Lesson 13: Boolean Algebra

1. Introduction

Set identities are very similar to the laws for manipulating logical expressions.
Consider for example

$P \vee F = P$	$A \cup \emptyset = A$
$P \wedge F = F$	$A \cap \emptyset = \emptyset$
$P \vee T = T$	$A \cup U = U$
$P \wedge T = P$	$A \cap U = A$
$P \wedge P = P$	$A \cap A = A$
$P \wedge \sim P = F$	$A \cap \sim A = \emptyset$

We can see the correspondence

Logic	**Sets**
statement	set
F	empty set Ø
T	universal set U
disjunction V	union ∪
conjunction Λ	intersection ∩
Negation ~	Set complement

The correspondence reflects the fact that both propositional calculus and sets with the operations union, intersection and complementation are special cases of **Boolean algebra**

2. Definition of Boolean algebra

Boolean algebra: A set of objects **S** with two operations **"+"** and **"."** such that

a. for all **a** and **b** in S, **a + b** and **a . b** are also in S.
b. the following axioms hold:

1. For all **a** and **b** in S,

$a + b = b + a$	Commutative laws
$a . b = b . a$	

2. For all **a** , **b**, and **c** in S,

$(a + b) + c = a + (b + c)$	Associative laws
$(a . b) . c = a . (b . c)$	

3. For all **a** , **b**, and **c** in S,

$a + (b . c) = (a + b) . (a + c)$	Distributive laws
$a . (b + c) = (a . b) + (a . c)$	

4. There are two distinct elements in S: **0** and **1**, such that for all **a** in S,

$a + 0 = a$	0 is the identity element for "+"
$a . 1 = a$	1 is the identity element for "**.**"

5. For each **a** in S there exists an element ~**a** such that

$a + \sim a = 1$	Complement laws
$a . \sim a = 0$	

Using the above axioms we can prove that $x = x + x$, and $x = x . x$

a. $x = x . x$

$x = x .1$	By axiom 4
$= x (x + \sim x)$	By axiom 5 ($1 = a + \sim a$)
$= x . x + x . \sim x$	By axiom 3 (distributive law)
$= x . x + 0$	By axiom 5 ($0 = a . \sim a$)
$= x . x$	By axiom 4 ($a + 0 = a$)

b. $x = x + x$

$x = x + 0$	By axiom 4
$= x + (x . \sim x)$	By axiom 5 ($0 = a . \sim a$)
$= (x + x) . (x + \sim x)$	By axiom 3 (distributive law)
$= (x + x) . 1$	By axiom 5 ($1 = a + \sim a$)
$= x + x$	By axiom 4 ($a . 1 = a$)

George Boole,

1815-1864, English mathematician and logician. He became professor at Queen's College, Cork, in 1849. Boole wrote *An Investigation of the Laws of Thought* (1854) and works on calculus and differential equations. He developed a form of symbolic logic, called Boolean algebra, that is of fundamental importance in the study of the foundations of pure mathematics and is also at the basis of computer technology.

A quote from http://www.encyclopedia.com/articles/01674.html

Application: in abstract algebra and in computer science - circuit design.

More about George Boole can be found at
http://www-groups.dcs.st-andrews.ac.uk/ ~history/Mathematicians/Boole.html

The article *The Calculus of Logic*,
(http://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/CalcLogic.html)
describing the new algebra, is available online so that everybody can enjoy the crystal clear line of thought of an extremely clever person.
The article was first published in *Cambridge and Dublin Mathematical Journal*, vol. 3 (1848), pp. 183-98.

Problems

1. Rewrite the following logical expressions as set expressions and as Boolean expressions:

   $P \wedge Q$
   $(P \wedge Q) \vee (\sim P \wedge R)$
   $P \wedge Q \wedge R$
   $(P \vee Q) \wedge R$
   $\sim P \vee Q$
   $\sim(P \wedge Q) \vee \sim(P \wedge R)$

2. Rewrite the following set expressions as logical expressions and as Boolean expressions

   $A \cap B$
   $(A \cap B) \cup (\sim A \cap C)$
   $A \cap B \cap C$
   $(A \cup B) \cap C$
   $\sim A \cup B$
   $\sim(A \cap B) \cup \sim(A \cap C)$

3. Rewrite the following Boolean expressions as logical expressions and as expressions involving sets

   xy
   $xy + \sim xz$
   xyz
   $(x + y)z$
   $\sim x + y$
   $\sim(xy) + \sim(xz)$

Solution

Logic	**Sets**	**Boolean expressions**
$P \wedge Q$	$A \cap B$	xy
$(P \wedge Q) \vee (\sim P \wedge R)$	$(A \cap B) \cup (\sim A \cap C)$	$xy + \sim xz$
$P \wedge Q \wedge R$	$A \cap B \cap C$	xyz
$(P \vee Q) \wedge R$	$(A \cup B) \cap C$	$(x + y)z$
$\sim P \vee Q$	$\sim A \cup B$	$\sim x + y$
$\sim(P \wedge Q) \vee \sim(P \wedge R)$	$\sim(A \cap B) \cup \sim (A \cap C)$	$\sim(xy) + \sim(xz)$

Correspondence

Set identities	**Logical expressions**	**Boolean algebra**	
$A \cup \sim A = U$	$P \vee \sim P = T$	$x + (\sim x) = 1$	Complementation Law
$A \cap \sim A = \emptyset$	$P \wedge \sim P = F$	$x \cdot (\sim x) = 0$	Exclusion Law
$A \cap U = A$ $A \cup \emptyset = A$	$P \wedge T = P$ $P \vee F = P$	$x \cdot 1 = x$ $x + 0 = x$	Identity Laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	$P \vee T = T$ $P \wedge F = F$	$x + 1 = 1$ $x \cdot 0 = 0$	Domination Laws
$A \cup A = A$ $A \cap A = A$	$P \vee P = P$ $P \wedge P = P$	$x + x = x$ $x \cdot x = x$	Idempotent Laws
$\sim(\sim A) = A$	$\sim(\sim P) = P$	$\sim(\sim x) = x$	Double Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	$P \vee Q = Q \vee P$ $P \wedge Q = Q \wedge P$	$x + y = y + x$ $x.y = y.x$	Commutative Laws
$(A \cup B) \cup C =$ $A \cup (B \cup C)$ $(A \cap B) \cap C =$ $A \cap (B \cap C)$	$(P \vee Q) \vee R =$ $P \vee (Q \vee R)$ $(P \wedge Q) \wedge R =$ $P \wedge (Q \wedge R)$	$(x + y) + z = x + (y + z)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	Associative Laws
$A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$	$P \vee (Q \wedge R) =$ $(P \vee Q) \wedge (P \wedge R)$ $P \wedge (Q \vee R) =$ $(P \wedge Q) \vee (P \wedge R)$	$x + (y \cdot z) =$ $(x + y) \cdot (x + z)$ $x \cdot (y + z) = x \cdot y + x \cdot z$	Distributive Laws
$\sim(A \cap B) = \sim A \cup \sim B$ $\sim(A \cup B) = \sim A \cap \sim B$	$\sim(P \wedge Q) = \sim P \vee \sim Q$ $\sim(P \vee Q) = \sim P \wedge \sim Q$	$\sim(x \cdot y) = \sim x + \sim y$ $\sim(x + y) = \sim x \cdot \sim y$	De Morgan's Laws