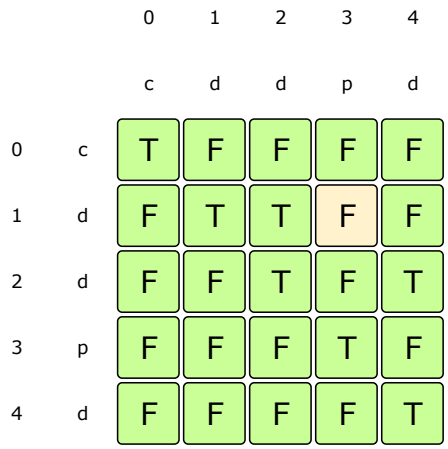
		0	1	2	3	4
		c	d	d	p	d
0	c	T	F	F	F	F
1	d	F	T	F	F	F
2	d	F	F	T	F	T
3	p	F	F	F	T	F
4	d	F	F	F	F	T

startIndex:2, endIndex:4 => since st[startIndex] == st[endIndex] and dp[startIndex+1][endIndex-1] is true => true



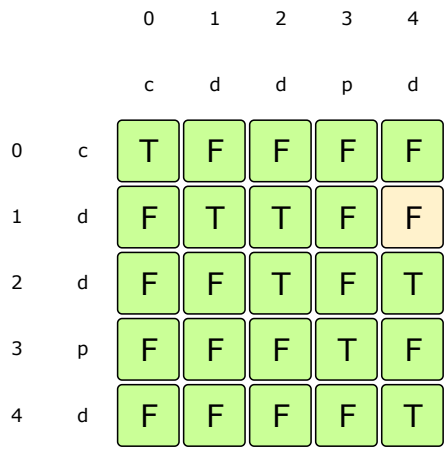
startIndex:1, endIndex:2 => since st[startIndex] == st[endIndex] and it is a two character string => true

5 of 8



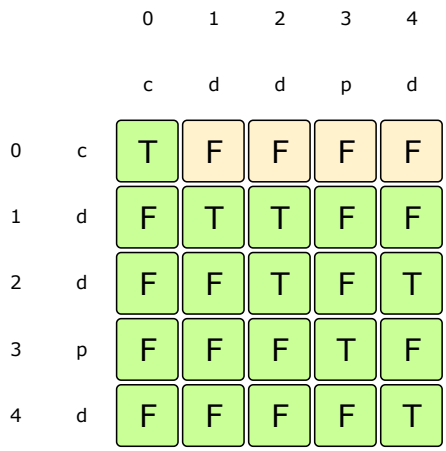
startIndex:1, endIndex:3 => since st[startIndex] != st[endIndex] => false

6 of 8



startIndex:1, endIndex:4 => since st[startIndex] == st[endIndex] but dp[startIndex+1][endIndex-1] is false => false

7 of 8



startIndex:0, endIndex:1-4 => since st[startIndex] != st[endIndex] => false

8 of 8

Here is the code for our bottom-up dynamic programming approach:

Java | JS | Python3 | C++

```python
def find_LPS_length(st):
  n = len(st)
  # dp[i][j] will be 'true' if the string from index 'i' to index 'j' is a palindrome
  dp = [[False for _ in range(n)] for _ in range(n)]

  # every string with one character is a palindrome
  for i in range(n):
    dp[i][i] = True

  maxLength = 1
  for startIndex in range(n - 1, -1, -1):
    for endIndex in range(startIndex + 1, n):
      if st[startIndex] == st[endIndex]:
        # if it's a two character string or if the remaining string is a palindrome too
        if endIndex - startIndex == 1 or dp[startIndex + 1][endIndex - 1]:
          dp[startIndex][endIndex] = True
          maxLength = max(maxLength, endIndex - startIndex + 1)

  return maxLength


def main():
  print(find_LPS_length("abdbca"))
  print(find_LPS_length("cddpd"))
  print(find_LPS_length("pqr"))


main()
```

The time and space complexity of the above algorithm is $O(n^2)$, where 'n' is the length of the input string.

Manacher's Algorithm

The best-known algorithm to find the longest palindromic substring which runs in linear time $O(n)$ is Manacher's Algorithm (https://en.wikipedia.org/wiki/Longest_palindromic_substring). However, it is a non-trivial algorithm that doesn't use DP. Please take a look to familiarize yourself with this algorithm, however, no one expects you to come up with such an algorithm in a 45 minutes coding interview.