Calculus

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For all lovers of mathematics and science.

Introduction

Calculus is the study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Contents

1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y.

Definition 1.1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the Cartesian product of X and Y.

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X, there exists a y in Y such that (x, y) is in R.

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R, then $y = z$.

$$(x, y), (x, z) \in R \implies y = z$$

□

The real line is the one-dimensional (1D) Eculidean space defined as the set of all real numbers $\mathbb{R}$. The xy-plane in the Cartesian coordinate system is the two-dimensional (2D) Eculidean space defined as the set of all ordered pairs of real numbers $(x, y) \in \mathbb{R}^2$. A function of a real variable is a function whose domain is the set of real numbers $\mathbb{R}$. A real function is a real-valued function of a real variable whose domain and codomain is $\mathbb{R}$.

Definition 1.1.2. A function f is injective, or one-to-one if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$. □

Definition 1.1.3. A function f is surjective, or onto if for all y in range Y, there exists an x in domain X such that $f(x) = y$. □

Definition 1.1.4. A function f is bijective if f is injective and surjective. □

Definition 1.1.5. Let f be an injective function with domain X and range Y. Then its inverse function f^{-1} has domain Y and range X and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in Y. □

1.1.2 Intuitive Definition of a Limit

Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.1.6. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.

$$\lim_{x \to a} f(x) = L \qquad \square$$

Definition 1.1.7. The left-hand limit of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \to a^-} f(x) = L \qquad \square$$

Definition 1.1.8. The right-hand limit of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \to a^+} f(x) = L \qquad \square$$

The limit exists if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit does not exist.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Definition 1.2.1.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.2.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.2.3.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Problem 1.2.1. Prove that $\lim_{x\to 3}(4x-5)=7$.

Solution. Let $\varepsilon>0$ be given, we want to find a number $\delta>0$ such that

$$0<|x-3|<\delta \implies |(4x-5)-7|<\varepsilon$$

We simplify to get $|(4x-5)-7|=|4x-12|=4|x-3|$ so we have

$$4|x-3|<\varepsilon \iff |x-3|<\frac{\varepsilon}{4}$$

Let $\delta=\varepsilon/4$, we have

$$0<|x-3|<\frac{\varepsilon}{4} \implies 4|x-3|<\varepsilon \implies |(4x-5)-7|<\varepsilon$$

Therefore, by the definition of a limit, it is proved that $\lim_{x\to 3}(4x-5)=7$. ■

Problem 1.2.2. Prove that $\lim_{x\to 3}x^2=9$.

Solution. Let $\varepsilon>0$ be given, we want to find a number $\delta>0$ such that

$$0<|x-3|<\delta \implies |x^2-9|<\varepsilon$$

We simplify to get

$$|x^2-9|=|x+3|\,|x-3|<\varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3|<C\,|x-3|<\varepsilon \iff |x-3|<\frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3|<1$ such that $|x+3|<7$ so $C=7$. Let $\delta=\min\{1,\varepsilon/7\}$, we have

$$\begin{aligned}
0<|x-3|<1 &\iff |x+3|<7\\
0<|x-3|<\frac{\varepsilon}{7} &\iff 7\,|x-3|<\varepsilon\\
|x+3|\,|x-3|<7\,|x-3|<\varepsilon &\implies |x^2-9|<\varepsilon
\end{aligned}$$

Therefore, it is proved that $\lim_{x\to 3}x^2=9$. ■

Problem 1.2.3. Prove that $\lim_{x\to 0^+}\sqrt{x}=0$.

Solution. Let $\varepsilon>0$ be given, we want to find a number $\delta>0$ such that

$$0<x<\delta \implies |\sqrt{x}-0|<\varepsilon$$

We simplify to get $\sqrt{x}<\varepsilon \iff x<\varepsilon^2$. Let $\delta=\varepsilon^2$, we have

$$0<x<\varepsilon^2 \implies |\sqrt{x}-0|<\varepsilon$$

It is proved that $\lim_{x\to 0^+}\sqrt{x}=0$. ■

1.3 Evaluating Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

exist. We have the following properties of limits called the limit laws to compute limits.

Theorem 1.3.1.

$$\lim_{x\to a} c = c$$ □

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |c-c| < \varepsilon$$

We have $|c-c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that $\lim_{x\to a} c = c$. ■

Theorem 1.3.2.

$$\lim_{x\to a} x = a$$ □

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |x-a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x-a| < \delta = \varepsilon \implies |x-a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that $\lim_{x\to a} x = a$. ■

Theorem 1.3.3 Constant Multiple Law.

$$\lim_{x\to a}[c\,f(x)] = c\lim_{x\to a} f(x) = cL$$ □

Proof. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x\to a}[0\cdot f(x)] = \lim_{x\to a} 0 = 0 = 0\cdot\lim_{x\to a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |cf(x) - c\lim_{x\to a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x-a| < \delta \implies |cf(x) - c\lim_{x\to a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x\to a}[c\,f(x)] = c\lim_{x\to a} f(x) = cL \qquad \blacksquare$$

Theorem 1.3.4 Sum and Difference Law.

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \square$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) + g(x) - (L+M)| < \varepsilon$$

By the triangle inequality $|a+b| \leq |a| + |b|$, we have

$$|f(x) + g(x) - (L+M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number δ_2 such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L$$

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x-a| < \delta \implies 0 < |x-a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x-a| < \delta \implies |f(x) + g(x) - (L+M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned}
\lim_{x\to a}[f(x) - g(x)] &= \lim_{x\to a}[f(x) + (-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x) \\
&= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) \\
&= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M
\end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \blacksquare$$

Theorem 1.3.5 Product Law.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x)-L]g(x) + L[g(x)-M]| \\ &\leq |[f(x)-L]g(x)| + |L[g(x)-M]| = |f(x)-L|\,|g(x)| + |L|\,|g(x)-M|\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1+|M|)}$$

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x-a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x-a| < \delta \implies 0 < |x-a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x)-L|\,|g(x)| + |L|\,|g(x)-M| \\ &< \frac{\varepsilon}{2(1+|M|)}(1+|M|) + (1+|L|)\frac{\varepsilon}{2(1+|L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \blacksquare$$

Theorem 1.3.6 Quotient Law.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \iff \lim_{x\to a} g(x) = M \neq 0 \qquad \square$$

Proof. First we prove that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Notice that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|}\frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$, then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \left(\frac{1}{g(x)} \right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \qquad \blacksquare$$

Theorem 1.3.7 Power Law.

$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n, n \in \mathbb{R} \qquad \square$$

Theorem 1.3.8 Root Law.

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, n \in \mathbb{R}$$ □

Theorem 1.3.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$ □

Thus, we have the following limits

$$\lim_{\theta \to 0} \sin \theta = 0 \qquad \lim_{\theta \to 0} \cos \theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ if the limits exist.

Problem 1.3.1. Show that $\lim_{x \to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim_{x \to 0} |x| = 0$. ■

Theorem 1.3.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$$

then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \iff L \leq M$$ □

Proof. We use the technique of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x \to a} [g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x - a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$ ■

Theorem 1.3.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

□

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there exists a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x \to a} g(x) = L$$

■

If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1 \qquad\qquad \theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta}$$

so we have the following inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $(\sin\theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.3.2. Evaluate

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

Solution. We have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit then

$$\begin{aligned}\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} &= \lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)\right] = \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) \\ &= 1\left(\frac{(-1)(0)}{1 + 1}\right) = 0\end{aligned}$$

■

1.4 Continuity

1.4.1 Continuous Functions

Let $f(x)$ be a function and the number a is in the domain of f so $f(a)$ is defined. If the limit exists, then we have the following definition.

Definition 1.4.1. A function f is **continuous** at the number a if

$$\lim_{x \to a} f(x) = f(a) \qquad \square$$

A function f is continuous from the left at a if the left-hand limit equals $f(a)$ and it is continuous from the right at a if the right-hand limit equals $f(a)$. A function f is continuous on an interval if it is continuous at every number in the interval. If f is not continuous at a, then it is a discontinuous function at a. If f and g are continuous functions at a and c is a constant, then the following functions are also continuous at a.

$$f+g \qquad f-g \qquad cf \qquad f \cdot g \qquad \frac{f}{g} \iff g(x) \neq 0$$

Theorem 1.4.1. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$. $\square$

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients a_i are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. $\blacksquare$

Theorem 1.4.2. Let f be any rational function, then f is continuous on its domain. $\square$

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. $\blacksquare$

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

Theorem 1.4.3. If f is continuous at b and $\lim_{x \to a} g(x) = b$, then the limit of the composite function $f \circ g$ is

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b) \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, then we have $\lim_{y \to b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$ ■

Theorem 1.4.4. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. □

Proof. Since g is continuous at a, we have $\lim_{x \to a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a. ■

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

Theorem 1.4.5 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number c in the open interval (a, b) such that $f(c) = N$. □

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then there exists a root of $f(x)$ in (a, b) which follows immediately from the intermediate value theorem.

1.5 Limits and Infinity

1.5.1 Infinite Limits

Definition 1.5.1. The limit of $f(x)$ as x approaches a is **infinity** if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

$$\lim_{x \to a} f(x) = \infty$$ □

Definition 1.5.2. The limit of $f(x)$ as x approaches a is negative infinity if the values of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a but not equal to a.

$$\lim_{x \to a} f(x) = -\infty$$ □

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Definition 1.5.3. The vertical asymptote of the curve $y = f(x)$ is the line $x = a$ if one of the infinite limits is infinity or negative infinity. □

1.5.2 Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞). The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x\to\infty} f(x) = L$$ □

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x\to-\infty} f(x) = L$$ □

Problem 1.5.1. Evaluate $\lim_{x\to\infty} \sin x$ and $\lim_{x\to\infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to \infty$ so the limits do not exist. ■

Definition 1.5.6. The horizontal asymptote of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L. □

1.5.3 Infinite Limits at Infinity

Definition 1.5.7. The values of $f(x)$ become arbitrarily large for sufficiently large x.

$$\lim_{x\to\infty} f(x) = \infty$$ □

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

1.5.4 Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.8.

$$\lim_{x\to a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M$$ □

Problem 1.5.2. Prove that
$lim_{x\to a} \frac{1}{x^2} = \infty$.

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that $\lim_{x\to a} \frac{1}{x^2} = \infty$. ■

Let f be a function defined on some interval (a, ∞).

Definition 1.5.9.

$$\lim_{x\to\infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon$$ □

Problem 1.5.3. Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that $\lim_{x\to\infty} \frac{1}{x} = 0$. ■

Definition 1.5.10.

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M$$ □

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

2.1.1 Derivatives and Rates of Change

Definition 2.1.1. The tangent line of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

where $h = x - a$, if this limit exists. $\square$

Definition 2.1.2. The velocity at time $t = a$ of a position function $s = f(t)$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \square$$

Definition 2.1.3. The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. $\square$

The slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, the derivative of f at a. The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The instantaneous rate of change of $y = f(x)$ with respect to x at $x = x_0$ in the interval $[x_0, x_1]$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of an object, then the velocity of the object at time $t = a$ is $v(a) = f'(a)$ and the speed of the object is $|f'(a)|$, the magnitude of the velocity.

2.1.2 The Derivative as a Function

Definition 2.1.4. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \square$$

The following notations of the derivative of $y = f(x)$ with respect to x are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$

where $f'(x)$ is Newton's notation and dy/dx is Leibniz's notation. The notation d/dx is the differential operator that indicates the operation of differentiation. The notations of the derivative of $f(x)$ at a are

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left[\frac{dy}{dx}\right]_{x=a}$$

Problem 2.1.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

■

2.1.3 Differentiable Functions

Definition 2.1.5. A function f is **differentiable** at a if $f'(a)$ exists. f is differentiable on an open interval if it is differentiable at every number in the interval. □

Theorem 2.1.1. If f is differentiable at a, then f is continuous at a. □

Proof. Given that f is differentiable at a, we want to show that

$$\lim_{x\to a} f(x) = f(a)$$

Since $f'(a)$ exists, we have

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Then we have

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0$$

Then we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(a)+f(x)-f(a)) = \lim_{x\to a} f(a) + \lim_{x\to a}(f(x)-f(a)) = f(a)+0 = f(a)$$

Therefore, it is proved that f is continuous at a. ■

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h\to 0}\frac{|0+h|-|0|}{h}$$

if the limit exists but

$$\begin{aligned} \lim_{h\to 0^-}\frac{|0+h|-|0|}{h} &= \lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = -1 \\ \lim_{h\to 0^+}\frac{|0+h|-|0|}{h} &= \lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1 \end{aligned}$$

thus the limit does not exist so $f'(0)$ does not exist. If a function is differentiable, then it has no sharp points, it is continuous, and it has no vertical tangent lines.

2.1.4 Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $f'(x)$ is differentiable, then the second derivative of f is

$$y'' = f''(x) = \frac{d^2y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. Let $s = s(t)$ be a position function of an object with respect to time t. The velocity function $v(t)$ of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the acceleration. Thus the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

In general, the nth derivative of $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Problem 2.1.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. We apply the binomial theorem by Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

For the second derivative we have

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2 + 2hx + h^2 - x^2)}{h} = \lim_{h\to 0} \frac{6hx + 3h^2}{h} \\ &= \lim_{h\to 0} (6x + 3h) = 6x \end{aligned}$$

■

2.2 Differentiation

2.2.1 Differentiation Formulas

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the following differentiation formulas.

Theorem 2.2.1. Let $f(x) = c$ where c is a constant, then

$$\frac{d}{dx}(c) = 0 \qquad \square$$

Proof.

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0 \qquad \blacksquare$$

Theorem 2.2.2 Power Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R} \qquad \square$$

Proof. We prove the power rule for $n \in \mathbb{N}$.

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand $(x+h)^n$ then we have

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h \to 0}(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}) = nx^{x-1} \end{aligned}$$

because every term has a factor of h except nx^{n-1}. $\blacksquare$

Note the special case when $n = 1$, then we have

$$\frac{d}{dx}(x) = 1$$

Problem 2.2.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \qquad \blacksquare$$

Theorem 2.2.3 Constant Multiple Rule. If c is a constant, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \qquad \square$$

Proof.

$$\begin{aligned} \frac{d}{dx}[cf(x)] &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0}\left[c\left(\frac{f(x+h) - f(x)}{h}\right)\right] \\ &= c\left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right) = c\frac{d}{dx}f(x) \end{aligned} \qquad \blacksquare$$

Theorem 2.2.4 Sum and Difference Rule.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \qquad \square$$

Proof. We prove the sum rule.

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)
\end{aligned}$$

Then we prove the difference rule.

$$\begin{aligned}
\frac{d}{dx}[f(x)-g(x)] &= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-[f(x)-g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)-g(x+h)+g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)-[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)
\end{aligned}$$

■

2.2.2 Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.2.5 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

□

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+f(x+h)g(x)-f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x+h)g(x+h)-f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x)\right] + \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}\right] \\
&= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]g(x) + \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]
\end{aligned}$$

■

Theorem 2.2.6 Quotient Rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$$ □

Proof.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\cdot g(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - [f(x)g(x+h) - f(x)g(x)]}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)g(x)} \\
&= \left(\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x)\right] - \lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]\right)\frac{1}{[g(x)]^2} \\
&= \left[\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x) - f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right]\frac{1}{[g(x)]^2} \\
&= \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}
\end{aligned}$$ ■

2.2.3 Trigonometric Functions

Theorem 2.2.7.

$$\frac{d}{dx}\sin x = \cos x$$ □

Proof. We use the **angle sum identity** of the sine function

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

then we have

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Note that we are taking the limit with respect to h so $\sin x$ and $\cos x$ are constants then we have

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right] = \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} \\
&= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\
&= (\sin x)(0) + (\cos x)(1) = \cos x
\end{aligned}$$ ■

Theorem 2.2.8.

$$\frac{d}{dx}\cos x = -\sin x$$ □

Proof. We use the angle sum identity of the cosine function

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

then we have

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x\sin h}{h}\right) \\
&= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\
&= (\cos x)(0) - (\sin x)(1) = -\sin x
\end{aligned}$$

■

Theorem 2.2.9.

$$\frac{d}{dx}\tan x = \sec^2 x$$

□

Proof.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

■

Then we can derive the following derivatives:

$$\begin{aligned}
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\csc x\cot x \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x\tan x \\
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x
\end{aligned}$$

2.2.4 Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.2.10 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

□

Proof. We know that by definition if $y = f(x)$, then $\Delta y = f(a + \Delta x) - f(a)$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let ε be the difference between the difference quotient and the derivative, then we have

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

Thus for a differentiable function f, if we define $\varepsilon = 0$ when $\Delta x = 0$, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx. Suppose that $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. Then we have

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1]\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2]\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. We substitute the expression for Δu then we have

$$\begin{aligned} \Delta y &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\Delta x \\ \frac{\Delta y}{\Delta x} &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \end{aligned}$$

Since $\Delta u \to 0$ as $\Delta x \to 0$, then $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] = f'(b)g'(a) = f'(g(a))g'(a)$$

thus the chain rule is proved. ■

2.3 Implicit Differentiation

2.3.1 Implicit Differentiation

An explicit function $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. An implicit function is defined implicitly by a relation between x and y. An example of implicit functions is the equation of the circle $x^2 + y^2 = r^2$ where the raidus r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y in an implicit function.

Problem 2.3.1. Find dy/dx of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then we have

$$\begin{aligned} \frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Since $y = f(x)$, we use the chain rule then we have

$$\begin{aligned} 2x + \frac{d}{dy}(y^2)\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

We solve for dy/dx then we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.3.2. Find dy/dx of the folium of Descartes $x^3 + y^3 = 6xy$.

Solution.

$$\begin{aligned} x^3+y^3 &= 6xy \\ \frac{d}{dx}x^3 + \frac{d}{dy}y^3\left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y + 6x\frac{dy}{dx} \\ 3x^2+3y^2\frac{dy}{dx} &= 6y+6x\frac{dy}{dx} \\ x^2+y^2\frac{dy}{dx} &= 2y+2x\frac{dy}{dx} \\ (y^2-2x)\frac{dy}{dx} &= 2y-x^2 \\ \frac{dy}{dx} &= \frac{2y-x^2}{y^2-2x} \end{aligned}$$

■

Problem 2.3.3. Find y' if $\sin(x+y) = y^2\cos x$.

Solution.

$$\begin{aligned} \sin(x+y) &= y^2\cos x \\ \cos(x+y)(1+\frac{dy}{dx}) &= 2y\frac{dy}{dx}\cos x - y^2\sin x \\ [2y\cos x - \cos(x+y)]\frac{dy}{dx} &= \cos(x+y)+y^2\sin x \\ \frac{dy}{dx} &= \frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)} \end{aligned}$$

■

Problem 2.3.4. Find y'' if $x^4+y^4=16$.

Solution. First we find y' then we have

$$\begin{aligned} x^4 + y^4 &= 16 \\ 4x^3 + 4y^3 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x^3}{y^3} \end{aligned}$$

Then we find y'' and we have

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) &= -3x^2y^{-3} + (-x^3)(-3y^{-4})\frac{dy}{dx} = 3x^3y^{-4}(-x^3y^{-3}) - 3x^2y^{-3} \\ &= -3x^6y^{-7} - 3x^2y^{-3} = -3x^2y^{-7}(x^4+y^4) = -48\frac{x^2}{y^7} \end{aligned}$$

■

2.4 Derivatives of Inverse Functions

2.4.1 Differentiation and Inverse Functions

Theorem 2.4.1. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous. □

Theorem 2.4.2. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

□

2.4.2 Derivatives of Logarithmic Functions

The **Euler's number** e named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function $y = e^x$ and the natural logarithmic function $y = \ln x$.

Definition 2.4.1 Euler's Number. The Euler's number e is defined as

$$e = \lim_{x \to 0}(1+x)^{1/x} = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

□

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.4.3. The exponential function $f(x) = \log_a x$ is differentiable and

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e = \frac{1}{x \ln a}$$

□

Proof. First we have

$$\frac{d}{dx}\log_a x = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a\left(\dfrac{x+h}{x}\right)}{h} = \lim_{h\to 0}\left[\frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right]$$
$$= \frac{1}{x}\left[\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h}\right] = \frac{1}{x}\left[\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)}\right] = \frac{1}{x}\log_a e$$

We know from the change of base formula that

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Therefore,

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$$

■

Theorem 2.4.4. The derivative of the natural logarithmic function $f(x) = \ln x$ is

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

□

Proof.

$$\frac{d}{dx}\ln x = \frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x}$$

■

Problem 2.4.1. Find $f'(x)$ if $f(x) = \ln|x|$.

Solution. Since $f(x) = \ln x$ for $x > 0$ and $f(x) = \ln(-x)$ for $x < 0$, it follows that

$$f'(x) = \frac{d}{dx}\ln x = \frac{1}{x}, \quad x > 0$$
$$f'(x) = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0$$

Therefore, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ for all $x \neq 0$. ■

2.4.3 Derivatives of Exponential Functions

Theorem 2.4.5. The exponential function $f(x) = a^x, a > 0$ is differentiable and

$$\frac{d}{dx}a^x = a^x\ln a$$

□

Proof. Let $y = a^x \iff \log_a y = x$ then by implicit differentiation we have

$$\log_a y = x$$
$$\frac{1}{y\ln a}\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = y\ln a = a^x\ln a$$

■

Theorem 2.4.6. The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx}e^x = e^x \qquad \square$$

Proof.

$$\frac{d}{dx}e^x = e^x \ln e = e^x \qquad \blacksquare$$

Problem 2.4.2. Prove the power rule $\dfrac{d}{dx}x^n = nx^{n-1}$ for $n \in \mathbb{R}$.

Solution. The rule is true when $x = 0$ which is trivial so it remains to prove the cases for $x \neq 0$. Let $y = x^n$ such that $y > 0$, by implicit differentiation we have

$$\begin{aligned}
\ln y &= \ln x^n = n \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n = nx^{n-1}
\end{aligned}$$

Similarly, let $y = x^n$ such that $y < 0$ then we have

$$\begin{aligned}
\ln|y| &= \ln(-x^n) = \ln[((-1)^{1/n}x)^n] = n\ln[(-1)^{1/n}x] \\
\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{(-1)^{1/n}x}(-1)^{1/n} = \frac{n}{x} \\
\frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n = nx^{n-1}
\end{aligned} \qquad \blacksquare$$

2.4.4 Inverse Trigonometric Functions

Theorem 2.4.7.

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \qquad \square$$

Proof. Let $y = \arcsin x \iff \sin y = x$ and $-\pi/2 \le y \le \pi/2$ such that $-1 \le x \le 1$. Note the approximate value of π is $\pi \approx 3.14159$. Then we have

$$\begin{aligned}
\frac{d}{dx}\sin y &= \frac{d}{dx}x \\
\cos y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y}
\end{aligned}$$

Since $-\pi/2 \le y \le \pi/2$ so $\cos y \ge 0$ thus $\cos y = \sqrt{1-\sin^2 y}$. Therefore, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$. $\blacksquare$

Theorem 2.4.8.

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \qquad \square$$

Proof. Let $y = \arccos x \iff \cos y = x$ and $0 \leq y \leq \pi$ such that $-1 \leq x \leq 1$. Then we have

$$\begin{aligned} \frac{d}{dx}\cos y &= \frac{d}{dx}x \\ -\sin y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y} \end{aligned}$$

Since $0 \leq y \leq \pi$ so $\sin y \geq 0$ thus $\sin y = \sqrt{1-\cos^2 y}$. Therefore, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$. ■

Theorem 2.4.9.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$ □

Proof. Let $y = \arctan x \iff \tan y = x$. Then we have

$$\begin{aligned} \frac{d}{dx}\tan y &= \frac{d}{dx}x \\ \sec^2 y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \end{aligned}$$

Then we can show that

$$\begin{aligned} \tan y &= x \\ 1+\frac{\sin^2 y}{\cos^2 y} &= 1+x^2 \\ 1+\frac{1-\cos^2 y}{\cos^2 y} &= 1+x^2 \\ \sec^2 y &= 1+x^2 \\ \cos^2 y &= \frac{1}{1+x^2} \end{aligned}$$

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. ■

2.5 Indeterminate Forms and L'Hôpital's Rule

2.5.1 Indeterminate Forms and L'Hôpital's Rule

Consider the limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then the limit is an **indeterminate form** of type $0/0$. If $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$, then the limit is an indeterminate form of type ∞/∞. We have the **L'Hôpital's Rule** discoverd by **Johann Bernoulli** (1667–1748) and named after **Guillaume de l'Hôpital** (1661–1704) to evaluate limits of indeterminate forms of type $0/0$ and ∞/∞.

Theorem 2.5.1 L'Hôpital's Rule. Suppose that f and g are differentiable and $g'(x) \neq 0$ near a, except possibly at a. If

$$\lim_{x\to a} f(x) = 0 \qquad\qquad \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \qquad\qquad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, is ∞, or is $-\infty$. □

L'Hôpital's rule is also valid for one-sided limits and for limits at infinity or negative infinity.

Problem 2.5.1. Find $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$.

Solution. Notice that the limit is an indeterminate form of $0/0$. We apply L'Hôpital's rule then we have

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 1$$ ■

Problem 2.5.2. Find $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$.

Proof. Notice that the limit is an indeterminate form of $0/0$. We apply L'Hôpital's rule then we have

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

The limit still is an indeterminate form of $0/0$ so we apply L'Hôpital's rule again then we have

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Similarly, we apply L'Hôpital's rule again then we have

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{2\sec^4 x + 4\sec^2 x \tan^2 x}{6} = \frac{2(1) + 4(1)(0)}{6} = \frac{1}{3}$$ ■

Note that L'Hôpital's rule only applies if the limit is one of the indeterminate forms of type $0/0$ or ∞/∞. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then the limit $\lim_{x\to a} f(x)g(x)$ is an indeterminate form of type $0 \cdot \infty$. We can express the product as a quotient then use L'Hôpital's rule to calculate the limit.

Problem 2.5.3. Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0 \qquad \blacksquare$$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ is an indeterminate form of type $\infty - \infty$. We can convert the difference into a quotient to apply L'Hôpital's rule to calculate the limit.

Problem 2.5.4. Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Solution.

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin x} = 0 \qquad \blacksquare$$

Consider the limit of the form

$$\lim_{x\to a}[f(x)^{g(x)}]$$

We have the following indeterminate forms arise from the limit:

- Type 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
- Type ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
- Type 1^∞ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Let $y = f(x)^{g(x)}$, then we can write $\ln y = \ln f(x)^{g(x)}$ and $y = e^{g(x)\ln f(x)}$ to find the limit.

Problem 2.5.5. Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution. Let $y = (1+\sin 4x)^{\cot x}$, then $\ln y = \ln[(1+\sin 4x)^{\cot x}] = \cot x \ln(1+\sin 4x)$. We apply L'Hôpital's rule then we have

$$\begin{aligned}\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{[1/(1+\sin 4x)](\cos 4x)(4)}{\sec^2 x} \\ &= \lim_{x\to 0^+} \frac{4\cos 4x \cos^2 x}{1+\sin 4x} = \frac{4(1)(1)}{1+0} = 4\end{aligned}$$

Therefore, we have

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4 \qquad \blacksquare$$

Problem 2.5.6. Calculate $\lim_{x\to 0^+} x^x$.

Solution. We have shown that $\lim_{x\to 0^+} x\ln x = 0$ and therefore we have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1 \qquad \blacksquare$$

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Pierre de Fermat (1601–1665)

Theorem 3.1.1 Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. □

3.2 The Mean Value Theorem

Michel Rolle (1652–1719)

Theorem 3.2.1 Rolle's Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$. □

Joseph-Louis Lagrange (1736–1813)

Theorem 3.2.2 Lagrange's Mean Value Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ □

Theorem 3.2.3 Cauchy's Mean Value Theorem. Suppose that the fucntions f and g are continuous on $[a, b]$ and differentiable on (a, b) and $g(x) \neq 0$ for all x in (a, b). Then there is a number c in a, b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$ □

3.3 Derivatives and Graphs

3.4 Antiderivatives

Definition 3.4.1. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I. □

Theorem 3.4.1. If F is an antiderivative of f on interval I, then the general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant. □

Problem 3.4.1. Find the general antiderivative of $f(x) = 1/x$.

Solution. We know that

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

for all $x \neq 0$. Then the antiderivative of f is $F(x) = \ln x + C_1$ if $x > 0$ or $F(x) = \ln(-x) + C_2$ if $x < 0$. Therefore, the general antiderivative of f is $F(x) = \ln |x| + C$. ■

4 Integrals

4.1 Integrals

4.1.1 Definite Integrals

A **Riemann sum** named after **Bernhard Riemann** (1826–1866) associated with a partition P of interval $[a, b]$ and a function f at the sample points x_i^* in the subinterval $[x_{i-1}, x_i]$ is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

Definition 4.1.1. If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if the limit exists such that f is integrable on $[a, b]$. □

Definition 4.1.2.

$$\int_a^b f(x)\,dx = I$$

if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \varepsilon$$

for all partitions P of $[a, b]$ with $\max \Delta x_i < \delta$ and for all x_i^* in $[x_{i-1}, x_i]$. □

Theorem 4.1.1. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$ and the definite integral $\displaystyle\int_a^b f(x)\,dx$ eixsts. □

Theorem 4.1.2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$. □

4.1.2 Properties of Definite Integrals

We have the following properties of indefinite integrals. Suppose all of the following integrals exist, and c is any constant.

- If $a > b$, then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- If $a = b$, then $\displaystyle\int_a^a f(x)\,dx = 0$.

- $\int_a^b c\,dx = c(b-a)$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

4.1.3 Indefinite Integrals

Definition 4.1.3. The **indefinite integral**

$$\int f(x)\,dx = F(x) + C$$

is the general antiderivative of f. □

4.2 Evaluating Integrals

4.3 The Fundamental Theorem of Calculus

Theorem 4.3.1 The Fundamental Theorem of Calculus (Newton-Leibniz Theorem). Suppose f is continuous on $[a, b]$. If the function F is defined by

$$F(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

then F is an antiderivative of f and

$$F'(x) = f(x), \quad a < x < b$$

and therefore

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If F is an antiderivative of f such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

□

The Fresnel integrals named after **Augustin-Jean Fresnel** (1788–1827) are

$$S(x) = \int_0^x \sin(t^2)\,dt \qquad\qquad C(x) = \int_0^x \cos(t^2)\,dt$$

and by the fundamental theorem of calculus the derivatives are

$$S'(x) = \sin(x^2) \qquad\qquad C'(x) = \cos(x^2)$$

4.4 The Substitution Rule

4.4.1 The Substitution Rule

Theorem 4.4.1 The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If g' is continuous on $[a, b]$ and f is continuous on I, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \square$$

Proof. Let $F' = f$, then by the chain rule we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int f(u)\,du \qquad \blacksquare$$

Problem 4.4.1. Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution. Let $u = 1 - 4x^2 \iff du = -8x\,dx$ so $xdx = -\dfrac{1}{8}\,du$. Therefore,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\int \frac{1}{8}u^{-(1/2)}\,du = -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C \qquad \blacksquare$$

Problem 4.4.2. Evaluate $\displaystyle\int \tan x\,dx$.

Solution. We have $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, let $u = \cos x \iff du = -\sin\,dx$ then

$$\int \tan x\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln\frac{1}{|\sec x|} + C = \ln|\sec x| + C \qquad \blacksquare$$

Problem 4.4.3. Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

Solution. Let $u = \ln x \iff du = \dfrac{1}{x}\,dx$, then

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} \qquad \blacksquare$$

4.4.2 Symmetry

Theorem 4.4.2. Suppose f is continuous on $[-a, a]$, then

- If $f(-x) = f(x)$ so f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_a^a f(x)\,dx$.
- If $f(-x) = -f(x)$ so f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$. $\square$

5 Techniques of Integration

5.1 Integration by Parts

5.1.1 Integration by Parts

Theorem 5.1.1 Integration by Parts Formula. If f and g are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If $u = f(x)$ and $v = g(x)$, then

$$\int u\,dv = uv - \int v\,du$$

and

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$ □

Proof. We know from the product rule that

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

and therefore

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$ ■

Problem 5.1.1. Evaluate $\int \ln x\,dx$.

Solution. Let $u = \ln x \iff du = \frac{1}{x}\,dx$ and $dv = dx \iff v = x$, then

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$ ■

Problem 5.1.2. Evaluate $\int \arctan x\,dx$.

Solution. Let $u = \arctan x \iff du = \frac{dx}{1+x^2}$ and $dv = dx \iff v = x$, then

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

Let $u = 1+x^2 \iff du = 2x\,dx$ so $x\,dx = \frac{1}{2}\,du$, then

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C$$

Therefore,

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$ ■

Problem 5.1.3. Evaluate $\int e^x \sin x \, dx$.

Solution. Let $u = \sin x \iff du = \cos x \, dx$ and $dv = e^x \, dx \iff v = e^x$, then

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Let $u = \cos x \iff du = -\sin x \, dx$ and $dv = e^x \, dx \iff v = e^x$, then

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Therefore,

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\ \int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) \end{aligned}$$

■

Problem 5.1.4. Prove the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution. Let $u = \sin^{n-1} x \iff du = (n-1) \sin^{n-2} x \cos x \, dx$ and $dv = \sin x \, dx \iff v = -\cos x$ then

$$\begin{aligned} \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \end{aligned}$$

Therefore,

$$\begin{aligned} (n-1+1) \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \\ \int \sin^n x \, dx &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \end{aligned}$$

■

5.2 Trigonometric Integrals and Substitutions

5.2.1 Trigonometric Integrals

We can use trigonometric identities to integrate trigonometric integrals.

Problem 5.2.1. Evaluate $\int \sin 5x \sin 2x \, dx$.

Solution.

$$\begin{aligned}\int \sin 5x \sin 2x\,dx &= \int \frac{1}{2}(\cos(5x-2x)-\cos(5x+2x))\,dx = \frac{1}{2}\int(\cos 3x - \cos 7x)\,dx \\ &= \frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C = \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C\end{aligned}$$ ■

Problem 5.2.2. Evaluate $\int \sin 3x \cos x\,dx$.

Solution.

$$\begin{aligned}\int \sin 3x \cos x\,dx &= \int \frac{1}{2}(\sin(3x+x)+\sin(3x-x))\,dx = \frac{1}{2}\int(\sin 4x + \sin 2x)\,dx \\ &= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right) + C = -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C\end{aligned}$$ ■

Problem 5.2.3. Evaluate $\int \cos^3 x\,dx$.

Solution.

$$\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int(1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$ ■

Problem 5.2.4. Evaluate $\int \sin^5 x \cos^2 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^5 x \cos^2 x\,dx &= \int (1-\cos^2 x)^2\cos^2 x \sin x\,dx = -\int (1-u^2)^2u^2\,du \\ &= -\int (u^2 - 2u^4 + u^6)\,du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$ ■

Problem 5.2.5. Evaluate $\int \cos^2 x\,dx$.

Solution.

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}\int(1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$ ■

Problem 5.2.6. Evaluate $\int_0^\pi \sin^2 x\,dx$.

Solution.

$$\begin{aligned}\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2x)\,dx \\ &= \frac{1}{2}\left(\Big[x\Big]_0^\pi - \frac{1}{2}\Big[\sin 2x\Big]_0^\pi\right) = \frac{\pi}{2}\end{aligned}$$ ■

Problem 5.2.7. Evaluate $\int \sin^4 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right) + C \\ &= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C\end{aligned}$$

■

In general, an integral of powers of $\sin x$ and $\cos x$ is in the form

$$\int \sin^m x \cos^n x\,dx$$

where $m, n \in \mathbb{Z}$ and $m, n \geq 0$. If m is odd, then we save a factor of $\sin x$ and express the rest in terms of $\cos x$ for substitution. If n is odd, then we save a factor of $\cos x$ and express the rest in terms of $\sin x$ for substitution. If m and n are even, then we use the power reduction formulas.

Problem 5.2.8. Evaluate $\int \tan^6 x \sec^4 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^6 x \sec^4 x\,dx &= \int \tan^6 x(\tan^2 x + 1)\sec^2 x\,dx = \int u^6(u^2+1)\,du = \int u^8 + u^6 \\ &= \frac{u^9}{9} + \frac{u^7}{7} = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C\end{aligned}$$

■

Problem 5.2.9. Evaluate $\int \tan^5 x \sec^7 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^5 x \sec^7 x\,dx &= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x\,dx = \int (u^2-1)^2 u^6\,du \\ &= \int (u^{10} - 2u^8 + u^6)\,dx = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C\end{aligned}$$

■

In general, an integral of powers of $\tan x$ and $\sec x$ is in the form

$$\int \tan^m x \sec^n x\,dx$$

where $m, n \in \mathbb{Z}$ and $m, n \geq 0$. If m is odd, then we save a factor of $\sec x \tan x$ and express the rest in terms of $\sec x$ for substitution. If n is even, then we save a factor of $\sec^2 x$ and express the rest in terms of $\tan x$ for substitution.

Problem 5.2.10. Evaluate $\int \sec x\,dx$.

Solution. We have

$$\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Let $u = \sec x + \tan x \iff du = (\sec x \tan x + \sec^2 x)\,dx$, then

$$\int \sec x\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \quad \blacksquare$$

Problem 5.2.11. Evaluate $\int \tan^3 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan x(\sec^2 x - 1)\,dx = \int \tan x \sec^2\,dx - \int \tan x\,dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned} \quad \blacksquare$$

Problem 5.2.12. Evaluate $\int \sec^3 x\,dx$.

Solution. Let $u = \sec x \iff du = \sec x \tan x$ and $dv = \sec^2 x\,dx \iff v = \tan x$, then

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$$

We have

$$\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$$

Therefore,

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ \int \sec^3 x\,dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C\end{aligned} \quad \blacksquare$$

5.2.2 Trigonometric Substitutions

If an integral has the form $\int \sqrt{a^2 - x^2}\,dx$, then we can use the substitution $x = a\sin\theta$ where $-\pi/2 \le \theta \le \pi/2$ to get

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = |a|\cos\theta$$

If an integral has the form $\int \sqrt{a^2 + x^2}\,dx$, then we can use the substitution $x = a\tan\theta$ where $-\pi/2 < \theta < \pi/2$ to get

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(\tan^2\theta + 1)} = \sqrt{a^2\sec^2\theta} = |a|\sec\theta$$

If an integral has the form $\int \sqrt{x^2 - a^2}\,dx$, then we can use the substitution $x = a\sec\theta$ where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$ to get

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a|\tan\theta$$

Problem 5.2.13. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Solution. Let $x = 3\sin\theta \iff dx = 3\cos\theta\,d\theta$, then

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C\end{aligned}$$ ■

Problem 5.2.14. Evaluate $\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution. Let $x = 2\tan\theta \iff dx = 2\sec^2\theta\,d\theta$ and $-\pi/2 < \theta < \pi/2$, then

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta}{4\tan^2\theta(2\sec\theta)}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} \\ &= -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C\end{aligned}$$ ■

Problem 5.2.15. Find the area enclosed by a circle with radius r.

Solution. The area is $A = 4\int_0^r \sqrt{r^2-x^2}\,dx$. Let $x = r\sin\theta \iff dx = r\cos\theta\,d\theta$ and $0 \le \theta \le \pi/2$, then

$$\begin{aligned}A &= 4\int_0^{\pi/2} r\cos\theta(r\cos\theta)\,d\theta = 4r^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4r^2\left(\frac{1}{2}\left(\left[\theta\right]_0^{\pi/2} + \frac{1}{2}\left[\sin 2\theta\right]_0^{\pi/2}\right)\right) \\ &= 4r^2\left(\frac{\pi}{4}\right) = \pi r^2\end{aligned}$$ ■

Problem 5.2.16. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. The area is $A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$. Let $x = a\sin\theta \iff dx = a\cos\theta\,d\theta$ and $0 \le \theta \le \pi/2$, then

$$A = 4\int_0^{\pi/2} \frac{b}{a}(a\cos\theta)(a\cos\theta)\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\left(\frac{\pi}{4}\right) = \pi ab$$ ■

5.3 Partial Fractions

5.3.1 Partial Fractions

Consider a rational function $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials. If f is improper, that is, $\deg(P) \ge \deg(Q)$, then we simplify to get $f(x) = S(x) + \frac{R(x)}{Q(x)}$ where S and R are polynomials. Then we can factor Q to be irreducible and express $\dfrac{R(x)}{Q(x)}$ as the a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

There are four possible cases:

1. Q is a product of distinct linear factors. Then

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

   and therefore
$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

2. Q is a product of linear factors where some are repeated. Suppose that the first linear factor is repeated r times, then we have

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

   for the first repeated linear factor and similarly for other repeated linear factors.

3. Q has irreducible quadratic factors without repeated factors. Then for every quadratic factor we have
$$\frac{Ax+B}{ax^2+bx+c}$$

4. Q has a repeated irreducible quadratic factor. Suppose that the first quadratic factor is repeated r times, then we have

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

   for the first repeated quadratic factor and similarly for the others.

Problem 5.3.1. Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Solution. We simplify to get $2x^3+3x^2-2x = x(2x-1)(x+2)$ then

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$
$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$$x = 0 \iff -1 = -2A \iff A = \frac{1}{2}$$
$$x = \frac{1}{2} \iff \frac{1}{4} = \frac{5}{4}B \iff B = \frac{1}{5}$$
$$x = -2 \iff -1 = 10C \iff C = -\frac{1}{10}$$

and therefore

$$\begin{aligned}\int \frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{1}{2}\int\frac{dx}{x} + \frac{1}{5}\int\frac{dx}{2x-1} - \frac{1}{10}\int\frac{dx}{x+2}\\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K\end{aligned}$$ ■

Problem 5.3.2. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution. We simplify to get

$$\begin{aligned}x^4-2x^2+4x+1 &= (x^2-1)^2+4x = (x+1)^2(x-1)^2+4x\\ x^3-x^2-x+1 &= x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2\end{aligned}$$

and so

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{(x+1)(x-1)^2}\,dx$$

Then

$$\begin{aligned}\frac{4x}{(x+1)(x-1)^2} &= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\\ 4x &= A(x-1)^2 + B(x+1)(x-1) + C(x+1)\\ x=-1 \iff -4=4A &\iff A=-1\\ x=1 \iff 4=2C &\iff C=2\\ x=0 \iff 0=-1-B+2 &\iff B=1\end{aligned}$$

and therefore

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int (x+1)\,dx - \int\frac{dx}{x+1} + \int\frac{dx}{x-1} + 2\int\frac{dx}{(x-1)^2}\\ &= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + K\end{aligned}$$ ■

Problem 5.3.3. Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$.

Solution. We do partial fraction decomposition then

$$\begin{aligned}\frac{2x^2-x+4}{x^3+4x} &= \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}\\ 2x^2-x+4 &= A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A\end{aligned}$$

$$A=1,\quad B=1,\quad C=-1$$

and so

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int\frac{dx}{x} + \int\frac{x-1}{x^2+4}\,dx = \ln|x| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$$ ■

Problem 5.3.4. Evaluate $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$

Solution. We simplify to get

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int \frac{4x^2-4x+3+x-1}{4x^2-4x+3}\,dx = \int \left(1+\frac{x-1}{(2x-1)^2+2}\right)dx$$

then

$$\begin{aligned}\int \frac{x-1}{(2x-1)^2+2}\,dx &= \frac{1}{2}\int \frac{\frac{1}{2}(u+1)-1}{u^2+2}\,dx = \frac{1}{4}\int \frac{u-1}{u^2+2}\,dx \\ &= \frac{1}{4}\left(\frac{1}{2}\ln(u^2+2) - \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}u\right)\right) \\ &= \frac{1}{8}\ln((2x-1)^2+2) - \frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x-1)\right)\end{aligned}$$

and so

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = x + \frac{1}{8}\ln(4x^2-4x+3) - \frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x-1)\right) + C \qquad \blacksquare$$

Problem 5.3.5. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

Solution. We do partial fraction decomposition then

$$\begin{aligned}\frac{1-x+2x^2-x^3}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\ 1-x+2x^2-x^3 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x\end{aligned}$$

$$x = 0 \iff A = 1$$

and so

$$\begin{aligned}1-x+2x^2-x^3 &= x^4+2x^2+1+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \\ -x-x^3 &= (B+1)x^4 + Cx^3 + (B+D)x^2 + (C+E)x\end{aligned}$$

$$B=-1 \quad C=-1 \quad D=1 \quad E=0$$

Therefore

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{dx}{x} - \int \frac{x+1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) - \arctan x - \frac{1}{2(x^2+1)} + K \qquad \blacksquare\end{aligned}$$

5.4 Improper Integrals

6 Applications of Integration

6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

An infinite **sequence** a_n is a list of numbers defined by

$$a_1, a_2, \ldots, a_n, \ldots$$

where a_n is the nth term of the sequence.

7.2 Series

An infinite **series** is the sum of the terms of an infinite sequence

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

7.3 Convergence Tests

7.4 Power Series

A **power series** centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = a_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and the coefficient c_n is a constant.

7.5 Taylor Series

The **Taylor series** of a function f centered at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

For the special case of $a = 0$, the Taylor series becomes the **Maclaurin series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

A **differential equation** is an equation that relates some unknown functions and their derivatives. An **ordinary differential equation** (ODE) is a differential equation that relates one or more functions of a single variable and their ordinary derivatives. The **order** of a differential equation is the highest order of the derivative in the equation. Newton's second law of motion, the force F acting on an object with mass m and acceleration a is $F = ma$ can be written as an ordinary differential equation

$$F = m\frac{dv}{dt}$$

which is a first order differential equation, or

$$F = m\frac{d^2s}{dt^2}$$

which is a second order differential equation. By Newton's second law of motion, the net force F acting on a falling object with mass m, velocity v, and air resistance force γv can be modeled by the differential equation

$$F = m\frac{dv}{dt} = mg - \gamma v$$

where g is the acceleration due to gravity and γ is the drag coefficient. A function f is a **solution** of a differential equation if the function and its derivatives satisfy the equation for all values of x in some open interval $a < x < b$. It is possible that there are many solutions of a differential equation. An initial condition is a condition $y(x_0) = y_0$ or $y^{(n)}(x_0) = y_n$ on the solution. An **initial value problem** is solving a differential equation with initial conditions. The interval of validity is the largest possible interval on which the solution is valid and contains x_0 in the initial conditions. The general solution of a differential equation is the set of all solutions and the particular solution is the solution that satisfies the initial conditions. An explicit solution is any solution in the form $y = y(x)$, otherwise it is an implicit solution. The particular solution of the differential equation

$$\frac{dy}{dx} = y$$

with initial condition $y(0) = 1$ is $y = e^x$ since

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x = y$$

and $y(0) = e^0 = 1$. The **existence** and **uniqueness** problem asks that given a differential equation, does there exist a solution and if any is there only one solution.

9.2 Direction Fields and Euler's Method

9.2.1 Direction Fields

Suppose we are given a first order differential equation of the form $y' = F(x, y)$. If a solution curve, or an integral curve, passes through the point (x_0, y_0), then its slope at the point is $y'(x_0) = F(x_0, y_0)$. If we draw short line segments with slopes $F(x, y)$ at serval points (x, y), then the result is a **direction field**, or a **slope field**.

9.2.2 Euler's Method

9.3 Separable Equations

9.3.1 Separation of Variables

A **separable equation** is a first order differential equation that can be written in the differential form

$$M(x)\,dx + N(y)\,dy = 0$$

or can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

We can separate the variables if $g(y) \neq 0$, let $h(y) = 1/g(y)$ then

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$

We write the equation in the differential form

$$h(y)\,dy = f(x)\,dx$$

Then we integrate both sides of the equation:

$$\int h(y)\,dy = \int f(x)\,dx$$

Now we have an implicit solution of the differential equation and sometimes we can solve for an explicit solution. We can justify the method of separation of variables by using the chain rule to show that

$$\begin{aligned}
\frac{d}{dx}\int h(y)\,dy &= \frac{d}{dx}\int f(x)\,dx \\
\frac{d}{dy}\int h(y)\,dy\frac{dy}{dx} &= f(x) \\
h(y)\frac{dy}{dx} &= f(x) \\
\frac{dy}{dx} &= \frac{f(x)}{h(y)} = f(x)g(y)
\end{aligned}$$

Problem 9.3.1. Solve the differential equation $\dfrac{dy}{dx} = -xy$.

Solution. Notice that $y = 0$ is a trivial solution then we solve the differential equation for non-trivial solutions $y \neq 0$. We use separation of variables then

$$\begin{aligned}\frac{dy}{dx} &= -xy \\ \int \frac{dy}{y} &= -\int x\,dx \\ \ln|y| + C_1 &= -\frac{x^2}{2} + C_2\end{aligned}$$

Let $C = C_2 - C_1$, then

$$\begin{aligned}\ln|y| &= -\frac{x^2}{2} + C_2 - C_1 = -\frac{x^2}{2} + C \\ |y| &= e^{(-x^2/2)+C} = e^C e^{-x^2/2} \\ y &= \pm e^C e^{-x^2/2} = Ae^{-x^2/2}\end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.2. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ with the initial condition $y(0) = 2$.

Solution. We use separation of variables to find the general solution then

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2}{y^2} \\ \int y^2\,dy &= \int x^2\,dx \\ \frac{y^3}{3} + C_1 &= \frac{x^3}{3} + C_2 \\ \frac{y^3}{3} &= \frac{x^3}{3} + C, \quad C = C_2 - C_1 \\ y^3 &= x^3 + 3C \\ y &= \sqrt[3]{x^3 + K}, \quad K = 3C\end{aligned}$$

We consider the initial condition $y(0) = 2$ to find the particular solution then

$$y(0) = \sqrt[3]{0 + K} \iff 2 = \sqrt[3]{K} \iff K = 8$$

The solution of the initial value problem is $y = \sqrt[3]{x^3 + 8}$. ■

Problem 9.3.3. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Solution. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{6x^2}{2y + \cos y} \\ \int (2y + \cos y)\,dy &= \int 6x^2\,dx \\ y^2 + \sin y &= 2x^3 + C\end{aligned}$$

where $C \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.4. Solve the differential equation $y' = x^2 y$.

Solution. We have

$$\begin{aligned}\frac{dy}{dx} &= x^2 y \\ \int \frac{dy}{y} &= \int x^2\, dx \\ \ln|y| &= \frac{x^3}{3} + C \\ |y| &= e^{(x^3/3)+C} \\ y &= Ae^{x^3/3}\end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

9.3.2 Homogeneous Equations

A **homogeneous** equation is in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

We can transform a homogeneous equation into a separable equation by a change of variable. Consider the equation

$$\frac{dy}{dx} = \frac{y-4x}{x-y}$$

and we can show that

$$\frac{dy}{dx} = \frac{(y/x)-4}{1-(y/x)}$$

thus the equation is homogeneous. Let $v = y/x \iff y = vx$ so

$$\begin{aligned}\frac{dy}{dx} = \frac{dv}{dx}x + v &= \frac{v-4}{1-v} \\ \frac{dv}{dx}x &= \frac{v-4}{1-v} - v = \frac{v-4-v(1-v)}{1-v} = \frac{v^2-4}{1-v}\end{aligned}$$

and the equation is separable then

$$\int \frac{1-v}{v^2-4}\, dv = \int \frac{dx}{x}$$

Since

$$\int \frac{1-v}{v^2-4}\, dv = \int \frac{dv}{v^2-4} - \int \frac{v}{v^2-4}\, dv$$

then

$$\frac{1}{v^2-4} = \frac{1}{(v+2)(v-2)} = \frac{A}{v+2} + \frac{B}{v-2}$$
$$1 = A(v-2) + B(v+2)$$
$$v = -2 \iff -4A = 1 \iff A = -\frac{1}{4}$$
$$v = 2 \iff 4B = 1 \iff B = \frac{1}{4}$$

so

$$\int \frac{dv}{v^2-4} = -\frac{1}{4}\int \frac{dv}{v+2} + \frac{1}{4}\int \frac{dv}{v-2} = -\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2|$$

and

$$\int \frac{v}{v^2-4}\,dv = \frac{1}{2}\ln|v^2-4|$$

therefore

$$\int \frac{1-v}{v^2-4}\,dv = -\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2| - \frac{1}{2}\ln|v^2-4|$$

Then

$$-\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2| - \frac{1}{2}\ln|v^2-4| = \ln|x| + C_1$$
$$\ln|v-2| - \ln|v+2| - 2\ln|v^2-4| = 4\ln|x| + C_2, \quad C_2 = 4C_1$$
$$\ln\left|\frac{v-2}{v+2}\right| - \ln((v^2-4)^2) = \ln\left|\frac{v-2}{(v+2)(v+2)^2(v-2)^2}\right| = \ln(x^4) + C_2$$
$$\ln\left|\frac{1}{(v+2)^3(v-2)}\right| = \ln(x^4) + C_2$$
$$\frac{1}{(v+2)^3(v-2)} = C_3x^4, \quad C_3 = \pm e^{C_2}$$
$$(v+2)^3(v-2)x^4 = C, \quad C = 1/C_3$$
$$(vx+2x)^3(vx-2x) = C$$

Therefore the solution is

$$(y+2x)^3(y-2x) = C$$

Problem 9.3.5. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$$

Solution. Since

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

hence the equation is homogeneous. Let $y = vx$, then

$$\frac{dy}{dx} = \frac{dv}{dx}x + v = 1 + v + v^2$$
$$\frac{dv}{dx}x = 1 + v^2$$

and by separation of variables

$$\begin{aligned}\int \frac{dv}{1+v^2} &= \int \frac{dx}{x} \\ \arctan v &= \ln x + C \\ \arctan\left(\frac{y}{x}\right) - \ln x &= C\end{aligned} \qquad \blacksquare$$

9.4 Population Growth

Let $y = y(t)$ be a function representing the value of a quantity y at time t such that

$$\frac{dy}{dt} = ky$$

where k is a constant, then the differential equation is the law of natural growth if $k > 0$ or the law of natural decay if $k < 0$. Since the differential equation is separable, then

$$\begin{aligned}\frac{dy}{dt} &= ky \\ \int \frac{dy}{y} &= \int k\,dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt+C} = e^C e^{kt} \\ y &= \pm e^C e^{kt} = Ae^{kt}\end{aligned}$$

so $y = Ae^{kt}$ is the general solution of the differential equation. It follows that

$$\frac{dy}{dt} = kAe^{kt} = ky$$

and $y(0) = Ae^0 = A$ is the initial value of the function $y(t)$.

9.4.1 Logistic Growth

If M is the carrying capacity and $0 < y < M$, then the logistic differential equation is

$$\frac{dy}{dt} = ky(M-y)$$

We use separation of variables then

$$\begin{aligned}\frac{dy}{dt} &= ky(M-y) \\ \int \frac{dy}{y(M-y)} &= \int k\,dt\end{aligned}$$

and using partial fractions

$$\begin{aligned}\frac{1}{y(M-y)} &= \frac{A}{y} + \frac{B}{M-y} \\ 1 &= A(M-y) + By \\ y = 0 \iff A &= \frac{1}{M} \\ y = M \iff B &= \frac{1}{M}\end{aligned}$$

therefore

$$\begin{aligned}\frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)\,dy &= \int k\,dt \\ \frac{1}{M}(\ln|y| - \ln|M-y|) &= kt + C_1 \\ \ln\frac{y}{M-y} &= kMt + C_2 \\ \frac{y}{M-y} &= e^{kMt+C_2} = e^{C_2}e^{kMt} = Ae^{kMt}\end{aligned}$$

If the population at time $t = 0$ is $y(0) = y_0$, then $A = y_0/(M - y_0)$ and so

$$\begin{aligned}\frac{y}{M-y} &= \frac{y_0}{M-y_0}e^{kMt} \\ (M-y_0)y &= y_0e^{kMt}(M-y) = y_0Me^{kMt} - y_0e^{kMt}y \\ (M-y_0)y + y_0e^{kMt}y = (M - y_0 + y_0e^{kMt})y &= y_0Me^{kMt} \\ y &= \frac{y_0Me^{kMt}}{M-y_0+y_0e^{kMt}}\end{aligned}$$

then

$$y = \frac{y_0M}{(M-y_0+y_0e^{kMt})e^{-kMt}} = \frac{y_0M}{y_0+(M-y_0)e^{-kMt}}$$

is the solution of the differential equation and

$$\lim_{t\to\infty} y(t) = \frac{y_0M}{y_0+0} = M$$

We can show that

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}(kMy - ky^2) = kM\frac{dy}{dt} - 2ky\frac{dy}{dt} = k(M-2y)\frac{dy}{dt} = k(M-2y)ky(M-y) \\ &= k^2y(M-y)(M-2y)\end{aligned}$$

Then

$$k^2y(M-y)(M-2y) = 0$$

and

$$
\begin{array}{ccc}
k^2 y = 0 & M - y = 0 & M - 2y = 0 \\
y = 0 & y = M & y = \dfrac{M}{2}
\end{array}
$$

so

$$
\begin{aligned}
\left.\frac{dy}{dt}\right|_{y=0} &= k(0)(M-0) = 0 \\
\left.\frac{dy}{dt}\right|_{y=M} &= kM(M-M) = 0 \\
\left.\frac{dy}{dt}\right|_{y=M/2} &= k\frac{M}{2}\left(M - \frac{M}{2}\right) = \frac{kM^2}{4}
\end{aligned}
$$

We deduce that a population grows fastest when it reaches half its carrying capacity.

9.4.2 Mathematical Modeling

An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Suppose that the resisting force is proportional to the velocity for small values of v so

$$f(v) = -kv$$

where k is a positive constant. For large values of v a better model is

$$f(v) = -kv^2$$

where k is a positive constant.

Problem 9.4.1. According to Newton's law of universal gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x+R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's second law,

$$F = ma = m\frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that $v_0 = \sqrt{\dfrac{2gRh}{R+h}}$ and compute $v_e = \lim_{h\to\infty} v_0$, the escape velocity of the earth, using $R = 6,378$ km and $g = 9.8$ m/s^2.

Solution. By the chain rule

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

then

$$\begin{aligned} m\frac{dv}{dt} = mv\frac{dv}{dx} &= -\frac{mgR^2}{(x+R)^2} \\ v\,dv &= -\frac{gR^2}{(x+R)^2}\,dx \end{aligned}$$

Since the height is zero when $v = v_0$ and the height is maximum when $v = 0$ then

$$\begin{aligned} \int_{v_0}^{0} v\,dv &= -\int_0^h \frac{gR^2}{(x+R)^2}\,dx \\ \left[\frac{v^2}{2}\right]_{v_0}^{0} &= \left[\frac{gR^2}{x+R}\right]_0^h \\ -\frac{(v_0)^2}{2} &= gR^2\left(\frac{1}{R+h} - \frac{1}{R}\right) = gR^2\left(\frac{R-(R+h)}{R(R+h)}\right) = -\frac{gRh}{R+h} \\ v_0 &= = \sqrt{\frac{2gRh}{R+h}} \end{aligned}$$

The escape velocity of the earth is

$$\begin{aligned} v_e &= \lim_{h\to\infty}\sqrt{\frac{2gRh}{R+h}} = \sqrt{\lim_{h\to\infty}\frac{2gR}{(R/h)+1}} = \sqrt{\frac{2gR}{0+1}} = \sqrt{2gR} \\ &= \sqrt{2(9.8)(6.378\times 10^6)}\ \text{m/s} \approx 11.1807\ \text{km/s} \end{aligned}$$

■

The End