Calculus

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May 22, 2024

For all lovers of mathematics.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y. We have the following formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the Cartesian product of X and Y.

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X, there exists a y in Y such that (x, y) is in R.

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R, then $y = z$.

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

1.1.2 Intuitive Definition of a Limit

Let $f(x)$ be a function defined on some open interval that contains a, possibly not at a itself. Newton introduced the following working definition of a limit.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from left and right but $x \neq a$.

$$\lim_{x \to a} f(x) = L$$

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \to a^-} f(x) = L$$

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \to a^+} f(x) = L$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

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