Mathematical Derivation of the Closed-Form Image Matting

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For clarity of exposition, a grey-scale example is used to illustrate the methodology of the closed-form matting algorithm [1]. In this section, the derivation of optimized target is shown in Section I, and the formulation of the entry of the Matting Laplacian matrix is shown in Section II.

I. The Optimized Target

For a gray-scale image, the value (I_i) of each pixel can be represented as

$$I_i = \alpha_i F_i + (1 - \alpha_i) B_i \tag{1}$$

where F represents the foreground, B represents the background, and α denotes the alpha channel. Redoing (1), we can get

$$\alpha_i = \frac{1}{F_i - B_i} I_i + \left(-\frac{B_i}{F_i - B_i}\right) \tag{2}$$

Some assumptions on the nature of F, B and a are needed. Assume that both F and B are approximately constant over a small window around each pixel. This assumption allows us to rewrite (2), expressing α as a linear function of image I

$$\alpha_i \approx a I_i + b, \forall i \in w \tag{3}$$

where $a = \frac{1}{F-B}$, $b = -\frac{B}{F-B}$, and w is a small image window, whose size is 3×3 as usual. So the relation suggests finding α, a and b that minimizes the cost function

$$J(\alpha, a, b) = \sum_{j \in k} \left(\sum_{i \in w_k} (\alpha_i - a_j I_i - b_j)^2 + \epsilon a_j^2 \right) \tag{4}$$

where w_k is a small window around pixel j. The cost function includes a regularization term on a. One reason for this term is numerical stability [1].

The cost function can be written in matrix form as follows:

$$J(\alpha, a, b) = \sum_k \left\| \begin{bmatrix} I_1^j & 1 \\ I_2^j & 1 \\ \vdots & \vdots \\ I_w^j & 1 \\ \sqrt{\epsilon} & 0 \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix} - \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \\ \vdots \\ \alpha_w^j \\ 0 \end{bmatrix} \right\|^2 \quad j \in k \tag{5}$$

Let us define

$$G_k = \begin{bmatrix} I_1^j & 1 \\ I_2^j & 1 \\ \vdots & \vdots \\ I_w^j & 1 \\ \sqrt{\epsilon} & 0 \end{bmatrix}, \quad \overline{\alpha}_k = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \\ \vdots \\ \alpha_w^j \\ 0 \end{bmatrix} \quad j \in k \tag{6}$$

Then, the cost function changes to

$$J(\alpha, a, b) = \sum_k \left\| G_k \begin{bmatrix} a_k \\ b_k \end{bmatrix} - \overline{\alpha}_k \right\|^2 \tag{7}$$

For a given matte α, the optimal pair is

$$\begin{bmatrix} a_k^* \\ b_k^* \end{bmatrix} = \operatorname{argmin} \left\| G_k \begin{bmatrix} a_k \\ b_k \end{bmatrix} - \overline{\alpha}_k \right\|^2 \tag{8}$$

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Let $A = G_k,\ B = \overline{\alpha}_k,\ X = \begin{bmatrix} a_k \\ b_k \end{bmatrix}$, Hence, (8) changes to

$$X^* = \text{argmin}\, ||AX - B||^2 \tag{9}$$

where

$$\begin{aligned} ||AX - B||^2 &= (AX - B)^T(AX - B) \\ &= (X^T A^T - B^T)(AX - B) \\ &= X^T A^T AX - B^T AX - X^T A^T B + B^T B \\ &= X^T A^T AX - 2X^T A^T B + B^T B \end{aligned} \tag{10}$$

We compute the gradient of (10), and set it equal to 0 to obtain the optimal solution.

$$\begin{aligned} &\frac{\partial ||AX - B||^2}{\partial X} = 2A^T AX - 2A^T B = 0 \\ &\to A^T AX = A^T B \\ &\to X = (A^T A)^{-1} A^T B \end{aligned} \tag{11}$$

So, the optimal solution is

$$\begin{bmatrix} a_k^* \\ b_k^* \end{bmatrix} = (G_k^T G_k)^{-1} G_k^T \overline{\alpha}_k \tag{12}$$

Use the optimal solution $\begin{bmatrix} a_k^* \\ b_k^* \end{bmatrix}$ to replace (a, b) in the cost function $J(\alpha, a, b)$ in (7) as below:

$$\begin{aligned} J(\alpha) &= \sum_k \left|\left| G_k (G_k^T G_k)^{-1} G_k^T \overline{\alpha}_k - \overline{\alpha}_k \right|\right|^2 \\ &= \sum_k \left|\left| (I - G_k (G_k^T G_k)^{-1} G_k^T) \overline{\alpha}_k \right|\right|^2 \end{aligned} \tag{13}$$

Here, I is a identity matrix. Let $\overline{G}_k = I - G_k (G_k^T G_k)^{-1} G_k^T$, so $J(\alpha)$ can be written as:

$$\begin{aligned} J(\alpha) &= \sum_k ||\overline{G}_k \overline{\alpha}_k||^2 \\ &= \sum_k (\overline{G}_k \overline{\alpha}_k)^T \overline{G}_k \overline{\alpha}_k \\ &= \sum_k (\overline{\alpha}_k^T \overline{G}_k^T \overline{G}_k \overline{\alpha}_k) \end{aligned} \tag{14}$$

Let L represents the $\overline{G}_k^T \overline{G}_k$ and α refers to $\overline{\alpha}$, so $J(\alpha)$ is

$$J(\alpha) = \alpha^T L \alpha \tag{15}$$

The derivation of $L_{i,j}$ can be found in the next section.

So, the target is

$$\begin{aligned} &\min_\alpha J(\alpha) = \alpha^T L \alpha \\ &s.t.\ \ (\alpha - S)^T D_c (\alpha - S) = 0 \end{aligned} \tag{16}$$

Here S represents the scribbles image containing the specified alpha values for the constrained pixels and zero for all other pixels, and the dimension of S is $N \times 1$. D_c is a diagonal matrix, which at the position of the scribble takes value 1 and for others taken value 0. The dimension of D_c is $N \times N$.

The Lagrange function $L(\alpha, \lambda)$ for this problem is

$$\begin{aligned} L(\alpha, \lambda) &= \alpha^T L \alpha + \lambda(\alpha^T - S^T) D_c (\alpha - S) \\ &= \alpha^T L \alpha + \lambda(\alpha^T D_c \alpha - S^T D_c \alpha \\ &\quad - \alpha^T D_c S + S^T D_c S) \\ \text{therefore} \quad \frac{\partial L(\alpha, \lambda)}{\partial \alpha} &= 2L\alpha + \lambda(2D_c \alpha - 2D_c S) \\ &= (L + \lambda D_c) 2\alpha - 2\lambda D_c S \end{aligned} \tag{17}$$

Let the gradient be 0 to obtain the optimal solution,

$$\begin{aligned} &\frac{\partial L(\alpha,\lambda)}{\partial \alpha} = 0 \\ \rightarrow \quad & (L+\lambda D_c)2\alpha - 2\lambda D_c S = 0 \\ \rightarrow \quad & (L+\lambda D_c)\alpha - \lambda D_c S = 0 \\ \rightarrow \quad & (L+\lambda D_c)\alpha = \lambda D_c S \end{aligned} \tag{18}$$

Finally, the optimal solution α^* can be obtained by solving the following sparse linear system.

$$(L+\lambda D_c)\alpha - \lambda D_c S = 0 \tag{19}$$

where λ is some large number.

II. The Matting Laplacian Matrix

As known in previous section, L represents $\overline{G}_k^T \overline{G}_k$, where $\overline{G}_k = I - G_k(G_k^T G_k)^{-1} G_k^T$, here k represents the kth window, and I, α refers to the kth window.

$$\begin{aligned} & G_k = \begin{bmatrix} I_1 & 1 \\ I_2 & 1 \\ \vdots & \vdots \\ I_w & 1 \\ \sqrt{\epsilon} & 0 \end{bmatrix} \qquad G_k^T = \begin{bmatrix} I_1 & I_2 & \cdots & I_w & \sqrt{\epsilon} \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \\ & G_k^T G_k = \begin{bmatrix} I_1 & I_2 & \cdots & I_w & \sqrt{\epsilon} \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 & 1 \\ I_2 & 1 \\ \vdots & \vdots \\ I_w & 1 \\ \sqrt{\epsilon} & 0 \end{bmatrix} \\ & G_k^T G_k = \begin{bmatrix} \sum_{i=0}^{w} I_i^2 + \epsilon & \sum_{i=0}^{w} I_i \\ \sum_{i=0}^{w} I_i & w \end{bmatrix} \end{aligned} \tag{20}$$

As it is well known,

$$\begin{cases} \mu = \dfrac{1}{w} \sum_{i=0}^{w} I_i \\ \sigma^2 = \dfrac{1}{w} \sum_{i=0}^{w} (I_i - \mu)^2 \end{cases} \quad \rightarrow \quad \begin{cases} \sum_{i=0}^{w} I_i = \mu w \\ \sum_{i=0}^{w} I_i^2 = w\sigma^2 + w\mu^2 \end{cases} \tag{21}$$

So, (20) can be written as

$$G_k^T G_k = \begin{bmatrix} w\sigma^2 + w\mu^2 + \epsilon & w\mu \\ w\mu & w \end{bmatrix} \tag{22}$$

The inverse matrix $(G_k^T G_k)^{-1}$ is

$$\begin{aligned} & (G_k^T G_k)^{-1} \\ & = \frac{1}{w(w\sigma^2 + w\mu^2 + \epsilon) - \mu^2 w^2} \begin{bmatrix} w & -\mu w \\ -\mu w & w\sigma^2 + w\mu^2 + \epsilon \end{bmatrix} \\ & = \frac{1}{w^2\sigma^2 + w\epsilon} \begin{bmatrix} w & -\mu w \\ -\mu w & w\sigma^2 + w\mu^2 + \epsilon \end{bmatrix} \\ & = \frac{1}{w\sigma^2 + \epsilon} \begin{bmatrix} 1 & -\mu \\ -\mu & \sigma^2 + \mu^2 + \frac{\epsilon}{w} \end{bmatrix} \end{aligned} \tag{23}$$

Let $\frac{1}{w\sigma^2+\epsilon} = k_1$, $\sigma^2 + \mu^2 + \frac{\epsilon}{w} = k_2$, so $(G_k^T G_k)^{-1} = k_1 \begin{bmatrix} 1 & -\mu \\ -\mu & k_2 \end{bmatrix}$.

Therefore,

$$\begin{aligned} G_k(G_k^T G_k)^{-1} &= k_1 \begin{bmatrix} I_1 & 1 \\ I_2 & 1 \\ \vdots & \vdots \\ I_w & 1 \\ \sqrt{\epsilon} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\mu \\ -\mu & k_2 \end{bmatrix} \\ &= k_1 \begin{bmatrix} I_1 - \mu & -I_1\mu + k_2 \\ I_2 - \mu & -I_2\mu + k_2 \\ \vdots & \vdots \\ I_w - \mu & -I_w\mu + k_2 \\ \sqrt{\epsilon} & -\mu\sqrt{\epsilon} \end{bmatrix} \end{aligned} \tag{24}$$

Therefore,

$$
\begin{aligned}
&G_k(G_k^TG_k)^{-1}G_k^T \\
&= k_1 \begin{bmatrix} I_1-\mu & -I_1\mu+k_2 \\ I_2-\mu & -I_2\mu+k_2 \\ \vdots & \vdots \\ I_w-\mu & -I_w\mu+k_2 \\ \sqrt{\epsilon} & -\mu\sqrt{\epsilon} \end{bmatrix} \begin{bmatrix} I_1 & I_2 & \cdots & I_w & \sqrt{\epsilon} \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \\
&= k_1 \begin{bmatrix} I_1I_1-\mu I_1-\mu I_1+k_2 & I_1I_2-\mu I_2-\mu I_1+k_2 & \cdots & I_1I_w-\mu I_w-\mu I_1+k_2 & \sqrt{\epsilon}I_1-\mu\sqrt{\epsilon} \\ I_2I_1-\mu I_1-\mu I_2+k_2 & I_2I_2-\mu I_2-\mu I_2+k_2 & \cdots & I_2I_w-\mu I_w-\mu I_2+k_2 & \sqrt{\epsilon}I_2-\mu\sqrt{\epsilon} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\epsilon}I_1-\mu\sqrt{\epsilon} & \sqrt{\epsilon}I_2-\mu\sqrt{\epsilon} & \cdots & \sqrt{\epsilon}I_w-\mu\sqrt{\epsilon} & \epsilon \end{bmatrix}
\end{aligned} \tag{25}
$$

Since $\overline{G}_k = I - G_k(G_k^TG_k)^{-1}G_k^T$. Hence, the entry (i,j) of $\overline{G}_k(i,j)$ is

$$
\begin{aligned}
\overline{G}_k(i,j) &= \delta_{ij} - k_1(I_iI_j - \mu I_i - \mu I_j + k_2) \\
&= \delta_{ij} - (I_iI_j - \mu I_i - \mu I_j + \sigma^2 + \mu^2 + \frac{\epsilon}{w})\frac{1}{w\sigma^2+\epsilon} \\
&= \delta_{ij} - ((I_i-\mu)(I_j-\mu) + \sigma^2 + \frac{\epsilon}{w})\frac{1}{w\sigma^2+\epsilon} \\
&= \delta_{ij} - ((I_i-\mu)(I_j-\mu) + \frac{w\sigma^2+\epsilon}{w})\frac{1}{w\sigma^2+\epsilon} \\
&= \delta_{ij} - (\frac{1}{w\sigma^2+\epsilon}(I_i-\mu)(I_j-\mu) + \frac{1}{w}) \\
&= \delta_{ij} - \frac{1}{w}(1 + \frac{1}{\sigma^2+\frac{\epsilon}{w}}(I_i-\mu)(I_j-\mu))
\end{aligned} \tag{26}
$$

where δ_{ij} is the Kronecker delta,

$$
\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{27}
$$

$$
\begin{aligned}
&\text{because } \ \overline{G}_k = I - G_k(G_k^TG_k)^{-1}G_k^T \\
&\text{therefore } \ \overline{G}_k^T\overline{G}_k = (I - G_k(G_k^TG_k)^{-1}G_k^T)^T(I - G_k(G_k^TG_k)^{-1}G_k^T) \\
&= (I - G_k((G_k^TG_k)^{-1})^TG_k^T)(I - G_k(G_k^TG_k)^{-1}G_k^T) \\
&= I + G_k((G_k^TG_k)^{-1})^TG_k^TG_k(G_k^TG_k)^{-1}G_k^T - G_k((G_k^TG_k)^{-1})^TG_k^T - G_k(G_k^TG_k)^{-1}G_k^T \\
&= I + G_k((G_k^TG_k)^{-1})^TG_k^T - G_k((G_k^TG_k)^{-1})^TG_k^T - G_k(G_k^TG_k)^{-1}G_k^T \\
&= I - G_k(G_k^TG_k)^{-1}G_k^T \\
&= \overline{G}_k
\end{aligned} \tag{28}
$$

In the end, the $(i,j)th$ element in L matrix may be expressed as

$$
\delta_{ij} - \frac{1}{|w_k|}(1 + \frac{1}{\sigma^2+\frac{\epsilon}{|w_k|}}(I_i-\mu)(I_j-\mu)) \tag{29}
$$

REFERENCES

[1] A. Levin, D. Lischinski, and Y. Weiss, “A closed-form solution to natural image matting,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 30, no. 2, pp. 228–242, 2007.