Calculus

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Introduction

Calculus is the mathematical study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Contents

1 Functions and Limits

1.1 The Limit of a Function

Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E.

Definition 1.1.2. A function f is **injective** (or **one-to-one**) if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

Definition 1.1.3. A function f is **surjective** (or **onto**) if for all y in range Y, there exists an x in domain X such that $f(x) = y$.

Definition 1.1.4. A function f is **bijective** if f is injective and surjective.

Definition 1.1.5. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in B.

Intuitive Definition of a Limit

Suppose $f(x)$ is defined near the number a. (This means that $f(x)$ is defined on some open interval that contains the number a, except possibly at a itself.)

Definition 1.1.6. We write

$$\lim_{x \to a} f(x) = L$$

and say that the **limit** of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but $x \neq a$.

An alternative notation for the limit is $f(x) \to L$ as $x \to a$.

One-Sided Limits

Definition 1.1.7. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

Definition 1.1.8. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

Theorem 1.1.1. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The limit exists if and only if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a are equal to L, otherwise the limit does not exist.

1.2 The Precise Definition of a Limit

Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2.1. We say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.1. Prove that $\lim\limits_{x \to 3}(4x - 5) = 7$.

Solution. Let $\varepsilon > 0$ be a given positive number. We want to find a number δ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$. Note that $4|x - 3| < \varepsilon \iff |x - 3| < \varepsilon/4$. Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \to 3}(4x - 5) = 7$$

■

Problem 1.2.2. Prove that $\lim\limits_{x \to 3} x^2 = 9$.

Solution. Let ε be a given positive number. We want to find a number δ such that

$$0 < |x - 3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x + 3|\,|x - 3| < \varepsilon$$

Let C be a positive constant such that

$$|x + 3|\,|x - 3| < C\,|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x - 3| < 1$. Then we have $|x + 3| < 7$, and so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x - 3| < \delta$, then

$$|x^2 - 9| = |x + 3|\,|x - 3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim\limits_{x \to 3} x^2 = 9$. ■

Definition 1.2.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.2.3.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution. Let ε be a given positive number We want to find a number δ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

But $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$. If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$ so $|\sqrt{x} - 0| < \varepsilon$. This shows that $\lim_{x \to 0^+} \sqrt{x} = 0$. ■

1.3 Calculating Limits Using the Limit Laws

We have the following properties of limits called the **limit laws** to calculate limits. Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

exist. Then

1. Sum Law: $\lim_{x \to a} \Big[f(x) + g(x)\Big] = L + M$
2. Difference Law: $\lim_{x \to a} \Big[f(x) - g(x)\Big] = L - M$
3. Constant Multiple Law: $\lim_{x \to a} \Big[cf(x)\Big] = cL$
4. Product Law: $\lim_{x \to a} \Big[f(x)g(x)\Big] = LM$
5. Quotient Law: $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.
6. Power Law: $\lim_{x \to a} \Big[f(x)\Big]^n = \Big[\lim_{x \to a} f(x)\Big]^n$ where n is a positive integer.
7. Root Law: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim_{x \to a} f(x) > 0$.

8. $\lim_{x \to a} c = c$

9. $\lim_{x \to a} x = a$

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer.

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

Proof. Proof of limit law 8: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. It is proved that $\lim_{x \to a} c = c$. ■

Proof. Proof of limit law 9: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

It is proved that $\lim_{x \to a} x = a$. ■

Proof. Proof of the sum law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality,

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

We make $|f(x) - L| + |g(x) - M|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} \Big[f(x) + g(x) \Big] = L + M$$

■

Proof. Proof of the product law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows and use the triangle inequality:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= \Big|\big[f(x) - L\big]g(x) + L[g(x) - M]\Big| \\
&\leq \Big|\big[f(x) - L\big]g(x)\Big| + \Big|L\big[g(x) - M\big]\Big| \\
&= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x \to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$. Then we can combine the inequalities to get

$$\begin{aligned}
|f(x)g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\varepsilon}{2(1 + |L|)} \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that

$$\lim_{x \to a}\big[f(x)g(x)\big] = LM$$

■

Proof. Proof of the constant multiple law: If we take $g(x) = c$ then by the product law and limit law 8, we get

$$\lim_{x \to a} \Big[cf(x) \Big] = \lim_{x \to a} c \cdot \lim_{x \to a} f(x) = c \lim_{x \to a} f(x) = cL$$

We can prove the constant multiple law using the precise the definition. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} \Big[0 \cdot f(x) \Big] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x) = 0 \cdot L$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of a limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

This shows that $\lim_{x \to a} \Big[cf(x) \Big] = cL$. ■

Proof. Proof of the difference law: Using the sum law and the constant multiple law with $c = -1$, we have

$$\begin{aligned} \lim_{x \to a} \Big[f(x) - g(x) \Big] &= \lim_{x \to a} \Big[f(x) + (-1)g(x) \Big] = \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x) \\ &= \lim_{x \to a} f(x) + (-1) \lim_{x \to a} g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M \end{aligned}$$

■

Proof. Proof of the quotient law: First we prove that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

This shows that

$$0 < |x-a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \iff \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)}\right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

■

We have the following **direct substitution property** to calculate limits. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Problem 1.3.1. Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Solution.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

■

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided that this limit exists. When computing one-sided limits, we use the fact that the limit laws also hold for one-sided limits.

Problem 1.3.2. Show that $\lim_{x \to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim_{x \to 0} |x| = 0$. ■

The Squeeze Theorem

Theorem 1.3.1. If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

then $L \leq M$.

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x \to a} \Big[g(x) - f(x)\Big] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x - a| < \delta \implies \Big[g(x) - f(x)\Big] - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

But this contradicts $f(x) \leq g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leq M$. ■

Theorem 1.3.2 Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x\to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$, so

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x\to a} g(x) = L$. ■

Problem 1.3.3. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Solution. Since

$$-1 \le \sin\frac{1}{x} \le 1$$

then

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

We know that

$$\lim_{x\to 0}(-x^2) = 0 \qquad \lim_{x\to 0} x^2 = 0$$

By the squeeze theorem,

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

■

If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1$$

and $\theta \le \tan\theta$. Therefore we have

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$, so by the squeeze theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$$

But the function $(\sin\theta)/\theta$ is an even function, so its left and right limits must be equal. Hence we have

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.3.4. Find $\lim_{x \to 0} \dfrac{\sin 7x}{4x}$.

Solution.

$$\lim_{x \to 0} \frac{\sin 7x}{4x} = \lim_{x \to 0} \frac{7x \cdot \sin 7x}{4x \cdot 7x} = \frac{7}{4} \lim_{x \to 0} \frac{\sin 7x}{7x} = \frac{7}{4}$$

■

Problem 1.3.5. Evaluate $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta}$.

Solution. We have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit then

$$\begin{aligned} \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} &= \lim_{\theta \to 0}\left(\frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}\right) = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} \\ &= -\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\cos\theta + 1}\right) = -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1} = -1 \cdot \frac{0}{1+1} = 0 \end{aligned}$$

■

1.4 Continuity

Definition 1.4.1. A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Note that f is continuous at a requires that $f(a)$ is defined and the limit exists. We say that f is **discontinuous** at a if f is not continuous at a.

Definition 1.4.2. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 1.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem 1.4.1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Proof. Each of the five parts of this theorem follows from the corresponding limit law. We give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad\qquad \lim_{x \to a} g(x) = g(a)$$

Then

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} \Big[f(x) + g(x) \Big] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a)$$

This shows that $f+g$ is continuous at a. ■

Theorem 1.4.2. Any polynomial is continuous on $\mathbb{R} = (-\infty, \infty)$. Any rational function is continuous on its domain.

Proof. A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know that polynomials are continuous on $\mathbb{R}$ so the rational function f is continuous at every number in D. ■

Theorem 1.4.3. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions

- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 1.4.4. If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

Proof. Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1$$

Combining these two statements, we see that when $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

Theorem 1.4.5. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $f \circ g = f(g(x))$ is continuous at a.

Proof. Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we have

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $f(g(x))$ is continuous at a. ■

Theorem 1.4.6 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.

The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then by the intermediate value theorem there exists a root of $f(x)$ in (a, b).

1.5 Limits Involving Infinity

Infinite Limits

Definition 1.5.1. The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but $x \neq a$.

Another notation for the limit is $f(x) \to \infty$ as $x \to a$. We say that the limit of $f(x)$, as x approaches a, is infinity.

Definition 1.5.2.

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but $x \neq a$.

We say that the limit of $f(x)$, as x approaches a, is negative infinity. Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Definition 1.5.3. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation is $f(x) \to L$ as $x \to \infty$. We say that the limit of $f(x)$, as x approaches infinity, is L.

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

We say that the limit of $f(x)$, as x approaches negative infinity, is L.

Definition 1.5.6. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L$$

or

$$\lim_{x\to-\infty} f(x) = L$$

If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad\qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Problem 1.5.1. Evaluate $\lim_{x\to\infty} \sin x$.

Solution. As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often. Thus $\lim_{x\to\infty} \sin x$ does not exist. ■

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.7.

$$\lim_{x\to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.5.2. Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution. Let M be a given positive number We want to find a number δ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

But

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > M$$

This shows that $\lim_{x \to 0} \frac{1}{x^2} = \infty$. ■

Definition 1.5.8.

$$\lim_{x \to a} f(x) = -\infty$$

if for every negative number N there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) < N$$

Definition 1.5.9. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x > N \implies |f(x) - L| < \varepsilon$$

Definition 1.5.10. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x < N \implies |f(x) - L| < \varepsilon$$

Problem 1.5.3. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Solution. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, we can that $x > 0$ in computing the limit. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, so

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

■

Definition 1.5.11. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$x > N \implies f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

2 Derivatives

2.1 Derivatives and Rates of Change

Tangents

Definition 2.1.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

If $h = x - a$, then

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Let $s = f(t)$ be a **position function** that describes the motion of an object where s is the displacement of the object from the origin at time t. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

The **velocity** (or **instantaneous velocity**) of the object at time $t = a$ is the limit of the average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Derivatives

Definition 2.1.2. The **derivative of a function** f **at a number** a is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

The tangent line to $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$. The limit of these average rates of change is called the (**instantaneous**) **rate of change of y with respect to x** at $x = x_1$:
$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_2} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
We recognize this limit as being the derivative $f'(x_1)$. The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of a particle, then $f'(a)$ is the rate of change of the displacement s with respect to time t. $f'(a)$ is the velocity of the particle at time $t = a$. The **speed** of the particle is $|f'(a)|$, the absolute value of the velocity.

2.2 The Derivative as a Function

Definition 2.2.1. The **derivative of a function f** is the function
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Problem 2.2.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

■

Notations

The following notations for the derivative of the function $y = f(x)$ with respect to x are equivalent:
$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$
The symbols d/dx and D_x are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. The symbol dy/dx is callled the Leibniz notation. We can rewrite the definition of the derivative in Leibniz notation in the form
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
The following notations for the value of the derivative of $y = f(x)$ evaluated at the number a are equivalent:
$$y'(a) = f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left[\frac{dy}{dx} \right]_{x=a}$$

Differentiable Functions

Definition 2.2.2. A function f is **differentiable at** $\boldsymbol{a}$ if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) (or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$) if it is differentiable at every number in the interval.

Theorem 2.2.1. If f is differentiable at a, then f is continuous at a.

Proof. To prove that f is continuous at a, we want to show that

$$\lim_{x\to a} f(x) = f(a)$$

Given that f is differentiable at a so

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists. Then

$$\lim_{x\to a}\Big[f(x)-f(a)\Big] = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0$$

and

$$\lim_{x\to a} f(x) = \lim_{x\to a}\Big[f(a)+(f(x)-f(a))\Big] = \lim_{x\to a} f(a) + \lim_{x\to a}(f(x)-f(a)) = f(a)+0 = f(a)$$

Therefore f is continuous at a. ■

Note that there are functions that are continuous but not differentiable. The function $y=|x|$ is continuous at 0 but it is not differentiable at 0. Since

$$f'(0) = \lim_{h\to 0}\frac{|0+h|-|0|}{h}$$

if the limit exists. But

$$\begin{aligned}
\lim_{h\to 0^-}\frac{|0+h|-|0|}{h} &= \lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = -1\\
\lim_{h\to 0^+}\frac{|0+h|-|0|}{h} &= \lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1
\end{aligned}$$

then the limit does not exist so $f'(0)$ does not exist. Thus $y=|x|$ is differentiable at all x except 0.

Higher Order Derivatives

If $y=f(x)$ is a differentiable function and its derivative $y'=f'(x)$ is also a differentiable function, then the **second derivative** of $y=f(x)$ is

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$f''(x)$ is the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. In general, the second derivative is the rate of change of the rate of change. If $s = s(t)$ is the position function of an object, then its first derivative is the velocity $v(t)$ of the object as a function of time t:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2 s}{dt^2}$$

In general, the nth derivative of f is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Problem 2.2.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. The first derivative is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

The second derivative is

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h} = \lim_{h\to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h} \\ &= \lim_{h\to 0} \frac{6hx + 3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x \end{aligned}$$

■

2.3 Basic Differentiation Formulas

The derivative of the constant function $f(x) = c$ is

$$\frac{d}{dx}(c) = 0$$

Proof.

$$f'(x) = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

■

Power Functions

$$\frac{d}{dx}(x) = 1$$

Proof.

$$f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

■

The power rule: If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof. Let $f(x) = x^n$, then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand $(x+h)^n$ then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} (nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}) \\
&= nx^{n-1}
\end{aligned}$$

because every term except the first has h as a factor and therefore approaches 0. ■

The power rule (general version): If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Problem 2.3.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

■

The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P. The constant multiple rule: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\Big[cf(x)\Big] = c\frac{d}{dx}f(x)$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} c \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \frac{d}{dx} f(x) \end{aligned}$$

■

The sum rule: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x) + g(x)\Big] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \end{aligned}$$

■

The difference rule: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x) - g(x)\Big] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.3.1 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx} f(x)\right] g(x) + f(x) \left[\frac{d}{dx} g(x)\right]$$

Proof.

$$
\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+f(x+h)g(x)-f(x+h)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x+h)g(x+h)-f(x+h)g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x)\right]+\lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}\right]\\
&= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]g(x)+\lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \left[\frac{d}{dx}f(x)\right]g(x)+f(x)\left[\frac{d}{dx}g(x)\right]
\end{aligned}
$$

■

Theorem 2.3.2 Quotient Rule.

$$
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\left[\frac{d}{dx}f(x)\right]g(x)-f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}
$$

Proof.

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h}=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{h\cdot g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)-[f(x)g(x+h)-f(x)g(x)]}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\\
&= \left(\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x)\right]-\lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]\right)\frac{1}{[g(x)]^2}\\
&= \left[\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x)-f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right]\frac{1}{[g(x)]^2}\\
&= \frac{\left[\frac{d}{dx}f(x)\right]g(x)-f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}
\end{aligned}
$$

■

Trigonometric Functions

Theorem 2.3.3.

$$
\frac{d}{dx}\sin x=\cos x
$$

Proof. We use the **angle sum identity** of the sine function

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

then we have

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Note that we are taking the limit with respect to h so $\sin x$ and $\cos x$ are constants then we have

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right] = \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}\\ &= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\ &= (\sin x)(0) + (\cos x)(1) = \cos x\end{aligned}$$

■

Theorem 2.3.4.

$$\frac{d}{dx}\cos x = -\sin x$$

Proof. We use the angle sum identity of the cosine function

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

then we have

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= \lim_{h\to 0}\left(\frac{\cos x(\cos h-1)}{h} - \frac{\sin x\sin h}{h}\right)\\ &= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\ &= (\cos x)(0) - (\sin x)(1) = -\sin x\end{aligned}$$

■

Theorem 2.3.5.

$$\frac{d}{dx}\tan x = \sec^2 x$$

Proof.

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

■

Then we can derive the following derivatives:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$
$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.3.6 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Proof. We know that by definition if $y = f(x)$, then $\Delta y = f(a + \Delta x) - f(a)$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let ε be the difference between the difference quotient and the derivative, then we have

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} - f'(a)\right) = f'(a) - f'(a) = 0$$

Thus for a differentiable function f, if we define $\varepsilon = 0$ when $\Delta x = 0$, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx. Suppose that $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. Then we have

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1]\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2]\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. We substitute the expression for Δu then we have

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\Delta x$$
$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

Since $\Delta u \to 0$ as $\Delta x \to 0$, then $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}[f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] = f'(b)g'(a) = f'(g(a))g'(a)$$

thus the chain rule is proved. ■