# Modern Portfolio Theory and Black-Scholes Option Price

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***Abstract*— In this report, we will explore two important areas in the financial world. The first is modern portfolio theory, which provides a framework for portfolio designs. The second is Black-Scholes model, which propose a model of the relationship between option prices and other economic variables. We will first introduce and implement the basic ideas and structures of these two models. After that, we will implement enhancements variations of these two models, with insightful discussion and reliable evaluations.**

## I. INTRODUCTION OF MODERN PORTFOLIO THEORY

In finance market, a portfolio is a collection of investments. This collection of financial assets can be stocks, equity or cash. Components of a portfolio are chosen according to the investor's risk tolerance and financial objectives. The principle of designing a portfolio is to maximize the expected return while minimizing the risk.

Modern portfolio theory (MPT), or mean-variance analysis is introduced by the economist Harry Markowitz in 1952. He proposes a mathematical model for portfolio optimization, which introduces the idea of the efficient frontier. The efficient frontier is made up of a set of portfolios which yields the highest expected return for a given level of risk. [1][2]

Modern portfolio theory believes that investors will only take risks if it is compensated by larger profits. This also implies that investors who pursue higher expected returns have to accept higher risks. However, different investors have different risk aversion characteristics and therefore different portfolio strategies. Mean-variance analysis model or (MPT) has the following assumptions:

### A. mean-variance portfolio optimization model

- $r_i(t)$ is return on asset i at time t.
- Assume return $r_i(t)$ as a Gaussian random variable, with mean $\mu_i$ and variance $\sigma_i^2$
- Mean in return $r_i(t)$ is viewed as expected gain
- Variance in return $r_i(t)$ is viewed as risk
- Returns of different assets are correlated
- A portfolio invests in N assets with relative weights $\pi_i$
- Return on the portfolio is: $r_p = \sum_N^{i=1} \pi_i r_i$
- Returns on the portfolio has a multivariate Gaussian distribution $r_p \sim (\pi^T \mu, \pi^T \Sigma \pi)$

Under these conditions the efficient frontier can be obtained by solving a linear problem:

$$\max_{\pi} \pi^T \mu \quad subject \quad to \quad \pi^T \Sigma \pi = \mu_0 \tag{1}$$

For a given return, in order to minimum risk we have:

$$\max_{\pi} \pi^T \Sigma \pi \quad subject \quad to \quad \pi^T \mu = r_0 \tag{2}$$

We will continue to use these denotes specified above during the rest of this report.

Despite the theoretical importance of MPT, it has been criticized for the model's inconsistency with the real world.[3]

In mean-variance model, the return, risk, and correlation are mathematical statements about the future (based on expected values). However, the expected values derived from historical data cannot take account of new circumstances that did not exist in the past.[4]

Another criticism points out that the fluctuation of stock prices does not always behave in a normally distributed manner and its covariances are not always constant. This means that key parameters like the variance-covariance matrix needs to be estimated. And an accurate estimation of this matrix is paramount. Therefore, to utilize Monte-Carlo simulation with the Gaussian copula and well-specified marginal distributions in the modeling process are important and necessary.[5]

These limitations of the mean-variance model often lead to estimation errors of the model's key parameters. This results in many portfolios constructed using this method can be outperformed by a naive diversified model.

Since the first introduction of MPT, numerous work has been made to address some of these issues in order to improve the model. For instance, post-modern portfolio theory and Black-Litterman model optimization are two major extensions of the original theory.

## II. PORTFOLIO OPTIMIZATION

In this section, we will explore the mean-variance model and its developments. In task 1 to 3, we will implement different algorithms for the computing of the efficient frontier and discuss the difference between these two approaches. In task 4 to 7, we will apply these methods to real-life data and then implement their enhancements.

### A. Task 1 Drive Efficient Frontier from basic algebra

For two given assets with the expected returns m and covariance matrix c, as shown in equation 3.

$$m = \begin{bmatrix} 0.10 \\ 0.10 \end{bmatrix} C = \begin{bmatrix} 0.005 & 0.0 \\ 0.0 & 0.005 \end{bmatrix}$$

The efficient frontier is a straight line.This can be derived from equation 3.

$$\begin{aligned} For \quad \pi_1 + \pi_2 = 1, \quad \pi^T \mu = 0.1\pi_1 + 0.1\pi_2 = 0.1 \\ \pi^T \Sigma \pi = 0.005(2\pi^2 - 2\pi + 1) \end{aligned} \tag{3}$$

From the formula of the variance $\pi^T \Sigma \pi$ in equation 3, we know that the minimum variance can be achieved at $\pi = 0.5$.

Therefore, the optimal portfolio is to divide the investment evenly into the two assets. This can also be derived from the fact that the covariance matrix C suggests the two assets are uncorrelated and the return of any asset is the same. Hence, for the same return, we can minimize the risk by dividing the money evenly into two assets.

### *B. Task 2 Drive Efficient Frontier using MATLAB*

When given three assets, namely A, B and C, whose expected returns m and their corresponding covariances C are known, as shown in equation 4. It is clear from equation 4 that assets AB, BC are negatively correlated while assets AC are positively correlated.

$$m = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.15 \end{bmatrix} C = \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix}$$

MATLABs Financial Toolbox package[6] is used to draw the efficient portfolio frontier of the three asset model, and for the three two-asset portfolios. 100 random portfolios are also generated in the E V space, as shown in Fig. 1.

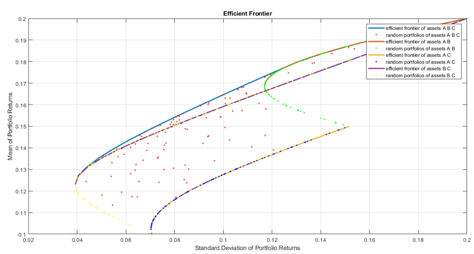


**Fig. 1: E-V space of efficient frontier and random portfolios of the three assets portfolios, and for the three two-asset portfolios.** For the three assets(ABC) portfolios, its efficient frontier is marked with blue and the random portfolios is marked with red. For the two assets(AB) portfolios, its efficient frontier is marked with deep red and the random portfolios is marked with green. For the two assets(AC) portfolios, its efficient frontier is marked with orange and the random portfolios is marked with deep blue. For the two assets(BC) portfolios, its efficient frontier is marked with purple and the random portfolios is marked with bright yellow.

As can be seen in Fig. 1, the efficient portfolio frontier of three assets ABC is always higher or equal to those three two-asset portfolios.Meaning for any given return, the three assets model always has a smaller risk(smaller variance). This phenomenon suggests that an more diverse investment leads to a lower volatility of the portfolio.

The efficient portfolio frontier of assets that are negatively correlated, namely AB and BC, outperform assets that are positively correlated (AC).This suggests negative correlations between two investments can be used in risk management to reduce the overall volatility of the portfolio. This strategy can be vital in times of high volatility.

Another intriguing phenomenon is that all random two assets portfolios are on the same quadratic curve as efficient frontier. This phenomenon can be explained by the fact that the degree of freedom in two assets portfolio is always one. When the weight of one asset is $\omega$, the weight of the other asset is $1-\omega$.

### *C. Task 3 Compare different algorithms*

In this section, we will compare three different algorithms for calculating efficient frontier. They are Portfolio function in MATLAB's financial toolbox, NaiveMV algorithm and NaiveMV algorithm using CVX.

The MATLAB's financial toolbox is a built-in toolbox from MATLAB. It is fast, stable and user-friendly.However, this also means that it is a black box for the user.

The NaiveMV algorithm is a better option for users who are interested in exploring the math behind Mean-Variance optimization. It can be viewed as a simplified version of frontcon function(now replaced by Portfolio function) provided by MATLAB.[7]

CVX package is a Matlab-based modeling system for convex optimization. CVX turns Matlab into a modeling language, allowing constraints and objectives to be specified using standard Matlab expression syntax.[8] [9] The advantage of CVX is its straightforward syntax and its great adaptability. To replace built-in functions of MATLAB(i.e. linprog and quadprog) with CVX functions in naiveMV, can help users to further understand the math in behind.

The results from these three algorithms are identical, as shown in Fig. 2. The differences between three algorithms fluctuate around $10^{-6}$.

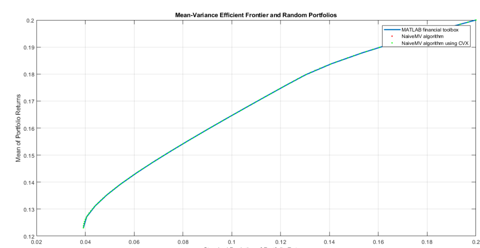


**Fig. 2: E-V space of efficient frontier using MATLAB's financial toolbox, NaiveMV algorithm and NaiveMV algorithm modified with CVX.** For the efficient frontier drew using the built-in financial toolbox is marked with blue color. For the efficient frontier drew using the NaiveMV algorithm is marked with blue color. For the efficient frontier drew using the NaiveMV algorithm modified with CVX is marked with blue color.

### *D. Task 4 mean-variance portfolio designing using FTSE 100*

In this section, we adopt three random stocks from FTSE 100 and use historical data to design an optimal mean-variance portfolio with the biggest Sharp ratio and a 1/N portfolio. Then we will evaluate and compare their out-of-sample performance. To obtain a reliable measure of the performance metrics, we repeat the weight optimization phase for many independent random stocks selections and average the performance metrics over these selections.

*1) The naive strategy design:* The naive strategy or 1/N strategy does not take any historical data into consideration nor performs any optimization. It holds a N risky assets portfolio with a weighting strategy of $\pi_i = 1/N$ in each asset. In task 4, a three assets portfolio has a set weight of $\pi_i = 1/3, (i = 1, 2, 3)$.

*2) The mean-variance portfolio design:* Daily data from 24th February 2015 to 24th February 2018 of FTSE 100 stocks are used in parameters estimation and performance testing.After obtaining the daily data of stocks in FTSE 100, stocks that have more than 5% value missing are removed. This missing of data can be caused by entering or leaving the FTSE 100 index within this three years period. 30 stocks are chosen from the remaining of FTSE 100 stocks.

Then we will calculate the daily return rate of each stocks. This can be formed as equation 4:

$$r_t = \frac{P_t - P_{t-1}}{P_t} \tag{4}$$

where $r_t$ is the daily return rate at time t, and $P_t$ is the stock price at time t.

Expected returns and covariance will be calculated using the daily close prices of these three stocks during the first one and a half year. The equations used to calculate expected returns and covariance are shown in equation 5 & 6.

$$\widehat{\mu} = \frac{1}{T}\sum_{T}^{t=1} r(t) \tag{5}$$

$$\widehat{\Sigma} = \frac{1}{T}\sum_{T}^{t=1}(r(t)-\widehat{\mu})(r(t)-\widehat{\mu})^T \tag{6}$$

Once we have the estimation of returns and covariance, we can use methods mentioned in task 2 and 3 to obtain a set of optimal portfolios. Although all portfolios on the efficient frontier are viable candidates, we can use Sharpe ratio to further examine their performance. The portfolio that has the biggest Sharp ratio will be chosen as test subject.

The Sharp ratio measures the excess return(over risk-free assets) per unit of deviation in an investment asset.[10] This metric tells us how well does one portfolio perform with the risk taken. Generally speaking, the bigger the sharp ratio is the greater the portfolio's risk-adjusted return will be. The ex-ante Sharpe ratio is :

$$s = \frac{mean(r-r_f))}{std(r-r_f)} \tag{7}$$

where r is asset's daily returns and $r_f$ is the risk free rate. The risk-free rate can be set as the interest rate of LIBOR, 3-Month Treasury Bill or 0%. In this task, we use 12 month LIBOR interest rate as our risk-free rate. The current 12 month LIBOR interest rate is -0.32971%. Therefore, the daily risk-free rate is:

$$r_f = \sqrt[256]{1-0.0032971} - 1 = -1.31 \times 10^{-5} \tag{8}$$

For every experiment, three random stocks will be selected from thirty stocks. Their expected returns, covariance and efficient frontier will be calculated using training data. The optimal portfolio that has the biggest sharp ratio will be selected as the testing portfolio. The test portfolio and simple 1/N portfolio will be tested on the test data set. Performance metrics such as the mean of daily return and variance, the Sharp ratio, and the maximum daily potential loss are used to assess the performance of different models.

A total of 1000 independent experiments were conducted. The box plots of the performance metrics of these experiments are shown in Fig. 3.

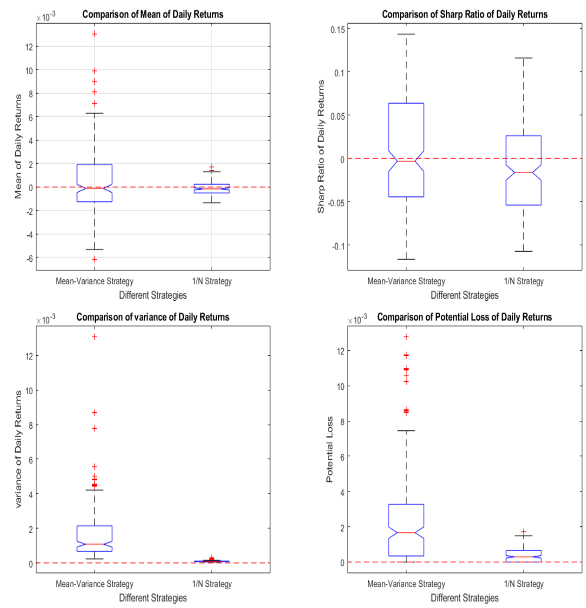


**Fig. 3: Box plot of the mean of daily return, the Sharp ratio, the variance and the value at risk**

The four panels in this figure show the fluctuation of performance metrics of the mean-variance strategy and the 1/N strategy in 1000 experiments. In every panel, the box shape on the left represents the mean-variance strategy, the box shape the right represents the 1/N strategy. The panels from top to bottom from left to right are the mean of daily return, the Sharp ratio, the variance and the value at risk respectively.

From Fig. 3 panels 1&2, we notice that the mean-variance portfolio strategy yields a slightly higher average mean of returns and Sharpe ratio but suffers from a more violent fluctuation compare with the naive strategy.

From Fig. 3 panels 3&4, we see a significantly higher variance and the value at risk was produced by the mean-variance portfolio strategy. This phenomenon again indicates that the mean-variance portfolio strategy is more unstable and risky than the naive strategy.

In conclusion, the mean-variance portfolio strategy might yield a slightly higher return, however, due to its volatility, this strategy is still inferior compared to the naive strategy.

### *E. Task 5 Shortsale-constrained portfolios*

In this section, we will discuss shortsale-constrained portfolios. In order to understand the effects of shortsale-constrained strategy, we will impose the constraint $\pi_i \geqslant 0$ on the weights in basic mean-variance portfolio strategy. This shortsale-constrained strategy will then be evaluated and compared to the basic mean-variance strategy, the naive strategy on the performance metrics used in previous section. To obtain a reliable measure of the performance metrics, the weight optimization phase will be repeated for a number of independent random stocks selections and average the performance metrics over these selections.

In the basic mean-variance portfolio strategy, we can rewrite equation 1 as the form of equation 9:

$$\max_{\pi} \pi^T \mu - \frac{\gamma}{2} \pi^T \Sigma \pi \tag{9}$$

where $\gamma$ can be viewed as the investors risk aversion.

When we impose the constrain $\pi_i \geqslant 0$ on the basic mean-variance portfolio strategy, we will obtain a Lagrangian, shown in equation 10:

$$L = \pi^T \mu - \frac{\gamma}{2} \pi^T \varepsilon \pi + \pi^T \lambda \tag{10}$$

where $\lambda$ is the $N1$ vector of Lagrange multipliers for the constraints on shortselling. Equation 10 can be rearranged into the form of $L = \pi^T \widetilde{\mu} - \frac{\gamma}{2} \pi^T \varepsilon \pi$, where $\widetilde{\mu} = \mu + \lambda$. The $\widetilde{\mu}$ can be interpreted as the adjusted mean of return vector. For asset that yields poor return $\lambda > 0$, which leads to $\widetilde{\mu} = \mu + \lambda \to 0$. Therefore short-selling constraint on a low return asset will present a shrinkage effect on the expected returns. Resulting in the expected return moves toward the average. [11]

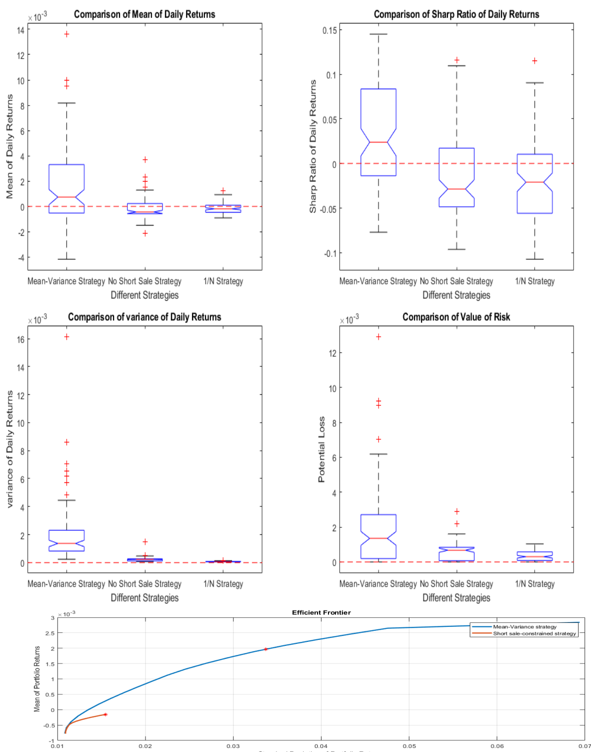


**Fig. 4: Box plot of the mean of daily return, the Sharp ratio, the variance and the value at risk. And efficient frontiers.**

The four panels in this figure show the fluctuation of performance metrics of the mean-variance strategy and the 1/N strategy in 1000 experiments. In panel 1-4, the box shape on the left represents the mean-variance strategy, the one in the middle represents the short sale-constrained strategy, the one on the right represents the 1/N strategy. The panels from top to bottom from left to right are the mean of daily return, the Sharp ratio, the variance and the value at risk respectively. The fifth panels is the efficient frontiers, in which the efficient frontier of mean-variance strategy marked in blue, the efficient frontier of the short sale-constrained strategy is marked in orange and the portfolio that has the biggest sharp ratio is marked as red star.

**TABLE I: Mean of performance metrics on out-of-sample data**

| Portfolio strategies ($10^{-3}$) | Mean-Variance | Shortsale-constrained | naive |
|---|---|---|---|
| in sample return | 2.4 | 0.43 | -0.22 |
| out-of-sample return | 1.2 | 0.13 | 0.08 |
| in sample Sharp ratio | 59.3 | 20.8 | -15.9 |
| out-of-sample Sharp ratio | 24.3 | -14.3 | -12.6 |
| in sample variance | 1.7 | 0.47 | 0.21 |
| out-of-sample variance | 1.8 | 0.24 | 0.08 |
| in sample VaR | 0.69 | 0.36 | 0.58 |
| out-of-sample VaR | 2.1 | 0.07 | 0.03 |

This table reports the out-of-sample daily return, the out-of-sample Sharp ratio, the out-of-sample variance and the out-of-sample Value at Risk for the Mean-Variance strategy, Shortsale-constrained strategy, and naive strategy.

From Fig. 4 panels 1 and table 1, we notice that in terms of the average mean of returns, the mean-variance portfolio strategy yields a slightly better result but suffers from a more violent fluctuation, while the 1/N strategy presents similar results with significantly smaller fluctuations.The short sale-constrained strategy produces the lowest average mean of returns and fluctuation.

From panels 2 and table 1, we notice that the mean-variance portfolio strategy yields a best Sharp ratio, the 1/N strategy presents similar results with a lower average Sharp ratio. Although the short sale-constrained strategy produces the lowest average sharp ratio, it is the most stable one among the three strategies.

From panels 3&4 and table 1, we witness a significantly higher variance and value at risk was produced by mean-variance portfolio strategy. Although the short sale-constrained strategy and the 1/N strategy presents similar results with significantly smaller average and fluctuations. It is worth noting that the 1/N strategy is still slightly better than the short sale-constrained strategy, in terms of stability and the value at risk.

In conclusion, compared to the mean-variance portfolio strategy, the short sale-constrained strategy is better.When compared to the naive strategy, however, the short sale-constrained strategy can provide a similar but still slightly inferior performance.

### F. *Task 6 Index tracking*

In this section, we will explore and compare different strategies of index tracking using the daily return rates of the FTSE 100. A total of 6 stocks will be selected from the 30 constituent used in previous sections. For the reliable measure of the performance of two strategies, a number of independent experiments will be conducted and the some performance metrics will be averaged.

In certain situations, investors want to get the same return as a stock index. However, it is often impractical and expensive to invest evenly to all stocks in one index. Therefore, we wish to find a subset of all the stocks in an index that can approximate the performance of that index. This problem is often referred as index tracking.

*1) Greedy forward selection:* The forward method begins with an empty set and adds features through iterations. In every iteration, the remaining unselected features will be evaluated respectively, the one feature that delivers the best performance will be selected into the new subset. Once a feature is added to the subset, it will remain in the subset. [12] [13] In this task, we will use the $l2$ norm of the difference of FTSE index return and the subset's return as performance metrics. We will run greedy forward algorithm six times to select six stocks. We use basic Mean-Variance strategy for the portfolio design.

*2) Spare index tracking:* Sparse and stable Markowitz portfolios is a method of portfolio selection within the mean-variance framework. Sparse and stable strategy introduce a penalty proportional to the sum of the absolute values of the weights in a portfolio. This leads to a sparse portfolio with fewer active assets and stabilizes the optimization process.[14]

When adding an $l1$ norm as the penalty to the original Markowitz objective function, we can formulate the equation as:

$$\min_{\pi}\left[\|o - r\pi\|_2^2 + \tau\,\|\pi\|\right] \tag{11}$$

where o is the objective, $\tau$ is the penalty added. This particular type of minimizing problem is also called lasso regression problem.[15][16] In this task, we choose the daily return rate of FTSE 100 as our objective and $\tau = 0.17$ to obtain a sparse portfolio that has six active positions.

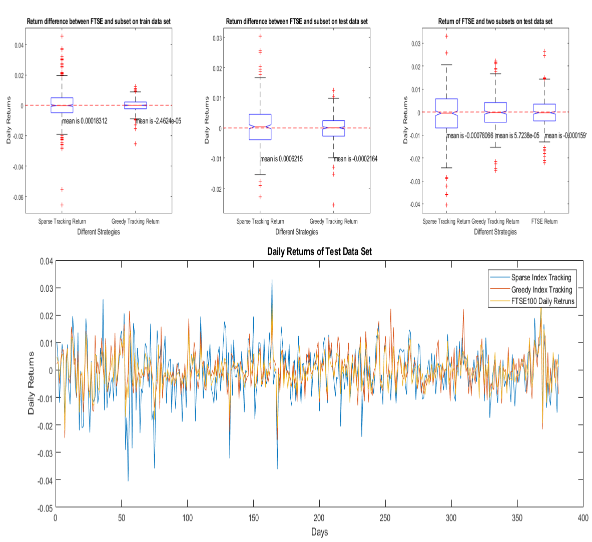


**Fig. 5: Box plot of the return difference between FTSE and two different strategies on training data set and test data set. And box plot of the daily return rate of the FTSE and two different strategies. And plot of daily returns of sparse tracking, greedy tracking and FTSE index on the test data set**

The first two panels in this figure show the fluctuation of the difference of daily return rate between the FTSE 100 and the two different strategies, in which the sparse index tracking is shown on the left and greedy index tracking is shown on the right. The last panel, shows a line chart of the daily returns, in which FTSE 100 is marked in yellow, sparse index tracking is marked in blue, and greedy index tracking is marked in red.

**TABLE II: Mean of performance metrics on out-of-sample data**

| Index Tracking strategies | Sparse Index Tracking | Greedy Index Tracking |
|---|---|---|
| in sample average return | -0.27 $\times 10^{-3}$ | 0.17$\times 10^{-3}$ |
| out-of-sample average return | 0.62$\times 10^{-3}$ | 0.21$\times 10^{-3}$ |
| in sample variance | 0.12 $\times 10^{-5}$ | 9.3 $\times 10^{-5}$ |
| out-of-sample variance | 4.83 $\times 10^{-5}$ | 1.7 $\times 10^{-5}$ |

This table reports the out-of-sample daily return, the out-of-sample Sharp ratio, the out-of-sample variance and the out-of-sample Value at Risk for the Mean-Variance strategy, Shortsale-constrained strategy, and naive strategy.

From Fig. 5 panels 1 and table 2, we notice that in terms of training data set, sparse index tracking portfolio yields similar results as the portfolio selected by the greedy forward selection algorithm.

From panels 2 and table 2, however, we notice that in terms of the testing data set, portfolio selected by greedy forward selection algorithm can track the FTSE 100 index slightly better than the sparse index tracking portfolio. This suggests in a chaotic system like the index market, it is very difficult to use historical data to predict the future. Therefore, any model that is too good at approximate the past may be proven to be over-fitting.

From panels 3&4, we notice that the FTSE 100 still out-perform sparse index tracking portfolio and greedy tracking portfolio in terms of minimizing the risk. However, it is possible to find a subset of FTSE 100 which can approximate the performance of this index.

However, this phenomenon does raise the question about whether it is the weights within each portfolio rather the choice of the stocks that determined the outcome of this two strategies. In order to understand the role of the stocks selected in each subset in index tracking, it is reasonable to implement the 1/N weighting strategy to the subsets.

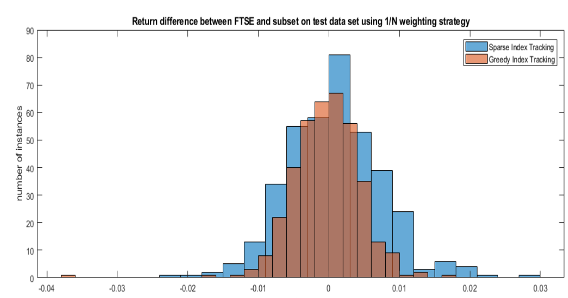


**Fig. 6: Histogram of return difference between FTSE and two different strategies using the 1/N weights on test data set**

This histogram shows a return difference between FTSE and two different strategies, in which sparse/Lasso strategy is marked in blue, greedy algorithm strategy is marked in red.

From Fig. 6, we notice that for the test data set when implementing the simple 1/N portfolio on subsets selected from these two strategies, subset selected using greedy algorithm still slightly outperform the parse index tracking strategy.

In conclusion, index-based portfolios can be achieved and used for minimizing the risk. In this experiments, two different methods have chosen two quite different subsets, which only shares two common choices out of six. The sparse index tracking may be the better choice in terms of finding the optimal solution. However, although greedy forward selection algorithm tends to reach the local optimal solution, this does not necessarily lead to worse performance in reality.

### G. Task 7 Transaction cost

In this section, we will discuss how transaction cost may influence the process of portfolio optimization. In the previous section, we try to add a $l1$-penalty to the original Markowitz objective function, which can lead to a sparse portfolio and a stable optimization process. By interpreting the $l1$ term as a model of transaction costs, fixed costs can be modeled into the equation 12:

$$\pi = arg\min_{\pi}\left[\|o - r\pi\|_2^2 + \tau\sum_i s_i\,|\pi_i|\right] \tag{12}$$

where $s_i$ represents the transaction cost for each security. However, this assumed that investors start with no assets which does not represent the reality. Therefore, adjustment can be made to use $\Delta_\pi$ instead of $pi$, which gives:

$$\Delta_\pi = arg\min_{\Delta_\pi}\left[\|\rho 1_T - r(\pi + \Delta_\pi)\|_2^2 + \tau\,\|\Delta_\pi\|\right] \tag{13}$$

However, when facing transaction costs that include a fixed fee, the transaction cost model can be reformulated as: $\mathbf{w}$ to $(\mathbf{w}+\mathbf{x})$, which give:

$$\phi_i(x_i) = \begin{cases} \beta_i^+ + \alpha_i^+ x_i, & x_i \geq 0 \\ \beta_i^- - \alpha_i^- x_i, & x_i \leq 0 \end{cases}$$

where $\alpha_i^+$ and $\alpha_i^-$ are the cost rates for buying and selling securities respectively, $x_i$ is the amounts of transacted in each asset. This is a linear model of the transaction costs problem. However, this function is not in a convex form, and therefore tradition convex optimization methods will struggle to solve it.[17] Lobo et al. propose a relaxation technique which enables a readily computable convex optimization solution. This is achieved by imposing a series of constraints, which simplify the concave problem into two convex linear equations that can be solved by quadratic programming.

These constrain can be summed up into diversification constrain, shortselling constraint, variance constraint, and shortfall risk constraint. Diversification constraints limit the portion of the total investment in one or a small group of assets. Shortselling constraint limits the maximum volume of short selling. The risk of loss in short selling is potentially infinite, therefore need to be restricted. Variance constraint can help reduce the volatility of a portfolio. Shortfall risk constraint also aim to reduce the risk by setting acceptable lower boundary of potential loss. Variance constraint and shortfall risk constraint are second-order cone constraints.[18]The form of second-order cone constraint is ($l_2$ norm) Euclidean distance, which creates a circular bound projected into a cone.

Because the linear combination of convex problems is still a convex problem. Therefore, we can combine the transaction cost and constraints into one convex optimization problem. One implementation of such optimization problem is in the following form:

$$\begin{aligned} &\text{maximize } \bar{a}^T(w + x^+ - x^-) \\ &\text{subject to } \mathbf{1}^T(x^+ - x^-) + \sum_{i=1}^{n}(\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leq 0 \\ &\qquad x_i^+ \geq 0, x_i^- \geq 0, i = 1, ..., n \\ &\qquad w_i + x_i^+ - x_i^- \geq s_i, i = 1, ..., n \\ &\qquad \Phi^{-1}(\eta_j)||\Sigma^{1/2}(\omega + x^+ - x^-)|| \leq \bar{a}^T(\omega + x^+ - x^-) \\ &\qquad - W_j^{low}, j = 1, 2 \end{aligned} \tag{14}$$

where $a$ is the expected return of each asset, $\Phi$ is the transaction cost function. The objection of this optimization problem is to maximize the return with respect to transaction costs. The constraints of this implementation include transaction cost, shortselling constraint, and shortfall risk constraint.

This method provides a flexible and readily computable model, which empower the user to easily modify and implement. For instance, we can extend our model used in the previous section to include transaction cost, with constraints like shortselling, variance constraint, and shortfall risk constraint. All these three constraints can help reduce the volatility and risk. Therefore, creating an optimal portfolio for investors who are sensitive to risk. For example, a pension fund.

## III. INTRODUCTION OF THE BLACK-SCHOLES MODEL

Options are contracts, which give its owner the right to sell or buy a certain amount of stocks at a specific price at some point in the future. It has a long history and is widely used to hedge against the risk in the financial market. [19]Although options have been traded for many years, the BlackScholes model published in 1973, was the first model to provide the mathematical understanding behind this trading.

The BlackScholes model is a mathematical model of the price variation over time of investment instruments in the financial market. This model provides us the mathematical mean to the theoretical estimate of the price of European-style options. The underlying assumption of the BlackScholes model implies that the price of any heavily traded asset mirrors a geometric Brownian motion. [20]

The BlackScholes equation, as shown in equation 15,[20] use current stock prices, expected interest rates, the call or put option's strike price, volatility and time to expiration to estimate the theoretical value of European-style options.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV + rS\frac{\partial V}{\partial S} = 0 \tag{15}$$

where $V$ is the price of the option, $t$ is time ,$S$ is the stock price at time t ,$r$ is the Risk-free interest rate and $\sigma$ is volatility.

## IV. The BlackScholes model

### A. solution of The BlackScholes equation

In this section, we will go through the basic mathematic derivation behind the BlackScholes equation. We will derive the solution for the call option price and prove that it satisfies the Black-Scholes differential equation.

The expression of the derivative of the cumulative normal distribution is:

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{16}$$

To show $S\mathcal{N}'(d_1) = Ke^{-r(T-t)}\mathcal{N}'(d_2)$ holds, we can substitute $\mathcal{N}'(x)$ into the formula $Ke^{-r(T-t)}\mathcal{N}'(d_2)$, and rewrite it as:

$$\begin{aligned}
Ke^{-r(T-t)}N'(d_2) &= Ke^{-r(T-t)} \times \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \\
&= Ke^{-r(T-t)} \times \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2} + d_1\sigma\sqrt{T-t} - \frac{\sigma^2}{2}(T-t)} \\
&= K\mathcal{N}'(d_1) e^{-(\frac{\sigma^2}{2}+r)(T-t)} \times e^{d_1\sigma\sqrt{T-t}} \\
&= K\mathcal{N}'(d_1) e^{-(\frac{\sigma^2}{2}+r)(T-t)} \times \frac{S}{K} e^{(r+\sigma^2)(T-t)} \\
&= S\mathcal{N}'(d_1)
\end{aligned}$$

Therefore, the following equation holds:

$$S\mathcal{N}'(d_1) = Ke^{-r(T-t)}\mathcal{N}'(d_2) \tag{17}$$

where the $d1$ and $d2$ in equation 17 is defined as:

$$d_1 = \frac{log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{18a}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{18b}$$

In equation 19, we will present the solution of the derivatives $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_1}{\partial S}$.

$$\frac{\partial d_1}{\partial S} = \frac{1}{SK\sigma\sqrt{T-t}} \tag{19a}$$

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_2}{\partial d_1}\frac{\partial d_1}{\partial S} = \frac{\partial d_1}{\partial S} \tag{19b}$$

When given the equation of the call option price as follow:

$$c = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \tag{20}$$

where K is the strike price and T is the time of maturity. We can derive the derivative of the call option price $\frac{\partial c}{\partial t}$ as:

$$\begin{aligned}
\frac{\partial c}{\partial t} &= \frac{\partial S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)}{\partial t} \\
&= S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - (Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\partial d_2}{\partial t} \ldots \\
&+ \quad Kre^{-r(T-t)}\mathcal{N}(d_2))
\end{aligned}$$

where $\frac{\partial d2}{\partial t}$ can be rewritten as the form of:

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}} \tag{21}$$

When substitute formula 21 and equation 17 into $\frac{\partial c}{\partial t}$:

$$\begin{aligned}
\frac{\partial c}{\partial t} &= S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - S\mathcal{N}'(d_1)(\frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}}) \\
&\quad - Kre^{-r(T-t)}\mathcal{N}(d_2) \\
&= -Kre^{-r(T-t)}\mathcal{N}(d_2) + S\mathcal{N}'(d_1)\frac{\sigma}{2\sqrt{T-t}})
\end{aligned} \tag{22}$$

To prove $\frac{\partial c}{\partial S} = \mathcal{N}(d_1)$ holds, we can substitute formula 19 into the derivative of equation 20:

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) + S\frac{\partial \mathcal{N}(d_1)}{\partial t} - Ke^{-r(T-t)}\frac{\partial \mathcal{N}(d_2)}{\partial t}$$

where owing to formula 19 and the chain rule $\frac{\partial \mathcal{N}(d_1)}{\partial S}$ and $\frac{\partial \mathcal{N}(d_1)}{\partial S}$ can be reformulate as:

$$\frac{\partial \mathcal{N}(d_1)}{\partial S} = \frac{\mathcal{N}'(d_2)}{SK\sigma\sqrt{T-t}} \tag{23a}$$

$$\frac{\partial \mathcal{N}(d_2)}{\partial S} = \frac{\mathcal{N}'(d_1)}{SK\sigma\sqrt{T-t}} \tag{23b}$$

when substitute 23 into $\frac{\partial c}{\partial t}$:

$$\begin{aligned}
\frac{\partial c}{\partial S} &= \mathcal{N}(d_1) + \frac{S\mathcal{N}'(d_1) - Ke^{-r(T-t)}\mathcal{N}'(d_2)}{SK\sigma\sqrt{T-t}} \\
&= \mathcal{N}(d_1)
\end{aligned} \tag{24}$$

When we take the second derivative of the call option price, we can formulate:

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial \mathcal{N}(d_1)}{\partial S} = \frac{\mathcal{N}'(d_2)}{SK\sigma\sqrt{T-t}} \tag{25}$$

When given the BlackScholes equation (Eq. 15) as shown below:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc + rS\frac{\partial c}{\partial S} = 0$$

where $\frac{\partial c}{\partial t}$ can be found in equation 22, $\frac{\partial c}{\partial S}$ can be found in equation 24, $\frac{\partial^2 c}{\partial S^2}$ can be found in equation 25. When substitute these equations into the BlackScholes equation and reformulate:

$$\begin{aligned}
-rKe^{-r(T-t)}\mathcal{N}(d_2) - \frac{S\mathcal{N}'(d_1)\sigma}{2\sqrt{T-t}} \ldots \\
+ \frac{\sigma S\mathcal{N}'(d_1)}{2\sqrt{T-t}} - rc + rS\mathcal{N}(d_1) = 0 \\
-rKe^{-r(T-t)}\mathcal{N}(d_2) - rc + rS\mathcal{N}(d_1) = 0 \\
-Ke^{-r(T-t)}\mathcal{N}(d_2) + S\mathcal{N}(d_1) = c
\end{aligned} \tag{26}$$

Equation 26 prove the call option price can be derived from BlackScholes differential equation.

### B. Testing the BlackScholes model on historical data

In this section, we will implement BlackScholes model with historical data, and evaluate the performance of BlackScholes model.

In order to implement the BlackScholes model, daily volatilities need to estimated using a time window of $T/4$. The formula is shown in equation 27.[21]

$$u_i = ln(\frac{S_i}{S_{i-1}})$$
$$\hat{\sigma} = \frac{std(u)}{\sqrt{\tau}} \tag{27}$$

Volatility is calculated using historical data and therefore represents the fluctuation rate of prices during the time window. This will cause a delay and smooth effect on the transmission of up-to-date price fluctuation into volatility. After the estimation of volatilities, we can use the BlackScholes equation to compute the theory value of call and put options and compare them to the real price. The theory and real value of call option are shown in Fig.7.

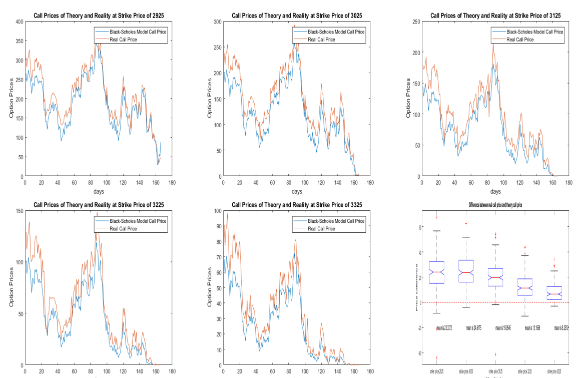

**Fig. 7: Line charts of real call price and B-S theory call price at different strike prices. Box plot of the difference between real call price and B-S theory call price at different strike prices.**
In the first five panels the real daily call price is marked in orange and B-S theory daily call price is marked in blue. The strike price of these five panels is 2925, 3025, 3125, 3225 and 3325 respectively. The last panel is a box plot of the values of real call prices minus B-S theory call prices, in which the zero base line is marked in red dash, the strike price from left to right is 2925, 3025, 3125, 3225 and 3325 respectively.

From Fig. 7 panel 1 to table 5, we notice that the real call price is constantly higher than the price given by the BlackScholes equation. This indicates that investors tend to be more optimistic about the market than what the historical data suggests, which leads to a systematically over-valued call option price.

However, From Fig. 7 panel 6, we can see that this level of optimism gradually decreases from lower strike price to higher strike price. This phenomenon shows that the market tends to be more irrational about the value of call options when the potential profit is high.

The theory and real value of put option are shown in Fig.8.

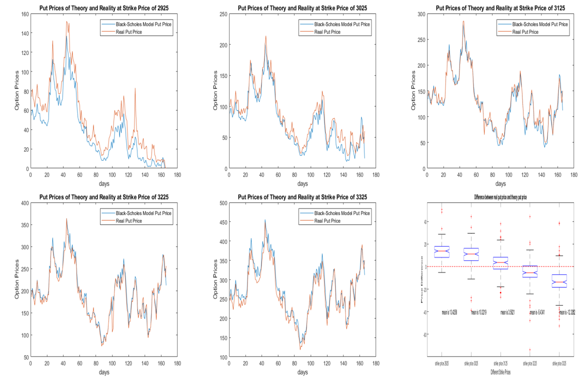

**Fig. 8: Line charts of real put price and B-S theory put price at different strike prices. Box plot of the difference between real put price and B-S theory put price at different strike prices.**
In the first five panels the real daily put price is marked in orange and B-S theory daily put price is marked in blue. The strike price of these five panels is 2925, 3025, 3125, 3225 and 3325 respectively. The last panel is a box plot of the values of real put prices minus B-S theory put prices, in which the zero base line is marked in red dash, the strike price from left to right is 2925, 3025, 3125, 3225 and 3325 respectively.

From Fig. 8 panel 1 to table 5, we notice that the real put price goes from higher than the theory price to lower than the theory price. This also indicates that investors tend to be more optimistic about the market than what the historical data suggests.

From Fig. 8 panel 6, we can see that the difference between real price and theory price goes from positive to negative with the increase of strike price. This suggests that in an optimistic atmosphere the market tends to be irrational when the profit margin is high and too cautious when the profit margin is low.

The theory and implied volatilities are shown in Fig.9.

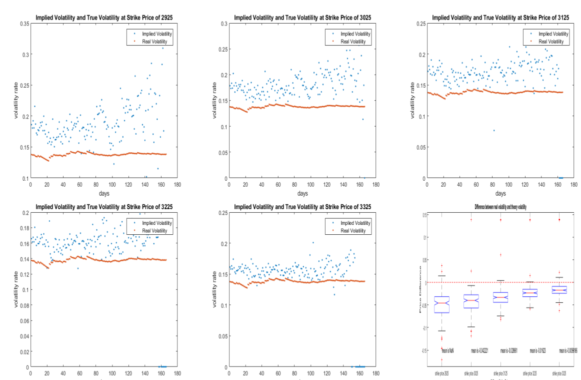

**Fig. 9: Line charts of real volatility and implied volatility at different strike prices. Box plot of the difference between real volatility and implied volatility at different strike prices.**
In the first five panels the real volatilities is marked in orange and B-S theory volatilities is marked in blue. The strike price of these five panels is 2925, 3025, 3125, 3225 and 3325 respectively. The last panel is a box plot of the values of real volatilities minus B-S theory volatilities, in which the zero base line is marked in red dash and the strike price from left to right is 2925, 3025, 3125, 3225 and 3325 respectively.

From Fig. 9 we notice that the implied volatilities is constantly higher than the volatilities estimated from the data.

From Fig. 9 panel 6, we can see that the difference between volatilities estimated from the data and implied volatilities drop as the profit margin decrease. This phenomenon is also being referred as the volatility smile, which is a curve of implied volatilities for a set of options that has the same time of maturity. Implied volatility is high when the option's strike price deviates significantly from its underlying asset's price. This peculiarity suggests the basic assumption of Black-Scholes model are flawed.[22] In particular, assumptions that volatility is constant and the underlying asset return obey normal distribution may deviate from reality.

### *C. Using RBF neural network to approximate relationship between option prices and primary economic variables*

In this section, we will implement a Radial Basis Functions(RBF) model and Monte Carlo experiments to approximate the complex relationship between primary economic variables and call/put option price. And evaluate the performance of this data-driven model against BlackScholes model in terms of pricing and delta-hedge options out-of-sample.

A Radial Basis Functions is a function which maps a point's Euclidean distance to a real value. A radial basis function network is a type of three-layer artificial neural network which uses RBF function as the activation function.[23]

In Hutchinson's paper, the inputs of the data-driven model are the value of the underlying assets $S(t)$ , the strike price $K$ and the time-to-maturity $T-t$. [24]These inputs can be further reduced to two input parameters, namely $\frac{S(t)}{K}$ and $T-t$. The volatility and risk-free interest rate can be ignored because in the paper's Monte Carlo experiment they remain constant. Therefore does not contribute to the learning process of the model. This reduction of input parameters can help reduce the volume of training data.

However, in our dataset, we calculate volatility in a time window of 56 days, which leads to variable volatilities. Hence, in our model, volatility $V_o$ will be included as an input. The inputs of RBF model is shown in equation 28.

$$Input = [\frac{S(t)}{K}, V_o, T-t] \tag{28}$$

Further experiments will includes other relevant economic variables as inputs to explore whether they will contributes to the accuracy of the RBF model.

The training target of this RBF model is call option price, put option price, call Delta and put delta. There are two main reasons to include delta as this model's objection. The first reason lies the importance of delta in hedging option's position in the financial market. The second is that because Delta $\frac{\partial c}{\partial S}$ is a derivative of underlying pricing formula, the accurate approximation of Delta can perform as a smoothness constraint on the RBF's optimization process.[25][26] The box plot of the option price's MSE with two different objection strategies are shown in Fig.10.

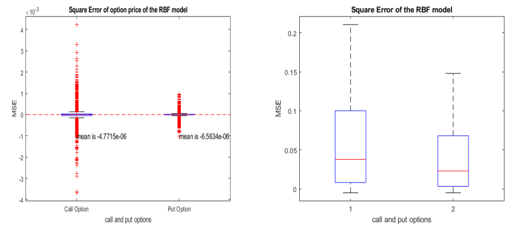


**Fig. 10: Box plot of the out-of-sample MSE of call/put option prices between RBF model's prediction and real value with and without Delta as objection.**
The first panel in this figure show out-of-sample MSE between RBF model with Delta as an objection and the Black-Scholes model.The second panel in this figure show out-of-sample MSE between FBF model without Delta as an objection and the Black-Scholes model. In both panels, the call option is shown on the left and the put option is shown on the right.

Fig. 10 shows to include Delta as objection can significantly reduce the mean square error of option prices from $10^{-1}$ to $10^{-6}$.

The Monte Carlo experiment used in RBF modeling contains a training phase and testing phase. The training phase is responsible for generating samples for the model to train on. And after the training phase, the model will be tested on the samples generated by the testing phase. The sample generation process is based on the data we used in the previous sections. For each strike price, we have a historical data of 166 days, which will be referred as one "training path". A total of 40 training paths are generated for training and testing.

After training the model, it will be tested on test dataset. The performance of the RBF model is shown in Fig. 11.

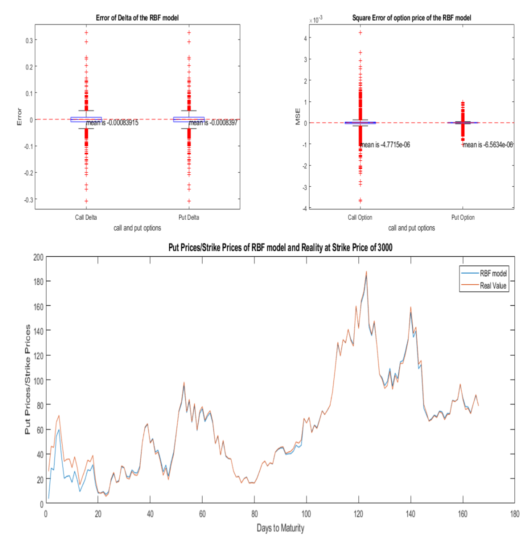


**Fig. 11: Box plots of the error of Delta and the option price's MSE. Price of FTSE 100 options at strike price of 3025 and its approximations from RBF model.**
The first two panels in this figure show out-of-sample option prices MSE and error of Delta between RBF model with Delta as an objection and the Black-Scholes model. In both panels, the call option is shown on the left and the put option is shown on the right. The third panels show prices of FTSE 100 options and its approximations from RBF model, in which the RBF model is marked in blue and real value is marked in orange.

From Fig. 11, we can see that the RBF model can approximate the relationship between option prices and underlying variables with remarkable accuracy. Panel 2 indicates the RBF model can extract Delta with great precisions. Therefore, the Hutchison's model can approximate the Black-Scholes model.

However, the model we test above is a slight variance compared to the original model. This is because we include volatility into the inputs. And my experiments indicate, by including more relevant variables in the inputs we can further increase the accuracy of this model.

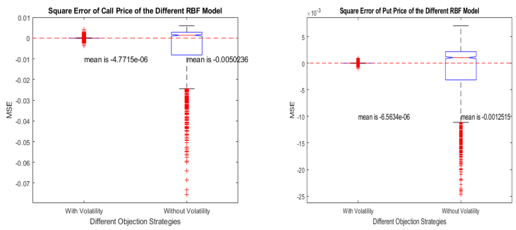


**Fig. 12: The out-of-sample MSE of call and put option prices from RBF model with and without volatility as inputs.**
The first panel show the out-of-sample MSE of call option prices from RBF model with and without volatility as inputs.The second panel show out-of-sample MSE of put option prices from these two models. In both panels, the RBF model with volatility input is shown on the left and the model without volatility input is shown on the right.

Fig. 12 shows to include volatility as a input can significantly reduce the mean square error of option prices from $10^{-3}$ to $10^{-6}$.

To include risk-free interest rate into inputs has a similar effect on the performance of accuracy, but with smaller improvements. Because to more inputs usually require more training data we only include volatility into the inputs. This also raises another question about whether the performance of this model can be further improved, if we include more parameters as inputs which may not necessarily in the Black-Scholes equation.

In conclusion, the claims made in Hutchison's paper are true. However, improvements mentioned above can be implemented to further improve the accuracy of the Hutchison's model.

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