Math 4100 Homework #1 **Name: Yaohui Wu**

Instructions: Show **ALL** of your work! Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation. Course notes, textbooks, etc. are allowed.

This is due at **5:00 PM on Friday, October 13**!

You may use LaTeX to type the answers. If you use Latex, upload your .tex file along with the PDF output.

If you choose not to use LaTeX, write your answers **legibly** on perforated paper or loose leaf paper.

You **MUST** attach this sheet as the first page of your solutions.

Put your EMPLID on each page.

Solutions MUST be in the proper numerical order! You must make a PDF scan of your work (include the cover page as the first page) as a single PDF file.

The work **must** be submitted on Dropbox: https://www.dropbox.com/request/QPyB46xQsp5l8xoZtk7v

If you use any resources other than your textbook, **cite the source!**

Justify your answers!

Question	Possible Points	Score
1	4	
2	4	
3	6	
4	6	
Total	20	

Question 1. For each of the following statements, prove the statement (showing all steps), or find a counterexample.

(a) Let $\mathbb{F}$ be a field. If $a, b, c \in \mathbb{F}$ satisfy $ac = bc$, then $a = b$.

Solution:

Proof. Counterexample: Suppose $\mathbb{F} = \mathbb{R}$, $a \neq b, a \neq 0, b \neq 0, c = 0$.
Then $ac = 0$ and $bc = 0$.
Therefore, it is proved that $ac = bc$ and $a \neq b$. QED

(b) Let $\mathbb{F}$ be a field. If $a, b \in \mathbb{F}$ are both non-zero, then $(ab)^{-1} = a^{-1}b^{-1}$.

Solution:

Proof. Suppose $aa^{-1} = 1$ with $a^{-1} \in \mathbb{F}$, $bb^{-1} = 1$ with $b^{-1} \in \mathbb{F}$, and $ab(ab)^{-1} = 1$ with $(ab)^{-1} \in \mathbb{F}$ by the definition of multiplicative inverses.
We have $a^{-1}ab(ab)^{-1} = a^{-1}$ and then $b(ab)^{-1} = a^{-1}$.
Then we have $(ab)^{-1}b = a^{-1}$ by the commutativity of multiplication in $\mathbb{F}$ so that we have $(ab)^{-1}bb^{-1} = a^{-1}b^{-1}$.
Therefore, it is proved that $(ab)^{-1} = a^{-1}b^{-1}$. QED

(c) Suppose that A, B, and C are matrices such that both $AB = BA$ and $AC = CA$. Then, $A(B + C) = (B + C)A$.

Solution:

Proof. We have $A(B + C) = AB + AC$ by the distributive property.
Since $AB = BA$ and $AC = CA$, then $A(B + C) = BA + CA$.
We also have $(B + C)A = BA + CA$ by the distributive property.
Therefore, it is proved that $A(B + C) = (B + C)A$. QED

(d) Suppose that A, B, and C are non-zero matrices such that $AB = AC$. Then, $B = C$.

Solution:

Proof. Counterexample: Suppose $AA^{-1} = I$, if $AB = AC$, then $A^{-1}AB = A^{-1}AC$.
Then we have $IB = IC$ so $B = C$ if and only if A is invertible or A^{-1} exists but it is possible that A is non-invertible or A^{-1} does not exist.
Consider a 2×2 matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $ad - bc \neq 0$.
Let A be the following non-zero 2×2 matrix: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$ad - bc = 1(0) - (0)(0) = 0$$

Thus, A^{-1} does not exist.
Let B and C be the following non-zero 2×2 matrices: $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so that $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Therefore, it is proved that $AB = AC$ and $B \neq C$. QED

Question 2. Let V be a vector space over $\mathbb{R}$. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ satisfy $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \overrightarrow{\mathbf{0}}$ if and only if $a = b = c = 0$.

Let $\mathbf{v}_1 = \mathbf{u} + 3\mathbf{v}$, $\mathbf{v}_2 = 2\mathbf{v} - 5\mathbf{w}$, and $\mathbf{v}_3 = 3\mathbf{u} - \mathbf{v} + 5\mathbf{w}$. Suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \overrightarrow{\mathbf{0}}$. Prove that $a_1 = a_2 = a_3 = 0$, or find non-zero values for a_1, a_2, and a_3 such that the equation holds.

Hint: Rewrite the equation in the form $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \overrightarrow{\mathbf{0}}$.

Solution:

Proof.

$$\begin{aligned} a_1(\mathbf{u} + 3\mathbf{v}) + a_2(2\mathbf{v} - 5\mathbf{w}) + a_3(3\mathbf{u} - \mathbf{v} + 5\mathbf{w}) &= \overrightarrow{\mathbf{0}} \\ a_1\mathbf{u} + 3a_1\mathbf{v} + 2a_2\mathbf{v} - 5a_2\mathbf{w} + 3a_3\mathbf{u} - a_3\mathbf{v} + 5a_3\mathbf{w} &= \overrightarrow{\mathbf{0}} \\ (a_1 + 3a_3)\mathbf{u} + (3a_1 + 2a_2 - a_3)\mathbf{v} + (-5a_2 + 5a_3)\mathbf{w} &= \overrightarrow{\mathbf{0}} \end{aligned}$$

Now we have the system of equations:

$$\begin{array}{rcrcrcl} a_1 & & & + & 3a_3 & = & 0 \\ 3a_1 & + & 2a_2 & - & a_3 & = & 0 \\ & & -5a_2 & + & 5a_3 & = & 0 \end{array}$$

Then we have the augmented matrix A:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 3 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right]$$

$$R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 2 & -10 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right]$$

$$\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right]$$

$$R_3 + 5R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -20 & 0 \end{array}\right]$$

$$-\frac{1}{20}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 3R_3, R_2 + 5R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The matrix A is now in reduced row echelon form and the solution to the system is:

$$a_1 = 0$$
$$a_2 = 0$$
$$a_3 = 0$$

Therefore, it is proved that $a_1 = a_2 = a_3 = 0$. QED

Question 3. Let $a, b, c \in \mathbb{R}$. Consider the following questions about systems of linear equations (over $\mathbb{R}$):

$$\begin{array}{rcrcrcrcl} 5x_1 & - & 13x_2 & + & 14x_3 & + & 27x_4 & = & a \\ x_1 & - & 3x_2 & + & 4x_3 & + & 7x_4 & = & b \\ 11x_1 & - & 19x_2 & + & 2x_3 & + & 21x_4 & = & c \end{array}$$

(a) Find a vector (a, b, c) with non-zero values of a, b, and c such that the system is inconsistent.

Solution: We have the following 3×4 augmented matrix A:

$$\left[\begin{array}{cccc|c} 5 & -13 & 14 & 27 & a \\ 1 & -3 & 4 & 7 & b \\ 11 & -19 & 2 & 21 & c \end{array}\right]$$

$$R_1 - 4R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & a-4b \\ 1 & -3 & 4 & 7 & b \\ 11 & -19 & 2 & 21 & c \end{array}\right]$$

$$R_2 - R_1, R_3 - 11R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & a-4b \\ 0 & -2 & 6 & 8 & -a+5b \\ 0 & -8 & 24 & 32 & -11a+44b+c \end{array}\right]$$

$$-\frac{1}{2}R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & a-4b \\ 0 & 1 & -3 & -4 & \frac{a}{2}-\frac{5b}{2} \\ 0 & -8 & 24 & 32 & -11a+44b+c \end{array}\right]$$

$$R_3 + 8R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & a-4b \\ 0 & 1 & -3 & -4 & \frac{a}{2}-\frac{5b}{2} \\ 0 & 0 & 0 & 0 & -7a+24b-c \end{array}\right]$$

Now A is in row echelon form and it is inconsistent when $-7a + 24b - c \neq 0$.
Let $a = 1$, $b = 1$, and $c = 1$, then we have $-7(1) + 24(1) - 1 = 16$.
Since $16 \neq 0$ so the system is inconsistent with vector $(1, 1, 1)$. QED

(b) If $a = 23$ and $c = 41$, find all values of b such that the system of equations is consistent. For these values of b, describe the set of all solutions. If there are infinitely many solutions, use s and t for the free variable and write all answers in terms of the free variables s and t (use s for the first free variable from left to right).

Solution: The system is consistent if $-7(23) + 24b - 41 = 0$ so $b = 5$.
Then we have the following augmented matrix A:

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & 3 \\ 0 & 1 & -3 & -4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & -5 & 2 \\ 0 & 1 & -3 & -4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now A is in reduced row echelon form and we have the following system of equations:

$$\begin{array}{ccccccccc} x_1 & & & - & 5x_3 & - & 5x_4 & = & 2 \\ & & x_2 & - & 4x_3 & - & 4x_4 & = & -1 \end{array}$$

After solving for x_1 and x_2, we have the set of solutions to the system:

$$\begin{array}{ccccccc} x_1 & = & 2 & + & 5x_3 & + & 5x_4 \\ x_2 & = & -1 & + & 4x_3 & + & 4x_4 \end{array}$$

Let $x_3 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$. The set of solutions to the system is:

$$\begin{array}{ccccccc} x_1 & = & 2 & + & 5s & + & 5t \\ x_2 & = & -1 & + & 4s & + & 4t \end{array}$$

QED

Question 4. Let $x, y \in \mathbb{R}$. Suppose that A is a 2×2 matrix given by $A = \begin{bmatrix} x & 3 \\ 2 & y \end{bmatrix}$. Find the values of x and y if

$$A^2 = \begin{bmatrix} 5x & -12 \\ -8 & -7y \end{bmatrix}$$

Solution:

$$A^2 = \begin{bmatrix} x & 3 \\ 2 & y \end{bmatrix} \begin{bmatrix} x & 3 \\ 2 & y \end{bmatrix} = \begin{bmatrix} x^2 + 6 & 3x + 3y \\ 2x + 2y & y^2 + 6 \end{bmatrix}$$

We have the system of equations:

$$\begin{aligned} x^2 + 6 &= 5x \\ 3x + 3y &= -12 \\ 2x + 2y &= -8 \\ y^2 + 6 &= -7y \end{aligned}$$

First we simplify the system and we get:

$$\begin{aligned} x^2 + 6 &= 5x \\ x + y &= -4 \\ y^2 + 6 &= -7y \end{aligned}$$

Then we solve for x and we get:

$$
\begin{aligned}
x^2 + 6 &= 5x \\
x^2 - 5x + 6 &= 0 \\
(x-2)(x-3) &= 0
\end{aligned}
$$

so $x = 2$ or $x = 3$. Then we solve for y and we get:

$$
\begin{aligned}
y^2 + 6 &= -7y \\
y^2 + 7y + 6 &= 0 \\
(y+1)(y+6) &= 0
\end{aligned}
$$

so $y = -1$ or $y = -6$.
Then we substitute values of x and y to check if it satisfies $x + y = -4$.
It is obvious that $x = 2$ and $y = -6$ is the solution. QED