Math 4100 Homework #4/#5 **Name: Yaohui Wu**

Instructions: Show **ALL** of your work! Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation. Course notes, textbooks, etc. are allowed.

This is due at **10:00 PM on Monday, December 11**! Late submissions will be accepted until **5:00 PM on Wednesday, December 13**.

You may use LaTeX to type the answers. If you use Latex, upload your .tex file along with the PDF output.

If you choose not to use LaTeX, write your answers **legibly** on perforated paper or loose leaf paper.

You **MUST** attach this sheet as the first page of your solutions.

Put your EMPLID on each page.

Solutions MUST be in the proper numerical order! You must make a PDF scan of your work (include the cover page as the first page) as a single PDF file.

The work **must** be submitted on Dropbox: https://www.dropbox.com/request/zejLlEefwDvv10GFesXG

If you use any resources other than your textbook, **cite the source!**

Justify your answers!

Question	Possible Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
Total	25	

Question 1. For each of the following statements, prove the statement (showing all steps), or find a counterexample.

(a) Let V be a vector space. Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$. Suppose that $\mathbf{v} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{v} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\mathbf{v} \notin \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_3\}$. Then, $\mathbf{v} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution. Suppose the statement is true, let $\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3$ for non-zero scalars a_1, a_2, a_3. If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent then $\dim(\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}) = 3$. It follows that $\boldsymbol{v} \notin \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ because $\dim(\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) = 2$. Similarly, $\boldsymbol{v} \notin \operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\boldsymbol{v} \notin \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$. However, $\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and this is a contradiction. Therefore, it is proved that the statement is false. ■

(b) Suppose that A and B are $n \times n$ matrices with $A = CBC^{-1}$ for some invertible matrix C. Then, $rk(A) = rk(B)$.

Solution. Since elementary row operations do not change the row space, we have

$$\operatorname{rowsp}(A) = \operatorname{rowsp}(CBC^{-1}) = \operatorname{rowsp}(CB) = \operatorname{rowsp}(B).$$

From the definition of the rank of a matrix we have

$$rank(A) = \dim(\operatorname{colsp}(A)) = \dim(\operatorname{rowsp}(A)) = \dim(\operatorname{rowsp}(B)) = rank(B).$$

Therefore, it is proved that the statement is true. ■

(c) Suppose A and B are $n \times n$ matrices and that λ is an eigenvalue for both A and B. Then, λ is an eigenvalue for $A + B$.

Solution. Let $\boldsymbol{u}$ be an eigenvector of A and $\boldsymbol{v}$ be eigenvector of B. We have

$$\begin{aligned} A\boldsymbol{u} + B\boldsymbol{v} &= \lambda\boldsymbol{u} + \lambda\boldsymbol{v} \\ A\boldsymbol{u} + B\boldsymbol{v} &= \lambda(\boldsymbol{u} + \boldsymbol{v}). \end{aligned}$$

If $\boldsymbol{u} \neq \boldsymbol{v}$ then λ is not an eigenvalue of $A + B$. Therefore, it is proved that the statement is false. ■

(d) Suppose A and B are $n \times n$ matrices and that $\mathbf{v}$ is an eigenvector for both A and B. Then, $\mathbf{v}$ is an eigenvector for $A + B$.

Solution. Let λ_1 be an eigenvalue of A and λ_2 be an eigenvalue of B. We have

$$\begin{aligned} A\boldsymbol{v} + B\boldsymbol{v} &= \lambda_1\boldsymbol{v} + \lambda_2\boldsymbol{v} \\ (A + B)\boldsymbol{v} &= (\lambda_1 + \lambda_2)\boldsymbol{v}. \end{aligned}$$

Therefore, it is proved that the statement is true. ■

(e) Suppose that A and B are $n \times n$ matrices with $A = CBC^{-1}$ for some invertible matrix C. If λ is an eigenvalue for B, then λ is an eigenvalue for A.

Solution. Let $\boldsymbol{v}$ be an eigenvector of B. We have

$$\begin{aligned}AC &= CB\\ AC\boldsymbol{v} &= CB\boldsymbol{v}\\ AC\boldsymbol{v} &= C(\lambda\boldsymbol{v})\\ A(C\boldsymbol{v}) &= \lambda(C\boldsymbol{v}).\end{aligned}$$

Therefore, it is proved that the statement is true. ■

Question 2. It can be shown that the eigenvalues of a real symmetric matrix are real numbers. A symmetric matrix whose eigenvalues are all positive is called a **positive definite** matrix. Determine if the following matrix is positive definite:

$$A = \begin{bmatrix} 10 & -4 & -1 \\ -4 & 10 & -1 \\ -1 & -1 & 7 \end{bmatrix}$$

Hint: One of the eigenvalues of this matrix is $\lambda = 8$.

Solution. If λ is an eigenvalue of A then we have

$$A - \lambda I = \begin{bmatrix} 10-\lambda & -4 & 1 \\ -4 & 10-\lambda & -1 \\ -1 & -1 & 7-\lambda \end{bmatrix}$$

so that $\det(A - \lambda I) = 0$. Then we use cofactor expansion along row 1 to get

$$\begin{aligned}\det(A-\lambda I) &= (10-\lambda)[(10-\lambda)(7-\lambda)-1] - (-4)[(-4)(7-\lambda)-1] + 4 - 10 + \lambda\\ &= (10-\lambda)^2(7-\lambda) + 18\lambda - 140\\ &= -\lambda^3 + 27\lambda^2 - 222\lambda + 560.\end{aligned}$$

It follows that

$$\begin{aligned}-\lambda^3 + 27\lambda^2 - 222\lambda + 560 &= 0\\ \lambda^3 - 27\lambda^2 + 222\lambda - 560 &= 0\\ (\lambda^2 - 13\lambda + 40)(\lambda - 14) &= 0\\ (\lambda-5)(\lambda-8)(\lambda-14) &= 0.\end{aligned}$$

The eigenvalues of A are

$$\lambda_1 = 5, \lambda_2 = 8, \lambda_3 = 14.$$

The eigenvalues of A are all positive so A is a positive definite matrix. ■

Question 3. Suppose that A and B are 3×3 matrices such that $A = CBC^{-1}$ where

$$C = \begin{bmatrix} 3 & 5 & 1 \\ 7 & 2 & 4 \\ 5 & 1 & 8 \end{bmatrix}$$

Suppose that $\mathbf{v} = (6, 11, -4)^T$ is an eigenvector for B with eigenvalue $\lambda = -5$. Let $\mathbf{w} = C\mathbf{v}$. Compute $A\mathbf{w}$.

Solution. We have

$$\begin{aligned}
A\boldsymbol{w} &= CBC^{-1}C\boldsymbol{v} \\
&= CB\boldsymbol{v} \\
&= C(-5)\boldsymbol{v} \\
&= (-5)C\boldsymbol{v} \\
&= -5\begin{bmatrix} 3 & 5 & 1 \\ 7 & 2 & 4 \\ 5 & 1 & 8 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ -4 \end{bmatrix} \\
&= -5\begin{bmatrix} 69 \\ 48 \\ 9 \end{bmatrix} \\
&= \begin{bmatrix} -345 \\ -240 \\ -45 \end{bmatrix}.
\end{aligned}$$

■

Question 4. Suppose that A is an $n \times n$ matrix such that $A^3 + 3A^2 + A = 5I_n$. Find all possible complex eigenvalues for A.

Solution. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $\boldsymbol{v}$ be an eigenvector of A. We have

$$\begin{aligned}
(A^3 + 3A^2 + A - 5I)\boldsymbol{v} &= \mathbf{0} \\
A^3\boldsymbol{v} + 3A^2\boldsymbol{v} + A\boldsymbol{v} - 5\boldsymbol{v} &= \mathbf{0} \\
\lambda A^2\boldsymbol{v} + 3\lambda A\boldsymbol{v} + \lambda\boldsymbol{v} - 5\boldsymbol{v} &= \mathbf{0} \\
\lambda^3\boldsymbol{v} + 3\lambda^2\boldsymbol{v} + \lambda\boldsymbol{v} - 5\boldsymbol{v} &= \mathbf{0} \\
(\lambda^3 + 3\lambda^2 + \lambda - 5)\boldsymbol{v} &= \mathbf{0} \\
\lambda^3 + 3\lambda^2 + \lambda - 5 &= 0 \\
(\lambda - 1)(\lambda^2 + 4\lambda + 5) &= 0.
\end{aligned}$$

It follows that

$$\lambda = 1$$

or

$$\begin{aligned}
\lambda^2 + 4\lambda + 5 &= 0 \\
\lambda^2 + 4\lambda + 4 + 5 &= 4 \\
(\lambda + 2)^2 &= -1 \\
\lambda + 2 &= \pm\sqrt{-1} \\
\lambda &= -2 \pm i.
\end{aligned}$$

Therefore, the eigenvalues of A are

$$\lambda_1 = 1, \lambda_2 = -2 + i, \lambda_3 = -2 - i.$$

■

Question 5. For the following 2×2 matrix A, find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix}$$

Solution. Let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $D = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ where $C, D \in \mathbb{R}^{2\times 2}$ so we have

$$\begin{aligned} AC &= CD \\ \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}. \end{aligned}$$

Then we have

$$\begin{cases} \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} \\ \begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = y \begin{bmatrix} b \\ d \end{bmatrix} \end{cases}$$

and we have

$$\begin{aligned} A\boldsymbol{v}_1 &= \lambda_1 \boldsymbol{v}_1 \\ A\boldsymbol{v}_2 &= \lambda_2 \boldsymbol{v}_2 \end{aligned}$$

for eigenvalues λ_1, λ_2 and eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. Now let

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 4 \\ 4 & -1-\lambda \end{bmatrix}$$

so that

$$\begin{aligned} \det(A - \lambda I) &= (5-\lambda)(-1-\lambda) - 16 \\ &= \lambda^2 - 4\lambda - 21. \end{aligned}$$

Since $\det(A - \lambda I) = 0$ then we solve for λ

$$\begin{aligned} \lambda^2 - 4\lambda - 21 &= 0 \\ \lambda^2 - 4 + 4 &= 25 \\ (\lambda - 2)^2 &= 25 \\ \lambda &\in \{-3, 7\} \end{aligned}$$

The eigenvalues are

$$\lambda_1 = -3, \lambda_2 = 7$$

and we have

$$\begin{aligned} (A + 3I)\boldsymbol{v}_1 &= \mathbf{0} \\ (A - 7I)\boldsymbol{v}_2 &= \mathbf{0}. \end{aligned}$$

Now we use elementary row operations to get RREF and solve for $\boldsymbol{v}_1$

$$\begin{aligned}
A + 3I &= \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \\
\xrightarrow{\frac{1}{8}R_1} &= \begin{bmatrix} 1 & \frac{1}{2} \\ 4 & 2 \end{bmatrix} \\
\xrightarrow{R_2 - 4R_1} &= \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \\
x_2 &= s, s \in \mathbb{R} \\
x_1 &= -\frac{1}{2}s \\
\boldsymbol{v}_1 &= \begin{bmatrix} -\frac{1}{2}s \\ s \end{bmatrix}.
\end{aligned}$$

Similarly, we solve for $\boldsymbol{v}_2$

$$\begin{aligned}
A - 7I &= \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \\
\xrightarrow{-\frac{1}{2}R_1} &= \begin{bmatrix} 1 & -2 \\ 4 & -8 \end{bmatrix} \\
\xrightarrow{R_2 - 4R_1} &= \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\
x_2 &= t, t \in \mathbb{R} \\
x_1 &= 2t \\
\boldsymbol{v}_2 &= \begin{bmatrix} 2t \\ t \end{bmatrix}.
\end{aligned}$$

Therefore, we showed that

$$\begin{aligned}
D &= \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} \\
C &= \begin{bmatrix} -\frac{1}{2}s & 2t \\ s & t \end{bmatrix}.
\end{aligned}$$

$\blacksquare$