Instrumentation Tutorial 5 Answers

Instrumentation (Flinders University)

INSTRUMENTATION

ENGR4732, SEMESTER 2 2014

TUTORIAL 5: EXTENDED KALMAN FILTERS

QUESTIONS

QUESTION 1

Consider robot with two dimensional motion with which we want to estimate its position and orientation. The robot has a bearing sensor that provides measurements with a 1σ uncertainty of 20 degrees. The robots motion is performed using the odometery model in "Motion Models" consisting of rotation command forward motion command and final rotation command.

The initial position and velocity are 0 m and 0 m/s respectively, with a 1σ uncertainty of 1 m and 10 degrees, for position and orientation respectively. Given the Extended Kalman Filter structure;

$$\widehat{\boldsymbol{x}}_{k|k-1} = f_{k-1}\left(\widehat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{u}_k\right)$$

$$\boldsymbol{P}_{k|k-1} = \widehat{\boldsymbol{G}}_{k-1}\boldsymbol{Q}_{k-1}\widehat{\boldsymbol{G}}_{k-1}^T + \widehat{\boldsymbol{F}}_{k-1}\boldsymbol{P}_{k-1|k-1}\widehat{\boldsymbol{F}}_{k-1}^T$$

$$\widehat{\boldsymbol{x}}_{k|k} = \widehat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k\left(\boldsymbol{z}_k - h_k\left(\widehat{\boldsymbol{x}}_{k|k-1}\right)\right)$$

$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{K}_k\boldsymbol{S}_k\boldsymbol{K}_k^T$$

where

$$\boldsymbol{S}_k = \boldsymbol{R}_k + \widehat{\boldsymbol{H}}_k\boldsymbol{P}_{k|k-1}\widehat{\boldsymbol{H}}_k^T$$

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1}\widehat{\boldsymbol{H}}_k^T\boldsymbol{S}_k^{-1}$$

$$\widehat{\boldsymbol{F}}_{k-1} = \left[\nabla_{\mathrm{x}_{k-1}} f_{k-1}^T(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k)\right]^T\Big|_{\boldsymbol{x}_{k-1}=\widehat{\boldsymbol{x}}_{k-1|k-1}}$$

$$\widehat{\boldsymbol{G}}_{k-1} = \left[\nabla_{\mathrm{x}_{k-1}} f_{k-1}^T(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k)\right]^T\Big|_{\boldsymbol{u}_k}$$

$$\widehat{\boldsymbol{H}}_k = \left[\nabla_{\mathrm{x}_k} h_k^T(\boldsymbol{x}_k)\right]^T\Big|_{\boldsymbol{x}_k=\widehat{\boldsymbol{x}}_{k|k-1}}$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

(a) Define the required state vector, $\boldsymbol{x}_k$.

(b) Define the motion model, $f_{k-1}(\boldsymbol{x}_k)$.

(c) Define the Jacobian of the motion model, $\widehat{\boldsymbol{F}}_{k-1}$.

(d) Define the Jacobian of the control model, $\widehat{\boldsymbol{G}}_{k-1}$.

(e) Define the measurement model, $h_k\left(\widehat{\boldsymbol{x}}_{k|k-1}\right)$.

(f) Define the Jacobian of the measurement model, $\widehat{\boldsymbol{H}}_k$.

(g) Define the measurement uncertainty, $\boldsymbol{R}_k$.

(h) Define the initial state estimate, $\langle \widehat{\boldsymbol{x}}_0, \boldsymbol{P}_0 \rangle$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

QUESTION 2

Given robot in *Question 1*, with an update period of T=0.1 sec, the vehicle performs the given command sequence 20 degrees anticlockwise rotation, forward 0.1 m, and a final rotation of 10 degrees clockwise.

The command uncertainties are;

- $\alpha_1 = 0.05$
- $\alpha_2 = 0.001$
- $\alpha_3 = 0.05$
- $\alpha_4 = 0.01$

The first bearing measurement is 90 degrees to a known landmark located at (x,y) = (0,1), find;

(a) The predicted state mean, $\hat{\boldsymbol{x}}_{k|k-1}$.

(b) The predicted state covariance, $\boldsymbol{P}_{k|k-1}$.

(c) The Kalman gain, $\boldsymbol{K}_k$.

(d) The corrected state mean, $\hat{\boldsymbol{x}}_{k|k}$.

(e) The corrected state covariance, $\boldsymbol{P}_{k|k}$.

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$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k\left(\boldsymbol{z}_k - h_k(\hat{\boldsymbol{x}}_{k|k-1})\right)$$

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(e) Define the measurement model, $h_k(\widehat{\boldsymbol{x}}_{k|k-1})$.

(f) Define the Jacobian of the measurement model, $\widehat{\boldsymbol{H}}_k$.

(g) Define the measurement uncertainty, $\boldsymbol{R}_k$.

(h) Define the initial state estimate, $\langle \widehat{\boldsymbol{x}}_0, \boldsymbol{P}_0 \rangle$

Answers

(a) $\boldsymbol{x}_k = \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix}$

(b) $f_{k-1}(\boldsymbol{x}_{k-1}) \equiv \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + d\sin(\theta_{k-1} + \theta_1) \\ y_{k-1} + d\cos(\theta_{k-1} + \theta_1) \\ \theta_{k-1} + \theta_1 + \theta_2 \end{bmatrix}$

(c) $\widehat{\boldsymbol{F}}_{k-1} = \begin{bmatrix} 1 & 0 & d\cos(\theta_{k-1} + \theta_1) \\ 0 & 1 & -d\sin(\theta_{k-1} + \theta_1) \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\widehat{\boldsymbol{G}}_{k-1} = \begin{bmatrix} d\cos(\theta_{k-1} + \theta_1) & \sin(\theta_{k-1} + \theta_1) & 0 \\ -d\sin(\theta_{k-1} + \theta_1) & \cos(\theta_{k-1} + \theta_1) & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(e) $h_k(\widehat{\boldsymbol{x}}_{k|k-1}) \equiv \psi_k = [atan2(y_l - y, x_l - x) - \theta_k]$

(f) $\widehat{\boldsymbol{H}}_k = \begin{bmatrix} \frac{y_l - y}{(y-y)^2 + (x_l - x)^2} & -\frac{x_l - x}{(y-y)^2 + (x_l - x)^2} & -1 \end{bmatrix}$

(g) $\boldsymbol{R}_k = \left[\left(\frac{\pi}{9}\right)^2 \right]$

(h) $\langle \widehat{\boldsymbol{x}}_0, \boldsymbol{P}_0 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & \left(\frac{\pi}{18}\right)^2 \end{bmatrix} \right\rangle$

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Answers

(a) $$f_{k-1}(\boldsymbol{x}_{k-1}) \equiv \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} 0 + 0.1\sin\left(0 + \left(-\frac{\pi}{9}\right)\right) \\ 0 + 0.1\cos\left(0 + \left(-\frac{\pi}{9}\right)\right) \\ 0 + \left(-\frac{\pi}{9}\right) + \left(\frac{\pi}{18}\right) \end{bmatrix} = \begin{bmatrix} -0.0342020 \\ 0.0939693 \\ -0.174533 \end{bmatrix}$$

(b) $$\boldsymbol{Q}_{k-1} = \begin{bmatrix} \alpha_1|\hat{\delta}_{rot1}| + \alpha_2\hat{\delta}_{trans} & 0 & 0 \\ 0 & \alpha_3\hat{\delta}_{trans} + \alpha_4(|\hat{\delta}_{rot1}| + |\hat{\delta}_{rot2}|) & 0 \\ 0 & 0 & \alpha_1|\hat{\delta}_{rot2}| + \alpha_2\hat{\delta}_{trans} \end{bmatrix}$$

$$= \begin{bmatrix} 0.05\left|-\frac{\pi}{9}\right| + 0.001 \times 0.1 & 0 & 0 \\ 0 & 0.05 \times 0.1 + 0.01\left(\left|-\frac{\pi}{9}\right| + \left|\frac{\pi}{18}\right|\right) & 0 \\ 0 & 0 & 0.05\left|\frac{\pi}{18}\right| + 0.001 \times 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.01755 & 0 & 0 \\ 0 & 0.01024 & 0 \\ 0 & 0 & 0.00883 \end{bmatrix}$$

$$\widehat{\boldsymbol{F}}_{k-1} = \begin{bmatrix} 1 & 0 & d\cos(\theta_{k-1} + \theta_1) \\ 0 & 1 & -d\sin(\theta_{k-1} + \theta_1) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1\cos\left(0 + -\frac{\pi}{9}\right) \\ 0 & 1 & -0.1\sin\left(0 + -\frac{\pi}{9}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0.09397 \\ 0 & 1 & 0.03420 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{\boldsymbol{G}}_{k-1} = \begin{bmatrix} d\cos(\theta_{k-1} + \theta_1) & \sin(\theta_{k-1} + \theta_1) & 0 \\ -d\sin(\theta_{k-1} + \theta_1) & \cos(\theta_{k-1} + \theta_1) & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1\cos\left(0 + -\frac{\pi}{9}\right) & \sin\left(0 + -\frac{\pi}{9}\right) & 0 \\ -0.1\sin\left(0 + -\frac{\pi}{9}\right) & \cos\left(0 + -\frac{\pi}{9}\right) & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.09397 & -0.34202 & 0 \\ 0.03420 & 0.93969 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}_{k|k-1} = \begin{bmatrix} 0.09397 & -0.34202 & 0 \\ 0.03420 & 0.93969 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01755 & 0 & 0 \\ 0 & 0.01024 & 0 \\ 0 & 0 & 0.00883 \end{bmatrix} \begin{bmatrix} 0.09397 & 0.03420 & 1 \\ -0.34202 & 0.93969 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0.09397 \\ 0 & 1 & 0.03420 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & \left(\frac{\pi}{18}\right)^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.09397 & 0.03420 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.00162 & -0.00314 & 0.00451 \\ -0.00314 & 1.00910 & 0.00164 \\ 0.00451 & 0.00164 & 0.05684 \end{bmatrix}$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.

(c) $\widehat{\boldsymbol{H}}_k = \begin{bmatrix} \frac{y_l - y}{(y-y)^2 + (x_l - x)^2} & -\frac{x_l - x}{(y-y)^2 + (x_l - x)^2} & -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1 - 0.0939693}{(1 - 0.0939693)^2 + (0.0342020)^2} & -\frac{0 - (-0.0342020)}{(1 - 0.0939693)^2 + (0.0342020)^2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.10214 & -0.04161 & -1 \end{bmatrix}$$

$$\boldsymbol{S}_k = \left[\left(\frac{\pi}{9}\right)^2\right] + \begin{bmatrix} 1.10214 & -0.04161 & -1 \end{bmatrix} \begin{bmatrix} 1.00162 & -0.00314 & 0.00451 \\ -0.00314 & 1.00910 & 0.00164 \\ 0.00451 & 0.00164 & 0.05684 \end{bmatrix} \begin{bmatrix} 1.10214 \\ -0.04161 \\ -1 \end{bmatrix}$$

$$= [1.36875]$$

$$\boldsymbol{K}_k = \begin{bmatrix} 1.00162 & -0.00314 & 0.00451 \\ -0.00314 & 1.00910 & 0.00164 \\ 0.00451 & 0.00164 & 0.05684 \end{bmatrix} \begin{bmatrix} 1.10214 \\ -0.04161 \\ -1 \end{bmatrix} \left[\frac{1}{1.36875}\right]$$

$$= \begin{bmatrix} 0.80521 \\ -0.03040 \\ -0.02226 \end{bmatrix}$$

(d) $\widehat{\boldsymbol{x}}_{k|k} = \widehat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k \left(\boldsymbol{z}_k - h_k\left(\widehat{\boldsymbol{x}}_{k|k-1}\right)\right)$

$$= \begin{bmatrix} -0.0342020 \\ 0.0939693 \\ -0.174533 \end{bmatrix} + \begin{bmatrix} 0.80521 \\ -0.03040 \\ -0.02226 \end{bmatrix} \left(\frac{\pi}{2} - \left[atan2\left(1 - 0.0939693, 0 - (-0.0342020)\right) - (-0.174533)\right]\right)$$

$$= \begin{bmatrix} -0.0342020 \\ 0.0939693 \\ -0.174533 \end{bmatrix} + \begin{bmatrix} 0.80521 \\ -0.03040 \\ -0.02226 \end{bmatrix} [-0.13680]$$

$$= \begin{bmatrix} -0.14436 \\ 0.09812 \\ -0.17149 \end{bmatrix}$$

(e) $\boldsymbol{P}_{k|k} =$

$$\begin{bmatrix} 1.00162 & -0.00314 & 0.00451 \\ -0.00314 & 1.00910 & 0.00164 \\ 0.00451 & 0.00164 & 0.05684 \end{bmatrix} - \begin{bmatrix} 0.80521 \\ -0.03040 \\ -0.02226 \end{bmatrix} [1.36875] \begin{bmatrix} 0.80521 & -0.03040 & -0.02226 \end{bmatrix}$$

$$= \begin{bmatrix} 0.11416 & 0.03037 & 0.02904 \\ 0.03037 & 1.00783 & 0.00072 \\ 0.02904 & 0.00072 & 0.05616 \end{bmatrix}$$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic robotics* vol. 1: MIT press Cambridge, 2005.