

Calculus

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For all lovers of mathematics.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x) dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y . We have the following formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the Cartesian product of X and Y .

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every x in X , there exists a y in Y such that (x, y) is in R .

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If (x, y) and (x, z) are in R , then $y = z$.

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

1.1.2 Intuitive Definition of a Limit

Let $f(x)$ be a function defined on some open interval that contains a , possibly not at a itself. Newton introduced the following working definition of a limit.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from left and right but $x \neq a$.

$$\lim_{x \rightarrow a} f(x) = L$$

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.

$$\lim_{x \rightarrow a^-} f(x) = L$$

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.

$$\lim_{x \rightarrow a^+} f(x) = L$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L , otherwise the limit **does not exist**.

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.3 Computing Limits

1.4 Continuity

1.5 Limits and Infinity

2 Derivatives

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2.3 Implicit Differentiation

2.4 Derivatives of Inverse Functions

2.5 Indeterminate Forms and l'Hospital's Rule

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3.3 Derivatives and Graphs

3.4 Antiderivatives

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4.2 Evaluating Definite Integrals

4.3 The Fundamental Theorem of Calculus

4.4 The Substitution Rule

5 Techniques of Integration

5.1 Integration by Parts

5.2 Trigonometric Integrals and Substitutions

5.3 Partial Fractions

5.4 Improper Integrals

6 Applications of Integration

6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

7.2 Series

7.3 Convergence Tests

7.4 Power Series

7.5 Taylor Series

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

9.2 Direction Fields and Euler's Method

9.3 Separable Equations