Calculus

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For all lovers of mathematics.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y. We have a formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the Cartesian product of X and Y.

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X, there exists a y in Y such that (x, y) is in R.

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R, then $y = z$.

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.

$$\lim_{x \to a} f(x) = L$$

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \to a^-} f(x) = L$$

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \to a^+} f(x) = L$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) developed a rigorous definition of a limit.

Definition 1.5.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.6.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.7.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.1. Prove that

$$\lim_{x \to 3} (4x - 5) = 7$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

■

Problem 1.2. Prove that
$$\lim_{x \to 3} x^2 = 9$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get
$$|x^2 - 9| = |x + 3|\,|x - 3| < \varepsilon$$

Let C be a positive constant such that

$$|x + 3|\,|x - 3| < C\,|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x - 3| < 1$ such that $|x + 3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$\begin{aligned}
&0 < |x - 3| < 1 \iff |x + 3| < 7 \\
&0 < |x - 3| < \frac{\varepsilon}{7} \iff 7\,|x - 3| < \varepsilon \\
&|x + 3|\,|x - 3| < 7\,|x - 3| < \varepsilon \implies |x^2 - 9| < \varepsilon
\end{aligned}$$

Therefore, it is proved that
$$\lim_{x \to 3} x^2 = 9$$

■

Problem 1.3. Prove that
$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, so we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

Therefore, it is proved that
$$\lim_{x \to 0^+} \sqrt{x} = 0$$

■

1.3 Computing Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

exists. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.1.

$$\lim_{x\to a} c = c$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |c-c| < \varepsilon$$

Since $|c-c| = 0 < \varepsilon$ so the trivial inequality is always true. Let $\delta > 0$ be any number and the proof is obvious. ■

Theorem 1.2.

$$\lim_{x\to a} x = a$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |x-a| < \varepsilon$$

Let $\delta = \varepsilon$ such that

$$0 < |x-a| < \delta = \varepsilon$$

and the proof is trivial. ■

Theorem 1.3 (Constant Multiple Law)**.** The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x\to a} [c\,f(x)] = c \lim_{x\to a} f(x) = cL$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - c \lim_{x\to a} f(x)| < \varepsilon$$

■

Theorem 1.4 (Sum and Difference Law)**.** The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the **triangle inequality** we get

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

■

Theorem 1.5 (Product Law)**.** The limit of a product is the product of the limits.

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

Theorem 1.6 (Quotient Law)**.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \iff \lim_{x \to a} g(x) = M \neq 0$$

Theorem 1.7 (Power Law)**.**

$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$$

Theorem 1.8 (Root Law)**.**

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

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2.1 Derivatives

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