

Calculus

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For all lovers of mathematics and science.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x) dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y . We have a formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the **Cartesian product** of X and Y .

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X , there exists a y in Y such that (x, y) is in R .

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R , then $y = z$.

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

□

The real line is the 1-dimensional **Euclidean space** defined as the set of real numbers \mathbb{R} . The xy -plane in the **Cartesian coordinate system** by **René Descartes** (1596–1650) is the 2-dimensional Euclidean space defined as the set of all ordered pairs of real numbers $(x, y) \in \mathbb{R}^2$. A **function of a real variable** is a function whose domain is the set of real numbers \mathbb{R} . A real function is a real-valued function of a real variable whose domain and codomain is \mathbb{R} .

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a , except possibly at a itself.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.

$$\lim_{x \rightarrow a} f(x) = L$$

□

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \rightarrow a^-} f(x) = L$$

□

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \rightarrow a^+} f(x) = L$$

□

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L , otherwise the limit **does not exist**.

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

Definition 1.5.

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \quad \square$$

Definition 1.6.

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \quad \square$$

Definition 1.7.

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \quad \square$$

Problem 1.1. Prove that

$$\lim_{x \rightarrow 3} (4x - 5) = 7$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that

$$\lim_{x \rightarrow 3} (4x - 5) = 7 \quad \blacksquare$$

Problem 1.2. Prove that

$$\lim_{x \rightarrow 3} x^2 = 9$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x + 3| |x - 3| < \varepsilon$$

Let C be a positive constant such that

$$|x + 3| |x - 3| < C |x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x - 3| < 1$ such that $|x + 3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$0 < |x - 3| < 1 \iff |x + 3| < 7$$

$$0 < |x - 3| < \frac{\varepsilon}{7} \iff 7|x - 3| < \varepsilon$$

$$|x + 3| |x - 3| < 7|x - 3| < \varepsilon \implies |x^2 - 9| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \rightarrow 3} x^2 = 9$$

■

Problem 1.3. Prove that

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

■

1.3 Evaluating Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a} g(x) = M$$

exist. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.1.

$$\lim_{x \rightarrow a} c = c$$

□

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} c = c \quad \blacksquare$$

Theorem 1.2.

$$\lim_{x \rightarrow a} x = a \quad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} x = a \quad \blacksquare$$

Theorem 1.3 Constant Multiple Law. The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL \quad \square$$

Proof. Note that if $c = 0$, then $c f(x) = 0$ and we have

$$\lim_{x \rightarrow a} [0 \cdot f(x)] = \lim_{x \rightarrow a} 0 = 0 = 0 \cdot \lim_{x \rightarrow a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c f(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |c f(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL \quad \blacksquare$$

Theorem 1.4 Sum and Difference Law. The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \quad \square$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the **triangle inequality** $|a + b| \leq |a| + |b|$, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

Since $\lim_{x \rightarrow a} f(x) = L$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number δ_2 such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} [f(x) + (-1)g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} (-1)g(x) \\ &= \lim_{x \rightarrow a} f(x) + (-1) \lim_{x \rightarrow a} g(x) \\ &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \quad \blacksquare$$

Theorem 1.5 Product Law. The limit of a product is the product of the limits.

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leq |[f(x) - L]g(x)| + |L[g(x) - M]| = |f(x) - L| |g(x)| + |L| |g(x) - M| \end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x \rightarrow a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Since $\lim_{x \rightarrow a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L| |g(x)| + |L| |g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)} (1 + |M|) + (1 + |L|) \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \blacksquare$$

Theorem 1.6 Quotient Law. The limit of a quotient is the quotient of the limits (if that the limit of the denominator is not 0).

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M} \iff \lim_{x \rightarrow a} g(x) = M \neq 0 \quad \square$$

Proof. First we prove that

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Notice that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \rightarrow a} g(x) = M$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of x ,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M| |g(x)|} < \frac{1}{|M|} \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2} \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$, then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|} |g(x) - M| < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} f(x) \left(\frac{1}{g(x)} \right) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \quad \blacksquare$$

Theorem 1.7 Power Law.

$$\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n, n \in \mathbb{R} \quad \square$$

Theorem 1.8 Root Law.

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, n \in \mathbb{R} \quad \square$$

Theorem 1.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

Thus, we have the following limits

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \qquad \lim_{\theta \rightarrow 0} \cos \theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ if the limits exist.

Problem 1.4. Show that

$$\lim_{x \rightarrow 0} |x| = 0$$

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, it is shown that

$$\lim_{x \rightarrow 0} |x| = 0 \quad \blacksquare$$

Theorem 1.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a , except possibly at a , and

$$\lim_{x \rightarrow a} f(x) = L \qquad \lim_{x \rightarrow a} g(x) = M$$

then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \iff L \leq M \quad \square$$

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x \rightarrow a} [g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a , we have

$$0 < |x - a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \quad \blacksquare$$

Theorem 1.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a , except possibly at a , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L \quad \square$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \rightarrow a} h(x) = L$, there exists a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x \rightarrow a} g(x) = L \quad \blacksquare$$

By **algebra**, **geometry**, and **trigonometry**, we can get the following result by the **Pythagorean theorem** $a^2 + b^2 = c^2$. If $0 < \theta < \pi/2$, then

$$\sin \theta < \theta \implies \frac{\sin \theta}{\theta} < 1 \qquad \theta < \tan \theta = \frac{\sin \theta}{\cos \theta} \implies \cos \theta < \frac{\sin \theta}{\theta}$$

so we have the following inequality

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Since $(\sin \theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Problem 1.5. Evaluate

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$$

Solution. By the **Pythagorean identity** $\sin^2 \theta + \cos^2 \theta = 1$, we have

$$\frac{\cos \theta - 1}{\theta} = \frac{\cos \theta - 1}{\theta} \left(\frac{\cos \theta + 1}{\cos \theta + 1} \right) = \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} = \frac{\sin \theta}{\theta} \left(\frac{-\sin \theta}{\cos \theta + 1} \right)$$

We take the limit and we have

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \rightarrow 0} \left[\frac{\sin \theta}{\theta} \left(\frac{-\sin \theta}{\cos \theta + 1} \right) \right] = \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \left(\lim_{\theta \rightarrow 0} \frac{-\sin \theta}{\cos \theta + 1} \right) = 1 \left(\frac{(-1)(0)}{1 + 1} \right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0 \quad \blacksquare$$

1.4 Continuity

Let $f(x)$ be a function and the number a is in the domain of f so $f(a)$ is defined. If the limit exists, then we have the following definition.

Definition 1.8. A function f is **continuous** at the number a if

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

A function f is continuous from the left at a if the left-hand limit equals $f(a)$ and it is continuous from the right at a if the right-hand limit equals $f(a)$. A function f is continuous on an interval if it is continuous at every number in the interval. If f is not continuous at a , then it is a discontinuous function at a . If f and g are continuous functions at a and c is a constant, then the following functions are also continuous at a .

$$f + g \quad f - g \quad cf \quad f \cdot g \quad \frac{f}{g} \iff g(x) \neq 0$$

Theorem 1.12. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$. \square

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients a_i are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. \blacksquare

Theorem 1.13. Let f be any rational function, then f is continuous on its domain. \square

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. \blacksquare

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

Theorem 1.14. If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then the limit of the composite function $f \circ g$ is

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b , then we have $\lim_{y \rightarrow b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \rightarrow a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \blacksquare$$

Theorem 1.15. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a . \square

Proof. Since g is continuous at a , we have $\lim_{x \rightarrow a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x \rightarrow a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a . \blacksquare

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

Theorem 1.16 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number c in the open interval (a, b) such that $f(c) = N$. \square

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b) , then there exists a root of $f(x)$ in (a, b) which follows immediately from the intermediate value theorem.

1.5 Limits and Infinity

1.5.1 Infinite Limits

Definition 1.9. The limit of $f(x)$ as x approaches a is **infinity** if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a .

$$\lim_{x \rightarrow a} f(x) = \infty \quad \square$$

Definition 1.10. The limit of $f(x)$ as x approaches a is **negative infinity** if the values of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a but not equal to a .

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \square$$

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x \rightarrow a^-} f(x) = \infty \quad \lim_{x \rightarrow a^+} f(x) = \infty \quad \lim_{x \rightarrow a^-} f(x) = -\infty \quad \lim_{x \rightarrow a^+} f(x) = -\infty$$

Definition 1.11. The **vertical asymptote** of the curve $y = f(x)$ is the line $x = a$ if one of the infinite limits is infinity or negative infinity. \square

1.5.2 Limits at Infinity

Definition 1.12. Let f be a function defined on some interval (a, ∞) . The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x \rightarrow \infty} f(x) = L \quad \square$$

Definition 1.13. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \square$$

Problem 1.6. Evaluate $\lim_{x \rightarrow \infty} \sin x$ and $\lim_{x \rightarrow \infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \rightarrow \infty$ so the limits do not exist. \blacksquare

Definition 1.14. The **horizontal asymptote** of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L . \square

1.5.3 Infinite Limits at Infinity

Definition 1.15. The values of $f(x)$ become arbitrarily large for sufficiently large x .

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \square$$

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

1.5.4 Precise Definitions

Let f be a function defined on some open interval that contains the number a , except possibly at a itself.

Definition 1.16.

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \quad \square$$

Problem 1.7. Prove that

$$\lim_{x \rightarrow a} \frac{1}{x^2} = \infty$$

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that

$$\lim_{x \rightarrow a} \frac{1}{x^2} = \infty \quad \blacksquare$$

Let f be a function defined on some interval (a, ∞) .

Definition 1.17.

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon \quad \square$$

Problem 1.8. Prove that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left| \frac{1}{x} - 0 \right| < \varepsilon$$

Since $x \rightarrow \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

■

Definition 1.18.

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M$$

□

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

Definition 2.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

where $h = x - a$, if this limit exists.

□

Definition 2.2. The **velocity** at time $t = a$ of a **position function** $s = f(t)$ is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

□

2.1.1 Derivatives and Rates of Change

Definition 2.3. The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

□

The slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, the derivative of f at a . The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The **instantaneous rate of change** of $y = f(x)$ with respect to x at $x = x_0$ in the interval $[x_0, x_1]$ is

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of an object, then the velocity of the object at time $t = a$ is $v(a) = f'(a)$ and the **speed** of the object is $|f'(a)|$, the **magnitude** of the velocity.

2.1.2 The Derivative as a Function

Definition 2.4. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \square$$

The following notations of the derivative of $y = f(x)$ with respect to x are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

where $f'(x)$ is Newton's notation and dy/dx is Leibniz's notation. The notation d/dx is the **differential operator** that indicates the operation of **differentiation**. The notations of the derivative of $f(x)$ at a are

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left[\frac{dy}{dx} \right]_{x=a}$$

Problem 2.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution. We have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

The derivative of $f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \blacksquare$$

2.1.3 Differentiable Functions

Definition 2.5. A function f is **differentiable** at a if $f'(a)$ exists. f is differentiable on an open interval if it is differentiable at every number in the interval. \square

Theorem 2.1. If f is differentiable at a , then f is continuous at a . \square

Proof. Given that f is differentiable at a , we want to show that

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Since $f'(a)$ exists, we have

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} (x - a) \right) = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) \lim_{x \rightarrow a} (x - a) = f'(a) \cdot 0 = 0$$

Then we have

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (f(a) + f(x) - f(a)) = \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} (f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore, it is proved that f is continuous at a . ■

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h \rightarrow 0} \frac{|0 + h| - |0|}{h}$$

if the limit exists but

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \\ \lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} &= \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

thus the limit does not exist so $f'(0)$ does not exist. If a function is differentiable, then it is **smooth**, continuous, and it has no vertical tangent lines.

2.1.4 Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $f'(x)$ is differentiable, then the **second derivative** of f is

$$y'' = f''(x) = \frac{d^2 y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. Let $s = s(t)$ be a position function of an object with respect to time t . The velocity function $v(t)$ of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the **acceleration**. Thus the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2 s}{dt^2}$$

In general, the n th derivative of $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Problem 2.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. We apply the **binomial theorem** by Newton

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

For the second derivative we have

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h \rightarrow 0} \frac{3(x^2 + 2hx + h^2 - x^2)}{h} = \lim_{h \rightarrow 0} \frac{6hx + 3h^2}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

Therefore, the first and the second derivatives of $f(x) = x^3$ are

$$f'(x) = 3x^2 \qquad f''(x) = 6x \quad \blacksquare$$

2.2 Differentiation

2.2.1 Differentiation Formulas

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the following differentiation formulas.

Theorem 2.2. Let $f(x) = c$ where c is a constant, then

$$\frac{d}{dx}(c) = 0 \quad \square$$

Proof.

$$f'(x) = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0 \quad \blacksquare$$

Theorem 2.3 Power Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}, n \in \mathbb{R} \quad \square$$

Note the special case when $n = 1$, then we have

$$\frac{d}{dx}(x) = 1$$

Theorem 2.4 Constant Multiple Rule. If c is a constant, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x) \quad \square$$

Theorem 2.5 Sum and Difference Rule.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \quad \square$$

2.2.2 Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.6 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = f(x) \left[\frac{d}{dx}g(x) \right] + \left[\frac{d}{dx}f(x) \right] g(x) \quad \square$$

Theorem 2.7 Quotient Rule.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{\left[\frac{d}{dx}f(x) \right] g(x) - f(x) \left[\frac{d}{dx}g(x) \right]}{[g(x)]^2} \quad \square$$

2.2.3 Trigonometric Functions

Theorem 2.8.

$$\frac{d}{dx} \sin x = \cos x \quad \square$$

Theorem 2.9.

$$\frac{d}{dx} \cos x = -\sin x \quad \square$$

Theorem 2.10.

$$\frac{d}{dx} \tan x = \sec^2 x \quad \square$$

2.2.4 Chain Rule

We have the chain rule by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.11 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \square$$

2.3 Implicit Differentiation

A function $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. Some functions are defined implicitly by a relation between x and y . The equation of the unit circle $x^2 + y^2 = 1$ is an implicit function. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of implicit differentiation to find the derivative of y in an implicit function.

Problem 2.3. Find dy/dx of $x^2 + y^2 = 1$

Solution. We differentiate on both sides of the equation then we have

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0\end{aligned}$$

Since $y = f(x)$, we use the chain rule then we have

$$\begin{aligned}2x + \frac{d}{dy}(y^2) \frac{dy}{dx} &= 0 \\ 2x + 2y \frac{dy}{dx} &= 0\end{aligned}$$

We solve for dy/dx then we have

$$\frac{dy}{dx} = \frac{-x}{y}$$

■

Problem 2.4. Find dy/dx of the folium of Descartes $x^3 + y^3 = 6xy$.

2.4 Derivatives of Inverse Functions

Theorem 2.12. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous. □

Theorem 2.13. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \square$$

2.4.1 Logarithmic and Exponential Functions

The **Euler's number** e named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function $y = e^x$.

Definition 2.6 Euler's Number. The Euler's number e is

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x} \quad \square$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.14. The exponential function $f(x) = \log_a x$ is differentiable and

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e = \frac{1}{x \ln a} \quad \square$$

Theorem 2.15. The derivative of the natural logarithmic function $f(x) = \ln x$ is

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \square$$

Theorem 2.16. The exponential function $f(x) = a^x, a > 0$ is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a \quad \square$$

Theorem 2.17. The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx} e^x = e^x \quad \square$$

2.4.2 Inverse Trigonometric Functions

Theorem 2.18.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1 \quad \square$$

Theorem 2.19.

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1 \quad \square$$

Theorem 2.20.

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \quad \square$$

2.5 L'Hôpital's Rule

3 Applications of Differentiation

3.1 Maximum and Minimum Values

3.2 The Mean Value Theorem

3.3 Derivatives and Graphs

3.4 Antiderivatives

4 Integrals

4.1 Integrals

4.2 Evaluating Integrals

4.3 The Fundamental Theorem of Calculus

4.4 The Substitution Rule

5 Techniques of Integration

5.1 Integration by Parts

5.2 Trigonometric Integrals and Substitutions

5.3 Partial Fractions

5.4 Improper Integrals

6 Applications of Integration

6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

7.2 Series

7.3 Convergence Tests

7.4 Power Series

7.5 Taylor Series

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

9.2 Direction Fields and Euler's Method

9.3 Separable Equations