Calculus

Yaohui Wu

July 21, 2024

Introduction

Calculus is the mathematical study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Contents

1 Functions and Limits

1.1 The Limit of a Function

Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E.

Definition 1.1.2. A function f is **injective** (or one-to-one) if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

Definition 1.1.3. A function f is **surjective** (or onto) if for all y in range Y, there exists an x in domain X such that $f(x) = y$.

Definition 1.1.4. A function f is **bijective** if f is injective and surjective.

Definition 1.1.5. Let f be an injective function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in B.

Intuitive Definition of a Limit

Let $f(x)$ be a function defined near the number a, that is, $f(x)$ is defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.1.6. The **limit** of $f(x)$ as x approaches a equals L, and we write

$$\lim_{x \to a} f(x) = L$$

or $f(x) \to L$ as $x \to a$ if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a but $x \neq a$.

One-Sided Limits

Definition 1.1.7. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a with $x < a$.

Definition 1.1.8. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a equals L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a with $x > a$.

Theorem 1.1.1. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The limit **exists** if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a equal L, otherwise the limit **does not exist**.

Infinite Limits and Vertical Asymptotes

Let f be a function defined on both sides of a, except possibly at a itself.

Definition 1.1.9.

$$\lim_{x \to a} f(x) = \infty$$

or $f(x) \to \infty$ as $x \to a$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but $x \neq a$.

Definition 1.1.10.

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but $x \neq a$.

Similar definitions can be given for the one-sided infinite limits

$$\begin{array}{ll} \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Definition 1.1.11. The vertical line $x = a$ is the **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \to a} f(x) = \infty & \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

1.2 The Precise Definition of a Limit

The Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2.1. The limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.1. Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

We simplify to get $|(4x-5)-7| = |4x-12| = 4|x-3|$ so we have

$$4|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{4}$$

Let $\delta = \varepsilon/4$, we have

$$0 < |x-3| < \frac{\varepsilon}{4} \implies 4|x-3| < \varepsilon \implies |(4x-5)-7| < \varepsilon$$

Therefore, by the definition of a limit, it is proved that $\lim_{x\to 3}(4x-5) = 7$. ■

Problem 1.2.2. Prove that $\lim_{x\to 3} x^2 = 9$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2-9| = |x+3|\,|x-3| < \varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3| < C\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3| < 1$ then $|x+3| < 7$ so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x-3| < \delta$, then

$$|x^2-9| = |x+3|\,|x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

It is proved that $\lim_{x\to 3} x^2 = 9$. ■

One-Sided Limits

Definition 1.2.2.

$$\lim_{x\to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.2.3.

$$\lim_{x\to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.3. Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Solution. Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

We simplify to get $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$. If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$ so $|\sqrt{x} - 0| < \varepsilon$. It is proved that $\lim_{x\to 0^+} \sqrt{x} = 0$. ■

Infinite Limits

Definition 1.2.4.

$$\lim_{x \to a} f(x) = \infty$$

if for every positive number M there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.2.4. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Definition 1.2.5.

$$\lim_{x \to a} f(x) = -\infty$$

if for every negative number N there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) < N$$

1.3 Calculating Limits Using the Limit Laws

Properties of Limits

We have the following properties of limits called the **limit laws** to calculate limits. Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

exist. Then

1. Sum Law: $\lim_{x \to a} \big[f(x) + g(x)\big] = L + M$
2. Difference Law: $\lim_{x \to a} \big[f(x) - g(x)\big] = L - M$
3. Constant Multiple Law: $\lim_{x \to a} \big[cf(x)\big] = cL$
4. Product Law: $\lim_{x \to a} \big[f(x)g(x)\big] = LM$
5. Quotient Law: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$.
6. Power Law: $\lim_{x \to a} \big[f(x)\big]^n = \Big[\lim_{x \to a} f(x)\Big]^n$ where n is a positive integer.
7. Root Law: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim_{x \to a} f(x) > 0$.
8. $\lim_{x \to a} c = c$

9. $\lim_{x \to a} x = a$

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer.

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

Proof. Proof of limit law 1, the sum law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

We make $|f(x) - L| + |g(x) - M|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Therefore, by the definition of the limit, it is proved that

$$\lim_{x \to a} \Big[f(x) + g(x)\Big] = L + M$$

■

Proof. Proof of limit law 2, the difference law: Using the sum law and the constant multiple law with $c = -1$, we have

$$\begin{aligned} \lim_{x \to a} \Big[f(x) - g(x)\Big] &= \lim_{x \to a} \Big[f(x) + (-1)g(x)\Big] = \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x) \\ &= \lim_{x \to a} f(x) + (-1) \lim_{x \to a} g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M \end{aligned}$$

■

Proof. Proof of limit law 3, the constant multiple law: If we take $g(x) = c$ then by the product law and limit law 8, we get

$$\lim_{x\to a}\Big[cf(x)\Big] = \lim_{x\to a} c \cdot \lim_{x\to a} f(x) = c\lim_{x\to a} f(x) = cL$$

We can prove the constant multiple law using the precise the definition. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x\to a}\Big[0 \cdot f(x)\Big] = \lim_{x\to a} 0 = 0 = 0 \cdot \lim_{x\to a} f(x) = 0 \cdot L$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |cf(x) - cL| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of the limit, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x-a| < \delta \implies |cf(x) - cL| < \varepsilon$$

It is proved that $\lim_{x\to a}\Big[cf(x)\Big] = cL$. ■

Proof. Proof of limit law 4, the product law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows and use the triangle inequality:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= \Big|\Big[f(x) - L\Big]g(x) + L[g(x) - M]\Big| \\
&\le \Big|\Big[f(x) - L\Big]g(x)\Big| + \Big|L\Big[g(x) - M\Big]\Big| \\
&= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$. Therefore we can combine the inequalities to get

$$\begin{aligned}
|f(x)g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\varepsilon}{2(1 + |L|)} \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

It is proved that

$$\lim_{x \to a}\Big[f(x)g(x)\Big] = LM$$

■

Proof. Proof of limit law 5, the quotient law: First we prove that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

Observe that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x \to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

This shows that

$$0 < |x - a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \iff \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\left(f(x) \cdot \frac{1}{g(x)}\right) = \lim_{x\to a} f(x) \lim_{x\to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

■

Proof. Proof of limit law 8: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. It is proved that $\lim_{x\to a} c = c$. ■

Proof. Proof of limit law 9: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

It is proved that $\lim_{x\to a} x = a$. ■

Evaluating Limits by Direct Substitution

We have the following **direct substitution property** to calculate limits. If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Problem 1.3.1. Find $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1}$.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided that this limit exists.

Using One-Sided Limits

When computing one-sided limits, we use the fact that the limit laws also hold for one-sided limits.

Problem 1.3.2. Show that $\lim_{x\to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim_{x\to 0} |x| = 0$. ■

The Squeeze Theorem

Theorem 1.3.1. If $f(x) \le g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then $L \le M$.

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a}\Big[g(x) - f(x)\Big] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < L - M$$

Since $b \le |b|$ for any number b, we have

$$0 < |x - a| < \delta \implies g(x) - f(x) - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

but this is a contradiction since $f(x) \le g(x)$. Then the inequality $L > M$ must be false therefore $L \le M$. ■

Theorem 1.3.2 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta - 1$ and $0 < |x - a| < \delta_2$. Then

$$\begin{gathered} L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon \\ L - \varepsilon < g(x) < L + \varepsilon \end{gathered}$$

and so $|g(x) - L < \varepsilon|$. Therefore $\lim_{x \to a} g(x) = L$. ∎

If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1 \qquad\qquad \theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta}$$

so we have the following inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $(\sin\theta)/\theta$ is an even function, its left and right limits must be equal. Therefore, we have the following limit by the squeeze theorem.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.3.3. Evaluate

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

Solution. We have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit then

$$\begin{aligned} \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} &= \lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)\right] = \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{-\sin\theta}{\cos\theta + 1}\right) \\ &= 1\left(\frac{(-1)(0)}{1 + 1}\right) = 0 \end{aligned}$$

∎

1.4 Continuity

Continuity of a Function

Definition 1.4.1. A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Note that f is continuous at a requires that $f(a)$ is defined and the limit exists. We say that f is **discontinuous** at a if f is not continuous at a. If we can remove the discontinuity by redefining f at a, then it is a **removable discontinuity**. If f has a vertical asymptote, then it has an **infinite discontinuity**. If the left-hand limit is not equal to the right-hand limit, then f has a **jump discontinuity**.

Definition 1.4.2. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 1.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval.

Properties of Continuous Functions

Theorem 1.4.1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Theorem 1.4.2. Let $P(x)$ be any polynomial, then $P(x)$ is continuous on $\mathbb{R} = (-\infty, \infty)$.

Proof. A polynomial $P(x)$ is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. ■

Theorem 1.4.3. Let f be any rational function, then f is continuous on its domain.

Proof. A rational function f is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We know that polynomials are continuous so a rational function is continuous on its domain. ■

Theorem 1.4.4. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 1.4.5. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

Proof. Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, then we have $\lim_{y\to b} f(y) = f(b)$. There exists $\delta_1 > 0$ such that

$$0 < |y-b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \varepsilon$$

It is proved that

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

■

Theorem 1.4.6. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at a.

Proof. Since g is continuous at a, we have $\lim_{x\to a} g(x) = g(a)$. Since f is continuous at $g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

Therefore, $f(g(x))$ is continuous at a. ■

The Intermediate Value Theorem

Theorem 1.4.7 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.

If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then there exists a root of $f(x)$ in (a, b) by the intermediate value theorem.

1.5 Limits at Infinity and Horizontal Asymptotes

Limits at Infinity and Horizontal Asymptotes

Evaluating Limits at Infinity

Infinite Limits at Infinity

Precise Definitions

Limits at Infinity

Definition 1.5.1. Let f be a function defined on some interval (a, ∞). The limit of $f(x)$ as x approaches infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

$$\lim_{x \to \infty} f(x) = L$$

Definition 1.5.2. Let f be a function defined on some interval $(-\infty, a)$. The limit of $f(x)$ as x approaches negative infinity is L if the values of $f(x)$ can be made as close to L as we like by taking x sufficiently small.

$$\lim_{x \to -\infty} f(x) = L$$

Problem 1.5.1. Evaluate $\lim_{x \to \infty} \sin x$ and $\lim_{x \to \infty} \cos x$.

Solution. The values of $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to \infty$ so the limits do not exist. ■

Definition 1.5.3. The horizontal asymptote of the curve $y = f(x)$ is the line $y = L$ if one of the limits at infinity is L.

Infinite Limits at Infinity

Definition 1.5.4. The values of $f(x)$ become arbitrarily large for sufficiently large x.

$$\lim_{x \to \infty} f(x) = \infty$$

Similar definitions can be given for other infinite limits at infinity or negative infinity.

$$\lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.5.

$$\lim_{x\to a} f(x) = \infty$$

if for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.5.2. Prove that
$lim_{x\to a}\frac{1}{x^2} = \infty$.

Solution. Let $M > 0$ be given, we want to find a $\delta > 0$ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

We have

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then we have

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Therefore, by definition, it is proved that $\lim\limits_{x\to a} \dfrac{1}{x^2} = \infty$. ∎

Let f be a function defined on some interval (a, ∞).

Definition 1.5.6.

$$\lim_{x\to\infty} f(x) = L$$

if for every $\varepsilon > 0$, there is an N such that

$$x > N \implies |f(x) - L| < \varepsilon$$

Problem 1.5.3. Prove that $\lim_{x\to\infty}\frac{1}{x} = 0$.

Proof. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, it is reasonable to assume that $x > 0$ in computing the limit. Then we have $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, then we have

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition, it is proved that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$. ∎

Definition 1.5.7.

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $M > 0$, there is an $N > 0$ such that

$$x > N \implies f(x) > M$$

Similar definitions apply for limits involving negative infinity.

2 Derivatives

2.1 Derivatives

Derivatives and Rates of Change

Definition 2.1.1. The tangent line of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

where $h = x - a$, if this limit exists.

Definition 2.1.2. The velocity at time $t = a$ of a position function $s = f(t)$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Definition 2.1.3. The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

The slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, the derivative of f at a. The equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The instantaneous rate of change of $y = f(x)$ with respect to x at $x = x_0$ in the interval $[x_0, x_1]$ is

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of an object, then the velocity of the object at time $t = a$ is $v(a) = f'(a)$ and the speed of the object is $|f'(a)|$, the magnitude of the velocity.

The Derivative as a Function

Definition 2.1.4. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The following notations of the derivative of $y = f(x)$ with respect to x are equivalent.

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$

where $f'(x)$ is Newton's notation and dy/dx is Leibniz's notation. The notation d/dx is the differential operator that indicates the operation of differentiation. The notations of the derivative of $f(x)$ at a are

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left[\frac{dy}{dx}\right]_{x=a}$$

Problem 2.1.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

■

Differentiable Functions

Definition 2.1.5. A function f is **differentiable** at a if $f'(a)$ exists. f is differentiable on an open interval if it is differentiable at every number in the interval.

Theorem 2.1.1. If f is differentiable at a, then f is continuous at a.

Proof. Given that f is differentiable at a, we want to show that

$$\lim_{x \to a} f(x) = f(a)$$

Since $f'(a)$ exists, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then we have

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} (x - a) \right) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) \lim_{x \to a} (x - a) = f'(a) \cdot 0 = 0$$

Then we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} (f(a) + f(x) - f(a)) = \lim_{x \to a} f(a) + \lim_{x \to a} (f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore, it is proved that f is continuous at a. ■

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but not differentiable at 0 since

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

if the limit exists but

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$
$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

thus the limit does not exist so $f'(0)$ does not exist. If a function is differentiable, then it has no sharp points, it is continuous, and it has no vertical tangent lines.

Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $f'(x)$ is differentiable, then the second derivative of f is

$$y'' = f''(x) = \frac{d^2y}{dx^2}$$

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. Let $s = s(t)$ be a position function of an object with respect to time t. The velocity function $v(t)$ of the object is

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity is the acceleration. Thus the acceleration function $a(t)$ is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

In general, the nth derivative of $y = f(x)$ is

$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Problem 2.1.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. We apply the binomial theorem by Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

For the first derivative we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

For the second derivative we have

$$f''(x) = \lim_{h\to0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to0} \frac{3(x^2+2hx+h^2-x^2)}{h} = \lim_{h\to0} \frac{6hx+3h^2}{h}$$
$$= \lim_{h\to0}(6x+3h) = 6x$$

■

2.2 Differentiation

Differentiation Formulas

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the following differentiation formulas.

Theorem 2.2.1. Let $f(x) = c$ where c is a constant, then

$$\frac{d}{dx}(c) = 0$$

Proof.

$$f'(x) = \lim_{h\to0} \frac{c-c}{h} = \lim_{h\to0} 0 = 0$$

■

Theorem 2.2.2 Power Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}$$

Proof. We prove the power rule for $n \in \mathbb{N}$.

$$\frac{d}{dx}(x^n) = \lim_{h\to0} \frac{(x+h)^n - x^n}{h}$$

We use the binomial theorem to expand $(x+h)^n$ then we have

$$\frac{d}{dx}(x^n) = \lim_{h\to0} \frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h}$$
$$= \lim_{h\to0}(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}) = nx^{x-1}$$

because every term has a factor of h except nx^{n-1}. ■

Note the special case when $n = 1$, then we have

$$\frac{d}{dx}(x) = 1$$

Problem 2.2.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

■

Theorem 2.2.3 Constant Multiple Rule. If c is a constant, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof.

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0}\left[c\left(\frac{f(x+h)-f(x)}{h}\right)\right] \\ &= c\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) = c\frac{d}{dx}f(x)\end{aligned}$$

■

Theorem 2.2.4 Sum and Difference Rule.

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Proof. We prove the sum rule.

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\end{aligned}$$

Then we prove the difference rule.

$$\begin{aligned}\frac{d}{dx}[f(x)-g(x)] &= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-[f(x)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)-g(x+h)+g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)-[g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) - \frac{d}{dx}g(x)\end{aligned}$$

■

Product and Quotient Rules

Let $f(x)$ and $g(x)$ be differentiable functions, then we have the product rule by Leibniz and the quotient rule.

Theorem 2.2.5 Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x+h)g(x+h) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}g(x)\right] + \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h}\right] \\
&= \left[\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\right]g(x) + \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \\
&= \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]
\end{aligned}$$

■

Theorem 2.2.6 Quotient Rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\cdot g(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - [f(x)g(x+h) - f(x)g(x)]}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)g(x)} \\
&= \left(\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}g(x)\right] - \lim_{h\to 0}\left[f(x)\frac{g(x+h) - g(x)}{h}\right]\right)\frac{1}{[g(x)]^2} \\
&= \left[\left(\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\right)g(x) - f(x)\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right)\right]\frac{1}{[g(x)]^2} \\
&= \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}
\end{aligned}$$

■

Trigonometric Functions

Theorem 2.2.7.

$$\frac{d}{dx}\sin x = \cos x$$

Proof. We use the **angle sum identity** of the sine function

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

then we have

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Note that we are taking the limit with respect to h so $\sin x$ and $\cos x$ are constants then we have

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\left[\frac{\sin x(\cos h-1)}{h}+\frac{\cos x\sin h}{h}\right] = \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\ &= \left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)+\left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\ &= (\sin x)(0)+(\cos x)(1) = \cos x\end{aligned}$$

■

Theorem 2.2.8.

$$\frac{d}{dx}\cos x = -\sin x$$

Proof. We use the angle sum identity of the cosine function

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

then we have

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= \lim_{h\to 0}\left(\frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}\right)\\ &= \left(\lim_{h\to 0}\cos x\right)\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)-\left(\lim_{h\to 0}\sin x\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\ &= (\cos x)(0)-(\sin x)(1) = -\sin x\end{aligned}$$

■

Theorem 2.2.9.

$$\frac{d}{dx}\tan x = \sec^2 x$$

Proof.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

■

Then we can derive the following derivatives:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$
$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Chain Rule

We have the **chain rule** formulated by **James Gregory** (1638–1675) to find the derivative of a composite function.

Theorem 2.2.10 Chain Rule. If f and g are differentiable functions and $F = f(g(x))$, then F is differentiable and F' is

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Proof. We know that by definition if $y = f(x)$, then $\Delta y = f(a + \Delta x) - f(a)$ and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

Let ε be the difference between the difference quotient and the derivative, then we have

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a)\right) = f'(a) - f'(a) = 0$$

Thus for a differentiable function f, if we define $\varepsilon = 0$ when $\Delta x = 0$, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx. Suppose that $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. Then we have

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1]\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2]\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. We substitute the expression for Δu then we have

$$\begin{aligned} \Delta y &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\Delta x \\ \frac{\Delta y}{\Delta x} &= [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \end{aligned}$$

Since $\Delta u \to 0$ as $\Delta x \to 0$, then $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] = f'(b)g'(a) = f'(g(a))g'(a)$$

thus the chain rule is proved. ■

2.3 Implicit Differentiation

Implicit Differentiation

An explicit function $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. An implicit function is defined implicitly by a relation between x and y. An example of implicit functions is the equation of the circle $x^2 + y^2 = r^2$ where the raidus r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y in an implicit function.

Problem 2.3.1. Find dy/dx of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then we have

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Since $y = f(x)$, we use the chain rule then we have

$$\begin{aligned} 2x + \frac{d}{dy}(y^2)\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

We solve for dy/dx then we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.3.2. Find dy/dx of the folium of Descartes $x^3 + y^3 = 6xy$.

Solution.

$$\begin{aligned}
x^3 + y^3 &= 6xy \\
\frac{d}{dx}x^3 + \frac{d}{dy}y^3\left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y + 6x\frac{dy}{dx} \\
3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\
x^2 + y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\
(y^2 - 2x)\frac{dy}{dx} &= 2y - x^2 \\
\frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}$$

■

Problem 2.3.3. Find y' if $\sin(x + y) = y^2 \cos x$.

Solution.

$$\begin{aligned}
\sin(x + y) &= y^2 \cos x \\
\cos(x + y)(1 + \frac{dy}{dx}) &= 2y\frac{dy}{dx}\cos x - y^2 \sin x \\
[2y \cos x - \cos(x + y)]\frac{dy}{dx} &= \cos(x + y) + y^2 \sin x \\
\frac{dy}{dx} &= \frac{\cos(x + y) + y^2 \sin x}{2y \cos x - \cos(x + y)}
\end{aligned}$$

■

Problem 2.3.4. Find y'' if $x^4 + y^4 = 16$.

Solution. First we find y' then we have

$$\begin{aligned}
x^4 + y^4 &= 16 \\
4x^3 + 4y^3\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^3}{y^3}
\end{aligned}$$

Then we find y'' and we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -3x^2y^{-3} + (-x^3)(-3y^{-4})\frac{dy}{dx} = 3x^3y^{-4}(-x^3y^{-3}) - 3x^2y^{-3} \\
&= -3x^6y^{-7} - 3x^2y^{-3} = -3x^2y^{-7}(x^4 + y^4) = -48\frac{x^2}{y^7}
\end{aligned}$$

■

2.4 Derivatives of Inverse Functions

Differentiation and Inverse Functions

Theorem 2.4.1. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem 2.4.2. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \iff \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

Derivatives of Logarithmic Functions

The **Euler's number** e named after **Leonhard Euler** (1707–1783) is the base of the natural exponential function $y = e^x$ and the natural logarithmic function $y = \ln x$.

Definition 2.4.1 Euler's Number. The Euler's number e is defined as

$$e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.4.3. The exponential function $f(x) = \log_a x$ is differentiable and

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$$

Proof. First we have

$$\begin{aligned}\frac{d}{dx}\log_a x &= \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a\left(\dfrac{x+h}{x}\right)}{h} = \lim_{h\to 0}\left[\frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right] \\ &= \frac{1}{x}\left[\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h}\right] = \frac{1}{x}\left[\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)}\right] = \frac{1}{x}\log_a e\end{aligned}$$

We know from the change of base formula that

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Therefore,

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$$

∎

Theorem 2.4.4. The derivative of the natural logarithmic function $f(x) = \ln x$ is

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Proof.

$$\frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{x \ln e} = \frac{1}{x}$$

■

Problem 2.4.1. Find $f'(x)$ if $f(x) = \ln |x|$.

Solution. Since $f(x) = \ln x$ for $x > 0$ and $f(x) = \ln(-x)$ for $x < 0$, it follows that

$$f'(x) = \frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0$$
$$f'(x) = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0$$

Therefore, $\frac{d}{dx} \ln |x| = \frac{1}{x}$ for all $x \neq 0$. ■

Derivatives of Exponential Functions

Theorem 2.4.5. The exponential function $f(x) = a^x, a > 0$ is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a$$

Proof. Let $y = a^x \iff \log_a y = x$ then by implicit differentiation we have

$$\begin{aligned} \log_a y &= x \\ \frac{1}{y \ln a} \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y \ln a = a^x \ln a \end{aligned}$$

■

Theorem 2.4.6. The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx} e^x = e^x$$

Proof.

$$\frac{d}{dx} e^x = e^x \ln e = e^x$$

■

Problem 2.4.2. Prove the power rule $\frac{d}{dx} x^n = n x^{n-1}$ for $n \in \mathbb{R}$.

Solution. The rule is true when $x = 0$ which is trivial so it remains to prove the cases for $x \neq 0$. Let $y = x^n$ such that $y > 0$, by implicit differentiation we have

$$\begin{aligned}
\ln y &= \ln x^n = n \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n = nx^{n-1}
\end{aligned}$$

Similarly, let $y = x^n$ such that $y < 0$ then we have

$$\begin{aligned}
\ln|y| &= \ln(-x^n) = \ln[((-1)^{1/n}x)^n] = n\ln[(-1)^{1/n}x] \\
\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{(-1)^{1/n}x}(-1)^{1/n} = \frac{n}{x} \\
\frac{dy}{dx} &= nx^{-1}y = nx^{-1}x^n = nx^{n-1}
\end{aligned}$$

∎

Inverse Trigonometric Functions

Theorem 2.4.7.

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arcsin x \iff \sin y = x$ and $-\pi/2 \leq y \leq \pi/2$ such that $-1 \leq x \leq 1$. Note the approximate value of π is $\pi \approx 3.14159$. Then we have

$$\begin{aligned}
\frac{d}{dx}\sin y &= \frac{d}{dx}x \\
\cos y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y}
\end{aligned}$$

Since $-\pi/2 \leq y \leq \pi/2$ so $\cos y \geq 0$ thus $\cos y = \sqrt{1-\sin^2 y}$. Therefore, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$. ∎

Theorem 2.4.8.

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arccos x \iff \cos y = x$ and $0 \leq y \leq \pi$ such that $-1 \leq x \leq 1$. Then we have

$$\begin{aligned}
\frac{d}{dx}\cos y &= \frac{d}{dx}x \\
-\sin y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= -\frac{1}{\sin y}
\end{aligned}$$

Since $0 \le y \le \pi$ so $\sin y \ge 0$ thus $\sin y = \sqrt{1-\cos^2 y}$. Therefore, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$. ■

Theorem 2.4.9.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Proof. Let $y = \arctan x \iff \tan y = x$. Then we have

$$\begin{aligned} \frac{d}{dx}\tan y &= \frac{d}{dx}x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \end{aligned}$$

Then we can show that

$$\begin{aligned} \tan y &= x \\ 1 + \frac{\sin^2 y}{\cos^2 y} &= 1 + x^2 \\ 1 + \frac{1-\cos^2 y}{\cos^2 y} &= 1 + x^2 \\ \sec^2 y &= 1 + x^2 \\ \cos^2 y &= \frac{1}{1+x^2} \end{aligned}$$

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. ■

2.5 Indeterminate Forms and L'Hôpital's Rule

Indeterminate Forms and L'Hôpital's Rule

Consider the limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then the limit is an **indeterminate form** of type $0/0$. If $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$, then the limit is an indeterminate form of type ∞/∞. We have the **L'Hôpital's Rule** discoverd by **Johann Bernoulli** (1667–1748) and named after **Guillaume de l'Hôpital** (1661–1704) to evaluate limits of indeterminate forms of type $0/0$ and ∞/∞.

Theorem 2.5.1 L'Hôpital's Rule. Suppose that f and g are differentiable and $g'(x) \neq 0$ near a, except possibly at a. If

$$\lim_{x \to a} f(x) = 0 \qquad\qquad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \qquad\qquad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, is ∞, or is $-\infty$.

L'Hôpital's rule is also valid for one-sided limits and for limits at infinity or negative infinity.

Problem 2.5.1. Find $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

Solution. Notice that the limit is an indeterminate form of $0/0$. We apply L'Hôpital's rule then we have

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

■

Problem 2.5.2. Find $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}$.

Proof. Notice that the limit is an indeterminate form of $0/0$. We apply L'Hôpital's rule then we have

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

The limit still is an indeterminate form of $0/0$ so we apply L'Hôpital's rule again then we have

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$$

Similarly, we apply L'Hôpital's rule again then we have

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{2\sec^4 x + 4\sec^2 x\tan^2 x}{6} = \frac{2(1)+4(1)(0)}{6} = \frac{1}{3}$$

■

Note that L'Hôpital's rule only applies if the limit is one of the indeterminate forms of type $0/0$ or ∞/∞. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then the limit $\lim_{x\to a} f(x)g(x)$ is an indeterminate form of type $0\cdot\infty$. We can express the product as a quotient then use L'Hôpital's rule to calculate the limit.

Problem 2.5.3. Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

■

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ is an indeterminate form of type $\infty - \infty$. We can convert the difference into a quotient to apply L'Hôpital's rule to calculate the limit.

Problem 2.5.4. Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Solution.

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin x} = 0$$

■

Consider the limit of the form

$$\lim_{x\to a}[f(x)^{g(x)}]$$

We have the following indeterminate forms arise from the limit:

- Type 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
- Type ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
- Type 1^∞ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Let $y = f(x)^{g(x)}$, then we can write $\ln y = \ln f(x)^{g(x)}$ and $y = e^{g(x)\ln f(x)}$ to find the limit.

Problem 2.5.5. Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution. Let $y = (1+\sin 4x)^{\cot x}$, then $\ln y = \ln[(1+\sin 4x)^{\cot x}] = \cot x \ln(1+\sin 4x)$. We apply L'Hôpital's rule then we have

$$\begin{aligned}\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} &= \lim_{x\to 0^+} \frac{[1/(1+\sin 4x)](\cos 4x)(4)}{\sec^2 x} \\ = \lim_{x\to 0^+} \frac{4\cos 4x \cos^2 x}{1+\sin 4x} &= \frac{4(1)(1)}{1+0} = 4\end{aligned}$$

Therefore, we have

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

■

Problem 2.5.6. Calculate $\lim_{x\to 0^+} x^x$.

Solution. We have shown that $\lim_{x\to 0^+} x\ln x = 0$ and therefore we have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

■

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Absolute and Local Extreme Values

Critical Numbers and the Closed Interval Method

Theorem 3.1.1 Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

3.2 The Mean Value Theorem

Rolle's Theorem

Theorem 3.2.1 Rolle's Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem

Theorem 3.2.2 Lagrange's Mean Value Theorem. Suppose that f is a continuous function on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 3.2.3 Cauchy's Mean Value Theorem. Suppose that the fucntions f and g are continuous on $[a, b]$ and differentiable on (a, b) and $g(x) \neq 0$ for all x in (a, b). Then there is a number c in a, b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

3.3 Derivatives and Graphs

3.4 Antiderivatives

The Antiderivative of a Function

Definition 3.4.1. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 3.4.1. If F is an antiderivative of f on interval I, then the general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

Problem 3.4.1. Find the general antiderivative of $f(x) = 1/x$.

Solution. We know that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

for all $x \neq 0$. Then the antiderivative of f is $F(x) = \ln x + C_1$ if $x > 0$ or $F(x) = \ln(-x) + C_2$ if $x < 0$. Therefore, the general antiderivative of f is $F(x) = \ln|x| + C$. ■

4 Integrals

4.1 Integrals

Definite Integrals

A **Riemann sum** named after **Bernhard Riemann** (1826–1866) associated with a partition P of interval $[a, b]$ and a function f at the sample points x_i^* in the subinterval $[x_{i-1}, x_i]$ is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

Definition 4.1.1. If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if the limit exists such that f is integrable on $[a, b]$.

Definition 4.1.2.

$$\int_a^b f(x)\,dx = I$$

if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \varepsilon$$

for all partitions P of $[a, b]$ with $\max \Delta x_i < \delta$ and for all x_i^* in $[x_{i-1}, x_i]$.

Theorem 4.1.1. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$ and the definite integral $\int_a^b f(x)\,dx$ eixsts.

Theorem 4.1.2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

Properties of Definite Integrals

We have the following properties of indefinite integrals. Suppose all of the following integrals exist, and c is any constant.

- If $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- If $a = b$, then $\int_a^a f(x)\,dx = 0$.
- $\int_a^b c\,dx = c(b-a)$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Indefinite Integrals

Definition 4.1.3. The **indefinite integral**

$$\int f(x)\,dx = F(x) + C$$

is the general antiderivative of f.

4.2 Evaluating Integrals

4.3 The Fundamental Theorem of Calculus

Theorem 4.3.1 The Fundamental Theorem of Calculus. Suppose f is continuous on $[a, b]$. If the function F is defined by

$$F(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

then F is an antiderivative of f and

$$F'(x) = f(x), \quad a < x < b$$

and therefore

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If F is an antiderivative of f such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

4.4 The Substitution Rule

The Substitution Rule

Theorem 4.4.1 The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If g' is continuous on $[a, b]$ and f is continuous on I, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof. Let $F' = f$, then by the chain rule we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int f(u)\,du$$

■

Problem 4.4.1. Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution. Let $u = 1 - 4x^2 \iff du = -8x\,dx$ so $xdx = -\dfrac{1}{8}\,du$. Therefore,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\int \frac{1}{8}u^{-(1/2)}\,du = -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

■

Problem 4.4.2. Evaluate $\displaystyle\int \tan x\,dx$.

Solution. We have $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, let $u = \cos x \iff du = -\sin\,dx$ then

$$\int \tan x\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln\frac{1}{|\sec x|} + C = \ln|\sec x| + C$$

■

Problem 4.4.3. Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

Solution. Let $u = \ln x \iff du = \dfrac{1}{x}\,dx$, then

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

■

Symmetry

Theorem 4.4.2. Suppose f is continuous on $[-a, a]$, then

- If $f(-x) = f(x)$ so f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{a}^{a} f(x)\,dx$.
- If $f(-x) = -f(x)$ so f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

5 Techniques of Integration

5.1 Integration by Parts

Theorem 5.1.1 Integration by Parts Formula. If f and g are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If $u = f(x)$ and $v = g(x)$, then

$$\int u\,dv = uv - \int v\,du$$

and

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Proof. We know from the product rule that

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

and therefore

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

■

Problem 5.1.1. Evaluate $\displaystyle\int \ln x\,dx$.

Solution. Let $u = \ln x \iff du = \dfrac{1}{x}\,dx$ and $dv = dx \iff v = x$, then

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

■

Problem 5.1.2. Evaluate $\displaystyle\int \arctan x\,dx$.

Solution. Let $u = \arctan x \iff du = \dfrac{dx}{1+x^2}$ and $dv = dx \iff v = x$, then

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

Let $u = 1 + x^2 \iff du = 2x\,dx$ so $x\,dx = \dfrac{1}{2}\,du$, then

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C$$

Therefore,

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

■

Problem 5.1.3. Evaluate $\int e^x \sin x\,dx$.

Solution. Let $u = \sin x \iff du = \cos x\,dx$ and $dv = e^x\,dx \iff v = e^x$, then

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

Let $u = \cos x \iff du = -\sin x\,dx$ and $dv = e^x\,dx \iff v = e^x$, then

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

Therefore,

$$\begin{aligned}
\int e^x \sin x\,dx &= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx \\
\int e^x \sin x\,dx &= \frac{1}{2}e^x(\sin x - \cos x)
\end{aligned}$$

■

Problem 5.1.4. Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution. Let $u = \sin^{n-1} x \iff du = (n-1)\sin^{n-2} x\cos x\,dx$ and $dv = \sin x\,dx \iff v = -\cos x$ then

$$\begin{aligned}
\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx
\end{aligned}$$

Therefore

$$(n-1+1)\int \sin^n x\,dx = n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$$
$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

■

Problem 5.1.5. Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution. Let $u = \cos^{n-1} x \iff du = -(n-1)\cos^{n-2} x\sin x\,dx$ and $dv = \cos x\,dx \iff v = \sin x$ then

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx\\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx\\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

Therefore

$$(n-1+1)\int \cos^n x\,dx = n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

■

Problem 5.1.6. Show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution. We use the reduction formula then

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \left[-\frac{1}{n}\sin^{n-1} x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\\ &= -\frac{1}{n}\left(\sin^{n-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin^{n-1}(0)\cos(0)\right) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\end{aligned}$$

and since $\sin(0) = 0$ and $\cos(\pi/2) = 0$ so

$$\int_0^{\pi/2} \sin^n x\,dx = -\frac{1}{n}(0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

■

Problem 5.1.7. Show that

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution. By the reduction formula

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n+1} x\, dx &= \frac{2n}{2n+1} \int_0^{\pi/2} \sin^{2n-1} x\, dx = \frac{(2n)(2n-2)}{(2n+1)(2n-1)} \int_0^{\pi/2} \sin^{2n-3} x\, dx \\ &= \frac{(2n)(2n-2)\cdots(6)(4)(2)}{(2n+1)(2n-1)\cdots(7)(5)(3)} \int_0^{\pi/2} \sin x\, dx\end{aligned}$$

and since

$$\int_0^{\pi/2} \sin x\, dx = \Big[-\cos x \Big]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 0 + 1 = 1$$

hence

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

■

Problem 5.1.8. Show that

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}$$

where $n \in \mathbb{N}$ and $n \geq 2$.

Solution.

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n} x\, dx &= \frac{2n-1}{2n} \int_0^{\pi/2} \sin^{2n-2} x\, dx = \frac{(2n-1)(2n-3)}{(2n)(2n-2)} \int_0^{\pi/2} \sin^{2n-4} x\, dx \\ &= \frac{(2n-1)(2n-3)\cdots(5)(3)(1)}{(2n)(2n-2)\cdots(6)(4)(2)} \int_0^{\pi/2} dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}\end{aligned}$$

■

Problem 5.1.9. Prove the reduction formula

$$\int (\ln x)^n\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$$

Solution. Let $u = (\ln x)^n \iff du = \dfrac{n}{x}(\ln x)^{n-1}$ and $dv = dx \iff v = x$ then

$$\int (\ln x)^n\, dx = x(\ln x)^n - \int x \cdot \frac{n}{x}(\ln x)^{n-1}\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$$

■

Problem 5.1.10. Prove the reduction formula

$$\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$$

Solution. Let $u = x^n \iff du = nx^{n-1}\, dx$ and $dv = e^x\, dx \iff v = e^x$ then

$$\int x^n e^x\, dx = x^n e^x - \int nx^{n-1} e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$$

■

5.2 Trigonometric Integrals and Substitutions

Trigonometric Integrals

We can use trigonometric identities to integrate trigonometric integrals.

Problem 5.2.1. Evaluate $\int \sin 5x \sin 2x \, dx$.

Solution.

$$\begin{aligned}\int \sin 5x \sin 2x \, dx &= \int \frac{1}{2}(\cos(5x - 2x) - \cos(5x + 2x)) \, dx = \frac{1}{2} \int (\cos 3x - \cos 7x) \, dx \\ &= \frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C = \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C\end{aligned}$$

∎

Problem 5.2.2. Evaluate $\int \sin 3x \cos x \, dx$.

Solution.

$$\begin{aligned}\int \sin 3x \cos x \, dx &= \int \frac{1}{2}(\sin(3x + x) + \sin(3x - x)) \, dx = \frac{1}{2} \int (\sin 4x + \sin 2x) \, dx \\ &= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right) + C = -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C\end{aligned}$$

∎

Problem 5.2.3. Evaluate $\int \cos^3 x \, dx$.

Solution.

$$\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int (1 - u^2) \, du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

∎

Problem 5.2.4. Evaluate $\int \sin^5 x \cos^2 x \, dx$.

Solution.

$$\begin{aligned}\int \sin^5 x \cos^2 x \, dx &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx = -\int (1 - u^2)^2 u^2 \, du \\ &= -\int (u^2 - 2u^4 + u^6) \, du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$

∎

Problem 5.2.5. Evaluate $\int \cos^2 x \, dx$.

Solution.

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$$

■

Problem 5.2.6. Evaluate $\int_0^\pi \sin^2 x\,dx$.

Solution.

$$\begin{aligned}\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2x)\,dx \\ &= \frac{1}{2}\left(\Big[x\Big]_0^\pi - \frac{1}{2}\Big[\sin 2x\Big]_0^\pi\right) = \frac{\pi}{2}\end{aligned}$$

■

Problem 5.2.7. Evaluate $\int \sin^4 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx \\ &= \frac{1}{4}\left(x-\sin 2x+\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right)+C \\ &= \frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right)+C\end{aligned}$$

■

In general, an integral of powers of $\sin x$ and $\cos x$ is in the form

$$\int \sin^m x\cos^n x\,dx$$

where $m,n\in\mathbb{Z}$ and $m,n\geq 0$. If m is odd, then we save a factor of $\sin x$ and express the rest in terms of $\cos x$ for substitution. If n is odd, then we save a factor of $\cos x$ and express the rest in terms of $\sin x$ for substitution. If m and n are even, then we use the power reduction formulas.

Problem 5.2.8. Evaluate $\int \tan^6 x\sec^4 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^6 x\sec^4 x\,dx &= \int \tan^6 x(\tan^2 x+1)\sec^2 x\,dx = \int u^6(u^2+1)\,du = \int u^8+u^6 \\ &= \frac{u^9}{9}+\frac{u^7}{7} = \frac{1}{9}\tan^9 x+\frac{1}{7}\tan^7 x+C\end{aligned}$$

■

Problem 5.2.9. Evaluate $\int \tan^5 x \sec^7 x\, dx$.

Solution.

$$\begin{aligned}\int \tan^5 x \sec^7 x\, dx &= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x\, dx = \int (u^2-1)^2 u^6\, du \\ &= \int (u^{10} - 2u^8 + u^6)\, dx = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C\end{aligned}$$

■

In general, an integral of powers of $\tan x$ and $\sec x$ is in the form

$$\int \tan^m x \sec^n x\, dx$$

where $m, n \in \mathbb{Z}$ and $m, n \geq 0$. If m is odd, then we save a factor of $\sec x \tan x$ and express the rest in terms of $\sec x$ for substitution. If n is even, then we save a factor of $\sec^2 x$ and express the rest in terms of $\tan x$ for substitution.

Problem 5.2.10. Evaluate $\int \sec x\, dx$.

Solution. We have

$$\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x \iff du = (\sec x \tan x + \sec^2 x)\, dx$, then

$$\int \sec x\, dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

■

Problem 5.2.11. Evaluate $\int \tan^3 x\, dx$.

Solution.

$$\begin{aligned}\int \tan^3 x\, dx &= \int \tan x (\sec^2 x - 1)\, dx = \int \tan x \sec^2\, dx - \int \tan x\, dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned}$$

■

Problem 5.2.12. Evaluate $\int \sec^3 x\, dx$.

Solution. Let $u = \sec x \iff du = \sec x \tan x$ and $dv = \sec^2 x\,dx \iff v = \tan x$, then

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$$

We have

$$\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$$

Therefore,

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
\int \sec^3 x\,dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C
\end{aligned}$$

■

Trigonometric Substitutions

If an integral has the form $\int \sqrt{a^2 - x^2}\,dx$, then we can use the substitution $x = a\sin\theta$ where $-\pi/2 \le \theta \le \pi/2$ to get

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = |a|\cos\theta$$

If an integral has the form $\int \sqrt{a^2 + x^2}\,dx$, then we can use the substitution $x = a\tan\theta$ where $-\pi/2 < \theta < \pi/2$ to get

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(\tan^2\theta + 1)} = \sqrt{a^2\sec^2\theta} = |a|\sec\theta$$

If an integral has the form $\int \sqrt{x^2 - a^2}\,dx$, then we can use the substitution $x = a\sec\theta$ where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$ to get

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a|\tan\theta$$

Problem 5.2.13. Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$

Solution. Let $x = 3\sin\theta \iff dx = 3\cos\theta\,d\theta$, then

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta \\
&= -\cot\theta - \theta + C = -\frac{\sqrt{9 - x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

■

Problem 5.2.14. Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 4}}$.

Solution. Let $x = 2\tan\theta \iff dx = 2\sec^2\theta\, d\theta$ and $-\pi/2 < \theta < \pi/2$, then

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{4\tan^2\theta(2\sec\theta)}\, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{du}{u^2}$$
$$= -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

■

Problem 5.2.15. Find the area enclosed by a circle with radius r.

Solution. The area is $A = 4\int_0^r \sqrt{r^2 - x^2}\, dx$. Let $x = r\sin\theta \iff dx = r\cos\theta\, d\theta$ and $0 \le \theta \le \pi/2$, then

$$A = 4\int_0^{\pi/2} r\cos\theta(r\cos\theta)\, d\theta = 4r^2\int_0^{\pi/2} \cos^2\theta\, d\theta = 4r^2\left(\frac{1}{2}\left(\Big[\theta\Big]_0^{\pi/2} + \frac{1}{2}\Big[\sin 2\theta\Big]_0^{\pi/2}\right)\right)$$
$$= 4r^2\left(\frac{\pi}{4}\right) = \pi r^2$$

■

Problem 5.2.16. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. The area is $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$. Let $x = a\sin\theta \iff dx = a\cos\theta\, d\theta$ and $0 \le \theta \le \pi/2$, then

$$A = 4\int_0^{\pi/2} \frac{b}{a}(a\cos\theta)(a\cos\theta)\, d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta = 4ab\left(\frac{\pi}{4}\right) = \pi ab$$

■

5.3 Partial Fractions

Partial Fractions

Consider a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials. If f is improper, that is, $\deg(P) \ge \deg(Q)$, then we simplify to get $f(x) = S(x) + \dfrac{R(x)}{Q(x)}$ where S and R are polynomials. Then we can factor Q to be irreducible and express $\dfrac{R(x)}{Q(x)}$ as the a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

There are four possible cases:

1. Q is a product of distinct linear factors. Then

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

and therefore

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

2. Q is a product of linear factors where some are repeated. Suppose that the first linear factor is repeated r times, then we have

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

for the first repeated linear factor and similarly for other repeated linear factors.

3. Q has irreducible quadratic factors without repeated factors. Then for every quadratic factor we have

$$\frac{Ax + B}{ax^2 + bx + c}$$

4. Q has a repeated irreducible quadratic factor. Suppose that the first quadratic factor is repeated r times, then we have

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

for the first repeated quadratic factor and similarly for the others.

Problem 5.3.1. Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

Solution. We simplify to get $2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$ then

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$
$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

$$\begin{aligned} x = 0 &\iff -1 = -2A \iff A = \frac{1}{2} \\ x = \frac{1}{2} &\iff \frac{1}{4} = \frac{5}{4}B \iff B = \frac{1}{5} \\ x = -2 &\iff -1 = 10C \iff C = -\frac{1}{10} \end{aligned}$$

and therefore

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} &= \frac{1}{2}\int \frac{dx}{x} + \frac{1}{5}\int \frac{dx}{2x - 1} - \frac{1}{10}\int \frac{dx}{x + 2} \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x - 1| - \frac{1}{10}\ln|x + 2| + K \end{aligned}$$

■

Problem 5.3.2. Evaluate $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution. We simplify to get

$$\begin{aligned} x^4-2x^2+4x+1 &= (x^2-1)^2+4x = (x+1)^2(x-1)^2+4x \\ x^3-x^2-x+1 &= x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2 \end{aligned}$$

and so

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{(x+1)(x-1)^2}\,dx$$

Then

$$\begin{aligned} \frac{4x}{(x+1)(x-1)^2} &= \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \\ 4x &= A(x-1)^2+B(x+1)(x-1)+C(x+1) \\ x=-1 \iff -4=4A &\iff A=-1 \\ x=1 \iff 4=2C &\iff C=2 \\ x=0 \iff 0=-1-B+2 &\iff B=1 \end{aligned}$$

and therefore

$$\begin{aligned} \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int (x+1)\,dx - \int \frac{dx}{x+1} + \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} \\ &= \frac{x^2}{2}+x-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}+K \end{aligned}$$

■

Problem 5.3.3. Evaluate $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$.

Solution. We do partial fraction decomposition then

$$\begin{aligned} \frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} &= \frac{A}{x}+\frac{Bx+C}{x^2+4} \\ 2x^2-x+4 = A(x^2+4)+(Bx+C)x &= (A+B)x^2+Cx+4A \end{aligned}$$

$$A=1, \quad B=1, \quad C=-1$$

and so

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{dx}{x} + \int \frac{x-1}{x^2+4}\,dx = \ln|x| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right)+K$$

■

Problem 5.3.4. Evaluate $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$.

Solution. We simplify to get

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int \frac{4x^2-4x+3+x-1}{4x^2-4x+3}\,dx = \int \left(1+\frac{x-1}{(2x-1)^2+2}\right)dx$$

then

$$\begin{aligned}\int \frac{x-1}{(2x-1)^2+2}\,dx &= \frac{1}{2}\int \frac{\frac{1}{2}(u+1)-1}{u^2+2}\,dx = \frac{1}{4}\int \frac{u-1}{u^2+2}\,dx \\ &= \frac{1}{4}\left(\frac{1}{2}\ln(u^2+2)-\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}u\right)\right) \\ &= \frac{1}{8}\ln((2x-1)^2+2)-\frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x-1)\right)\end{aligned}$$

and so

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = x+\frac{1}{8}\ln(4x^2-4x+3)-\frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x-1)\right)+C$$

■

Problem 5.3.5. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

Solution. We do partial fraction decomposition then

$$\begin{aligned}\frac{1-x+2x^2-x^3}{x(x^2+1)^2} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2} \\ 1-x+2x^2-x^3 &= A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\end{aligned}$$

$$x=0 \iff A=1$$

and so

$$\begin{aligned}1-x+2x^2-x^3 &= x^4+2x^2+1+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \\ -x-x^3 &= (B+1)x^4+Cx^3+(B+D)x^2+(C+E)x\end{aligned}$$

$$B=-1 \quad C=-1 \quad D=1 \quad E=0$$

Therefore

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{dx}{x}-\int \frac{x+1}{x^2+1}\,dx+\int \frac{x}{(x^2+1)^2}\,dx \\ &= \ln|x|-\frac{1}{2}\ln(x^2+1)-\arctan x-\frac{1}{2(x^2+1)}+K\end{aligned}$$

■

5.4 Improper Integrals

An **improper integral** is an extension of a definite integral $\int_a^b f(x)\,dx$ where the interval is infinite or $f(x)$ has an infinite discontinuity on the definite interval $[a,b]$.

Infinite Intervals

Definition 5.4.1. If $\int_t^a f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

if the limit exists. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

if the limit exists. The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are **convergent** if the corresponding limit exists and **divergent** if the limit does not exist. If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

for any real number a.

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Discontinuous Integrands

Definition 5.4.2. If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if the limit exists. If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if the limit exists. The improper integral $\int_a^b f(x)\,dx$ is **convergent** if the corresponding limit exists and **divergent** if the limit does not exist. If f has a discontinuity at c where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Theorem 5.4.1 Comparison Theorem. Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

6 Applications of Integration

6.1 Areas Between Curves

Area Between Curves: Integrating With Respect to x

Area Between Curves: Integrating With Respect to y

6.2 Volumes

Definition of Volume

Volumes of Solids of Revolution

Finding Volume Using Cross-Sectional Area

Volumes by Cylindrical Shells

6.3 Arc Length

Arc Length of a Curve

The Arc Length Function

6.4 Area of a Surface of Revolution

7 Sequences and Series

7.1 Sequences

Infinite Sequences

An infinite **sequence** $\{a_n\}$ is a list of numbers is a definite number

$$a_1, a_2, a_3 \ldots, a_n, a_{n+1}, \ldots$$

where a_n is the nth term of the sequence. Consider the sequence

$$a_n = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots\right\}$$

we can rewrite it as

$$a_n = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \cdots\right\}$$

so the formula for the nth term is

$$a_n = \frac{1}{2^n}$$

for $n = 1, 2, 3, \cdots$ and

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}$$

The Limit of a Sequence

The intuitive definition of a limit of a sequence is:

Definition 7.1.1. A sequence $\{a_n\}$ has the limit L so

$$\lim_{n\to\infty} a_n = L$$

or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, then the sequence converges, otherwise it diverges.

The precise definition of a limit of a sequence is:

Definition 7.1.2. A sequence $\{a_n\}$ has the limit L so

$$\lim_{n\to\infty} a_n = L$$

or $a_n \to L$ as $n \to \infty$ if for every $\varepsilon > 0$ there is an integer N such that

$$n > N \implies |a_n - L| < \varepsilon$$

The precise definition of an infinite limit of a sequence is:

Definition 7.1.3. $\lim\limits_{n\to\infty} a_n = \infty$ if for every positive number M there is an integer N such that

$$n > N \implies a_n > M$$

An analogous definition applies for $\lim\limits_{n\to\infty} a_n = -\infty$

Properties of Convergent Sequences

Theorem 7.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$.

Theorem 7.1.2. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 7.1.3. If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

Monotonic and Bounded Sequences

Definition 7.1.4. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. A sequence is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Definition 7.1.5. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. A sequence is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If a sequence is bounded above and below, then it is a **bounded sequence**.

Theorem 7.1.4 Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent. In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

The proof of the monotonic sequence theorem is based on the **completeness axiom** of the set of real numbers $\mathbb{R}$. The completeness axiom states if S is a nonempty set of real numbers that has an upper bound M so $x \leq M$ for all $x \in S$, then S has a **least upper bound** b. (This means that b is an upper bound of S, but if M is any other upper bound, then $b \leq M$.) The completeness axiom is an expression of the fact that there is no gap or hole in the real number line.

Proof. Suppose $\{a_n\}$ is an incerasing sequence. Since $\{a_n\}$ is an bounded, the set $S = \{a_n \mid n \geq 1\}$ has an upper bound. By the completeness axiom, it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S since L is the least upper bound. Therefore we have $a_N > L - \varepsilon$ for some integer N but the sequence is incerasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$\begin{aligned} a_n &> L - \varepsilon \\ a_n + \varepsilon &> L \\ L &< a_n + \varepsilon \\ 0 \leq L - a_n &< \varepsilon \end{aligned}$$

since $a_n \leq L$. Thus

$$\begin{aligned} |L - a_n| &< \varepsilon \\ |a_n - L| &< \varepsilon \end{aligned}$$

whenever $n > N$ so $\lim\limits_{n\to\infty} a_n = L$. ∎

A similar proof (using the greatest lower bound) holds if $\{a_n\}$ is decreasing.

7.2 Series

Infinite Series

The sum of the terms of an infinite sequence $\{a_n\}$ is the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Definition 7.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots$, let s_n be the nth partial sum:

$$s_n = \sum_{i=1}^{n} a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = s$$

The number s is the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is divergent.

Sum of a Geometric Series

Consider the **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots, \quad a \neq 0$$

with common ratio r. If $r = 1$, then $s_n = a + a + \cdots + a = na$. Thus $\lim_{n\to\infty} s_n = \pm\infty$ so the limit does not exist and the geometric series diverges when $r = 1$. If $r \neq 1$, then we have

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \\ (1-r)s_n &= a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

If $-1 < r < 1$, since $\lim_{n\to\infty} r^n = 0$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1-r)$. If $r \leq -1$ or $r > 1$, then the sequence $\{r^n\}$ is divergent so $\lim_{n\to\infty} s_n$ does not exist thus the geometric series diverges. The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, then the geometric series is divergent.

Problem 7.2.1. Find the sum of the series $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$.

Test for Divergence

Problem 7.2.2. Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Theorem 7.2.1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Theorem 7.2.2 Test for Divergence. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Properties of Convergent Series

7.3 Convergence Tests

The Integral Test

The Direct Comparison Test

The Limit Comparison Test

The Ratio Test

Theorem 7.3.1 The Ratio Test. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then no conclusion can be drawn about the convergence or divergence of the series.

The Root Test

7.4 Power Series

Power Series

A **power series** centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = a_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and the coefficient c_n is a constant. The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series converges.

Interval of Convergence

Theorem 7.4.1. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all $x \in \mathbb{R}$.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in cases 3 is the radius of convergence of the power series. The radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case 2. The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Representations of Functions as Power Series

We start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Differentiation and Integration of Power Series

We can differentiate and integrate a power series term by term.

Theorem 7.4.2. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and continuous on the interval $(a-R, a+R)$. Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

and

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series are both R.

Functions Defined by Power Series

7.5 Taylor Series

Taylor Series and Maclaurin Series

The **Taylor series** of a function f centered at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

For the special case of $a = 0$, the Taylor series becomes the **Maclaurin series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

Problem 7.5.1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution. We have

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

and since $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = e^0 = 1$ for all n therefore

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $a_n = x^n/n!$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

and

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

so by the ratio test the series converges for all $x \in \mathbb{R}$ and the radius of convergence is $R = \infty$. ■

When Is a Function Represented by Its Taylor Series?

The nth degree Taylor polynomial T_n of f at a is the partial sum

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Let $R_n(x) = f(x) - T_n(x)$ so that $f(x) = T_n(x) + R_n(x)$, then $R_n(x)$ is the remainder of the Taylor series. If $\lim_{n\to\infty} R_n(x) = 0$, then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x)$$

Theorem 7.5.1. If $f(x) = T_n(x) + R_n(x)$ where T_n is the nth degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem 7.5.2 Taylor's Formula. If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder term in the Taylor series is the expression

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

which is the Lagrange's form of the remainder term.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for all $x \in \mathbb{R}$.

Taylor Series of Important Functions

$$e^x = \sum_{n=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

If k is any real number and $|x| < 1$, then the binomial series is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Problem 7.5.2. Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

New Taylor Series from Old

Multiplication and Division of Power Series

8 Parametric Equations and Polar Coordinates

8.1 Curves Defined by Parametric Equations

Parametric Equations

Suppose that x and y are both given as functions of the variable t, the **parameter**, by the equations

$$x = f(t) \qquad\qquad y = g(t)$$

which are the **parametric equations**. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve which is the parametric curve. In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

The Cycloid

8.2 Calculus with Parametric Curves

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), y = g(t)$, where y is a differentiable function of x. Then the chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

It follows that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Areas

If the curve $y = F(x)$ from a to b is traced out once by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, then the area under the curve is

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

Arc Length

Theorem 8.2.1. If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once at t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Surface Area

Suppose that a curve C is given by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$ where f', g' are continuous, $g(t) \geq 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about the x-axis, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

8.3 Polar Coordinates

The Polar Coordinate System

We choose a point O in the plane called the **pole** (or origin). Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates. If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and line OP. Then the point P is represented by the ordered pair (r, θ) called the polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ. We extend the meaning of polar coordinates (r, θ) to the case r is negative by agreeing that, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Note that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Relationship between Polar and Cartesian Coordinates

Suppose that point P has Cartesian coordinates (x, y) and polar coordinates (r, θ). To find Cartesian coordinates (x, y) when polar coordinates (r, θ) are known, we use the equations

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

To find polar coordinates (x, y) when Cartesian coordinates (r, θ) are known, we use the equations

$$r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x}$$

Polar Curves

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Symmetry

8.4 Calculus in Polar Coordinates

Area

Arc Length

Let θ be a parameter, then the parametric equations for a polar curve $r = f(\theta)$ are

$$x = r\cos\theta = f(\theta)\cos\theta \qquad\qquad y = r\sin\theta = f(\theta)\sin\theta$$

To find the length of the polar curve $r = f(\theta), a \le \theta \le b$, we differentiate with respect to θ (using the product rule):

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad\qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Then we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, the length of a curve with polar equation $r = f(\theta), \alpha \le \theta \le \beta$, is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Tangents

To find a tangent line for polar curve $r = f(\theta)$, we find the slope of the parametric curve. We have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\cos\theta - r\sin\theta}{\frac{dr}{d\theta}\sin\theta + r\cos\theta}$$

9 Differential Equations

9.1 Ordinary Differential Equations

A **differential equation** is an equation that relates some unknown functions and their derivatives. An **ordinary differential equation** (ODE) is a differential equation that relates one or more functions of a single variable and their ordinary derivatives. The **order** of a differential equation is the highest order of the derivative in the equation. Newton's second law of motion, the force F acting on an object with mass m and acceleration a is $F = ma$ can be written as an ordinary differential equation

$$F = m\frac{dv}{dt}$$

which is a first order differential equation, or

$$F = m\frac{d^2s}{dt^2}$$

which is a second order differential equation. By Newton's second law of motion, the net force F acting on a falling object with mass m, velocity v, and air resistance force γv can be modeled by the differential equation

$$F = m\frac{dv}{dt} = mg - \gamma v$$

where g is the acceleration due to gravity and γ is the drag coefficient. A function f is a **solution** of a differential equation if the function and its derivatives satisfy the equation for all values of x in some open interval $a < x < b$. It is possible that there are many solutions of a differential equation. An initial condition is a condition $y(x_0) = y_0$ or $y^{(n)}(x_0) = y_n$ on the solution. An **initial value problem** is solving a differential equation with initial conditions. The interval of validity is the largest possible interval on which the solution is valid and contains x_0 in the initial conditions. The general solution of a differential equation is the set of all solutions and the particular solution is the solution that satisfies the initial conditions. An explicit solution is any solution in the form $y = y(x)$, otherwise it is an implicit solution. The particular solution of the differential equation

$$\frac{dy}{dx} = y$$

with initial condition $y(0) = 1$ is $y = e^x$ since

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x = y$$

and $y(0) = e^0 = 1$. The **existence** and **uniqueness** problem asks that given a differential equation, does there exist a solution and if any is there only one solution.

9.2 Direction Fields and Euler's Method

Direction Fields

Suppose we are given a first order differential equation of the form $y' = F(x, y)$. If a solution curve, or an integral curve, passes through the point (x_0, y_0), then its slope at the point is $y'(x_0) = F(x_0, y_0)$. If we draw short line segments with slopes $F(x, y)$ at serval points (x, y), then the result is a **direction field**, or a **slope field**.

Euler's Method

9.3 Separable Equations

Separable Equations

A **separable equation** is a first order differential equation that can be written in the differential form

$$M(x)\,dx + N(y)\,dy = 0$$

or can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

We can separate the variables if $g(y) \neq 0$, let $h(y) = 1/g(y)$ then

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}$$

We write the equation in the differential form

$$h(y)\,dy = f(x)\,dx$$

Then we integrate both sides of the equation:

$$\int h(y)\,dy = \int f(x)\,dx$$

Now we have an implicit solution of the differential equation and sometimes we can solve for an explicit solution. We can justify the method of separation of variables by using the chain rule to show that

$$\begin{aligned}
\frac{d}{dx}\int h(y)\,dy &= \frac{d}{dx}\int f(x)\,dx \\
\frac{d}{dy}\int h(y)\,dy\frac{dy}{dx} &= f(x) \\
h(y)\frac{dy}{dx} &= f(x) \\
\frac{dy}{dx} &= \frac{f(x)}{h(y)} = f(x)g(y)
\end{aligned}$$

Problem 9.3.1. Solve the differential equation $\dfrac{dy}{dx} = -xy$.

Solution. Notice that $y = 0$ is a trivial solution then we solve the differential equation for non-trivial solutions $y \neq 0$. We use separation of variables then

$$\begin{aligned}
\frac{dy}{dx} &= -xy \\
\int \frac{dy}{y} &= -\int x\,dx \\
\ln|y| + C_1 &= -\frac{x^2}{2} + C_2
\end{aligned}$$

Let $C = C_2 - C_1$, then

$$\begin{aligned}
\ln|y| &= -\frac{x^2}{2} + C_2 - C_1 = -\frac{x^2}{2} + C \\
|y| &= e^{(-x^2/2)+C} = e^C e^{-x^2/2} \\
y &= \pm e^C e^{-x^2/2} = Ae^{-x^2/2}
\end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.2. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ with the initial condition $y(0) = 2$.

Solution. We use separation of variables to find the general solution then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2}{y^2} \\
\int y^2\,dy &= \int x^2\,dx \\
\frac{y^3}{3} + C_1 &= \frac{x^3}{3} + C_2 \\
\frac{y^3}{3} &= \frac{x^3}{3} + C, \quad C = C_2 - C_1 \\
y^3 &= x^3 + 3C \\
y &= \sqrt[3]{x^3 + K}, \quad K = 3C
\end{aligned}$$

We consider the initial condition $y(0) = 2$ to find the particular solution then

$$y(0) = \sqrt[3]{0 + K} \iff 2 = \sqrt[3]{K} \iff K = 8$$

The solution of the initial value problem is $y = \sqrt[3]{x^3 + 8}$. ■

Problem 9.3.3. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

Solution. We have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{6x^2}{2y + \cos y} \\
\int (2y + \cos y)\,dy &= \int 6x^2\,dx \\
y^2 + \sin y &= 2x^3 + C
\end{aligned}$$

where $C \in \mathbb{R}$ is an arbitrary constant. ■

Problem 9.3.4. Solve the differential equation $y' = x^2 y$.

Solution. We have

$$\begin{aligned}
\frac{dy}{dx} &= x^2 y \\
\int \frac{dy}{y} &= \int x^2\,dx \\
\ln|y| &= \frac{x^3}{3} + C \\
|y| &= e^{(x^3/3)+C} \\
y &= Ae^{x^3/3}
\end{aligned}$$

where $A \in \mathbb{R}$ is an arbitrary constant. ■

Homogeneous Equations

A **homogeneous** equation is in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

We can transform a homogeneous equation into a separable equation by a change of variable. Consider the equation

$$\frac{dy}{dx} = \frac{y-4x}{x-y}$$

and we can show that

$$\frac{dy}{dx} = \frac{(y/x)-4}{1-(y/x)}$$

thus the equation is homogeneous. Let $v = y/x \iff y = vx$ so

$$\begin{aligned}
\frac{dy}{dx} = \frac{dv}{dx}x + v &= \frac{v-4}{1-v} \\
\frac{dv}{dx}x &= \frac{v-4}{1-v} - v = \frac{v-4-v(1-v)}{1-v} = \frac{v^2-4}{1-v}
\end{aligned}$$

and the equation is separable then

$$\int \frac{1-v}{v^2-4}\,dv = \int \frac{dx}{x}$$

Since

$$\int \frac{1-v}{v^2-4}\,dv = \int \frac{dv}{v^2-4} - \int \frac{v}{v^2-4}\,dv$$

then

$$\begin{aligned}
\frac{1}{v^2-4} = \frac{1}{(v+2)(v-2)} &= \frac{A}{v+2} + \frac{B}{v-2} \\
1 &= A(v-2) + B(v+2) \\
v = -2 \iff -4A = 1 \iff A &= -\frac{1}{4} \\
v = 2 \iff 4B = 1 \iff B &= \frac{1}{4}
\end{aligned}$$

so

$$\int \frac{dv}{v^2-4} = -\frac{1}{4}\int \frac{dv}{v+2} + \frac{1}{4}\int \frac{dv}{v-2} = -\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2|$$

and

$$\int \frac{v}{v^2-4}\,dv = \frac{1}{2}\ln|v^2-4|$$

therefore

$$\int \frac{1-v}{v^2-4}\,dv = -\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2| - \frac{1}{2}\ln|v^2-4|$$

Then

$$\begin{aligned}
-\frac{1}{4}\ln|v+2| + \frac{1}{4}\ln|v-2| - \frac{1}{2}\ln|v^2-4| &= \ln|x| + C_1 \\
\ln|v-2| - \ln|v+2| - 2\ln|v^2-4| &= 4\ln|x| + C_2, \quad C_2 = 4C_1 \\
\ln\left|\frac{v-2}{v+2}\right| - \ln((v^2-4)^2) = \ln\left|\frac{v-2}{(v+2)(v+2)^2(v-2)^2}\right| &= \ln(x^4) + C_2 \\
\ln\left|\frac{1}{(v+2)^3(v-2)}\right| &= \ln(x^4) + C_2 \\
\frac{1}{(v+2)^3(v-2)} &= C_3x^4, \quad C_3 = \pm e^{C_2} \\
(v+2)^3(v-2)x^4 &= C, \quad C = 1/C_3 \\
(vx+2x)^3(vx-2x) &= C
\end{aligned}$$

Therefore the solution is

$$(y+2x)^3(y-2x) = C$$

Problem 9.3.5. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$$

Solution. Since

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

hence the equation is homogeneous. Let $y = vx$, then

$$\begin{aligned}
\frac{dy}{dx} = \frac{dv}{dx}x + v &= 1 + v + v^2 \\
\frac{dv}{dx}x &= 1 + v^2
\end{aligned}$$

and by separation of variables

$$\begin{aligned}
\int \frac{dv}{1+v^2} &= \int \frac{dx}{x} \\
\arctan v &= \ln x + C \\
\arctan\left(\frac{y}{x}\right) - \ln x &= C
\end{aligned}$$

■

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles.

Problem 9.3.6. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Solution. The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis. We differentiate $x = ky^2$ and have

$$\begin{aligned} 1 &= 2ky\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{2ky} \end{aligned}$$

and since $k = x/y^2$ so we get

$$\frac{dy}{dx} = \frac{1}{2(x/y^2)y} = \frac{y}{2x}$$

which is the slope of the tangent line at any point (x, y) on one of the parabolas. On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\begin{aligned} \int y\,dy &= -\int 2x\,dx \\ \frac{y^2}{2} &= -x^2 + C \\ x^2 + \frac{y^2}{2} &= C \end{aligned}$$

where C is an arbitrary positive constant. Thus the orthogonal trajectories are the family of ellipses given by the equation $x^2 + (y^2/2) = C$. ■

Mixing Problems

9.4 Population Growth

Let $y = y(t)$ be a function representing the value of a quantity y at time t such that

$$\frac{dy}{dt} = ky$$

where k is a constant, then the differential equation is the law of natural growth if $k > 0$ or the law of natural decay if $k < 0$. Since the differential equation is separable, then

$$\begin{aligned}
\frac{dy}{dt} &= ky \\
\int \frac{dy}{y} &= \int k\,dt \\
\ln|y| &= kt + C \\
|y| &= e^{kt+C} = e^C e^{kt} \\
y &= \pm e^C e^{kt} = Ae^{kt}
\end{aligned}$$

so $y = Ae^{kt}$ is the general solution of the differential equation. It follows that

$$\frac{dy}{dt} = kAe^{kt} = ky$$

and $y(0) = Ae^0 = A$ is the initial value of the function $y(t)$.

Logistic Growth

If M is the carrying capacity and $0 < y < M$, then the logistic differential equation is

$$\frac{dy}{dt} = ky(M - y)$$

We use separation of variables then

$$\begin{aligned}
\frac{dy}{dt} &= ky(M - y) \\
\int \frac{dy}{y(M - y)} &= \int k\,dt
\end{aligned}$$

and using partial fractions

$$\begin{aligned}
\frac{1}{y(M - y)} &= \frac{A}{y} + \frac{B}{M - y} \\
1 &= A(M - y) + By \\
y = 0 \iff A &= \frac{1}{M} \\
y = M \iff B &= \frac{1}{M}
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{1}{M} \int \left(\frac{1}{y} + \frac{1}{M - y} \right) dy &= \int k\,dt \\
\frac{1}{M}(\ln|y| - \ln|M - y|) &= kt + C_1 \\
\ln \frac{y}{M - y} &= kMt + C_2 \\
\frac{y}{M - y} &= e^{kMt + C_2} = e^{C_2} e^{kMt} = Ae^{kMt}
\end{aligned}$$

If the population at time $t = 0$ is $y(0) = y_0$, then $A = y_0/(M - y_0)$ and so

$$\begin{aligned}
\frac{y}{M-y} &= \frac{y_0}{M-y_0}e^{kMt} \\
(M-y_0)y &= y_0 e^{kMt}(M-y) = y_0 M e^{kMt} - y_0 e^{kMt} y \\
(M-y_0)y + y_0 e^{kMt} y = (M - y_0 + y_0 e^{kMt})y &= y_0 M e^{kMt} \\
y &= \frac{y_0 M e^{kMt}}{M - y_0 + y_0 e^{kMt}}
\end{aligned}$$

then

$$y = \frac{y_0 M}{(M - y_0 + y_0 e^{kMt})e^{-kMt}} = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}$$

is the solution of the differential equation and

$$\lim_{t\to\infty} y(t) = \frac{y_0 M}{y_0 + 0} = M$$

We can show that

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}(kMy - ky^2) = kM\frac{dy}{dt} - 2ky\frac{dy}{dt} = k(M-2y)\frac{dy}{dt} = k(M-2y)ky(M-y) \\
&= k^2 y(M-y)(M-2y)
\end{aligned}$$

then

$$k^2 y(M-y)(M-2y) = 0$$

and

$$\begin{aligned}
k^2 y &= 0 & M - y &= 0 & M - 2y &= 0 \\
y &= 0 & y &= M & y &= \frac{M}{2}
\end{aligned}$$

so

$$\begin{aligned}
\left.\frac{dy}{dt}\right|_{y=0} &= k(0)(M-0) = 0 \\
\left.\frac{dy}{dt}\right|_{y=M} &= kM(M-M) = 0 \\
\left.\frac{dy}{dt}\right|_{y=M/2} &= k\frac{M}{2}\left(M - \frac{M}{2}\right) = \frac{kM^2}{4}
\end{aligned}$$

We deduce that a population grows fastest when it reaches half its carrying capacity.

Problem 9.4.1. An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Suppose that the resisting force is proportional to the velocity for small values of v so

$$f(v) = -kv$$

where k is a positive constant. For large values of v a better model is

$$f(v) = -kv^2$$

where k is a positive constant. Determine v and s at any time t for each of the two models. What is the total distance that the object travels in each case?

Solution. For small values of v we have

$$\begin{aligned}
m\frac{dv}{dt} &= -kv \\
\int \frac{dv}{v} &= -\int \frac{k}{m}\,dt \\
\ln(|v|) &= -\frac{k}{m}t + C \\
|v| &= e^{-(k/m)t+C} = e^C e^{-(k/m)t} \\
v &= \pm e^C e^{-(k/m)t} = Ae^{-(k/m)t}
\end{aligned}$$

and since $v(0) = A = v_0$ hence $v = v_0 e^{-(k/m)t}$. Then

$$s = \int v_0 e^{-(k/m)t}\,dt = -\frac{m}{k}v_0 e^{-(k/m)t} + C$$

and since $s(0) = -(m/k)v_0 + C = s_0 \iff C = s_0 + (m/k)v_0$ hence

$$s = -\frac{m}{k}v_0 e^{-(k/m)t} + s_0 + \frac{m}{k}v_0 = s_0 + \frac{m}{k}v_0(1 - e^{-(k/m)t})$$

The total distance traveled by the object is

$$\begin{aligned}
\lim_{t\to\infty}(s(t) - s(0)) &= \lim_{t\to\infty}\left(s_0 + \frac{m}{k}v_0(1 - e^{-(k/m)t}) - s_0\right) = \lim_{t\to\infty}\frac{m}{k}v_0(1 - e^{-(k/m)t}) \\
&= \frac{m}{k}v_0(1 - 0) = \frac{m}{k}v_0
\end{aligned}$$

For large values of v we have

$$\begin{aligned}
m\frac{dv}{dt} &= -kv^2 \\
\int \frac{dv}{v^2} &= -\int \frac{k}{m}\,dt \\
-\frac{1}{v} &= -\frac{k}{m}t + C_1 \\
\frac{1}{v} &= \frac{k}{m}t + C = \frac{kt + Cm}{m} \\
v &= \frac{m}{kt + Cm}
\end{aligned}$$

and since $v(0) = m/(Cm) = v_0 \iff C = 1/v_0$ hence

$$v = \frac{m}{kt + (m/v_0)}$$

Then

$$s = \int \frac{m}{kt + (m/v_0)}\,dt = \frac{m}{k}\int \frac{du}{u} = \frac{m}{k}\ln(|u|) + C = \frac{m}{k}\ln\left(\left|kt + \frac{m}{v_0}\right|\right) + C$$

and since $s(0) = (m/k)\ln(|m/v_0|) + C = s_0 \iff C = s_0 - (m/k)\ln(|m/v_0|)$ hence

$$\begin{aligned} s &= \frac{m}{k}\ln\left(\left|kt + \frac{m}{v_0}\right|\right) + s_0 - \frac{m}{k}\ln\left(\left|\frac{m}{v_0}\right|\right) = s_0 + \frac{m}{k}\left(\ln\left(\left|kt + \frac{m}{v_0}\right|\right) - \ln\left(\left|\frac{m}{v_0}\right|\right)\right) \\ &= s_0 + \frac{m}{k}\ln\left(\left|\frac{kt + (m/v_0)}{m/v_0}\right|\right) = s_0 + \frac{m}{k}\ln\left(\left|\frac{k}{m}v_0 t + 1\right|\right) \end{aligned}$$

Since m is the mass so m is positive and given k is positive therefore

$$s = s_0 + \frac{m}{k}\ln\left(\frac{k}{m}v_0 t + 1\right)$$

and the distance traveled by the object is

$$\lim_{t\to\infty}(s(t) - s(0)) = \lim_{t\to\infty}\left(s_0 + \frac{m}{k}\ln\left(\frac{k}{m}v_0 t + 1\right) - s_0\right) = \lim_{t\to\infty}\frac{m}{k}\ln\left(\frac{k}{m}v_0 t + 1\right) = \infty \quad \blacksquare$$

Problem 9.4.2. According to Newton's law of universal gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x+R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's second law,

$$F = ma = m\frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that $v_0 = \sqrt{\dfrac{2gRh}{R+h}}$ and compute $v_e = \lim_{h\to\infty} v_0$, the escape velocity of the earth, using $R = 6{,}378$ km and $g = 9.8\ \text{m/s}^2$.

Solution. By the chain rule

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

then

$$\begin{aligned} m\frac{dv}{dt} = mv\frac{dv}{dx} &= -\frac{mgR^2}{(x+R)^2} \\ v\,dv &= -\frac{gR^2}{(x+R)^2}\,dx \end{aligned}$$

Since the height is zero when $v = v_0$ and the height is maximum when $v = 0$ then

$$\begin{aligned}
\int_{v_0}^{0} v\,dv &= -\int_0^h \frac{gR^2}{(x+R)^2}\,dx \\
\left[\frac{v^2}{2}\right]_{v_0}^{0} &= \left[\frac{gR^2}{x+R}\right]_0^h \\
-\frac{(v_0)^2}{2} &= gR^2\left(\frac{1}{R+h} - \frac{1}{R}\right) = gR^2\left(\frac{R-(R+h)}{R(R+h)}\right) = -\frac{gRh}{R+h} \\
v_0 &= = \sqrt{\frac{2gRh}{R+h}}
\end{aligned}$$

The escape velocity of the earth is

$$\begin{aligned}
v_e &= \lim_{h\to\infty}\sqrt{\frac{2gRh}{R+h}} = \sqrt{\lim_{h\to\infty}\frac{2gR}{(R/h)+1}} = \sqrt{\frac{2gR}{0+1}} = \sqrt{2gR} \\
&= \sqrt{2(9.8)(6.378\times 10^6)}\ \text{m/s} \approx 11.1807\ \text{km/s}
\end{aligned}$$

■

The End