MTH 4140 Homework 2

Yaohui Wu

March 11, 2024

Problem 1

Section 1.3 Problem 8

Solution. We apply the Havel-Hakimi algorithm then we have

(a) $(5,5,4,3,2,2,2,1) \implies (4,3,2,1,1,2,1) \implies (4,3,2,2,1,1,1) \implies (2,1,1,0,1,1)$
$\implies (2,1,1,1,1,0) \implies (0,0,1,1,0) \implies (1,1,0,0,0) \implies (0,0,0,0)$

(b) $(5,5,4,4,2,2,1,1) \implies (4,3,3,1,1,1,1) \implies (2,2,0,0,1,1) \implies (2,2,1,1,0,0) \implies$
$(1,0,1,0,0) \implies (1,1,0,0,0) \implies (0,0,0,0)$

(c) $(5,5,5,3,2,2,1,1) \implies (4,4,2,1,1,1,1) \implies (3,1,0,0,1,1) \implies (3,1,1,1,0,0) \implies$
$(0,0,0,0,0)$

(d) $(5,5,5,4,2,1,1,1) \implies (4,4,3,1,0,1,1) \implies (4,4,3,1,1,1,0)$
$implies(3,2,0,0,1,0) \implies (3,2,1,0,0,0) \implies (1,0,-1,0,0)$

Therefore, it is proved that $(5,5,4,3,2,2,2,1), (5,5,4,4,2,2,1,1), (5,5,5,3,2,2,1,1)$ are graphic sequences but $(5,5,5,4,2,1,1,1)$ is not a graphic sequence. ■

Problem 2

Section 1.3 Problem 9

Solution. Let G be a graph of 26 vertices from two sets of 13 vertices s.t. the vertices represent the teams, the sets represent the divisions, and there is an edge between two vertices if two teams play against each other. Let H be a subgraph of G with 13 vertices in the same set s.t. it represent 13 teams in the same division. Suppose the proposition is true, each of the 13 vertices in H will have degree 9. Then the sum of the degrees in H must be $13 \cdot 9$ which must be odd. However, the handshaking lemma states that for a graph $G = (V, E)$ we have $\sum_{deg(v) \in V} = 2|E|$, i.e. the sum of the degrees is twice the number of edges which must be even. There is a contradiction and the proposition is false, that which was to be demonstrated. Therefore, it is proved that for a league with two divisions of 13 teams each, it is impossible to schedule a season with each team playing nine games against teams within its division and four games against teams in the other division. ■

Problem 3

Section 1.3 Problem 18

Solution. Suppose the proposition is true for the sake of contradiction. Let G be a k-regular bipartite graph that has a cut edge e s.t. $G - e$ is not connected. Let A and B be the two independent sets of vertices in G. We remove e from G then $G - e$ becomes a disjoint union of 2 bipartite graphs G_1 and G_2. The endpoints of e are the vertex $u \in A$ in G_1 and the vertex $v \in B$ in G_2. G_1 has a vertices with $a - 1$ vertices having degree k and the vertex u having degree $k - 1$ in A. G_1 also has b vertices with degree k in B. Then we have

$$\begin{aligned} e(G_1) = (a-1)k + (k-1) &= b \cdot k \\ a \cdot k - k + k - 1 &= b \cdot k \\ a \cdot k - 1 &= b \cdot k \\ (a-b)k &= 1 \end{aligned}$$

Since we are given that $k \geq 2$ so we have $(a - b)k \neq 1$ hence there is a contradicition and the proposition is false. Therefore, it is proved that for $k \geq 2$, a k-regular bipartite graph has no cut-edge. ■

Problem 4

Section 1.3 Problem 45

Solution. The Kőnig's theorem states that a graph is bipartite if and only if it has no odd cycles. Let G be the Petersen graph, we proved that G has 12 C_5 and every edge of G belongs to 4 C_5. Thus in order to remove 12 C_5 from G, we have to remove at least $\frac{12}{4} = 3$ edges. Since G has 15 edges hence we will have at most $15 - 3 = 12$ edges in the bipartite subgraph of G after we removed all of the odd cycles. Therefore, it is proved that the maximum number of edges in a bipartite subgraph of the Petersen graph is 12. ■

Problem 5

Section 1.3 Problem 61

Solution. Let $(d_1, d_2, \ldots, d_{n(G)})$ be the degree sequence of the graph G with $n(G)$ vertices. Then the degree sequence of $\overline{G}$ is $(n - 1 - d_{n(G)}, \ldots, n(G) - 1 - d_2, n(G) - 1 - d_1)$. Since $G \cong \overline{G}$ then the two degree sequences of G and $\overline{G}$ are exactly the same. Thus we have

$$(d_1, d_2, \ldots, d_{n(G)}) = (n(G) - 1 - d_{n(G)}, \ldots, n(G) - 1 - d_2, n(G) - 1 - d_1)$$

Given that $n(G) \equiv 1 \pmod 4$ hence we have that $n(G)$ is an odd number. Consider the middle elements in both degree sequences, they are equal thus we have

$$\begin{aligned} d_{\frac{n(G)+1}{2}} &= n(G) - 1 - d_{\frac{n(G)+1}{2}} \\ 2 \cdot d_{\frac{n(G)+1}{2}} &= n(G) - 1 \\ d_{\frac{n(G)+1}{2}} &= \frac{n(G)-1}{2} \end{aligned}$$

Therefore, it is proved that if $G \cong \overline{G}$ and that $n(G) \equiv 1 \pmod 4$ then G has at least one vertex with degree $\frac{n(G)-1}{2}$. ■