Section 8.1 recurrence relations

8. This is very similar to Exercise 7, except that we need to go one level deeper.

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

c) We will compute a_3 through a_7 using the recurrence relation:

$$
\begin{aligned}
a_3 &= a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1 \\
a_4 &= a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3 \\
a_5 &= a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8 \\
a_6 &= a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20 \\
a_7 &= a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47
\end{aligned}
$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

12. This is identical to Exercise 11, one level deeper.

a) Let a_n be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in a_{n-1} ways) or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in a_{n-2} ways) or else start with a step of three stairs and then climb $n-3$ stairs (and this can be done in a_{n-3} ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b) The initial conditions are $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb two stairs (one step twice or two steps at once). Note that the recurrence relation is the same as that for Exercise 9.

c) Each term in our sequence $\{a_n\}$ is the sum of the previous three terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 13$, $a_6 = 24$, $a_7 = 44$, $a_8 = 81$. Thus a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

Section 8.2 linear recurrence

2. **a)** linear, homogeneous, with constant coefficients; degree 2
b) linear with constant coefficients but not homogeneous
c) not linear
d) linear, homogeneous, with constant coefficients; degree 3
e) linear and homogeneous, but not with constant coefficients
f) linear with constant coefficients, but not homogeneous
g) linear, homogeneous, with constant coefficients; degree 7

4. For each problem, we first write down the characteristic equation and find its roots. Using this we write down the general solution. We then plug in the initial conditions to obtain a system of linear equations. We solve these equations to determine the arbitrary constants in the general solution, and finally we write down the unique answer.

a) $r^2 - r - 6 = 0 \qquad r = -2, 3$

$a_n = \alpha_1(-2)^n + \alpha_2 3^n$

$3 = \alpha_1 + \alpha_2$

$6 = -2\alpha_1 + 3\alpha_2$

$\alpha_1 = 3/5 \qquad \alpha_2 = 12/5$

$a_n = (3/5)(-2)^n + (12/5)3^n$

b) $r^2 - 7r + 10 = 0 \qquad r = 2, 5$

$a_n = \alpha_1 2^n + \alpha_2 5^n$

$2 = \alpha_1 + \alpha_2$

$1 = 2\alpha_1 + 5\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = -1$

$a_n = 3 \cdot 2^n - 5^n$

c) $r^2 - 6r + 8 = 0 \qquad r = 2, 4$

$a_n = \alpha_1 2^n + \alpha_2 4^n$

$4 = \alpha_1 + \alpha_2$

$10 = 2\alpha_1 + 4\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = 1$

$a_n = 3 \cdot 2^n + 4^n$

d) $r^2 - 2r + 1 = 0 \qquad r = 1, 1$

$a_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$

$4 = \alpha_1$

$1 = \alpha_1 + \alpha_2$

$\alpha_1 = 4 \qquad \alpha_2 = -3$

$a_n = 4 - 3n$

#48

I have no idea why I asked for this problem. It deals with nonhomogeneous recurrence relations. It may have been a typo of mine, I think I was aiming for 38, sorry! Everyone will get credit for this problem. In case you were wondering, the solution is as follows.

48. **a)** This is just a matter of keeping track of what all the symbols mean. First note that $Q(n+1) = Q(n)f(n)/g(n+1)$. Now the left-hand side of the desired equation is $b_n = g(n+1)Q(n+1)a_n = Q(n)f(n)a_n$. The right-hand side is $b_{n-1} + Q(n)h(n) = g(n)Q(n)a_{n-1} + Q(n)h(n) = Q(n)(g(n)a_{n-1} + h(n))$. That the two sides are the same now follows from the original recurrence relation, $f(n)a_n = g(n)a_{n-1} + h(n)$. Note that the initial condition for $\{b_n\}$ is $b_0 = g(1)Q(1)a_0 = g(1)(1/g(1))a_0 = a_0 = C$, since it is conventional to view an empty product as the number 1.

b) Since $\{b_n\}$ satisfies the trivial recurrence relation shown in part **(a)**, we see immediately that

$$\begin{aligned} b_n &= Q(n)h(n) + b_{n-1} = Q(n)h(n) + Q(n-1)h(n-1) + b_{n-2} = \cdots \\ &= \sum_{i=1}^{n} Q(i)h(i) + b_0 = \sum_{i=1}^{n} Q(i)h(i) + C\,. \end{aligned}$$

The value of a_n follows from the definition of b_n given in part **(a)**.

Section 8.3 divide and conquer

12. An exact formula comes from the proof of Theorem 1, namely $f(n) = [f(1) + c/(a-1)]n^{\log_b a} - c/(a-1)$, where $a = 2$, $b = 3$, and $c = 4$ in this exercise. Therefore the answer is $f(n) = 5n^{\log_3 2} - 4$.

14. If there is only one team, then no rounds are needed, so the base case is $R(1) = 0$. Since it takes one round to cut the number of teams in half, we have $R(n) = 1 + R(n/2)$.

22. **a)** $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$

b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2$, $b = 2$, $c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this says that our function is $O(\log n \cdot \log\log n)$.

Section 8.5 inclusion exclusion

16. $4 \cdot 100 - 6 \cdot 50 + 4 \cdot 25 - 5 = 195$

22. The base case is $n = 2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. Look at a situation with $n+1$ sets, and temporarily consider $A_n \cup A_{n+1}$ as one set. Then by the inductive hypothesis we have

$$|A_1 \cup \cdots \cup A_{n+1}| = \sum_{i<n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n} |A_i \cap A_j|$$
$$- \sum_{i<n} |A_i \cap (A_n \cup A_{n+1})| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_{n-1} \cap (A_n \cup A_{n+1})| \,.$$

Next we apply the distributive law to each term on the right involving $A_n \cup A_{n+1}$, giving us

$$\sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum |(A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1})| \,.$$

Now we apply the basis step to rewrite each of these terms as

$$\sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}| \,,$$

which gives us precisely the summation we want.

Section 8.6 Applications of Inclusion Exclusion

4. $C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) = 20$

Section 9.1 Relations

6. a) Since $1+1 \neq 0$, this relation is not reflexive. Since $x+y = y+x$, it follows that $x+y=0$ if and only if $y+x=0$, so the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.
b) Since $x = \pm x$ (choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1.
c) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x-y) = y - x$. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x,y) \in R$ and $(y,z) \in R$, then $x-y$ and $y-z$ are rational numbers. Therefore their sum $x-z$ is rational, and that means that $(x,z) \in R$.

d) Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2,1) \in R$, but $(1,2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x,y) and (y,x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4,2) \in R$ and $(2,1) \in R$, but $(4,1) \notin R$.

e) This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,3)$ and $(3,2)$ are both in R. It is not transitive; for example, $(1,0) \in R$ and $(0,-2) \in R$, but $(1,-2) \notin R$.

f) This is not reflexive, since $(1,1) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,0)$ and $(0,2)$ are both in R. It is not transitive; for example, $(1,0) \in R$ and $(0,-2) \in R$, but $(1,-2) \notin R$.

g) This is not reflexive, since $(2,2) \notin R$. It is not symmetric, since $(1,2) \in R$ but $(2,1) \notin R$. It is antisymmetric, because if $(x,y) \in R$ and $(y,x) \in R$, then $x = 1$ and $y = 1$, so $x = y$. It is transitive, because if $(x,y) \in R$ and $(y,z) \in R$, then $x = 1$ (and $y = 1$, although that doesn't matter), so $(x,z) \in R$.

h) This is not reflexive, since $(2,2) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,1)$ and $(1,2)$ are both in R. It is not transitive; for example, $(3,1) \in R$ and $(1,7) \in R$, but $(3,7) \notin R$.

10. We give the simplest example in each case.

a) the empty set on $\{a\}$ (vacuously symmetric and antisymmetric)

b) $\{(a,b),(b,a),(a,c)\}$ on $\{a,b,c\}$

50. a) Since R contains all the pairs (x,x), so does $R \cup S$. Therefore $R \cup S$ is reflexive.

b) Since R and S each contain all the pairs (x,x), so does $R \cap S$. Therefore $R \cap S$ is reflexive.

c) Since R and S each contain all the pairs (x,x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x,x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

e) Since R and S each contain all the pairs (x,x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

Section 9.5 equivalence relations

2. a) This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).

b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).

c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

d) This is not an equivalence relation since it is clearly not transitive.

e) Again, just as in part **(c)**, this is not transitive.

48. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

b) $\{(a,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(f,e),(f,f),(g,g)\}$

c) $\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),$ $(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

d) $\{(a,a),(a,c),(a,e),(a,g),(c,a),(c,c),(c,e),(c,g),(e,a),(e,c),(e,e),(e,g),(g,a),(g,c),(g,e),(g,g),$ $(b,b),(b,d),(d,b),(d,d),(f,f)\}$

Section 9.6 partial orders

4. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S,R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S,R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b, then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b, and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

24. This picture is a four-dimensional cube. We draw the sets with k elements at level k: the empty set at level 0 (the bottom), the entire set at level 4 (the top).