Question 2

Growth Functions

(Growth Functions)
Consider the following two functions:

$$f(n) = 6n^2$$

$$g(n) = n^2 \log(n)$$

(a) Is $f(n) \in O(n)$? (big-Oh) Answer YES or NO, and then give an explanation as to why.

(b) Is $f(n) \in \Omega(n)$? (big-Omega) Answer YES or NO, and then give an explanation as to why.

(c) Is $f(n) \in \Theta(n)$? (big-Theta) Answer YES or NO, and then give an explanation as to why.

All of these should have used g(n), I.e. f(n) in O(g(n)), but many people did not see the email so I have to prepare two answers, one for the question as I intended, and one for the question as it is actually written.

AS INTENDED:

a) f(n) in O(g(n)): YES 6 points

6n^2 <= 6n^2(log(n)
when n > the base (assume the base is 2, but it can be any constant greater than 1))
hence, witnesses k = 2, C = 6 (or anything larger than these)

b) f(n) in Omega(g(n)): NO 6 points

Proof by contradiction.
Assume Exists C and k s.t. for all n >=k,
C*n^2*log(n) <= 6n^2
log(n) <= 6n^2/ C*n^2 = 6/C
log(n) has no constant upper bound, hence, a contradiction.

c) f(n) in Theta(g(n)): NO 2 points

We showed in b) that it is not Omega, and hence it cannot be Theta.

AS WRITTEN:

a) f(n) in O(n)): NO 6 points

Proof by contradiction.
Assume Exists C and k s.t. for all n >=k,
6n^2 <= C(n)
This implies n <= C/6, as n grows beyond C/6 this reaches a contradiction.

b) f(n) in Omega(n)): YES 6 points

C*n <= 6n^2, for C = 6 and n > 1

c) f(n) in Theta(n)): NO 2 points

From (a), f(n) is not O(n).

(Growth Functions)

Consider the following two functions:

$$f(n) = \frac{3}{2}n^2 + 7n - 4$$

$$g(n) = 8n^2$$

(a) Is $f(n) \in O(n)?$ (big-Oh) Answer YES or NO, and then give an explanation as to why.

(b) Is $f(n) \in \Omega(n)?$ (big-Omega) Answer YES or NO, and then give an explanation as to why.

(c) Is $f(n) \in \Theta(n)?$ (big-Theta) Answer YES or NO, and then give an explanation as to why.

AS INTENDED:

a) f(n) in O(g(n)): YES   6 points

(3/2)n^2 + 7n – 4 <= 2n^2 + 7n^2 <= 9 n^2 for n >= 1, hence, C = 9 and k = 1 are witnesses.

b) f(n) in Omega(g(n)): YES 6 points

(8n^2)/8 <= n^2 <=   (3/2)n^2 + 7n – 4 for n >= 1
hence, C = 1/8 and k = 1 are witnesses

c) f(n) in Theta(g(n)): YES 2 points
f(n) is both O(g(n)) and Omega(g(n)), hence, it is Theta(g(n))

AS WRITTEN:

a) f(n) in O(n)): NO 6 points

Proof by contradiction. If exists k, C s.t. (3/2)n^2 + 7n – 4 <= Cn
it implies (3/2)n <= C - 7 - 4n, which is false when n > C, hence contradiction.

b) f(n) in Omega(n)):   YES 6 points

1*n <= (3/2)n^2 + 7n – 4 for all n > 1, hence k = 1 and c = 1 are your witnesses.

c) f(n) in Theta(n)): NO 6 points

Because f(n) is not O(n). (although it is Omega(n)).

(Growth Functions)

Consider the following two functions:

$$f(n) = n^4$$

$$g(n) = n^3 \log(n)$$

(a) Is $f(n) \in O(n)$? (big-Oh) Answer YES or NO, and then give an explanation as to why.

(b) Is $f(n) \in \Omega(n)$? (big-Omega) Answer YES or NO, and then give an explanation as to why

(c) Is $f(n) \in \theta(n)$? (big-Theta) Answer YES or NO, and then give an explanation as to why.

AS INTENDED:

a) f(n) in O(g(n)): NO 6 points

Proof by contradiction. If exist k, C, s.t. n^4 <= C n^3 log (n) for n > k this implies n/log(n) <= C for all n, n > k. There is no such constant. Hence, contradiction

b) f(n) in Omega(g(n)): YES 6 points

n^4 >= n^3 log (n) for n >= 1, witnesses C = 1 k = 1

c) f(n) in Theta(g(n)): NO 2 points

Since it is not O(g(n)) it cannot be Theta(g(n))

AS WRITTEN:

a) f(n) in O(n)): NO 6 points

Proof by contradiction. If exist k, C, s.t. n^4 <= C n for n > k this implies n^3 <= C, which is not true when n > 1. Hence, contradiction

b) f(n) in Omega(n)): YES 6 points

1*n <= n^4 for n >=1 hence C = 1, k = 1 are witnesses

c) f(n) in Theta(n)): NO 6 points

Since it is not O(n) it cannot be Theta(n)

Question 3

Growth Functions Theory

(Growth Functions Theory)

Prove whether or not each of the following statements are true. For those that you believe are false, prove this by giving a counterexample (i.e., particular functions for f(n) and g(n) for which the statements are not true). For those that you believe to be true, use the formal definitions of big-O, big-Omega, and big-Theta to prove it (only the definitions, do not use a theorem proven in class without proving it yourself). In all problems, you are given that for all n, f(n) >= 0 and g(n) >= 0.

(a) If $f(n) \in \Omega(g(n))$ then $g(n) \in O(f(n))$

(b) If $f(n) \in O(g(n))$ then $g(n) \in O(f(n))$

(c) If $f(n) \in \theta(g(n))$ and $g(n) \in \theta(h(n))$ then $f(n) \in \theta(h(n))$

5 points, 5 points, 4 points.

a) TRUE.

Given: Exists C k s.t. f(n) >= C*g(n) for all n > k
That implies: Exists C k s.t. (1/C)*f(n) >= g(n) for all n > k, which is the definition of g(n) in O(f(n))

b) FALSE.

n^3 in O(n^4) but n^4 not in O(n^3)

c) TRUE.

Exists C1, C1’, k1, s.t. C1*g(n) <= f(n) <= C1’*g(n) for all n > k1

Exists C2, C2’, k2, s.t. C2*h(n) <= g(n) <= C2’*h(n) for all n > k2

Hence

C1, C1’, k, s.t. C1*C2*h(n) <= f(n) <= h2*C1’C2’ for all n > max(k1,k2)

Hence, witnesses are C1*C2, C1’*C2’, max(k1,k2)

Question 4

(Algorithm Complexity)

(Algorithm Complexity)

In the algorithms below, show the "best" big-Oh notation to describe the (worst-case) time complexity of the given algorithm.

Give a brief (one sentence or two) explanation justifying your choice.

a) A binary search of n numbers

b) A search to find the smallest number in a list of unordered numbers

c) An algorithm that prints to the screen all the ways in which the numbers 1, 2, 3, . . . n, can be written in a row (assuming printing a number to the screen takes constant time)

d) An algorithm that prints all the bit strings of length n except those bit strings that contain exactly three zeroes (assuming that printing one bit takes constant time).

3 points each a, and b, 4 points each c and d.

a. log(n), the input is halved after each recursive step

b. n you may have to traverse the whole list

c. n*n! there are n! permutations, so it takes n! time to print each, and each has n numbers.

d. n*2^n there are an exponential number of strings, only C(n,3) have exactly three zeroes, C(n,3) = n!/(n-3!)(3!) which is O(n^3), so the result is still O(n*2^n)

Question 5

(Induction)

Consider proving using induction the following predicate P(n), where n is a positive integer.

$$\sum_{j=n}^{2n-1} (2j+1) = 3n^2$$

a) Prove the basis

b) Show the inductive step (don't prove it yet just show it)

c) Prove the inductive case.

1. Basis: n = 1 (2 points)

   (Sum : j = 1 to 2*1-1 : (2j+1)) = (Sum j = 1 to 1: (2j+1)) = 2*1+1 = 3

   3n^2 = 3*1^2 = 3

2. Induction Step: (4 points)

   Let k >=1, show

   (Sum : j = k to 2k - 1 : (2j + 1)) = 3k^2

   implies

   (Sum : j = k+1 to 2(k+1) - 1 : (2j + 1)) = 3(k+1)^2

3. Prove induction Step: (8 points)

   (Sum : j = k+1 to 2(k+1) - 1 : (2j + 1)) = 3(k+1)^2

   =

   (Sum : j = k+1 to 2k +1 : (2j + 1))

   =

   (Sum : j = k to 2k - 1 : (2j + 1)) - (2k + 1) + (2(2k) + 1) + (2(2k+1))

   = {ind. hyp.}

   3k^2 - (2k + 1) + (2(2k) + 1) + (2(2k+1))

   =

   3k^2 - 2k - 1 + 4k + 2 + 4k+2

   =

   3k^2 + 6k + 3

   =

   3(k+1)^2

(Induction)

Consider proving using induction the following predicate P(n), where n is a positive integer.

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$$

a) Prove the basis

b) Show the inductive step (don't prove it yet just show it)

c) Prove the inductive case.

a) Basis: n = 1 (2 points)

   1*1! = 1

   (1+1)! - 1 = 1

b) Induction Step: (4 points)

   1*1! + 2*2! … + n*n! = (n+1)! - 1

   implies

   1*1! + 2*2! … + n*n! + (n+1)*(n+1)! = (n+2)! - 1

c) Prove Inductive step (8 points)

   1*1! + 2*2! … + n*n! + (n+1)*(n+1)!

   = {by ind hyp}

   (n+1)! - 1 + (n+1)*(n+1)!

   = (n+1)!*(1 + n + 1) - 1

   = (n+1)!*(n+2) - 1

   = (n+2)! - 1

‘

(Induction)

Consider proving using induction the following predicate P(n), where n is a non-negative integer.

$$\sum_{j=0}^{n} (j+1) = (n+1)(n+2)/2$$

a) Prove the basis

b) Show the inductive step (don't prove it yet just show it)

c) Prove the inductive case.

d) Basis: n = 1 (2 points)

   n = 0 we get

   (0+1) = (0 + 1)(0 +2)/2

   1 = 1*2/2

   1 = 1

e) Induction Step: (4 points)

   (Sum j = 0 to k: (j+1)) = (k+1)(k+2)/2

   implies

   (Sum j = 0 to k+1: (j+1)) = (k+1+1)(k+1+2)/2

f) Prove induction Step: (8 points)

   (Sum j = 0 to k+1: (j+1))

   = (Sum j = 0 to k: (j+1)) + (k+1+1)

   = {from induction hyp}

   (k+1)(k+2)/2 + (k+1+1)

   = (k+1)(k+2)/2 + k + 2

   = ((k+1)(k+2) + 2k + 4)/2

   = (k^2 + 3k + 2 + 2k + 4)/2

   = (k^2 + 5k + 6)/2

   = (k+2)(k+3)/2 = (k+1+1)(k+1+2)/2

Question 6

(Permutations and Combinations)

(Permutations and Combinations)

a) How many ways are there to arrange the letters of the word bookkeeper? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

b) How may ways are there to choose 18 cookies if there are 7 varieties of cookies? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

a)

Permutation of n objects, where n1 indistinguishable objects of type 1, n2 indistinguishable objects of type 2, etc.

bookkeeper = 1 b, 2 o, 2 k, 3 e, 1 p, 1 r

Total letters: 10

Permutations: $10!/(1!2!2!3!1!1!) = 10!/(3!2!2!)$

b)

r-combinations with repetition allowed: $(n+r-1)!/(r!*(n-1)!)$

n = number of categories = 7

r = number of chosen objects = 18

Total ways: $(7+18-1)!/(18!*(7-1)!)$

$=(24!)/(18!*6!)$

(Permutations and Combinations)

(Permutations and Combinations)

a) How many ways are there to arrange the letters of the word aggressiveness? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

b) How may ways are there to choose 20 cookies if there are 6 varieties of cookies? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

7 points each.

a) aggressiveness = 14 letters, 1a, 2g, 1r, 3e, 4s, 1i, 1v, 1n
   #Ways = $14!/(2!3!4!)$
b) combination with repetition, $r = 20$, $n = 6$,
   $C(20+6-1, 20) = C(20+6-1, 5) = C(25, 5) = C(25, 20) = 25!/(20!5!)$

(Permutations and Combinations)

(Permutations and Combinations)

a) How many ways are there to arrange the letters of the word stubbornness? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

b) How may ways are there to choose 15 cookies if there are 4 varieties of cookies? Show me the result using factorials (no need to calculate the actual number). Show your intermediate work.

7 points each

a) stubbornness = 12 letters   3s 2b 1t 1u 1o 1r 1e 2n
   #ways = $12!/3!2!2!$
b) combination with repetition, $n = 4$ $r = 15$
   $C(n+r-1,r) = C(18, 15) = C(18, 3) = 18!/(3!15!)$

Question 7

(Pigeonhole Principle)

(Pigeonhole Principle)

Assume computer passwords are 5, 6, or 7 characters long. Each character can be one of the following: a lower case letter, an upper case letter, or a digit.

Assume each password must have at least one lower case letter, at least one upper case letter, and least THREE digits.

a) Let S be the set of passwords. What is the minimum cardinality of S that will guarantee that at least two passwords in S have the same length AND the same number of lower case letters?

b) Let S be the set of passwords. What is the minimum cardinality of S to guarantee that at least two passwords in S have the same length, the same number of lower case letters, AND the same number of upper case letters?

14 points

a)

L = lower case, U = upper case, D = digit, Fixed: 1L 1U 3D

length 5: one box (no flexibility if only five characters)

length 6: 0L ,1L, two boxes, depending on the number of additional lower case letters

length 7: 0L, 1L, 2L, three boxes, depending on the number of additional lower case letters

Total boxes: 1 + 2 + 3 = 6

|S| > 6 (at least 7)

length 5: one box

length 6: 0L & 0U (just extra digits), 0L & 1U, 1L & 0U boxes

length 7: 0L & 0U (just extra digits), 0L & 1U, 1L & 0U, 1L & 1U, 2L & 0U, 0U & 2L boxes

Total boxes = 1 + 3 + 6 = 10

|S| > 10 (at least 11)

(Pigeonhole Principle)

(Pigeonhole Principle)

Assume computer passwords are 6, 7, or 8 characters long. Each character can be one of the following: a lower case letter, an upper case letter, or a digit.

Assume each password must have at least one lower case letter, at least one upper case letter, and at least TWO digits.

a) Let S be the set of passwords. What is the minimum cardinality of S that will guarantee that at least two passwords in S have the same length AND the same number of lower case letters?

b) Let S be the set of passwords. What is the minimum cardinality of S to guarantee that at least two passwords in S have the same length, the same number of lower case letters, AND the same number of upper case letters?

14 points

a) 10 points

L = lower case, U = upper case, D = digit, Fixed: 1L 1U 2D = 4

length 6: 0L ,1L, 2L three boxes, depending on the number of additional lower case letters

length 7: 0L, 1L, 2L, 3L four boxes, depending on the number of additional lower case letters

length 8: 0L, 1L, 2L, 3L, 4L five boxes, depending on the number of additional lower case letters

Total boxes: 3 + 4 + 5 = 12

|S| > 12 (at least 13)

b) 4 points

length 6: two positions undecided, 0 0 , 1 0, 0 1, 1 1, 2 0, 0 2, six boxes

length 7: three positions undecided, 0 0, 1 0, 2 0, 3 0, 0 1, 1 1, 2 1, 0 2, 1 2, 0, 3 ten boxes

You can also think of it as choose three from three categories with repetition, r = 3 n = 3

C(3 + 3 - 1, 2) = C(5,2) = 5!/2!3! = 5*4/2 = 10

length 8: four positions undecided,

0 0, 1 0, 2 0, 3 0, 4 0,

0 1, 1 1, 2 1, 3 1,

0 2, 1 2, 2 2,

0, 3, 1 3

0 4

boxes = 15
You can also think about it as choose four from three categories with repetition, r = 4, n = 3
C(3 + 4 - 1, 2) = C(6,2) = 6!/(4!2!) = 6*5/2 = 15

Total boxes = 6 + 10 + 15 = 31

|S| > 31 (at least 32)

(Pigeonhole Principle)

(Pigeonhole Principle)

Assume computer passwords are 6, 7, or 8 characters long. Each character can be one of the following: a lower case letter, an upper case letter, or a digit.

Assume each password must have at least two lower case letters, at least two upper case letters, and at least two digits.

a) Let S be the set of passwords. What is the minimum cardinality of S that will guarantee that at least two passwords in S have the same length AND the same number of lower case letters?

b) Let S be the set of passwords. What is the minimum cardinality of S to guarantee that at least two passwords in S have the same length, the same number of lower case letters, AND the same number of upper case letters?

14 points

a) 10 points

L = lower case, U = upper case, D = digit, Fixed: 2L 2U 2D = 6

length 6: only one box (no freedom)

length 7: 0L, 1L, two boxes, depending on the number of additional lower case letters

length 8: 0L, 1L, 2L, three boxes, depending on the number of additional lower case letters

Total boxes: 1 + 2 + 3 = 6

|S| > 6 (at least 7)

b) 4 points

length 6: only one box (no freedom)

length 7: 0L 0U, 1L 0U, 0L 1U, three boxes,
depending on the number of additional lower and upper case letters

length 8: 0L 0U, 0L 1U, 0L 2U, 1L 0U, 1L 1U, 2L 0U six boxes

Total boxes = 1 + 3 + 6 = 10

|S| > 10 (at least 11)

Question 7

Advanced Counting

(Advanced Counting)

For the questions below, give me the result using factorials (no need to compute the actual number). Show your intermediate work! Assume we have 14 people in 7 different rooms.

a) How many ways are there to assign people to the different rooms if we don't care about the differences between each person? (i.e. we don't care about names or any other characteristic of a person)

b) How many ways are there to assign people to the different rooms if we do care about the differences between each person (i.e., each person has a distinct name and we do care about names)

14 points

a) We can think of choosing the rooms, 14 times, with repetition, the number of rooms (or categories) is 7
   C(14 + 7 - 1, 6) = **C(20, 6) = 20!/(6!14!)**
b) Distribute distinguishable objects into distinguishable boxes, note that the order of the objects (people) inside each box (room) does not matter.
   7^14

Advanced Counting

(Advanced Counting)

For the questions below, give me the result using factorials (no need to compute the actual number). Show your intermediate work! Assume we have 19 people in 5 different rooms.

a) How many ways are there to assign people to the different rooms if we don't care about the differences between each person? (i.e. we don't care about names or any other characteristic of a person)

b) How many ways are there to assign people to the different rooms if we do care about the differences between each person (i.e., each person has a distinct name and we do care about names)

14 points

a) We can think of choosing the rooms, 19 times, with repetition, the number of rooms (or categories) is 5
   C(19 + 5 - 1, 19) = **C(23, 19) = 23!/(19!4!)**
b) Distribute distinguishable objects into distinguishable boxes, note that the order of the objects (people) inside each box (room) does not matter.
   5^19

Advanced Counting

(Advanced Counting)

For the questions below, give me the result using factorials (no need to compute the actual number). Show your intermediate work! Assume we have 16 people in 9 different rooms.

a) How many ways are there to assign people to the different rooms if we do care about the differences between each person (i.e., each person has a distinct name and we do care about names)

b) How many ways are there to assign people to the different rooms if we don't care about the differences between each person? (i.e. we don't care about names or any other characteristic of a person)

14 points

a) Distribute distinguishable objects into distinguishable boxes, note that the order of the objects (people) inside each box (room) does not matter.
   9^16
b) We can think of choosing the rooms, 16 times, with repetition, the number of rooms (or categories) is 9
   C(16 + 9 - 1, 16) = **C(24, 16) = 24!/(16!8!)**