Calculus

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May 23, 2024

For all lovers of mathematics.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y. We have a formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the Cartesian product of X and Y.

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X, there exists a y in Y such that (x, y) is in R.

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R, then $y = z$.

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

A **function of a real variable** is a function whose domain is the set of real numbers $\mathbb{R}$. A real function is a real-valued function of a real variable whose domain and codomain is $\mathbb{R}$.

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.

$$\lim_{x \to a} f(x) = L$$

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \to a^-} f(x) = L$$

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \to a^+} f(x) = L$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

Definition 1.5.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.6.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.7.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.1. Prove that

$$\lim_{x \to 3}(4x - 5) = 7$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that

$$\lim_{x \to 3}(4x - 5) = 7$$

■

Problem 1.2. Prove that

$$\lim_{x \to 3} x^2 = 9$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x+3|\,|x-3| < \varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3| < C\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3| < 1$ such that $|x+3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$0 < |x-3| < 1 \iff |x+3| < 7$$
$$0 < |x-3| < \frac{\varepsilon}{7} \iff 7\,|x-3| < \varepsilon$$
$$|x+3|\,|x-3| < 7\,|x-3| < \varepsilon \implies |x^2 - 9| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \to 3} x^2 = 9$$

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Problem 1.3. Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

■

1.3 Computing Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$$

exist. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.1.

$$\lim_{x \to a} c = c$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} c = c$$

■

Theorem 1.2.

$$\lim_{x \to a} x = a$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} x = a$$

■

Theorem 1.3 Constant Multiple Law. The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL$$

Proof. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} [0 \cdot f(x)] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL$$

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Theorem 1.4 Sum and Difference Law. The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the **triangle inequality** $|a + b| \leq |a| + |b|$, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similary, there is a number δ_2 such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned}\lim_{x\to a}[f(x) - g(x)] &= \lim_{x\to a}[f(x) + (-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) \\ &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M\end{aligned}$$

Therefore, it is proved

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

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Theorem 1.5 Product Law. The limit of a product is the product of the limits.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leq |[f(x) - L]g(x)| + |L[g(x) - M]| = |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\varepsilon}{2(1+|M|)}(1+|M|) + (1+|L|)\frac{\varepsilon}{2(1+|L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

It is proved that

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$$

■

Theorem 1.6 Quotient Law. The limit of a quotient is the quotient of the limits (if that the limit of the denominator is not 0).

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \iff \lim_{x\to a} g(x) = M \neq 0$$

Proof. First we prove that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

Notice that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x\to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x-a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|}\frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a} f(x)\left(\frac{1}{g(x)}\right) = \lim_{x\to a} f(x) \lim_{x\to a}\frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

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Theorem 1.7 Power Law.

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n, n \in \mathbb{R}$$

Theorem 1.8 Root Law.

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, n \in \mathbb{R}$$

Theorem 1.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Thus, we have the following limits

$$\lim_{\theta\to 0}\sin\theta = 0 \qquad\qquad \lim_{\theta\to 0}\cos\theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if the limits exist.

Problem 1.4. Show that

$$\lim_{x\to 0}|x| = 0$$

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0$$

Therefore, it is shown that

$$\lim_{x\to 0}|x| = 0$$

■

Theorem 1.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \iff L \leq M$$

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a}[g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x-a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x-a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

■

Theorem 1.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

By **algebra**, **geometry**, and **trigonometry**, we can get the following result by the **Pythagorean theorem** $a^2 + b^2 = c^2$. If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1 \qquad\qquad \theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta}$$

so we have the following inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $(\sin\theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.5. Evaluate

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

Solution. By the **Pythagorean identity** $\sin^2\theta + \cos^2\theta = 1$, we have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit and we have

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)\right] = \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) = 0\left(\frac{-1 \cdot 0}{1 + 1}\right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

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