Calculus

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Introduction

Calculus is the mathematical study of continuous change established by Issac Newton and Gottfried Wilhelm Leibniz. **Single variable calculus** studies **derivatives** and **integrals** of functions of one variable and their relationship stated by the **Fundamental Theorem of Calculus**.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Contents

1 Functions and Limits

1.1 The Limit of a Function

Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. A symbol that represents a number in the range of f is called a **dependent variable**. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**.

Increasing and Decreasing Functions

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2)$$

whenever $x_1 < x_2$ in I. It is called **decreasing** on I if

$$f(x_1) > f(x_2)$$

whenever $x_1 < x_2$ in I.

Inverse Functions

Definition 1.1.2. A function f is **injective** (or **one-to-one**) if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

Definition 1.1.3. A function f is **surjective** (or **onto**) if for all y in range Y, there exists an x in domain X such that $f(x) = y$.

Definition 1.1.4. A function f is **bijective** if f is injective and surjective (or one-to-one and onto).

Definition 1.1.5. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for all y in B.

We usually reverse the roles of x and y and write

$$f^{-1}(x) = y \iff f(y) = x$$

Intuitive Definition of a Limit

Suppose $f(x)$ is defined near the number a. (This means that $f(x)$ is defined on some open interval that contains the number a, except possibly at a itself.)

Definition 1.1.6. We write

$$\lim_{x \to a} f(x) = L$$

and say that the **limit** of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but $x \neq a$.

An alternative notation for the limit is $f(x) \to L$ as $x \to a$.

One-Sided Limits

Definition 1.1.7. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x < a$.

Definition 1.1.8. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a and $x > a$.

Theorem 1.1.1. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

The limit exists if and only if the left-hand limit and the right-hand limit of $f(x)$ as x approaches a are equal to L, otherwise the limit does not exist.

1.2 The Precise Definition of a Limit

Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.2.1. We say that limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Problem 1.2.1. Prove that $\lim_{x\to 3}(4x-5)=7$.

Solution. Let $\varepsilon>0$ be a given positive number. We want to find a number δ such that

$$0<|x-3|<\delta \implies |(4x-5)-7|<\varepsilon$$

But $|(4x-5)-7|=|4x-12|=4|x-3|$. Note that $4|x-3|<\varepsilon \iff |x-3|<\varepsilon/4$. Let $\delta=\varepsilon/4$, we have

$$0<|x-3|<\frac{\varepsilon}{4} \implies 4|x-3|<\varepsilon \implies |(4x-5)-7|<\varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x-5)=7$$

■

Problem 1.2.2. Prove that $\lim_{x\to 3}x^2=9$.

Solution. Let ε be a given positive number. We want to find a number δ such that

$$0<|x-3|<\delta \implies |x^2-9|<\varepsilon$$

We simplify to get

$$|x^2-9|=|x+3|\,|x-3|<\varepsilon$$

Let C be a positive constant such that

$$|x+3|\,|x-3|<C\,|x-3|<\varepsilon \iff |x-3|<\frac{\varepsilon}{C}$$

Since we are interested only in values of x that are close to 3, it is reasonable to assume that $|x-3|<1$. Then we have $|x+3|<7$, and so $C=7$. Let $\delta=\min\{1,\varepsilon/7\}$. If $0<|x-3|<\delta$, then

$$|x^2-9|=|x+3|\,|x-3|<7\cdot\frac{\varepsilon}{7}=\varepsilon$$

This shows that $\lim_{x\to 3}x^2=9$. ■

Definition 1.2.2.

$$\lim_{x\to a^-}f(x)=L$$

if for every number $\varepsilon>0$ there is a number $\delta>0$ such that

$$a-\delta<x<a \implies |f(x)-L|<\varepsilon$$

Definition 1.2.3.

$$\lim_{x\to a^+}f(x)=L$$

if for every number $\varepsilon>0$ there is a number $\delta>0$ such that

$$a<x<a+\delta \implies |f(x)-L|<\varepsilon$$

Problem 1.2.3. Prove that $\lim_{x\to 0^+}\sqrt{x}=0$.

Solution. Let ε be a given positive number We want to find a number δ such that

$$0<x<\delta \implies |\sqrt{x}-0|<\varepsilon$$

But $\sqrt{x}<\varepsilon \iff x<\varepsilon^2$. Let $\delta=\varepsilon^2$. If $0<x<\delta$, then $\sqrt{x}<\sqrt{\delta}=\sqrt{\varepsilon^2}=\varepsilon$ so $|\sqrt{x}-0|<\varepsilon$. This shows that $\lim_{x\to 0^+}\sqrt{x}=0$. ■

1.3 Calculating Limits Using the Limit Laws

We have the following properties of limits called the **limit laws** to calculate limits. Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. Sum Law: $\lim_{x\to a} \left[f(x) + g(x)\right] = L + M$
2. Difference Law: $\lim_{x\to a} \left[f(x) - g(x)\right] = L - M$
3. Constant Multiple Law: $\lim_{x\to a} \left[cf(x)\right] = cL$
4. Product Law: $\lim_{x\to a} \left[f(x)g(x)\right] = LM$
5. Quotient Law: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.
6. Power Law: $\lim_{x\to a} \left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer.
7. Root Law: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim_{x\to a} f(x) > 0$.
8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^n = a^n$ where n is a positive integer.
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

Proof. Proof of limit law 8: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |c - c| < \varepsilon$$

We have $|c - c| = 0 < \varepsilon$ so the trivial inequality is always true for any number $\delta > 0$. It is proved that $\lim_{x\to a} c = c$. ■

Proof. Proof of limit law 9: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon$$

Let $\delta = \varepsilon$, we have

$$0 < |x - a| < \delta = \varepsilon \implies |x - a| < \varepsilon$$

It is proved that $\lim_{x\to a} x = a$. ■

Proof. Proof of the sum law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

By the triangle inequality,

$$|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M|$$

We make $|f(x) - L| + |g(x) - M|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and so

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then

$$0 < |x - a| < \delta \implies |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} \left[f(x) + g(x)\right] = L + M$$

■

Proof. Proof of the product law: Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows and use the triangle inequality:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= \left|\left[f(x) - L\right]g(x) + L[g(x) - M]\right| \\ &\leq \left|\left[f(x) - L\right]g(x)\right| + \left|L\left[g(x) - M\right]\right| \\ &= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \end{aligned}$$

We want to make both of the terms less than $\varepsilon/2$. Since $\lim_{x \to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$. Then we can combine the inequalities to get

$$\begin{aligned}
|f(x)g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\varepsilon}{2(1 + |L|)} \\
&< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that

$$\lim_{x \to a} \left[f(x)g(x)\right] = LM$$

■

Proof. Proof of the constant multiple law: If we take $g(x) = c$ then by the product law and limit law 8, we get

$$\lim_{x \to a} \left[cf(x)\right] = \lim_{x \to a} c \cdot \lim_{x \to a} f(x) = c \lim_{x \to a} f(x) = cL$$

We can prove the constant multiple law using the precise the definition. Note that if $c = 0$, then $cf(x) = 0$ and we have

$$\lim_{x \to a} \left[0 \cdot f(x)\right] = \lim_{x \to a} 0 = 0 = 0 \cdot \lim_{x \to a} f(x) = 0 \cdot L$$

Let $\varepsilon > 0$ and $c \neq 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

We simplify to get

$$|f(x) - L| < \frac{\varepsilon}{|c|}$$

By the definition of a limit, there is a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{|c|}$$

Let $\delta = \delta_1$, we have

$$0 < |x - a| < \delta \implies |cf(x) - cL| < \varepsilon$$

This shows that $\lim_{x \to a} \left[cf(x)\right] = cL$. ■

Proof. Proof of the difference law: Using the sum law and the constant multiple law with $c = -1$, we have

$$\begin{aligned}\lim_{x\to a}\Big[f(x) - g(x)\Big] &= \lim_{x\to a}\Big[f(x) + (-1)g(x)\Big] = \lim_{x\to a} f(x) + \lim_{x\to a}(-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M\end{aligned}$$

■

Proof. Proof of the quotient law: First we prove that

$$\lim_{x\to a}\frac{1}{g(x)} = \frac{1}{M}$$

Let $\varepsilon > 0$ be given, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

Observe that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{|g(x) - M|}{|Mg(x)|}$$

Since $\lim\limits_{x\to a} g(x) = M$, there is a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2}$$

and therefore

$$|M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| = |g(x) - M| + |g(x)| < \frac{|M|}{2} + |g(x)|$$

This shows that

$$0 < |x-a| < \delta_1 \implies \frac{|M|}{2} < |g(x)| \iff \frac{1}{|g(x)|} < \frac{2}{|M|}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|}\cdot\frac{2}{|M|} = \frac{2}{M^2}$$

Also, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} = \frac{1}{|Mg(x)|}|g(x) - M| < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

which is what we want to show. We apply the product law to prove the quotient law

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left(f(x)\cdot\frac{1}{g(x)}\right) = \lim_{x\to a} f(x)\lim_{x\to a}\frac{1}{g(x)} = L\cdot\frac{1}{M} = \frac{L}{M}$$

■

We have the following **direct substitution property** to calculate limits. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Problem 1.3.1. Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$.

Solution.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2$$

■

If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, provided that this limit exists. When computing one-sided limits, we use the fact that the limit laws also hold for one-sided limits.

Problem 1.3.2. Show that $\lim\limits_{x\to 0} |x| = 0$.

Solution. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, it is shown that $\lim\limits_{x\to 0} |x| = 0$. ■

The Squeeze Theorem

Theorem 1.3.1. If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = L \qquad\qquad \lim_{x\to a} g(x) = M$$

then $L \leq M$.

Proof. We use the method of proof by contradiction. Suppose that $L > M$, then we have

$$\lim_{x\to a} \Big[g(x) - f(x)\Big] = M - L$$

Therefore, for any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < \varepsilon$$

Note that $L - M > 0$ by the hypothesis. Let $\varepsilon = L - M$, there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \Big|\Big[g(x) - f(x)\Big] - (M - L)\Big| < L - M$$

Since $a \leq |a|$ for any number a, we have

$$0 < |x-a| < \delta \implies \Big[g(x) - f(x)\Big] - (M-L) < L - M$$

which simplifies to

$$0 < |x-a| < \delta \implies g(x) < f(x)$$

But this contradicts $f(x) \leq g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leq M$. ■

Theorem 1.3.2 Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim\limits_{x\to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |h(x) - L| < \varepsilon \implies L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$, so

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim\limits_{x\to a} g(x) = L$. ■

Problem 1.3.3. Show that $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$.

Solution. Since

$$-1 \leq \sin\frac{1}{x} \leq 1$$

then

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

We know that

$$\lim_{x\to 0}(-x^2) = 0 \qquad\qquad \lim_{x\to 0} x^2 = 0$$

By the squeeze theorem,

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

■

Note the approximate value of π is $\pi \approx 3.14159$. If $0 < \theta < \pi/2$, then

$$\sin\theta < \theta \implies \frac{\sin\theta}{\theta} < 1$$

and $\theta \leq \tan\theta$. Therefore we have

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0}\cos\theta = 1$, so by the squeeze theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

But the function $(\sin\theta)/\theta$ is an even function, so its left and right limits must be equal. Hence we have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Problem 1.3.4. Find $\lim_{x\to 0}\frac{\sin 7x}{4x}$.

Solution.

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \lim_{x\to 0}\frac{7x\cdot\sin 7x}{4x\cdot 7x} = \frac{7}{4}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{4}$$

■

Problem 1.3.5. Evaluate $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}$.

Solution. We have

$$\frac{\cos\theta-1}{\theta} = \frac{\cos\theta-1}{\theta}\left(\frac{\cos\theta+1}{\cos\theta+1}\right) = \frac{\cos^2\theta-1}{\theta(\cos\theta+1)} = \frac{-\sin^2\theta}{\theta(\cos\theta+1)} = \frac{\sin\theta}{\theta}\left(\frac{-\sin\theta}{\cos\theta+1}\right)$$

We take the limit then

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right) = \lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)} = \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}\\ &= -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right) = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1} = -1\cdot\frac{0}{1+1} = 0\end{aligned}$$

■

1.4 Continuity

Definition 1.4.1. A function f is **continuous at a number** a if

$$\lim_{x\to a} f(x) = f(a)$$

Note that f is continuous at a requires that $f(a)$ is defined and the limit exists. We say that f is **discontinuous** at a if f is not continuous at a.

Definition 1.4.2. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 1.4.3. A function f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem 1.4.1. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Proof. Each of the five parts of this theorem follows from the corresponding limit law. We give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad\qquad \lim_{x \to a} g(x) = g(a)$$

Then

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} \big[f(x) + g(x) \big] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

This shows that $f + g$ is continuous at a. ∎

Theorem 1.4.2. Any polynomial is continuous on $\mathbb{R} = (-\infty, \infty)$. Any rational function is continuous on its domain.

Proof. A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants. $P(x)$ is the sum of power functions with a constant multiple and therefore it is continuous. A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know that polynomials are continuous on $\mathbb{R}$ so the rational function f is continuous at every number in D. ∎

Theorem 1.4.3. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 1.4.4. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b)$$

Proof. Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y\to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$0 < |y-b| < \delta_1 \implies |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |g(x) - b| < \delta_1$$

Combining these two statements, we see that when $0 < |x-a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x\to a} f(g(x)) = f(b)$. ∎

Theorem 1.4.5. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $f \circ g = f(g(x))$ is continuous at a.

Proof. Since g is continuous at a, we have

$$\lim_{x\to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we have

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $f(g(x))$ is continuous at a. ∎

Theorem 1.4.6 Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.

The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. If a continuous function $f(x)$ has values of opposite sign in an interval (a, b), then by the intermediate value theorem there exists a root of $f(x)$ in (a, b).

1.5 Limits Involving Infinity

Infinite Limits

Definition 1.5.1. The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but $x \neq a$.

Another notation for the limit is $f(x) \to \infty$ as $x \to a$. We say that the limit of $f(x)$, as x approaches a, is infinity.

Definition 1.5.2.

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but $x \neq a$.

We say that the limit of $f(x)$, as x approaches a, is negative infinity. Similar definitions can be given for the one-sided infinite limits

$$\begin{array}{ll} \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Definition 1.5.3. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \to a} f(x) = \infty & \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

Limits at Infinity

Definition 1.5.4. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation is $f(x) \to L$ as $x \to \infty$. We say that the limit of $f(x)$, as x approaches infinity, is L.

Definition 1.5.5. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

We say that the limit of $f(x)$, as x approaches negative infinity, is L.

Definition 1.5.6. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = L$$

If n is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

Problem 1.5.1. Evaluate $\lim_{x \to \infty} \sin x$.

Solution. As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often. Thus $\lim_{x \to \infty} \sin x$ does not exist. ■

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Precise Definitions

Let f be a function defined on some open interval that contains the number a, except possibly at a itself.

Definition 1.5.7.

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) > M$$

Problem 1.5.2. Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution. Let M be a given positive number We want to find a number δ such that

$$0 < |x| < \delta \implies \frac{1}{x^2} > M$$

But

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

Let $\delta = 1/\sqrt{M}$, then

$$0 < |x| < \delta = \frac{1}{\sqrt{M}} \implies \frac{1}{x^2} > M$$

This shows that $\lim_{x\to 0} \frac{1}{x^2} = \infty$. ■

Definition 1.5.8.

$$\lim_{x\to a} f(x) = -\infty$$

if for every negative number N there is a positive number δ such that

$$0 < |x - a| < \delta \implies f(x) < N$$

Definition 1.5.9. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x > N \implies |f(x) - L| < \varepsilon$$

Definition 1.5.10. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$x < N \implies |f(x) - L| < \varepsilon$$

Problem 1.5.3. Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Solution. Given $\varepsilon > 0$, we want to find an N such that

$$x > N \implies \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $x \to \infty$, we can that $x > 0$ in computing the limit. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let $N = 1/\varepsilon$, so

$$x > N = \frac{1}{\varepsilon} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$$

Therefore, by definition,

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

■

Definition 1.5.11. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$x > N \implies f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

2 Derivatives

2.1 Derivatives and Rates of Change

Tangents

Definition 2.1.1. The **tangent line** of the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

If $h = x - a$, then

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Let $s = f(t)$ be a **position function** that describes the motion of an object where s is the displacement of the object from the origin at time t. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

The **velocity** (or **instantaneous velocity**) of the object at time $t = a$ is the limit of the average velocities:

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Derivatives

Definition 2.1.2. The **derivative of a function** f **at a number** a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

The tangent line to $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$. The limit of these average rates of change is called the (**instantaneous**) **rate of change of y with respect to x** at $x = x_1$:

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_2} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$. The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. If $s = f(t)$ is a position function of a particle, then $f'(a)$ is the rate of change of the displacement s with respect to time t. $f'(a)$ is the velocity of the particle at time $t = a$. The **speed** of the particle is $|f'(a)|$, the absolute value of the velocity.

2.2 The Derivative as a Function

Definition 2.2.1. The **derivative of a function** f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Problem 2.2.1. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

■

Notations

The following notations for the derivative of the function $y = f(x)$ with respect to x are equivalent:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

The symbols $\frac{d}{dx}$ and D_x are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. The symbol $\frac{dy}{dx}$ is callled the Leibniz notation. We can rewrite the definition of the derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The following notations for the value of the derivative of $y = f(x)$ evaluated at the number a are equivalent:

$$y'(a) = f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left[\frac{dy}{dx}\right]_{x=a}$$

Differentiable Functions

Definition 2.2.2. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) (or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$) if it is differentiable at every number in the interval.

Theorem 2.2.1. If f is differentiable at a, then f is continuous at a.

Proof. To prove that f is continuous at a, we want to show that

$$\lim_{x \to a} f(x) = f(a)$$

Given that f is differentiable at a so

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Then

$$\lim_{x \to a} \big[f(x) - f(a)\big] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} \big[f(a) + (f(x) - f(a))\big] = \lim_{x \to a} f(a) + \lim_{x \to a}(f(x) - f(a)) = f(a) + 0 = f(a)$$

Therefore f is continuous at a. ∎

Note that there are functions that are continuous but not differentiable. The function $y = |x|$ is continuous at 0 but it is not differentiable at 0. Since

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h}$$

if the limit exists. But

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$
$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

then the limit does not exist so $f'(0)$ does not exist. Thus $y = |x|$ is differentiable at all x except 0.

Higher Order Derivatives

If $y = f(x)$ is a differentiable function and its derivative $y' = f'(x)$ is also a differentiable function, then the **second derivative** of $y = f(x)$ is

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$f''(x)$ is the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$, which is the rate of change of the slope of the original curve $y = f(x)$. In general, the second derivative is the rate of change of the rate of change. If $s = s(t)$ is the position function of an object, then its first derivative is the velocity $v(t)$ of the object as a function of time t:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

In general, the nth derivative of f is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Problem 2.2.2. Find the first and the second derivatives of $f(x) = x^3$.

Solution. The first derivative is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

The second derivative is

$$f''(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2+2hx+h^2) - 3x^2}{h} = \lim_{h\to 0} \frac{3x^2+6hx+3h^2-3x^2}{h}$$

$$= \lim_{h\to 0} \frac{6hx+3h^2}{h} = \lim_{h\to 0}(6x+3h) = 6x$$

■

2.3 Basic Differentiation Formulas

The derivative of the constant function $f(x) = c$ is

$$\frac{d}{dx}(c) = 0$$

Proof. If $f(x) = c$ where c is a constant, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

■

Power Functions

$$\frac{d}{dx}(x) = 1$$

Proof. If $f(x) = x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

■

The **Power Rule**: If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof. Let $f(x) = x^n$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

We use the Binomial Theorem to expand $(x+h)^n$ then

$$f'(x) = \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right)$$

$$= nx^{n-1}$$

because every term except the first has h as a factor and therefore approaches 0. ■

The Power Rule (general version): If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Problem 2.3.1. Differentiate $f(x) = 1/x$.

Solution.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

■

The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

The **Constant Multiple Rule**: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\Big[cf(x)\Big] = c\frac{d}{dx}f(x)$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{d}{dx}f(x) \end{aligned}$$

■

The **Sum Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{\Big[f(x+h)+g(x+h)\Big]-\Big[f(x)+g(x)\Big]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \end{aligned}$$

■

The Sum Rule can be extended to the sum of any number of functions. By writing $f-g$ as $f+(-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The **Difference Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)-g(x)\Big] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof. Let $F(x) = f(x) - g(x)$. Then

$$
\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{\big[f(x+h) - g(x+h)\big] - \big[f(x) - g(x)\big]}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x) - \big[g(x+h) - g(x)\big]}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\end{aligned}
$$

■

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial.

The Sine and Cosine Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof. If $f(x) = \sin x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right] = \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\
&= \sin x \cdot \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h \to 0} \left(\frac{\sin h}{h}\right) \\
&= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x
\end{aligned}
$$

■

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof. If $f(x) = \cos x$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left[\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h}\right] = \lim_{h\to 0}\left[\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\right] \\
&= \cos x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \sin x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\
&= (\cos x)\cdot 0 - (\sin x)\cdot 1 = -\sin x
\end{aligned}$$

■

If $f(x) = \sin x$, then

$$f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad f^{(4)}(x) = \sin x$$

In general, if $f(x) = \sin x$, then for $n = 0, 1, 2, \cdots$ we have

$$f^{(4n)}(x) = \sin x \qquad f^{(4n+1)}(x) = \cos x \qquad f^{(4n+2)}(x) = -\sin x \qquad f^{(4n+3)}(x) = -\cos x$$

If $f(x) = \cos x$, then

$$f'(x) = -\sin x \qquad f''(x) = -\cos x \qquad f'''(x) = \sin x \qquad f^{(4)}(x) = \cos x$$

In general, if $f(x) = \cos x$, then for $n = 0, 1, 2, \cdots$ we have

$$f^{(4n)}(x) = \cos x \qquad f^{(4n+1)}(x) = -\sin x \qquad f^{(4n+2)}(x) = -\cos x \qquad f^{(4n+3)}(x) = \sin x$$

2.4 The Product and Quotient Rules

The Product Rule

The **Product Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = \left[\frac{d}{dx}\Big[f(x)\Big]\right]g(x) + f(x)\frac{d}{dx}\Big[g(x)\Big]$$

Proof. Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x+h)g(x+h) - f(x+h)g(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x) + f(x+h)\frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0} g(x) + \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

■

The Quotient Rule

The **Quotient Rule**: If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}\big[f(x)\big]\right]g(x) - f(x)\frac{d}{dx}\big[g(x)\big]}{\big[g(x)\big]^2}$$

Proof. Let $F(x) = \dfrac{f(x)}{g(x)}$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)+f(x)g(x)-f(x)g(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)-f(x)g(x+h)+f(x)g(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)-\big[f(x)g(x+h)-f(x)g(x)\big]}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{\dfrac{f(x+h)-f(x)}{h}g(x) - f(x)\dfrac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\
&= \frac{\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}\cdot\lim\limits_{h\to 0}g(x) - \lim\limits_{h\to 0}f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0}g(x+h)\cdot\lim\limits_{h\to 0}g(x)} \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{\big[g(x)^2\big]}
\end{aligned}$$

■

Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left[\frac{d}{dx}(\sin x)\right]\cos x - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

■

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\left[\frac{d}{dx}(1)\right]\sin x - 1 \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x
\end{aligned}$$

■

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\left[\frac{d}{dx}(1)\right]\cos x - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x
\end{aligned}$$

■

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\left[\frac{d}{dx}(\cos x)\right]\sin x - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{(-\sin x) \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x
\end{aligned}$$

■

2.5 The Chain Rule

The **Chain Rule**: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Comments on the proof of the Chain Rule: Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du} \cdot \frac{du}{dx} \end{aligned}$$

Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous. The only flaw in this reasoning is that it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we cannot divide by 0. Nonetheless, this reasoning does at least suggest that the Chain Rule is true. Note that $\frac{dy}{dx}$ is the derivative of y with respect to x, whereas $\frac{dy}{du}$ is the derivative of y with respect to u.
The Power Rule combined with the Chain Rule: If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}\Big[g(x)\Big]^n = n\Big[g(x)\Big]^{n-1} \cdot g'(x)$$

Suppose that $y = f(u), u = g(x)$, and $x = h(t)$ where f, g, h are differentiable functions. Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$$

Proof of the Chain Rule

If we denote ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But

$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \implies \Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx.
The proof of the Chain Rule:

Proof. Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can write

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = \Big[g'(a) + \varepsilon_1\Big]\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(b) + \varepsilon_2 \Delta u = \Big[f'(b) + \varepsilon_2\Big]\Delta x$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$ If we now substitute the expression for Δu, we get

$$\Delta y = \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big]\Delta x$$

so

$$\frac{\Delta y}{\Delta x} = \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big]$$

As $\Delta x \to 0$, it shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \Big[f'(b) + \varepsilon_2\Big]\Big[g'(a) + \varepsilon_1\Big] = f'(b)g'(a) = f'(g(a))g'(a)$$

This proves the Chain Rule. ∎

2.6 Implicit Differentiation

An explicit function $y = f(x)$ is defined by expressing one variable explicitly in terms of another variable. Some functions are defined implicitly by a relation between x and y. For example, the equation of the circle is defined by $x^2 + y^2 = r^2$ where the raidus r is a constant. In some cases it is possible to solve an implicit function to get an explicit function. We can use the method of **implicit differentiation** to find the derivative of y. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Problem 2.6.1. Find $\dfrac{dy}{dx}$ of the unit circle $x^2 + y^2 = 1$.

Solution. We differentiate on both sides of the equation then

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Since $y = f(x)$, we use the chain rule then

$$\begin{aligned} 2x + \frac{d}{dy}(y^2)\frac{dy}{dx} &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

We solve this equation for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}$$

■

Problem 2.6.2. Find $\dfrac{dy}{dx}$ and the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Solution. We have

$$\begin{aligned} x^3 + y^3 &= 6xy \\ \frac{d}{dx}x^3 + \frac{d}{dy}y^3\left(\frac{dy}{dx}\right) &= \left[\frac{d}{dx}(6x)\right]y + 6x\frac{dy}{dx} \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\ x^2 + y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\ (y^2 - 2x)\frac{dy}{dx} &= 2y - x^2 \\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

When $x = 3$ and $y = 3$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

The equation of the tangent line is

$$y - 3 = -1(x - 3)$$

■

Problem 2.6.3. Find y' if $\sin(x+y) = y^2 \cos x$.

Solution.

$$\begin{aligned}
\sin(x+y) &= y^2 \cos x \\
\cos(x+y)(1+\frac{dy}{dx}) &= 2y\frac{dy}{dx}\cos x - y^2 \sin x \\
\left[2y\cos x - \cos(x+y)\right]\frac{dy}{dx} &= \cos(x+y) + y^2 \sin x \\
\frac{dy}{dx} &= \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}
\end{aligned}$$

■

Problem 2.6.4. Find y'' if $x^4 + y^4 = 16$.

Solution. First we find y' then we have

$$\begin{aligned}
x^4 + y^4 &= 16 \\
4x^3 + 4y^3\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^3}{y^3}
\end{aligned}$$

Then we find y'' and we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -3x^2y^{-3} + (-x^3)(-3y^{-4})\frac{dy}{dx} = 3x^3y^{-4}(-x^3y^{-3}) - 3x^2y^{-3} \\
&= -3x^6y^{-7} - 3x^2y^{-3} = -3x^2y^{-7}(x^4+y^4) = -48\frac{x^2}{y^7}
\end{aligned}$$

■

2.7 Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity. The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem 2.7.1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution. Let V be the volume of the balloon and let r be its radius. The rate of increase of the volume with respect to time is $\frac{dV}{dt}$, and the rate of increase of the radius is $\frac{dr}{dt}$. Then, we are given that

$$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

and we want to find $\frac{dr}{dt}$ when $r = 25$ cm. We relate V and r by the formula of the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Then by the Chain Rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

If $r = 25$ and $\frac{dV}{dt} = 100$, then

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(25)^2} \cdot 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $\frac{1}{25\pi}$ cm/s. ■

2.8 Linear Approximations and Differentials

We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Applications to Physics

The linear approximations

$$\sin\theta \approx \theta \qquad\qquad \cos\theta \approx 1$$

are used in physics when θ is close to 0.

Differentials

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable. The differential dy is then defined by

$$dy = f'(x)\,dx$$

so dy is an independent variable.

2.9 Derivatives of Inverse Functions

The Calculus of Inverse Functions

Theorem 2.9.1. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem 2.9.2. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

In general, for any number x we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, in Leibniz notation we have

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

Derivatives of Logarithmic Functions

The **Euler's number** e is the base of the natural exponential function $y = e^x$. It is also the base of the the natural logarithmic function $y = \ln x$.

Definition 2.9.1 Euler's Number. The Euler's number e is defined as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{1/x}$$

Note that the approximate value of e is $e \approx 2.71828$.

Theorem 2.9.3. The exponential function $f(x) = \log_a x$ is differentiable and

$$f'(x) = \frac{1}{x}\log_a e$$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a\left(\dfrac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) = \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h} \\
&= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)} = \frac{1}{x}\log_a e
\end{aligned}$$

■

We know from the change of base formula that

$$\log_a e = \frac{\ln e}{\ln a} = \frac{1}{\ln a}$$

and therefore

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e = \frac{1}{x \ln a}$$

The derivative of the natural logarithmic function $f(x) = \ln x$ is:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof.

$$\frac{d}{dx}(\ln x) = \frac{1}{x \ln e} = \frac{1}{x}$$

■

In general, if $u = f(x)$, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Problem 2.9.1. Find $f'(x)$ if $f(x) = \ln|x|$.

Solution. Since $f(x) = \ln x$ for $x > 0$ and $f(x) = \ln(-x)$ for $x < 0$, it follows that

$$f'(x) = \frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$
$$f'(x) = \frac{d}{dx}\Big[\ln(-x)\Big] = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0$$

Therefore $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for all $x \neq 0$. ■

Logarithmic Differentiation

The steps in **logarithmic differentiation** are

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.

2. Differentiate implicitly with respect to x.

3. Solve the resulting equation for y'.

The proof of the Power Rule (general version):

Proof. Let $y = x^n$ and we use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x|, \quad x \neq 0$$

Therefore

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{n}{x}$$

Hence

$$\frac{dy}{dx} = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

■

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of the derivative.

Derivatives of Exponential Functions

Theorem 2.9.4. The exponential function $f(x) = a^x, a > 0$, is differentiable and

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Proof. We know that the logarithmic function $y = \log_a x$ is differentiable (and its derivative is nonzero) so its inverse function $y = a^x$ is differentiable. If $y = a^x$, then $\log_a y = x$. By implicit differentiation we have

$$\begin{aligned} \log_a y &= x \\ \frac{1}{y\ln a} \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y\ln a = a^x \ln a \end{aligned}$$

■

The derivative of the natural exponential function $f(x) = e^x$ is:

$$\frac{d}{dx}(e^x) = e^x$$

Proof.

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x$$

■

In general, if $u = f(x)$, then

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arcsin x$. Then $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$. By implicit differentiation we have

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\ \cos y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

Now $\cos y \geq 0$ since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y}$$

Therefore

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

∎

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Proof. Let $y = \arccos x$. Then $\cos y = x$ and $0 \leq y \leq \pi$. By implicit differentiation we have

$$\begin{aligned} \frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) \\ -\sin y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y} \end{aligned}$$

Now $\sin y \geq 0$ since $0 \leq y \leq \pi$, so

$$\sin y = \sqrt{1 - \cos^2 y}$$

Therefore

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

∎

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Proof. Let $y = \arctan x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We have

$$\begin{aligned}
\frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\
\sec^2 y \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{1+x^2}
\end{aligned}$$

Therefore

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

■

$$\begin{aligned}
\frac{d}{dx}(\operatorname{arccsc} x) &= -\frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}(\operatorname{arcsec} x) &= \frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx}(\operatorname{arccot} x) &= \frac{1}{x\sqrt{1+x^2}}
\end{aligned}$$

2.10 Hyperbolic Functions

Definition of **hyperbolic functions**:

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} & \operatorname{csch} x &= \frac{1}{\sinh x} \\
\cosh x &= \frac{e^x + e^{-x}}{2} & \operatorname{sech} x &= \frac{1}{\cosh x} \\
\tanh x &= \frac{\sinh x}{\cosh x} & \coth x &= \frac{\cosh x}{\sinh x}
\end{aligned}$$

Hyperbolic Identities:

- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \operatorname{sech}^2 x$
- $\sinh(x+y) = \sinh x \cosh x + \cosh x \sinh y$
- $\cosh(x+y) = \cosh x \cosh x + \sinh x \sinh y$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Inverse Hyperbolic Functions

$$\begin{aligned}
\operatorname{arsinh} x &= \ln(x + \sqrt{x^2+1}), \quad x \in \mathbb{R} \\
\operatorname{arcosh} x &= \ln(x + \sqrt{x^2-1}), \quad x \geq 1 \\
\operatorname{artanh} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 \leq x \leq 1
\end{aligned}$$

Derivatives of inverse hyperbolic functions:

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{arcsch} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arsech} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1-x^2}$$

2.11 Indeterminate Forms and l'Hospital's Rule

In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{0}{0}$. In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then this limit may or may not exist and is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

Theorem 2.11.1 l'Hospital's Rule. Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \lim_{x \to a} g(x) = \pm\infty$$

In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

It is important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. For the special case in which $f(a) = g(a) = 0$, f' and g' is continuous, and $g'(a) \neq 0$, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Problem 2.11.1. Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Solution. Notice that we have an indeterminate form of type $\frac{0}{0}$. We apply l'Hospital's rule then

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

■

Problem 2.11.2. Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution. Notice that we have an indeterminate form of type $\frac{\infty}{\infty}$. Then l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since we still have an indeterminate form of $\frac{\infty}{\infty}$, a second application of l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

■

Problem 2.11.3. Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

Solution. Notice that we have an indeterminate form of type $\frac{0}{0}$. We apply l'Hospital's rule then

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} = \lim_{x\to 0}\frac{2\sec^4 x + 4\sec^2 x\tan^2 x}{6}$$
$$= \frac{2(1)+4(1)(0)}{6} = \frac{1}{3}$$

■

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then the limit

$$\lim_{x\to a} f(x)g(x)$$

is called an **indeterminate form of type** $0\cdot\infty$. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g}$$

or

$$fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

Problem 2.11.4. Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

■

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}\Big[f(x) - g(x)\Big]$$

is called an **indeterminate form of type** $\infty - \infty$. We can convert the difference into a quotient so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Problem 2.11.5. Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Solution.

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0$$

■

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}$$

- Type 0^0: $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
- Type ∞^0: $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
- Type 1^∞: $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Note that the form 0^∞ is not indeterminate. Each of these three cases can be treated either by taking the natural logarithm: let $y = \left[f(x)\right]^{g(x)}$, then $\ln y = g(x)\ln f(x)$ or by writing the function as an exponential:

$$\left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$$

In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0\cdot\infty$.

Problem 2.11.6. Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution. Let

$$y = (1+\sin 4x)^{\cot x}$$

Then

$$\ln y = \ln\left[(1+\sin 4x)^{\cot x}\right] = \cot x\ln(1+\sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\dfrac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

Then

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

■

Problem 2.11.7. Find $\lim_{x\to 0^+} x^x$.

Solution. We used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x\ln x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

■

3 Applications of Differentiation

3.1 Maximum and Minimum Values

Definition 3.1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Definition 3.1.2. The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

Theorem 3.1.1 Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem 3.1.2 Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof. Suppose that f has a local maximum at c. Then $f(c) \geq f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geq f(c+h)$$

and theorefore

$$f(c+h) - f(c) \leq 0$$

If $h > 0$ and h is sufficiently small, we have

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Taking the right-hand limit, we get

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

and so we have shown that $f'(c) \leq 0$. If $h < 0$, then

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

Taking the left-hand limit, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

We have shown that $f'(c) \leq 0$ and $f'(c) \geq 0$. Therefore $f'(c) = 0$. The case of a local minimum can be proved in a similar manner. ■

Definition 3.1.3. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local maximum or minimum at c, then c is a critical number of f. The Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of these values is the absolute maximum value; the smallest of these values is the absolute minimum value.

3.2 The Mean Value Theorem

Theorem 3.2.1 Rolle's Theorem. Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. There are three cases:
Case 1: $f(x) = k$, a constant
Then $f'(x) = 0$ so the number c can be any number in (a, b).
Case 2: $f(x) > f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c. Therefore $f'(c) = 0$ by Fermat's Theorem.
Case 3: $f(x) < f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a minimum value somewhere in $[a, b]$, and since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Then $f'(c) = 0$ by Fermat's Theorem. ■

Theorem 3.2.2 Lagrange's Mean Value Theorem. Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. We apply Rolle's Theorem to a new function h defined as a difference between f and the function whose graph is the secant line AB where A is the point $(a, f(a))$ and B is the point $(b, f(b))$. The slope of the secant line AB is

$$m = \frac{f(b) - f(a)}{b - a}$$

Then the equation of the secant line AB is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

So,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on $[a, b]$ because it is the sum of f and a first degree polynomial, both of which is continuous.

2. The function h is differentiable on (a, b) because both f and a first degree polynomial are differentiable. In fact, we can compute h' directly:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

   Note that $f(a)$ and $\dfrac{f(b) - f(a)}{b - a}$ are constants.

3.
$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$
$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - \Big[f(b) - f(a)\Big] = 0$$

Therefore $h(a) = h(b)$. Since h satisfies the hypotheses of Rolle's Theorem, there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

■

In general, Lagrange's Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval. The main significance of Lagrange's Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative.

Theorem 3.2.3. If $f'(x) = 0$ for all x in an interval (a, b), then f is a constant on (a, b).

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying Lagrange's Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so

$$f(x_2) - f(x_1) = 0$$

or

$$f(x_2) = f(x_1)$$

Therefore f has the same value at any two numbers x_1 and x_2 in (a, b). This means that f is a constant on (a, b). ■

Corollary 3.2.3.1. If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is a constant on (a, b); that is; $f(x) = g(x) + c$ where c is a constant.

Proof. Since $f'(x) = g'(x)$, we have $f'(x) - g'(x) = 0$. Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus F is a constant; that is, $f - g$ is a constant. ■

Theorem 3.2.4 Cauchy's Mean Value Theorem. Suppose that the fucntions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

3.3 Derivatives and the Shapes of Graphs

Increasing/Decreasing Test:

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. Given $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. By Lagrange's Mean Value Theorem, there is a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, thus

$$f(x_2) - f(x_1) > 0$$

or

$$f(x_1) < f(x_2)$$

This shows that f is increasing. Part 2 is proved similarly. ∎