

# Calculus

Yaohui Wu

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*For all lovers of mathematics.*

## Introduction

**Calculus** is the study of continuous change established by **Issac Newton** (1643–1727) and **Gottfried Wilhelm Leibniz** (1646–1716) in the 17th century. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x) dx = F(b) - F(a)$$

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# 1 Functions and Limits

## 1.1 The Limit of a Function

### 1.1.1 Functions

A function  $f : X \mapsto Y$  is a rule that assigns each element  $x$  in set  $X$  to exactly one element  $y$  in set  $Y$ . We have a formal definition of a function.

**Definition 1.1.** A **function**  $f$  is a binary relation  $R$  between domain  $X$  and codomain  $Y$  that satisfies:

- $R$  is a subset of the Cartesian product of  $X$  and  $Y$ .

$$R \subset \{(x, y) \mid x \in X, y \in Y\}$$

- For every  $x$  in  $X$ , there exists a  $y$  in  $Y$  such that  $(x, y)$  is in  $R$ .

$$\forall x \in X, \exists y \in Y, (x, y) \in R$$

- If  $(x, y)$  and  $(x, z)$  are in  $R$ , then  $y = z$ .

$$(x, y) \in R \wedge (x, z) \in R \implies y = z$$

### 1.1.2 Intuitive Definition of a Limit

Newton and Leibniz introduced a working definition of a limit. Let  $f(x)$  be a function defined on some open interval that contains the number  $a$ , except possibly at  $a$  itself.

**Definition 1.2.** The **limit** of  $f(x)$  as  $x$  approaches  $a$  equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  from the left and the right but  $x \neq a$ .

$$\lim_{x \rightarrow a} f(x) = L$$

**Definition 1.3.** The **left-hand limit** of  $f(x)$  as  $x$  approaches  $a$  from the left equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  where  $x < a$ .

$$\lim_{x \rightarrow a^-} f(x) = L$$

**Definition 1.4.** The **right-hand limit** of  $f(x)$  as  $x$  approaches  $a$  from the right equals  $L$  if we can make  $f(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to  $a$  where  $x > a$ .

$$\lim_{x \rightarrow a^+} f(x) = L$$

The limit **exists** if the left-hand limit and the right-hand limit of  $f(x)$  as  $x$  approaches  $a$  equal  $L$ , otherwise the limit **does not exist**.

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

## 1.2 The Precise Definition of a Limit

### 1.2.1 Epsilon-Delta Definition of a Limit

**Augustin-Louis Cauchy** (1789–1857) and **Karl Weierstrass** (1815–1897) developed a rigorous definition of a limit.

**Definition 1.5.**

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\varepsilon > 0$ , there is a number  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

**Definition 1.6.**

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number  $\varepsilon > 0$ , there is a number  $\delta > 0$  such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

**Definition 1.7.**

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number  $\varepsilon > 0$ , there is a number  $\delta > 0$  such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

**Problem 1.1.** Prove that

$$\lim_{x \rightarrow 3} (4x - 5) = 7$$

*Solution.* Let  $\varepsilon > 0$  be given, we want to find a number  $\delta > 0$  such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

■

**1.3 Computing Limits**

**1.4 Continuity**

**1.5 Limits and Infinity**

## **2 Derivatives**

**2.1 Derivatives**

**2.2 Differentiation Formulas**

**2.3 Implicit Differentiation**

**2.4 Derivatives of Inverse Functions**

**2.5 Indeterminate Forms and l'Hospital's Rule**

## **3 Applications of Differentiation**

**3.1 Maximum and Minimum Values**

**3.2 The Mean Value Theorem**

**3.3 Derivatives and Graphs**

**3.4 Antiderivatives**

## **4 Integrals**

**4.1 Definite Integrals**

**4.2 Evaluating Definite Integrals**

**4.3 The Fundamental Theorem of Calculus**

**4.4 The Substitution Rule**

## **5 Techniques of Integration**

### **5.1 Integration by Parts**

### **5.2 Trigonometric Integrals and Substitutions**

### **5.3 Partial Fractions**

### **5.4 Improper Integrals**

## **6 Applications of Integration**

### **6.1 Areas**

### **6.2 Volumes**

### **6.3 Arc Length**

## **7 Sequences and Series**

### **7.1 Sequences**

### **7.2 Series**

### **7.3 Convergence Tests**

### **7.4 Power Series**

### **7.5 Taylor Series**

## **8 Parametric Equations and Polar Coordinates**

### **8.1 Calculus of Parametric Equations**

### **8.2 Calculus in Polar Coordinates**

## **9 Differential Equations**

### **9.1 Ordinary Differential Equations**

### **9.2 Direction Fields and Euler's Method**

### **9.3 Separable Equations**