Basics of Counting

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Counting

- Counting is the process of determining the number of objects in a specified set.
- This set is typically specified by some properties, often as a model of a real-life collection of objects
  - set of UTD IP addresses
  - set of 12-credit hour schedules possible from undergrad CS courses

Why do we count?

- Determining the size of a set has implications for the *complexity* of tasks involving that set
  - how much storage is needed (physical or in computer memory)
  - how much additional work will be required
- In the particular case of algorithms, counting the number of steps taken allows us to estimate the computational burden of that algorithm
- Often, counting the size of a set enables us to calculate probabilities involving that set, which enables probabilistic reasoning and decision theory. E.g.,
  - Should one fold in Poker?
  - What is the probability that you win the lottery?

Basic Counting Techniques

The Product Rule

Product Rule

Assume a procedure can be broken down into a sequence of two tasks. Assume there are n_1 ways to do the first task and n_2 ways to do the second task. Then, there are $n_1 \cdot n_2$ ways to do the procedure.

Product Rule Example

A new company with only two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees? (that is, don't place the two employees in the same office).

SOLUTION: The procedure consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Note, obviously, if we first assign an office to Patel and then an office to Sanchez the result is the same $12 \cdot 11 = 132$.

Extended Product Rule

Extended Product Rule

Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, m$, can be done in n_i ways regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \ldots n_{m-1} \cdot n_m$ ways to carry out the procedure.

This version of the product rule can be proved by mathematical induction from the product rule for two tasks.

Example of Extended Product Rule

How many bit strings of length seven are there?

SOLUTION: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

I.e., each bit is a task to be done, and there are two ways (a 0 or a 1) to do the task.

Another Example

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Note: there are 26 letters and 10 digits.

Solution: By the product rule, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$$

different possible license plates.

Example: counting functions

COUNTING FUNCTIONS: How many functions are there from a set with m elements to a set with n elements?

SOLUTION: Since a function represents a choice of one of the n elements of the co-domain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdot n \cdots n = n^m$ such functions.

COUNTING ONE-TO-ONE FUNCTIONS: How many one-to-one functions are there from a set with m elements to one with n elements?

SOLUTION: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n \cdot (n-1) \cdot (n-2) \ldots (n-m+1)$ such functions.

Example: telephone numbering plan

The *North American Numbering Plan* (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code.

There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was NYX-NNX-XXX.
- In the new plan, the format is NXX-NXX-XXX.

How many different telephone numbers are possible under the old plan and the new plan?

continued ...

SOLUTION: Use the Product Rule.

- old plan:
  - There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX.
  - There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX.
  - There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format XXXX.
- new plan:
  - There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX.
  - There are $8 \cdot 10 \cdot 10 = 800$ office codes with the format NXX.
  - There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format XXXX.

Number of old plan telephone numbers:
$160 \cdot 640 \cdot 10,000 = 1,024,000,000.$

Number of new plan telephone numbers:
$800 \cdot 800 \cdot 10,000 = 6,400,000,000.$

Example: counting subsets of a finite set

COUNTING SUBSETS OF A FINITE SET: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. (mathematical induction can also be used to prove this same result.)

SOLUTION: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Example: cartesian product cardinality

Cardinality of a Cartesian product

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

Why?

The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \ldots A_m$ is done by choosing an element in A_1, an element in A_2 , . . . , and an element in A_m.

By the product rule, it follows that:

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

The Sum Rule

Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example of Sum Rule

The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

SOLUTION: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

Sum Rule in terms of sets

The Sum Rule can be phrased in terms of sets:

$$|A \cup B| = |A| + |B|$$

as long as A and B are disjoint.

More generally

$$|A_1 \cup A_2 \cup \ldots \cup \ldots A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

when $|A_i \cap A_j = \emptyset|$ for all i and j.

The case where the sets have elements in common will be covered in another chapter.

Combining the Sum and Product Rule

Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels

Note: examples of labels are H, B0, C5, F, C, B, A, A9

SOLUTION: Use the product rule.

$$26 + 26 \cdot 10 = 286$$

Example: Counting Passwords (more complex)

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$
- To find each of P_6, P_7, and P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$\begin{aligned}
P_6 &= 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,5 \\
P_7 &= 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,3 \\
P_8 &= 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,6
\end{aligned}$$

Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

Example: Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

<table>
<tr><th>Bit Number</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th></th><th>8</th><th>16</th><th>24</th><th>31</th></tr>
<tr><td>Class A</td><td>0</td><td colspan="5">netid</td><td colspan="4">hostid</td></tr>
<tr><td>Class B</td><td>1</td><td>0</td><td colspan="6">netid</td><td colspan="3">hostid</td></tr>
<tr><td>Class C</td><td>1</td><td>1</td><td>0</td><td colspan="6">netid</td><td colspan="2">hostid</td></tr>
<tr><td>Class D</td><td>1</td><td>1</td><td>1</td><td>0</td><td colspan="6">Multicast Address</td></tr>
<tr><td>Class E</td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td><td colspan="5">Address</td></tr>
</table>

- **Class A Addresses:** used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses:** used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses:** used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned for individual computers. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.

continued ...

How many different IPv4 addresses are available for computers on the Internet?

SOLUTION: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, $x_A : 2^7 - 1 = 127$ netids. $2^{24} - 2 = 16,777,214$ hostids.

$$x_A = 127 \cdot 16,777,214 = 2,130,706,178$$

- To find, $x_B : 2^{14} = 16,384$ netids. $2^{16} - 2 = 16,534$ hostids.

$$x_B = 16,384 \cdot 16,534 = 1,073,709,056.$$

- To find, $x_C : 2^{21} = 2,097,152$ netids. $2^8 - 2 = 254$ hostids.

$$x_C = 2,097,152 \cdot 254 = 532,676,608$$

continued ...

- Hence, the total number of available IPv4 addresses is

$$\begin{aligned} x &= x_A + x_B + x_C \\ &= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\ &= 3,737,091,842. \end{aligned}$$

Not enough today! The newer IPv6 protocol solves the problem of too few addresses.

Subtraction Rule

Subtraction Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

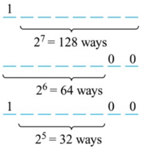
Also known as the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Counting Bit Strings

How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

SOLUTION:



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$.
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$.
- Number of bit strings of length eight that start with a 1 bit and end with bits 00: $2^5 = 32$

Hence, the number is 128 $+$ 64 - 32 $=$ 160.

Division Rule

Division Rule

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

- Restated in terms of sets:
  If the finite set A, $|A| = n$, is the union of x pairwise disjoint subsets each with d elements, then $x = |A|/d$.
- Restated in terms of functions:
  If f is a function from A to B, where both are finite sets, and for every value $y \in$ B there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Division Rule Example

How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

SOLUTION:

- Number the seats around the table from 1 to 4 proceeding clockwise.
- There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4.
- Thus there are $4! = 24$ ways to order the four people.
- But if we rotate the table we have the same seating arrangement. One particular seating can be rotated in 4 possible ways.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Tree Diagrams

Tree Diagrams

We can solve many counting problems through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.

Example of Tree Diagrams

Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

SOLUTION: Draw the tree diagram. The store must stock 17 T-shirts.