MTH 4140 Graph Theory

Yaohui Wu

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Homework 4

Introduction to Graph Theory

Chapter 4 Connectivity and Paths

Section 4.1 Cuts and Connectivity

Problem 1. Excercise 4.1.1

Solution. (a) A graph is k-connected if its connectivity is at least k. If a graph G is 2-connected, then its connectivity is at least 2. Given G has connectivity 4 so we have $\kappa(G) = 4 \geq 2$ which is true. The statement "Every graph with connectivity 4 is 2-connected." is true.
(b) A 3-connected graph G has connectivity at least 3 so $\kappa(G) \geq 3$. Consider the graph K_5, we know that $\kappa(G) \leq \delta(G)$ so $\kappa(K_5) \leq 4$. We know that $\kappa(K_n) = n - 1$ so we get that $\kappa(K_5) = 4 > 3$. This is a counterexample since K_5 is 3-connected but it has connectivity 4. The statement "Every 3-connected graph has connectivity 3." is false.
(c) A graph is k-edge-connected if every disconnecting set has at least k edges. Consider a k-connected graph G, we know that $\kappa(G) \leq \kappa'(G)$ from Whitney's theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity $\kappa'(G)$ is the minimum size of a disconnecting set and it is at least k so G is k-edge-connected. The statement "Every k-connected graph is k-edge-connected." is true.
(d) Consider a k-edge-connected graph G with connectivity $\kappa(G)$ and
edge-connectivity $\kappa'(G)$. We have $\kappa(G) \leq \kappa'(G)$ and $k \leq \kappa'(G)$. If $k > \kappa(G)$, then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney's theorem still holds so G is not k-connected.
The statement "Every k-edge-connected graph is k-connected." is false. ■

Problem 2. Excercise 4.1.4

Solution. It is proved a graph G is k-connected if $G + K_r$ is $k + r$-connected. ■

Problem 3. Excercise 4.1.8(b)

Solution. Let G be the graph on the right, we see that G is a 4-regular graph so the minimum degree of G is 4. Since every vertex is connected to four other vertices, we must remove 4 incidental edges of any vertex at minimum to disconnect the graph. Similarly, we must remove 4 adjacent vertices of any vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of G is 4 and the connectivity of G is 4. The connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, and the minimum degree $\delta(G)$ are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

Problem 4. Excercise 4.1.11

Solution. Let S be a minimum vertex cut and since $\kappa(G) \leq \kappa'(G)$ by Whitney's theorem, we only need an edge cut of size $|S| = \kappa(G)$. Let H_1 and H_2 be two components of $G - S$. Since S is minimum, every vertex v in S has a neighbor in H_1 and a neighbor in H_2. Given $\Delta(G) \leq 3$, v cannot have two neighbors in H_1 and two neighbors in H_2. For each $v \in S$, delete the incidental edge in the component where v has only one neighbor. If all incidnetal edges to both components satisfy the condition, then we simply delete all incidental edges to one of the components. Thus, we get a disconnecting set of edges S' with minimum size $|S'| = \kappa'(G) = |S| = \kappa(G)$. It is proved that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$. ■

Chapter 5 Coloring of Graphs

Section 5.1 Vertex Coloring and Upper Bounds

Problem 5. Excercise 5.1.14

Solution. We can use the same color to color the vertices in the maximum independent set. Then we can use different colors to color the rest of the vertices. Therefore, we get a proper color of the graph. It is proved that for every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$ is true. ■

Section 5.2 Structure of k-chromatic Graphs

Problem 6. Excercise 5.2.1

Solution. Suppose G is not a complete graph such that $\chi(G - x - y) = \chi(G) - 2$. Let x, y be two non-adjacent vertices so we can color them with the same color. It follows that

$$\chi(G) = \chi(G - x - y) + 1$$

which implies that

$$\chi(G - x - y) = \chi(G) - 1$$

Hence, there is a contradiction thus every pair of distinct vertices in G must be adjacent which implies that G is a complete graph. Therefore, if $\chi(G - x - y) = \chi(G) - 2$, then G is a complete graph. ■

Problem 7. Excercise 5.2.22

Solution. ■

Section 5.3 Enumerative Aspects

Problem 8. Excercise 5.3.3

Solution. It is obvious that if $k = 2$, then we have

$$k^4 - 4k^3 + 3k^2 = (2)^4 - 4(2)^3 + 3(2)^2 = 16 - 32 + 12 = -4$$

The polynomial cannot count the proper 2-colorings of any graph. Therefore, it is proved that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial. ■

Problem 9. Excercise 5.3.4(a)

Solution. We can use mathematical induction to prove that the formula is true. The least vertices a simple cycle has is 3. If $n = 3$, then $\chi(C_3; k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k$ by counting the number of ways to choose the proper color of each vertex. The polynomial gives $(k-1)^3 + (-1)^3(k-1) = k^3 - 3k^2 + 2k$ when $n = 3$ so it is true. For every $n \geq 3$, assume that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$ is true. If we remove an edge from a cycle of n vertices, then we get a path of n vertices. If we contract an edge from a cycle of n vertices, then we get a cycle of $n-1$ vertices. The chromatic recurrence gives $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$ and the chromatic polynomial of a tree T with n vertices is $\chi(T; k) = k(k-1)^{n-1}$. The chromatic polynomial of C_{n+1} is

$$\begin{aligned}
\chi(C_{n+1}; k) &= \chi(C_{n+1} - e; k) - \chi(C_{n+1} \cdot e; k) = \chi(P_{n+1}; k) - \chi(C_n; k) \\
&= k(k-1)^{(n+1)-1} - [(k-1)^n + (-1)^n(k-1)] \\
&= k(k-1)^n - (k-1)^n + (-1)^{n+1}(k-1) \\
&= (k-1)(k-1)^n + (-1)^{n+1}(k-1) \\
&= (k-1)^{n+1} + (-1)^{n+1}(k-1)
\end{aligned}$$

which implies that the formula is true. It is proved that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$. ■

Problem 10. Excercise 5.3.5

Solution. If $n = 1$, then we have a path of 2 vertices so $\chi(G_1; k) = k(k-1)$ and the polynomial gives

$$(k^2 - 3k + 3)^{n-1}k(k-1) = (k^2 - 3k + 3)^{1-1}k(k-1) = k(k-1)$$

which is true. Assume that the polynomial is true for $n \geq 1$. Observe that $\chi(G_n; k) = (k^2 - 3k + 3)\chi(G_{n-1}; k)$ by counting. Therefore, we have

$$\begin{aligned}
\chi(G_{n+1}; k) &= (k^2 - 3k + 3)\chi(G_n; k) = (k^2 - 3k + 3)(k^2 - 3k + 3)^{n-1}k(k-1) \\
&= (k^2 - 3k + 3)^n k(k-1)
\end{aligned}$$

It is proved that $\chi(G_n; k) = (k^2 - 3k + 3)^{n-1}k(k-1)$. ■

Problem 11. Excercise 5.3.11

Solution. The sum of the coefficients of a chromatic polynomial implies that $k = 1$ so the sum is the number of proper colorings with exactly one color which must be 0 if the graph has edges. It is proved that the sum of the coefficients of $\chi(G; k)$ is 0 unless G has no edges. $\blacksquare$