REAL ANALYSIS

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1 The Real Number System

1.1 The Set of Real Numbers

We define:

1. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$.
2. The set of integers is $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$.
3. The set of rational numbers is $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\right\}$.
4. The set of real numbers is $\mathbb{R}$.

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Definition 1.1.1 (Well Ordering Property of $\mathbb{N}$)**.** Every nonempty subset of $\mathbb{N}$ has a least element.

Definition 1.1.2. An ordered set is a set S with a relation $<$ such that

1. (Trichtomy) If $x, y \in S$, then one and only one of the statements $x < y, x = y, y < x$ is true.
2. (Transitivity) If $x, y, z \in S$, $x < y$, and $y < z$, then $x < z$.

Definition 1.1.3. Suppose S is an ordered set and $E \subset S$. If there exists a $b \in S$ such that $x \leq b$ for all $x \in E$, then E is bounded above and b is an upper bound of E. If there exists an upper bound a such that $a \leq b$ for all upper bounds b of E, then a is the least upper bound or the supremum of E and we write $\sup E = a$.

Similar definition applies for a set that is bounded below by a lower bound. The greatest lower bound or the infimum of E is denoted by $\inf E$.

Definition 1.1.4. A set is bounded if it is bounded above and below.

Definition 1.1.5 (Dedekind Completeness)**.** An ordered set S has the least-upper-bound property if every nonempty subset $E \subset S$ that is bounded above has a least upper bound, that is, $\sup E$ exists in S.

Theorem 1.1.1. *There exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.*

1.2 Archimedian Property

Theorem 1.2.1. *If $x \in \mathbb{R}$ and $x \leq \varepsilon$ for all real numbers $\varepsilon > 0$, then $x \leq 0$.*

Theorem 1.2.2. *Suppose $x, y \in \mathbb{R}$.*

1. *(Archimedian property) If $x > 0$, then there exists an $n \in \mathbb{N}$ such that $nx > y$.*
2. *($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x < y$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.*

If $A \subset \mathbb{R}$ and $\sup A \in A$, then the supremum is the maximum of A denoted by $\max A$. Similarly, the infimum is the minimum of A denoted by $\min A$.

Theorem 1.2.3 (Triangle Inequality). *If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.*

2 Sequences and Series

2.1 Sequences

Definition 2.1.1. A sequence $\{x_n\}_{n=1}^{\infty}$ is a function with domain $\mathbb{N}$ and codomain $\mathbb{R}$.

Definition 2.1.2. A sequence $\{x_n\}$ is bounded if there exists a $B \in \mathbb{R}$ such that $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Definition 2.1.3. A sequence is convergent and has the limit L and we write

$$\lim_{n\to\infty} x_n = L$$

if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$.

If a sequence is not convergent, then it is divergent.

Theorem 2.1.1. *A convergent sequence has a unique limit.*

Theorem 2.1.2. *If a sequence is convergent, then it is bounded.*

Theorem 2.1.3 (Squeeze Theorem). *If $a_n \leq b_n \leq c_n$ for all $n \geq N$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then*

$$\lim_{n\to\infty} b_n = L.$$

2.2 Monotonic Sequences

Definition 2.2.1. A sequence $\{x_n\}$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. $\{x_n\}$ is decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotonic if it is increasing or decreasing.

Theorem 2.2.1 (Monotone Convergence Theorem). *A monotonic sequence is convergent if and only if it is bounded. If $\{x_n\}$ is increasing and bounded above, then*

$$\lim_{n\to\infty} x_n = \sup x_n.$$

If $\{x_n\}$ is decreasing and bounded below, then

$$\lim_{n\to\infty} x_n = \inf x_n.$$

2.3 Subsequences

Definition 2.3.1. Let n_k be a sequence with $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. The sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Theorem 2.3.1. *If $\{x_n\}$ is convergent, then every subsequence $\{x_{n_k}\}$ is convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_k}.$$

2.4 Bolzano-Weierstrass Theorem

Theorem 2.4.1 (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

2.5 Cauchy Sequences

Definition 2.5.1. A sequence $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq \mathbb{N}$.

Theorem 2.5.1. *If a sequence is a Cauchy sequence, then it is bounded.*

Theorem 2.5.2 (Cauchy Completeness). *A sequence is convergent if and only if it is a Cauchy sequence.*