MTH 4140 Homework 1

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Problem 1

Section 1.1 Problem 9

Solution. Let the graph on the left be G and the graph on the right be H. It is equivalent to show that $G \cong \overline{H}$. If we label every vertex in G and H using the same set of labels then two vertices are adjacent in G if and only if they are adjacent in $\overline{H}$. Therefore, it is proved that $G \cong \overline{H}$ so $\overline{G} \cong H$. ■

Problem 2

Section 1.1 Problem 10

Solution. We have a disconnected graph G and its complement $\overline{G}$. Let u, v be two vertices of different components in G s.t. there is no path from u to v. Then u, v are adjacent in $\overline{G}$ and all other vertices in $\overline{G}$ are adjacent to one of u, v or both since u, v have no common neighbors. There is a path for any two vertices in $\overline{G}$ using u, v. Therefore, it is proved that the complement of a simple disconnected graph must be connected. ■

Problem 3

Section 1.1 Problem 17

Solution. We know that $G \cong H \iff \overline{G} \cong \overline{H}$ so it is equivalent to count all 2-regular graphs on 7 vertices. Only the graph C_7 and the graph with only C_3, C_4 satisfy this. Therefore, the number of isomorphic classes of simple 7-vertex 4-regular graphs is 2. ■

Problem 4

Section 1.1 Problem 25

Solution. Suppose the Peterson graph has C_7 then there are 3 remaining vertices outside of C_7. The Peterson graph is 3-regular so each of the 7 vertices on C_7 has one extra edge outside of C_7. The Peterson graph has girth 5 so the 7 extra edges have to have one of the 3 remaining vertices as an endpoint. By the pigeonhole principle, at least 3 of the extra edges end on the same vertex. WLOG, we will have a C_3 but the Peterson graph has girth 5 so there is a contradiction. Therefore, it is proved that the Peterson graph does not have a cycle of length 7. ■

Problem 5

Section 1.1 Problem 26

Solution. If G has girth 4 then there are two adjacent vertices u, v with no common neighbors. Since G is k-regular and u, v each has degree $k-1$, the sum of the degrees is at least $k-1+1+1+k-1=2k$. Therefore, it is proved that G has at least $2k$ vertices. If each vertex in the independent set $\{x_1, \ldots, x_k\}$ is adjacent to all vertices of the independent set $\{y_1, \ldots, y_k\}$ then we have exactly $2k$ vertices. All such graphs with excatly $2k$ vertices must be complete bipartite graphs $K_{n,n}$ where $n = k$. ■