MTH 4140 Graph Theory

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Homework 4

Introduction to Graph Theory

Chapter 4 Connectivity and Paths

Section 4.1 Cuts and Connectivity

Problem 1. Excercise 4.1.1

Solution. (a) A graph is k-vertex-connected if its vertex-connectivity is at least k. If a graph G is 2-connected, then its vertex-connectivity is at least 2. Given G has vertex-connectivity 4 so we have $\kappa(G) = 4 \geq 2$ which is true. The statement "Every graph with connectivity 4 is 2-connected." is true.
(b) A 3-connected graph G has vertex-connectivity at least 3 so $\kappa(G) \geq 3$. Consider the graph K_5, we know that $\kappa(G) \leq \delta(G)$ so $\kappa(K_5) \leq 4$. We know that $\kappa(K_n) = n - 1$ so we get that $\kappa(K_5) = 4 > 3$. This is a counterexample since K_5 is 3-connected but it has vertex-connectivity 4. The statement "Every 3-connected graph has connectivity 3." is false.
(c) A graph is k-edge-connected if every disconnecting set has at least k edges. Consider a k-vertex-connected graph G, we know that $\kappa(G) \leq \kappa'(G)$ from Whitney's theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity $\kappa'(G)$ is the minimum size of a disconnecting set and it is at least k so G is k-edge-connected. The statement "Every k-connected graph is k-edge-connected." is true.
(d) Consider a k-edge-connected graph G with vertex-connectivity $\kappa(G)$ and edge-connectivity $\kappa'(G)$. We have $\kappa(G) \leq \kappa'(G)$ and $k \leq \kappa'(G)$. If $k > \kappa(G)$, then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney's theorem still holds so G is not k-vertex-connected.
The statement "Every k-edge-connected graph is k-connected." is false. ∎

Problem 2. Excercise 4.1.8(b)

Solution. Let G be the graph on the right, we see that G is a 4-regular graph so the minimum degree of G is 4. Since every vertex is connected to four other vertices, we must remove all four incidental edges of an arbitrary vertex at minimum to disconnect the graph. Similarly, we must remove all four adjacent vertices of an arbitrary vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of G is 4 and the vertex-connectivity of G is 4. The vertex-connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, and the minimum degree $\delta(G)$ are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

Problem 3. Excercise 4.1.11

Solution. We can show the following by Whitney's theorem

$$\begin{aligned} &\kappa(G) \leq \kappa'(G) \leq \delta(G) \leq \Delta(G) \leq 3 \\ &\kappa(G) \leq \kappa'(G) \leq 3 \end{aligned}$$

It is proved that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$. ■

Chapter 5 Coloring of Graphs

Section 5.1 Vertex Coloring and Upper Bounds

Section 5.2 Structure of k-chromatic Graphs

Section 5.3 Enumerative Aspects