

## A General Identification Condition for Causal Effects

Jin Tian and Judea Pearl

Cognitive Systems Laboratory  
Computer Science Department  
University of California, Los Angeles, CA 90024  
{jtian, judea}@cs.ucla.edu

### Abstract

This paper concerns the assessment of the effects of actions or policy interventions from a combination of: (i) nonexperimental data, and (ii) substantive assumptions. The assumptions are encoded in the form of a directed acyclic graph, also called “causal graph”, in which some variables are presumed to be unobserved. The paper establishes a necessary and sufficient criterion for the identifiability of the causal effects of a singleton variable on all other variables in the model, and a powerful sufficient criterion for the effects of a singleton variable on any set of variables.

### Introduction

This paper explores the feasibility of inferring cause effect relationships from various combinations of data and theoretical assumptions. The assumptions considered will be represented in the form of an acyclic causal diagram which contains both arrows and bi-directed arcs (Pearl 1995; 2000). The arrows represent the potential existence of direct causal relationships between the corresponding variables, and the bi-directed arcs represent spurious dependencies due to unmeasured confounders. Our main task will be to decide whether the assumptions represented in any given diagram are sufficient for assessing the strength of causal effects from nonexperimental data and, if sufficiency is proven, to express the target causal effect in terms of estimable quantities.

It is well known that, in the absence of unmeasured confounders, all causal effects are *identifiable*, that is, the joint response of any set  $Y$  of variables to intervention on a set  $T$  of treatment variables, denoted  $P_t(y)$ ,<sup>1</sup> can be estimated consistently from nonexperimental data (Robins 1987; Spirtes, Glymour, & Scheines 1993; Pearl 1993). If some confounders are not measured, then the question of identifiability arises, and whether the desired quantity can be estimated depends critically on the precise locations (in the diagram) of those confounders vis a vis the sets  $T$  and  $Y$ . Sufficient graphical conditions for ensuring the identification of  $P_t(y)$  were established by several

authors (Spirtes, Glymour, & Scheines 1993; Pearl 1993; 1995) and are summarized in (Pearl 2000, Chapters 3 and 4). For example, a criterion called “back-door” permits one to determine whether a given causal effect  $P_t(y)$  can be obtained by “adjustment”, that is, whether a set  $C$  of covariates exists such that

$$P_t(y) = \sum_c P(y|c, t)P(c) \quad (1)$$

When there exists no set of covariates that is sufficient for adjustment, causal effects can sometimes be estimated by invoking multi-stage adjustments, through a criterion called “front-door” (Pearl 1995). More generally, identifiability can be decided using *do*-calculus derivations (Pearl 1995), that is, a sequence of syntactic transformations capable of reducing expressions of the type  $P_t(y)$  to subscript-free expressions. Using *do*-calculus as a guide, (Galles & Pearl 1995) devised a graphical criterion for identifying  $P_x(y)$  (where  $X$  and  $Y$  are singletons) that combines and expands the “front-door” and “back-door” criteria (see (Pearl 2000, pp. 114-8)).

This paper develops new graphical identification criteria that generalize and simplify existing criteria in several ways. We show that  $P_x(v)$ , where  $X$  is a singleton and  $V$  is the set of all variables excluding  $X$ , is identifiable if and only if there is no consecutive sequence of confounding arcs between  $X$  and  $X$ ’s immediate successors in the diagram.<sup>2</sup> When interest lies in the effect of  $X$  on a subset  $S$  of outcome variables, not on the entire set  $V$ , it is possible that  $P_x(s)$  would be identifiable even though  $P_x(v)$  is not. To this end, the paper gives a sufficient criterion for identifying  $P_x(s)$ , which is an extension of the criterion for identifying  $P_x(v)$ . It says that  $P_x(s)$  is identifiable if there is no consecutive sequence of confounding arcs between  $X$  and  $X$ ’s children in the subgraph composed of the ancestors of  $S$ . Other than this requirement, the diagram may have an arbitrary structure, including any number of confounding arcs between  $X$  and  $S$ . This simple criterion is shown to cover all criteria reported in the literature (with  $X$  singleton), including the “back-door”, “front-door”, and those developed by (Galles & Pearl 1995).

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<sup>1</sup>(Pearl 1995; 2000) used the notation  $P(y|set(t))$ ,  $P(y|do(t))$ , or  $P(y|\hat{t})$  for the post-intervention distribution, while (Lauritzen 2000) used  $P(y||t)$ .

<sup>2</sup>A variable  $Z$  is an “immediate successor” (or a “child”) of  $X$  if there exists an arrow  $X \rightarrow Z$  in the diagram.

## Notation, Definitions, and Problem Formulation

The use of causal models for encoding distributional and causal assumptions is now fairly standard (see, for example, (Pearl 1988; Spirtes, Glymour, & Scheines 1993; Greenland, Pearl, & Robins 1999; Lauritzen 2000; Pearl 2000)). The simplest such model, called *Markovian*, consists of a directed acyclic graph (DAG) over a set  $V = \{V_1, \dots, V_n\}$  of vertices, representing variables of interest, and a set  $E$  of directed edges, or arrows, that connect these vertices. The interpretation of such a graph has two components, probabilistic and causal. The probabilistic interpretation views the arrows as representing probabilistic dependencies among the corresponding variables, and the missing arrows as representing conditional independence assertions: Each variable is independent of all its non-descendants given its direct parents in the graph.<sup>3</sup> These assumptions amount to asserting that the joint probability function  $P(v) = P(v_1, \dots, v_n)$  factorizes according to the product

$$P(v) = \prod_i P(v_i | pa_i) \quad (2)$$

where  $pa_i$  are (values of) the parents of variable  $V_i$  in the graph.<sup>4</sup>

The causal interpretation views the arrows as representing causal influences between the corresponding variables. In this interpretation, the factorization of (2) still holds, but the factors are further assumed to represent autonomous data-generation processes, that is, each conditional probability  $P(v_i | pa_i)$  represents a stochastic process by which the values of  $V_i$  are chosen in response to the values  $pa_i$  (previously chosen for  $V_i$ 's parents), and the stochastic variation of this assignment is assumed independent of the variations in all other assignments. Moreover, each assignment process remains invariant to possible changes in the assignment processes that govern other variables in the system. This modularity assumption enables us to predict the effects of interventions, whenever interventions are described as specific modifications of some factors in the product of (2). The simplest such intervention involves fixing a set  $T$  of variables to some constants  $T = t$ , which yields the post-intervention distribution

$$P_t(v) = \begin{cases} \prod_{\{i | V_i \notin T\}} P(v_i | pa_i) & v \text{ consistent with } t. \\ 0 & v \text{ inconsistent with } t. \end{cases} \quad (3)$$

Eq. (3) represents a truncated factorization of (2), with factors corresponding to the manipulated variables removed. This truncation follows immediately from (2) since, assuming modularity, the post-intervention probabilities  $P(v_i | pa_i)$

<sup>3</sup>We use family relationships such as “parents,” “children,” “ancestors,” and “descendants,” to describe the obvious graphical relationships. For example, the parents  $PA_i$  of node  $V_i$  are the set of nodes that are directly connected to  $V_i$  via arrows pointing to  $V_i$ .

<sup>4</sup>We use uppercase letters to represent variables or sets of variables, and use corresponding lowercase letters to represent their values (instantiations).

corresponding to variables in  $T$  are either 1 or 0, while those corresponding to unmanipulated variables remain unaltered.<sup>5</sup> If  $T$  stands for a set of treatment variables and  $Y$  for an outcome variable in  $V \setminus T$ , then Eq. (3) permits us to calculate the probability  $P_t(y)$  that event  $Y = y$  would occur if treatment condition  $T = t$  were enforced uniformly over the population. This quantity, often called the *causal effect* of  $T$  on  $Y$ , is what we normally assess in a controlled experiment with  $T$  randomized, in which the distribution of  $Y$  is estimated for each level  $t$  of  $T$ .

We see from Eq. (3) that the model needed for predicting the effect of interventions requires the specification of three elements

$$M = \langle V, G, P(v_i | pa_i) \rangle$$

where (i)  $V = \{V_1, \dots, V_n\}$  is a set of variables, (ii)  $G$  is a directed acyclic graph with nodes corresponding to the elements of  $V$ , and (iii)  $P(v_i | pa_i)$ ,  $i = 1, \dots, n$ , is the conditional probability of variable  $V_i$  given its parents in  $G$ . Since  $P(v_i | pa_i)$  is estimable from nonexperimental data whenever  $V_i$  is observed, we see that, given the causal graph  $G$ , all causal effects are estimable from the data as well.<sup>6</sup>

Our ability to estimate  $P_t(v)$  from nonexperimental data is severely curtailed when some variables in a Markovian model are unobserved, or, equivalently, if two or more variables in  $V$  are affected by unobserved confounders; the presence of such confounders would not permit the decomposition in (2). Let  $V$  and  $U$  stand for the sets of observed and unobserved variables, respectively. Assuming that no  $U$  variable is a descendant of any  $V$  variable (called a *semi-Markovian model*), the observed probability distribution,  $P(v)$ , becomes a mixture of products:

$$P(v) = \sum_u \prod_i P(v_i | pa_i, u^i) P(u) \quad (4)$$

where  $pa_i$  and  $u^i$  stand for the sets of the observed and unobserved parents of  $V_i$ , and the summation ranges over all the  $U$  variables. The post-intervention distribution, likewise, will be given as a mixture of truncated products

$$P_t(v) = \begin{cases} \sum_u \prod_{\{i | V_i \notin T\}} P(v_i | pa_i, u^i) P(u) & v \text{ consistent with } t. \\ 0 & v \text{ inconsistent with } t. \end{cases} \quad (5)$$

and, the question of identifiability arises, i.e., whether it is possible to express  $P_t(v)$  as a function of the observed distribution  $P(v)$ .

Formally, our semi-Markovian model consists of five elements

$$M = \langle V, U, G_{VU}, P(v_i | pa_i, u^i), P(u) \rangle$$

<sup>5</sup>Eq. (3) was named “Manipulation Theorem” in (Spirtes, Glymour, & Scheines 1993), and is also implicit in Robins’ (1987)  $G$ -computation formula.

<sup>6</sup>It is in fact enough that the parents of each variable in  $T$  be observed (Pearl 2000, p. 78).

where  $G_{VU}$  is a causal graph consisting of variables in  $V \times U$ . Clearly, given  $M$  and any two sets  $T$  and  $S$  in  $V$ ,  $P_t(s)$  can be determined unambiguously using (5). The question of identifiability is whether a given causal effect  $P_t(s)$  can be determined uniquely from the distribution  $P(v)$  of the observed variables, and is thus independent of the unknown quantities,  $P(u)$  and  $P(v_i|pa_i, u^i)$ , that involve elements of  $U$ .

In order to analyze questions of identifiability, it is convenient to represent our modeling assumptions in the form of a graph  $G$  that does not show the elements of  $U$  explicitly but, instead, represents the confounding effects of  $U$  using bidirected edges. A bidirected edge between nodes  $V_i$  and  $V_j$  represents the presence (in  $G_{VU}$ ) of a divergent path  $V_i \leftarrow U_k \rightarrow V_j$  going strictly through elements of  $U$ . The presence of such bidirected edges in  $G$  represents unmeasured factors (or confounders) that may influence two variables in  $V$ ; we assume that substantive knowledge permits us to decide if such confounders can be ruled out from the model. See Figure 1 for an example graph with bidirected edges.

**Definition 1 (Causal-Effect Identifiability)** *The causal effect of a set of variables  $T$  on a disjoint set of variables  $S$  is said to be identifiable from a graph  $G$  if the quantity  $P_t(s)$  can be computed uniquely from any positive probability of the observed variables—that is, if  $P_t^{M_1}(s) = P_t^{M_2}(s)$  for every pair of models  $M_1$  and  $M_2$  with  $P^{M_1}(v) = P^{M_2}(v) > 0$  and  $G(M_1) = G(M_2) = G$ .*

In other words, the quantity  $P_t(s)$  can be determined from the observed distribution  $P(v)$  alone; the details of  $M$  are irrelevant.

### The Identification of $P_x(v)$

Let  $X$  be a singleton variable. In this section we study the problem of identifying the causal effect of  $X$  on  $V' = V \setminus \{X\}$ , (namely, on all other variables in  $V$ ), a quantity denoted by  $P_x(v)$ .

#### The easiest case

**Theorem 1** *If there is no bidirected edge connected to  $X$ , then  $P_x(v)$  is identifiable and is given by*

$$P_x(v) = P(v|x, pa_x)P(pa_x) \quad (6)$$

*Proof:* Since there is no bidirected edge connected to  $X$ , then the term  $P(x|pa_x, u^x) = P(x|pa_x)$  in Eq. (4) can be moved ahead of the summation, giving

$$\begin{aligned} P(v) &= P(x|pa_x) \sum_u \prod_{\{i|V_i \neq X\}} P(v_i|pa_i, u^i) P(u) \\ &= P(x|pa_x) P_x(v). \end{aligned} \quad (7)$$

Hence,

$$P_x(v) = P(v)/P(x|pa_x) = P(v|x, pa_x)P(pa_x). \quad (8)$$

□

Theorem 1 also follows from Theorem 3.2.5 of (Pearl 2000) which states that for any disjoint sets  $S$  and  $T$  in a Markovian model  $M$ , if the parents of  $T$  are measured, then  $P_t(s)$  is identifiable.

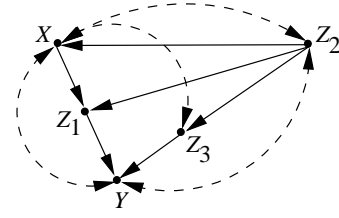


Figure 1:

#### A more interesting case

The case where there is no bidirected edge connected to any child of  $X$  is also easy to handle. Letting  $Ch_x$  denote the set of  $X$ 's children, we have the following theorem.

**Theorem 2** *If there is no bidirected edge connected to any child of  $X$ , then  $P_x(v)$  is identifiable and is given by*

$$P_x(v) = \left( \prod_{\{i|V_i \in Ch_x\}} P(v_i|pa_i) \right) \sum_x \frac{P(v)}{\prod_{\{i|V_i \in Ch_x\}} P(v_i|pa_i)} \quad (9)$$

*Proof:* Let  $S = V \setminus (Ch_x \cup \{X\})$  and  $A = \prod_{\{i|V_i \in S\}} P(v_i|pa_i, u^i)$ . Since there is no bidirected edge connected to any child of  $X$ , the factors corresponding to the variables in  $Ch_x$  can be moved ahead of the summation in Eqs. (4) and (5). We have

$$P(v) = \left( \prod_{\{i|V_i \in Ch_x\}} P(v_i|pa_i) \right) \sum_u P(x|pa_x, u^x) \cdot A \cdot P(u), \quad (10)$$

and

$$P_x(v) = \left( \prod_{\{i|V_i \in Ch_x\}} P(v_i|pa_i) \right) \sum_u A \cdot P(u). \quad (11)$$

The variable  $X$  does not appear in the factors of  $A$ , hence we augment  $A$  with the term  $\sum_x P(x|pa_x, u^x) = 1$ , and write

$$\begin{aligned} \sum_u A \cdot P(u) &= \sum_x \sum_u P(x|pa_x, u^x) \cdot A \cdot P(u) \\ &= \sum_x \frac{P(v)}{\prod_{\{i|V_i \in Ch_x\}} P(v_i|pa_i)}. \end{aligned} \quad (\text{by (10)}) \quad (12)$$

Substituting this expression into Eq. (11) leads to Eq. (9). □

The usefulness of Theorem 2 can be demonstrated in the model of Figure 1. Although the diagram is quite complicated, Theorem 2 is applicable, and readily gives

$$\begin{aligned} P_x(z_1, z_2, z_3, y) &= P(z_1|x, z_2) \sum_{x'} \frac{P(x', z_1, z_2, z_3, y)}{P(z_1|x', z_2)} \\ &= P(z_1|x, z_2) \sum_{x'} P(y, z_3|x', z_1, z_2) P(x', z_2). \end{aligned} \quad (13)$$

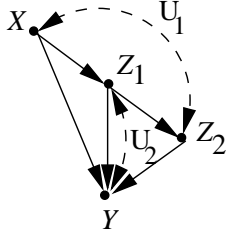


Figure 2:

### The general case

When there are bidirected edges connected to the children of  $X$ , it may still be possible to identify  $P_x(v)$ . To illustrate, consider the graph in Figure 2, for which we have

$$P(v) = \sum_{u_1} P(x|u_1)P(z_2|z_1, u_1)P(u_1) \cdot \sum_{u_2} P(z_1|x, u_2)P(y|x, z_1, z_2, u_2)P(u_2), \quad (14)$$

and

$$P_x(v) = \sum_{u_1} P(z_2|z_1, u_1)P(u_1) \cdot \sum_{u_2} P(z_1|x, u_2)P(y|x, z_1, z_2, u_2)P(u_2). \quad (15)$$

Let

$$Q_1 = \sum_{u_1} P(x|u_1)P(z_2|z_1, u_1)P(u_1), \quad (16)$$

and

$$Q_2 = \sum_{u_2} P(z_1|x, u_2)P(y|x, z_1, z_2, u_2)P(u_2). \quad (17)$$

Eq. (14) can then be written as

$$P(v) = Q_1 \cdot Q_2, \quad (18)$$

and Eq. (15) as

$$P_x(v) = Q_2 \sum_x Q_1. \quad (19)$$

Thus, if  $Q_1$  and  $Q_2$  can be computed from  $P(v)$ , then  $P_x(v)$  is identifiable and given by Eq. (19). In fact, it is enough to show that  $Q_1$  can be computed from  $P(v)$  (i.e., identifiable);  $Q_2$  would then be given by  $P(v)/Q_1$ . To show that  $Q_1$  can indeed be obtained from  $P(v)$ , we sum both sides of Eq. (14) over  $y$ , and get

$$P(x, z_1, z_2) = Q_1 \cdot \sum_{u_2} P(z_1|x, u_2)P(u_2). \quad (20)$$

Summing both sides of (20) over  $z_2$ , we get

$$P(x, z_1) = P(x) \sum_{u_2} P(z_1|x, u_2)P(u_2), \quad (21)$$

hence,

$$\sum_{u_2} P(z_1|x, u_2)P(u_2) = P(z_1|x). \quad (22)$$

From Eqs. (22) and (20),

$$Q_1 = P(x, z_1, z_2)/P(z_1|x) = P(z_2|x, z_1)P(x), \quad (23)$$

and from Eq. (18),

$$Q_2 = P(v)/Q_1 = P(y|x, z_1, z_2)P(z_1|x). \quad (24)$$

Finally, from Eq. (19), we obtain

$$P_x(v) = P(y|x, z_1, z_2)P(z_1|x) \sum_{x'} P(z_2|x', z_1)P(x'). \quad (25)$$

From the preceding example, we see that because the two bidirected arcs in Figure 2 do not share a common node, the set of factors (of  $P(v)$ ) containing  $U_1$  is disjoint of those containing  $U_2$ , and  $P(v)$  can be decomposed into a product of two terms, each being a summation of products. This decomposition, to be treated next, plays an important role in the general identifiability problem.

**C-components** Let a path composed entirely of bidirected edges be called a *bidirected path*. The set of variables  $V$  can be partitioned into disjoint groups by assigning two variables to the same group if and only if they are connected by a bidirected path. Assume that  $V$  is thus partitioned into  $k$  groups  $S_1, \dots, S_k$ , and denote by  $N_j$  the set of  $U$  variables that are parents of those variables in  $S_j$ . Clearly, the sets  $N_1, \dots, N_k$  form a partition of  $U$ . Define

$$Q_j = \sum_{n_j} \prod_{\{i|V_i \in S_j\}} P(v_i|pa_i, u^i)P(n_j), \quad j = 1, \dots, k. \quad (26)$$

The disjointness of  $N_1, \dots, N_k$  implies that  $P(v)$  can be decomposed into a product of  $Q_j$ 's:

$$P(v) = \prod_{j=1}^k Q_j. \quad (27)$$

We will call each  $S_j$  a *c-component* (abbreviating “confounded component”) of  $V$  in  $G$  or a *c-component* of  $G$ , and  $Q_j$  the *c-factor* corresponding to the c-component  $S_j$ . For example, in the model of Figure 2,  $V$  is partitioned into the c-components  $S_1 = \{X, Z_2\}$  and  $S_2 = \{Z_1, Y\}$ , the corresponding c-factors are given in equations (16) and (17), and  $P(v)$  is decomposed into a product of c-factors as in (18).

Let  $Pa(S)$  denote the union of a set  $S$  and the set of parents of  $S$ , that is,  $Pa(S) = S \cup (\cup_{V_i \in S} Pa_i)$ . We see that  $Q_j$  is a function of  $Pa(S_j)$ . Moreover, each  $Q_j$  can be interpreted as the post-intervention distribution of the variables in  $S_j$ , under an intervention that sets all other variables to constants, or

$$Q_j = P_{v \setminus s_j}(s_j) \quad (28)$$

The importance of the c-factors stems from that all c-factors are identifiable, as shown in the following lemma.

**Lemma 1** Let a topological order over  $V$  be  $V_1 < \dots < V_n$ , and let  $V^{(i)} = \{V_1, \dots, V_i\}$ ,  $i = 1, \dots, n$ , and  $V^{(0)} = \emptyset$ . For any set  $C$ , let  $G_C$  denote the subgraph of  $G$  composed only of variables in  $C$ . Then

(i) Each c-factor  $Q_j$ ,  $j = 1, \dots, k$ , is identifiable and is given by

$$Q_j = \prod_{\{i|V_i \in S_j\}} P(v_i|v^{(i-1)}). \quad (29)$$

(ii) Each factor  $P(v_i|v^{(i-1)})$  can be expressed as

$$P(v_i|v^{(i-1)}) = P(v_i|pa(T_i) \setminus \{v_i\}), \quad (30)$$

where  $T_i$  is the c-component of  $G_{V^{(i)}}$  that contains  $V_i$ .

*Proof:* We prove (i) and (ii) simultaneously by induction on the number of variables  $n$ .

Base:  $n = 1$ ; we have one c-component  $Q_1 = P(v_1)$ , which is identifiable and is given by Eq. (29), and Eq. (30) is satisfied.

Hypothesis: When there are  $n$  variables, all c-factors are identifiable and are given by Eq. (29), and Eq. (30) holds for all  $V_i \in V$ .

Induction step: When there are  $n + 1$  variables in  $V$ , assuming that  $V$  is partitioned into c-components  $S_1, \dots, S_l, S'$ , with corresponding c-factors  $Q_1, \dots, Q_l, Q'$ , and that  $V_{n+1} \in S'$ , we have

$$P(v) = Q' \prod_i Q_i. \quad (31)$$

Summing both sides of (31) over  $v_{n+1}$  leads to

$$P(v^{(n)}) = \left( \sum_{v_{n+1}} Q' \right) \prod_i Q_i. \quad (32)$$

It is clear that each  $S_i, i = 1, \dots, l$ , is a c-component of  $G_{V^{(n)}}$ . By the induction hypothesis, each  $Q_i, i = 1, \dots, l$ , is identifiable and is given by Eq. (29). From Eq. (31),  $Q'$  is identifiable as well, and is given by

$$Q' = \frac{P(v)}{\prod_i Q_i} = \prod_{\{i|V_i \in S'\}} P(v_i|v^{(i-1)}), \quad (33)$$

which is clear from Eq. (29) and the chain decomposition  $P(v) = \prod_i P(v_i|v^{(i-1)})$ .

By the induction hypothesis, Eq. (30) holds for  $i$  from 1 to  $n$ . Next we prove that it holds for  $V_{n+1}$ . In Eq. (33),  $Q'$  is a function of  $Pa(S')$ , and each term  $P(v_i|v^{(i-1)})$ ,  $V_i \in S'$  and  $V_i \neq V_{n+1}$ , is a function of  $Pa(T_i)$  by Eq. (30), where  $T_i$  is a c-component of the graph  $G_{V^{(i)}}$  and therefore is a subset of  $S'$ . Hence we obtain that  $P(v_{n+1}|v^{(n)})$  is a function only of  $Pa(S')$  and is independent of  $C = V \setminus Pa(S')$ , which leads to

$$\begin{aligned} & P(v_{n+1}|pa(S') \setminus \{v_{n+1}\}) \\ &= \sum_c P(v_{n+1}|v^{(n)}) P(c|pa(S') \setminus \{v_{n+1}\}) \\ &= P(v_{n+1}|v^{(n)}) \sum_c P(c|pa(S') \setminus \{v_{n+1}\}) \\ &= P(v_{n+1}|v^{(n)}) \end{aligned} \quad (34)$$

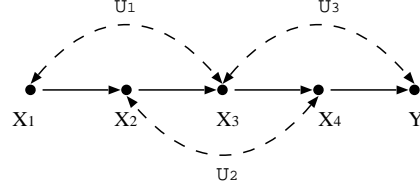


Figure 3:

The proposition (ii) in Lemma 1 can also be proved by using d-separation criterion (Pearl 1988) to show that  $V_i$  is independent of  $V^{(i)} \setminus Pa(T_i)$  given  $Pa(T_i) \setminus \{V_i\}$ .

We show the use of Lemma 1 by an example shown in Figure 3, which has two c-components  $S_1 = \{X_2, X_4\}$  and  $S_2 = \{X_1, X_3, Y\}$ .  $P(v)$  decomposes into

$$P(x_1, x_2, x_3, x_4, y) = Q_1 Q_2, \quad (35)$$

where

$$Q_1 = \sum_{u_2} P(x_2|x_1, u_2) P(x_4|x_3, u_2) P(u_2), \quad (36)$$

$$Q_2 = \sum_{u_1, u_3} P(x_1|u_1) P(x_3|x_2, u_1, u_3) P(y|x_4, u_3) \cdot P(u_1) P(u_3). \quad (37)$$

By Lemma 1, both  $Q_1$  and  $Q_2$  are identifiable. The only admissible order of variables is  $X_1 < X_2 < X_3 < X_4 < Y$ , and Eq. (29) gives

$$Q_1 = P(x_4|x_1, x_2, x_3) P(x_2|x_1), \quad (38)$$

$$Q_2 = P(y|x_1, x_2, x_3, x_4) P(x_3|x_1, x_2) P(x_1). \quad (39)$$

We can also check that the expressions obtained in Eq.s (23) and (24) for Figure 2 satisfy Lemma 1.

**The identification criterion for  $P_x(v)$**  Let  $X$  belong to the c-component  $S^X$  with corresponding c-factor  $Q^X$ . Let  $Q_x^X$  denote the c-factor  $Q^X$  with the term  $P(x|pa_x, u^x)$  removed, that is,

$$Q_x^X = \sum_{n^X} \prod_{\{i|V_i \neq X, V_i \in S^X\}} P(v_i|pa_i, u^i) P(n^X). \quad (40)$$

We have

$$P(v) = Q^X \prod_i Q_i, \quad (41)$$

$$P_x(v) = Q_x^X \prod_i Q_i. \quad (42)$$

Since all  $Q_i$ 's are identifiable,  $P_x(v)$  is identifiable if and only if  $Q_x^X$  is identifiable, and we have the following theorem.

**Theorem 3**  $P_x(v)$  is identifiable if and only if there is no bidirected path connecting  $X$  to any of its children. When

$P_x(v)$  is identifiable, it is given by

$$P_x(v) = \frac{P(v)}{Q^X} \sum_x Q^X, \quad (43)$$

where  $Q^X$  is the c-factor corresponding to the c-component  $S^X$  that contains  $X$ .

*Proof: (if)* If there is no bidirected path connecting  $X$  to any of its children, then none of  $X$ 's children is in  $S^X$ . Under this condition, removing the term  $P(x|pa_x, u^x)$  from  $Q^X$  is equivalent to summing  $Q^X$  over  $X$ , and we can write

$$Q_x^X = \sum_x Q^X. \quad (44)$$

Hence from Eq.s (42) and (41), we obtain

$$P_x(v) = \left( \sum_x Q^X \right) \prod_i Q_i = \left( \sum_x Q^X \right) \frac{P(v)}{Q^X}, \quad (45)$$

which proves the identifiability of  $P_x(v)$ .

**(only if)** Sketch: Assuming that there is a bidirected path connecting  $X$  to a child of  $X$ , one can construct two models (by specifying all conditional probabilities) such that  $P(v)$  has the same values in both models while  $P_x(v)$  takes different values. The proof is lengthy and is given in (Tian & Pearl 2002).  $\square$

We demonstrate the use of Theorem 3 by identifying  $P_{x_1}(x_2, x_3, x_4, y)$  in Figure 3. The graph has two c-components  $S_1 = \{X_2, X_4\}$  and  $S_2 = \{X_1, X_3, Y\}$ , with corresponding c-factors given in (38) and (39). Since  $X_1$  is in  $S_2$  and its child  $X_2$  is not in  $S_2$ , Theorem 3 ensures that  $P_{x_1}(x_2, x_3, x_4, y)$  is identifiable and is given by

$$\begin{aligned} P_{x_1}(x_2, x_3, x_4, y) &= Q_1 \sum_{x_1} Q_2 \\ &= P(x_4|x_1, x_2, x_3)P(x_2|x_1) \\ &\quad \sum_{x'_1} P(y|x'_1, x_2, x_3, x_4)P(x_3|x'_1, x_2)P(x'_1). \end{aligned} \quad (46)$$

### A Criterion for Identifying $P_x(s)$

Let  $X$  be a singleton variable and  $S \subseteq V$  be any set of variables. Clearly, whenever  $P_x(v)$  is identifiable, so is  $P_x(s)$ . However, there are obvious cases where  $P_x(v)$  is not identifiable and still  $P_x(s)$  is identifiable for some subsets  $S$  of  $V$ . In this section we give a criterion for identifying  $P_x(s)$ .

Let  $An(S)$  denote the union of a set  $S$  and the set of ancestors of the variables in  $S$ , and let  $G_{An(S)}$  denote the subgraph of  $G$  composed only of variables in  $An(S)$ . Summing both sides of Eq. (4) over  $V \setminus An(S)$ , we have that the marginal distribution  $P(an(S))$  decomposes exactly according to the graph  $G_{An(S)}$ . Therefore, if  $P_x(s)$  is identifiable in  $G_{An(S)}$ , then it is computable from  $P(an(S))$ , and thus is computable from  $P(v)$ . A direct extension of Theorem 3 then leads to the following sufficient criterion for identifying  $P_x(s)$ .

**Theorem 4**  $P_x(s)$  is identifiable if there is no bidirected path connecting  $X$  to any of its children in  $G_{An(S)}$ .

When the condition in Theorem 4 is satisfied, we can compute  $P_x(an(S))$  by applying Theorem 3 in  $G_{An(S)}$ , and  $P_x(s)$  can be obtained by marginalizing over  $P_x(an(S))$ .

This simple criterion can classify correctly all the examples treated in the literature with  $X$  singleton, including those contrived by (Galles & Pearl 1995). In fact, for  $X$  and  $S$  being singletons, it is shown in the Appendix that if there is a bidirected path connecting  $X$  to one of its children such that every node on the path is in  $An(S)$ , then none of the “back-door”, “front-door”, and (Galles & Pearl 1995) criteria is applicable. However, this criterion is *not necessary* for identifying  $P_x(s)$ . Examples exist in which  $P_x(s)$  is identifiable but Theorem 4 is not applicable.<sup>7</sup> An improved criterion that covers those cases is described in (Tian & Pearl 2002).

## Conclusion

We developed new graphical criteria for identifying the causal effects of a singleton variable on a set of variables. Theorem 4 has important ramifications to the theory and practice of observational studies. It implies that the key to identifiability lies not in blocking back-door paths between  $X$  and  $S$  but, rather, in blocking back-door paths between  $X$  and its immediate successors on the pathways to  $S$ . The potential of finding and measuring intermediate variables that satisfy this condition opens new vistas in experimental design.

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## Appendix

In this appendix we show that Theorem 4 covers the criterion in (Galles & Pearl 1995) (which will be called the G-P criterion). The G-P criterion is for identifying  $P_x(y)$  with  $X$  and  $Y$  being singletons, and it includes the “front-door” and “back-door” criteria as special cases (see (Pearl 2000, pp. 114-8)). We will prove that if there is a bidirected path connecting  $X$  to one of its children such that every node on the path is an ancestor of  $Y$ , then the G-P criterion is not applicable. There are four conditions in the G-P criterion, among which Condition 1 is a special case of Condition 3, and Condition 2 is trivial. Therefore we only need to consider Condition 3 and 4.

*Proof:* Assume that there is a bidirected path  $p$  from  $X$  to its child  $Y_1$  such that every node on  $p$  is an ancestor of  $Y$ , and that there is a directed path  $q$  from  $Y_1$  to  $Y$ . We will show by contradiction that neither Condition 3 nor Condition 4 is applicable for identifying  $P_x(y)$ . For any set  $Z$ , a node will be called  $Z$ -active if it is in  $Z$  or any of its descendants is in  $Z$ , otherwise it will be called  $Z$ -inactive.

**(Condition 3)** Assume that there exists a set  $Z$  that blocks all back-door paths from  $X$  to  $Y$  so that  $P_x(z)$  is identifiable.

<sup>7</sup>This implies that, contrary to claims, the criterion developed in (Galles & Pearl 1995) is *not* complete.

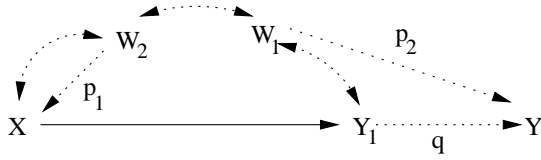


Figure 4:

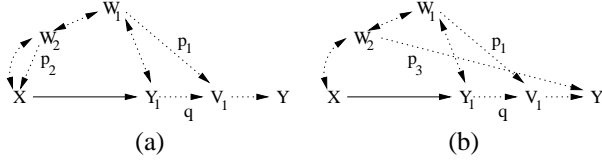


Figure 5:

able.<sup>8</sup> If every internal node on  $p$  is an ancestor of  $X$ , or if every nonancestor of  $X$  on  $p$  is  $Z$ -active, then let  $W_1 = Y_1$ , otherwise let  $W_1$  be the  $Z$ -inactive non-ancestor of  $X$  that is closest to  $X$  on  $p$  (see Figure 4). If every internal node on the subpath  $p(W_1, X)$ <sup>9</sup> is  $Z$ -active, then let  $W_2 = X$ , otherwise let  $W_2$  be the  $Z$ -inactive node that is closest to  $W_1$  on  $p(W_1, X)$ . From the definition of  $W_1$  and  $W_2$ ,  $W_2$  must be an ancestor of  $X$  (or be  $X$  itself), and let  $p_1$  be any directed path from  $W_2$  to  $X$ . (i) If  $W_1 \neq Y_1$ , letting  $p_2$  be any directed path from  $W_1$  to  $Y$ , then from the definition of  $W_1$  and  $W_2$  the path  $p' = (p_1(X, W_2), p(W_2, W_1), p_2(W_1, Y))$  is a back-door path from  $X$  to  $Y$  that is not blocked by  $Z$  (see Figure 4) since  $W_2$  is  $Z$ -inactive, all internal nodes on  $p(W_2, W_1)$  is  $Z$ -active, and  $W_1$  is  $Z$ -inactive. (ii) If  $W_1 = Y_1$ , there are two situations:

(a)  $Z$  consists entirely of nondescendants of  $X$ . Then the path  $p'' = (p_1(X, W_2), p(W_2, Y_1), q(Y_1, Y))$  is a back-door path from  $X$  to  $Y$  that is not blocked by  $Z$ .

(b)  $Z$  contains a variable  $Y'$  on  $q(Y_1, Y)$  so that  $P_x(z)$  is identifiable. By the definition of  $W_1$ , every node on  $p$  is an ancestor of  $Z$ .  $P_x(z)$  can not be identified by Theorem 4, and the G-P criterion is not applicable for identifying  $P_x(z)$  if  $Z$  contains more than one variable. If  $Z$  contains only one variable  $Y'$ , then every node on  $p$  is an ancestor of  $Y'$ . If  $P_x(y')$  is identifiable by Condition 3 of the G-P criterion (Condition 4 is not applicable as proved later), then from the preceding analysis there is a  $Y''$  on the path  $q(Y_1, Y')$  such that every node on  $p$  is an ancestor of  $Y''$  and  $P_x(y'')$  is identifiable. By induction, in the end we have every node on  $p$  is an ancestor of  $Y_1$  and  $P_x(y_1)$  is identifiable, which does not hold from the preceding analysis.

**(Condition 4)** Assume that there exist sets  $Z_1$  and  $Z_2$  that satisfy all (i)–(iv) conditions in Condition 4. Since  $Z_1$  has to block the path  $((X, Y_1), q(Y_1, Y))$ , let  $V_1$  be the variable in  $Z_1$  that is closest to  $Y_1$  on the path  $q$  (see Figure 5(a)). If none of the internal node on  $p$  is in  $An(V_1) \setminus An(X)$  (the set of ancestors of  $V_1$  that are not ancestors of  $X$ ) or if every

variable in  $An(V_1) \setminus An(X)$  on  $p$  is  $Z_2$ -active, then let  $W_1 = Y_1$ , otherwise let  $W_1$  be the  $Z_2$ -inactive variable in  $An(V_1) \setminus An(X)$  that is closest to  $X$  on  $p$ . Let  $p_1$  be any directed path from  $W_1$  to  $V_1$ . If every internal node on the subpath  $p(W_1, X)$  is  $Z_2$ -active, then let  $W_2 = X$ , otherwise let  $W_2$  be the  $Z_2$ -inactive node that is closest to  $W_1$  on  $p(W_1, X)$ . Since  $W_2$  must be an ancestor of  $Y$ , from the definition of  $W_1$  and  $W_2$ , there are two possible situations:

(a)  $W_2$  is an ancestor of  $X$  or  $W_2 = X$ . Let  $p_2$  be any directed path from  $W_2$  to  $X$  (see Figure 5(a)). From the definition of  $W_1$  and  $W_2$ , the path  $p' = (p_2(X, W_2), p(W_2, W_1), p_1(W_1, V_1))$  is a back-door path from  $X$  to  $V_1 \in Z_1$  that is not blocked by  $Z_2$  that does not contain any descendant of  $X$  (see Figure 5(a)).

(b)  $W_2$  is an ancestor of  $Y$  but not ancestor of  $V_1$  ( $W_2 \in An(Y) \setminus An(V_1)$ ). Let  $p_3$  be any directed path from  $W_2$  to  $Y$  (see Figure 5(b)). From the definition of  $W_1$  and  $W_2$ , the path  $p'' = (p_1(V_1, W_1), p(W_1, W_2), p_3(W_2, Y))$  is a back-door path from  $V_1 \in Z_1$  to  $Y$  that is not blocked by  $Z_2$  (see Figure 5(b)).  $\square$

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<sup>8</sup> A path from  $X$  to  $Y$  is said to be a *back-door* path if it contains an arrow into  $X$ .

<sup>9</sup> We use  $p(W_1, X)$  to represent the subpath of  $p$  from  $W_1$  to  $X$ .