

# MTH 4140 Graph Theory

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## Homework 4

### Introduction to Graph Theory

#### Chapter 4 Connectivity and Paths

##### Section 4.1 Cuts and Connectivity

###### Problem 1. Exercise 4.1.1

*Solution.* (a) A graph is  $k$ -connected if its connectivity is at least  $k$ . If a graph  $G$  is 2-connected, then its connectivity is at least 2. Given  $G$  has connectivity 4 so we have  $\kappa(G) = 4 \geq 2$  which is true. The statement “Every graph with connectivity 4 is 2-connected.” is true.

(b) A 3-connected graph  $G$  has connectivity at least 3 so  $\kappa(G) \geq 3$ . Consider the graph  $K_5$ , we know that  $\kappa(G) \leq \delta(G)$  so  $\kappa(K_5) \leq 4$ . We know that  $\kappa(K_n) = n - 1$  so we get that  $\kappa(K_5) = 4 > 3$ . This is a counterexample since  $K_5$  is 3-connected but it has connectivity 4. The statement “Every 3-connected graph has connectivity 3.” is false.

(c) A graph is  $k$ -edge-connected if every disconnecting set has at least  $k$  edges. Consider a  $k$ -connected graph  $G$ , we know that  $\kappa(G) \leq \kappa'(G)$  from Whitney’s theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity  $\kappa'(G)$  is the minimum size of a disconnecting set and it is at least  $k$  so  $G$  is  $k$ -edge-connected. The statement “Every  $k$ -connected graph is  $k$ -edge-connected.” is true.

(d) Consider a  $k$ -edge-connected graph  $G$  with connectivity  $\kappa(G)$  and edge-connectivity  $\kappa'(G)$ . We have  $\kappa(G) \leq \kappa'(G)$  and  $k \leq \kappa'(G)$ . If  $k > \kappa(G)$ , then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney’s theorem still holds so  $G$  is not  $k$ -connected.

The statement “Every  $k$ -edge-connected graph is  $k$ -connected.” is false. ■

###### Problem 2. Exercise 4.1.4

*Solution.* It is proved a graph  $G$  is  $k$ -connected if  $G + K_r$  is  $k + r$ -connected. ■

**Problem 3.** Exercise 4.1.8(b)

*Solution.* Let  $G$  be the graph on the right, we see that  $G$  is a 4-regular graph so the minimum degree of  $G$  is 4. Since every vertex is connected to four other vertices, we must remove 4 incidental edges of any vertex at minimum to disconnect the graph. Similarly, we must remove 4 adjacent vertices of any vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of  $G$  is 4 and the connectivity of  $G$  is 4. The connectivity  $\kappa(G)$ , edge-connectivity  $\kappa'(G)$ , and the minimum degree  $\delta(G)$  are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

**Problem 4.** Exercise 4.1.11

*Solution.* It is proved that  $\kappa'(G) = \kappa(G)$  when  $G$  is a simple graph with  $\Delta(G) \leq 3$ . ■

## Chapter 5 Coloring of Graphs

### Section 5.1 Vertex Coloring and Upper Bounds

**Problem 5.** Exercise 5.1.14

*Solution.* We can use the same color to color the vertices in the maximum independent set. Then we can use different colors to color the rest of the vertices. Therefore, we get a proper color of the graph. It is proved that for every graph  $G$ ,  $\chi(G) \leq n(G) - \alpha(G) + 1$  is true. ■

### Section 5.2 Structure of $k$ -chromatic Graphs

**Problem 6.** Exercise 5.2.1

*Solution.* Suppose  $G$  is not a complete graph such that  $\chi(G - x - y) = \chi(G) - 2$ . Let  $x, y$  be two non-adjacent vertices so we can color them with the same color. It follows that

$$\chi(G) = \chi(G - x - y) + 1$$

which implies that

$$\chi(G - x - y) = \chi(G) - 1$$

Hence, there is a contradiction thus every pair of distinct vertices in  $G$  must be adjacent which implies that  $G$  is a complete graph. Therefore, if  $\chi(G - x - y) = \chi(G) - 2$ , then  $G$  is a complete graph. ■

**Problem 7.** Exercise 5.2.22

*Solution.* ■

## Section 5.3 Enumerative Aspects

### Problem 8. Exercise 5.3.3

*Solution.* It is obvious that if  $k = 2$ , then we have

$$k^4 - 4k^3 + 3k^2 = (2)^4 - 4(2)^3 + 3(2)^2 = 16 - 32 + 12 = -4$$

The polynomial cannot count the proper 2-colorings of any graph. Therefore, it is proved that  $k^4 - 4k^3 + 3k^2$  is not a chromatic polynomial. ■

### Problem 9. Exercise 5.3.4(a)

*Solution.* We can use mathematical induction to prove that the formula is true. The least vertices a simple cycle has is 3. If  $n = 3$ , then  $\chi(C_3; k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k$  by counting the number of ways to choose the proper color of each vertex. The polynomial gives  $(k-1)^3 + (-1)^3(k-1) = k^3 - 3k^2 + 2k$  when  $n = 3$  so it is true. For every  $n \geq 3$ , assume that  $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$  is true. If we remove an edge from a cycle of  $n$  vertices, then we get a path of  $n$  vertices. If we contract an edge from a cycle of  $n$  vertices, then we get a cycle of  $n-1$  vertices. The chromatic recurrence gives  $\chi(G; k) = \chi(G-e; k) - \chi(G \cdot e; k)$  and the chromatic polynomial of a tree  $T$  with  $n$  vertices is  $\chi(T; k) = k(k-1)^{n-1}$ . The chromatic polynomial of  $C_{n+1}$  is

$$\begin{aligned}\chi(C_{n+1}; k) &= \chi(C_{n+1} - e; k) - \chi(C_{n+1} \cdot e; k) = \chi(P_{n+1}; k) - \chi(C_n; k) \\ &= k(k-1)^{(n+1)-1} - [(k-1)^n + (-1)^n(k-1)] \\ &= k(k-1)^n - (k-1)^n + (-1)^{n+1}(k-1) \\ &= (k-1)(k-1)^n + (-1)^{n+1}(k-1) \\ &= (k-1)^{n+1} + (-1)^{n+1}(k-1)\end{aligned}$$

which implies that the formula is true. It is proved that  $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$ . ■

### Problem 10. Exercise 5.3.5

*Solution.* ■

### Problem 11. Exercise 5.3.11

*Solution.* ■