Calculus

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For all lovers of mathematics and science.

Introduction

Calculus is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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1 Functions and Limits

1.1 The Limit of a Function

1.1.1 Functions

A function $f : X \mapsto Y$ is a rule that assigns each element x in set X to exactly one element y in set Y. We have a formal definition of a function.

Definition 1.1. A **function** f is a binary relation R between domain X and codomain Y that satisfies:

- R is a subset of the **Cartesian product** of X and Y.
$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$
- For every x in X, there exists a y in Y such that (x, y) is in R.
$$\forall x \in X, \exists y \in Y, (x, y) \in R$$
- If (x, y) and (x, z) are in R, then $y = z$.
$$(x, y) \in R \wedge (x, z) \in R \implies y = z \qquad \square$$

The real line is the 1-dimensional **Eculidean space** defined as the set of real numbers $\mathbb{R}$. The xy-plane in the **Cartesian coordinate system** by **René Descartes** (1596–1650) is the 2-dimensional Eculidean space defined as the set of all ordered pairs of real numbers $(x, y) \in \mathbb{R}^2$. A **function of a real variable** is a function whose domain is the set of real numbers $\mathbb{R}$. A real function is a real-valued function of a real variable whose domain and codomain is $\mathbb{R}$.

1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2. The **limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a from the left and the right but $x \neq a$.
$$\lim_{x \to a} f(x) = L \qquad \square$$

Definition 1.3. The **left-hand limit** of $f(x)$ as x approaches a from the left equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x < a$.
$$\lim_{x \to a^-} f(x) = L \qquad \square$$

Definition 1.4. The **right-hand limit** of $f(x)$ as x approaches a from the right equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a where $x > a$.
$$\lim_{x \to a^+} f(x) = L \qquad \square$$

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.
$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

1.2 The Precise Definition of a Limit

1.2.1 Epsilon-Delta Definition of a Limit

Augustin-Louis Cauchy (1789–1857) and **Karl Weierstrass** (1815–1897) formalized a rigorous definition of a limit.

Definition 1.5.

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.6.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon \qquad \square$$

Definition 1.7.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon \qquad \square$$

Problem 1.1. Prove that

$$\lim_{x \to 3} (4x - 5) = 7$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$ so we have

$$4|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that

$$\lim_{x \to 3} (4x - 5) = 7 \qquad \blacksquare$$

Problem 1.2. Prove that

$$\lim_{x \to 3} x^2 = 9$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x + 3|\,|x - 3| < \varepsilon$$

Let C be a positive constant such that

$$|x + 3|\,|x - 3| < C\,|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x - 3| < 1$ such that $|x + 3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$, we have

$$\begin{gathered} 0 < |x - 3| < 1 \iff |x + 3| < 7 \\ 0 < |x - 3| < \frac{\varepsilon}{7} \iff 7\,|x - 3| < \varepsilon \\ |x + 3|\,|x - 3| < 7\,|x - 3| < \varepsilon \implies |x^2 - 9| < \varepsilon \end{gathered}$$

Therefore, it is proved that

$$\lim_{x \to 3} x^2 = 9$$

■

Problem 1.3. Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$, we have

$$0 < x < \varepsilon^2 \implies |\sqrt{x} - 0| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

■

1.3 Computing Limits

1.3.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

exist. We have the following properties of limits called the **limit laws** to compute limits.

Theorem 1.1.

$$\lim_{x \to a} c = c$$

□

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} c = c \qquad \blacksquare$$

Theorem 1.2.

$$\lim_{x \to a} x = a \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} x = a \qquad \blacksquare$$

Theorem 1.3 Constant Multiple Law. The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL \qquad \square$$

Proof. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} [0 \cdot f(x)] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x)$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - c \lim_{x \to a} f(x)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = cL \qquad \blacksquare$$

Theorem 1.4 Sum and Difference Law. The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \square$$

Proof. First we prove the sum law. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the **triangle inequality** $|a + b| \le |a| + |b|$, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \le |f(x) - L| + |g(x) - M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number δ_2 such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L$$

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1, \delta_2$$

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we have

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

We prove the difference law using the sum law and the constant multiple law with $c = -1$.

$$\begin{aligned}\lim_{x\to a}[f(x) - g(x)] &= \lim_{x\to a}[f(x) + (-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) \\ &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M\end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \qquad \blacksquare$$

Theorem 1.5 Product Law. The limit of a product is the product of the limits.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

By the triangle inequality, we have

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| = |[f(x) - L]g(x) + L[g(x) - M]| \\
&\le |[f(x) - L]g(x)| + |L[g(x) - M]| = |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2(1+|M|)}$$

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$$

Also, there is a number $\delta_3 > 0$ such that

$$0 < |x-a| < \delta_3 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \le |g(x) - M| + |M| < 1 + |M|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ such that

$$0 < |x-a| < \delta \implies 0 < |x-a| < \delta_1, \delta_2, \delta_3$$

so we can combine the inequalities to get

$$\begin{aligned}
|f(x)g(x) - LM| &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\
&< \frac{\varepsilon}{2(1+|M|)}(1+|M|) + (1+|L|)\frac{\varepsilon}{2(1+|L|)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Therefore, it is proved that

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M \qquad \blacksquare$$

Theorem 1.6 Quotient Law. The limit of a quotient is the quotient of the limits (if that the limit of the denominator is not 0).

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \iff \lim_{x\to a} g(x) = M \neq 0 \qquad \square$$

Proof. First we prove that

$$\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{M}$$

Let $\varepsilon>0$ be given, we want to find a number $\delta>0$ such that

$$0<|x-a|<\delta \implies \left|\frac{1}{g(x)}-\frac{1}{M}\right|<\varepsilon$$

Notice that

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\frac{|M-g(x)|}{|Mg(x)|}=\frac{|g(x)-M|}{|Mg(x)|}$$

Since $\lim_{x\to a}g(x)=M$, there is a number δ_1 such that

$$0<|x-a|<\delta_1 \implies |g(x)-M|<\frac{|M|}{2}$$

and therefore

$$|M|=|M-g(x)+g(x)|\le|M-g(x)|+|g(x)|=|g(x)-M|+|g(x)|<\frac{|M|}{2}+|g(x)|$$

It is shown that

$$0<|x-a|<\delta_1 \implies \frac{|M|}{2}<|g(x)| \implies \frac{1}{|g(x)|}<\frac{2}{|M|}$$

It follows that for these values of x,

$$\frac{1}{|Mg(x)|}=\frac{1}{|M|\,|g(x)|}<\frac{1}{|M|}\frac{2}{|M|}=\frac{2}{M^2}$$

Also, there is a number $\delta_2>0$ such that

$$0<|x-a|<\delta_2 \implies |g(x)-M|<\frac{M^2}{2}\varepsilon$$

Let $\delta=\min\{\delta_1,\delta_2\}$, if $0<|x-a|<\delta$, then

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\frac{|M-g(x)|}{|Mg(x)|}=\frac{1}{|Mg(x)|}|g(x)-M|<\frac{2}{M^2}\frac{M^2}{2}\varepsilon=\varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}f(x)\left(\frac{1}{g(x)}\right)=\lim_{x\to a}f(x)\lim_{x\to a}\frac{1}{g(x)}=L\cdot\frac{1}{M}=\frac{L}{M}$$ ■

Theorem 1.7 Power Law.

$$\lim_{x\to a}[f(x)]^n=[\lim_{x\to a}f(x)]^n, n\in\mathbb{R}$$ □

Theorem 1.8 Root Law.

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, n \in \mathbb{R} \qquad \square$$

Theorem 1.9 Direct Substitution Property. If f is a polynomial function, rational function, or trigonometric function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a) \qquad \square$$

Thus, we have the following limits

$$\lim_{\theta\to 0}\sin\theta = 0 \qquad\qquad \lim_{\theta\to 0}\cos\theta = 1$$

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if the limits exist.

Problem 1.4. Show that

$$\lim_{x\to 0}|x| = 0$$

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0$$

Therefore, it is shown that

$$\lim_{x\to 0}|x| = 0 \qquad \blacksquare$$

Theorem 1.10. If $f(x) \leq g(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \iff L \leq M \qquad \square$$

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a}[g(x) - f(x)] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - f(x) - (M - L)| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x-a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x-a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since given $f(x) \leq g(x)$. Then the inequality $L > M$ must be false so $L \leq M$ must be true. Therefore, it is proved that

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \qquad \blacksquare$$

Theorem 1.11 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a, except possibly at a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L \qquad \square$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x\to a} h(x) = L$, there exists a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \implies L - \varepsilon < g(x) < L + \varepsilon \implies |g(x) - L| < \varepsilon$$

which is what we want to prove. Therefore, it is proved that

$$\lim_{x\to a} g(x) = L \qquad \blacksquare$$

By **algebra**, **geometry**, and **trigonometry**, we can get the following result by the **Pythagorean theorem** $a^2 + b^2 = c^2$. If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1 \qquad\qquad \theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta}$$

so we have the following inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $(\sin\theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.5. Evaluate

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

Solution. By the **Pythagorean identity** $\sin^2\theta + \cos^2\theta = 1$, we have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit and we have

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)\right] = \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) = 1\left(\frac{(-1)(0)}{1+1}\right) = 0$$

Therefore, it is shown that

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

■

1.4 Continuity

Let $f(x)$ be a function and the number a is in the domain of f so $f(a)$ is defined. If the limit exists, then we have the following definition.

Definition 1.8. A function f is **continuous** at the number a if

$$\lim_{x \to a} f(x) = f(a)$$

□

A function f is continuous from the left at a if the left-hand limit equals $f(a)$ and it is continuous from the right at a if the right-hand limit equals $f(a)$. A function f is continuous on an interval if it is continuous at every number in the interval. If f is not continuous at a, then it is a discontinuous function at a. If f and g are continuous functions at a and c is a constant, then the following functions are also continuous at a.

$$f + g \qquad f - g \qquad cf \qquad f \cdot g \qquad \frac{f}{g} \iff g(x) \neq 0$$

Theorem 1.12. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$. □

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients a_i are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. ■

Theorem 1.13. Let f be any rational function, then f is continuous on its domain. □

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. ■

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, logarithmic functions, and exponential functions are continuous on their domain.

Theorem 1.14. If f is a one-to-one continuous function defined on an interval $[a, b]$, then its inverse function f^{-1} is also continuous. $\square$

Theorem 1.15. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then the limit of the composite function $f \circ g$ is

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b) \qquad \square$$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, then we have $\lim_{y\to b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

Therefore, it is proved that

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b) \qquad \blacksquare$$

Theorem 1.16. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. $\square$

Proof. Since g is continuous at a, we have $\lim_{x\to a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a. $\blacksquare$

An important property of continuous functions is formulated by the following theorem proved by **Bernard Bolzano** (1781–1848).

Theorem 1.17 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ such that

$$\min\{f(a), f(b)\} < N < \max\{f(a), f(b)\}$$

Then there exists a number c in the open interval (a, b) such that $f(c) = N$. $\square$

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then there exists a root of $f(x)$ in (a, b) which follows immediately from the intermediate value theorem.

1.5 Limits and Infinity

1.5.1 Infinite Limits

Definition 1.9. The limit of $f(x)$ as x approaches a is **infinity** if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

$$\lim_{x \to a} f(x) = \infty \qquad \square$$

Definition 1.10. The limit of $f(x)$ as x approaches a is **negative infinity** if the values of $f(x)$ can be made arbitrarily small by taking x sufficiently close to a but not equal to a.

$$\lim_{x \to a} f(x) = -\infty \qquad \square$$

Similar definitions can be given for one-sided infinite limits.

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Definition 1.11. The **vertical asymptote** of the curve $y = f(x)$ is the line $x = a$ if one of the infinite limits is infinity or negative infinity. $\square$

1.5.2 Limits at Infinity

Definition 1.12. Let f be a function defined on some interval (a, ∞). The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x \to \infty} f(x) = L \qquad \square$$

Definition 1.13. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x \to -\infty} f(x) = L \qquad \square$$

Problem 1.6. Evaluate $\lim_{x \to \infty} \sin x$ and $\lim_{x \to \infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to \infty$ so the limits do not exist. $\blacksquare$

Definition 1.14. The **horizontal asymptote** of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L. $\square$

1.5.3 Infinite Limits at Infinity

Definition 1.15. The values of $f(x)$ become arbitrarily large for sufficiently large x.

$$\lim_{x \to \infty} f(x) = \infty \qquad \square$$

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

1.5.4 Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.16.

$$\lim_{x\to a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \qquad \square$$

Problem 1.7. Prove that

$$\lim_{x\to a} \frac{1}{x^2} = \infty$$

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that

$$\lim_{x\to a} \frac{1}{x^2} = \infty \qquad \blacksquare$$

Let f be a function defined on some interval (a, ∞).

Definition 1.17.

$$\lim_{x\to \infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon \qquad \square$$

Problem 1.8. Prove that

$$\lim_{x\to \infty} \frac{1}{x} = 0$$

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \blacksquare$$

Definition 1.18.

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M \qquad \square$$

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

2.2 Differentiation Formulas

2.3 Implicit Differentiation

2.4 Derivatives of Inverse Functions

2.5 Indeterminate Forms and l'Hospital's Rule

3 Applications of Differentiation

3.1 Maximum and Minimum Values

3.2 The Mean Value Theorem

3.3 Derivatives and Graphs

3.4 Antiderivatives

4 Integrals

4.1 Definite Integrals

4.2 Evaluating Definite Integrals

4.3 The Fundamental Theorem of Calculus

4.4 The Substitution Rule

5 Techniques of Integration

5.1 Integration by Parts

5.2 Trigonometric Integrals and Substitutions

5.3 Partial Fractions

5.4 Improper Integrals

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6.1 Areas

6.2 Volumes

6.3 Arc Length

7 Sequences and Series

7.1 Sequences

7.2 Series

7.3 Convergence Tests

7.4 Power Series

7.5 Taylor Series

8 Parametric Equations and Polar Coordinates

8.1 Calculus of Parametric Equations

8.2 Calculus in Polar Coordinates

9 Differential Equations

9.1 Ordinary Differential Equations

9.2 Direction Fields and Euler's Method

9.3 Separable Equations