Homework 3
Statistics 201B
Due 9:30am Nov 6

Please turn in your solutions for problems 4-8. The others are for practice.

1. Suppose that $X_1, \cdots, X_n$ are i.i.d. Poisson random variables with parameter θ. Show that $T = \sum_{i=1}^n X_i$ is sufficient by definition (i.e., computing the conditional distribution given $T = t$).

2. Prove that a necessary and sufficient condition for a statistic T to be sufficient for a family $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ is that for any fixed θ and θ_0, the ratio $f_\theta(x)/f_{\theta_0}(x)$ is a function only of $T(x), \theta$ and θ_0.

3. Let $X_1, \cdots, X_n$ be independent Bernoulli variables with $p_i = P(X_i = 1)$ for $i = 1, ..., n$. Let $t_1, ..., t_n$ be a sequence of known constants related to p_i by:
$$\log(\frac{p_i}{1 - p_i}) = \alpha + \beta t_i,$$
where α and β are unknown parameters. Determine a sufficient statistic for the family of joint distributions indexed by $\theta = (\alpha, \beta)$.

4. Let $X_1, \cdots, X_n$ be i.i.d. $Uniform[\alpha, \beta]$, where α and β are unknown. Suppose that we wish to estimate the mean $\theta = \frac{\alpha+\beta}{2}$ under the quadratic loss $L(\theta, a) = (\theta - a)^2$. The sample mean $\bar{X}_n$ is one reasonable estimate of θ. Show that the estimator
$$\delta(X_1, ..., X_n) := E(\bar{X}_n | X_{(1)}, X_{(n)})$$
improves upon $\bar{X}_n$, and moreover that
$$\delta(X_1, ..., X_n) = \frac{X_{(1)} + X_{(n)}}{2}.$$

5. Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$.

   (a) Let $\lambda_0 > 0$. Find the size α Wald test for
   $$H_0 : \lambda = \lambda_0 \text{ versus } H_1 : \lambda \neq \lambda_0.$$

(b) (Computer experiment) Let $\lambda_0 = 1$, $n = 20$, and $\alpha = 0.05$. Simulate $X_1, \ldots, X_n \sim Poisson(\lambda_0)$ and perform the Wald test. Repeat many times and count how often you reject the null hypothesis. How close is the type I error rate to 0.05?

6. Let $X \sim Binomial(n, p)$. Construct the likelihood ratio test for

$$H_0 : p = p_0 \text{ versus } H_1 : p \neq p_0.$$

Compare it to the Wald test.

7. There is a theory that people can postpone their death until after an important event. To test the theory, Philips and King (1988) collected data on deaths around the Jewish holiday of Passover. Of 1919 daths, 922 died the week before the holiday and 997 died the week after. Think of this as a binomial and test the null hypothesis that $\theta = 1/2$. Report and interpret the p-value.

8. Let $X_1, \ldots, X_n$ be *iid* with density $f(x; \beta) = \beta e^{-\beta x}$ for $x > 0$ and $\beta > 0$. Find the asymptotic (large sample) likelihood ratio test of size α for $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$.

9. Let X and Y be two random variables with joint distribution F. Suppose we observe pairs $(x_1, y_1), \ldots, (x_n, y_n)$, a random sample from F. Without making any assumptions about F, form a statistic for testing $H_0 : P(X > Y) = 0.5$. How would you calculate the p-value?