Math 4100 Homework #2&3 **Name: Yaohui Wu**

Instructions: Show **ALL** of your work! Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation. Course notes, textbooks, etc. are allowed.

This is due at **5:00 PM on Friday, November 10**!

You may use LaTeX to type the answers. If you use Latex, upload your .tex file along with the PDF output.

If you choose not to use LaTeX, write your answers **legibly** on perforated paper or loose leaf paper.

You **MUST** attach this sheet as the first page of your solutions.

Put your EMPLID on each page.

Solutions MUST be in the proper numerical order! You must make a PDF scan of your work (include the cover page as the first page) as a single PDF file.

The work **must** be submitted on Dropbox: https://www.dropbox.com/request/oycXLwSWqV3W6qolkPOq

If you use any resources other than your textbook, **cite the source!**

Justify your answers!

Question	Possible Points	Score
1	4	
2	4	
3	6	
4	6	
Total	20	

Question 1. For each of the following statements, prove the statement (showing all steps), or find a counterexample.

(a) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

Solution. The statement is false. Suppose the statement is true, let A be a 2×2 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let B be a 2×2 matrix $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, we have $\det(A) = 0$ and $\det(B) = 0$. Then we know that $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\det(A+B) = 1$. Since $\det(A)+\det(B) = 0+0 = 0$ so $\det(A+B) \neq \det(A)+\det(B)$ and this is a contradiction. Therefore, it is proved that the statement is false. ■

(b) If A is a non-zero $n \times n$ matrix with real number entries, then $\det(AA^T) > 0$.

Solution. The statement is false. Suppose the statement is true, let A be a 2×2 matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that we have $\det(A) = 0$. Thus, we have $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\det(A^T) = 0$. Since $\det(AA^T) = \det(A) \times \det(A^T) = 0 \times 0 = 0$ so $\det(AA^T) \not> 0$ but this is a contradiction. Therefore, it is proved that the statement is false. ■

(c) Suppose that C is an invertible $n \times n$ matrix. If A and B are $n \times n$ matrices such that $AC = CB$, then $\det(A) = \det(B)$.

Solution. The statement is true. Given C is invertible and $AC = CB$ then we know that

$$\begin{aligned} ACC^{-1} &= CBC^{-1} \\ AI &= CBC^{-1} \\ A &= CBC^{-1} \end{aligned}$$

It follows that

$$\begin{aligned} \det(A) &= \det(CBC^{-1}) \\ &= \det(C) \times \det(B) \times \det(C^{-1}) \\ &= \det(C) \times \det(C^{-1}) \times \det(B) \\ &= \det(C) \times (\det(C))^{-1} \times \det(B) \\ &= \det(B) \end{aligned}$$

Therefore, it is proved that if A, B, and C are $n \times n$ matrices, and C is invertible such that $AC = CB$, then $\det(A) = \det(B)$. ■

(d) Suppose that $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is an $n \times n$ matrix, and $\mathbf{v} \in \mathbb{C}^n$ satisfies $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}$. Then, $A^3 - \lambda^3 I_n$ is non-invertible.

Solution. The statement is false. Let $\boldsymbol{v} = \mathbf{0}$ be the only solution to $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and let $\lambda = 0$ such that $A^3 - \lambda^3 I = A^3$. By definition A is non-invertible if and only if there exists a $\boldsymbol{v} \neq 0$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Since $\boldsymbol{v} = \mathbf{0}$ is the only solution to $A\boldsymbol{v} = \lambda\boldsymbol{v}$ so that A is invertible and $\det(A) \neq 0$. It follows that

$$\begin{aligned} \det(A^3 - \lambda^3 I) &= \det(A^3) \\ &= (\det(A))^3 \end{aligned}$$

If $\det(A) \neq 0$ then $(\det(A))^3 \neq 0$ so $\det(A^3 - \lambda^3 I) \neq 0$ and $A^3 - \lambda^3 I$ is invertible. Therefore, it is proved that the statement is false. ■

Question 2. For each of the following, find examples of 3×3 matrices A and B satisfying the property. Show that the matrices you give satisfy the given property.

(a) $\det(A) < 0$, $\det(B) < 0$, $\det(A + B) > 0$

Solution. Let A be the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and let B be the matrix $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Since A and B are both upper and lower triangular, we have $\det(A) = 2 \times 1 \times (-1) = -2$ and $\det(B) = (-1) \times 1 \times 2 = -2$. Then, we have $A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and it is upper and lower triangular so that $\det(A + B) = 1 \times 2 \times 1 = 2$. Therefore, it is proved that A and B satisfies $\det(A) < 0$, $\det(B) < 0$, and $\det(A + B) > 0$. ■

(b) $\det(A) > 0$, $\det(B) > 0$, $\det(A + B) = 0$.

Solution. Let A be the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and let B be the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ so that we have $A+B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then we have $A = I$ so that $\det(A) = \det(I) = 1$, $\det(B) = 1 \times (-1) \times (-1) = 1$, and $\det(A+B) = 2 \times 0 \times 0 = 0$. Therefore, it is proved that A and B satisfies $\det(A) > 0$, $\det(B) > 0$, and $\det(A + B) = 0$. ■

(c) $\det(A) > 0$, $\det(B) = 0$, $\det(A + B) < 0$.

Solution. Let A be the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and let B be the matrix $B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ such that we have $A + B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It follows that $\det(A) = 1$, $\det(B) = -2 \times 0 \times 0 = 0$ and $\det(A+B) = (-1) \times 1 \times 1 = -1$. Therefore, it is proved that A and B satisfies $\det(A) > 0$, $\det(B) = 0$, and $\det(A + B) < 0$. ■

(d) $\det\left(A^2 + B^2\right) < 0$.

Solution. Let A be the matrix $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where we defined i as $i^2 = -1$ such that $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let B be the matrix $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ such that $B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then we have

$A^2 + B^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\det(A^2 + B^2) = -1 \times 2 \times 1 = -2$. Therefore, it is proved that A and B satisfies $\det(A^2 + B^2) < 0$. ■

Question 3. Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} 4 & 3 & 7 \\ 5 & 1 & 4 \\ 7 & -2 & 9 \end{bmatrix}$$

Suppose that B is an upper triangular matrix

$$B = \begin{bmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{bmatrix}$$

such that $AC = CB$ for some **invertible** matrix C. Compute the value adg.

Solution. We know that

$$\begin{aligned} ACC^{-1} &= CBC^{-1} \\ AI &= CBC^{-1} \\ A &= CBC^{-1} \end{aligned}$$

such that

$$\begin{aligned} \det(A) &= \det(CBC^{-1}) \\ &= \det(C) \times \det(B) \times \det(C^{-1}) \\ &= \det(C) \times \det(C^{-1}) \times \det(B) \\ &= \det(C) \times (\det(C))^{-1} \times \det(B) \\ &= \det(B) \end{aligned}$$

Since B is upper triangular so $\det(A) = \det(B) = a \times d \times g$. Note that for any $n \times n$ matrix A, we can compute $\det(A)$ by cofactor expansion. Let a_{ij} be the entry at row i and column j of A. Let M_{ij} be a $(n-1) \times (n-1)$ submatrix given by removing row i and column j of A such that the $\det(M_{ij})$ is the (i, j) minor of A. Let $C_{ij} = (-1)^{i+j} \det(M_{ij})$ be the (i, j) cofactor of A. The cofactor expansion along row i where $1 \le i \le n$ is

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij}) \\ &= a_{i1}(-1)^{i+1} \det(M_{i1}) + a_{i2}(-1)^{i+2} \det(M_{i2}) + \cdots + a_{in}(-1)^{i+n} \det(M_{in}) \end{aligned}$$

Then we compute $\det(A)$ using cofactor expansion along row 1.

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{3} a_{1j}C_{1j} \\
&= a_{11}(-1)^{1+1}\det(M_{11}) + a_{12}(-1)^{1+2}\det(M_{12}) + a_{13}(-1)^{1+3}\det(M_{13}) \\
&= 4\begin{vmatrix} 1 & 4 \\ -2 & 9 \end{vmatrix} - 3\begin{vmatrix} 5 & 4 \\ 7 & 9 \end{vmatrix} + 7\begin{vmatrix} 5 & 1 \\ 7 & -2 \end{vmatrix} \\
&= 4[1 \times 9 - 4 \times (-2)] - 3[5 \times 9 - 4 \times 7] + 7[5 \times (-2) - 1 \times 7] \\
&= 4(17) - 3(17) + 7(-17) \\
&= (4 - 3 - 7)17 \\
&= -102
\end{aligned}$$

Therefore, it is proved that $\det(A) = \det(B) = a \times d \times g = -102$. ■

Question 4. Suppose that B is the following matrix:

$$B = \begin{bmatrix} 9 & 5 & 2 \\ 5 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

Use determinants to prove that there is no 3×3 matrix A *with real number entries* such that $B = AA^T$.

Solution. Suppose that there exists a 3×3 matrix $A \in \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned}
\det(B) &= \det(AA^T) \\
&= \det(A)\det(A^T)
\end{aligned}$$

and since $\det(A) = \det(A^T)$ then we have

$$\begin{aligned}
\det(B) &= \det(A)\det(A) \\
&= (\det(A))^2
\end{aligned}$$

We use cofactor expansion along row 1 to compute $\det(B)$

$$\begin{aligned}
\det(B) &= 9\begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} \\
&= 9[3 \times 2 - 4 \times 4] - 5[5 \times 2 - 4 \times 2] + 2[5 \times 4 - 3 \times 2] \\
&= 9(-10) - 5(2) + 2(14) \\
&= -72
\end{aligned}$$

If $A \in \mathbb{R}^{3\times 3}$ then $\det(A) \in \mathbb{R}$ so that $(\det(A))^2 \geq 0$. However, $(\det(A))^2 = \det(B) = -72$ and this is a contradiction. Therefore, it is proved that there does not exist a 3×3 matrix $A \in \mathbb{R}^{3\times 3}$ such that $B = AA^T$. ■