MTH 4140 Graph Theory

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Homework 4

Introduction to Graph Theory

Chapter 4 Connectivity and Paths

Section 4.1 Cuts and Connectivity

Problem 1. Excercise 4.1.1

Solution. (a) A graph is k-connected if its connectivity is at least k. If a graph G is 2-connected, then its connectivity is at least 2. Given G has connectivity 4 so we have $\kappa(G) = 4 \geq 2$ which is true. The statement "Every graph with connectivity 4 is 2-connected." is true.
(b) A 3-connected graph G has connectivity at least 3 so $\kappa(G) \geq 3$. Consider the graph K_5, we know that $\kappa(G) \leq \delta(G)$ so $\kappa(K_5) \leq 4$. We know that $\kappa(K_n) = n - 1$ so we get that $\kappa(K_5) = 4 > 3$. This is a counterexample since K_5 is 3-connected but it has connectivity 4. The statement "Every 3-connected graph has connectivity 3." is false.
(c) A graph is k-edge-connected if every disconnecting set has at least k edges. Consider a k-connected graph G, we know that $\kappa(G) \leq \kappa'(G)$ from Whitney's theorem. We can deduce that

$$k \leq \kappa(G) \leq \kappa'(G)$$

The edge-connectivity $\kappa'(G)$ is the minimum size of a disconnecting set and it is at least k so G is k-edge-connected. The statement "Every k-connected graph is k-edge-connected." is true.
(d) Consider a k-edge-connected graph G with connectivity $\kappa(G)$ and
edge-connectivity $\kappa'(G)$. We have $\kappa(G) \leq \kappa'(G)$ and $k \leq \kappa'(G)$. If $k > \kappa(G)$, then we have

$$\kappa(G) < k \leq \kappa'(G)$$

and Whitney's theorem still holds so G is not k-connected.
The statement "Every k-edge-connected graph is k-connected." is false. ■

Problem 2. Excercise 4.1.8(b)

Solution. Let G be the graph on the right, we see that G is a 4-regular graph so the minimum degree of G is 4. Since every vertex is connected to four other vertices, we must remove 4 incidental edges of any vertex at minimum to disconnect the graph. Similarly, we must remove 4 adjacent vertices of any vertex at minimum to disconnect the graph. Therefore, the edge-connectivity of G is 4 and the connectivity of G is 4. The connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, and the minimum degree $\delta(G)$ are

$$\kappa(G) = \kappa'(G) = \delta(G) = 4$$

which follows immediately from Whitney's theorem. ■

Problem 3. Excercise 4.1.11

Solution. We can show the following by Whitney's theorem

$$1 \leq \kappa(G) \leq \kappa'(G) \leq \delta(G) \leq \Delta(G) \leq 3$$

If $\delta(G) = 1$, then we have

$$1 \leq \kappa(G) \leq \kappa'(G) \leq 1$$

which implies that

$$\kappa(G) = \kappa'(G) = 1$$

If $\delta(G) = 2$, then we have

$$1 \leq \kappa(G) \leq \kappa'(G) \leq \delta(G) = 2 \leq \Delta(G) \leq 3$$

If $\Delta(G) = 2$ so G is a cycle, then it is 2-regular. We must remove 2 incidental edges or 2 adjacent vertices of any vertex at minimum to disconnect a cycle. If $\Delta(G) = 3$ so G is not a cycle. It follows that

$$\kappa(G) = \kappa'(G)$$

when $\delta(G) = 2$. If $\delta(G) = 3$, then we have

$$1 \leq \kappa(G) \leq \kappa'(G) \leq \delta(G) = \Delta(G) = 3$$

Since G is 3-regular, we must remove 3 vertices or 3 edges at minimum to disconnect G so

$$\kappa(G) = \kappa'(G)$$

when $\delta(G) = 3$. It is proved that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$. ■

Chapter 5 Coloring of Graphs

Section 5.1 Vertex Coloring and Upper Bounds

Section 5.2 Structure of k-chromatic Graphs

Section 5.3 Enumerative Aspects