

Calculus

Yaohui Wu

May 31, 2024

For all lovers of mathematics and science.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x) dx = F(b) - F(a)$$

Contents

1	Functions and Limits	4
1.1	The Limit of a Function	4
1.1.1	Functions	4
1.1.2	Intuitive Definition of a Limit	4
1.2	The Precise Definition of a Limit	5
1.2.1	Epsilon-Delta Definition of a Limit	5
1.3	Evaluating Limits	6
1.3.1	Limit Laws	6
1.4	Continuity	13
1.4.1	Continuous Functions	13
1.5	Limits and Infinity	14
1.5.1	Infinite Limits	14
1.5.2	Limits at Infinity	15
1.5.3	Infinite Limits at Infinity	15
1.5.4	Precise Definitions	15
2	Derivatives	17
2.1	Derivatives	17
2.1.1	Derivatives and Rates of Change	17
2.1.2	The Derivative as a Function	17

2.1.3	Differentiable Functions	18
2.1.4	Higher Order Derivatives	19
2.2	Differentiation	19
2.2.1	Differentiation Formulas	19
2.2.2	Product and Quotient Rules	21
2.2.3	Trigonometric Functions	22
2.2.4	Chain Rule	23
2.3	Implicit Differentiation	24
2.3.1	Implicit Differentiation	24
2.4	Derivatives of Inverse Functions	25
2.4.1	Differentiation and Inverse Functions	25
2.4.2	Logarithmic and Exponential Functions	25
2.4.3	Inverse Trigonometric Functions	26
2.5	Indeterminate Forms and L'Hôpital's Rule	27
2.5.1	Indeterminate Forms	27
2.5.2	L'Hôpital's Rule	28
3	Applications of Differentiation	28
3.1	Maximum and Minimum Values	28
3.2	The Mean Value Theorem	28
3.3	Derivatives and Graphs	29
3.4	Antiderivatives	29
4	Integrals	29
4.1	Integrals	29
4.1.1	Definite Integrals	29
4.1.2	Indefinite Integrals	29
4.2	Evaluating Integrals	29
4.3	The Fundamental Theorem of Calculus	29
4.4	The Substitution Rule	30
4.4.1	The Substitution Rule	30
4.4.2	Symmetry	30
5	Techniques of Integration	31
5.1	Integration by Parts	31
5.2	Trigonometric Integrals and Substitutions	31
5.3	Partial Fractions	31
5.4	Improper Integrals	31
6	Applications of Integration	31
6.1	Areas	31
6.2	Volumes	31
6.3	Arc Length	31

7	Sequences and Series	31
7.1	Sequences	31
7.2	Series	31
7.3	Convergence Tests	31
7.4	Power Series	31
7.5	Taylor Series	31
8	Parametric Equations and Polar Coordinates	31
8.1	Calculus of Parametric Equations	31
8.2	Calculus in Polar Coordinates	31
9	Differential Equations	31
9.1	Ordinary Differential Equations	31
9.2	Direction Fields and Euler's Method	32
9.2.1	Direction Fields	32
9.2.2	Euler's Method	32
9.3	Separable Equations	32
9.3.1	Separation of Variables	32

1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y . We have a formal definition of a function.

Definition 1.1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the **Cartesian product** of X and Y .

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X , there exists a y in Y such that (x, y) is in R .

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R , then $y = z$.

$$(x, y), (x, z) \in R \implies y = z \quad \square$$

The real line is the one-dimensional (1D) **Euclidean space** defined as the set of all real numbers \mathbb{R} . The xy -plane in the **Cartesian coordinate system** by **René Descartes** (1596–1650) is the two-dimensional (2D) Euclidean space defined as the set of all ordered pairs of real numbers $(x, y) \in \mathbb{R}^2$. A **function of a real variable** is a function whose domain is the set of real numbers \mathbb{R} . A real function is a real-valued function of a real variable whose domain and codomain is \mathbb{R} .

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a , except possibly at a itself.

Definition 1.1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.

$$\lim_{x \rightarrow a} f(x) = L \quad \square$$

Definition 1.1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \rightarrow a^-} f(x) = L \quad \square$$

Definition 1.1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \rightarrow a^+} f(x) = L \quad \square$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L , otherwise the limit **does not exist**.

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

Definition 1.2.1.

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \quad \square$$

Definition 1.2.2.

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \quad \square$$

Definition 1.2.3.

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \quad \square$$

Problem 1.2.1. Prove that $\lim_{x \rightarrow 3} (4x - 5) = 7$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that $\lim_{x \rightarrow 3} (4x - 5) = 7$. ■

Problem 1.2.2. Prove that $\lim_{x \rightarrow 3} x^2 = 9$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x + 3| |x - 3| < \varepsilon$$

Let C be a positive constant such that

$$|x + 3| |x - 3| < C |x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x - 3| < 1$ such that $|x + 3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$0 < |x - 3| < 1 \iff |x + 3| < 7$$

$$0 < |x - 3| < \frac{\varepsilon}{7} \iff 7|x - 3| < \varepsilon$$

$$|x + 3| |x - 3| < 7|x - 3| < \varepsilon \implies |x^2 - 9| < \varepsilon$$

Therefore, it is proved that $\lim_{x \rightarrow 3} x^2 = 9$. ■

Problem 1.2.3. Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

It is proved that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$. ■

1.3 Evaluating Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a} g(x) = M$$

exist. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.3.1.

$$\lim_{x \rightarrow a} c = c$$

□

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that $\lim_{x \rightarrow a} c = c$. ■

Theorem 1.3.2.

$$\lim_{x \rightarrow a} x = a$$

□

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that $\lim_{x \rightarrow a} x = a$. ■

Theorem 1.3.3 Constant Multiple Law.

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL$$

□

Proof. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \rightarrow a} [0 \cdot f(x)] = \lim_{x \rightarrow a} 0 = 0 = 0 \cdot \lim_{x \rightarrow a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL$$

■

Theorem 1.3.4 Sum and Difference Law.

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$$

□

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the **triangle inequality** $|a + b| \leq |a| + |b|$, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

Since $\lim_{x \rightarrow a} f(x) = L$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number δ_2 such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L$$

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} [f(x) + (-1)g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} (-1)g(x) \\ &= \lim_{x \rightarrow a} f(x) + (-1) \lim_{x \rightarrow a} g(x) \\ &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \quad \blacksquare$$

Theorem 1.3.5 Product Law.

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leq |[f(x) - L]g(x)| + |L[g(x) - M]| = |f(x) - L| |g(x)| + |L| |g(x) - M| \end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x \rightarrow a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Since $\lim_{x \rightarrow a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L| |g(x)| + |L| |g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)} (1 + |M|) + (1 + |L|) \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \blacksquare$$

Theorem 1.3.6 Quotient Law.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M} \iff \lim_{x \rightarrow a} g(x) = M \neq 0 \quad \square$$

Proof. First we prove that

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Notice that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \rightarrow a} g(x) = M$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of x ,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M| |g(x)|} < \frac{1}{|M|} \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2} \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$, then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|} |g(x) - M| < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} f(x) \left(\frac{1}{g(x)} \right) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \quad \blacksquare$$

Theorem 1.3.7 Power Law.

$$\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n, n \in \mathbb{R} \quad \square$$

Theorem 1.3.8 Root Law.

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, n \in \mathbb{R} \quad \square$$

Theorem 1.3.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

Thus, we have the following limits

$$\lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$\lim_{\theta \rightarrow 0} \cos \theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ if the limits exist.

Problem 1.3.1. Show that $\lim_{x \rightarrow 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, it is shown that $\lim_{x \rightarrow 0} |x| = 0$. ■

Theorem 1.3.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a , except possibly at a , and

$$\lim_{x \rightarrow a} f(x) = L \qquad \lim_{x \rightarrow a} g(x) = M$$

then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \iff L \leq M \quad \square$$

Proof. We use the technique of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x \rightarrow a} [g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a , we have

$$0 < |x - a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \quad \blacksquare$$

Theorem 1.3.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a , except possibly at a , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L \quad \square$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \rightarrow a} h(x) = L$, there exists a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x \rightarrow a} g(x) = L \quad \blacksquare$$

We can derive the following results using the **Pythagorean theorem** $a^2 + b^2 = c^2$. If $0 < \theta < \pi/2$, then

$$\sin \theta < \theta \implies \frac{\sin \theta}{\theta} < 1 \qquad \theta < \tan \theta = \frac{\sin \theta}{\cos \theta} \implies \cos \theta < \frac{\sin \theta}{\theta}$$

so we have the following inequality

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Since $(\sin \theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Problem 1.3.2. Evaluate

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$$

Solution. By the **Pythagorean identity** $\sin^2 \theta + \cos^2 \theta = 1$, we have

$$\frac{\cos \theta - 1}{\theta} = \frac{\cos \theta - 1}{\theta} \left(\frac{\cos \theta + 1}{\cos \theta + 1} \right) = \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} = \frac{\sin \theta}{\theta} \left(\frac{-\sin \theta}{\cos \theta + 1} \right)$$

We take the limit then we have

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \rightarrow 0} \left[\frac{\sin \theta}{\theta} \left(\frac{-\sin \theta}{\cos \theta + 1} \right) \right] = \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \left(\lim_{\theta \rightarrow 0} \frac{-\sin \theta}{\cos \theta + 1} \right) = 1 \left(\frac{(-1)(0)}{1 + 1} \right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0 \quad \blacksquare$$

1.4 Continuity

1.4.1 Continuous Functions

Let $f(x)$ be a function and the number a is in the domain of f so $f(a)$ is defined. If the limit exists, then we have the following definition.

Definition 1.4.1. A function f is **continuous** at the number a if

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

A function f is continuous from the left at a if the left-hand limit equals $f(a)$ and it is continuous from the right at a if the right-hand limit equals $f(a)$. A function f is continuous on an interval if it is continuous at every number in the interval. If f is not continuous at a , then it is a discontinuous function at a . If f and g are continuous functions at a and c is a constant, then the following functions are also continuous at a .

$$f + g \quad f - g \quad cf \quad f \cdot g \quad \frac{f}{g} \iff g(x) \neq 0$$

Theorem 1.4.1. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$. \square

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients a_i are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. \blacksquare

Theorem 1.4.2. Let f be any rational function, then f is continuous on its domain. \square

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. \blacksquare

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

Theorem 1.4.3. If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then the limit of the composite function $f \circ g$ is

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b , then we have $\lim_{y \rightarrow b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \rightarrow a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \blacksquare$$

Theorem 1.4.4. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a . \square

Proof. Since g is continuous at a , we have $\lim_{x \rightarrow a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x \rightarrow a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a . \blacksquare

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

Theorem 1.4.5 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number c in the open interval (a, b) such that $f(c) = N$. \square

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b) , then there exists a root of $f(x)$ in (a, b) which follows immediately from the intermediate value theorem.

1.5 Limits and Infinity

1.5.1 Infinite Limits

Definition 1.5.1. The limit of $f(x)$ as x approaches a is **infinity** if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a .

$$\lim_{x \rightarrow a} f(x) = \infty \quad \square$$

Definition 1.5.2. The limit of $f(x)$ as x approaches a is **negative infinity** if the values of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a but not equal to a .

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \square$$

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x \rightarrow a^-} f(x) = \infty \quad \lim_{x \rightarrow a^+} f(x) = \infty \quad \lim_{x \rightarrow a^-} f(x) = -\infty \quad \lim_{x \rightarrow a^+} f(x) = -\infty$$

Definition 1.5.3. The **vertical asymptote** of the curve $y = f(x)$ is the line $x = a$ if one of the infinite limits is infinity or negative infinity. \square

1.5.2 Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞) . The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x \rightarrow \infty} f(x) = L \quad \square$$

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \square$$

Problem 1.5.1. Evaluate $\lim_{x \rightarrow \infty} \sin x$ and $\lim_{x \rightarrow \infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \rightarrow \infty$ so the limits do not exist. ■

Definition 1.5.6. The **horizontal asymptote** of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L . □

1.5.3 Infinite Limits at Infinity

Definition 1.5.7. The values of $f(x)$ become arbitrarily large for sufficiently large x .

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \square$$

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

1.5.4 Precise Definitions

Let f be a function defined on some open interval that contains the number a , except possibly at a itself.

Definition 1.5.8.

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \quad \square$$

Problem 1.5.2. Prove that $\lim_{x \rightarrow a} \frac{1}{x^2} = \infty$.

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that $\lim_{x \rightarrow a} \frac{1}{x^2} = \infty$. ■

Let f be a function defined on some interval (a, ∞) .

Definition 1.5.9.

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon$$

□

Problem 1.5.3. Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left| \frac{1}{x} - 0 \right| < \varepsilon$$

Since $x \rightarrow \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. ■

Definition 1.5.10.

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M$$

□

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

2.1.1 Derivatives and Rates of Change

Definition 2.1.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

where $h = x - a$, if this limit exists. □

Definition 2.1.2. The **velocity** at time $t = a$ of a **position function** $s = f(t)$ is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

□

Definition 2.1.3. The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists. □

The slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, the derivative of f at a . The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The **instantaneous rate of change** of $y = f(x)$ with respect to x at $x = x_0$ in the interval $[x_0, x_1]$ is

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of an object, then the velocity of the object at time $t = a$ is $v(a) = f'(a)$ and the **speed** of the object is $|f'(a)|$, the **magnitude** of the velocity.

2.1.2 The Derivative as a Function

Definition 2.1.4. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

□

The following notations of the derivative of $y = f(x)$ with respect to x are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

where $f'(x)$ is Newton's notation and dy/dx is Leibniz's notation. The notation d/dx is the **differential operator** that indicates the operation of **differentiation**. The notations of the derivative of $f(x)$ at a are

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left[\frac{dy}{dx} \right]_{x=a}$$

Problem 2.1.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned} \quad \blacksquare$$

2.1.3 Differentiable Functions

Definition 2.1.5. A function f is **differentiable** at a if $f'(a)$ exists. f is differentiable on an open interval if it is differentiable at every number in the interval. \square

Theorem 2.1.1. If f is differentiable at a , then f is continuous at a . \square

Proof. Given that f is differentiable at a , we want to show that

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Since $f'(a)$ exists, we have

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} (x - a) \right) = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) \lim_{x \rightarrow a} (x - a) = f'(a) \cdot 0 = 0$$

Then we have

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (f(a) + f(x) - f(a)) = \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} (f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore, it is proved that f is continuous at a . \blacksquare

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$$

if the limit exists but

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \\ \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} &= \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

thus the limit does not exist so $f'(0)$ does not exist. If a function is differentiable, then it is **smooth**, continuous, and it has no vertical tangent lines.

2.1.4 Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $f'(x)$ is differentiable, then the **second derivative** of f is

$$y'' = f''(x) = \frac{d^2 y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. Let $s = s(t)$ be a position function of an object with respect to time t . The velocity function $v(t)$ of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the **acceleration**. Thus the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2 s}{dt^2}$$

In general, the n th derivative of $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Problem 2.1.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. We apply the **binomial theorem** by Newton

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

For the second derivative we have

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h \rightarrow 0} \frac{3(x^2 + 2hx + h^2 - x^2)}{h} = \lim_{h \rightarrow 0} \frac{6hx + 3h^2}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned} \quad \blacksquare$$

2.2 Differentiation

2.2.1 Differentiation Formulas

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the following differentiation formulas.

Theorem 2.2.1. Let $f(x) = c$ where c is a constant, then

$$\frac{d}{dx}(c) = 0 \quad \square$$

Proof.

$$f'(x) = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0 \quad \blacksquare$$

Theorem 2.2.2 Power Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R} \quad \square$$

Proof. We prove the power rule for $n \in \mathbb{N}$.

$$\frac{d}{dx}(x^n) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand $(x+h)^n$ then we have

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} (nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}) = nx^{n-1} \end{aligned}$$

because every term has a factor of h except nx^{n-1} . \blacksquare

Note the special case when $n = 1$, then we have

$$\frac{d}{dx}(x) = 1$$

Problem 2.2.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx} \left(\frac{1}{x} \right) = \frac{d}{dx} x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \quad \blacksquare$$

Theorem 2.2.3 Constant Multiple Rule. If c is a constant, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x) \quad \square$$

Proof.

$$\begin{aligned} \frac{d}{dx}[cf(x)] &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \rightarrow 0} \left[c \left(\frac{f(x+h) - f(x)}{h} \right) \right] \\ &= c \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) = c \frac{d}{dx}f(x) \quad \blacksquare \end{aligned}$$

Theorem 2.2.4 Sum and Difference Rule.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \quad \square$$

Proof. We prove the sum rule.

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)
\end{aligned}$$

Then we prove the difference rule.

$$\begin{aligned}
\frac{d}{dx}[f(x) - g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) - g(x+h) - [f(x) - g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - [g(x+h) - g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)
\end{aligned}$$

2.2.2 Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.2.5 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x) \right] g(x) + f(x) \left[\frac{d}{dx}g(x) \right]$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x+h)g(x+h) - f(x+h)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} g(x) \right] + \lim_{h \rightarrow 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} \right] \\
&= \left[\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right] g(x) + \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \left[\frac{d}{dx}f(x) \right] g(x) + f(x) \left[\frac{d}{dx}g(x) \right]
\end{aligned}$$

Theorem 2.2.6 Quotient Rule.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{\left[\frac{d}{dx} f(x) \right] g(x) - f(x) \left[\frac{d}{dx} g(x) \right]}{[g(x)]^2}$$

□

Proof.

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) - [f(x)g(x+h) - f(x)g(x)]}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \\ &= \left(\lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} g(x) \right] - \lim_{h \rightarrow 0} \left[f(x) \frac{g(x+h) - g(x)}{h} \right] \right) \frac{1}{[g(x)]^2} \\ &= \left[\left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) g(x) - f(x) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \right] \frac{1}{[g(x)]^2} \\ &= \frac{\left[\frac{d}{dx} f(x) \right] g(x) - f(x) \left[\frac{d}{dx} g(x) \right]}{[g(x)]^2} \end{aligned}$$

■

2.2.3 Trigonometric Functions

Theorem 2.2.7.

$$\frac{d}{dx} \sin x = \cos x$$

□

Proof. We use the **angle sum identity** of the sine function

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

then we have

$$\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Note that we are taking the limit with respect to h so $\sin x$ and $\cos x$ are constants then we have

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right] = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \left(\lim_{h \rightarrow 0} \sin x \right) \left(\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) + \left(\lim_{h \rightarrow 0} \cos x \right) \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \\ &= (\sin x)(0) + (\cos x)(1) = \cos x \end{aligned}$$

■

Theorem 2.2.8.

$$\frac{d}{dx} \cos x = -\sin x$$

□

Proof. We use the angle sum identity of the cosine function

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

then we have

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right) \\ &= \left(\lim_{h \rightarrow 0} \cos x \right) \left(\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) - \left(\lim_{h \rightarrow 0} \sin x \right) \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned} \quad \blacksquare$$

Theorem 2.2.9.

$$\frac{d}{dx} \tan x = \sec^2 x \quad \square$$

Proof. We have

$$\begin{aligned} \frac{d}{dx} \tan x &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x) \cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned} \quad \blacksquare$$

2.2.4 Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.2.10 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \square$$

Proof. We know that by definition $\Delta y = f(a + \Delta x) - f(a)$ and

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let ε be the difference between the difference quotient and the derivative, then we have

$$\lim_{\Delta x \rightarrow 0} \varepsilon = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) \quad \blacksquare$$

2.3 Implicit Differentiation

2.3.1 Implicit Differentiation

An **explicit function** $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. An **implicit function** is defined implicitly by a relation between x and y . An example of implicit functions is the equation of the circle $x^2 + y^2 = r^2$ where the radius r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y in an implicit function.

Problem 2.3.1. Find dy/dx of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then we have

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0\end{aligned}$$

Since $y = f(x)$, we use the chain rule then we have

$$\begin{aligned}2x + \frac{d}{dy}(y^2) \frac{dy}{dx} &= 0 \\ 2x + 2y \frac{dy}{dx} &= 0\end{aligned}$$

We solve for dy/dx then we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.3.2. Find dy/dx of the folium of Descartes $x^3 + y^3 = 6xy$.

Solution.

$$\begin{aligned}x^3 + y^3 &= 6xy \\ \frac{d}{dx}x^3 + \frac{d}{dy}y^3 \left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y + 6x\frac{dy}{dx} \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\ x^2 + y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\ (y^2 - 2x)\frac{dy}{dx} &= 2y - x^2 \\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}\end{aligned}$$

■

2.4 Derivatives of Inverse Functions

2.4.1 Differentiation and Inverse Functions

Theorem 2.4.1. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous. \square

Theorem 2.4.2. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \square$$

2.4.2 Logarithmic and Exponential Functions

The **Euler's number** e named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function $y = e^x$.

Definition 2.4.1 Euler's Number. The Euler's number e is

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x} \quad \square$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.4.3. The exponential function $f(x) = \log_a x$ is differentiable and

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e = \frac{1}{x \ln a} \quad \square$$

Theorem 2.4.4. The derivative of the natural logarithmic function $f(x) = \ln x$ is

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \square$$

Theorem 2.4.5. The exponential function $f(x) = a^x$, $a > 0$ is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a \quad \square$$

Theorem 2.4.6. The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx} e^x = e^x \quad \square$$

Problem 2.4.1. Prove the power rule $\frac{d}{dx} x^n = nx^{n-1}$ for $n \in \mathbb{R}$.

Solution. The rule is true when $x = 0$ which is trivial so it remains to prove the cases for $x \neq 0$. Let $y = x^n$ such that $y > 0$, we use implicit differentiation and logarithmic differentiation then we have

$$\begin{aligned}\ln y &= \ln x^n = n \ln x \\ \frac{d}{dx} \ln y &= \frac{d}{dx} n \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \frac{n}{x} \\ \frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n \\ \frac{dy}{dx} &= nx^{n-1}\end{aligned}$$

Similarly, let $y = x^n$ such that $y < 0$ then we have

$$\begin{aligned}\ln |y| &= \ln(-x^n) = \ln[(-1)^{1/n}x^n] \\ \frac{d}{dx} \ln |y| &= \frac{d}{dx} [n \ln((-1)^{1/n}x)] \\ \frac{1}{y} \frac{dy}{dx} &= n \frac{1}{(-1)^{1/n}x} (-1)^{1/n} = \frac{n}{x} \\ \frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n \\ \frac{dy}{dx} &= nx^{n-1}\end{aligned}$$

■

2.4.3 Inverse Trigonometric Functions

Theorem 2.4.7.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

□

Proof. Let $y = \arcsin x \iff \sin y = x$ and $-\pi/2 \leq y \leq \pi/2$ such that $-1 \leq x \leq 1$. Note the approximate value of π is $\pi \approx 3.14159$. Then we have

$$\begin{aligned}\frac{d}{dx} \sin y &= \frac{d}{dx} x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}\end{aligned}$$

Since $-\pi/2 \leq y \leq \pi/2$ so $\cos y \geq 0$ thus $\cos y = \sqrt{1 - \sin^2 y}$. Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. ■

Theorem 2.4.8.

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

□

Proof. Let $y = \arccos x \iff \cos y = x$ and $0 \leq y \leq \pi$ such that $-1 \leq x \leq 1$. Then we have

$$\begin{aligned}\frac{d}{dx} \cos y &= \frac{d}{dx} x \\ -\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y}\end{aligned}$$

Since $0 \leq y \leq \pi$ so $\sin y \geq 0$ thus $\sin y = \sqrt{1 - \cos^2 y}$. Therefore, $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$. ■

Theorem 2.4.9.

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \quad \square$$

Proof. Let $y = \arctan x \iff \tan y = x$. Then we have

$$\begin{aligned}\frac{d}{dx} \tan y &= \frac{d}{dx} x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y\end{aligned}$$

Then we can show that

$$\begin{aligned}\tan y &= x \\ 1 + \frac{\sin^2 y}{\cos^2 y} &= 1 + x^2 \\ 1 + \frac{1 - \cos^2 y}{\cos^2 y} &= 1 + x^2 \\ \sec^2 y &= 1 + x^2 \\ \cos^2 y &= \frac{1}{1 + x^2}\end{aligned}$$

Therefore, $\frac{dy}{dx} = \frac{1}{1 + x^2}$. ■

2.5 Indeterminate Forms and L'Hôpital's Rule

2.5.1 Indeterminate Forms

Consider the limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

If $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$, then the limit is an **indeterminate form** of type $0/0$. If $f(x) \rightarrow \infty$ or $-\infty$ and $g(x) \rightarrow \infty$ or $-\infty$, then the limit is an indeterminate form of type ∞/∞ .

2.5.2 L'Hôpital's Rule

We have the **L'Hôpital's Rule** discovered by **Johann Bernoulli** (1667–1748) and named after **Guillaume de l'Hôpital** (1661–1704) to evaluate limits of indeterminate forms of type $0/0$ and ∞/∞ .

Theorem 2.5.1 L'Hôpital's Rule. Suppose that f and g are differentiable and $g'(x) \neq 0$ near a , except possibly at a . If

$$\lim_{x \rightarrow a} f(x) = 0$$

$$\lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a} g(x) = \pm\infty$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, is ∞ , or is $-\infty$. □

L'Hôpital's rule is also valid for one-sided limits and for limits at infinity or negative infinity.

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Pierre de Fermat (1601–1665)

Theorem 3.1.1 Fermat's Theorem. If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$. □

3.2 The Mean Value Theorem

Michel Rolle (1652–1719)

Theorem 3.2.1 Rolle's Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b) , and $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$. □

Joseph-Louis Lagrange (1736–1813)

Theorem 3.2.2 Lagrange's Mean Value Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b) , and $f(a) \neq f(b)$. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

3.3 Derivatives and Graphs

3.4 Antiderivatives

4 Integrals

4.1 Integrals

4.1.1 Definite Integrals

A **Riemann sum** named after **Bernhard Riemann** (1826–1866) associated with a partition P and a function f is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n$$

Definition 4.1.1. If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad \square$$

4.1.2 Indefinite Integrals

Definition 4.1.2. The **indefinite integral**

$$\int f(x) dx$$

is the antiderivative of f . □

4.2 Evaluating Integrals

4.3 The Fundamental Theorem of Calculus

Theorem 4.3.1 The Fundamental Theorem of Calculus (Newton-Leibniz Theorem). Suppose f is continuous on $[a, b]$. If the function F is defined by

$$F(x) = \int_a^x f(t) dt, a \leq x \leq b$$

then F is an antiderivative of f and

$$F'(x) = f(x), a < x < b$$

and therefore

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

If F is an antiderivative of f such that $F' = f$, then

$$\int_a^b f(x) dx = F(b) - F(a) \quad \square$$

The **Fresnel integrals** named after **Augustin-Jean Fresnel** (1788–1827) are

$$S(x) = \int_0^x \sin(t^2) dt \qquad C(x) = \int_0^x \cos(t^2) dt$$

4.4 The Substitution Rule

4.4.1 The Substitution Rule

Theorem 4.4.1 The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

If g' is continuous on $[a, b]$ and f is continuous on I , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \quad \square$$

Problem 4.4.1. Evaluate $\int \tan x dx$.

Solution. We have $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$, let $u = \cos x \iff du = -\sin x dx$ then we have

$$\int \tan x dx = - \int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C = \ln \frac{1}{|\sec x|} + C = \ln |\sec x| + C \quad \blacksquare$$

4.4.2 Symmetry

Theorem 4.4.2. Suppose f is continuous on $[-a, a]$, then

- If $f(-x) = f(x)$ so f is even, then $\int_{-a}^a f(x) dx = 2 \int_a^a f(x) dx$.
- If $f(-x) = -f(x)$ so f is odd, then $\int_{-a}^a f(x) dx = 0$. \square

5 Techniques of Integration

5.1 Integration by Parts

5.2 Trigonometric Integrals and Substitutions

5.3 Partial Fractions

5.4 Improper Integrals

6 Applications of Integration

6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

7.2 Series

7.3 Convergence Tests

7.4 Power Series

7.5 Taylor Series

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

A **differential equation** is an equation that relates some unknown functions and their derivatives. Differential equations are important in mathematics and science. An **ordinary differential equation** (ODE) is a differential equation that relates one or more functions of a single variable and their ordinary derivatives. The **order** of a differential equation is the highest order of the derivative in the equation. **Newton's second law of motion**

$$F = ma$$

is an ordinary differential equation since we can write it in the form

$$F = m \frac{dv}{dt}$$

which is a first order differential equation, or

$$F = m \frac{d^2 s}{dt^2}$$

which is a second order differential equation. A function f is a **solution** of a differential equation if the function and its derivatives satisfy the equation for all values of x in some open interval $a < x < b$. It is possible that there are many solutions of a differential equation. An **initial condition** is a condition $y(x_0) = y_0$ or $y^{(n)}(x_0) = y_n$ on the solution. An **initial value problem** is solving a differential equation with initial conditions. The **interval of validity** is the largest possible interval on which the solution is valid and contains x_0 in the initial conditions. The **general solution** of a differential equation is a family of solutions in the most general form. The actual solution is the solution that satisfies the initial conditions. An explicit solution is any solution in the form $y = y(x)$, otherwise it is an implicit solution. The **existence** and **uniqueness** problem asks that given a differential equation, does there exist a solution and if any is there only one solution.

9.2 Direction Fields and Euler's Method

9.2.1 Direction Fields

9.2.2 Euler's Method

9.3 Separable Equations

9.3.1 Separation of Variables

A **separable equation** is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

We can separate the variables if $g(y) \neq 0$, let $h(y) = 1/g(y)$ then we have

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$

We write the equation in the differential form

$$h(y) dy = f(x) dx$$

so that there is a function of y on one side of the equation and a function of x on the other side. Then we integrate both sides of the equation with respect to the variable of the function.

$$\int h(y) dy = \int f(x) dx$$

Now we have an implicit solution and in some cases we can solve for an explicit solution. We use the chain rule then we have

$$\begin{aligned}\frac{d}{dx} \int h(y) dy &= \frac{d}{dx} \int f(x) dx \\ \frac{d}{dy} \int h(y) dy \frac{dy}{dx} &= f(x) \\ h(y) \frac{dy}{dx} &= f(x)\end{aligned}$$

Therefore, we solved the differential equation by **separation of variables**.

Problem 9.3.1. Solve the differential equation $\frac{dy}{dx} = -xy$.

Solution. Notice that $y = 0$ is a trivial solution then we solve the differential equation assuming that $y \neq 0$. We use separation of variables then we have

$$\begin{aligned}\frac{dy}{dx} &= -xy \\ \frac{dy}{y} &= -x dx \\ \int \frac{dy}{y} &= - \int x dx \\ \ln |y| + C_1 &= -\frac{x^2}{2} + C_2\end{aligned}$$

Let $C = C_2 - C_1$ where $C \in \mathbb{R}$, then we have

$$\begin{aligned}\ln |y| &= -\frac{x^2}{2} + C \\ e^{\ln |y|} &= e^{-(x^2/2)+C} \\ |y| &= e^C e^{-x^2/2} \\ y &= A e^{-x^2/2}\end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.2. Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ with the initial condition $y(0) = 2$.

Solution. We use separation of variables to find the general solution then we have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{x^2}{y^2} \\
 y^2 dy &= x^2 dx \\
 \int y^2 dy &= \int x^2 dx \\
 \frac{y^3}{3} + C_1 &= \frac{x^3}{3} + C_2 \\
 \frac{y^3}{3} &= \frac{x^3}{3} + C, \quad C = C_2 - C_1 \\
 y^3 &= x^3 + 3C \\
 y &= \sqrt[3]{x^3 + 3C}
 \end{aligned}$$

We consider the initial condition to find the actual solution then we have

$$y(0) = 2 = \sqrt[3]{(0)^3 + 3C} \implies 2 = \sqrt[3]{3C} \implies 3C = 8$$

Thus, the solution to the differential equation is $y = \sqrt[3]{x^3 + 8}$. ■

Problem 9.3.3. Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$.

Solution. We have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{6x^2}{2y + \cos y} \\
 \int (2y + \cos y) dy &= \int 6x^2 dx \\
 y^2 + \sin y &= 2x^3 + C
 \end{aligned}$$

where $C \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.4. Solve the differential equation $y' = x^2 y$.

Solution. We have

$$\begin{aligned}
 \frac{dy}{dx} &= x^2 y \\
 \int \frac{dy}{y} &= \int x^2 dx \\
 \ln |y| &= \frac{x^3}{3} + C \\
 |y| &= e^{(x^3/3)+C} \\
 y &= Ae^{x^3/3}
 \end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.5. Newton's law of universal gravitation states that

$$F = G \frac{m_1 m_2}{r^2}$$

Then the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t , R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's second law, we have

$$F = ma = m \frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity v_0 . Let h be the maximum height above the surface reached by the object. Show that $v_0 = \sqrt{\frac{2gRh}{R + h}}$ and compute v_e , the escape velocity of the earth, using $R = 6,378$ km and $g = 9.8$ m/s².