MTH 4140 Graph Theory

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Homework 4

Introduction to Graph Theory

Chapter 4 Connectivity and Paths

Section 4.1 Cuts and Connectivity

Problem 1. Excercise 4.1.1

Solution. (a) A graph is k-connected if its connectivity is at least k. If a graph G is 2-connected, then its connectivity is at least 2. Given G has connectivity 4 so we have $\kappa(G) = 4 \geq 2$ which is true. The statement "Every graph with connectivity 4 is 2-connected." is true.
(b) A 3-connected graph G has connectivity at least 3 so $\kappa(G) \geq 3$. Consider the graph K_5, we know that $\kappa(G) \leq \delta(G)$ so $\kappa(K_5) \leq 4$. We know that $\kappa(K_n) = n - 1$ so we get that $\kappa(K_5) = 4 > 3$. This is a counterexample since K_5 is 3-connected but it has connectivity 4. The statement "Every 3-connected graph has connectivity 3." is false.
(c) A graph is k-edge-connected if every disconnecting set has at least k edges. Consider a k-connected graph G, we know that $\kappa(G) \leq \kappa'(G)$ from Whitney's theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity $\kappa'(G)$ is the minimum size of a disconnecting set and it is at least k so G is k-edge-connected. The statement "Every k-connected graph is k-edge-connected." is true.
(d) Consider a k-edge-connected graph G with connectivity $\kappa(G)$ and
edge-connectivity $\kappa'(G)$. We have $\kappa(G) \leq \kappa'(G)$ and $k \leq \kappa'(G)$. If $k > \kappa(G)$, then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney's theorem still holds so G is not k-connected.
The statement "Every k-edge-connected graph is k-connected." is false. ∎

Problem 2. Excercise 4.1.8(b)

Solution. Let G be the graph on the right, we see that G is a 4-regular graph so the minimum degree of G is 4. Since every vertex is connected to four other vertices, we must remove 4 incidental edges of any vertex at minimum to disconnect the graph. Similarly, we must remove 4 adjacent vertices of any vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of G is 4 and the connectivity of G is 4. The connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, and the minimum degree $\delta(G)$ are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

Problem 3. Excercise 4.1.11

Solution. It is proved that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$. ■

Chapter 5 Coloring of Graphs

Section 5.1 Vertex Coloring and Upper Bounds

Problem 4. Excercise 5.1.14

Solution. We can use the same color to color the vertices in the maximum independent set. Then we can use different colors to color the rest of the vertices. Therefore, we get a proper color of the graph. It is proved that for every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$ is true. ■

Section 5.2 Structure of k-chromatic Graphs

Problem 5. Excercise 5.2.1

Solution. Suppose G is not a complete graph such that $\chi(G - x - y) = \chi(G) - 2$. Let x, y be two non-adjacent vertices so we can color them with the same color. It follows that

$$\chi(G) = \chi(G - x - y) + 1$$

which implies that

$$\chi(G - x - y) = \chi(G) - 1$$

Hence, there is a contradiction thus every pair of distinct vertices in G must be adjacent which implies that G is a complete graph. Therefore, if $\chi(G - x - y) = \chi(G) - 2$, then G is a complete graph. ■

Section 5.3 Enumerative Aspects

Problem 6. Excercise 5.3.3

Solution. ■

Problem 7. Excercise 5.3.4(a)

Solution. ■