Calculus

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For all lovers of mathematics and science.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y. We have a formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the **Cartesian product** of X and Y.
$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$
- For every x in X, there exists a y in Y such that (x, y) is in R.
$$\forall x \in X, \exists y \in Y, (x, y) \in R$$
- If (x, y) and (x, z) are in R, then $y = z$.
$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

□

The real line is the 1-dimensional **Eculidean space** defined as the set of real numbers $\mathbb{R}$. The xy-plane in the **Cartesian coordinate system** by **René Descartes** (1596–1650) is the 2-dimensional Eculidean space defined as the set of all ordered pairs of real numbers $(x, y) \in \mathbb{R}^2$. A **function of a real variable** is a function whose domain is the set of real numbers $\mathbb{R}$. A real function is a real-valued function of a real variable whose domain and codomain is $\mathbb{R}$.

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.
$$\lim_{x \to a} f(x) = L$$
□

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.
$$\lim_{x \to a^-} f(x) = L$$
□

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.
$$\lim_{x \to a^+} f(x) = L$$
□

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.
$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

Definition 1.5.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.6.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.7.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Problem 1.1. Prove that

$$\lim_{x \to 3} (4x - 5) = 7$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that

$$\lim_{x \to 3} (4x - 5) = 7 \qquad \blacksquare$$

Problem 1.2. Prove that

$$\lim_{x \to 3} x^2 = 9$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x+3|\,|x-3| < \varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3| < C\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3| < 1$ such that $|x+3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$\begin{gathered} 0 < |x-3| < 1 \iff |x+3| < 7 \\ 0 < |x-3| < \frac{\varepsilon}{7} \iff 7\,|x-3| < \varepsilon \\ |x+3|\,|x-3| < 7\,|x-3| < \varepsilon \implies |x^2-9| < \varepsilon \end{gathered}$$

Therefore, it is proved that

$$\lim_{x \to 3} x^2 = 9$$

■

Problem 1.3. Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

■

1.3 Evaluating Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$$

exist. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.1.

$$\lim_{x \to a} c = c$$

□

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} c = c \qquad \blacksquare$$

Theorem 1.2.

$$\lim_{x \to a} x = a \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} x = a \qquad \blacksquare$$

Theorem 1.3 Constant Multiple Law. The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL \qquad \square$$

Proof. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} [0 \cdot f(x)] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL \qquad \blacksquare$$

Theorem 1.4 Sum and Difference Law. The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \square$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)+g(x)-(L+M)| < \varepsilon$$

By the **triangle inequality** $|a+b| \le |a|+|b|$, we have

$$|f(x)+g(x)-(L+M)| = |f(x)-L+g(x)-M| \le |f(x)-L| + |g(x)-M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |f(x)-L| < \frac{\varepsilon}{2}$$

Similarly, there is a number δ_2 such that

$$0 < |x-a| < \delta_2 \implies |h(x)-L| < \varepsilon \implies L$$

$$0 < |x-a| < \delta_2 \implies |g(x)-M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x-a| < \delta \implies 0 < |x-a| < \delta_1, \delta_2$$

$$|f(x)-L| + |g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x-a| < \delta \implies |f(x)+g(x)-(L+M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned}\lim_{x\to a}[f(x)-g(x)] &= \lim_{x\to a}[f(x)+(-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) \\ &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M\end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \blacksquare$$

Theorem 1.5 Product Law. The limit of a product is the product of the limits.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x)-L]g(x) + L[g(x)-M]| \\ &\leq |[f(x)-L]g(x)| + |L[g(x)-M]| = |f(x)-L|\,|g(x)| + |L|\,|g(x)-M| \end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1+|M|)}$$

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x-a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x-a| < \delta \implies 0 < |x-a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)-L|\,|g(x)| + |L|\,|g(x)-M| \\ &< \frac{\varepsilon}{2(1+|M|)}(1+|M|) + (1+|L|)\frac{\varepsilon}{2(1+|L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \blacksquare$$

Theorem 1.6 Quotient Law. The limit of a quotient is the quotient of the limits (if that the limit of the denominator is not 0).

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \iff \lim_{x\to a} g(x) = M \neq 0 \qquad \square$$

Proof. First we prove that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Notice that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|} \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$, then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|} |g(x) - M| < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \left(\frac{1}{g(x)} \right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \qquad \blacksquare$$

Theorem 1.7 Power Law.

$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n, n \in \mathbb{R} \qquad \square$$

Theorem 1.8 Root Law.

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, n \in \mathbb{R} \qquad \square$$

Theorem 1.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a) \qquad \square$$

Thus, we have the following limits

$$\lim_{\theta\to 0} \sin\theta = 0 \qquad\qquad \lim_{\theta\to 0} \cos\theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if the limits exist.

Problem 1.4. Show that

$$\lim_{x\to 0} |x| = 0$$

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, it is shown that

$$\lim_{x\to 0} |x| = 0 \qquad \blacksquare$$

Theorem 1.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \iff L \leq M \qquad \square$$

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a}[g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x-a| < \delta \implies g(x) - f(x) - (M-L) < L - M$$

which simplifies to

$$0 < |x-a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \qquad \blacksquare$$

Theorem 1.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L \qquad \square$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x\to a} h(x) = L$, there exists a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x\to a} g(x) = L \qquad \blacksquare$$

By **algebra**, **geometry**, and **trigonometry**, we can get the following result by the **Pythagorean theorem** $a^2 + b^2 = c^2$. If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1 \qquad\qquad \theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta}$$

so we have the following inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $(\sin\theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.5. Evaluate

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

Solution. By the **Pythagorean identity** $\sin^2\theta + \cos^2\theta = 1$, we have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit and we have

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)\right] = \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) = 1\left(\frac{(-1)(0)}{1+1}\right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

■

1.4 Continuity

1.4.1 Continuous Functions

Let $f(x)$ be a function and the number a is in the domain of f so $f(a)$ is defined. If the limit exists, then we have the following definition.

Definition 1.8. A function f is **continuous** at the number a if

$$\lim_{x \to a} f(x) = f(a)$$

□

A function f is continuous from the left at a if the left-hand limit equals $f(a)$ and it is continuous from the right at a if the right-hand limit equals $f(a)$. A function f is continuous on an interval if it is continuous at every number in the interval. If f is not continuous at a, then it is a discontinuous function at a. If f and g are continuous functions at a and c is a constant, then the following functions are also continuous at a.

$$f+g \qquad f-g \qquad cf \qquad f \cdot g \qquad \frac{f}{g} \iff g(x) \neq 0$$

Theorem 1.12. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$. □

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients a_i are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. ■

Theorem 1.13. Let f be any rational function, then f is continuous on its domain. □

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. $\blacksquare$

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

Theorem 1.14. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then the limit of the composite function $f \circ g$ is

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$ $\square$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, then we have $\lim_{y\to b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$ $\blacksquare$

Theorem 1.15. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. $\square$

Proof. Since g is continuous at a, we have $\lim_{x\to a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a. $\blacksquare$

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

Theorem 1.16 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number c in the open interval (a, b) such that $f(c) = N$. $\square$

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then there exists a root of $f(x)$ in (a, b) which follows immediately from the intermediate value theorem.

1.5 Limits and Infinity

1.5.1 Infinite Limits

Definition 1.9. The limit of $f(x)$ as x approaches a is **infinity** if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

$$\lim_{x\to a} f(x) = \infty$$ □

Definition 1.10. The limit of $f(x)$ as x approaches a is **negative infinity** if the values of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a but not equal to a.

$$\lim_{x\to a} f(x) = -\infty$$ □

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Definition 1.11. The **vertical asymptote** of the curve $y = f(x)$ is the line $x = a$ if one of the infinite limits is infinity or negative infinity. □

1.5.2 Limits at Infinity

Definition 1.12. Let f be a function defined on some interval (a, ∞). The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x\to\infty} f(x) = L$$ □

Definition 1.13. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x\to-\infty} f(x) = L$$ □

Problem 1.6. Evaluate $\lim_{x\to\infty} \sin x$ and $\lim_{x\to\infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to \infty$ so the limits do not exist. ■

Definition 1.14. The **horizontal asymptote** of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L. □

1.5.3 Infinite Limits at Infinity

Definition 1.15. The values of $f(x)$ become arbitrarily large for sufficiently large x.

$$\lim_{x\to\infty} f(x) = \infty$$ □

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

1.5.4 Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.16.

$$\lim_{x\to a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \qquad \square$$

Problem 1.7. Prove that

$$\lim_{x\to a} \frac{1}{x^2} = \infty$$

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that

$$\lim_{x\to a} \frac{1}{x^2} = \infty \qquad \blacksquare$$

Let f be a function defined on some interval (a, ∞).

Definition 1.17.

$$\lim_{x\to\infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon \qquad \square$$

Problem 1.8. Prove that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \blacksquare$$

Definition 1.18.

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M \qquad \square$$

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

2.1.1 Derivatives and Rates of Change

Definition 2.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

where $h = x - a$, if this limit exists. $\square$

Definition 2.2. The **velocity** at time $t = a$ of a **position function** $s = f(t)$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \square$$

Definition 2.3. The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. $\square$

The slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, the derivative of f at a. The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The **instantaneous rate of change** of $y = f(x)$ with respect to x at $x = x_0$ in the interval $[x_0, x_1]$ is

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of an object, then the velocity of the object at time $t = a$ is $v(a) = f'(a)$ and the **speed** of the object is $|f'(a)|$, the **magnitude** of the velocity.

2.1.2 The Derivative as a Function

Definition 2.4. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \square$$

The following notations of the derivative of $y = f(x)$ with respect to x are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$

where $f'(x)$ is Newton's notation and dy/dx is Leibniz's notation. The notation d/dx is the **differential operator** that indicates the operation of **differentiation**. The notations of the derivative of $f(x)$ at a are

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left[\frac{dy}{dx}\right]_{x=a}$$

Problem 2.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution. We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

The derivative of $f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \blacksquare$$

2.1.3 Differentiable Functions

Definition 2.5. A function f is **differentiable** at a if $f'(a)$ exists. f is differentiable on an open interval if it is differentiable at every number in the interval. $\square$

Theorem 2.1. If f is differentiable at a, then f is continuous at a. $\square$

Proof. Given that f is differentiable at a, we want to show that

$$\lim_{x \to a} f(x) = f(a)$$

Since $f'(a)$ exists, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} (x - a) \right) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) \lim_{x \to a} (x - a) = f'(a) \cdot 0 = 0$$

Then we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(a) + f(x) - f(a)) = \lim_{x\to a} f(a) + \lim_{x\to a}(f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore, it is proved that f is continuous at a. ■

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h\to 0} \frac{|0+h| - |0|}{h}$$

if the limit exists but

$$\begin{aligned}
\lim_{h\to 0^-} \frac{|0+h| - |0|}{h} &= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \\
\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} &= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1
\end{aligned}$$

thus the limit does not exist so $f'(0)$ does not exist. If a function is differentiable, then it is **smooth**, continuous, and it has no vertical tangent lines.

2.1.4 Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $f'(x)$ is differentiable, then the **second derivative** of f is

$$y'' = f''(x) = \frac{d^2y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. Let $s = s(t)$ be a position function of an object with respect to time t. The velocity function $v(t)$ of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the **acceleration**. Thus the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

In general, the nth derivative of $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Problem 2.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. We apply the **binomial theorem** by Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

For the second derivative we have

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2 + 2hx + h^2 - x^2)}{h} = \lim_{h\to 0} \frac{6hx + 3h^2}{h} \\ &= \lim_{h\to 0}(6x + 3h) = 6x \end{aligned}$$

Therefore, the first and the second derivatives of $f(x) = x^3$ are

$$f'(x) = 3x^2 \qquad\qquad f''(x) = 6x$$

■

2.2 Differentiation

2.2.1 Differentiation Formulas

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the following differentiation formulas.

Theorem 2.2. Let $f(x) = c$ where c is a constant, then

$$\frac{d}{dx}(c) = 0$$

□

Proof.

$$f'(x) = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

■

Theorem 2.3 Power Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}, n \in \mathbb{R}$$

□

Proof. We prove the power rule for $n \in \mathbb{N}$.

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand $(x+h)^n$ then we have

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h\to 0}(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}) = nx^{x-1} \end{aligned}$$

because every term has a factor of h except nx^{n-1}.

■

Note the special case when $n = 1$, then we have

$$\frac{d}{dx}(x) = 1$$

Problem 2.3. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \qquad \blacksquare$$

Theorem 2.4 Constant Multiple Rule. If c is a constant, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \qquad \square$$

Proof.

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0}\left[c\left(\frac{f(x+h)-f(x)}{h}\right)\right] \\ &= c\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) = c\frac{d}{dx}f(x) \qquad \blacksquare\end{aligned}$$

Theorem 2.5 Sum and Difference Rule.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \qquad \square$$

Proof. We prove the sum rule.

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\end{aligned}$$

Then we prove the difference rule.

$$\begin{aligned}\frac{d}{dx}[f(x)-g(x)] &= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-[f(x)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)-g(x+h)+g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)-[g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) - \frac{d}{dx}g(x) \qquad \blacksquare\end{aligned}$$

2.2.2 Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.6 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = f(x)\left[\frac{d}{dx}g(x)\right] + \left[\frac{d}{dx}f(x)\right]g(x) \qquad \square$$

Theorem 2.7 Quotient Rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2} \qquad \square$$

2.2.3 Trigonometric Functions

Theorem 2.8.

$$\frac{d}{dx}\sin x = \cos x \qquad \square$$

Proof. We use the **angle sum identity** of the sine function

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

then we have

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Note that we are taking the limit with respect to h so $\sin x$ and $\cos x$ are constants then we have

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right] = \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}\\ &= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\ &= (\sin x)(0) + (\cos x)(1) = \cos x \qquad \blacksquare\end{aligned}$$

Theorem 2.9.

$$\frac{d}{dx}\cos x = -\sin x \qquad \square$$

Proof. We use the angle sum identity of the cosine function

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

then we have

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}\right) \\
&= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)-\left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\
&= (\cos x)(0)-(\sin x)(1) = -\sin x \qquad \blacksquare
\end{aligned}$$

Theorem 2.10.

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \square$$

Proof.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}(\sin x)\cos x-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\sin^2 x+\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \qquad \blacksquare
\end{aligned}$$

2.2.4 Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.11 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x))\cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \square$$

2.3 Implicit Differentiation

2.3.1 Implicit Differentiation

An **explicit function** $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. An **implicit function** is defined implicitly by a relation between x and y. An example of implicit functions is the equation of the circle $x^2 + y^2 = r^2$ where the raidus r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y in an implicit function.

Problem 2.4. Find dy/dx of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then we have

$$\begin{aligned}\frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(1)\\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= 0\end{aligned}$$

Since $y = f(x)$, we use the chain rule then we have

$$\begin{aligned}2x+\frac{d}{dy}(y^2)\frac{dy}{dx} &= 0\\ 2x+2y\frac{dy}{dx} &= 0\end{aligned}$$

We solve for dy/dx then we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.5. Find dy/dx of the folium of Descartes $x^3+y^3=6xy$.

Solution.

$$\begin{aligned}x^3+y^3 &= 6xy\\ \frac{d}{dx}x^3+\frac{d}{dy}y^3\left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y+6x\frac{dy}{dx}\\ 3x^2+3y^2\frac{dy}{dx} &= 6y+6x\frac{dy}{dx}\\ x^2+y^2\frac{dy}{dx} &= 2y+2x\frac{dy}{dx}\\ (y^2-2x)\frac{dy}{dx} &= 2y-x^2\\ \frac{dy}{dx} &= \frac{2y-x^2}{y^2-2x}\end{aligned}$$

■

2.4 Derivatives of Inverse Functions

2.4.1 Differentiation and Inverse Functions

Theorem 2.12. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous. □

Theorem 2.13. If is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

□

2.4.2 Logarithmic and Exponential Functions

The **Euler's number** e named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function $y = e^x$.

Definition 2.6 Euler's Number. The Euler's number e is

$$e = \lim_{x \to 0} (1 + x)^{1/x} \qquad \square$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.14. The exponential function $f(x) = \log_a x$ is differentiable and

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e = \frac{1}{x \ln a} \qquad \square$$

Theorem 2.15. The derivative of the natural logarithmic function $f(x) = \ln x$ is

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \square$$

Problem 2.6. Prove the power rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

for $n \in \mathbb{R}$.

Solution. Let $y = x^n$, we use implicit and logarithmic differentiation then we have

$$\begin{aligned}
\ln y &= \ln x^n \\
\ln y &= n \ln x \\
\frac{d}{dx} \ln y &= \frac{d}{dx}(n \ln x) \\
\frac{1}{y} \frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= nx^{-1} y \\
\frac{dy}{dx} &= nx^{-1} x^n \\
\frac{dy}{dx} &= nx^{n-1}
\end{aligned} \qquad \blacksquare$$

Theorem 2.16. The exponential function $f(x) = a^x, a > 0$ is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a \qquad \square$$

Theorem 2.17. The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx} e^x = e^x \qquad \square$$

2.4.3 Inverse Trigonometric Functions

Theorem 2.18.
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$
□

Theorem 2.19.
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$
□

Theorem 2.20.
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
□

2.5 Indeterminate Forms and L'Hôpital's Rule

2.5.1 Indeterminate Forms

Consider the limit of the form
$$\lim_{x\to a}\frac{f(x)}{g(x)}$$
If $f(x) \to 0$ and $g(x) \to 0$, then the limit is an **indeterminate form** of type $0/0$. If $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$, then the limit is an indeterminate form of type ∞/∞.

2.5.2 L'Hôpital's Rule

We have the **L'Hôpital's Rule** discoverd by **Johann Bernoulli** (1667–1748) and named after **Guillaume de l'Hôpital** (1661–1704) to evaluate limits of indeterminate forms of type $0/0$ and ∞/∞.

Theorem 2.21 L'Hôpital's Rule. Suppose that f and g are differentiable and $g'(x) \neq 0$ near a, except possibly at a. If

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \qquad \lim_{x\to a} g(x) = \pm\infty$$

Then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$
if the limit on the right side exists, is ∞, or is $-\infty$. □

L'Hôpital's rule is also valid for one-sided limits and for limits at infinity or negative infinity.

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Pierre de Fermat (1601–1665)

Theorem 3.1 Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. $\square$

3.2 The Mean Value Theorem

Michel Rolle (1652–1719)

Theorem 3.2 Rolle's Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$. $\square$

Joseph-Louis Lagrange (1736–1813)

Theorem 3.3 Lagrange's Mean Value Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \square$$

3.3 Derivatives and Graphs

3.4 Antiderivatives

4 Integrals

4.1 Integrals

4.1.1 Definite Integrals

A **Riemann sum** named after **Bernhard Riemann** (1826–1866) associated with a partition P and a function f is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

Definition 4.1. If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \qquad \square$$

4.1.2 Indefinite Integrals

Definition 4.2. The **indefinite integral**

$$\int f(x)\,dx$$

is the antiderivative of f. $\square$

4.2 Evaluating Integrals

4.3 The Fundamental Theorem of Calculus

Theorem 4.1 The Fundamental Theorem of Calculus (Newton-Leibniz Theorem). Suppose f is continuous on $[a, b]$. If the function F is defined by

$$F(x) = \int_a^x f(t)\,dt, a \leq x \leq b$$

then F is an antiderivative of f and

$$F'(x) = f(x), a < x < b$$

and therefore

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If F is an antiderivative of f such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$\square$

The **Fresnel integrals** named after **Augustin-Jean Fresnel** (1788–1827) are

$$S(x) = \int_0^x \sin(t^2)\,dt \qquad\qquad C(x) = \int_0^x \cos(t^2)\,dt$$

4.4 The Substitution Rule

5 Techniques of Integration

5.1 Integration by Parts

5.2 Trigonometric Integrals and Substitutions

5.3 Partial Fractions

5.4 Improper Integrals

6 Applications of Integration

6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

7.2 Series

7.3 Convergence Tests

7.4 Power Series

7.5 Taylor Series

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

A **differential equation** is an equation that relates some unknown functions and their derivatives. Differential equations are important in mathematics and science. An **ordinary differential equation** (ODE) is a differential equation that relates one or more functions of a single variable and their ordinary derivatives. The **order** of a differential equation is the highest order of the derivative in the equation. **Newton's second law of motion**

$$F = ma$$

is an ordinary differential equation since we can write it in the form

$$F = m\frac{dv}{dt}$$

which is a first order differential equation, or

$$F = m\frac{d^2s}{dt^2}$$

which is a second order differential equation. A function f is a **solution** of a differential equation if the function and its derivatives satisfy the equation for all values of x in some open interval $a < x < b$. It is possible that there are many solutions of a differential equation. An **initial condition** is a condition $y(x_0) = y_0$ or $y^{(n)}(x_0) = y_n$ on the solution. An **initial value problem** is solving a differential equation with initial conditions. The **interval of validity** is the largest possible interval on which the solution is valid and contains x_0 in the initial conditions. The **general solution** of a differential equation is a family of solutions in the most general form. The actual solution is the solution that satisfies the initial conditions. An explicit solution is any solution in the form $y = y(x)$, otherwise it is an implicit solution. The **existence** and **uniqueness** problem asks that given a differential equation, does there exist a solution and if any is there only one solution.

9.2 Direction Fields and Euler's Method

9.2.1 Direction Fields

9.2.2 Euler's Method

9.3 Separable Equations

9.3.1 Separation of Variables

A **separable equation** is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

We can separate the variables if $g(y) \neq 0$, let $h(y) = 1/g(y)$ then we have

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$

We write the equation in the differential form

$$h(y)\,dy = f(x)\,dx$$

so that there is a function of y on one side of the equation and a function of x on the other side. Then we integrate both sides of the equation with respect to the variable of the function.

$$\int h(y)\,dy = \int f(x)\,dx$$

Now we have an implicit solution and in some cases we can solve for an explicit solution. We use the chain rule then we have

$$\begin{aligned}\frac{d}{dx}\int h(y)\,dy &= \frac{d}{dx}\int f(x)\,dx\\ \frac{d}{dy}\int h(y)\,dy\frac{dy}{dx} &= f(x)\\ h(y)\frac{dy}{dx} &= f(x)\end{aligned}$$

Therefore, we solved the differential equation by **separation of variables**.

Problem 9.1. Solve the differential equation $\frac{dy}{dx} = -xy$.

Solution. Notice that $y = 0$ is a trivial solution then we solve the differential equation assuming that $y \neq 0$. We use separation of variables then we have

$$\begin{aligned}\frac{dy}{dx} &= -xy\\ \frac{dy}{y} &= -x\,dx\\ \int\frac{dy}{y} &= -\int x\,dx\\ \ln|y| + C_1 &= -\frac{x^2}{2} + C_2\end{aligned}$$

Let $C = C_2 - C_1$, then we have

$$\begin{aligned}\ln|y| &= -\frac{x^2}{2} + C\\ e^{\ln|y|} &= e^{-(x^2/2)+C}\\ |y| &= e^{C}e^{-x^2/2}\\ y &= Ae^{-x^2/2}\end{aligned}$$

where $A = \pm e^{C}$ is an arbitrary constant. ■

Problem 9.2. Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ with the initial condition $y(0) = 2$.

Solution. We use separation of variables to find the general solution then we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2}{y^2}\\ y^2\,dy &= x^2\,dx\\ \int y^2\,dy = \frac{y^3}{3} + C_1 &= \frac{x^3}{3} + C_2 = \int x^2\,dx\\ \frac{y^3}{3} &= \frac{x^3}{3} + C, \quad C = C_2 - C_1\\ y^3 &= x^3 + 3C\\ y &= \sqrt[3]{x^3 + 3C}\end{aligned}$$

We consider the initial condition to find the actual solution then we have

$$y(0) = 2 = \sqrt[3]{(0)^3 + 3C} \implies 2 = \sqrt[3]{3C} \implies 3C = 8$$

Thus, the solution to the differential equation is $y = \sqrt[3]{x^3 + 24}$. ■

Problem 9.3. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Problem 9.4. Solve the differential equation $y' = x^2 y$.