MTH 4140 Homework 1

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February 20, 2024

Problem 1

Section 1.1 Problem 8

Solution. Let the graph on the left be G and we label its vertices using $\{a, b, c, d, e, f, g, h\}$. The vertices for the outer square are $\{a, b\}$ from left to right on the top and $\{c, d\}$ from left to right on the bottom respectively. Similarly, the vertices for the inner square are $\{e, f\}$ from left to right on the top and $\{g, h\}$ from left to right on the bottom respectively. We can decompose G into 4 copies of K_3 with the center being the vertex of degree 3. They are $\{a, e, f, g\}$ with center e, $\{a, b, d, f\}$ with center b, $\{d, f, g, h\}$ with center h, and $\{a, c, d, g\}$ with center c. We can also decompose G into 4 copies of P_4. They are the a, e-path $\{a, b, f, e\}$, the b, f-path $\{b, d, h, f\}$, the d, h-path $\{d, c, g, h\}$, and the c, g-path $\{c, a, e, g\}$. Therefore, it is shown that G decomposes into copies of K_3 and also into copies of P_4. ■

Problem 2

Section 1.1 Problem 9

Solution. Let the graph on the left be G and the graph on the right be H. It is equivalent to show that $G \cong \overline{H}$. If we label every vertex in G and H using the same set of labels then two vertices are adjacent in G if and only if they are adjacent in $\overline{H}$. Therefore, we have $G \cong \overline{H}$ so it is proved that $\overline{G} \cong H$. ■

Problem 3

Section 1.1 Problem 10

Solution. We have a disconnected graph G and its complement $\overline{G}$. Let u, v be two vertices of different components in G s.t. there is no path from u to v. Then u, v are adjacent in $\overline{G}$ and all other vertices in $\overline{G}$ are adjacent to one of u, v or both since u, v have no common neighbors. There is a path for any two vertices in $\overline{G}$ using u, v. Therefore, it is proved that the complement of a simple disconnected graph must be connected. ■

Problem 4

Section 1.1 Problem 11

Solution. Let's label the vertices in the middle as $\{a, b, c, d\}$ from left to right respectively, the top vertex as e, and the bottom vertex as f. We can see that the graph G has a clique of size 4 with vertices $\{a, b, e, f\}$. Suppose G has a clique of size greater than 4 then there must be at least 5 vertices with a degree of at least 4. However, G only has 4 vertices with a degree of at least 4 so there is a contradiction. Therefore, it is shown that the maximum size of a clique in G is 4. All of the independent sets in G are $\{a, c\}, \{a, d\}, \{b, d\}, \{e\}, \{f\}$. There is not a set of three or more vertices s.t. any two vertices in the set are not adjacent. Therefore, it is shown that the maximum size of an independent set in G is 2. ■

Problem 5

Section 1.1 Problem 12

Solution. The Petersen graph is not bipartite because it has odd cycles. The independent set of the Petersen graph must have vertices that are 2-element subsets of $\{1, 2, 3, 4, 5\}$ with a common element. The 2-element subsets of the largest independent set are the combination of any element $n \in \{1, 2, 3, 4, 5\}$ and the rest of the 4 elements in $\{1, 2, 3, 4, 5\} - \{n\}$. Therefore, the size of the largest independent set of the Petersen graph is 4. ■

Problem 6

Section 1.1 Problem 17

Solution. We know that $G \cong H \iff \overline{G} \cong \overline{H}$ so it is equivalent to count all 2-regular graphs on 7 vertices. Only the graph C_7 and the graph with only C_3, C_4 satisfy this. Therefore, the number of isomorphic classes of simple 7-vertex 4-regular graphs is 2. ■

Problem 7

Section 1.1 Problem 19

Solution. Let the graphs be G, H, F from left to right respectively. There is a corresponding relationship between the vertices of H and F. If we label all vertices in one of the two graphs then we can also label all of the vertices in the other graph to have the same properties of adjacency. However, this is not true for G because G is bipartite and H, F are not bipartite. Therefore, it is shown that $H \cong F$. ■

Problem 8

Section 1.1 Problem 25

Solution. Suppose the Petersen graph has C_7 then there are 3 remaining vertices outside of C_7. The Petersen graph is 3-regular so each of the 7 vertices on C_7 has one extra edge outside of C_7. The Petersen graph has girth 5 so the 7 extra edges have to have one of the 3 remaining vertices as an endpoint. By the pigeonhole principle, at least 3 of the extra edges end on the same vertex. WLOG, we will have a C_3 but the Petersen graph has girth 5 so there is a contradiction. Therefore, it is proved that the Petersen graph does not have a cycle of length 7. ∎

Problem 9

Section 1.1 Problem 26

Solution. If G has girth 4 then there are two adjacent vertices u, v with no common neighbors. Since G is k-regular and u, v each has degree $k-1$, the sum of the degrees is at least $k-1+1+1+k-1=2k$. Therefore, it is proved that G has at least $2k$ vertices. If each vertex in the independent set $\{x_1, \ldots, x_k\}$ is adjacent to all vertices of the independent set $\{y_1, \ldots, y_k\}$ then we have exactly $2k$ vertices. All such graphs with exactly $2k$ vertices must be complete bipartite graphs $K_{n,n}$ where $n=k$. ∎

Problem 10

Section 1.2 Problem 3

Solution. The graph G has 4 components and they are $\{1\}, \{11\}, \{13\}$, and $\{2,3,4,5,6,7,8,9,10,12,14,15\}$. We have a 7, 5-path $\{7,14,2,4,8,10,12,3,6,9,15,5\}$ because any two consecutive integers both have a greatest common factor greater than 1. This path travels every vertex in the largest component of G so it is the path with the longest length. The maximum length of a path in G is 12. ∎

Problem 11

Problem A

Solution. Let the three husbands be A, B, C and their wives be a, b, c respectively. The steps to cross the river from L to R are

1. $A, a \to R$
2. $L \leftarrow A$
3. $b, c \to R$

4. $L \leftarrow a$
5. $B, C \rightarrow R$
6. $L \leftarrow B, b$
7. $A, B \rightarrow R$
8. $L \leftarrow c$
9. $a, c \rightarrow R$
10. $L \leftarrow B$
11. $B, b \rightarrow R$

■

Problem 12

Problem B

Solution. Let K_n be a simple connected complete graph of n vertices s.t. every vertex has degree $n-1$ and K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. K_n has the maximum number of edges that a simple connected graph of n vertices can have because any two vertices in K_n are adjacent hence there is an edge for any pair of vertices. If G has the maximum number of edges then it must have two components, K_9 and an isolated vertex v. The number of edges of G is $\frac{9(9-1)}{2} = 36$ by the handshaking lemma. Therefore, it is proved that G has at most 36 edges. ■