MTH 4320 Homework 3

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Problem 1

Solution. Let the graph on the left be G and the graph of the drawing be H. Notice that the set of vertices $\{a, c, f, g, h\}$ in G is an independent set, where no two vertices in the set are adjacent. We can label the vertices in H as $\{a, c, f, h\}$ on the left side and $\{g, e, d, b\}$ on the right side from top to bottom respectively. Then we observe that vertex a is adjacent to the vertices $\{b, d, e\}$ denoted by $a \sim \{b, d, e\}$. Similarly, we observe that $c \sim \{b, d, g\}$, $f \sim \{b, e, g\}$, and $h \sim \{d, e, g\}$. Hence we have that any two vertices with the same labels are adjacent in G if and only if they are adjacent in H. Therefore, it is shown that G and H are identical or $G \cong H$, G is isomorphic to H. ■

Problem 2

Solution. The algorithm is

1. Label the vertices in graph G as $V = \{v_1, \ldots, v_n\}$. This takes $O(|V|)$ time.
2. For every vertex v_i in V starting from v_1
    - Assign a direction to every edge incidental to v_i that has not been assigned a direction.
    - If the other endpoint has a greater value of i then we assign the edge to that direction else we assign it to the opposite direction.
3. Labeling all of the edges takes $O(|E|)$ time.

The algorithm make sure that the directions of the edges of every vertex go to the adjacent vertices with a label of higher value. Therefore, there must not be any cycles in the directed graph. The running time of the algorithm is $O(|V|) + O(|E|) = O(|V| + |E|)$. ■

Problem 3

Solution. For a binary sequence of length $k = 3$, G can be a cube with 8 vertices being the corners having edges connecting the other 3 adjacent corners. In general, for any positive

integer n we have that $G(n)$ is a connected graph with only one component. We can prove this proposition using proof by induction. Let the statement be $P(k)$ for all positive integer k where k is the length of the binary sequence.
Base case: For $k = 1$ we have two vertices 0 and 1 connected by an edge so $P(1)$ is true.
Induction steps: Assume that $P(k)$ is true so $G(k)$ is connected. When we add one more bit to the binary sequence of $G(k)$, we can choose any position and insert the same bit to maintain the graph $G(k)$. Then we can change the new bit of each new binary sequence to create some new vertices connected to the existing vertices of $G(k)$ to form the graph $G(k+1)$. Hence $G(k+1)$ is constructed from $G(k)$ and $G(k)$ is connected so $G(k+1)$ is connected and $P(k+1)$ is true.
Therefore, it is proved by mathematical induction that for any positive integer n the graph $G(n)$ is connected so it has only one component. ■