MTH 4140 Graph Theory

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Homework 3

Introduction to Graph Theory

Chapter 2 Trees and Distances

Section 2.2 Spanning Trees and Enumeration

Problem 1. Exercise 2.2.1

Solution. (a) The trees with Prüfer codes that has only one value are the stars. (b) The trees with Prüfer codes that has exactly two values are the double stars. (c) The trees with Prüfer codes that have distinct values in all positions are the paths. ■

Problem 2. Exercise 2.2.8(a)

Solution. We can choose two leaves in $\binom{n}{2}$ ways since they do not appear in the Prüfer code. There are $(n-2)!$ permutations of the labels in the Prüfer code. There are

$$\binom{n}{2}(n-2)! = \frac{n(n-1)(n-2)!}{2} = \frac{n!}{2}$$

trees with only two leaves. Trees that have only two leaves are the paths. There are $n!$ permutations of the vertices in a path with vertex set $[n]$. We adjust for overcount of isomorphic paths so there are $\frac{n!}{2}$ paths. The number of trees with vertex set $[n]$ that have only two leaves is $\frac{n!}{2}$. ■

Section 2.3 Optimization and Trees

Problem 3. Exercise 2.3.3

Solution. We run Kruskal's algorithm to get a minimum spanning tree (MST) of the graph built from the given adjacency matrix. The total weight of the MST is

$$3 + 3 + 7 + 8 = 21$$

The least cost of making all the cities reachable from each other is 21. ■

Problem 4. Exercise 2.3.7

Solution. Suppose a graph G with distinct edge weights has two different minimum spanning trees T and T'. Let e be the lightest edge of the symmetric difference. Since the edge weights are distinct, so e is in exactly one of the two trees. WLOG, if e is in $E(T)$, then there exists an edge e' in $E(T')$ in the symmetric difference such that $T' - e' + e$ is a spanning tree. Thus, we have $w(e) < w(e')$ so

$$w(T' - e' + e) < w(T')$$

which is a contradiction since we assumed that T' is an MST. It is proved that a weighted connected graph G with distinct edge weights has only one minimum spanning tree. ∎

Problem 5. Exercise 2.3.26

Solution. Let a_n be the recurrence relation of the number of binary trees with $n+1$ leaves. There is only one binary tree with one leaf when $n = 0$ so $a_0 = 1$. In general when $n > 0$, for each tree there are k leaves in the left subtree and $n - k + 1$ leaves in the right subtree from the root where $1 \leq k \leq n$. The recurrence relation is

$$\begin{aligned} a_0 &= 1 \\ a_n &= \sum_{k=1}^{n} a_{k-1} a_{n-k}, n > 0 \end{aligned}$$

∎

Chapter 3 Matchings and Factors

Section 3.1 Matchings and Covers

Problem 6. Exercise 3.1.2

Solution. Suppose that C_n has a maximal matching M of size k. There are k matched edges that map to $n-2k$ unsaturated vertices in M. Since the matching is maximal, every matched edge maps to at most one unsaturated vertex which implies that this is a set of at most 3 vertices. By the Pigeonhole principle, there are $\left\lceil \frac{n}{3} \right\rceil$ such sets in a cycle C_n of n vertices. The minimum size of a maximal matching in the cycle C_n is $\left\lceil \frac{n}{3} \right\rceil$. ∎

Problem 7. Exercise 3.1.8

Solution. Suppose that a tree T has two different perfect matchings M and M'. Consider the symmetric difference of the edge sets of M and M' denoted by $M \triangle M'$. If a vertex is incidental to the same edge in both perfect matchings, then it is an isolated vertex in the symmetric difference. Otherwise, the vertex is incidental to two different edges in the symmetric difference. Hence, every vertex of $M \triangle M'$ must has degree 0 or 2. We know that every component of $M \triangle M'$ must be a path or an even cycle. Since T is a tree and trees have no cycles so every component of $M \triangle M'$ must be an isolated vertex with degree 0. This implies that the two perfect matchings are the same but there is a contradiction. Therefore, it is proved that every tree has at most one perfect matching. ∎

Problem 8. Exercise 3.1.16

Solution. Let M_k be the number of perfect matchings of the k-dimensional hypercube Q_k for $k \geq 2$. We can use proof by induction to show that $M_k \leq 2^{2^{k-2}}$. When $k = 2$, we have

$$M_2 \leq 2^{2^{2-2}} = 2$$

which is true. We assume that $M_k \leq 2^{2^{k-2}}$ for $k > 2$. The number of perfect matchings in Q_{k+1} is the the number of ways to choose the perfect matchings in each of the two disjoint Q_k from Q_{k+1}. Thus, we have

$$M_{k+1} \leq M_k^2 \leq (2^{2^{k-2}})^2 \leq 2^{2^{k-1}} = 2^{2^{(k+1)-2}}$$

By mathematical induction, we can deduce that

$$M_k \leq 2^{2^{k-2}}$$

is true for $k \geq 2$. It is proved that for $k \geq 2$, Q_k has at least $2^{2^{k-2}}$ perfect matchings. ■