MTH 4140 Graph Theory

Yaohui Wu

April 25, 2024

Homework 3

Introduction to Graph Theory

Chapter 2 Trees and Distances

Section 2.2 Spanning Trees and Enumeration

Problem 1. Exercise 2.2.1

Solution. (a) The trees with Prüfer codes that has only one value are the stars. (b) The trees with Prüfer codes that has exactly two values are the double stars. (c) The trees with Prüfer codes that have distinct values in all positions are the paths. ■

Problem 2. Exercise 2.2.8(a)

Section 2.3 Optimization and Trees

Problem 3. Exercise 2.3.3

Solution. We run Kruskal's algorithm to get a minimum spanning tree (MST) of the graph built from the given adjacency matrix. The total weight of the MST is

$$3 + 3 + 7 + 8 = 21$$

The least cost of making all the cities reachable from each other is 21. ■

Problem 4. Exercise 2.3.7

Problem 5. Exercise 2.3.26

Chapter 3 Matchings and Factors

Section 3.1 Matchings and Covers

Problem 6. Exercise 3.1.2

Problem 7. Exercise 3.1.8

Solution. Suppose that a tree T has two different perfect matchings M and M'. Consider the symmetric difference of the edge sets of M and M' denoted by $M \triangle M'$. If a vertex is incidental to the same edge in both perfect matchings, then it is an isolated vertex in the symmetric difference. Otherwise, the vertex is incidental to two different edges in the symmetric difference. Hence, every vertex of $M \triangle M'$ must has degree 0 or 2. We know that every component of $M \triangle M'$ must be a path or an even cycle. Since T is a tree and trees have no cycles so every component of $M \triangle M'$ must be an isolated vertex with degree 0. This implies that the two perfect matchings are the same but there is a contradiction. Therefore, it is proved that every tree has at most one perfect matching. ■

Problem 8. Exercise 3.1.16