Calculus

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Introduction

Calculus is the mathematical study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **Fundamental Theorem of Calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Contents

1 Functions and Limits

1.1 The Limit of a Function

Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. A symbol that represents a number in the range of f is called a **dependent variable**.

If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

The graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**.

Increasing and Decreasing Functions

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2)$$

whenever $x_1 < x_2$ in I. It is called **decreasing** on I if

$$f(x_1) > f(x_2)$$

whenever $x_1 < x_2$ in I.

Inverse Functions

Definition 1.1.2. A function f is **injective** (or **one-to-one**) if

$$f(x_1) \neq f(x_2)$$

whenever $x_1 \neq x_2$.

Definition 1.1.3. A function f is **surjective** (or **onto**) if for all y in range Y, there exists an x in domain X such that $f(x) = y$.

Definition 1.1.4. A function f is **bijective** if f is injective and surjective (or one-to-one and onto).

Definition 1.1.5. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in B.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y and write

$$f^{-1}(x) = y \iff f(y) = x$$

We have the following **cancellation equations**:

$$f^{-1}(f(x)) = x$$

for every x in A and

$$f(f^{-1}(x)) = x$$

for every x in B.
Find the Inverse of a One-to-One Function f:

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Mathematical Modeling

A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

A mathematical model is never a completely accurate representation of a physical situation, it is an idealization.

Intuitive Definition of a Limit

Suppose $f(x)$ is defined near the number a. (This means that $f(x)$ is defined on some open interval that contains the number a, except possibly at a itself.)

Definition 1.1.6. We write

$$\lim_{x \to a} f(x) = L$$

and say that the **limit** of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but $x \neq a$.

An alternative notation for the limit is $f(x) \to L$ as $x \to a$.

One-Sided Limits

Definition 1.1.7. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

Definition 1.1.8. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

Theorem 1.1.1. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The limit exists if and only if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a are equal to L, otherwise the limit does not exist.

1.2 The Precise Definition of a Limit

Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2.1. We say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.1. Prove that $\lim_{x \to 3}(4x - 5) = 7$.

Solution. Let $\varepsilon > 0$ be a given positive number. We want to find a number δ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$. Note that $4|x - 3| < \varepsilon \iff |x - 3| < \varepsilon/4$. Let $\delta = \varepsilon/4$, we have

$$0 < |x - 3| < \frac{\varepsilon}{4} \implies 4|x - 3| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \to 3}(4x - 5) = 7$$

$\blacksquare$

Problem 1.2.2. Prove that $\lim_{x \to 3} x^2 = 9$.

Solution. Let ε be a given positive number. We want to find a number δ such that

$$0 < |x-3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x+3|\,|x-3| < \varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3| < C\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3| < 1$. Then we have $|x+3| < 7$, and so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x-3| < \delta$, then

$$|x^2 - 9| = |x+3|\,|x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x \to 3} x^2 = 9$. ∎

Definition 1.2.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.2.3.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution. Let ε be a given positive number We want to find a number δ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

But $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$. If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$ so $|\sqrt{x} - 0| < \varepsilon$. This shows that $\lim_{x \to 0^+} \sqrt{x} = 0$. ∎

1.3 Calculating Limits Using the Limit Laws

We have the following properties of limits called the **limit laws** to calculate limits. Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. Sum Law: $\lim\limits_{x\to a}\left[f(x)+g(x)\right] = L + M$
2. Difference Law: $\lim\limits_{x\to a}\left[f(x)-g(x)\right] = L - M$
3. Constant Multiple Law: $\lim\limits_{x\to a}\left[cf(x)\right] = cL$
4. Product Law: $\lim\limits_{x\to a}\left[f(x)g(x)\right] = LM$
5. Quotient Law: $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.
6. Power Law: $\lim\limits_{x\to a}\left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$ where n is a positive integer.
7. Root Law: $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim\limits_{x\to a} f(x) > 0$.
8. $\lim\limits_{x\to a} c = c$
9. $\lim\limits_{x\to a} x = a$
10. $\lim\limits_{x\to a} x^n = a^n$ where n is a positive integer.
11. $\lim\limits_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

Proof. Proof of limit law 8: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |c-c| < \varepsilon$$

We have $|c-c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. It is proved that $\lim\limits_{x\to a} c = c$. ■

Proof. Proof of limit law 9: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |x-a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x-a| < \delta = \varepsilon \implies |x-a| < \varepsilon$$

It is proved that $\lim\limits_{x\to a} x = a$. ■

Proof. Proof of the sum law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality,

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

We make $|f(x) - L| + |g(x) - M|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} \left[f(x) + g(x)\right] = L + M$$

■

Proof. Proof of the product law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows and use the triangle inequality:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= \left|\left[f(x) - L\right]g(x) + L[g(x) - M]\right| \\
&\leq \left|\left[f(x) - L\right]g(x)\right| + \left|L\left[g(x) - M\right]\right| \\
&= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x \to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \le |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$. Then we can combine the inequalities to get

$$\begin{aligned} |f(x)g(x) - LM| &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that

$$\lim_{x \to a} \left[f(x)g(x)\right] = LM$$

■

Proof. Proof of the constant multiple law: If we take $g(x) = c$ then by the product law and limit law 8, we get

$$\lim_{x \to a} \left[cf(x)\right] = \lim_{x \to a} c \cdot \lim_{x \to a} f(x) = c \lim_{x \to a} f(x) = cL$$

We can prove the constant multiple law using the precise the definition. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} \left[0 \cdot f(x)\right] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x) = 0 \cdot L$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of a limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

This shows that $\lim_{x \to a} \left[cf(x)\right] = cL$. ■

Proof. Proof of the difference law: Using the sum law and the constant multiple law with $c = -1$, we have

$$\begin{aligned}\lim_{x\to a}\left[f(x) - g(x)\right] &= \lim_{x\to a}\left[f(x) + (-1)g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x)\\ &= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M\end{aligned}$$

■

Proof. Proof of the quotient law: First we prove that

$$\lim_{x\to a}\frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

Observe that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x\to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \le |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

This shows that

$$0 < |x-a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \iff \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|}\cdot\frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left(f(x)\cdot\frac{1}{g(x)}\right) = \lim_{x\to a} f(x)\lim_{x\to a}\frac{1}{g(x)} = L\cdot\frac{1}{M} = \frac{L}{M}$$

■

We have the following **direct substitution property** to calculate limits. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Problem 1.3.1. Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

Solution.
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

■

If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$, provided that this limit exists. When computing one-sided limits, we use the fact that the limit laws also hold for one-sided limits.

Problem 1.3.2. Show that $\lim\limits_{x \to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim\limits_{x \to 0} |x| = 0$. ■

The Squeeze Theorem

Theorem 1.3.1. If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

then $L \leq M$.

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x \to a} \Big[g(x) - f(x)\Big] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x - a| < \delta \implies \Big[g(x) - f(x)\Big] - (M - L) < L - M$$

which simplifies to

$$0 < |x - a| < \delta \implies g(x) < f(x)$$

But this contradicts $f(x) \leq g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leq M$. ∎

Theorem 1.3.2 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. ∎

Problem 1.3.3. Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

Solution. Since

$$-1 \leq \sin \frac{1}{x} \leq 1$$

then

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

We know that

$$\lim_{x \to 0} (-x^2) = 0 \qquad \lim_{x \to 0} x^2 = 0$$

By the squeeze theorem,

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

∎

Note the approximate value of π is $\pi \approx 3.14159$. If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1$$

and $\theta \leq \tan\theta$. Therefore we have

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0}\cos\theta = 1$, so by the squeeze theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

But the function $(\sin\theta)/\theta$ is an even function, so its left and right limits must be equal. Hence we have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Problem 1.3.4. Find $\lim_{x\to 0}\frac{\sin 7x}{4x}$.

Solution.

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \lim_{x\to 0}\frac{7x\cdot\sin 7x}{4x\cdot 7x} = \frac{7}{4}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{4}$$

■

Problem 1.3.5. Evaluate $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}$.

Solution. We have

$$\frac{\cos\theta - 1}{\theta} = \frac{\cos\theta - 1}{\theta}\left(\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta + 1}\right)$$

We take the limit then

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta - 1}{\theta}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta + 1)}\\ &= -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta + 1}\right) = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1} = -1\cdot\frac{0}{1+1} = 0\end{aligned}$$

■

1.4 Continuity

Definition 1.4.1. A function f is **continuous at a number** a if

$$\lim_{x\to a} f(x) = f(a)$$

Note that f is continuous at a requires that $f(a)$ is defined and the limit exists. We say that f is **discontinuous** at a if f is not continuous at a.

Definition 1.4.2. A function f is **continuous from the right at a number** a if

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x\to a^-} f(x) = f(a)$$

Definition 1.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem 1.4.1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a)\neq 0$.

Proof. Each of the five parts of this theorem follows from the corresponding limit law. We give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \qquad\qquad \lim_{x\to a} g(x) = g(a)$$

Then

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a}\big[f(x)+g(x)\big] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a)+g(a) = (f+g)(a)$$

This shows that $f+g$ is continuous at a. ■

Theorem 1.4.2. Any polynomial is continuous on $\mathbb{R}=(-\infty,\infty)$. Any rational function is continuous on its domain.

Proof. A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D=\{x\in\mathbb{R}\mid Q(x)\neq 0\}$. We know that polynomials are continuous on $\mathbb{R}$ so the rational function f is continuous at every number in D. ■

Theorem 1.4.3. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 1.4.4. If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

Proof. Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \delta_1$$

Combining these two statements, we see that when $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

Theorem 1.4.5. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $f \circ g = f(g(x))$ is continuous at a.

Proof. Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we have

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $f(g(x))$ is continuous at a. ■

Theorem 1.4.6 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.

The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then by the intermediate value theorem there exists a root of $f(x)$ in (a, b).

1.5 Limits Involving Infinity

Infinite Limits

Definition 1.5.1. The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but $x \neq a$.

Another notation for the limit is $f(x) \to \infty$ as $x \to a$. We say that the limit of $f(x)$, as x approaches a, is infinity.

Definition 1.5.2.

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but $x \neq a$.

We say that the limit of $f(x)$, as x approaches a, is negative infinity. Similar definitions can be given for the one-sided infinite limits

$$\begin{array}{ll} \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Definition 1.5.3. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \to a} f(x) = \infty & \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation is $f(x) \to L$ as $x \to \infty$. We say that the limit of $f(x)$, as x approaches infinity, is L.

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

We say that the limit of $f(x)$, as x approaches negative infinity, is L.

Definition 1.5.6. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = L$$

If n is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

Problem 1.5.1. Evaluate $\lim\limits_{x \to \infty} \sin x$.

Solution. As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often. Thus $\lim\limits_{x \to \infty} \sin x$ does not exist. ■

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.7.

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.5.2. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution. Let M be a given positive number We want to find a number δ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

But

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > M$$

This shows that $\lim_{x \to 0} \frac{1}{x^2} = \infty$. ■

Definition 1.5.8.

$$\lim_{x \to a} f(x) = -\infty$$

if for every negative number N there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) < N$$

Definition 1.5.9. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x > N \implies |f(x) - L| < \varepsilon$$

Definition 1.5.10. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x < N \implies |f(x) - L| < \varepsilon$$

Problem 1.5.3. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Solution. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left| \frac{1}{x} - 0 \right| < \varepsilon$$

Since $x \to \infty$, we can that $x > 0$ in computing the limit. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, so

$$x > N = \frac{1}{\varepsilon} \implies \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

■

Definition 1.5.11. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$x > N \implies f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

2 Derivatives

2.1 Derivatives and Rates of Change

Tangents

Definition 2.1.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

If $h = x - a$, then

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let $s = f(t)$ be a **position function** that describes the motion of an object where s is the displacement of the object from the origin at time t. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

The **velocity** (or **instantaneous velocity**) of the object at time $t = a$ is the limit of the average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Derivatives

Definition 2.1.2. The **derivative of a function** f **at a number** a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

The tangent line to $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of** y **with respect to** x over the interval $[x_1, x_2]$. The limit of these average rates of change is called the (**instantaneous**) **rate of change of** y **with respect to** x at $x = x_1$:

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_2} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$. The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of a particle, then $f'(a)$ is the rate of change of the displacement s with respect to time t. $f'(a)$ is the velocity of the particle at time $t = a$. The **speed** of the particle is $|f'(a)|$, the absolute value of the velocity.

2.2 The Derivative as a Function

Definition 2.2.1. The **derivative of a function** f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Problem 2.2.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

■

Notations

The following notations for the derivative of the function $y = f(x)$ with respect to x are equivalent:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

The symbols $\frac{d}{dx}$ and D_x are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. The symbol $\frac{dy}{dx}$ is callled Leibniz's notation. We can rewrite the definition of the derivative in Leibniz's notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The following notations for the value of the derivative of $y = f(x)$ evaluated at the number a are equivalent:

$$y'(a) = f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiable Functions

Definition 2.2.2. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) (or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$) if it is differentiable at every number in the interval.

Theorem 2.2.1. If f is differentiable at a, then f is continuous at a.

Proof. To prove that f is continuous at a, we want to show that

$$\lim_{x \to a} f(x) = f(a)$$

Given that f is differentiable at a so

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Then

$$\lim_{x \to a} \Big[f(x) - f(a)\Big] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} \Big[f(a) + (f(x) - f(a))\Big] = \lim_{x \to a} f(a) + \lim_{x \to a}(f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore f is continuous at a. ∎

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but it is not differentiable at 0. Since

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h}$$

if the limit exists. But

$$\begin{aligned}
\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} &= \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1 \\
\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} &= \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1
\end{aligned}$$

then the limit does not exist so $f'(0)$ does not exist. Thus $y = |x|$ is differentiable at all x except 0.

Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $y' = f'(x)$ is also a differentiable function, then the **second derivative** of $y = f(x)$ is

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$f''(x)$ is the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. In general, the second derivative is the rate of change of the rate of change. If $s = s(t)$ is the position function of an object, then its first derivative is the velocity $v(t)$ of the object as a function of time t:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

In general, the nth derivative of f is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Problem 2.2.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. The first derivative is

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0}(3x^2 + 3xh + h^2) = 3x^2
\end{aligned}$$

The second derivative is

$$f''(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2+2hx+h^2) - 3x^2}{h} = \lim_{h\to 0} \frac{3x^2+6hx+3h^2-3x^2}{h}$$
$$= \lim_{h\to 0} \frac{6hx+3h^2}{h} = \lim_{h\to 0}(6x+3h) = 6x$$

■

2.3 Basic Differentiation Formulas

The derivative of the constant function $f(x) = c$ is

$$\frac{d}{dx}(c) = 0$$

Proof. If $f(x) = c$ where c is a constant, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

■

Power Functions

$$\frac{d}{dx}(x) = 1$$

Proof. If $f(x) = x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

■

The **Power Rule**: If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof. Let $f(x) = x^n$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

We use the Binomial Theorem to expand $(x+h)^n$ then

$$f'(x) = \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n}{h}$$
$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$
$$= \lim_{h\to 0}(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1})$$
$$= nx^{n-1}$$

because every term except the first has h as a factor and therefore approaches 0. ■

The Power Rule (general version): If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Problem 2.3.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

■

The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

The **Constant Multiple Rule**: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\Big[cf(x)\Big] = c\frac{d}{dx}f(x)$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{d}{dx}f(x) \end{aligned}$$

■

The **Sum Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{\Big[f(x+h)+g(x+h)\Big]-\Big[f(x)+g(x)\Big]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \end{aligned}$$

■

The Sum Rule can be extended to the sum of any number of functions. By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The **Difference Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)-g(x)\Big] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof. Let $F(x) = f(x) - g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{\Big[f(x+h) - g(x+h)\Big] - \Big[f(x) - g(x)\Big]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x) - \Big[g(x+h) - g(x)\Big]}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx} f(x) - \frac{d}{dx} g(x) \end{aligned}$$

■

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial.

The Sine and Cosine Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof. If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right] = \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\ &= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h}\right) \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x \end{aligned}$$

■

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof. If $f(x) = \cos x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left[\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right] = \lim_{h\to 0} \left[\cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right) \right] \\
&= \cos x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) - \sin x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right) \\
&= (\cos x) \cdot 0 - (\sin x) \cdot 1 = -\sin x
\end{aligned}
$$

■

If $f(x) = \sin x$, then

$$f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad f^{(4)}(x) = \sin x$$

In general, if $f(x) = \sin x$, then for $n = 0, 1, 2, \cdots$ we have

$$f^{(4n)}(x) = \sin x \qquad f^{(4n+1)}(x) = \cos x \qquad f^{(4n+2)}(x) = -\sin x \qquad f^{(4n+3)}(x) = -\cos x$$

If $f(x) = \cos x$, then

$$f'(x) = -\sin x \qquad f''(x) = -\cos x \qquad f'''(x) = \sin x \qquad f^{(4)}(x) = \cos x$$

In general, if $f(x) = \cos x$, then for $n = 0, 1, 2, \cdots$ we have

$$f^{(4n)}(x) = \cos x \qquad f^{(4n+1)}(x) = -\sin x \qquad f^{(4n+2)}(x) = -\cos x \qquad f^{(4n+3)}(x) = \sin x$$

2.4 The Product and Quotient Rules

The Product Rule

The **Product Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = \left[\frac{d}{dx}\Big[f(x)\Big]\right] g(x) + f(x)\frac{d}{dx}\Big[g(x)\Big]$$

Proof. Let $F(x) = f(x)g(x)$. Then

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x+h)g(x+h) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x) + f(x+h) \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x) + \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

■

The Quotient Rule

The **Quotient Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}\big[f(x)\big]\right]g(x) - f(x)\frac{d}{dx}\big[g(x)\big]}{\big[g(x)\big]^2}$$

Proof. Let $F(x) = \dfrac{f(x)}{g(x)}$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h) + f(x)g(x) - f(x)g(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - \big[f(x)g(x+h) - f(x)g(x)\big]}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{\dfrac{f(x+h)-f(x)}{h}g(x) - f(x)\dfrac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\
&= \frac{\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}\cdot \lim\limits_{h\to 0}g(x) - \lim\limits_{h\to 0}f(x)\cdot \lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0}g(x+h)\cdot \lim\limits_{h\to 0}g(x)} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)^2\big]}
\end{aligned}$$

■

Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left[\frac{d}{dx}(\sin x)\right]\cos x - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

■

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\left[\frac{d}{dx}(1)\right]\sin x - 1 \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x
\end{aligned}$$

■

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\left[\frac{d}{dx}(1)\right]\cos x - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x
\end{aligned}$$

■

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\left[\frac{d}{dx}(\cos x)\right]\sin x - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{(-\sin x) \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x
\end{aligned}$$

■

2.5 The Chain Rule

The **Chain Rule**: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Comments on the proof of the Chain Rule: Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du} \cdot \frac{du}{dx}\end{aligned}$$

Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous. The only flaw in this reasoning is that it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we cannot divide by 0. Nonetheless, this reasoning does at least suggest that the Chain Rule is true. Note that $\frac{dy}{dx}$ is the derivative of y with respect to x, whereas $\frac{dy}{du}$ is the derivative of y with respect to u.
The Power Rule combined with the Chain Rule: If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}\big[g(x)\big]^n = n\big[g(x)\big]^{n-1} \cdot g'(x)$$

Suppose that $y = f(u), u = g(x)$, and $x = h(t)$ where f, g, h are differentiable functions. Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$$

Proof of the Chain Rule

If we denote ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But

$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \implies \Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx.
The proof of the Chain Rule:

Proof. Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can write

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = \Big[g'(a) + \varepsilon_1\Big]\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b) + \varepsilon_2 \Delta u = \Big[f'(b) + \varepsilon_2\Big]\Delta x$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$ If we now substitute the expression for Δu, we get

$$\Delta y = \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big]\Delta x$$

so

$$\frac{\Delta y}{\Delta x} = \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big]$$

As $\Delta x \to 0$, it shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big] = f'(b)g'(a) = f'(g(a))g'(a)$$

This proves the Chain Rule. ■

2.6 Implicit Differentiation

An explicit function $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. Some functions are defined implicitly by a relation between x and y. For example, the equation of the circle is defined by $x^2 + y^2 = r^2$ where the raidus r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Problem 2.6.1. Find $\dfrac{dy}{dx}$ of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0\end{aligned}$$

Since $y = f(x)$, we use the chain rule then

$$\begin{aligned}2x + \frac{d}{dy}(y^2)\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0\end{aligned}$$

We solve this equation for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.6.2. Find $\dfrac{dy}{dx}$ and the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Solution. We have

$$\begin{aligned}x^3 + y^3 &= 6xy \\ \frac{d}{dx}x^3 + \frac{d}{dy}y^3\left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y + 6x\frac{dy}{dx} \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\ x^2 + y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\ (y^2 - 2x)\frac{dy}{dx} &= 2y - x^2 \\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}\end{aligned}$$

When $x = 3$ and $y = 3$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

The equation of the tangent line is

$$y - 3 = -1(x - 3)$$

■

Problem 2.6.3. Find y' if $\sin(x+y) = y^2 \cos x$.

Solution.

$$\begin{aligned}
\sin(x+y) &= y^2 \cos x \\
\cos(x+y)(1+\frac{dy}{dx}) &= 2y\frac{dy}{dx}\cos x - y^2 \sin x \\
\Big[2y\cos x - \cos(x+y)\Big]\frac{dy}{dx} &= \cos(x+y) + y^2 \sin x \\
\frac{dy}{dx} &= \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}
\end{aligned}$$

■

Problem 2.6.4. Find y'' if $x^4 + y^4 = 16$.

Solution. First we find y' then we have

$$\begin{aligned}
x^4 + y^4 &= 16 \\
4x^3 + 4y^3\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^3}{y^3}
\end{aligned}$$

Then we find y'' and we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -3x^2y^{-3} + (-x^3)(-3y^{-4})\frac{dy}{dx} = 3x^3y^{-4}(-x^3y^{-3}) - 3x^2y^{-3} \\
&= -3x^6y^{-7} - 3x^2y^{-3} = -3x^2y^{-7}(x^4+y^4) = -48\frac{x^2}{y^7}
\end{aligned}$$

■

2.7 Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity. The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem 2.7.1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution. Let V be the volume of the balloon and let r be its radius. The rate of increase of the volume with respect to time is $\frac{dV}{dt}$, and the rate of increase of the radius is $\frac{dr}{dt}$. Then, we are given that

$$\frac{dV}{dt} = 100 \text{ cm}^3\text{/s}$$

and we want to find $\frac{dr}{dt}$ when $r = 25$ cm. We relate V and r by the formula of the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Then by the Chain Rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

If $r = 25$ and $\frac{dV}{dt} = 100$, then

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(25)^2} \cdot 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $\frac{1}{25\pi}$ cm/s. ■

2.8 Linear Approximations and Differentials

We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Applications to Physics

The linear approximations

$$\sin\theta \approx \theta \qquad\qquad \cos\theta \approx 1$$

are used in physics when θ is close to 0.

Differentials

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable. The differential dy is then defined by

$$dy = f'(x)\,dx$$

so dy is an independent variable.

2.9 Derivatives of Inverse Functions

The Calculus of Inverse Functions

Theorem 2.9.1. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem 2.9.2. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Proof. Let $y = f^{-1}(x)$, then $f(y) = x$. Since f is differentiable, it is continuous, so f^{-1} is continuous. Thus if $x \to a$, then $f^{-1}(x) \to f^{-1}(a)$, that is, $y \to b$. Therefore

$$\begin{aligned}
(f^{-1})'(a) &= \lim_{x\to a} \frac{f^{-1}(x) - f^{-1}(a)}{x-a} = \lim_{y\to b} \frac{y-b}{f(y)-f(b)} \\
&= \lim_{y\to b} \frac{1}{\dfrac{f(y)-f(b)}{y-b}} = \frac{1}{\lim\limits_{y\to b} \dfrac{f(y)-f(b)}{y-b}} \\
&= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}
\end{aligned}$$

■

In general, for any number x we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, in Leibniz's notation we have

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, and using the Chain Rule, we get

$$f'(y)\frac{dy}{dx} = 1$$

Therefore

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$

Derivatives of Logarithmic Functions

The **Euler's number** e is the base of the natural exponential function $y = e^x$. It is also the base of the the natural logarithmic function $y = \ln x$.

Definition 2.9.1 Euler's Number. The Euler's number e is defined as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{1/x}$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.9.3. The exponential function $f(x) = \log_a x$ is differentiable and

$$f'(x) = \frac{1}{x}\log_a e$$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a\left(\dfrac{x+h}{x}\right)}{h} \\ &= \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) = \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h} \\ &= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)} = \frac{1}{x}\log_a e \end{aligned}$$

■

We know from the change of base formula that

$$\log_a e = \frac{\ln e}{\ln a} = \frac{1}{\ln a}$$

and therefore

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$$

The derivative of the natural logarithmic function $f(x) = \ln x$ is:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof.

$$\frac{d}{dx}(\ln x) = \frac{1}{x\ln e} = \frac{1}{x}$$

■

In general, if $u = f(x)$, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

Problem 2.9.1. Find $f'(x)$ if $f(x) = \ln|x|$.

Solution. Since $f(x) = \ln x$ for $x > 0$ and $f(x) = \ln(-x)$ for $x < 0$, it follows that

$$f'(x) = \frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$
$$f'(x) = \frac{d}{dx}\Big[\ln(-x)\Big] = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0$$

Therefore $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for all $x \neq 0$. ∎

Logarithmic Differentiation

The steps in **logarithmic differentiation** are

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The proof of the Power Rule (general version):

Proof. Let $y = x^n$ and we use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x|, \quad x \neq 0$$

Therefore

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{n}{x}$$

Hence

$$\frac{dy}{dx} = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

∎

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of the derivative.

Derivatives of Exponential Functions

Theorem 2.9.4. The exponential function $f(x) = a^x, a > 0$, is differentiable and

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Proof. We know that the logarithmic function $y = \log_a x$ is differentiable (and its derivative is nonzero) so its inverse function $y = a^x$ is differentiable. If $y = a^x$, then $\log_a y = x$. By implicit differentiation we have

$$\begin{aligned}
\log_a y &= x \\
\frac{1}{y \ln a} \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= y \ln a = a^x \ln a
\end{aligned}$$

■

The derivative of the natural exponential function $f(x) = e^x$ is:

$$\frac{d}{dx}(e^x) = e^x$$

Proof.

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x$$

■

In general, if $u = f(x)$, then

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arcsin x$. Then $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$. By implicit differentiation we have

$$\begin{aligned}
\frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\
\cos y \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y}
\end{aligned}$$

Now $\cos y \geq 0$ since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y}$$

Therefore

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

■

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arccos x$. Then $\cos y = x$ and $0 \le y \le \pi$. By implicit differentiation we have

$$\begin{aligned} \frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) \\ -\sin y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y} \end{aligned}$$

Now $\sin y \ge 0$ since $0 \le y \le \pi$, so

$$\sin y = \sqrt{1 - \cos^2 y}$$

Therefore

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

■

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Proof. Let $y = \arctan x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We have

$$\begin{aligned} \frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\ \sec^2 y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{1+x^2} \end{aligned}$$

Therefore

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

■

$$\begin{aligned} \frac{d}{dx}(\operatorname{arccsc} x) &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\operatorname{arcsec} x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\operatorname{arccot} x) &= \frac{1}{x\sqrt{1+x^2}} \end{aligned}$$

2.10 Hyperbolic Functions

Definition of **hyperbolic functions**:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic Identities:

- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \operatorname{sech}^2 x$
- $\sinh(x + y) = \sinh x \cosh x + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh x + \sinh x \sinh y$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Inverse Hyperbolic Functions

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}), \quad x \in \mathbb{R}$$
$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$
$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right), \quad -1 \leq x \leq 1$$

Derivatives of inverse hyperbolic functions:

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{arcsch} x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{arsech} x) = -\frac{1}{x\sqrt{1 - x^2}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1 - x^2}$$

2.11 Indeterminate Forms and l'Hospital's Rule

In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{0}{0}$. In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then this limit may or may not exist and is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

Theorem 2.11.1 l'Hospital's Rule. Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad\qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad\qquad \lim_{x\to a} g(x) = \pm\infty$$

In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

It is important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. For the special case in which $f(a) = g(a) = 0$, f' and g' is continuous, and $g'(a) \neq 0$, we have

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Problem 2.11.1. Find $\lim_{x\to 1} \frac{\ln x}{x-1}$.

Solution. Notice that we have an indeterminate form of type $\frac{0}{0}$. We apply l'Hospital's rule then

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 1$$

■

Problem 2.11.2. Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution. Notice that we have an indeterminate form of type $\frac{\infty}{\infty}$. Then l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Since we still have an indeterminate form of $\frac{\infty}{\infty}$, a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

■

Problem 2.11.3. Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

Solution. Notice that we have an indeterminate form of type $\frac{0}{0}$. We apply l'Hospital's rule then

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x\to 0} \frac{2\sec^4 x + 4\sec^2 x \tan^2 x}{6} \\ &= \frac{2(1) + 4(1)(0)}{6} = \frac{1}{3}\end{aligned}$$

■

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then the limit

$$\lim_{x\to a} f(x)g(x)$$

is called an **indeterminate form of type** $0\cdot\infty$. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g}$$

or

$$fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

Problem 2.11.4. Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

■

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}\left[f(x) - g(x)\right]$$

is called an **indeterminate form of type** $\infty - \infty$. We can convert the difference into a quotient so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Problem 2.11.5. Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Solution.

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0$$

■

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}$$

- Type 0^0: $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
- Type ∞^0: $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
- Type 1^∞: $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Note that the form 0^∞ is not indeterminate. Each of these three cases can be treated either by taking the natural logarithm: let $y = \left[f(x)\right]^{g(x)}$, then $\ln y = g(x)\ln f(x)$ or by writing the function as an exponential:

$$\left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$$

In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0\cdot\infty$.

Problem 2.11.6. Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution. Let

$$y = (1+\sin 4x)^{\cot x}$$

Then

$$\ln y = \ln\left[(1+\sin 4x)^{\cot x}\right] = \cot x\ln(1+\sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

Then

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

■

Problem 2.11.7. Find $\lim\limits_{x\to 0^+} x^x$.

Solution. We used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x\ln x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

■

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Definition 3.1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Definition 3.1.2. The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

Theorem 3.1.1 Extreme Value Theorem. If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Theorem 3.1.2 Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof. Suppose that f has a local maximum at c. Then $f(c) \geq f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geq f(c+h)$$

and theorefore

$$f(c+h) - f(c) \leq 0$$

If $h > 0$ and h is sufficiently small, we have

$$\frac{f(c+h)-f(c)}{h} \leq 0$$

Taking the right-hand limit, we get

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq \lim_{h\to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$$

and so we have shown that $f'(c) \leq 0$. If $h < 0$, then

$$\frac{f(c+h)-f(c)}{h} \geq 0$$

Taking the left-hand limit, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

We have shown that $f'(c) \leq 0$ and $f'(c) \geq 0$. Therefore $f'(c) = 0$. The case of a local minimum can be proved in a similar manner. ■

Definition 3.1.3. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local maximum or minimum at c, then c is a critical number of f. The Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of these values is the absolute maximum value; the smallest of these values is the absolute minimum value.

3.2 Lagrange's Mean Value Theorem

Theorem 3.2.1 Rolle's Theorem. Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. There are three cases:
Case 1: $f(x) = k$, a constant
Then $f'(x) = 0$ so the number c can be any number in (a, b).
Case 2: $f(x) > f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c. Therefore $f'(c) = 0$ by Fermat's Theorem.
Case 3: $f(x) < f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a minimum value somewhere in $[a, b]$, and since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Then $f'(c) = 0$ by Fermat's Theorem. ∎

Theorem 3.2.2 Lagrange's Mean Value Theorem. Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. We apply Rolle's Theorem to a new function h defined as a difference between f and the function whose graph is the secant line AB where A is the point $(a, f(a))$ and B is the point $(b, f(b))$. The slope of the secant line AB is

$$m = \frac{f(b) - f(a)}{b - a}$$

Then the equation of the secant line AB is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

So,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on $[a, b]$ because it is the sum of f and a first degree polynomial, both of which is continuous.

2. The function h is differentiable on (a, b) because both f and a first degree polynomial are differentiable. In fact, we can compute h' directly:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Note that $f(a)$ and $\frac{f(b) - f(a)}{b - a}$ are constants.

3.
$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$
$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - \left[f(b) - f(a)\right] = 0$$

Therefore $h(a) = h(b)$. Since h satisfies the hypotheses of Rolle's Theorem, there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

■

In general, Lagrange's Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval. The main significance of Lagrange's Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative.

Theorem 3.2.3. If $f'(x) = 0$ for all x in an interval (a, b), then f is a constant on (a, b).

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying Lagrange's Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so

$$f(x_2) - f(x_1) = 0$$

or

$$f(x_2) = f(x_1)$$

Therefore f has the same value at any two numbers x_1 and x_2 in (a, b). This means that f is a constant on (a, b). ■

Corollary 3.2.3.1. If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is a constant on (a, b); that is; $f(x) = g(x) + c$ where c is a constant.

Proof. Since $f'(x) = g'(x)$, we have $f'(x) - g'(x) = 0$. Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus F is a constant; that is, $f - g$ is a constant. ■

Theorem 3.2.4 Cauchy's Mean Value Theorem. Suppose that the fucntions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

3.3 Derivatives and the Shapes of Graphs

First Derivatives

Increasing/Decreasing Test:

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. Given $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. By Lagrange's Mean Value Theorem, there is a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, thus

$$f(x_2) - f(x_1) > 0$$

or

$$f(x_1) < f(x_2)$$

This shows that f is increasing. Part 2 is proved similarly. ■

The First Derivative Test: Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, then f has no local maximum or minimum at c.

Second Derivatives

Definition 3.3.1. If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Definition 3.3.2. A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to downward or from concave downward to concave upward at P.

Concavity Test:

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

The Second Derivative Test: Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

3.4 Optimization Problems

Problem 3.4.1. A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution. We want to maximize the area A of the rectangle. Let x and y be the length and width of the rectangle. Then the area is

$$A = xy$$

Given that the total length of fencing is 2400 m so

$$x + 2y = 2400$$

Thus

$$y = 1200 - \frac{1}{2}x$$

and so

$$A = x\left(1200 - \frac{1}{2}x\right) = 1200x - \frac{1}{2}x^2$$

Note that $x \geq 0$ and $x \leq 2400$. So the function we want to maximize is

$$A(x) = 1200x - \frac{1}{2}x^2, \quad 0 \leq x \leq 2400$$

The derivative is $A'(x) = 1200 - x$, so to find the critical numbers we solve the equation

$$1200 - x = 0$$

which gives $x = 1200$. Since $A(0) = 0, A(1200) = 720000, A(2400) = 0$, the Closed Interval Method gives the maximum value as $A(1200) = 720000$. Thus the rectangular field should have a length of 1,200 m and a width of 600 m. ■

3.5 Newton's Method

We have **Newton's Method** to find an approximate numerical value of the root of an equation. Let r be the root of the equation. We start with a first approximation x_1. Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and we label the x-intercept of L as x_2. The slope of L is $f(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r. Next we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write

$$\lim_{n \to \infty} x_n = r$$

3.6 Antiderivatives

Definition 3.6.1. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 3.6.1. If F is an antiderivative of f on interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Problem 3.6.1. Find the most general antiderivative of $f(x) = \dfrac{1}{x}$.

Solution. We know that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

for all $x \neq 0$. Then the antiderivative of f is $F(x) = \ln x + C_1$ if $x > 0$ or $F(x) = \ln(-x) + C_2$ if $x < 0$. The general antiderivative of f is $F(x) = \ln|x| + C$ on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. ■

Problem 3.6.2. Find the most general antiderivative of $f(x) = x^n, \quad n \neq -1$.

Solution. If $n \neq -1$, then by the Power Rule,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

■

4 Integrals

4.1 The Definite Integral

The Area Problem

Definition 4.1.1. The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$$

The Distance Problem

In general, suppose an object moves with velocity $v = f(t)$, where $a \leq t \leq b$, and $f(t) > 0$. The exact total distance d traveled is

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$

The Definite Integral

In general, we start with any function f defined on $[a, b]$ and we divide $[a, b]$ into n smaller subintervals by choosing partition points $x_0, x_1, x_2, \ldots, x_n$ so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The resulting collection of subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

is called a **partition** P of the $[a,b]$. We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$. Thus

$$\Delta x_i = x_i - x_{i-1}$$

Then we choose **sample points** $x_1^*, x_2^*, \ldots, x_n^*$ in the subintervals with x_i^* in the ith subinterval $[x_{i-1}, x_i]$. A **Riemann sum** associated with a partition P and a function f is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

Definition 4.1.2. If f is a function defined on $[a,b]$, the **definite integral of f from a to b** is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a,b]$.

Definition 4.1.3.

$$\int_a^b f(x)\,dx = I$$

means that for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \varepsilon$$

for all partitions P of $[a,b]$ with $\max \Delta x_i < \delta$ and for all possible choices of x_i^* in $[x_{i-1}, x_i]$.

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx indicates that the independent variable is x. The procedure of calculating an integral is called **integration**. The definite integral is a number so it does not depend on x. In fact, we can use any letter in place without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt$$

Theorem 4.1.1. If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$; that is, the definite integral $\int_a^b f(x)\,dx$ eixsts.

Theorem 4.1.2. If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

The Midpoint Rule

The Midpoint Rule:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x\Big[f(\overline{x}_1) + \cdots + f(\overline{x}_n)\Big]$$

where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$.

Properties of The Definite Integrals

If $a > b$, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral: Suppose all of the following integrals exist.

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is any constant.
2. $\displaystyle\int_a^b \Big[f(x) + g(x)\Big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.
4. $\displaystyle\int_a^b \Big[f(x) - g(x)\Big]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Comparison Properties of the Integral:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

4.2 Evaluating Definite Integrals

Theorem 4.2.1 Evaluation Theorem. If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

Proof. We divide the interval $[a, b]$ into n subintervals with endpoints $x_0 = a, x_1, x_2, \cdots, x_n = b$ and with length $\Delta x = \dfrac{b-a}{n}$. Let F be any antiderivative of f. Then

$$\begin{aligned}
&F(b) - F(a)\\
&= F(x_n) - F(x_0)\\
&= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_2) - F(x_1) + F(x_1) - F(x_0)\\
&= \sum_{i=1}^{n} \Big[F(x_i) - F(x_{i-1})\Big]
\end{aligned}$$

Now F is continuous because it's differentiable and so we can apply Lagrange's Mean Value Theorem to F on each subinterval $[x_{i-1}, x_i]$. Thus there exists a number x_i^* between x_{i-1} and x_i such that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)\Delta x$$

Therefore

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Now we take the limit of each side of this equation as $n \to \infty$. Then

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

■

When applying the Evaluation Theorem we use the notation

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b = F(x)\Big|_a^b$$

Indefinite Integrals

The notation

$$\int f(x)\,dx$$

is used for an antiderivative of f and is called an **indefinite intergral**. Thus

$$\int f(x)\,dx = F(x)$$

means that

$$F'(x) = f(x)$$

Applications

Theorem 4.2.2 Net Change Theorem. The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

4.3 The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** establishes a connection between the two branches of calculus: differential calculus and integral calculus. It says that differentiation and integration are inverse processes and gives the precise inverse relationship between the derivative and the integral. The first part of the Fundamental Theorem of Calculus deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)\,dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b.

Theorem 4.3.1 Fundamental Theorem of Calculus Part 1. If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

Proof. If x and $x + h$ are in the open interval (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

and so, for $h \neq 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and the absolute maximum values of f in $[x, x+h]$. Then

$$m \cdot h \le \int_x^{x+h} f(t)\,dt \le M \cdot h$$

that is,

$$f(u) \cdot h \le \int_x^{x+h} f(t)\,dt \le f(v) \cdot h$$

Since $h > 0$, we have

$$f(u) \le \frac{1}{h} \int_x^{x+h} f(t)\,dt \le f(v)$$

Thus

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v)$$

This inequality can be proved in a similar manner for the case $h < 0$. Let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x + h$. Thus

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$$

and

$$\lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. By the Squeeze Theorem,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

If $x = a$ or b, then we can interpret this equation as a one-sided limit. Since g is differentiable, this shows that g is continuous on $[a, b]$. ■

Using Leibniz's notation for derivatives, we can write the Fundamental Theorem of Calculus Part 1 as

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

By the Fundamental Theorem of Calculus Part 1, the derivative of the Fresnel function

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

is

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$$

Problem 4.3.1. Find $\displaystyle \frac{d}{dx} \int_1^{x^4} \sec t\,dt$.

Solution. We use the Chain Rule and the Fundamental Theorem of Calculus Part 1. Let $u = x^4$. Then

$$\frac{d}{dx} \int_1^{x^4} \sec t\,dt = \frac{d}{du} \int_1^{u} \sec t\,dt \cdot \frac{du}{dx} = \sec u \cdot \frac{du}{dx} = \sec(x^4) \cdot 4x^3$$

■

Differentiation and Integration as Inverse Processes

Theorem 4.3.2 Fundamental Theorem of Calculus. Suppose f is continuous on $[a, b]$.

1. If
$$g(x) = \int_a^x f(t)\,dt$$
then
$$g'(x) = f(x)$$

2.
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
where F is any antiderivative of f, that is, $F' = f$.

The Fundamental Theorem of Calculus is the most important theorem in calculus. It is one of the greatest accomplishments of the human mind.

Average Value of a Function

We define the **average value of** f on the interval $[a, b]$ as

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Theorem 4.3.3 Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Proof. Let $F(x) = \displaystyle\int_a^x f(t)\,dt$ for $a \leq x \leq b$. By Lagrange's Mean Value Theorem for derivatives, there is a number c between a and b such that

$$F(b) - F(a) = F'(c)(b-a)$$

But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus Part 1. Therefore

$$\int_a^b f(t)\,dt - 0 = f(c)(b-a)$$

■

4.4 The Substitution Rule

In general, suppose we have an integral of the form

$$\int f(g(x))g'(x)\,dx$$

Observe that if $F' = f$, then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}\Big[F(g(x))\Big] = F'(g(x)) \cdot g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then we have

$$\int F'(g(x)g'(x)\,dx = F(g(x))) + C = F(u) + C = \int F'(u)\,du$$

or, writing $F' = f$, we get

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Thus we proved the following rule.
The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that if $u = g(x)$, then $du = g'(x)\,dx$. It is permissible to operate with dx and du after integral signs as if they were differentials.

Problem 4.4.1. Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution. Let $u = 1 - 4x^2 \iff du = -8x\,dx$ so $xdx = -\frac{1}{8}\,du$. Therefore,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\int \frac{1}{8}u^{-(1/2)}\,du = -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

■

Problem 4.4.2. Evaluate $\displaystyle\int \tan x\,dx$.

Solution. We have $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, let $u = \cos x \iff du = -\sin\,dx$ then

$$\int \tan x\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln\frac{1}{|\sec x|} + C = \ln|\sec x| + C$$

■

Definite Integrals

The Substitution Rule for Definite Integrals: If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Note that we must change the limits of integration when the variable is changed for definite integrals.

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ and so, by the Fundamental Theorem of Calculus, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a))$$

But, applying the Fundamental Theorem of Calculus a second time, we also have

$$\int_{g(a)}^{g(b)} \int f(u)\,du = \big[F(u)\big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

$\blacksquare$

Problem 4.4.3. Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

Solution. Let $u = \ln x \iff du = \dfrac{1}{x}\,dx$, then

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

$\blacksquare$

Symmetry

Integrals of Symmetric Functions: Suppose f is continuous on $[-a, a]$.

1. If f is even, then $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

2. If f is odd, then $\displaystyle\int_{-a}^a f(x)\,dx = 0$.

Proof. We split the integral in two:

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

We make the substitution $u = -x$. Then $du = -dx$ and when $x = -a, u = a$. Therefore

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\,du$$

and so

$$\int_{-a}^a f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

1. If f is even, then $f(-u) = f(u)$ so

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

2. If f is odd, then $f(-u) = -f(u)$ so

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0$$

■

5 Techniques of Integration

5.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts. The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

In the notation fo indefinite integrals this equation becomes

$$f(x)g(x) = \int \Big[f'(x)g(x) + f(x)g'(x)\Big]\,dx$$

or

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

and this equation is called the **formula for integration by parts**. Let $u = f(x)$ and $v = f(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\int u\,dv = uv - \int v\,du$$

Problem 5.1.1. Evaluate $\displaystyle\int \ln x\,dx$.

Solution. Let $u = \ln x \iff du = \dfrac{1}{x}\,dx$ and $dv = dx \iff v = x$. Then

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

■

Problem 5.1.2. Evaluate $\displaystyle\int e^x \sin x\, dx$.

Solution. Let $u = \sin x \iff du = \cos x\, dx$ and $dv = e^x\, dx \iff v = e^x$. then

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$$

Let $u = \cos x \iff du = -\sin x\, dx$ and $dv = e^x\, dx \iff v = e^x$. Then

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$$

Therefore

$$\begin{aligned}\int e^x \sin x\, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \\ \int e^x \sin x\, dx &= \frac{1}{2} e^x (\sin x - \cos x)\end{aligned}$$

■

The formula of integration by parts for definite integrals is

$$\int_a^b f(x)g'(x)\, dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\, dx$$

or

$$\int_a^b v\, du = \Big[uv\Big]_a^b - \int_a^b v\, du$$

Problem 5.1.3. Evaluate $\displaystyle\int_0^1 \arctan x\, dx$.

Solution. Let $u = \arctan x \iff du = \dfrac{dx}{1+x^2}$ and $dv = dx \iff v = x$, then

$$\int_0^1 \arctan x\, dx = \Big[x \arctan x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$

We have $\Big[x \arctan x\Big]_0^1 = \arctan(1) = \dfrac{\pi}{4}$. Let $u = 1 + x^2 \iff du = 2x\, dx$ so $x\, dx = \dfrac{1}{2}\, du$, then

$$\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2}\int_1^2 \frac{du}{u} = \left[\frac{1}{2}\ln|u|\right]_1^2 = \frac{1}{2}\ln(2)$$

Therefore

$$\int_0^1 \arctan x\, dx = \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

■

Problem 5.1.4. Prove the reduction formula

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

where $n \geq 2$ is an integer.

Solution. Let $u = \sin^{n-1} x \iff du = (n-1)\sin^{n-2} x \cos x\, dx$ and $dv = \sin x\, dx \iff v = -\cos x$. Then

$$\begin{aligned}\int \sin^n x\, dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx\end{aligned}$$

Therefore

$$\begin{aligned}(n-1+1)\int \sin^n x\, dx = n\int \sin^n x\, dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx \\ \int \sin^n x\, dx &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx\end{aligned}$$

■

Problem 5.1.5. Prove the reduction formula

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

where $n \geq 2$ is an integer.

Solution. Let $u = \cos^{n-1} x \iff du = -(n-1)\cos^{n-2} x \sin x\, dx$ and $dv = \cos x\, dx \iff v = \sin x$. Then

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx\end{aligned}$$

Therefore

$$\begin{aligned}(n-1+1)\int \cos^n x\, dx = n\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx \\ \int \cos^n x\, dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx\end{aligned}$$

■

Problem 5.1.6. Show that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

where $n \geq 2$ is an integer.

Solution. We use the reduction formula then

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \left[-\frac{1}{n}\sin^{n-1} x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \\ &= -\frac{1}{n}\left(\sin^{n-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin^{n-1}(0)\cos(0)\right) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\end{aligned}$$

and since $\sin(0) = 0$ and $\cos(\pi/2) = 0$ so

$$\int_0^{\pi/2} \sin^n x\,dx = -\frac{1}{n}(0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

■

Problem 5.1.7. Show that

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}$$

where $n \geq 2$ is an integer.

Solution. By the reduction formula

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n+1} x\,dx &= \frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx = \frac{(2n)(2n-2)}{(2n+1)(2n-1)}\int_0^{\pi/2} \sin^{2n-3} x\,dx \\ &= \frac{(2n)(2n-2)\cdots(6)(4)(2)}{(2n+1)(2n-1)\cdots(7)(5)(3)}\int_0^{\pi/2} \sin x\,dx\end{aligned}$$

and since

$$\int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 0 + 1 = 1$$

hence

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}$$

■

Problem 5.1.8. Show that

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}\frac{\pi}{2}$$

where $n \geq 2$ is an integer.

Solution.

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n} x\,dx &= \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2} x\,dx = \frac{(2n-1)(2n-3)}{(2n)(2n-2)}\int_0^{\pi/2} \sin^{2n-4} x\,dx \\ &= \frac{(2n-1)(2n-3)\cdots(5)(3)(1)}{(2n)(2n-2)\cdots(6)(4)(2)}\int_0^{\pi/2} dx = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}\frac{\pi}{2}\end{aligned}$$

■

Problem 5.1.9. Prove the reduction formula

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

Solution. Let $u = (\ln x)^n \iff du = \frac{n}{x}(\ln x)^{n-1}$ and $dv = dx \iff v = x$ then

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int x \cdot \frac{n}{x}(\ln x)^{n-1} \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

$\blacksquare$

Problem 5.1.10. Prove the reduction formula

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

Solution. Let $u = x^n \iff du = nx^{n-1} \, dx$ and $dv = e^x \, dx \iff v = e^x$ then

$$\int x^n e^x \, dx = x^n e^x - \int nx^{n-1} e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$\blacksquare$

5.2 Trigonometric Integrals and Substitutions

Trigonometric Integrals

We can use trigonometric identities to integrate trigonometric integrals.

Problem 5.2.1. Evaluate $\int \sin 5x \sin 2x \, dx$.

Solution.

$$\begin{aligned}\int \sin 5x \sin 2x \, dx &= \int \frac{1}{2}(\cos(5x - 2x) - \cos(5x + 2x)) \, dx = \frac{1}{2} \int (\cos 3x - \cos 7x) \, dx \\ &= \frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C = \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C\end{aligned}$$

$\blacksquare$

Problem 5.2.2. Evaluate $\int \sin 3x \cos x \, dx$.

Solution.

$$\begin{aligned}\int \sin 3x \cos x \, dx &= \int \frac{1}{2}(\sin(3x + x) + \sin(3x - x)) \, dx = \frac{1}{2} \int (\sin 4x + \sin 2x) \, dx \\ &= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right) + C = -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C\end{aligned}$$

$\blacksquare$

Problem 5.2.3. Evaluate $\int \cos^3 x \, dx$.

Solution.

$$\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int (1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

■

Problem 5.2.4. Evaluate $\displaystyle\int \sin^5 x \cos^2 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^5 x \cos^2 x\,dx &= \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx = -\int (1-u^2)^2 u^2\,du \\ &= -\int (u^2 - 2u^4 + u^6)\,du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$

■

Problem 5.2.5. Evaluate $\displaystyle\int \cos^2 x\,dx$.

Solution.

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

■

Problem 5.2.6. Evaluate $\displaystyle\int_0^\pi \sin^2 x\,dx$.

Solution.

$$\begin{aligned}\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2x)\,dx \\ &= \frac{1}{2}\left(\Big[x\Big]_0^\pi - \frac{1}{2}\Big[\sin 2x\Big]_0^\pi\right) = \frac{\pi}{2}\end{aligned}$$

■

Problem 5.2.7. Evaluate $\displaystyle\int \sin^4 x\,dx$.

Solution.

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2\,dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right) + C \\ &= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C\end{aligned}$$

■

In general, an integral of powers of $\sin x$ and $\cos x$ is in the form

$$\int \sin^m x \cos^n x \, dx$$

where m and n are non-negative integers. If m is odd, then we save a factor of $\sin x$ and express the rest in terms of $\cos x$ for substitution. If n is odd, then we save a factor of $\cos x$ and express the rest in terms of $\sin x$ for substitution. If m and n are even, then we use the power reduction formulas.

Problem 5.2.8. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution.

$$\begin{aligned}\int \tan^6 x \sec^4 x \, dx &= \int \tan^6 x(\tan^2 x + 1)\sec^2 x \, dx = \int u^6(u^2+1)\, du = \int u^8 + u^6 \\ &= \frac{u^9}{9} + \frac{u^7}{7} = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C\end{aligned}$$

■

Problem 5.2.9. Evaluate $\int \tan^5 x \sec^7 x \, dx$.

Solution.

$$\begin{aligned}\int \tan^5 x \sec^7 x \, dx &= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x \, dx = \int (u^2-1)^2 u^6 \, du \\ &= \int (u^{10} - 2u^8 + u^6)\, dx = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C \\ &= \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C\end{aligned}$$

■

In general, an integral of powers of $\tan x$ and $\sec x$ is in the form

$$\int \tan^m x \sec^n x \, dx$$

where m and n are non-negative integers. If m is odd, then we save a factor of $\sec x \tan x$ and express the rest in terms of $\sec x$ for substitution. If n is even, then we save a factor of $\sec^2 x$ and express the rest in terms of $\tan x$ for substitution.

Problem 5.2.10. Evaluate $\int \sec x \, dx$.

Solution. We have

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x \iff du = (\sec x \tan x + \sec^2 x)\, dx$, then

$$\int \sec x \, dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

■

Problem 5.2.11. Evaluate $\int \tan^3 x\,dx$.

Solution.

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan x(\sec^2 x - 1)\,dx = \int \tan x \sec^2\,dx - \int \tan x\,dx\\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned}$$

■

Problem 5.2.12. Evaluate $\int \sec^3 x\,dx$.

Solution. Let $u = \sec x \iff du = \sec x \tan x$ and $dv = \sec^2 x\,dx \iff v = \tan x$, then

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$$

We have

$$\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$$

Therefore

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ \int \sec^3 x\,dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C\end{aligned}$$

■

Trigonometric Substitutions

If an integral has the form $\int \sqrt{a^2 - x^2}\,dx$, then we can use the substitution $x = a\sin\theta$ where $-\pi/2 \le \theta \le \pi/2$ to get

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = |a|\cos\theta$$

If an integral has the form $\int \sqrt{a^2 + x^2}\,dx$, then we can use the substitution $x = a\tan\theta$ where $-\pi/2 < \theta < \pi/2$ to get

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(\tan^2\theta + 1)} = \sqrt{a^2\sec^2\theta} = |a|\sec\theta$$

If an integral has the form $\int \sqrt{x^2 - a^2}\,dx$, then we can use the substitution $x = a\sec\theta$ where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$ to get

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a|\tan\theta$$

Problem 5.2.13. Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Solution. Let $x = 3\sin\theta \iff dx = 3\cos\theta\,d\theta$, then

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} = \int \cot^2\theta\,d\theta = \int(\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C\end{aligned}$$

■

Problem 5.2.14. Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$.

Solution. Let $x = 2\tan\theta \iff dx = 2\sec^2\theta\,d\theta$ and $-\pi/2 < \theta < \pi/2$, then

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta}{4\tan^2\theta(2\sec\theta)}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} \\ &= -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C\end{aligned}$$

■

Problem 5.2.15. Find the area enclosed by a circle with radius r.

Solution. The area is $A = 4\displaystyle\int_0^r \sqrt{r^2-x^2}\,dx$. Let $x = r\sin\theta \iff dx = r\cos\theta\,d\theta$ and $0 \le \theta \le \pi/2$, then

$$\begin{aligned}A &= 4\int_0^{\pi/2} r\cos\theta(r\cos\theta)\,d\theta = 4r^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4r^2\left(\frac{1}{2}\left(\Big[\theta\Big]_0^{\pi/2} + \frac{1}{2}\Big[\sin 2\theta\Big]_0^{\pi/2}\right)\right) \\ &= 4r^2\left(\frac{\pi}{4}\right) = \pi r^2\end{aligned}$$

■

Problem 5.2.16. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. The area is $A = 4\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$. Let $x = a\sin\theta \iff dx = a\cos\theta\,d\theta$ and $0 \le \theta \le \pi/2$, then

$$A = 4\int_0^{\pi/2} \frac{b}{a}(a\cos\theta)(a\cos\theta)\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\left(\frac{\pi}{4}\right) = \pi ab$$

■

5.3 Partial Fractions

Consider a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials. If f is improper, that is, $\deg(P) \geq \deg(Q)$, then we simplify to get $f(x) = S(x) + \dfrac{R(x)}{Q(x)}$ where S and R are polynomials. Then we can factor Q to be irreducible and express $\dfrac{R(x)}{Q(x)}$ as the a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i}$$

or

$$\frac{Ax+B}{(ax^2+bx+c)^j}$$

There are four possible cases:

1. Q is a product of distinct linear factors. Then

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

and therefore

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Problem 5.3.1. Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Solution. We simplify to get $2x^3+3x^2-2x = x(2x-1)(x+2)$ then

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)
\end{aligned}$$

$$\begin{aligned}
x = 0 &\iff -1 = -2A \iff A = \frac{1}{2} \\
x = \frac{1}{2} &\iff \frac{1}{4} = \frac{5}{4}B \iff B = \frac{1}{5} \\
x = -2 &\iff -1 = 10C \iff C = -\frac{1}{10}
\end{aligned}$$

and therefore

$$\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{1}{2}\int \frac{dx}{x} + \frac{1}{5}\int \frac{dx}{2x-1} - \frac{1}{10}\int \frac{dx}{x+2} \\
&= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K
\end{aligned}$$

■

2. Q is a product of linear factors where some are repeated. Suppose that the first linear factor is repeated r times, then we have

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

for the first repeated linear factor and similarly for other repeated linear factors.

Problem 5.3.2. Evaluate $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution. We simplify to get

$$\begin{aligned} x^4-2x^2+4x+1 &= (x^2-1)^2+4x = (x+1)^2(x-1)^2+4x \\ x^3-x^2-x+1 &= x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2 \end{aligned}$$

and so

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{(x+1)(x-1)^2}\,dx$$

Then

$$\begin{aligned} \frac{4x}{(x+1)(x-1)^2} &= \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \\ 4x &= A(x-1)^2+B(x+1)(x-1)+C(x+1) \\ x=-1 \iff -4=4A &\iff A=-1 \\ x=1 \iff 4=2C &\iff C=2 \\ x=0 \iff 0=-1-B+2 &\iff B=1 \end{aligned}$$

and therefore

$$\begin{aligned} \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int (x+1)\,dx - \int \frac{dx}{x+1}+\int\frac{dx}{x-1}+2\int\frac{dx}{(x-1)^2} \\ &= \frac{x^2}{2}+x-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}+K \end{aligned}$$

■

3. Q has irreducible quadratic factors without repeated factors. Then for every quadratic factor we have

$$\frac{Ax+B}{ax^2+bx+c}$$

Problem 5.3.3. Evaluate $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$.

Solution. We do partial fraction decomposition then

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

$$A = 1, \quad B = 1, \quad C = -1$$

and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx = \int \frac{dx}{x} + \int \frac{x - 1}{x^2 + 4}\, dx = \ln|x| + \frac{1}{2}\ln|x^2 + 4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$$

■

Note that

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Problem 5.3.4. Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx$.

Solution. We simplify to get

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx = \int \frac{4x^2 - 4x + 3 + x - 1}{4x^2 - 4x + 3}\, dx = \int \left(1 + \frac{x - 1}{(2x - 1)^2 + 2}\right) dx$$

then

$$\begin{aligned}
\int \frac{x - 1}{(2x - 1)^2 + 2}\, dx &= \frac{1}{2}\int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2}\, dx = \frac{1}{4}\int \frac{u - 1}{u^2 + 2}\, dx \\
&= \frac{1}{4}\left(\frac{1}{2}\ln(u^2 + 2) - \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}u\right)\right) \\
&= \frac{1}{8}\ln((2x - 1)^2 + 2) - \frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x - 1)\right)
\end{aligned}$$

and so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx = x + \frac{1}{8}\ln(4x^2 - 4x + 3) - \frac{\sqrt{2}}{8}\arctan\left(\frac{\sqrt{2}}{2}(2x - 1)\right) + C$$

■

4. Q has a repeated irreducible quadratic factor. Suppose that the first quadratic factor is repeated r times, then we have

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

for the first repeated quadratic factor and similarly for the others.

Problem 5.3.5. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$.

Solution. We do partial fraction decomposition then

$$\begin{aligned}\frac{1-x+2x^2-x^3}{x(x^2+1)^2} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}\\ 1-x+2x^2-x^3 &= A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\end{aligned}$$

$$x=0 \iff A=1$$

and so

$$\begin{aligned}1-x+2x^2-x^3 &= x^4+2x^2+1+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\\ -x-x^3 &= (B+1)x^4+Cx^3+(B+D)x^2+(C+E)x\end{aligned}$$

$$B=-1 \quad C=-1 \quad D=1 \quad E=0$$

Therefore

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int \frac{dx}{x}-\int \frac{x+1}{x^2+1}\,dx+\int \frac{x}{(x^2+1)^2}\,dx\\ &= \ln|x|-\frac{1}{2}\ln(x^2+1)-\arctan x-\frac{1}{2(x^2+1)}+K\end{aligned}$$

■

5.4 Approximate Integration

Most of the functions we study here in calculus are **elementary functions**. These are the polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric and inverse trigonometric functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x)=e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x)=\int_0^x e^{t^2}\,dt$$

that we know that by the Fundamental Theorem of Calculus that

$$F'(x)=e^{x^2}$$

Thus $f(x)=e^{x^2}$ has an antiderivative F, but it can be proved that F is not an elementary function. This means that we cannot evaluate $\int e^{x^2}\,dx$ in terms of elementary functions. The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions do not have elementary antiderivatives.
Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.
Error Bounds: Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

and

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Simpson's Rule

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x)2+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]$$

where n is even and $\Delta x = \dfrac{b-a}{n}$.
Error Bounds for Simpson's Rule: Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

5.5 Improper Integrals

We extend the concept of a definite integral $\int_a^b f(x)\,dx$ to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an **improper integral**.

Infinite Intervals

Definition 5.5.1. 1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

2. If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx$$

for any real number a.

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

Discontinuous Integrands

Definition 5.5.2. 1. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if the limit exists.

2. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if the limit exists.

The improper integral $\int_a^b f(x)\,dx$ is **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Comparison Test for Improper Integrals

Theorem 5.5.1 Comparison Theorem. Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

6 Applications of Integration

6.1 Areas between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. We define the **area** A of S as as the limiting value of the sum of the areas of the approximating rectangles.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Big[f(x_i^*) - g(x_i^*)\Big]\Delta x$$

The area A of the region bounded by the curves $y = f(x), y = g(x)$ and the lines $x = a, x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b \Big[f(x) - g(x)\Big]\, dx$$

If a region is bounded by curves with equations $x = f(y), x = g(y), y = c, y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

$$A = \int_c^d \Big[f(y) - g(y)\Big]\, dy$$

6.2 Volumes

Definition 6.2.1. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\, dx$$

In general, we calculate the volume of a **solid of revolution** obtained by revolving a region about a line by using the basic defining formula

$$V = \int_a^b A(x)\, dx$$

or

$$V = \int_c^d A(y)\, dy$$

Volumes by Cylindrical Shells

The volume of the solid S obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx$$

where $0 \leq a < b$.

6.3 Arc Length

Suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \le x \le b$. We define the **length** L of the curve C with equation $y = f(x), a \le x \le b$, as the limit of the lengths of the inscribed polygons (if the limit exists):

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

We can derive an integral formula for L in the case where f has a continuous derivative. Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\,\Delta x$$

The **Arc Length Formula**: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x), a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx$$

In Leibniz's notation:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

If a curve has the equation $x = g(y), c \le y \le d$, and $g'(y)$ is continuous, then the Arc Length Formula is:

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

The Arc Length Function

If a smooth curve C has the equation $y = f(x), a \le x \le b$, let $s(x)$ be the distance along C from the initial starting point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the **arc length function**, and

$$s(x) = \int_a^x \sqrt{1 + \left[f'(t)\right]^2}\,dt$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dx} = \sqrt{1 + \left[f'(x)\right]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

and this equation is sometimes written in the symmetric form

$$(ds)^2 = (dx)^2 + (dy)^2$$

If we write $L = \int ds$, then we can solve to get

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

6.4 Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line.
If f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$, about the x-axis as

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx$$

In Leibniz's notation:

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y), c \leq y \leq d$, then the formula for surface area is

$$S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

These formulas can be summarized symbolically as

$$S = \int 2\pi y\, ds$$

For rotation about the y-axis, the surface area formula is

$$S = \int 2\pi x\, ds$$

7 Series

7.1 Sequences

An infinite **sequence** $\{a_n\}$ is a list of numbers written in a definite order:

$$a_1, a_2, a_3 \ldots, a_n, a_{n+1}, \ldots$$

where a_n is the nth term of the sequence. Consider the sequence

$$a_n = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots\right\}$$

We can rewrite it as

$$a_n = \left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \cdots \right\}$$

so the formula for the nth term is

$$a_n = \frac{1}{2^n}$$

for $n = 1, 2, 3, \cdots$ and

$$\{a_n\} = \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}$$

Definition 7.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L$$

or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, then the sequence is **convergent**, otherwise it is **divergent**.

The precise definition of a limit of a sequence is:

Definition 7.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there is an corresponding integer N such that

$$n > N \implies |a_n - L| < \varepsilon$$

Theorem 7.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$.

In particular, we have

$$\lim_{n\to\infty} \frac{1}{n^r} = 0$$

when $r > 0$. If a_n becomes large as n becomes large, we use the notation

$$\lim_{n\to\infty} a_n = \infty$$

Definition 7.1.3.

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive number M there is a positive integer N such that

$$n > N \implies a_n > M$$

If $\lim\limits_{n\to\infty} a_n = \infty$ then the sequence $\{a_n\}$ is divergent and we say that $\{a_n\}$ diverges to ∞. If a_n and b_n are convergent sequences and c is a constant, then

1. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$

3. $\lim_{n\to\infty} c \cdot a_n = c \lim_{n\to\infty} a_n$

4. $\lim_{n\to\infty} c = c$

5. $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$.

7. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

Theorem 7.1.2. If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 7.1.3. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 7.1.4 Continuity and Convergence Theorem. If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = 0$$

if $-1 < r < 1$ and

$$\lim_{n\to\infty} r^n = 1$$

if $r = 1$.

Definition 7.1.4. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. it is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Definition 7.1.5. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M$$

for all $n \geq 1$. It is **bounded below** if there is a number m such that

$$m \leq a_n$$

for all $n \geq 1$. If it is bounded above and below, then it is a **bounded sequence**.

Theorem 7.1.5 Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent.

The proof of the Monotonic Sequence Theorem is based on the **Completeness Axiom** of the set of real numbers $\mathbb{R}$. The Completeness Axiom states if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all $x \in S$), then S has a **least upper bound** b. (This means that b is an upper bound of S, but if M is any other upper bound, then $b \leq M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

Proof. Suppose $\{a_n\}$ is an incerasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for S (since L is the least upper bound). Therefore

$$a_N > L - \varepsilon$$

for some integer N. But the sequence is incerasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$ we have

$$\begin{aligned} a_n &> L - \varepsilon \\ a_n + \varepsilon &> L \\ L &< a_n + \varepsilon \end{aligned}$$

so

$$0 \leq L - a_n < \varepsilon$$

since $a_n \leq L$. Thus

$$|L - a_n| < \varepsilon$$

which implies that

$$|a_n - L| < \varepsilon$$

whenever $n > N$ so $\lim_{n\to\infty} a_n = L$. A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing. ■

The proof of the Monotonic Sequence Theorem shows that an increasing sequence that is bounded above is convergent and a decreasing sequence that is bounded below is convergent.

7.2 Series

In general, if we try to add the terms of an infinite sequence $\{a_n\}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by

$$\sum_{n=1}^{\infty} a_n$$

Definition 7.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots$, let s_n be its nth partial sum:

$$s_n = \sum_{k=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is **convergent** and we write $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = s$$

The number s is the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is **divergent**.

Thus the sum of a series is the limit of the sequence of partial sums. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_i$$

Consider the **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0$$

with common ratio r.

If $r = 1$, then $s_n = a + a + \cdots + a = na$. Thus $\lim\limits_{n\to\infty} s_n = \pm\infty$ so the limit does not exist and the geometric series diverges when $r = 1$.

If $r \neq 1$, then we have

$$\begin{aligned}
s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\
rs_n &= ar + ar^2 + ar^3 \cdots + ar^{n-1} + ar^n \\
s_n - rs_n &= a - ar^n \\
(1-r)s_n &= a(1-r^n) \\
s_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

If $-1 < r < 1$, since $\lim\limits_{n\to\infty} r^n = 0$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $\dfrac{a}{1-r}$.

If $r \leq -1$ or $r > 1$, then the sequence $\{r^n\}$ is divergent so $\lim\limits_{n\to\infty} s_n$ does not exist thus the geometric series diverges.

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, then the geometric series is divergent.

Problem 7.2.1. Find the sum of the series $\sum\limits_{n=0}^{\infty} x^n$, where $|x| < 1$.

Solution. Notice that

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$$

This is the geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

■

Problem 7.2.2. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Solution. From the definition of convergent series we compute the partial sums.

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Thus we have

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Notice that the terms cancel in pairs. This series is an example of a **telescoping series**. Because of all the cancellations, the sum collapses into just two terms. Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

■

Problem 7.2.3. Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution. For this particular series it is convenient to consider the partial sums

$$s_2, s_4, s_8, s_{16}, s_{32}, \dots$$

and show that they become large.

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}
\end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges. ■

Theorem 7.2.1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

■

With any series $\sum_{n=1}^{\infty} a_n$ we associate two sequences: the sequence $\{s_n\}$ of its partial sums and the sequence a_n of its terms. If $\sum_{n=1}^{\infty} a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

Theorem 7.2.2 Test for Divergence. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The Test of Divergence works because if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

Theorem 7.2.3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the following series:

1. $\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$ where c is a constant.

2. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

A finite number of terms does not affect the convergence or divergence of a series. If it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

7.3 The Integral and Comparison Tests

Testing with an Integral

Theorem 7.3.1 Integral Test. Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent.

1. If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Testing by Comparing

Theorem 7.3.2 Comparison Test. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

1. If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

2. If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

7.4 Other Convergence Tests

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative.

If the terms of an alternating series decrease to 0 in absolute value, then the series converges.

Theorem 7.4.1 Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \quad b_n > 0$$

satisfies

$$b_{n+1} \leq b_n$$

for all n and

$$\lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Theorem 7.4.2 Alternating Series Estimation Theorem. If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$0 \leq b_{n+1} \leq b_n$$

and

$$\lim_{n \to \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Absolute Convergence

Definition 7.4.1. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition 7.4.2. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem 7.4.3. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

Theorem 7.4.4 Ratio Test.

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Theorem 7.4.5 Root Test.

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

7.5 Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

A power series is a series in which each term is a power function. A **trigonometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trigonometric functions.

In general, a power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = a_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and the coefficient c_n is a constant. The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series converges.

Theorem 7.5.1. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all $x \in \mathbb{R}$.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in cases 3 is the **radius of convergence** of the power series. The radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case 2. The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

7.6 Representing Functions as Power Series

We start with the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

and find power series representations of similar functions.

Differentiation and Integration of Power Series

We can differentiate and integrate a function represented by the sum of a power series term by term. This is called **term-by-term differentiation and integration**.

Theorem 7.6.1. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

and

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of these power series are both R.

In Leibniz's notation:

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n(x-a)^n\right]$$

$$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad |x|<1, \quad R=1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \quad R=1$$

7.7 Taylor and Maclaurin Series

Theorem 7.7.1. If f has a power series expansion at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a|<R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The **Taylor series** of a function f centered at a is

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^2 + \cdots$$

For the special case $a=0$, the Taylor series becomes the **Maclaurin series**

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Problem 7.7.1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution. Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = e^0 = 1$ for all n therefore

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let $a_n = \dfrac{x^n}{n!}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

and

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

so by the Ratio Test the series converges for all $x \in \mathbb{R}$ and the radius of convergence is $R = \infty$. ■

The nth degree **Taylor polynomial** T_n of f at a is

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Let

$$R_n(x) = f(x) - T_n(x)$$

so that

$$f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is the **remainder** of the Taylor series. If $\lim\limits_{n\to\infty} R_n(x) = 0$, then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \Big[f(x) - R_n(x)\Big] = f(x)$$

Theorem 7.7.2. If $f(x) = T_n(x) + R_n(x)$ where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem 7.7.3 Taylor's Formula. If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder term in the Taylor series is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

This expression for $R_n(x)$ is **Lagrange's form of the remainder term**.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for every real number $x \in \mathbb{R}$.

Problem 7.7.2. Prove that e^x is equal to the sum of its Taylor series.

Solution. If $f(x) = e^x$, then $f^{n+1}(x) = e^x$, so the remainder term in Taylor's Formula is

$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$$

where z is a number strictly between 0 and x. Note that z depends on n. If $x > 0$, then $0 < z < x$, so $e^z < e^x$. Therefore

$$0 < R_n(x) = \frac{e^z}{(n+1)!}x^{n+1} < e^x \frac{x^{n+1}}{(n+1)!} \to 0$$

so $R_n(x) \to 0$ as $n \to \infty$ by the Squeeze Theorem. If $x < 0$, then $x < z < 0$, so $e^z < e^0 = 1$ and

$$|R_n(x)| < \frac{|x|^{n+1}}{(n+1)!} \to 0$$

Again $R_n(x) \to 0$. Thus e^x is equal to the sum of its Taylor series. ■

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad R = \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad R = \infty$$

If k is any real number and $|x| < 1$, then the **binomial series** is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where the **binomial coefficient** is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Problem 7.7.3. Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Solution. For all $x \in \mathbb{R}$ we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Then we integrate term by term:

$$\begin{aligned}\int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots\end{aligned}$$

This series converges for all $x \in \mathbb{R}$. ■

Power series can be added, subtracted, multiplied, and divided like polynomials.

7.8 Applications of Taylor Polynomial

Approximation Functions by Polynomials

The Taylor polynomial of f at a can be used as an approximation to f:

$$f(x) \approx T_n(x)$$

The size of the error of the approximation is:

$$|R_n(x)| = |f(x) - T_n(x)|$$

Applications to Physics

Physicist uses a Taylor polynomial as an approximation to the function.

Problem 7.8.1. In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0 c^2$$

Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics:

$$K = \frac{1}{2} m_0 v^2$$

Solution. Using the expressions given for K and m, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$= m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

With $x = -\frac{v^2}{c^2}$, then we have the binomial series with $k = -\frac{1}{2}$. Notice that $|x| < 1$ because $v < c$. Therefore we have

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

and

$$K = m_0c^2\left(\frac{1}{2}\cdot\frac{v^2}{c^2} + \frac{3}{8}\cdot\frac{v^2}{c^2} + \frac{5}{16}\cdot\frac{v^2}{c^2}\right)$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0c^2\left(\frac{1}{2}\cdot\frac{v^2}{c^2}\right) = \frac{1}{2}m_0v^2$$

■

The End