Homework 2

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1. I assume all 100 booklets' weight $\overset{iid}{\sim} N(1, 0.0025)$

Then because the weight of those 100 booklets are independent, the total weight of them also satisfies normal distribution which is $N(100, 0.25)$

then use normalization

$$\frac{N(100, 0.25) - 100}{\sqrt{0.25}} \sim N(0, 1)$$

$$\therefore \quad \frac{100.4 - 100}{\sqrt{0.25}} = 0.8$$

$\therefore$ P (100 booklets weigh more than 100.4 ounces)

$$= 1 - \underset{}{\overset{(\Phi(0.8))}{78.81\%}} = 21.19\%$$

2. (1) $E(\bar{X}_n^* \mid X_1 \cdots X_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^* \mid X_1 \cdots X_n\right)$

$= (X_1^* \mid X_1 \cdots X_n) = E_{ECDF}(X_1^*)$ because bootstrap samples are independent from $X_1 \cdots X_n$ $= \sum_{i=1}^{n} X_i P(X_1^* = X_i)$ but sample from $X_1 \cdots X_n$

$= \sum_{i=1}^{n} (X_i \times \frac{1}{n}) = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$

(2) $V(\bar{X}_n^* \mid X_1 \cdots X_n) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i^* \mid X_1 \cdots X_n\right)$

$= V\left(\frac{1}{n}\sum_{i=1}^{n} X_i^*\right) = \frac{1}{n^2} \times V\left(\sum_{i=1}^{n} X_i^*\right) = \frac{1}{n^2} \times n\, V(X_1^*) = \frac{1}{n} V(X_1^*)$

$= \frac{1}{n} E[X_1^* - E(X_1^*)]^2 = \frac{1}{n} \times \sum_{i=1}^{n} [X_i - E(X_1^*)]^2 \times P(X_1^* = X_i)$

$= \frac{1}{n} \times \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \times \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

(3) $\underbrace{E(\bar{X}_n^*) = E(E(\bar{X}_n^* \mid X_1 \cdots X_n)) = E(\bar{X}_n) = E(X_1)}_{\text{if } X_1 \cdots X_n \text{ are i.i.d}}$

if $E(X_1) = \mu$ then $E(\bar{X}_n^*) = E(\bar{X}_n) = E(X_1) = \mu$

(4) $V(\bar{X}_n^*) = V(E(\bar{X}_n^* \mid X_1 \cdots X_n)) + E(V(\bar{X}_n^* \mid X_1 \cdots X_n))$

$= V(\bar{X}_n) + E\left(\frac{1}{n^2}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2\right)$

if $X_1 \cdots X_n$ iid and $Var(X_i) = \sigma^2$

then: $= \frac{\sigma^2}{n} + \frac{1}{n^2} E\left(\sum_{i=1}^{n} (X_i - \bar{X}_n)^2\right)$

$= \frac{\sigma^2}{n} + \frac{1}{n^2}\left(\sum_{i=1}^{n} E(X_i^2) - n E(\bar{X}_n^2)\right)$

$= \frac{\sigma^2}{n} + \frac{1}{n^2}\left(n(V(X_i) + (E X_i)^2) - n(V(\bar{X}_n) + E^2(\bar{X}_n))\right)$

$= \frac{\sigma^2}{n} + \frac{1}{n^2}\left(n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right)$

$= \frac{\sigma^2}{n} + \frac{1}{n^2}(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2) = \frac{\sigma^2}{n} + \frac{n-1}{n^2}\sigma^2 = \frac{2n-1}{n^2}\sigma^2$

3. $L_n(\theta) = \prod_{i=1}^{n} P_\theta(X = x_i) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$

$l_n(\theta) = \log\left[\theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}\right]$

$= \sum_{i=1}^{n} x_i (\log \theta) + (n - \sum_{i=1}^{n} x_i) \log(1-\theta)$

$\frac{\partial}{\partial \theta} l_n(\theta) = \sum_{i=1}^{n} x_i \times \frac{1}{\theta} + (n - \sum_{i=1}^{n} x_i) \times \frac{1}{1-\theta} \times (-1) = 0$

$(\theta - 1) \sum_{i=1}^{n} x_i + \theta (n - \sum_{i=1}^{n} x_i) = 0$

$n\theta = \sum_{i=1}^{n} x_i \qquad \theta = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}_n$

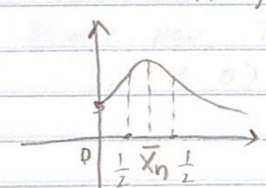
when $\theta < \bar{X}_n$: $\frac{\partial}{\partial \theta} l_n(\theta) = \frac{1}{\theta} \sum_{i=1}^{n} x_i + \frac{1}{\theta - 1} (n - \sum_{i=1}^{n} x_i)$

$= \frac{-1}{\theta(\theta-1)} \sum_{i=1}^{n} x_i + \frac{n}{\theta - 1} = \frac{n\theta - \sum_{i=1}^{n} x_i}{\theta(\theta-1)}$

$\because 0 < \theta \le \frac{1}{2}$ $\therefore \theta(\theta - 1) < 0$ $\because \theta < \bar{X}_n$ $\therefore \frac{\partial}{\partial \theta} l_n(\theta) > 0$

$\because$ using the same method, get when $\theta > \bar{X}_n$, $\frac{\partial}{\partial\theta} l_n(\theta) < 0$

$\therefore$ The graph of $l_n(\theta)$ looks approximately like this.



$\therefore$ When $\bar{X}_n \ge \frac{1}{2}$, then MLE of θ is $\frac{1}{2}$

when $\bar{X}_n < \frac{1}{2}$, then MLE of θ is $\bar{X}_n$

4. Now we only have one x. So, $L(\theta) = f(x;\theta)$

So, the MLE of θ will maximize $L(\theta)$ at different X value respectively.

$\therefore$

x	$\max f(x;\theta)$	θ with $\max f(x;\theta)$
0	$\frac{1}{3}$	1
1	$\frac{1}{3}$	1
2	$\frac{1}{4}$	2 and 3
3	$\frac{1}{2}$	3
4	$\frac{1}{4}$	3

$\because \theta \in \{1, 2, 3\}$ $\therefore$ the θ with $\max f(x;\theta)$ at each x value is the MLE of θ.

5. when $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, the density for $X_1 \dots X_n$ is

$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ then $\log[f(x;\mu,\sigma^2)] = -\frac{(x-\mu)^2}{2\sigma^2} - \log(\sqrt{2\pi}\sigma)$

$= -\frac{(x-\mu)^2}{2\sigma^2} - \log\sigma - \log\sqrt{2\pi}$

Then $\frac{\partial^2}{\partial \mu^2} \log f(x;\mu,\sigma) = \frac{\partial^2}{\partial \mu^2}\left(-\frac{(x-\mu)^2}{2\sigma^2} - \log\sigma - \log\sqrt{2\pi}\right) = -\frac{1}{\sigma^2}$

$\frac{\partial^2}{\partial\sigma\partial\mu}(\log f) = \frac{\partial^2}{\partial\mu\partial\sigma} \log f(x;\mu,\sigma) = \frac{\partial^2}{\partial\mu\partial\sigma}\left(-\frac{(x-\mu)^2}{2\sigma^2} - \log\sigma - \log\sqrt{2\pi}\right) = -\frac{2(x-\mu)}{\sigma^3}$

$\frac{\partial}{\partial\sigma^2} \log f(x;\mu,\sigma) = \frac{\partial^2}{\partial\sigma^2}\left(-\frac{(x-\mu)^2}{2\sigma^2} - \log\sigma - \log\sqrt{2\pi}\right) = -\frac{3(x-\mu)^2}{\sigma^4} + \frac{1}{\sigma^2}$

Then $I_n(\theta) = n I_1(\theta) = n \begin{pmatrix} -E_\theta(-\frac{1}{\sigma^2}) & -E_\theta(-\frac{2(x-\mu)}{\sigma^3}) \\ -E_\theta(-\frac{2(x-\mu)^2}{\sigma^3}) & -E_\theta(\frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}) \end{pmatrix}$

$= n \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & -\frac{1}{\sigma^2} + \frac{3}{\sigma^2} = \frac{2}{\sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$

because $E(X_i - \mu) = 0$

because $E(X_i-\mu)^2 = Var(X_i) = \sigma^2$

Because now $I_n(\theta)$ is a 2×2 matrix

$J_n(\mu,\sigma) = I_n(\mu,\sigma)^{-1} = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{pmatrix}$

6. $\because X_1 \dots X_n \overset{iid}{\sim} Unif(0,\theta)$

$\therefore f(x;\theta) = \frac{1}{\theta} I\{0<x<\theta\}$

$\therefore L_n(\theta) = \frac{1}{\theta^n}\prod_{i=1}^n I\{0<X_i<\theta\}$ To maximizes $L_n(\theta)$ here

I need to get the smallest θ possible.

$\because \prod_{i=1}^n I\{0<X_i<\theta\} = \prod_{i=1}^n (0 < X_{(1)} < X_{(2)} \cdots < X_{(n)} < \theta)$

$\therefore$ the minimum θ will be $X_{(n)}$ ($\max(X_i)$)

To show that the MLE of θ is consistent, I need to show that $\hat{\theta}_n \xrightarrow{P} \theta$ then I need to show that:

$P\{|\hat{\theta}_n - \theta| > \varepsilon\} \xrightarrow{D} 0$ for any $\varepsilon > 0$

$\because \hat{\theta}_n$ is the minimum θ possible $\therefore \hat{\theta}_n - \theta \le 0$

$\therefore P\{\theta - \hat{\theta}_n > \varepsilon\} = P\{\hat{\theta}_n < \theta - \varepsilon\} = P\{\max(X_i) < \theta - \varepsilon\}$

$= [P\{X_1 < \theta-\varepsilon\}]^n = [\frac{\theta-\varepsilon}{\theta}]^n = (1-\frac{\varepsilon}{\theta})^n$ $\because 0 < 1 - \frac{\varepsilon}{\theta} < 1$

$\therefore (1-\frac{\varepsilon}{\theta})^n \longrightarrow 0$ as $n\to\infty$ $\therefore P\{\theta - \hat{\theta}_n > \varepsilon\} \to 0$ as $n \to \infty$ for any $\varepsilon > 0$

$\therefore \hat{\theta}_n \xrightarrow{P} \theta$

7 (a) $\psi = P(Y_1 = 1) = P(X_1 > 0) = P(X_1 - \theta > -\theta)$

$\because X_1 \sim N(\theta, 1) \quad \therefore X_1 - \theta \sim N(0,1)$

$\therefore \psi = 1 - \Phi(-\theta) = \Phi(\theta)$

$\because X_1 \cdots X_n \overset{iid}{\sim} N(\theta, 1)$

$\therefore l_n(\theta) = \log\left(\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{\sum_{i=1}^n (X_i-\theta)^2}{2}}\right)$

$= n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{\sum_{i=1}^n (X_i-\theta)^2}{2}$

$\frac{\partial}{\partial\theta} l_n(\theta) = -\frac{1}{2} \times \sum_{i=1}^n 2(X_i - \theta) \times (-1) = \sum_{i=1}^n (X_i - \theta) = 0$

$\sum_{i=1}^n X_i - n\theta = 0 \quad n\theta = \sum_{i=1}^n X_i \quad \theta = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}_n$

$\therefore$ MLE of θ is $\bar{X}_n$

$\therefore$ MLE of ψ is $\Phi(\bar{X}_n)$

(b) $\because \Phi(x)$ is the CDF of standard normal distribution.

$\therefore \Phi'(x)$ is the PDF of standard normal distribution

$\therefore \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$\because \frac{\hat{\psi} - \psi}{se(\hat{\psi})} \xrightarrow{D} N(0,1) \quad \therefore \frac{\Phi(\bar{X}_n) - \psi}{se(\hat{\psi})} \xrightarrow{D} N(0,1)$

$se(\hat{\psi}) \approx se(\hat{\theta})|\Phi'(\hat{\theta})| = se(\bar{X}_n)|\Phi'(\hat{\theta})| = \sqrt{\frac{1}{n}}\,|\Phi'(\hat{\theta})|$

$= \frac{\Phi'(\hat{\theta})}{\sqrt{n}} = \frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{\hat{\theta}^2}{2}} = \frac{e^{-\frac{\bar{X}_n^2}{2}}}{\sqrt{2n\pi}}$

$\therefore \frac{\Phi(\bar{X}_n) - \psi}{\frac{e^{-\frac{\bar{X}_n^2}{2}}}{\sqrt{2n\pi}}} \xrightarrow{D} N(0,1)$

CI: $Z_{0.025} \le \frac{\Phi(\bar{X}_n) - \psi}{\frac{e^{-\frac{\bar{X}_n^2}{2}}}{\sqrt{2n\pi}}} \le -Z_{0.025}$

$C_n: \Phi(\bar{X}_n) \pm Z_{0.025} \frac{e^{-\frac{\bar{X}_n^2}{2}}}{\sqrt{2n\pi}}$

(c) Use Weak law of large numbers

$\tilde{\psi} = \frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{P} E(Y_1) = E(I\{X_1 > 0\}) = P\{X_1 > 0\} = \psi$

$\therefore \tilde{\psi}$ is a consistent estimator of ψ

(d) $\sqrt{n}(\tilde{\psi} - \psi) \xrightarrow{D} N(0, n\,Var(\tilde{\psi}))$ $\quad n Var(\tilde{\psi}) = n \times \frac{1}{n^2} \times n \times Var(Y_i)$

$= Var(Y_i)$

$\therefore \sqrt{n}\,(\tilde{\psi} - \psi) \xrightarrow{D} N(0, V(Y_i))$

$\because Y_i = I\{X_i > 0\} \quad \therefore Y_i \sim Ber(P\{X_i > 0\}) \quad \therefore Y_i \sim Ber(\Phi(\theta))$

$\therefore V(Y_i) = \Phi(\theta)(1 - \Phi(\theta))$

$\therefore V(\tilde{\psi}) = \dfrac{\Phi(\theta)(1-\Phi(\theta))}{n}$

$\because se(\hat{\psi}) \approx \dfrac{\Phi'(\theta)}{\sqrt{n}} \quad \therefore V(\hat{\psi}) \approx \dfrac{(\Phi'(\theta))^2}{n}$

$\therefore$ the asymptotic relative efficiency of $\tilde{\psi}$ to $\hat{\psi}$ is

$$\frac{V(\tilde{\psi})}{V(\hat{\psi})} = \frac{\Phi(\theta)(1-\Phi(\theta))}{(\Phi'(\theta))^2} = \frac{\Phi(\theta)(1-\Phi(\theta))}{\left(\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}\right)^2}$$

$$= \frac{\Phi(\theta)(1-\Phi(\theta))}{\frac{e^{-\theta^2}}{2\pi}} = 2\pi e^{\theta^2}\Phi(\theta)(1-\Phi(\theta))$$

e) $\because E(X_i) = \theta$, $Var(X_i) = 1$ but X_i is not normal distribution

$\therefore$ One possibility is that
$$X_i = \begin{cases} \theta+1 & \frac{1}{2} \\ \theta-1 & \frac{1}{2} \end{cases} \quad \text{(Probability)}$$

Prove: $E(X_i) = (\theta+1)\times\frac{1}{2} + (\theta-1)\times\frac{1}{2} = \theta$

$Var(X_i) = E(X_i^2) - (EX_i)^2 = (\theta+1)^2\times\frac{1}{2} + (\theta-1)^2\times\frac{1}{2} - \theta^2 = 1$

$\therefore$ This example is valid

$$\psi = P(Y_i = 1) = P(X_i > 0) = \begin{cases} 0 & \theta \le -1 \\ \frac{1}{2} & -1 < \theta \le 1 \\ 1 & \theta > 1 \end{cases}$$

if $\theta \le -1$

then: $P(|\hat{\psi} - \underbrace{\psi}_{=0}| > \varepsilon) = P(|\hat{\psi}| > \varepsilon) = P(\Phi(\bar{X}_n) > \varepsilon)$

$= P(\bar{X}_n > \Phi^{-1}(\varepsilon))$ according to Central Limit Theorem

$= P(\sqrt{n}(\bar{X}_n - \theta) > \sqrt{n}(\Phi^{-1}(\varepsilon) - \theta))$

$= 1 - \Phi(\sqrt{n}(\Phi^{-1}(\varepsilon) - \theta))$

when ε is a small positive number $\Phi^{-1}(\varepsilon)$ will be negative

then $\sqrt{n}(\Phi^{-1}(\varepsilon) - \theta) \to -\infty$ as $n \to \infty$

then $\Phi(\sqrt{n}(\Phi^{-1}(\varepsilon) - \theta)) \to 0$ as $n \to \infty$

then $1 - \Phi(\sqrt{n}(\Phi^{-1}(\varepsilon) - \theta)) \to 1$ as $n \to \infty$

$\therefore \hat{\psi} \overset{P}{\not\to} \psi$

$\because \hat{\psi} = \Phi(\bar{X}_n)$, $\bar{X}_n \xrightarrow{P} E(X_1) = \theta$ according to weak law of large numbers $\therefore \Phi(\bar{X}_n) \xrightarrow{P} \Phi(\theta)$ which is $\hat{\psi} \xrightarrow{P} \Phi(\theta)$

$\therefore \hat{\psi}$ always $\xrightarrow{P} \Phi(\theta)$ no matter $X_1 \cdots X_n$ are normal or not.

8. (a) $\because \alpha$ is known $\therefore f(x;\beta) = \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$

$$\ell_n(\beta) = \log\left(\prod_{i=1}^n \frac{\beta^\alpha x_i^{\alpha-1} e^{-\beta x_i}}{\Gamma(\alpha)}\right) = \log\left[\left(\frac{1}{\Gamma(\alpha)}\right)^n \beta^{n\alpha}\right] + (\alpha-1)\sum_{i=1}^n \log X_i - \beta\sum_{i=1}^n X_i$$

$$= -n\log(\Gamma(\alpha)) + n\alpha\log\beta + (\alpha-1)\sum_{i=1}^n \log X_i - \beta\sum_{i=1}^n X_i$$

$$\frac{\partial}{\partial\beta}\ell_n(\beta) = \frac{n\alpha}{\beta} - \sum_{i=1}^n X_i = 0 \iff \beta = \frac{\alpha}{\frac{1}{n}\sum_{i=1}^n X_i} = \frac{\alpha}{\bar{X}_n} > 0$$

It satisfies the range of β. $\therefore \beta = \dfrac{\alpha}{\bar{X}_n}$ will maximize $\ell_n(\beta)$

$\therefore$ The MLE of β is $\dfrac{\alpha}{\bar{X}_n}$.

(b) $\because$ Gamma distribution belongs to exponential family

$$\therefore I_n(\beta) = -E_\beta\left(\frac{\partial^2}{\partial\beta^2}\ell_n(\beta)\right) = -E_\beta\left((-1)\times n\alpha\times\frac{1}{\beta^2}\right) = -E_\beta\left(-\frac{n\alpha}{\beta^2}\right)$$

$$= E_\beta\left(\frac{n\alpha}{\beta^2}\right) = \frac{n\alpha}{\beta^2}$$

$$V(\hat{\beta}_n) = \frac{1}{I_n(\hat{\beta}_n)} = \frac{\hat{\beta}_n^2}{n\alpha} = \frac{\left(\frac{\alpha}{\bar{X}_n}\right)^2}{n\alpha} = \frac{\alpha^2}{n\alpha\bar{X}_n^2} = \frac{\alpha}{n\bar{X}_n^2}$$

$$\therefore \frac{\beta - \hat{\beta}_n}{\sqrt{V(\hat{\beta}_n)}} = \frac{\beta - \frac{\alpha}{\bar{X}_n}}{\sqrt{\frac{\alpha}{n\bar{X}_n^2}}} \xrightarrow{D} N(0,1)$$

$\therefore$ The 95% confidence interval for β is:

$$\frac{\alpha}{\bar{X}_n} \pm Z_{1-\frac{0.95}{2}}\sqrt{\frac{\alpha}{n\bar{X}_n^2}} \Rightarrow \frac{\alpha}{\bar{X}_n} \pm Z_{0.025}\sqrt{\frac{\alpha}{n\bar{X}_n^2}}$$

(c) First, delete the code in betaexample.R that generates the 100 x samples ~ Beta(5, 4). Then add the code in this question on top to import the precipitation data of Berkeley. Second, because in this question it says " Assume the values for each year are iid with distribution Gamma(α, β). So change dbeta into dgamma. Finally, change all x=x in the code into x=winter.precip. After all the changes above, I can run the code.

	MLE	MLE	
I get	α	β	by typing "mle" in the Console
	6.204	0.004651	

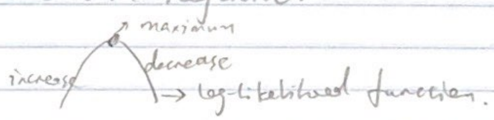
The fisher information matrix is $\begin{pmatrix} 14.514 & -18128.85 \\ -18128.85 & 26322909.88 \end{pmatrix}$ by typing "op $ hessian" in the Console.

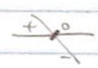
	α	β
By typing "lower" in the console I get	4.800	0.003608
	α	β
By typing "upper" in the Console I get	7.608	0.005694

So the 95% CI for α is $(4.800, 7.608)$

So the 95% CI for β is $(0.03608, 0.005694)$

The evidence for this algorithm to find a global optimum is that the second derivative of log-likelihood function to α and β are both negative.



∴ Second derivative must be  negative for α and β.

Note: The plots and output in R are on the next page.

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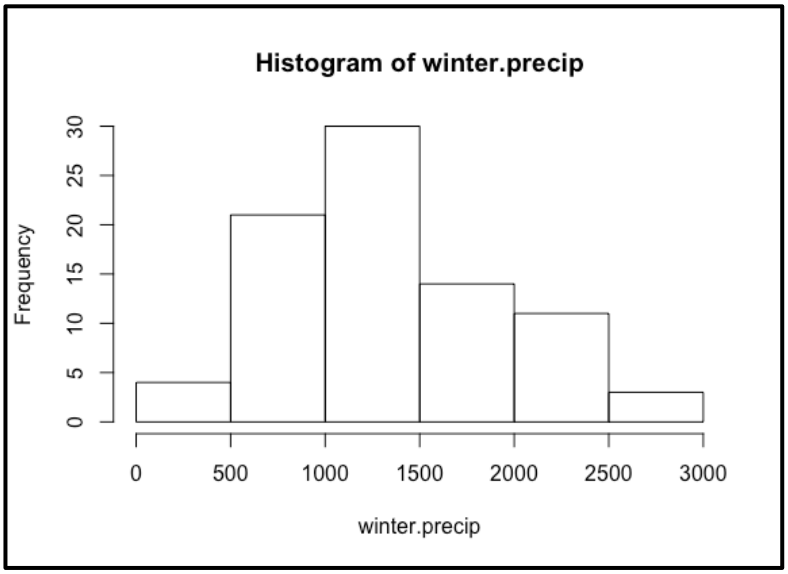
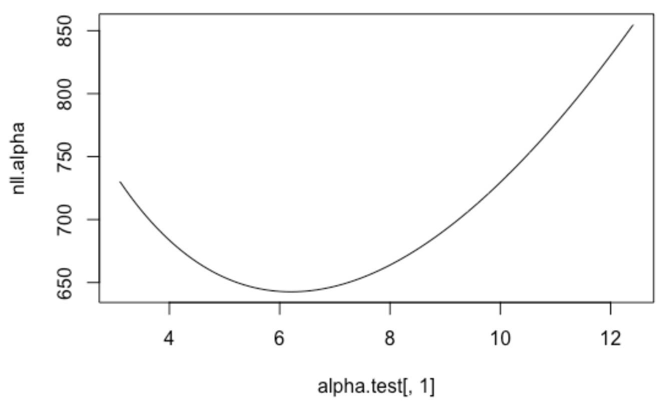
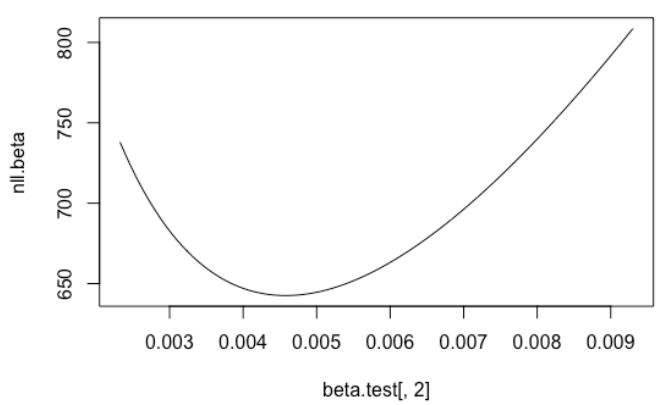
8 (c)







```
> mle
      alpha        beta
6.204123823 0.004650913
```

```
> op$hessian
            alpha        beta
alpha      14.514   -18128.85
beta   -18128.853 26322409.89
```

```
> lower
      alpha        beta
4.799790126 0.003608113
> upper
      alpha        beta
7.608457520 0.005693713
```