Instrumentation Tutorial 1 Answers

Instrumentation (Flinders University)

INSTRUMENTATION

ENGR4732, SEMESTER 2 2013

TUTORIAL 1: BAYESIAN FILTERING

QUESTIONS

QUESTION 1

A robot uses a range sensor that can measure ranges from *0m* and *3m*. For simplicity, assume that actual ranges are distributed uniformly in this interval. Unfortunately, the sensor can be faulty. When the sensor is faulty, it constantly outputs a range below 1m, regardless of the actual range in the sensor's measurement cone. We know that the prior probability for a sensor to be faulty is $p = 0.01$.

Suppose the robot queried its sensor N times, and every single time the measurement value is below *1m*. What is the posterior probability of a sensor fault, for $N = 1,2, \dots ,10$. Formulate the corresponding probabilistic model.

QUESTION 2

Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		Tomorrow will be…		
		sunny	cloudy	rainy
Today it is…	sunny	0.8	0.2	0
	cloudy	0.4	0.4	0.2
	rainy	0.2	0.6	0.2

(a) Suppose Day 1 is a sunny day. What is the probability of the following sequence of days: Day2 = *cloudy*, Day3 = *cloudy*, Day4 = *rainy*?

(b) Write a simulator that can randomly generate sequences of "weathers" from this state transition function.

(c) Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

(d) Can you devise a closed-form solution to calculating the stationary distribution based on the state transition matrix above?

(e) What is the entropy of the stationary distribution?

(f) Using Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from previous questions in this exercise.)

(g) Suppose we added seasons to our model. The state transition function above would only apply to the Summer, whereas different ones would apply to Winter, Spring, and Fall. Would this violate the Markov property of this process? Explain your answer.

QUESTION 3

Suppose that we cannot observe the weather directly, but instead rely on a sensor. The problem is that our sensor is noisy. Its measurements are governed by the following measurement model:

		Our sensor tells us…		
		sunny	cloudy	rainy
The actual weather is…	sunny	0.6	0.4	0
	cloudy	0.3	0.7	0
	rainy	0	0	1

(a) Suppose Day 1 is sunny (this is known for a fact), and in the subsequent four days our sensor observes *cloudy, cloudy, rainy, sunny*. What is the probability that Day 5 is indeed sunny as predicted by our sensor?

(b) Once again, suppose Day 1 is known to be sunny. At Days 2 through 4, the sensor measures *sunny, sunny, rainy*. For each of the Days 2 through 4, what is the most likely weather on that day? Answer the question in two ways: one in which only the data available to the day in question is used, and one in hindsight, where data from future days is also available.

(c) Consider the same situation (Day 1 is sunny, the measurements for Days 2, 3, and 4 are *sunny, sunny, rainy*). What is the most likely sequence of weather for Days 2 through 4? What is the probability of this most likely sequence?

QUESTION 4

In this exercise we will apply Bayes rule to Gaussians. Suppose we are a mobile robot who lives on a long straight road. Our location, *x,* will simply be the position along this road. Now suppose that initially, we believe to be at location $x_{init} = 1{,}000m$, but we happen to know that this estimate is uncertain. Based on this uncertainty, we model our initial belief by a Gaussian with variance $\sigma^2_{init} = 900m^2$.

To find out more about our location, we query a GPS receiver. The GPS tells us our location is $z_{GPS} = 1{,}100m$. This GPS receiver is known to have an error variance of $\sigma^2_{GPS} = 100m^2$.

(a) Write the probability density functions of the prior $p(x)$ and the measurement $p(z \mid x)$.

(b) Using Bayes rule, what is the posterior $p(x \mid z)$? Can you prove it to be Gaussian?

(c) How likely was the measurement $z_{GPS} = 1,\ 100m$ given our prior, and knowledge of the error probability of our GPS receiver?

Hint: This is an exercise in manipulating quadratic expressions.

INSTRUMENTATION

ENGR4732, SEMESTER 2 2014

TUTORIAL 1: BAYESIAN FILTERING

ANSWERS

QUESTION 1

A robot uses a range sensor that can measure ranges from *0m* and *3m*. For simplicity, assume that actual ranges are distributed uniformly in this interval. Unfortunately, the sensor can be faulty. When the sensor is faulty, it constantly outputs a range below 1m, regardless of the actual range in the sensor's measurement cone. We know that the prior probability for a sensor to be faulty is $p = 0.01$.

Suppose the robot queried its sensor N times, and every single time the measurement value is below *1m*. What is the posterior probability of a sensor fault, for $N = 1,2, \ldots ,10$. Formulate the corresponding probabilistic model.

ANSWER

$$bel(X_0 = \mathbf{faulty}) = 0.01$$

$$bel(X_0 = \mathbf{ok}) = 0.99$$

$$p(Z_t < \mathbf{1} \mid X_t = \mathbf{faulty}) = 1$$

$$p(Z_t \geq \mathbf{1} \mid X_t = \mathbf{faulty}) = 0$$

$$p(Z_t < \mathbf{1} \mid X_t = \mathbf{ok}) = {}^1/_3$$

$$p(Z_t \geq \mathbf{1} \mid X_t = \mathbf{ok}) = {}^2/_3$$

Assume persistent fault: $p(X_t = \mathbf{faulty} \mid X_{t-1} = \mathbf{faulty}) = 1$

$$p(X_t = \mathbf{ok} \mid X_{t-1} = \mathbf{faulty}) = 0$$

$$p(X_t = \mathbf{faulty} \mid X_{t-1} = \mathbf{ok}) = 0$$

$$p(X_t = \mathbf{ok} \mid X_{t-1} = \mathbf{ok}) = 1$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

t = 0

$$bel(X_0 = \mathbf{faulty}) = 0.01$$

$$bel(X_0 = \mathbf{ok}) = 0.99$$

t = 1

$$bel(X_1 = \mathbf{faulty}) = \eta\, p(Z_1 < \mathbf{1} \mid X_1 = \mathbf{faulty})\, bel(X_0 = \mathbf{faulty})$$
$$bel(X_1 = \mathbf{faulty}) = \eta \times 1 \times 0.01 = \eta 0.01$$

$$bel(X_1 = \mathbf{ok}) = \eta\, p(Z_1 < \mathbf{1} \mid X_1 = \mathbf{ok})\, bel(X_0 = \mathbf{ok})$$
$$bel(X_1 = \mathbf{ok}) = \eta \times 0.33 \times 0.99 = \eta 0.33$$

$$0.01\eta + 0.33\eta = 1 \Rightarrow \eta = \frac{1}{0.34} = 2.9412$$

$$bel(X_1 = \mathbf{faulty}) = \eta 0.01 = \ 2.9412 \times 0.01 = 0.029412$$

$$bel(X_1 = \mathbf{ok}) = \eta 0.33 = 2.9412 \times 0.33 = 0.970588$$

t = 2

$$bel(X_2 = \mathbf{faulty}) = \eta\, p(Z_2 < \mathbf{1} \mid X_2 = \mathbf{faulty})\, bel(X_1 = \mathbf{faulty})$$
$$bel(X_2 = \mathbf{faulty}) = \eta \times 1 \times 0.029412 = \eta 0.029412$$

$$bel(X_2 = \mathbf{ok}) = \eta\, p(Z_2 < \mathbf{1} \mid X_2 = \mathbf{ok})\, bel(X_1 = \mathbf{ok})$$
$$bel(X_2 = \mathbf{ok}) = \eta \times 0.33 \times 0.970588 = \eta 0.320294$$

$$0.029412\eta + 0.320294\eta = 1 \Rightarrow \eta = \frac{1}{0.3497} = 2.859544$$

$$bel(X_1 = \mathbf{faulty}) = \eta 0.029412 = \ 2.859544 \times 0.029412 = 0.084105$$

$$bel(X_1 = \mathbf{ok}) = \eta 0.320294 = 2.859544 \times 0.320294 = 0.915895$$

t = 3

$$bel(X_3 = \mathbf{faulty}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{faulty})\, bel(X_2 = \mathbf{faulty})$$
$$bel(X_3 = \mathbf{faulty}) = \eta \times 1 \times 0.084105 = \eta 0.084105$$

$$bel(X_3 = \mathbf{ok}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{ok})\, bel(X_1 = \mathbf{ok})$$
$$bel(X_3 = \mathbf{ok}) = \eta \times 0.33 \times 0.915895 = \eta 0.302245$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

$$0.084105\eta + 0.302245\eta = 1 \Rightarrow \eta = \frac{1}{0.3863} = 2.588324$$

$$bel(X_3 = \mathbf{faulty}) = \eta 0.084105 = 2.588324 \times 0.084105 = 0.217691$$

$$bel(X_3 = \mathbf{ok}) = \eta 0.302245 = 2.588324 \times 0.302245 = 0.782308$$

t = 4

$$bel(X_3 = \mathbf{faulty}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{faulty})\, bel(X_2 = \mathbf{faulty})$$
$$bel(X_3 = \mathbf{faulty}) = \eta \times 1 \times 0.217691 = \eta 0.217691$$

$$bel(X_3 = \mathbf{ok}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{ok})\, bel(X_1 = \mathbf{ok})$$
$$bel(X_3 = \mathbf{ok}) = \eta \times 0.33 \times 0.782308 = \eta 0.258162$$

$$0.217691\eta + 0.258162\eta = 1 \Rightarrow \eta = \frac{1}{0.475853} = 2.101491$$

$$bel(X_3 = \mathbf{faulty}) = \eta 0.217691 = 2.101491 \times 0.217691 = 0.457475$$

$$bel(X_3 = \mathbf{ok}) = \eta 0.258162 = 2.101491 \times 0.258162 = 0.542525$$

t = 5

$$bel(X_3 = \mathbf{faulty}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{faulty})\, bel(X_2 = \mathbf{faulty})$$
$$bel(X_3 = \mathbf{faulty}) = \eta \times 1 \times 0.457475 = \eta 0.457475$$

$$bel(X_3 = \mathbf{ok}) = \eta\, p(Z_3 < \mathbf{1} \mid X_3 = \mathbf{ok})\, bel(X_1 = \mathbf{ok})$$
$$bel(X_3 = \mathbf{ok}) = \eta \times 0.33 \times 0.542525 = \eta 0.179033$$

$$0.457475\eta + 0.179033\eta = 1 \Rightarrow \eta = \frac{1}{0.6365} = 1.57107$$

$$bel(X_3 = \mathbf{faulty}) = \eta 0.457475 = 1.57107 \times 0.457475 = 0.718726$$

$$bel(X_3 = \mathbf{ok}) = \eta 0.179033 = 1.57107 \times 0.179033 = 0.281274$$

.....

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

QUESTION 2

Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		Tomorrow will be…		
		sunny	Cloudy	rainy
Today it is…	sunny	0.8	0.2	0
	cloudy	0.4	0.4	0.2
	rainy	0.2	0.6	0.2

(a) Suppose Day 1 is a sunny day. What is the probability of the following sequence of days: Day2 = *cloudy*, Day3 = *cloudy*, Day4 = *rainy*?

(b) Write a simulator that can randomly generate sequences of "weathers" from this state transition function.

(c) Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

(d) Can you devise a closed-form solution to calculating the stationary distribution based on the state transition matrix above?

(e) What is the entropy of the stationary distribution?

(f) Using Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from previous questions in this exercise.)

(g) Suppose we added seasons to our model. The state transition function above would only apply to the Summer, whereas different ones would apply to Winter, Spring, and Fall. Would this violate the Markov property of this process? Explain your answer.

ANSWER

a) $p(X_1 = \mathbf{cloudy}, X_2 = \mathbf{cloudy}, X_3 = \mathbf{rain} \mid X_0 = \mathbf{sunny}) =$
$p(X_3 = \mathbf{rain} \mid X_2 = \mathbf{cloudy})p(X_2 = \mathbf{cloudy} \mid X_1 = \mathbf{cloudy})p(X_1 = \mathbf{cloudy} \mid X_0 = \mathbf{sunny})$

$$p(X_1 = \mathbf{cloudy}, X_2 = \mathbf{cloudy}, X_3 = \mathbf{rain} \mid X_0 = \mathbf{sunny}) = 0.2 \times 0.4 \times 0.2$$
$$= 0.016$$

b) simulator(X_0,N)

```
for (t = 1:N) {
        u_t ~ U([0,1])
        if (X_{t-1} = sunny) {
                if (ut > 0.8) {
                        X_t = cloudy
                } else {
                        X_t = sunny
```

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

```
            }

        }

        if (X_{t-1} = cloudy) {
            if (ut > 0.8) {
                X_t = rainy
            } elseif (ut > 0.4) {
                X_t = cloudy
            } else {
                X_t = sunny
            }

        }

        if (X_{t-1} = rainy) {
            if (ut > 0.8) {
                X_t = rainy
            } elseif (ut > 0.2) {
            X_t = cloudy
            } else {
            X_t = sunny
            }

        }

    }
```

c) simulator(,**any**,$N \to \infty$)
p(**sunny**) = count(X_N = **sunny**)/N
p(**cloudy**) = count(X_N = **cloudy**)/N
p(**rainy**) = count(X_N = **rainy**)/N

d) $p([\mathbf{sunny} \quad \mathbf{cloudy} \quad \mathbf{rainy}]) = \prod_k^\infty P^k = \prod_k^\infty \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}^k =$

$\begin{bmatrix} 0.6429 & 0.2857 & 0.0714 \\ 0.6429 & 0.2857 & 0.0714 \\ 0.6429 & 0.2857 & 0.0714 \end{bmatrix}$

e) $H(X) = -\sum_{i=1}^n p(x_i) \log_2 p(x_i) = -(0.6429 \log_2 0.6429 + 0.2857 \log_2 0.2857 + 0.0714 \log_2 0.0714) = 1.198\ bits$

f) $P(X_t | X_{t-1}) = P$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

then

$$P(X_{t-1}|X_t) = P^T$$

(g) Depends;

No, if;

If the state vector is extended to include the season and the season has its own transition table.

If the state vector is extended to include the time of year and as such can deterministically define the current season.

Yes, if;

These extension are not added, or if the season is added without a transition table. In this case, to determine the appropriate season (i.e. correct transition table) a complete history must be maintained to determine the duration of each season.

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

QUESTION 3

Suppose that we cannot observe the weather directly, but instead rely on a sensor. The problem is that our sensor is noisy. Its measurements are governed by the following measurement model:

		Our sensor tells us…		
		sunny	cloudy	rainy
The actual weather is…	sunny	0.6	0.4	0
	cloudy	0.3	0.7	0
	rainy	0	0	1

(a) Suppose Day 1 is sunny (this is known for a fact), and in the subsequent four days our sensor observes *cloudy, cloudy, rainy, sunny*. What is the probability that Day 5 is indeed sunny as predicted by our sensor?

(b) Once again, suppose Day 1 is known to be sunny. At Days 2 through 4, the sensor measures *sunny, sunny, rainy*. For each of the Days 2 through 4, what is the most likely weather on that day? Answer the question in two ways: one in which only the data available to the day in question is used, and one in hindsight, where data from future days is also available.

(c) Consider the same situation (Day 1 is sunny, the measurements for Days 2, 3, and 4 are *sunny, sunny, rainy*). What is the most likely sequence of weather for Days 2 through 4? What is the probability of this most likely sequence?

ANSWER

a)

$$p(x_5 \mid x_1, z_{2:5}) = \eta p(z_5 \mid x_5, x_1, z_{2:4}) p(x_5 \mid x_1, z_{2:4})$$
$$= \eta p(z_5 \mid x_5) p(x_5 \mid x_1, z_{2:4})$$

$$p(x_5 \mid x_1, z_{2:4}) = \sum_{x_4} p(x_4, x_5 \mid x_1, z_{2:4})$$
$$= \sum_{x_4} p(x_5 \mid x_4, x_1, z_{2:4}) p(x_4 \mid x_1, z_{2:4})$$

However, z_4 = **rainy** implies that x_4 = **rainy**.

$$= \sum_{x_4} p(x_5 \mid x_4) p(x_4 \mid x_1, z_{2:4})$$
$$= p(x_5 \mid x_4 = \boldsymbol{rainy}) \cdot 1$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

$$p(x_5 = \boldsymbol{sunny} |\, x_1\,, \mathrm{z}_{2:3}, z_4 = \boldsymbol{rainy}, z_4 = \boldsymbol{sunny})$$
$$= \eta p(z_5 = \boldsymbol{sunny} \mid x_5 = \boldsymbol{sunny}) p(x_5 = \boldsymbol{sunny} \mid x_4 = \boldsymbol{rainy})$$
$$= \frac{p(z_5 = \boldsymbol{sunny} \mid x_5 = \boldsymbol{sunny}) p(x_5 = \boldsymbol{sunny} \mid x_4 = \boldsymbol{rainy})}{\sum_{x_5'} p(z_5 = \boldsymbol{sunny} \mid x_5') p(x_5' \mid x_4 = \boldsymbol{rainy})}$$
$$= \frac{0.6 \cdot 0.2}{0.6 \cdot 0.2 + 0.3 \cdot 0.6 + 0}$$

b)

Day 2

$$p(x_2 \mid x_1\,, \mathrm{z}_2) = \eta p(z_2 \mid x_2,\, x_1) p(x_2 \mid x_1)$$
$$= \eta p(z_2 \mid x_2) p(x_2 \mid x_1)$$
$$= \eta \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix} = \eta \begin{pmatrix} 0.48 \\ 0.06 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 8/9 \\ 1/9 \\ 0 \end{pmatrix}$$

Day 2 data from future

$$p(x_2 \mid x_1\,, \mathrm{z}_{2:4}) = \eta p(x_2 \mid x_1) p(z_{2:4} \mid x_2,\, x_1)$$
$$= \eta p(x_2 \mid x_1) p(z_{2:4} \mid x_2)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid z_{3:4},\, x_2) p(z_{3:4} \mid x_2)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) p(z_{3:4} \mid x_2)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3, z_{3:4} \mid x_2)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_{3:4} \mid x_3,\, x_2)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_{3:4} \mid x_3)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) p(z_4 \mid z_3, x_3)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) p(z_4 \mid x_3)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) \sum_{x_4} p(x_4, z_4 \mid x_3)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) \sum_{x_4} p(x_4 \mid x_3) p(z_4 \mid x_4, x_3)$$
$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) \sum_{x_4} p(x_4 \mid x_3) p(z_4 \mid x_4)$$

$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) \sum_{x_4} p(x_4 \mid x_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) p(z_3 \mid x_3) \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix}$$

$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \sum_{x_3} p(x_3 \mid x_2) \begin{pmatrix} 0 \\ 0.06 \\ 0 \end{pmatrix}$$

$$= \eta p(x_2 \mid x_1) p(z_2 \mid x_2) \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}$$

$$= \eta \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}$$

$$= \eta \begin{pmatrix} 0.00576 \\ 0.00144 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}$$

Day 3

$$p(x_3 \mid x_1, z_{2:3}) = \eta p(z_3 \mid x_3, x_1, z_2) p(x_3 \mid x_1, x_2)$$

$$= \eta p(z_3 \mid x_3) p(x_3 \mid x_1, x_2)$$

$$= \eta p(z_3 \mid x_3) \sum_{x_2} p(x_3, x_2 \mid x_1, z_2)$$

$$= \eta p(z_3 \mid x_3) \sum_{x_2} p(x_3 \mid x_2) p(x_2 \mid x_1, z_2)$$

$$= \eta p(z_3 \mid x_3) \sum_{x_2} p(x_3 \mid x_2) \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}$$

$$= \eta p(z_3 \mid x_3) \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix}$$

$$= \eta \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8718 \\ 0.1282 \\ 0 \end{pmatrix}$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

Day 3 with future data

$$p(x_3 \mid x_1, \mathrm{z}_{2:4}) = \eta p(x_3 \mid x_1, z_{2:3}) p(z_4 \mid x_3, x_1, z_{2:3})$$

$$= \eta p(x_3 \mid x_1, z_{2:3}) p(z_4 \mid x_3)$$

$$= \eta \begin{pmatrix} 0.8718 \\ 0.1282 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Day 4

$$p(x_4 \mid x_1, \mathrm{z}_{2:4}) = p(x_4 \mid x_3, z_4)$$

$$= \eta p(z_4 \mid x_4) p(x_4 \mid x_3)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

c)

$$p(x_{2:4} \mid x_1, \mathrm{z}_{2:4}) = \eta p(z_{2:4} \mid x_1, x_{2:4}) p(x_{2:4} \mid x_1)$$

where

$$p(x_{2:4} \mid x_1) = p(x_4 \mid x_3) p(x_3 \mid x_2) p(x_2 \mid x_1)$$

and

$$p(z_{2:4} \mid x_1, x_{2:4}) = p(z_4 \mid x_4) p(z_3 \mid x_3) p(z_2 \mid x_2)$$

Hence

$$p(x_{2:4} \mid x_1, \mathrm{z}_{2:4}) = \eta p(z_4 \mid x_4) p(z_3 \mid x_3) p(z_2 \mid x_2) p(x_4 \mid x_3) p(x_3 \mid x_2) p(x_2 \mid x_1)$$

$$p(x_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy}) =$$

$$\eta p(z_4 = \boldsymbol{rainy} \mid x_4 = \boldsymbol{rainy}) p(z_3 = \boldsymbol{sunny} \mid x_3 = \boldsymbol{sunny}) p(z_2 = \boldsymbol{sunny} \mid x_2 = \boldsymbol{sunny})$$
$$p(x_4 = \boldsymbol{rainy} \mid x_3 = \boldsymbol{sunny}) p(x_3 = \boldsymbol{sunny} \mid x_2 = \boldsymbol{sunny}) p(x_2 = \boldsymbol{sunny} \mid x_1 = \boldsymbol{sunny})$$

$$= \eta \cdot 1 \cdot 0.6 \cdot 0.6 \cdot 0.14 \cdot 0.57 \cdot 0.57$$

$$= \eta 0.0168$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

$$p(x_{2:4} = \boldsymbol{sunny}, \boldsymbol{cloudy}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= \eta \cdot 1 \cdot 0.3 \cdot 0.6 \cdot 0.5 \cdot 0.29 \cdot 0.57$$

$$= \eta 0.0147$$

$$p(x_{2:4} = \boldsymbol{cloudy}, \boldsymbol{sunny}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= \eta \cdot 1 \cdot 0.6 \cdot 0.3 \cdot 0.14 \cdot 0.17 \cdot 0.29$$

$$= \eta 0.00245$$

$$p(x_{2:4} = \boldsymbol{cloudy}, \boldsymbol{cloudy}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= \eta \cdot 1 \cdot 0.3 \cdot 0.3 \cdot 0.5 \cdot 0.33 \cdot 0.29$$

$$= \eta 0.00429$$

$$\eta = \ 0.0168 + 0.0147 + 0.00245 + 0.00429 = 0.0382$$

$$p(x_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy}) =$$

$$= 0.439$$

$$p(x_{2:4} = \boldsymbol{sunny}, \boldsymbol{cloudy}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= 0.384$$

$$p(x_{2:4} = \boldsymbol{cloudy}, \boldsymbol{sunny}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= 0.064$$

$$p(x_{2:4} = \boldsymbol{cloudy}, \boldsymbol{cloudy}, \boldsymbol{rainy} \mid x_1 = \boldsymbol{sunny}, \mathrm{z}_{2:4} = \boldsymbol{sunny}, \boldsymbol{sunny}, \boldsymbol{rainy})$$

$$= 0.112$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.