MTH 4140 Graph Theory

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Homework 4

Introduction to Graph Theory

Chapter 4 Connectivity and Paths

Section 4.1 Cuts and Connectivity

Problem 1. Excercise 4.1.1

Solution. (a) A graph is k-connected if its connectivity is at least k. If a graph G is 2-connected, then its connectivity is at least 2. Given G has connectivity 4 so we have $\kappa(G) = 4 \geq 2$ which is true. The statement "Every graph with connectivity 4 is 2-connected." is true.
(b) A 3-connected graph G has connectivity at least 3 so $\kappa(G) \geq 3$. Consider the graph K_5, we know that $\kappa(G) \leq \delta(G)$ so $\kappa(K_5) \leq 4$. We know that $\kappa(K_n) = n - 1$ so we get that $\kappa(K_5) = 4 > 3$. This is a counterexample since K_5 is 3-connected but it has connectivity 4. The statement "Every 3-connected graph has connectivity 3." is false.
(c) A graph is k-edge-connected if every disconnecting set has at least k edges. Consider a k-connected graph G, we know that $\kappa(G) \leq \kappa'(G)$ from Whitney's theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity $\kappa'(G)$ is the minimum size of a disconnecting set and it is at least k so G is k-edge-connected. The statement "Every k-connected graph is k-edge-connected." is true.
(d) Consider a k-edge-connected graph G with connectivity $\kappa(G)$ and
edge-connectivity $\kappa'(G)$. We have $\kappa(G) \leq \kappa'(G)$ and $k \leq \kappa'(G)$. If $k > \kappa(G)$, then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney's theorem still holds so G is not k-connected.
The statement "Every k-edge-connected graph is k-connected." is false. ■

Problem 2. Excercise 4.1.8(b)

Solution. Let G be the graph on the right, we see that G is a 4-regular graph so the minimum degree of G is 4. Since every vertex is connected to four other vertices, we must remove 4 incidental edges of any vertex at minimum to disconnect the graph. Similarly, we must remove 4 adjacent vertices of any vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of G is 4 and the connectivity of G is 4. The connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, and the minimum degree $\delta(G)$ are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

Problem 3. Excercise 4.1.11

Solution. Let S be a minimum vertex cut and since $\kappa(G) \leq \kappa'(G)$ by Whitney's theorem, we only need an edge cut of size $|S| = \kappa(G)$. Let H_1 and H_2 be two components of $G - S$. Since S is minimum, every vertex v in S has a neighbor in H_1 and a neighbor in H_2. Given $\Delta(G) \leq 3$, v cannot have two neighbors in H_1 and two neighbors in H_2. For each $v \in S$, delete the incidental edge in the component where v has only one neighbor. If all incidnetal edges to both components satisfy the condition, then we simply delete all incidental edges to one of the components. Thus, we get a disconnecting set of edges S' with minimum size $|S'| = \kappa'(G) = |S| = \kappa(G)$. It is proved that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$. ■

Chapter 5 Coloring of Graphs

Section 5.1 Vertex Coloring and Upper Bounds

Problem 4. Excercise 5.1.14

Solution. We can use the same color for vertices in the maximum independent set. Then we can use different colors for rest of the vertices. Therefore, we get a proper coloring of the graph. It is proved that for every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$ is true. ■

Problem 5. Excercise 5.1.20

Solution. If G has no odd cycles, then it is bipartite so $\chi(G) \leq 2$. Assume that G has odd cycles and consider the contrapositive such that if $\chi(G) \geq 6$, then the odd cycles of G are disjoint. We can have an optimal coloring where the subgraph induced by vertices with colors 1,2,3 must have an odd cycle since they cannot be bipartite which makes the subgraph 2-colorable. Similarly, the subgraph induced by vertices with colors 4,5,6 must have an odd cycle and the two cycles are disjoint. It is proved that if G is a graph whose odd cycles are pairwise intersecting, then $\chi(G) \leq 5$. ■

Problem 6. 5.1.41

Solution. If $n = 1$, then $\chi(G) = \chi(\overline{G}) = 1$ so we have $1 + 1 = 2$ and the inequality is true. Assume that the inequality is true for $n \geq 1$, for a graph G with $n+1$ vertices, let $G' = G - v$ for $v \in V(G)$. We get $\chi(G') + \chi(\overline{G'}) \leq n(G') + 1$. When we add back v to get G, then the chromatic number of each graph increases by at most 1 so $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$ unless they both increase. If they both increase, then v must have $n - 1$ neighbors in total in the two graphs so $\chi(G') + \chi(\overline{G'}) \leq n(G') - 1$. When we add back v, it follows that $\chi(G') + \chi(\overline{G'}) \leq n(G') + 1$. It is proved that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. ■

Problem 7. Excercise 5.1.48

Solution. By Brook's theorem, we have $\chi(G) \leq \Delta(G) \leq 3$ so G is 3-colorable. Consider a proper colorings with colors 1,2,3 where the number of vertices with color 1 is the least. By the pigeonhole principle, there are at most $\left\lceil \frac{n}{3} \right\rceil$ vertices with color 1. Each vertex v with color 1 has at most three neighbors so by the pigeonhole principle, other vertices with different colors appears at most once in $N(v)$. If color 2 appears at most once, then we delete the edge from v to its neighbor with color 2 and change the color of v to color 2. Similarly, we do the same for color 3. This algorithm deletes at most $\left\lceil \frac{n}{3} \right\rceil$ vertices and we get a bipartite subgraph with two colors. It is proved that if we have a simple graph G with n vertices, m edges, $\Delta(G) \leq 3$, and G has no component of a complete graph of 4 vertices, then G contains a bipartite subgraph with at least $m - \frac{n}{3}$ edges. ■

Section 5.2 Structure of k-chromatic Graphs

Problem 8. Excercise 5.2.1

Solution. Suppose G is not a complete graph such that $\chi(G - x - y) = \chi(G) - 2$. Let x, y be two non-adjacent vertices so we can color them with the same color. It follows that

$$\chi(G) = \chi(G - x - y) + 1$$

which implies that

$$\chi(G - x - y) = \chi(G) - 1$$

Hence, there is a contradiction thus every pair of distinct vertices in G must be adjacent which implies that G is a complete graph. Therefore, if $\chi(G - x - y) = \chi(G) - 2$, then G is a complete graph. ■

Section 5.3 Enumerative Aspects

Problem 9. Excercise 5.3.3

Solution. It is obvious that if $k = 2$, then we have

$$k^4 - 4k^3 + 3k^2 = (2)^4 - 4(2)^3 + 3(2)^2 = 16 - 32 + 12 = -4$$

The polynomial cannot count the proper 2-colorings of any graph. Therefore, it is proved that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial. ■

Problem 10. Excercise 5.3.4(a)

Solution. We can use mathematical induction to prove that the formula is true. The least vertices a simple cycle has is 3. If $n = 3$, then $\chi(C_3; k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k$ by counting the number of ways to choose the proper color of each vertex. The polynomial gives $(k-1)^3 + (-1)^3(k-1) = k^3 - 3k^2 + 2k$ when $n = 3$ so it is true. For every $n \geq 3$, assume that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$ is true. If we remove an edge from a cycle of n vertices, then we get a path of n vertices. If we contract an edge from a cycle of n vertices, then we get a cycle of $n-1$ vertices. The chromatic recurrence gives $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$ and the chromatic polynomial of a tree T with n vertices is $\chi(T; k) = k(k-1)^{n-1}$. The chromatic polynomial of C_{n+1} is

$$\begin{aligned}
\chi(C_{n+1}; k) &= \chi(C_{n+1} - e; k) - \chi(C_{n+1} \cdot e; k) = \chi(P_{n+1}; k) - \chi(C_n; k) \\
&= k(k-1)^{(n+1)-1} - [(k-1)^n + (-1)^n(k-1)] \\
&= k(k-1)^n - (k-1)^n + (-1)^{n+1}(k-1) \\
&= (k-1)(k-1)^n + (-1)^{n+1}(k-1) \\
&= (k-1)^{n+1} + (-1)^{n+1}(k-1)
\end{aligned}$$

which implies that the formula is true. It is proved that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$. ■

Problem 11. Excercise 5.3.5

Solution. If $n = 1$, then we have a path of 2 vertices so $\chi(G_1; k) = k(k-1)$ and the polynomial gives

$$(k^2 - 3k + 3)^{n-1}k(k-1) = (k^2 - 3k + 3)^{1-1}k(k-1) = k(k-1)$$

which is true. Assume that the polynomial is true for $n \geq 1$. Observe that $\chi(G_n; k) = (k^2 - 3k + 3)\chi(G_{n-1}; k)$ by counting. Therefore, we have

$$\begin{aligned}
\chi(G_{n+1}; k) &= (k^2 - 3k + 3)\chi(G_n; k) = (k^2 - 3k + 3)(k^2 - 3k + 3)^{n-1}k(k-1) \\
&= (k^2 - 3k + 3)^n k(k-1)
\end{aligned}$$

It is proved that $\chi(G_n; k) = (k^2 - 3k + 3)^{n-1}k(k-1)$. ■

Problem 12. Excercise 5.3.11

Solution. The sum of the coefficients of a chromatic polynomial implies that $k = 1$ so the sum is the number of proper colorings with exactly one color which must be 0 if the graph has edges. It is proved that the sum of the coefficients of $\chi(G; k)$ is 0 unless G has no edges. ■