CS189 HW05

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2. Rayleigh Quotient

(a) $R(M, x) = \frac{x'Mx}{x'x}$ Assume $x'x = 1$ then $R(M, X) = x'Mx$

$\because$ M is symmetric $\therefore \underset{x'x=1}{R(M, x)} = x'V\Lambda V^T x$

$\therefore$ Upper bound: when $x = V_{max}$ $V'_{max} V \Lambda V^T V_{max} = \lambda_{max}$

Lower bound: when $x = V_{min}$ $V'_{min} V \Lambda V^T V_{min} = \lambda_{min}$

(b) $\max_{w: \|w\|_2 = 1} \|Xw\|_2^2 = \max_{w: \|w\|_2 = 1} w^T X^T X w = \max_w \frac{w^T X^T X w}{w^T w}$

$= \max R(X^T X, w) = \lambda_{max}$ of $X^T X$

(c) $\underset{\lambda}{\text{argmin}} \|Ax - \lambda x\|_2^2$

$= \underset{\lambda}{\text{argmin}} (Ax - \lambda x)^T (Ax - \lambda x) = x^T A^T A x - \lambda x^T A^T x - \lambda x^T A x + \lambda^2 x^T x$

$\nabla_\lambda \|Ax - \lambda x\|_2^2 = -x^T A^T x - x^T A x + 2x^T x \lambda = 0$

$2x^T x \lambda = x^T A^T x + x^T A x$

when A is asymmetric,

$2x^T x \lambda = 2x^T A x$

$\lambda = \frac{x^T A x}{x^T x} = R(A, x)$

When x is an eigenvector, λ must be eigenvalue of A.

3. Correlation Coefficient

(a) $|\rho(aX+c,\ bY+d)| = \left|\dfrac{Cov(aX+bY)}{\sqrt{Var(aX)\cdot Var(bY)}}\right| = \left|\dfrac{ab\ cov(X,Y)}{\sqrt{a^2\cdot b^2\cdot Var(X)\cdot Var(Y)}}\right|$

when $a\cdot b \neq 0$

$= \left|\dfrac{a\cdot b\ Cov(X,Y)}{|a\cdot b|\sqrt{Var(X)\cdot Var(Y)}}\right| = \left|\dfrac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}\right| = |\rho(X,Y)|$

(b) See the results, plots and code in Appendix

4. Canonical Correlation Analysis

(a) $$\max_{\|u\|=1\,\|v\|=1} \rho(Xu, Yv) = \max_{\|u\|=1\,\|v\|=1} \frac{u'X'Yv}{\sqrt{u'X'Xu \cdot v'Y'Yv}}$$

$$\max_{u,v} \; u'C_{xy}v$$

$$\text{s.t.} \quad u'C_{xx}u = 1 \qquad v'C_{yy}v = 1$$

where $\quad C_{xy} = X'Y \quad C_{xx} = X'X \quad C_{yy} = Y'Y$

Lagrangian:

$$L(u, v; a, b) = 2u'C_{xy}v - a(u'C_{xx}u - 1) - b(v'C_{yy}v - 1)$$

$$L_u = 2C_{xy}v - 2a\,C_{xx}u = 0 \Rightarrow C_{xy}v = a \cdot C_{xx}u$$

$$L_v = 2C_{xy}'u - 2b \cdot C_{yy}v = 0 \Rightarrow C_{xy}'u = b \cdot C_{yy}v$$

$$L_a = u'C_{xx}u - 1 = 0 \Rightarrow u'C_{xx}u = 1$$

$$L_b = v'C_{yy}v - 1 = 0 \Rightarrow v'C_{yy}v = 1$$

$$\Downarrow$$

$$u'C_{xy}v = a\,u'C_{xx}u = a \qquad \Rightarrow \quad C_{xy}v = \lambda C_{xx}u$$

$$v'C_{xy}'u = b\,v'C_{yy}v = b \qquad\quad C_{xy}'u = \lambda C_{yy}v$$

$$a = b = \lambda$$

$$\because C_{xy}' = C_{yx} \quad \therefore \quad C_{xy}v = \lambda C_{xx}u$$

$$C_{yx}u = \lambda C_{yy}v$$

$$\therefore \begin{pmatrix} & C_{xy} \\ C_{yx} & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

(b) Proof:
$$\begin{pmatrix} & C_{xy} \\ C_{yx} & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} & C_{xy} \\ C_{yx} & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} & C_{xy} \\ C_{yx} & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} C_{xx} & \\ & C_{yy} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix} & C_{xy}\\ C_{yx} & \end{pmatrix}\begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{\frac{1}{2}}\begin{pmatrix}u\\ v\end{pmatrix} = \lambda \begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{\frac{1}{2}}\begin{pmatrix}u\\ v\end{pmatrix}$$

After step 1 $\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{\frac{1}{2}}\begin{pmatrix}u\\ v\end{pmatrix}$

The equation becomes

$$\begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix} & C_{xy}\\ C_{yx} & \end{pmatrix}\begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \lambda\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$$

$$\begin{pmatrix} & C_{xx}^{-\frac{1}{2}} C_{xy} C_{yy}^{-\frac{1}{2}}\\ C_{xy}^{-\frac{1}{2}} C_{yx} C_{xx}^{-\frac{1}{2}} & \end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \lambda\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$$

After step 2: $(\tilde{u}, \tilde{v}, \lambda) = \mathrm{svd}(C_{xx}^{-\frac{1}{2}} C_{xy} C_{yy}^{-\frac{1}{2}})$

$\tilde{u}\lambda\tilde{v}' = C_{xx}^{-\frac{1}{2}} C_{xy} C_{yy}^{-\frac{1}{2}}$

$$\begin{pmatrix} & \tilde{u}\lambda\tilde{v}'\\ \tilde{v}\lambda\tilde{u}' & \end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix}\tilde{u}\lambda\tilde{v}'\tilde{v}\\ \tilde{v}\lambda\tilde{u}'\tilde{u}\end{pmatrix} = \begin{pmatrix}\lambda\tilde{u}\\ \lambda\tilde{v}\end{pmatrix} = \lambda\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$$

$\therefore$ We get $\tilde{u}, \tilde{v}$

To get back to u and v, we do step 3.

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{\frac{1}{2}}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix} C_{xx} & \\ & C_{yy}\end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$$

(c) $\rho(Xu, Yv) = \rho(X C_{xx}^{-\frac{1}{2}} \cdot C_{xx}^{\frac{1}{2}} u, Y C_{yy}^{-\frac{1}{2}} C_{yy}^{\frac{1}{2}} v) = \rho(\tilde{X}\tilde{u}, \tilde{Y}\tilde{v})$

$$\frac{u' C_{xy} v}{\sqrt{u' C_{xx} u \cdot v' C_{yy} v}}$$

$C_{\tilde{X}\tilde{Y}} = \tilde{X}'\tilde{Y} = C_{xx}^{-\frac{1}{2}} C_{xy} C_{yy}^{-\frac{1}{2}}$

$C_{\tilde{X}\tilde{X}} = \tilde{X}'\tilde{X} = (X C_{xx}^{-\frac{1}{2}})' X C_{xx}^{-\frac{1}{2}} = C_{xx}^{-\frac{1}{2}} X'X C_{xx}^{-\frac{1}{2}}$
$= C_{xx}^{-\frac{1}{2}} C_{xx} C_{xx}^{-\frac{1}{2}} = I$

$C_{\tilde{Y}\tilde{Y}} = \tilde{Y}'\tilde{Y} = (Y C_{yy}^{-\frac{1}{2}})' Y C_{yy}^{-\frac{1}{2}} = C_{yy}^{-\frac{1}{2}} Y'Y C_{yy}^{-\frac{1}{2}} = C_{yy}^{-\frac{1}{2}} C_{yy} C_{yy}^{-\frac{1}{2}}$
$= I$

$\therefore$ $\tilde{X}$ and $\tilde{Y}$ are whitened in their own spaces

(d) $\rho(Xu, Yv) = \rho(\tilde{X}\tilde{u}, \tilde{Y}\tilde{v}) = \dfrac{\tilde{u}' C_{\tilde{X}\tilde{Y}} \tilde{v}}{\sqrt{\tilde{u}'\tilde{u} \cdot \tilde{v}'\tilde{v}}}$

$$= \frac{\tilde{u}' C_{XX}^{-\frac{1}{2}} C_{XY} C_{YY}^{-\frac{1}{2}} \tilde{v}}{\sqrt{\tilde{u}'\tilde{u} \cdot \tilde{v}'\tilde{v}}}$$

Assume $\|\tilde{u}\|_2 = \|\tilde{v}\|_2 = 1$

~~then~~ $\max\limits_{\|\tilde{u}\|_2 = \|\tilde{v}\|_2 = 1} \rho(\tilde{X}\tilde{u}, \tilde{Y}\tilde{v}) = \max\limits_{\|\tilde{u}\|_2 = \|\tilde{v}\|_2 = 1} \tilde{u}' C_{XX}^{-\frac{1}{2}} C_{XY} C_{YY}^{-\frac{1}{2}} \tilde{v}$

$\because$ step 2 : $(\tilde{u}, \tilde{v}, \lambda) = \mathrm{svd}(C_{XX}^{-\frac{1}{2}} C_{XY} C_{YY}^{-\frac{1}{2}})$

$\therefore$ $\tilde{u} \lambda \tilde{v}' = C_{XX}^{-\frac{1}{2}} C_{XY} C_{YY}^{-\frac{1}{2}}$

$\lambda = \tilde{u}' C_{XX}^{-\frac{1}{2}} C_{XY} C_{YY}^{-\frac{1}{2}} \tilde{v}$

$\therefore$ $= \max\limits_{\|\tilde{u}\|_2 = \|\tilde{v}\|_2 = 1} \lambda = \lambda_1$

(e) The plots and code are shown in Appendix.