Calculus

Yaohui Wu

July 22, 2024

Introduction

Calculus is the mathematical study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **fundamental theorem of calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Contents

1 Functions and Limits **2**

1.1 The Limit of a Function 2

1.2 The Precise Definition of a Limit 3

1.3 Calculating Limits Using the Limit Laws 4

1.4 Continuity 10

1.5 Limits Involving Infinity 13

1 Functions and Limits

1.1 The Limit of a Function

Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E.

Definition 1.1.2. A function f is **injective** (or **one-to-one**) if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

Definition 1.1.3. A function f is **surjective** (or **onto**) if for all y in range Y, there exists an x in domain X such that $f(x) = y$.

Definition 1.1.4. A function f is **bijective** if f is injective and surjective.

Definition 1.1.5. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in B.

Intuitive Definition of a Limit

Suppose $f(x)$ is defined near the number a. (This means that $f(x)$ is defined on some open interval that contains the number a, except possibly at a itself.)

Definition 1.1.6. We write

$$\lim_{x \to a} f(x) = L$$

and say that the **limit** of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but $x \neq a$.

An alternative notation for the limit is $f(x) \to L$ as $x \to a$.

One-Sided Limits

Definition 1.1.7. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

Definition 1.1.8. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

Theorem 1.1.1. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The limit exists if and only if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a are equal to L, otherwise the limit does not exist.

1.2 The Precise Definition of a Limit

Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2.1. We say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.1. Prove that $\lim\limits_{x\to 3}(4x-5) = 7$.

Solution. Let $\varepsilon > 0$ be a given positive number. We want to find a number δ such that

$$0 < |x-3| < \delta \implies |(4x-5) - 7| < \varepsilon$$

But $|(4x-5)-7| = |4x-12| = 4|x-3|$. Note that $4|x-3| < \varepsilon \iff |x-3| < \varepsilon/4$. Let $\delta = \varepsilon/4$, we have

$$0 < |x-3| < \frac{\varepsilon}{4} \implies 4|x-3| < \varepsilon \implies |(4x-5)-7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x-5) = 7$$

■

Problem 1.2.2. Prove that $\lim\limits_{x\to 3} x^2 = 9$.

Solution. Let ε be a given positive number. We want to find a number δ such that

$$0 < |x-3| < \delta \implies |x^2 - 9| < \varepsilon$$

We simplify to get

$$|x^2 - 9| = |x+3|\,|x-3| < \varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3| < C\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3| < 1$. Then we have $|x+3| < 7$, and so $C = 7$. Let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x-3| < \delta$, then

$$|x^2-9| = |x+3|\,|x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim\limits_{x\to 3} x^2 = 9$. ■

Definition 1.2.2.

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Definition 1.2.3.

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.3. Prove that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$.

Solution. Let ε be a given positive number We want to find a number δ such that

$$0 < x < \delta \implies |\sqrt{x} - 0| < \varepsilon$$

But $\sqrt{x} < \varepsilon \iff x < \varepsilon^2$. Let $\delta = \varepsilon^2$. If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$ so $|\sqrt{x} - 0| < \varepsilon$. This shows that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$. ■

1.3 Calculating Limits Using the Limit Laws

We have the following properties of limits called the **limit laws** to calculate limits. Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \qquad\qquad \lim_{x \to a} g(x) = M$$

exist. Then

1. Sum Law: $\lim\limits_{x \to a}\Big[f(x) + g(x)\Big] = L + M$
2. Difference Law: $\lim\limits_{x \to a}\Big[f(x) - g(x)\Big] = L - M$
3. Constant Multiple Law: $\lim\limits_{x \to a}\Big[cf(x)\Big] = cL$
4. Product Law: $\lim\limits_{x \to a}\Big[f(x)g(x)\Big] = LM$
5. Quotient Law: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.
6. Power Law: $\lim\limits_{x \to a}\Big[f(x)\Big]^n = \Big[\lim\limits_{x \to a} f(x)\Big]^n$ where n is a positive integer.
7. Root Law: $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim\limits_{x \to a} f(x) > 0$.

8. $\lim_{x \to a} c = c$

9. $\lim_{x \to a} x = a$

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer.

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

Proof. Proof of limit law 8: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. It is proved that $\lim_{x \to a} c = c$. ∎

Proof. Proof of limit law 9: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

It is proved that $\lim_{x \to a} x = a$. ∎

Proof. Proof of the sum law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality, we have

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

We make $|f(x) - L| + |g(x) - M|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} \Big[f(x) + g(x) \Big] = L + M$$

∎

Proof. Proof of the product law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows and use the triangle inequality:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= \Big|\big[f(x) - L\big]g(x) + L[g(x) - M]\Big| \\
&\leq \Big|\big[f(x) - L\big]g(x)\Big| + \Big|L\big[g(x) - M\big]\Big| \\
&= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$. Then we can combine the inequalities to get

$$\begin{aligned}
|f(x)g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\varepsilon}{2(1 + |L|)} \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that

$$\lim_{x\to a}\big[f(x)g(x)\big] = LM$$

■

Proof. Proof of the constant multiple law: If we take $g(x) = c$ then by the product law and limit law 8, we get

$$\lim_{x \to a} \left[cf(x) \right] = \lim_{x \to a} c \cdot \lim_{x \to a} f(x) = c \lim_{x \to a} f(x) = cL$$

We can prove the constant multiple law using the precise the definition. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} \left[0 \cdot f(x) \right] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x) = 0 \cdot L$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of a limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

This shows that $\lim_{x \to a} \left[cf(x) \right] = cL$. ■

Proof. Proof of the difference law: Using the sum law and the constant multiple law with $c = -1$, we have

$$\begin{aligned} \lim_{x \to a} \left[f(x) - g(x) \right] &= \lim_{x \to a} \left[f(x) + (-1)g(x) \right] = \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x) \\ &= \lim_{x \to a} f(x) + (-1) \lim_{x \to a} g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M \end{aligned}$$

■

Proof. Proof of the quotient law: First we prove that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim_{x\to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \le |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

This shows that

$$0 < |x-a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \iff \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\left(f(x) \cdot \frac{1}{g(x)}\right) = \lim_{x\to a} f(x) \lim_{x\to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

■

We have the following **direct substitution property** to calculate limits. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Problem 1.3.1. Find $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided that this limit exists. When computing one-sided limits, we use the fact that the limit laws also hold for one-sided limits.

Problem 1.3.2. Show that $\lim_{x\to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim_{x\to 0} |x| = 0$. ■

The Squeeze Theorem

Theorem 1.3.1. If $f(x) \le g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then $L \le M$.

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a}\Big[g(x) - f(x)\Big] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M-L)\Big| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M-L)\Big| < L - M$$

Since $a \le |a|$ for any number a, we have

$$0 < |x-a| < \delta \implies \Big[g(x) - f(x)\Big] - (M-L) < L - M$$

which simplifies to

$$0 < |x-a| < \delta \implies g(x) < f(x)$$

But this contradicts $f(x) \le g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \le M$. ■

Theorem 1.3.2 Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x\to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$, so

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x\to a} g(x) = L$. ■

Problem 1.3.3. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1$$

and $\theta \le \tan\theta$. Therefore we have

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0}\cos\theta = 1$, so by the squeeze theorem, we have

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

But the function $(\sin\theta)/\theta$ is an even function, so its left and right limits must be equal. Hence we have

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Problem 1.3.4. Find $\lim_{x\to 0} \frac{\sin 7x}{4x}$

Problem 1.3.5. Evaluate $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$.

Solution. We have

$$\frac{\cos\theta-1}{\theta} = \frac{\cos\theta-1}{\theta}\left(\frac{\cos\theta+1}{\cos\theta+1}\right) = \frac{\cos^2\theta-1}{\theta(\cos\theta+1)} = \frac{-\sin^2\theta}{\theta(\cos\theta+1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta+1}\right)$$

We take the limit then

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta+1}\right)\right] = \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1}\right)\\ &= 1\left(\frac{(-1)(0)}{1+1}\right) = 0\end{aligned}$$

■

1.4 Continuity

Definition 1.4.1. A function f is **continuous at a number** a if

$$\lim_{x\to a} f(x) = f(a)$$

Note that f is continuous at a requires that $f(a)$ is defined and the limit exists. We say that f is **discontinuous** at a if f is not continuous at a.

Definition 1.4.2. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 1.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem 1.4.1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Proof. Each of the five parts of this theorem follows from the corresponding limit law. We give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad\qquad \lim_{x \to a} g(x) = g(a)$$

Then

$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a}\Big[f(x) + g(x)\Big] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a)$$

This shows that $f + g$ is continuous at a. ■

Theorem 1.4.2. Any polynomial is continuous on $\mathbb{R} = (-\infty, \infty)$. Any rational function is continuous on its domain.

Proof. A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know that polynomials are continuous on $\mathbb{R}$ so the rational function f is continuous at every number in D. ■

Theorem 1.4.3. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 1.4.4. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b)$$

Proof. Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y\to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$0 < |y-b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |g(x) - b| < \delta_1$$

Combining these two statements, we see that when $0 < |x-a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x\to a} f(g(x)) = f(b)$. ■

Theorem 1.4.5. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $f \circ g = f(g(x))$ is continuous at a.

Proof. Since g is continuous at a, we have

$$\lim_{x\to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $f(g(x))$ is continuous at a. ■

Theorem 1.4.6 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a,b) such that $f(c) = N$.

The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. If a continuous function $f(x)$ has values of opposite sign in an interval (a,b), then by the intermediate value theorem there exists a root of $f(x)$ in (a,b).

1.5 Limits Involving Infinity

Infinite Limits

Definition 1.5.1. The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but $x \neq a$.

Another notation for the limit is $f(x) \to \infty$ as $x \to a$. We say that the limit of $f(x)$, as x approaches a, is infinity.

Definition 1.5.2.

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but $x \neq a$.

We say that the limit of $f(x)$, as x approaches a, is negative infinity. Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Definition 1.5.3. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation is $f(x) \to L$ as $x \to \infty$. We say that the limit of $f(x)$, as x approaches infinity, is L.

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

We say that the limit of $f(x)$, as x approaches negative infinity, is L.

Definition 1.5.6. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = L$$

If n is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad\qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

Problem 1.5.1. Evaluate $\lim\limits_{x \to \infty} \sin x$.

Solution. As x increases, the values of $sinx$ oscillate between 1 and -1 infinitely often. Thus $\lim\limits_{x \to \infty} \sin x$ does not exist. ■

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to \infty} f(x) = \infty \qquad\qquad \lim_{x \to \infty} f(x) = -\infty \qquad\qquad \lim_{x \to -\infty} f(x) = -\infty$$

Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.7.

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.5.2. Prove that $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

Solution. Let M be a given positive number We want to find a number δ such that

$$0<|x|<\delta \implies \frac{1}{x^2}>M$$

But

$$\frac{1}{x^2}>M \iff x^2<\frac{1}{M} \iff \sqrt{x^2}<\sqrt{\frac{1}{M}} \iff |x|<\frac{1}{\sqrt{M}}$$

Let $\delta=1/\sqrt{M}$, then

$$0<|x|<\delta=\frac{1}{\sqrt{M}} \implies \frac{1}{x^2}>M$$

This shows that $\lim_{x\to 0}\frac{1}{x^2}=\infty$. ■

Definition 1.5.8.

$$\lim_{x\to a} f(x)=-\infty$$

if for every negative number N there is a positive number δ such that

$$0<|x-a|<\delta \implies f(x)<N$$

Definition 1.5.9. Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x)=L$$

means that for every $\varepsilon>0$ there is a corresponding number N such that

$$x>N \implies |f(x)-L|<\varepsilon$$

Definition 1.5.10. Let f be a function defined on some interval $(-\infty,a)$. Then

$$\lim_{x\to-\infty} f(x)=L$$

means that for every $\varepsilon>0$ there is a corresponding number N such that

$$x<N \implies |f(x)-L|<\varepsilon$$

Problem 1.5.3. Prove that $\lim_{x\to\infty}\frac{1}{x}=0$.

Solution. Given $\varepsilon>0$, we want to find an N such that

$$x>N \implies \left|\frac{1}{x}-0\right|<\varepsilon$$

Since $x\to\infty$, we can that $x>0$ in computing the limit. Then $1/x<\varepsilon \iff x>1/\varepsilon$. Let $N=1/\varepsilon$, so

$$x>N=\frac{1}{\varepsilon} \implies \left|\frac{1}{x}-0\right|=\frac{1}{x}<\varepsilon$$

Therefore, by definition,

$$\lim_{x\to\infty}\frac{1}{x}=0$$

■

Definition 1.5.11. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$x > N \implies f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.