

# Calculus

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*For all lovers of mathematics and science.*

## Introduction

**Calculus** is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x) dx = F(b) - F(a)$$

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# 1 Functions and Limits

## 1.1 The Limit of a Function

### 1.1.1 Functions

A function  $f : X \mapsto Y$  is a rule that assigns each element  $x$  in set  $X$  to exactly one element  $y$  in set  $Y$ . We have a formal definition of a function.

**Definition 1.1.1.** A **function**  $f$  is a binary relation  $R$  between domain  $X$  and codomain  $Y$  that satisfies:

- $R$  is a subset of the Cartesian product of  $X$  and  $Y$ .

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every  $x$  in  $X$ , there exists a  $y$  in  $Y$  such that  $(x, y)$  is in  $R$ .

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If  $(x, y)$  and  $(x, z)$  are in  $R$ , then  $y = z$ .

$$(x, y), (x, z) \in R \implies y = z$$

□

The real line is the one-dimensional (1D) Euclidean space defined as the set of all real numbers  $\mathbb{R}$ . The  $xy$ -plane in the Cartesian coordinate system by **René Descartes** (1596–1650) is the two-dimensional (2D) Euclidean space defined as the set of all ordered pairs of real numbers  $(x, y) \in \mathbb{R}^2$ . A function of a real variable is a function whose domain is the set of real numbers  $\mathbb{R}$ . A real function is a real-valued function of a real variable whose domain and codomain is  $\mathbb{R}$ .

### 1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let  $f(x)$  be a function defined on some open interval that contains the number  $a$ , except possibly at  $a$  itself.

**Definition 1.1.2.** The **limit** of  $f(x)$  as  $x$  approaches  $a$  equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  from the left and the right but  $x \neq a$ .

$$\lim_{x \rightarrow a} f(x) = L$$

□

**Definition 1.1.3.** The left-hand limit of  $f(x)$  as  $x$  approaches  $a$  from the left equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  where  $x < a$ .

$$\lim_{x \rightarrow a^-} f(x) = L$$

□

**Definition 1.1.4.** The right-hand limit of  $f(x)$  as  $x$  approaches  $a$  from the right equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  where  $x > a$ .

$$\lim_{x \rightarrow a^+} f(x) = L$$

□

The limit exists if the left-hand limit and the right-hand limit of  $f(x)$  as  $x$  approaches  $a$  equal  $L$ , otherwise the limit does not exist.

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

## 1.2 The Precise Definition of a Limit

### 1.2.1 Epsilon-Delta Definition of a Limit

**Augustin-Louis Cauchy** (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

**Definition 1.2.1.**

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \quad \square$$

**Definition 1.2.2.**

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \quad \square$$

**Definition 1.2.3.**

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \quad \square$$

**Problem 1.2.1.** Prove that  $\lim_{x \rightarrow 3} (4x - 5) = 7$ .

*Solution.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get  $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$  so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let  $\delta = \varepsilon/4$ , we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that  $\lim_{x \rightarrow 3} (4x - 5) = 7$ . ■

**Problem 1.2.2.** Prove that  $\lim_{x \rightarrow 3} x^2 = 9$ .

*Solution.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x + 3| |x - 3| < \varepsilon$$

Let  $C$  be a positive constant such that

$$|x + 3| |x - 3| < C |x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of  $x$  that are close to 3, it is reasonable to assume that  $|x - 3| < 1$  such that  $|x + 3| < 7$  so  $C = 7$ . Let  $\delta = \min\{1, \varepsilon/7\}$ , we have

$$0 < |x - 3| < 1 \iff |x + 3| < 7$$

$$0 < |x - 3| < \frac{\varepsilon}{7} \iff 7|x - 3| < \varepsilon$$

$$|x + 3| |x - 3| < 7|x - 3| < \varepsilon \implies |x^2 - 9| < \varepsilon$$

Therefore, it is proved that  $\lim_{x \rightarrow 3} x^2 = 9$ . ■

**Problem 1.2.3.** Prove that  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ .

*Solution.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get  $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$ . Let  $\delta = \varepsilon^2$ , we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

It is proved that  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ . ■

## 1.3 Evaluating Limits

### 1.3.1 Limit Laws

Suppose that  $c$  is a constant and the limits

$$\lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a} g(x) = M$$

exist. We have the following properties of limits called the limit laws to compute limits.

**Theorem 1.3.1.**

$$\lim_{x \rightarrow a} c = c$$

□

*Proof.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have  $|c - c| = 0 < \varepsilon$  so the trivial inequality is always true for any number  $\delta > 0$ . Therefore, by the definition of the limit, it is proved that  $\lim_{x \rightarrow a} c = c$ . ■

**Theorem 1.3.2.**

$$\lim_{x \rightarrow a} x = a$$

□

*Proof.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let  $\delta = \varepsilon$ , we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that  $\lim_{x \rightarrow a} x = a$ . ■

**Theorem 1.3.3 Constant Multiple Law.**

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL$$

□

*Proof.* Note that if  $c = 0$ , then  $cf(x) = 0$  and we have

$$\lim_{x \rightarrow a} [0 \cdot f(x)] = \lim_{x \rightarrow a} 0 = 0 = 0 \cdot \lim_{x \rightarrow a} f(x)$$

Let  $\varepsilon > 0$  and  $c \neq 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number  $\delta_1 > 0$  such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let  $\delta = \delta_1$ , we have

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \rightarrow a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x) = cL$$

■

**Theorem 1.3.4 Sum and Difference Law.**

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$$

□

*Proof.* First we prove the sum law. Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality  $|a + b| \leq |a| + |b|$ , we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

Since  $\lim_{x \rightarrow a} f(x) = L$ , there is a number  $\delta_1$  such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number  $\delta_2$  such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L$$

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let  $\delta = \min\{\delta_1, \delta_2\}$  such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with  $c = -1$ .

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} [f(x) + (-1)g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} (-1)g(x) \\ &= \lim_{x \rightarrow a} f(x) + (-1) \lim_{x \rightarrow a} g(x) \\ &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \quad \blacksquare$$

**Theorem 1.3.5 Product Law.**

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \square$$

*Proof.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leq |[f(x) - L]g(x)| + |L[g(x) - M]| = |f(x) - L| |g(x)| + |L| |g(x) - M| \end{aligned}$$

We want to make both of the terms less than  $\varepsilon/2$ . Since  $\lim_{x \rightarrow a} f(x) = L$ , there is a number  $\delta_1 > 0$  such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$



Since  $\lim_{x \rightarrow a} g(x) = M$ , there is a number  $\delta_2 > 0$  such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number  $\delta_3 > 0$  such that

$$0 < |x - a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Let  $\delta = \min\{\delta_1, \delta_2, \delta_3\}$  such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L| |g(x)| + |L| |g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)} (1 + |M|) + (1 + |L|) \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M \quad \blacksquare$$

**Theorem 1.3.6 Quotient Law.**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M} \iff \lim_{x \rightarrow a} g(x) = M \neq 0 \quad \square$$

*Proof.* First we prove that

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$$

Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Notice that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since  $\lim_{x \rightarrow a} g(x) = M$ , there is a number  $\delta_1$  such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

It is shown that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \implies \frac{1}{|g(x)|} < \frac{2}{|M|}$$

It follows that for these values of  $x$ ,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M| |g(x)|} < \frac{1}{|M|} \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number  $\delta_2 > 0$  such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ , if  $0 < |x - a| < \delta$ , then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|} |g(x) - M| < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} f(x) \left( \frac{1}{g(x)} \right) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \quad \blacksquare$$

**Theorem 1.3.7 Power Law.**

$$\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n, n \in \mathbb{R} \quad \square$$

**Theorem 1.3.8 Root Law.**

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, n \in \mathbb{R} \quad \square$$

**Theorem 1.3.9 Direct Substitution Property.** If  $f$  is a polynomial function, rational function, or trigonometric function and  $a$  is in the domain of  $f$ , then

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

Thus, we have the following limits

$$\lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$\lim_{\theta \rightarrow 0} \cos \theta = 1$$

If  $f(x) = g(x)$  when  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$  if the limits exist.

**Problem 1.3.1.** Show that  $\lim_{x \rightarrow 0} |x| = 0$ .

*Solution.* Since  $|x| = x$  for  $x > 0$ , we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For  $x < 0$  we have  $|x| = -x$  so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, it is shown that  $\lim_{x \rightarrow 0} |x| = 0$ . ■

**Theorem 1.3.10.** If  $f(x) \leq g(x)$  for all  $x$  in an open interval that contains  $a$ , except possibly at  $a$ , and

$$\lim_{x \rightarrow a} f(x) = L \qquad \lim_{x \rightarrow a} g(x) = M$$

then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \iff L \leq M \quad \square$$

*Proof.* We use the technique of proof by contradiction. Suppose that  $L > M$ , then we have

$$\lim_{x \rightarrow a} [g(x) - f(x)] = M - L$$

Therefore, for any number  $\varepsilon > 0$ , there is a  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that  $L - M > 0$  by the hypothesis. Let  $\varepsilon = L - M$ , there exists a  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since  $a \leq |a|$  for any number  $a$ , we have

$$0 < |x - a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given  $f(x) \leq g(x)$ . Then the inequality  $L > M$  must be false so  $L \leq M$  must be true. Therefore, it is proved that

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \quad \blacksquare$$

**Theorem 1.3.11 Squeeze Theorem.** If  $f(x) \leq g(x) \leq h(x)$  for all  $x$  in an open interval that contains  $a$ , except possibly at  $a$ , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L \quad \square$$

*Proof.* Let  $\varepsilon > 0$  be given. Since  $\lim_{x \rightarrow a} f(x) = L$ , there exists a  $\delta_1 > 0$  such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since  $\lim_{x \rightarrow a} h(x) = L$ , there exists a  $\delta_2 > 0$  such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . If  $0 < |x - a| < \delta$ , then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x \rightarrow a} g(x) = L \quad \blacksquare$$

If  $0 < \theta < \pi/2$ , then

$$\sin \theta < \theta \implies \frac{\sin \theta}{\theta} < 1 \qquad \theta < \tan \theta = \frac{\sin \theta}{\cos \theta} \implies \cos \theta < \frac{\sin \theta}{\theta}$$

so we have the following inequality

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Since  $(\sin \theta)/\theta$  is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

**Problem 1.3.2.** Evaluate

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$$

*Solution.* By the **Pythagorean identity**  $\sin^2 \theta + \cos^2 \theta = 1$ , we have

$$\frac{\cos \theta - 1}{\theta} = \frac{\cos \theta - 1}{\theta} \left( \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} = \frac{\sin \theta}{\theta} \left( \frac{-\sin \theta}{\cos \theta + 1} \right)$$

We take the limit then we have

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \rightarrow 0} \left[ \frac{\sin \theta}{\theta} \left( \frac{-\sin \theta}{\cos \theta + 1} \right) \right] = \left( \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \left( \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{\cos \theta + 1} \right) = 1 \left( \frac{(-1)(0)}{1 + 1} \right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0 \quad \blacksquare$$

## 1.4 Continuity

### 1.4.1 Continuous Functions

Let  $f(x)$  be a function and the number  $a$  is in the domain of  $f$  so  $f(a)$  is defined. If the limit exists, then we have the following definition.

**Definition 1.4.1.** A function  $f$  is **continuous** at the number  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \square$$

A function  $f$  is continuous from the left at  $a$  if the left-hand limit equals  $f(a)$  and it is continuous from the right at  $a$  if the right-hand limit equals  $f(a)$ . A function  $f$  is continuous on an interval if it is continuous at every number in the interval. If  $f$  is not continuous at  $a$ , then it is a discontinuous function at  $a$ . If  $f$  and  $g$  are continuous functions at  $a$  and  $c$  is a constant, then the following functions are also continuous at  $a$ .

$$f + g \quad f - g \quad cf \quad f \cdot g \quad \frac{f}{g} \iff g(x) \neq 0$$

**Theorem 1.4.1.** Let  $P(x)$  be any polynomial, then  $P(x)$  is continuous on  $\mathbb{R} = (-\infty, \infty)$ .  $\square$

*Proof.* A polynomial  $P(x)$  is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients  $a_i$  are constants.  $P(x)$  is the sum of power functions with a constant multiple and therefore it is continuous.  $\blacksquare$

**Theorem 1.4.2.** Let  $f$  be any rational function, then  $f$  is continuous on its domain.  $\square$

*Proof.* A rational function  $f$  is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P$  and  $Q$  are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain.  $\blacksquare$

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

**Theorem 1.4.3.** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$ , then the limit of the composite function  $f \circ g$  is

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \square$$

*Proof.* Let  $\varepsilon > 0$  be given, we want to find  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since  $f$  is continuous at  $b$ , then we have  $\lim_{y \rightarrow b} f(y) = f(b)$ . There exists  $\delta_1 > 0$  such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since  $\lim_{x \rightarrow a} g(x) = b$ , there exists  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b) \quad \blacksquare$$

**Theorem 1.4.4.** If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$ , then  $f \circ g$  is continuous at  $a$ .  $\square$

*Proof.* Since  $g$  is continuous at  $a$ , we have  $\lim_{x \rightarrow a} g(x) = g(a)$ . Since  $f$  is continuous at  $g(a)$ , we have

$$\lim_{x \rightarrow a} f(g(x)) = f(g(a))$$

Therefore,  $f(g(x))$  is continuous at  $a$ .  $\blacksquare$

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

**Theorem 1.4.5 Intermediate Value Theorem.** Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$  where  $f(a) \neq f(b)$  such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number  $c$  in the open interval  $(a, b)$  such that  $f(c) = N$ .  $\square$

If a continuous function  $f(x)$  has values of opposite sign in an interval  $(a, b)$ , then there exists a root of  $f(x)$  in  $(a, b)$  which follows immediately from the intermediate value theorem.

## 1.5 Limits and Infinity

### 1.5.1 Infinite Limits

**Definition 1.5.1.** The limit of  $f(x)$  as  $x$  approaches  $a$  is **infinity** if the values of  $f(x)$  can be made arbitrarily large by taking  $x$  sufficiently close to  $a$  but not equal to  $a$ .

$$\lim_{x \rightarrow a} f(x) = \infty \quad \square$$

**Definition 1.5.2.** The limit of  $f(x)$  as  $x$  approaches  $a$  is **negative infinity** if the values of  $f(x)$  can be made arbitrarily small by taking  $x$  sufficiently close to  $a$  but not equal to  $a$ .

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \square$$

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x \rightarrow a^-} f(x) = \infty \quad \lim_{x \rightarrow a^+} f(x) = \infty \quad \lim_{x \rightarrow a^-} f(x) = -\infty \quad \lim_{x \rightarrow a^+} f(x) = -\infty$$

**Definition 1.5.3.** The vertical asymptote of the curve  $y = f(x)$  is the line  $x = a$  if one of the infinite limits is infinity or negative infinity.  $\square$

### 1.5.2 Limits at Infinity

**Definition 1.5.4.** Let  $f$  be a function defined on some interval  $(a, \infty)$ . The limit of  $f(x)$  as  $x$  approaches infinity is  $L$  if the values of  $f(x)$  can be made as close to  $L$  as we like by taking  $x$  sufficiently large.

$$\lim_{x \rightarrow \infty} f(x) = L \quad \square$$

**Definition 1.5.5.** Let  $f$  be a function defined on some interval  $(-\infty, a)$ . The limit of  $f(x)$  as  $x$  approaches negative infinity is  $L$  if the values of  $f(x)$  can be made as close to  $L$  as we like by taking  $x$  sufficiently small.

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \square$$

**Problem 1.5.1.** Evaluate  $\lim_{x \rightarrow \infty} \sin x$  and  $\lim_{x \rightarrow \infty} \cos x$ .

*Solution.* The values of  $\sin x$  and  $\cos x$  oscillate between  $-1$  and  $1$  as  $x \rightarrow \infty$  so the limits do not exist. ■

**Definition 1.5.6.** The horizontal asymptote of the curve  $y = f(x)$  is the line  $y = L$  if one of the limits at infinity is  $L$ . □

### 1.5.3 Infinite Limits at Infinity

**Definition 1.5.7.** The values of  $f(x)$  become arbitrarily large for sufficiently large  $x$ .

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \square$$

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

### 1.5.4 Precise Definitions

Let  $f$  be a function defined on some open interval that contains the number  $a$ , except possibly at  $a$  itself.

**Definition 1.5.8.**

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for every  $M > 0$ , there is a  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies f(x) > M \quad \square$$

**Problem 1.5.2.** Prove that

$$\lim_{x \rightarrow a} \frac{1}{x^2} = \infty.$$

*Solution.* Let  $M > 0$  be given, we want to find a  $\delta > 0$  such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let  $\delta = 1/\sqrt{M}$ , then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that  $\lim_{x \rightarrow a} \frac{1}{x^2} = \infty$ . ■

Let  $f$  be a function defined on some interval  $(a, \infty)$ .

**Definition 1.5.9.**

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for every  $\varepsilon > 0$ , there is an  $N$  such that

$$x > N \implies |f(x) - L| < \varepsilon$$

□

**Problem 1.5.3.** Prove that  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ .

*Proof.* Given  $\varepsilon > 0$ , we want to find an  $N$  such that

$$x > N \implies \left| \frac{1}{x} - 0 \right| < \varepsilon$$

Since  $x \rightarrow \infty$ , it is reasonable to assume that  $x > 0$  in computing the limit. Then we have  $1/x < \varepsilon \iff x > 1/\varepsilon$ . Let  $N = 1/\varepsilon$ , then we have

$$x > N = \frac{1}{\varepsilon} \implies \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ . ■

**Definition 1.5.10.**

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if for every  $M > 0$ , there is an  $N > 0$  such that

$$x > N \implies f(x) > M$$

□

Similar definitions apply for limits involving negative infinity.



## 2 Derivatives

### 2.1 Derivatives

#### 2.1.1 Derivatives and Rates of Change

**Definition 2.1.1.** The tangent line of the curve  $y = f(x)$  at the point  $P(a, f(a))$  is the line through  $P$  with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

where  $h = x - a$ , if this limit exists. □

**Definition 2.1.2.** The velocity at time  $t = a$  of a position function  $s = f(t)$  is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

□

**Definition 2.1.3.** The **derivative** of a function  $f$  at a number  $a$  is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists. □

The slope of the tangent line to  $y = f(x)$  at the point  $(a, f(a))$  is  $f'(a)$ , the derivative of  $f$  at  $a$ . The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The instantaneous rate of change of  $y = f(x)$  with respect to  $x$  at  $x = x_0$  in the interval  $[x_0, x_1]$  is

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative  $f'(a)$  is the instantaneous rate of change of  $y = f(x)$  with respect to  $x$  when  $x = a$ . If  $s = f(t)$  is a position function of an object, then the velocity of the object at time  $t = a$  is  $v(a) = f'(a)$  and the speed of the object is  $|f'(a)|$ , the magnitude of the velocity.

#### 2.1.2 The Derivative as a Function

**Definition 2.1.4.** The derivative of a function  $f(x)$  is the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

□

The following notations of the derivative of  $y = f(x)$  with respect to  $x$  are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

where  $f'(x)$  is Newton's notation and  $dy/dx$  is Leibniz's notation. The notation  $d/dx$  is the differential operator that indicates the operation of differentiation. The notations of the derivative of  $f(x)$  at  $a$  are

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left[ \frac{dy}{dx} \right]_{x=a}$$

**Problem 2.1.1.** Find the derivative of  $f(x) = \sqrt{x}$ .

*Solution.*

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned} \quad \blacksquare$$

### 2.1.3 Differentiable Functions

**Definition 2.1.5.** A function  $f$  is **differentiable** at  $a$  if  $f'(a)$  exists.  $f$  is differentiable on an open interval if it is differentiable at every number in the interval.  $\square$

**Theorem 2.1.1.** If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .  $\square$

*Proof.* Given that  $f$  is differentiable at  $a$ , we want to show that

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Since  $f'(a)$  exists, we have

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} (x - a) \right) = \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} \right) \lim_{x \rightarrow a} (x - a) = f'(a) \cdot 0 = 0$$

Then we have

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (f(a) + f(x) - f(a)) = \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} (f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore, it is proved that  $f$  is continuous at  $a$ .  $\blacksquare$

Note that there are functions that are continuous but not differentiable. The function  $y = |x|$  is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$$

if the limit exists but

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \\ \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} &= \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

thus the limit does not exist so  $f'(0)$  does not exist. If a function is differentiable, then it has no sharp points, it is continuous, and it has no vertical tangent lines.

### 2.1.4 Higher Order Derivatives

If  $y = f(x)$  is a differentiable function and its derivative  $f'(x)$  is differentiable, then the second derivative of  $f$  is

$$y'' = f''(x) = \frac{d^2 y}{dx^2}$$

We can interpret  $f''(x)$  as the slope of the curve  $y = f'(x)$  at the point  $(x, f'(x))$ , which is the rate of change of the slope of the original curve  $y = f(x)$ . Let  $s = s(t)$  be a position function of an object with respect to time  $t$ . The velocity function  $v(t)$  of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the acceleration. Thus the acceleration function  $a(t)$  is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2 s}{dt^2}$$

In general, the  $n$ th derivative of  $y = f(x)$  is

$$f^{(n)}(x) = \frac{d^n y}{dx^n}$$

**Problem 2.1.2.** Find the first and the second derivatives of  $f(x) = x^3$ .

*Solution.* We apply the binomial theorem by Newton

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

For the second derivative we have

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h \rightarrow 0} \frac{3(x^2 + 2hx + h^2 - x^2)}{h} = \lim_{h \rightarrow 0} \frac{6hx + 3h^2}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned} \quad \blacksquare$$

## 2.2 Differentiation

### 2.2.1 Differentiation Formulas

Let  $f(x)$  and  $g(x)$  be differentiable functions, then we have the following differentiation formulas.

**Theorem 2.2.1.** Let  $f(x) = c$  where  $c$  is a constant, then

$$\frac{d}{dx}(c) = 0 \quad \square$$

*Proof.*

$$f'(x) = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0 \quad \blacksquare$$

**Theorem 2.2.2 Power Rule.**

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R} \quad \square$$

*Proof.* We prove the power rule for  $n \in \mathbb{N}$ .

$$\frac{d}{dx}(x^n) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand  $(x+h)^n$  then we have

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} (nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}) = nx^{n-1} \end{aligned}$$

because every term has a factor of  $h$  except  $nx^{n-1}$ .  $\blacksquare$

Note the special case when  $n = 1$ , then we have

$$\frac{d}{dx}(x) = 1$$

**Problem 2.2.1.** Differentiate  $f(x) = 1/x$ .

*Solution.*

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \quad \blacksquare$$

**Theorem 2.2.3 Constant Multiple Rule.** If  $c$  is a constant, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x) \quad \square$$

*Proof.*

$$\begin{aligned} \frac{d}{dx}[cf(x)] &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \rightarrow 0} \left[ c \left( \frac{f(x+h) - f(x)}{h} \right) \right] \\ &= c \left( \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) = c \frac{d}{dx}f(x) \quad \blacksquare \end{aligned}$$

**Theorem 2.2.4 Sum and Difference Rule.**

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \quad \square$$

*Proof.* We prove the sum rule.

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)
\end{aligned}$$

Then we prove the difference rule.

$$\begin{aligned}
\frac{d}{dx}[f(x) - g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) - g(x+h) - [f(x) - g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - [g(x+h) - g(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)
\end{aligned}$$

■

## 2.2.2 Product and Quotient Rules

Let  $f(x)$  and  $g(x)$  be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

**Theorem 2.2.5 Product Rule.**

$$\frac{d}{dx}[f(x)g(x)] = \left[ \frac{d}{dx}f(x) \right] g(x) + f(x) \left[ \frac{d}{dx}g(x) \right]$$

□

*Proof.*

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x+h)g(x+h) - f(x+h)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} g(x) \right] + \lim_{h \rightarrow 0} \left[ f(x+h) \frac{g(x+h) - g(x)}{h} \right] \\
&= \left[ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right] g(x) + \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \left[ \frac{d}{dx}f(x) \right] g(x) + f(x) \left[ \frac{d}{dx}g(x) \right]
\end{aligned}$$

■

**Theorem 2.2.6 Quotient Rule.**

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\left[ \frac{d}{dx} f(x) \right] g(x) - f(x) \left[ \frac{d}{dx} g(x) \right]}{[g(x)]^2}$$

□

*Proof.*

$$\begin{aligned} \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) - [f(x)g(x+h) - f(x)g(x)]}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \\ &= \left( \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} g(x) \right] - \lim_{h \rightarrow 0} \left[ f(x) \frac{g(x+h) - g(x)}{h} \right] \right) \frac{1}{[g(x)]^2} \\ &= \left[ \left( \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) g(x) - f(x) \left( \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \right] \frac{1}{[g(x)]^2} \\ &= \frac{\left[ \frac{d}{dx} f(x) \right] g(x) - f(x) \left[ \frac{d}{dx} g(x) \right]}{[g(x)]^2} \end{aligned}$$

■

**2.2.3 Trigonometric Functions**

**Theorem 2.2.7.**

$$\frac{d}{dx} \sin x = \cos x$$

□

*Proof.* We use the **angle sum identity** of the sine function

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

then we have

$$\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Note that we are taking the limit with respect to  $h$  so  $\sin x$  and  $\cos x$  are constants then we have

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \left[ \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right] = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \left( \lim_{h \rightarrow 0} \sin x \right) \left( \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) + \left( \lim_{h \rightarrow 0} \cos x \right) \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \\ &= (\sin x)(0) + (\cos x)(1) = \cos x \end{aligned}$$

■

**Theorem 2.2.8.**

$$\frac{d}{dx} \cos x = -\sin x$$

□

*Proof.* We use the angle sum identity of the cosine function

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

then we have

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right) \\ &= \left( \lim_{h \rightarrow 0} \cos x \right) \left( \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) - \left( \lim_{h \rightarrow 0} \sin x \right) \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned} \quad \blacksquare$$

**Theorem 2.2.9.**

$$\frac{d}{dx} \tan x = \sec^2 x \quad \square$$

*Proof.*

$$\begin{aligned} \frac{d}{dx} \tan x &= \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x) \cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned} \quad \blacksquare$$

## 2.2.4 Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

**Theorem 2.2.10 Chain Rule.** If  $f$  and  $g$  are differentiable functions and  $F = f(g(x))$ , then  $F$  is differentiable and  $F'$  is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \square$$

*Proof.* We know that by definition if  $y = f(x)$ , then  $\Delta y = f(a + \Delta x) - f(a)$  and

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let  $\varepsilon$  be the difference between the difference quotient and the derivative, then we have

$$\lim_{\Delta x \rightarrow 0} \varepsilon = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

Thus for a differentiable function  $f$ , if we define  $\varepsilon = 0$  when  $\Delta x = 0$ , then

$$\Delta y = f'(a)\Delta x + \varepsilon\Delta x$$

where  $\varepsilon \rightarrow 0$  as  $\Delta x \rightarrow 0$  and  $\varepsilon$  is a continuous function of  $\Delta x$ . Suppose that  $u = g(x)$  is differentiable at  $a$  and  $y = f(u)$  is differentiable at  $b = g(a)$ . Then we have

$$\Delta u = g'(a)\Delta x + \varepsilon_1\Delta x = [g'(a) + \varepsilon_1]\Delta x$$

where  $\varepsilon_1 \rightarrow 0$  as  $\Delta x \rightarrow 0$ . Similarly,

$$\Delta y = f'(b)\Delta u + \varepsilon_2\Delta u = [f'(b) + \varepsilon_2]\Delta u$$

where  $\varepsilon_2 \rightarrow 0$  as  $\Delta u \rightarrow 0$ . We substitute the expression for  $\Delta u$  then we have

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\Delta x$$

$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

Since  $\Delta u \rightarrow 0$  as  $\Delta x \rightarrow 0$ , then  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$  as  $\Delta x \rightarrow 0$ . Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] = f'(b)g'(a) = f'(g(a))g'(a)$$

thus the chain rule is proved. ■

## 2.3 Implicit Differentiation

### 2.3.1 Implicit Differentiation

An explicit function  $y = f(x)$  is defined by expressing one variable explicitly in terms of another variable. An implicit function is defined implicitly by a relation between  $x$  and  $y$ . An example of implicit functions is the equation of the circle  $x^2 + y^2 = r^2$  where the radius  $r$  is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of  $y$  in an implicit function.

**Problem 2.3.1.** Find  $dy/dx$  of the unit circle  $x^2 + y^2 = 1$ .

*Solution.* We differentiate on both sides of the equation then we have

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Since  $y = f(x)$ , we use the chain rule then we have

$$\begin{aligned} 2x + \frac{d}{dy}(y^2) \frac{dy}{dx} &= 0 \\ 2x + 2y \frac{dy}{dx} &= 0 \end{aligned}$$

We solve for  $dy/dx$  then we have

$$\frac{dy}{dx} = -\frac{x}{y}$$
■



**Problem 2.3.2.** Find  $dy/dx$  of the folium of Descartes  $x^3 + y^3 = 6xy$ .

*Solution.*

$$\begin{aligned}
 x^3 + y^3 &= 6xy \\
 \frac{d}{dx}x^3 + \frac{d}{dy}y^3 \left( \frac{dy}{dx} \right) &= \left[ \frac{d}{dx}(6x) \right] y + 6x \frac{dy}{dx} \\
 3x^2 + 3y^2 \frac{dy}{dx} &= 6y + 6x \frac{dy}{dx} \\
 x^2 + y^2 \frac{dy}{dx} &= 2y + 2x \frac{dy}{dx} \\
 (y^2 - 2x) \frac{dy}{dx} &= 2y - x^2 \\
 \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}
 \end{aligned}$$

**Problem 2.3.3.** Find  $y'$  if  $\sin(x + y) = y^2 \cos x$ .

*Solution.*

$$\begin{aligned}
 \sin(x + y) &= y^2 \cos x \\
 \cos(x + y) \left( 1 + \frac{dy}{dx} \right) &= 2y \frac{dy}{dx} \cos x - y^2 \sin x \\
 [2y \cos x - \cos(x + y)] \frac{dy}{dx} &= \cos(x + y) + y^2 \sin x \\
 \frac{dy}{dx} &= \frac{\cos(x + y) + y^2 \sin x}{2y \cos x - \cos(x + y)}
 \end{aligned}$$

**Problem 2.3.4.** Find  $y''$  if  $x^4 + y^4 = 16$ .

*Solution.* First we find  $y'$  then we have

$$\begin{aligned}
 x^4 + y^4 &= 16 \\
 4x^3 + 4y^3 \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= -\frac{x^3}{y^3}
 \end{aligned}$$

Then we find  $y''$  and we have

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( -\frac{x^3}{y^3} \right) = -3x^2y^{-3} + (-x^3)(-3y^{-4}) \frac{dy}{dx} = 3x^3y^{-4}(-x^3y^{-3}) - 3x^2y^{-3} \\
 &= -3x^6y^{-7} - 3x^2y^{-3} = -3x^2y^{-7}(x^4 + y^4) = -48 \frac{x^2}{y^7}
 \end{aligned}$$

## 2.4 Derivatives of Inverse Functions

### 2.4.1 Differentiation and Inverse Functions

**Theorem 2.4.1.** If  $f$  is a one-to-one continuous function defined on an interval, then its inverse function  $f^{-1}$  is also continuous.  $\square$

**Theorem 2.4.2.** If  $f$  is a one-to-one differentiable function with inverse function  $f^{-1}$  and  $f'(f^{-1}(a)) \neq 0$ , then the inverse function is differentiable at  $a$  and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \square$$

### 2.4.2 Derivatives of Logarithmic Functions

The **Euler's number**  $e$  named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function  $y = e^x$  and the natural logarithmic function  $y = \ln x$ .

**Definition 2.4.1 Euler's Number.** The Euler's number  $e$  is defined as

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x} \quad \square$$

Note that the approximate value of  $e$  is  $e \approx 2.71828$ .

**Theorem 2.4.3.** The exponential function  $f(x) = \log_a x$  is differentiable and

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e = \frac{1}{x \ln a} \quad \square$$

*Proof.* First we have

$$\begin{aligned} \frac{d}{dx} \log_a x &= \lim_{h \rightarrow 0} \frac{\log_a(x+h) - \log_a x}{h} = \lim_{h \rightarrow 0} \frac{\log_a \left( \frac{x+h}{x} \right)}{h} = \lim_{h \rightarrow 0} \left[ \frac{1}{x} \frac{x}{h} \log_a \left( 1 + \frac{h}{x} \right) \right] \\ &= \frac{1}{x} \left[ \lim_{h \rightarrow 0} \log_a \left( 1 + \frac{h}{x} \right)^{x/h} \right] = \frac{1}{x} \left[ \lim_{h \rightarrow 0} \log_a \left( 1 + \frac{h}{x} \right)^{1/(h/x)} \right] = \frac{1}{x} \log_a e \end{aligned}$$

We know from the change of base formula that

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Therefore,

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e = \frac{1}{x \ln a} \quad \blacksquare$$

**Theorem 2.4.4.** The derivative of the natural logarithmic function  $f(x) = \ln x$  is

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \square$$

*Proof.*

$$\frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{x \ln e} = \frac{1}{x}$$

■

**Problem 2.4.1.** Find  $f'(x)$  if  $f(x) = \ln |x|$ .

*Solution.* Since  $f(x) = \ln x$  for  $x > 0$  and  $f(x) = \ln(-x)$  for  $x < 0$ , it follows that

$$\begin{aligned} f'(x) &= \frac{d}{dx} \ln x = \frac{1}{x}, & x > 0 \\ f'(x) &= \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}, & x < 0 \end{aligned}$$

Therefore,  $\frac{d}{dx} \ln |x| = \frac{1}{x}$  for all  $x \neq 0$ .

■

### 2.4.3 Derivatives of Exponential Functions

**Theorem 2.4.5.** The exponential function  $f(x) = a^x$ ,  $a > 0$  is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a$$

□

*Proof.* Let  $y = a^x \iff \log_a y = x$  then by implicit differentiation we have

$$\begin{aligned} \log_a y &= x \\ \frac{1}{y \ln a} \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y \ln a = a^x \ln a \end{aligned}$$

■

**Theorem 2.4.6.** The derivative of the natural exponential function  $f(x) = e^x$  is

$$\frac{d}{dx} e^x = e^x$$

□

*Proof.*

$$\frac{d}{dx} e^x = e^x \ln e = e^x$$

■

**Problem 2.4.2.** Prove the power rule  $\frac{d}{dx} x^n = nx^{n-1}$  for  $n \in \mathbb{R}$ .

*Solution.* The rule is true when  $x = 0$  which is trivial so it remains to prove the cases for  $x \neq 0$ . Let  $y = x^n$  such that  $y > 0$ , by implicit differentiation we have

$$\begin{aligned} \ln y &= \ln x^n = n \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \frac{n}{x} \\ \frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n = nx^{n-1} \end{aligned}$$

Similarly, let  $y = x^n$  such that  $y < 0$  then we have

$$\begin{aligned}\ln |y| &= \ln(-x^n) = \ln[(-1)^{1/n} x^n] = n \ln[(-1)^{1/n} x] \\ \frac{1}{y} \frac{dy}{dx} &= n \frac{1}{(-1)^{1/n} x} (-1)^{1/n} = \frac{n}{x} \\ \frac{dy}{dx} &= nx^{-1} y = nx^{-1} x^n = nx^{n-1}\end{aligned}$$

■

#### 2.4.4 Inverse Trigonometric Functions

**Theorem 2.4.7.**

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

□

*Proof.* Let  $y = \arcsin x \iff \sin y = x$  and  $-\pi/2 \leq y \leq \pi/2$  such that  $-1 \leq x \leq 1$ . Note the approximate value of  $\pi$  is  $\pi \approx 3.14159$ . Then we have

$$\begin{aligned}\frac{d}{dx} \sin y &= \frac{d}{dx} x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}\end{aligned}$$

Since  $-\pi/2 \leq y \leq \pi/2$  so  $\cos y \geq 0$  thus  $\cos y = \sqrt{1 - \sin^2 y}$ . Therefore,  $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ . ■

**Theorem 2.4.8.**

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

□

*Proof.* Let  $y = \arccos x \iff \cos y = x$  and  $0 \leq y \leq \pi$  such that  $-1 \leq x \leq 1$ . Then we have

$$\begin{aligned}\frac{d}{dx} \cos y &= \frac{d}{dx} x \\ -\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y}\end{aligned}$$

Since  $0 \leq y \leq \pi$  so  $\sin y \geq 0$  thus  $\sin y = \sqrt{1 - \cos^2 y}$ . Therefore,  $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ . ■

**Theorem 2.4.9.**

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

□

*Proof.* Let  $y = \arctan x \iff \tan y = x$ . Then we have

$$\begin{aligned}\frac{d}{dx} \tan y &= \frac{d}{dx} x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y\end{aligned}$$

Then we can show that

$$\begin{aligned}\tan y &= x \\ 1 + \frac{\sin^2 y}{\cos^2 y} &= 1 + x^2 \\ 1 + \frac{1 - \cos^2 y}{\cos^2 y} &= 1 + x^2 \\ \sec^2 y &= 1 + x^2 \\ \cos^2 y &= \frac{1}{1 + x^2}\end{aligned}$$

Therefore,  $\frac{dy}{dx} = \frac{1}{1 + x^2}$ . ■

## 2.5 Indeterminate Forms and L'Hôpital's Rule

### 2.5.1 Indeterminate Forms and L'Hôpital's Rule

Consider the limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

If  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  as  $x \rightarrow a$ , then the limit is an **indeterminate form** of type  $0/0$ . If  $f(x) \rightarrow \infty$  or  $-\infty$  and  $g(x) \rightarrow \infty$  or  $-\infty$ , then the limit is an indeterminate form of type  $\infty/\infty$ . We have the **L'Hôpital's Rule** discovered by **Johann Bernoulli** (1667–1748) and named after **Guillaume de l'Hôpital** (1661–1704) to evaluate limits of indeterminate forms of type  $0/0$  and  $\infty/\infty$ .

**Theorem 2.5.1 L'Hôpital's Rule.** Suppose that  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  near  $a$ , except possibly at  $a$ . If

$$\lim_{x \rightarrow a} f(x) = 0$$

$$\lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a} g(x) = \pm\infty$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, is  $\infty$ , or is  $-\infty$ . □

L'Hôpital's rule is also valid for one-sided limits and for limits at infinity or negative infinity.

**Problem 2.5.1.** Find  $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$ .

*Solution.* Notice that the limit is an indeterminate form of  $0/0$ . We apply L'Hôpital's rule then we have

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \frac{1/x}{1} = \lim_{x \rightarrow 1} \frac{1}{x} = 1 \quad \blacksquare$$

**Problem 2.5.2.** Find  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$ .

*Proof.* Notice that the limit is an indeterminate form of  $0/0$ . We apply L'Hôpital's rule then we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

The limit still is an indeterminate form of  $0/0$  so we apply L'Hôpital's rule again then we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x}$$

Similarly, we apply L'Hôpital's rule again then we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 4 \sec^2 x \tan^2 x}{6} = \frac{2(1) + 4(1)(0)}{6} = \frac{1}{3} \quad \blacksquare$$

Note that L'Hôpital's rule only applies if the limit is one of the indeterminate forms of type  $0/0$  or  $\infty/\infty$ . If  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = \infty$  (or  $-\infty$ ), then the limit  $\lim_{x \rightarrow a} f(x)g(x)$  is an indeterminate form of type  $0 \cdot \infty$ . We can express the product as a quotient then use L'Hôpital's rule to calculate the limit.

**Problem 2.5.3.** Evaluate  $\lim_{x \rightarrow 0^+} x \ln x$ .

*Solution.*

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0 \quad \blacksquare$$

If  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = \infty$ , then the limit  $\lim_{x \rightarrow a} [f(x) - g(x)]$  is an indeterminate form of type  $\infty - \infty$ . We can convert the difference into a quotient to apply L'Hôpital's rule to calculate the limit.

**Problem 2.5.4.** Compute  $\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x)$ .

*Solution.*

$$\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) = \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \rightarrow (\pi/2)^-} \frac{\cos x}{-\sin x} = 0 \quad \blacksquare$$

Consider the limit of the form

$$\lim_{x \rightarrow a} [f(x)^{g(x)}]$$

We have the following indeterminate forms arise from the limit:

- Type  $0^0$  if  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$ .
- Type  $\infty^0$  if  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = 0$ .
- Type  $1^\infty$  if  $\lim_{x \rightarrow a} f(x) = 1$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$ .

Let  $y = f(x)^{g(x)}$ , then we can write  $\ln y = \ln f(x)^{g(x)}$  and  $y = e^{g(x) \ln f(x)}$  to find the limit.

**Problem 2.5.5.** Calculate  $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x}$ .

*Solution.* Let  $y = (1 + \sin 4x)^{\cot x}$ , then  $\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$ . We apply L'Hôpital's rule then we have

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{[1/(1 + \sin 4x)](\cos 4x)(4)}{\sec^2 x} \\ &= \lim_{x \rightarrow 0^+} \frac{4 \cos 4x \cos^2 x}{1 + \sin 4x} = \frac{4(1)(1)}{1 + 0} = 4 \end{aligned}$$

Therefore, we have

$$\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^4 \quad \blacksquare$$

**Problem 2.5.6.** Calculate  $\lim_{x \rightarrow 0^+} x^x$ .

*Solution.* We have shown that  $\lim_{x \rightarrow 0^+} x \ln x = 0$  and therefore we have

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^0 = 1 \quad \blacksquare$$

## 3 Applications of Differentiation

### 3.1 Maximum and Minimum Values

**Pierre de Fermat** (1601–1665)

**Theorem 3.1.1 Fermat's Theorem.** If  $f$  has a local maximum or minimum at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ . □

### 3.2 The Mean Value Theorem

**Michel Rolle** (1652–1719)

**Theorem 3.2.1 Rolle's Theorem.** Suppose that  $f$  is a continuous function on the closed interval  $[a, b]$ ,  $f$  is differentiable on the open interval  $(a, b)$ , and  $f(a) = f(b)$ . Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ . □

**Joseph-Louis Lagrange** (1736–1813)

**Theorem 3.2.2 Lagrange's Mean Value Theorem.** Suppose that  $f$  is a continuous function on the closed interval  $[a, b]$ ,  $f$  is differentiable on the open interval  $(a, b)$ , and  $f(a) = f(b)$ . Then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \square$$

**Theorem 3.2.3 Cauchy's Mean Value Theorem.** Suppose that the functions  $f$  and  $g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $g'(x) \neq 0$  for all  $x$  in  $(a, b)$ . Then there is a number  $c$  in  $a, b$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \square$$

### 3.3 Derivatives and Graphs

### 3.4 Antiderivatives

**Definition 3.4.1.** A function  $F$  is an **antiderivative** of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .  $\square$

**Theorem 3.4.1.** If  $F$  is an antiderivative of  $f$  on interval  $I$ , then the general antiderivative of  $f$  on  $I$  is  $F(x) + C$  where  $C$  is an arbitrary constant.  $\square$

**Problem 3.4.1.** Find the general antiderivative of  $f(x) = 1/x$ .

*Solution.* We know that

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

for all  $x \neq 0$ . Then the antiderivative of  $f$  is  $F(x) = \ln x + C_1$  if  $x > 0$  or  $F(x) = \ln(-x) + C_2$  if  $x < 0$ . Therefore, the general antiderivative of  $f$  is  $F(x) = \ln |x| + C$ .  $\blacksquare$

## 4 Integrals

### 4.1 Integrals

#### 4.1.1 Definite Integrals

A **Riemann sum** named after **Bernhard Riemann** (1826–1866) associated with a partition  $P$  of interval  $[a, b]$  and a function  $f$  at the sample points  $x_i^*$  in the subinterval  $[x_{i-1}, x_i]$  is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n$$

**Definition 4.1.1.** If  $f$  is a function defined on  $[a, b]$ , the **definite integral** of  $f$  from  $a$  to  $b$  is the number

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

if the limit exists such that  $f$  is integrable on  $[a, b]$ .  $\square$



**Definition 4.1.2.**

$$\int_a^b f(x) dx = I$$

if for every  $\varepsilon > 0$ , there is a number  $\delta > 0$  such that

$$\left| I - \sum_{i=1}^n f(x_i^*) \Delta x_i \right| < \varepsilon$$

for all partitions  $P$  of  $[a, b]$  with  $\max \Delta x_i < \delta$  and for all  $x_i^*$  in  $[x_{i-1}, x_i]$ . □

**Theorem 4.1.1.** If  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$  and the definite integral  $\int_a^b f(x) dx$  exists. □

**Theorem 4.1.2.** If  $f$  is integrable on  $[a, b]$ , then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where  $\Delta x = \frac{b-a}{n}$  and  $x_i = a + i\Delta x$ . □

**4.1.2 Properties of Definite Integrals**

We have the following properties of indefinite integrals. Suppose all of the following integrals exist, and  $c$  is any constant.

- If  $a > b$ , then  $\int_a^b f(x) dx = -\int_b^a f(x) dx$ .
- If  $a = b$ , then  $\int_a^a f(x) dx = 0$ .
- $\int_a^b c dx = c(b-a)$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$
- If  $f(x) \geq 0$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) dx \geq 0$ .
- If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) dx \geq \int_a^b g(x) dx$ .
- If  $m \leq f(x) \leq M$ , then  $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ .

### 4.1.3 Indefinite Integrals

**Definition 4.1.3.** The **indefinite integral**

$$\int f(x) dx = F(x) + C$$

is the general antiderivative of  $f$ . □

## 4.2 Evaluating Integrals

### 4.3 The Fundamental Theorem of Calculus

**Theorem 4.3.1** The Fundamental Theorem of Calculus (Newton-Leibniz Theorem). Suppose  $f$  is continuous on  $[a, b]$ . If the function  $F$  is defined by

$$F(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

then  $F$  is an antiderivative of  $f$  and

$$F'(x) = f(x), \quad a < x < b$$

and therefore

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

If  $F$  is an antiderivative of  $f$  such that  $F' = f$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad \square$$

The Fresnel integrals named after **Augustin-Jean Fresnel** (1788–1827) are

$$S(x) = \int_0^x \sin(t^2) dt \qquad C(x) = \int_0^x \cos(t^2) dt$$

and by the fundamental theorem of calculus the derivatives are

$$S'(x) = \sin(x^2) \qquad C'(x) = \cos(x^2)$$

## 4.4 The Substitution Rule

### 4.4.1 The Substitution Rule

**Theorem 4.4.1** The Substitution Rule. If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

If  $g'$  is continuous on  $[a, b]$  and  $f$  is continuous on  $I$ , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \quad \square$$

**Problem 4.4.1.** Evaluate  $\int \frac{x}{\sqrt{1-4x^2}} dx$ .

*Solution.* Let  $u = 1 - 4x^2 \iff du = -8x dx$  so  $x dx = -\frac{1}{8} du$ . Therefore,

$$\int \frac{x}{\sqrt{1-4x^2}} dx = -\int \frac{1}{8} u^{-(1/2)} du = -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C \quad \blacksquare$$

**Problem 4.4.2.** Evaluate  $\int \tan x dx$ .

*Solution.* We have  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$ , let  $u = \cos x \iff du = -\sin x dx$  then we have

$$\int \tan x dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln \frac{1}{|\sec x|} + C = \ln|\sec x| + C \quad \blacksquare$$

**Problem 4.4.3.** Calculate  $\int_1^e \frac{\ln x}{x} dx$ .

*Solution.* Let  $u = \ln x \iff du = \frac{1}{x} dx$ , then we have

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u du = \left[ \frac{u^2}{2} \right]_0^1 = \frac{1}{2} \quad \blacksquare$$

## 4.4.2 Symmetry

**Theorem 4.4.2.** Suppose  $f$  is continuous on  $[-a, a]$ , then

- If  $f(-x) = f(x)$  so  $f$  is even, then  $\int_{-a}^a f(x) dx = 2 \int_a^a f(x) dx$ .
- If  $f(-x) = -f(x)$  so  $f$  is odd, then  $\int_{-a}^a f(x) dx = 0$ .  $\square$

# 5 Techniques of Integration

## 5.1 Integration by Parts

**Theorem 5.1.1 Integration by Parts Formula.** If  $f$  and  $g$  are differentiable functions, then

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

If  $u = f(x)$  and  $v = g(x)$ , then

$$\int u dv = uv - \int v du$$

and

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du \quad \square$$

**Problem 5.1.1.** Evaluate  $\int \ln x \, dx$ .

*Solution.* Let  $u = \ln x \iff du = \frac{1}{x} dx$  and  $dv = dx \iff v = x$ , then we have

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C \quad \blacksquare$$

**Problem 5.1.2.** Evaluate  $\int \arctan x \, dx$ .

*Solution.* Let  $u = \arctan x \iff du = \frac{dx}{1+x^2}$  and  $dv = dx \iff v = x$ , then

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} dx$$

Let  $u = 1 + x^2 \iff du = 2x \, dx$  so  $x \, dx = \frac{1}{2} du$ , then

$$\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(1+x^2) + C$$

Therefore,

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C \quad \blacksquare$$

**Problem 5.1.3.** Evaluate  $\int e^x \sin x \, dx$ .

*Solution.* Let  $u = \sin x \iff du = \cos x \, dx$  and  $dv = e^x \, dx \iff v = e^x$ , then we have

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Let  $u = \cos x \iff du = -\sin x \, dx$  and  $dv = e^x \, dx \iff v = e^x$ , then we have

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Therefore,

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\ \int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) \end{aligned} \quad \blacksquare$$

## 5.2 Trigonometric Integrals and Substitutions

## 5.3 Partial Fractions

## 5.4 Improper Integrals

# 6 Applications of Integration

## 6.1 Areas

## 6.2 Volumes

## 6.3 Arc Length

# 7 Sequences and Series

## 7.1 Sequences

## 7.2 Series

## 7.3 Convergence Tests

## 7.4 Power Series

## 7.5 Taylor Series

# 8 Parametric Equations and Polar Coordinates

## 8.1 Calculus of Parametric Equations

## 8.2 Calculus in Polar Coordinates

# 9 Differential Equations

## 9.1 Ordinary Differential Equations

A **differential equation** is an equation that relates some unknown functions and their derivatives. Differential equations are important in mathematics and science. An **ordinary differential equation** (ODE) is a differential equation that relates one or more functions of a single variable and their ordinary derivatives. The **order** of a differential equation is the highest order of the derivative in the equation. **Newton's second law of motion**

$$F = ma$$

is an ordinary differential equation since we can write it in the form

$$F = m \frac{dv}{dt}$$

which is a first order differential equation, or

$$F = m \frac{d^2 s}{dt^2}$$

which is a second order differential equation. A function  $f$  is a **solution** of a differential equation if the function and its derivatives satisfy the equation for all values of  $x$  in some open interval  $a < x < b$ . It is possible that there are many solutions of a differential equation. An **initial condition** is a condition  $y(x_0) = y_0$  or  $y^{(n)}(x_0) = y_n$  on the solution. An **initial value problem** is solving a differential equation with initial conditions. The **interval of validity** is the largest possible interval on which the solution is valid and contains  $x_0$  in the initial conditions. The **general solution** of a differential equation is a family of solutions in the most general form. The actual solution is the solution that satisfies the initial conditions. An explicit solution is any solution in the form  $y = y(x)$ , otherwise it is an implicit solution. The **existence and uniqueness** problem asks that given a differential equation, does there exist a solution and if any is there only one solution.

## 9.2 Direction Fields and Euler's Method

### 9.2.1 Direction Fields

### 9.2.2 Euler's Method

## 9.3 Separable Equations

### 9.3.1 Separation of Variables

A **separable equation** is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

We can separate the variables if  $g(y) \neq 0$ , let  $h(y) = 1/g(y)$  then we have

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$

We write the equation in the differential form

$$h(y) dy = f(x) dx$$

so that there is a function of  $y$  on one side of the equation and a function of  $x$  on the other side. Then we integrate both sides of the equation with respect to the variable of the function.

$$\int h(y) dy = \int f(x) dx$$

Now we have an implicit solution and in some cases we can solve for an explicit solution. We use the chain rule then we have

$$\begin{aligned}\frac{d}{dx} \int h(y) dy &= \frac{d}{dx} \int f(x) dx \\ \frac{d}{dy} \int h(y) dy \frac{dy}{dx} &= f(x) \\ h(y) \frac{dy}{dx} &= f(x)\end{aligned}$$

Therefore, we solved the differential equation by **separation of variables**.

**Problem 9.3.1.** Solve the differential equation  $\frac{dy}{dx} = -xy$ .

*Solution.* Notice that  $y = 0$  is a trivial solution then we solve the differential equation assuming that  $y \neq 0$ . We use separation of variables then we have

$$\begin{aligned}\frac{dy}{dx} &= -xy \\ \frac{dy}{y} &= -x dx \\ \int \frac{dy}{y} &= - \int x dx \\ \ln |y| + C_1 &= -\frac{x^2}{2} + C_2\end{aligned}$$

Let  $C = C_2 - C_1$  where  $C \in \mathbb{R}$ , then we have

$$\begin{aligned}\ln |y| &= -\frac{x^2}{2} + C \\ e^{\ln |y|} &= e^{-(x^2/2)+C} \\ |y| &= e^C e^{-x^2/2} \\ y &= A e^{-x^2/2}\end{aligned}$$

where  $A \in \mathbb{R}$  is an arbitrary constant. ■

**Problem 9.3.2.** Solve the differential equation  $\frac{dy}{dx} = \frac{x^2}{y^2}$  with the initial condition  $y(0) = 2$ .

*Solution.* We use separation of variables to find the general solution then we have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{x^2}{y^2} \\
 y^2 dy &= x^2 dx \\
 \int y^2 dy &= \int x^2 dx \\
 \frac{y^3}{3} + C_1 &= \frac{x^3}{3} + C_2 \\
 \frac{y^3}{3} &= \frac{x^3}{3} + C, \quad C = C_2 - C_1 \\
 y^3 &= x^3 + 3C \\
 y &= \sqrt[3]{x^3 + 3C}
 \end{aligned}$$

We consider the initial condition to find the actual solution then we have

$$y(0) = 2 = \sqrt[3]{(0)^3 + 3C} \implies 2 = \sqrt[3]{3C} \implies 3C = 8$$

Thus, the solution to the differential equation is  $y = \sqrt[3]{x^3 + 8}$ . ■

**Problem 9.3.3.** Solve the differential equation  $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$ .

*Solution.* We have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{6x^2}{2y + \cos y} \\
 \int (2y + \cos y) dy &= \int 6x^2 dx \\
 y^2 + \sin y &= 2x^3 + C
 \end{aligned}$$

where  $C \in \mathbb{R}$  is an arbitrary constant. ■

**Problem 9.3.4.** Solve the differential equation  $y' = x^2 y$ .

*Solution.* We have

$$\begin{aligned}
 \frac{dy}{dx} &= x^2 y \\
 \int \frac{dy}{y} &= \int x^2 dx \\
 \ln |y| &= \frac{x^3}{3} + C \\
 |y| &= e^{(x^3/3)+C} \\
 y &= Ae^{x^3/3}
 \end{aligned}$$

where  $A \in \mathbb{R}$  is an arbitrary constant. ■



**Problem 9.3.5.** Newton's law of universal gravitation states that

$$F = G \frac{m_1 m_2}{r^2}$$

Then the gravitational force on an object of mass  $m$  that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where  $x = x(t)$  is the object's distance above the surface at time  $t$ ,  $R$  is the earth's radius, and  $g$  is the acceleration due to gravity. Also, by Newton's second law, we have

$$F = ma = m \frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity  $v_0$ . Let  $h$  be the maximum height above the surface reached by the object. Show that  $v_0 = \sqrt{\frac{2gRh}{R + h}}$  and compute  $v_e$ , the escape velocity of the earth, using  $R = 6,378$  km and  $g = 9.8$  m/s<sup>2</sup>.