Assignment #3

Section 8.1

#8.

a). Let a_n denote the number of bit strings of length n that contains three consecutive zeros.

For string ending with 1, there are a_{n-1} number of bit strings contains consecutive zeros.

For bit string ending with 10, there are a_{n-2} number of bit strings contains consecutive zeros.

For bit string ending with 100, there are a_{n-3} number of bit strings contains consecutive zeros.

For bit string ending with 000, there are 2^{n-3} number of bit strings contains consecutive zeros.

So. $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

b.). $a_0 = 0$, $a_1 = 0$, $a_2 = 0$

c). $a_3 = 2^0 = 1$

$a_4 = 1 + 2^1 = 3$

$a_5 = 1 + 3 + 2^2 = 8$

$a_6 = 1 + 3 + 8 + 2^3 = 20$

$a_7 = 3 + 8 + 20 + 2^4 = 47$

#12. Let a_n denote the number of ways to climb n stairs.

For final step of taking 1 step, there are a_{n-1} number of ways to climb $n-1$ stairs.

For final step of taking 2 steps, there are a_{n-2} number of ways to climb $n-2$ stairs.

For final step of taking 3 steps, there are a_{n-3} number of ways to climb $n-3$ stairs.

Overall, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ number of ways to climb n stairs.

b). $a_0 = 1,\ a_1 = 1,\ a_2 = 2$

c). $a_3 = 1+1+2=4$

$a_4 = 1+2+4=7$

$a_5 = 2+4+7=13$

$a_6 = 4+7+13=24$

$a_7 = 7+13+24=44$

$a_8 = 13+24+44=81$

Section 8.2

#2.

a) degree of 2

b). Not homogenous

c). Non-Linear

d). degree of 3

e). Not constant coefficient

f). Not homogenous

g). degree of 7

#4. a). $a_n = a_{n-1} + 6a_{n-2}$

$$r^2 = r^1 + 6$$

$$\Rightarrow (r+2)(r-3) = 0$$

$$\Rightarrow r = -2 \text{ or } r = 3$$

$$a_n = b_1 \cdot (-2)^n + b_2 \cdot 3^n$$

$$a_0 = b_1 + b_2 = 3$$

$$a_1 = -2b_1 + 3b_2 = 6$$

$$\Rightarrow b_2 = \frac{12}{5}, \quad b_1 = \frac{3}{5}$$

$$a_n = \frac{3}{5} \cdot (-2)^n + \frac{12}{5} \cdot 3^n$$

b). $a_n = 7a_{n-1} - 10a_{n-2}$

$$r^2 - 7r + 10 = 0$$

$$\Rightarrow (r-2)(r-5) = 0$$

$$\Rightarrow r = 2 \text{ or } r = 5$$

$$a_n = b_1 \cdot 2^n + b_2 \cdot 5^n$$

$$a_0 = b_1 + b_2 = 2$$

$$a_1 = 2b_1 + 5b_2 = 1$$

$$\Rightarrow b_2 = -1, \, b_1 = 3$$

$a_n = 3 \cdot 2^n - 5^n$

c). $a_n = 6a_{n-1} - 8a_{n-2}$

$r^2 - 6r + 8 = 0$

$\Rightarrow (r-2)(r-4) = 0$

$\Rightarrow r = 2$ or $r = 4$

$a_n = b_1 2^n + b_2 \cdot 4^n$

$a_0 = b_1 + b_2 = 4$

$a_1 = 2b_1 + 4b_2 = 10$

$\Rightarrow b_2 = 1, \ b_1 = 3$

$\cdot \ a_n = 3 \cdot 2^n + 4^n$

d). $a_n = 2a_{n-1} - a_{n-2}$

$r^2 - 2r + 1 = 0$

$\Rightarrow (r-1)^2 = 0$

$\Rightarrow r = 1$

$a_n = b_1 + b_2 \cdot n.$

$a_0 = b_1 = 4$

$a_1 = b_1 + b_2 = 1$

$a_n = 4 - 3n$

#48

a). $b_n = g(n+1)\,Q(n+1)\,a_n$

$$= g(n+1)\cdot \frac{f(1)\cdot f(2)\cdots f(n)}{g(1)\cdot g(2)\cdots g(n+1)}\cdot a_n$$

$$= \frac{f(1)\cdot f(2)\cdots f(n-1)}{g(1)\cdot g(2)\cdots g(n)}\cdot (g(n)\cdot a_{n-1} + h(n))$$

$$= \frac{f(1)\cdot f(2)\cdots f(n-1)}{g(1)\cdot g(2)\cdots g(n)}\cdot g(n)\cdot a_{n-1} + \frac{f(1)\cdot f(2)\cdots f(n-1)}{g(1)\cdot g(2)\cdots g(n)}\cdot h(n)$$

$$= Q(n)\cdot g(n)\cdot a_{n-1} + Q(n)\cdot h(n)$$

$$= b_{n-1} + Q(n)\cdot h(n)$$

b. $b_n = b_{n-1} + Q(n)\cdot h(n)$

$$= b_{n-2} + Q(n-1)\cdot h(n-1) + Q(n)\cdot h(n)$$

$$= \cdots$$

$$= b_1 + Q(2)\cdot h(2) + Q(3)\cdot h(3) + \cdots + Q(n)h(n)$$

$$= b_0 + Q(1)\cdot h(1) + Q(2)h(2) + \cdots + Q(n)h(n)$$

$$= b_0 + \sum_{i=1}^{n} Q(i)h(i)$$

$$b_0 = g(1)\cdot Q(1)\cdot a_0 = g(1)\cdot\frac{1}{g(1)}\,a_0 = C$$

$$a_n = \frac{b_n}{g(n+1)Q(n+1)} = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)\,Q(n+1)}$$

Section 8.3

#12 $f(n) = C_1 n^{\log_b a} + C_2$

$C_1 = f(1) + c/(a-1) = 1 + \frac{4}{1} = 5$

$C_2 = -c/(a-1) = \frac{-4}{1} = -4$

$f(n) = 5 \cdot n^{\log_3 2} - 4$

#14 Let $f(n)$ represent the number of rounds in the tournament of team size n.

Then. $f(n) = f(n/2) + 1$

#22.

a). $f(16) = 2 \cdot f(4) + \log 16$

$= 2 \cdot (2 \cdot f(2) + \log 4) + \log 16$

$= 2 \cdot (2 \times 1 + 2) + 4$

$= 12$

b.). Let $m = \log n$, then $n = 2^m$

$$f(2^m) = 2 \cdot f(2^{m/2}) + m$$

Let $g(m) = f(2^m)$,

then $g(m/2) = f(2^{m/2})$

then $g(m) = 2 \cdot g(m/2) + m$

where $a=2$, $b=2$, $c=1$, $d=1$

$a = b^d$, then $g(m) = O(m \cdot \log m)$

$$\Rightarrow f(n) = f(2^m) = g(m) = O(m \log m)$$

$$= O(\log n \log \log n)$$

Section 8.5

#16. $|A_1 \cup A_2 \cup A_3 \cup A_4| = A_1 + A_2 + A_3 + A_4 - A_1 \cap A_2 - A_1 \cap A_3$

$- A_1 \cap A_4 - A_2 \cap A_3 - A_2 \cap A_4 - A_3 \cap A_4$

$+ A_1 \cap A_2 \cap A_3 + A_2 \cap A_3 \cap A_4 + A_1 \cap A_3 \cap A_4$

$+ A_1 \cap A_2 \cap A_4 - A_1 \cap A_2 \cap A_3 \cap A_4$

$= 100 + 100 + 100 + 100 - 50 - 50 - 50$

$- 50 - 50 - 50 + 25 + 25 + 25 + 25$

$- 5$

$= 195$

#22

For $n=2$. two set A_1, A_2

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

suppose it is true for k sets.

we have $|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{1 \le i \le k} |A_i| - \sum_{1 \le i < j \le k} |A_i \cap A_j|$

$+ \cdots + (-1)^{k+1} |A_1 \cap A_2 \cap \cdots \cap A_k|$

Let $B_k = |A_1 \cup A_2 \cup \cdots \cup A_k|$

For $k+1$ sets, $|A_1 \cup A_2 \cup \cdots \cup A_{k+1}| = |B_k \cup A_{k+1}|$

$$= |B_k| + |A_{k+1}| - |B_k \cap A_{k+1}|$$

$$|B_k \cap A_{k+1}| = |(A_1 \cup A_2 \cup \cdots \cup A_k) \cap A_{k+1}|$$
$$= |(A_1 \cap A_{k+1}) \cup (A_2 \cap A_{k+1}) \cup \cdots \cup (A_k \cap A_{k+1})|$$

$$|B_k \cap A_{k+1}| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{1 \le i \le k} |A_i \cap A_{k+1}| - \sum_{1 \le i < j \le k} |A_i \cap A_j \cap A_{k+1}|$$
$$+ \cdots + (-1)^{k+1} |A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}|$$

$$\Rightarrow \quad |A_1 \cup A_2 \cup \cdots \cup A_{k+1}| = \sum_{1 \le i \le k} |A_i| - \sum_{1 \le i < j \le k} |A_i \cap A_j|$$
$$+ \cdots + (-1)^{k+1} |A_1 \cap A_2 \cap \cdots \cap A_k| + |A_{k+1}| + \sum_{1 \le i \le k} |A_i \cap A_{k+1}|$$
$$- \sum_{1 \le i < j \le k} |A_i \cap A_j \cap A_{k+1}| + \cdots + (-1)^{k+1} |A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}|$$

$$= \sum_{1 \le i \le k+1} |A_i| - \sum_{1 \le i < j \le k+1} |A_i \cap A_j|$$

$$+ \cdots + (-1)^{k+1+1} |A_1 \cap A_2 \cap \cdots \cap A_{k+1}|$$

as desired.

Section 8.6

#4. Let a solution have property P_1 if $x_1 > 3$ and P_2 if $x_2 > 4$ and P_3 if $x_3 > 5$ and P_4 if $x_4 > 8$. The number of solutions satisfying inequalities $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 5$ $x_4 \le 8$ is

$$N(P_1' \cdot P_2' \cdot P_3' \cdot P_4') = N - N(P_1) - N(P_2) - N(P_3)$$

$$- N(P_4) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4)$$

$$+ N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3)$$

$$- N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4)$$

$$+ N(P_1P_2P_3P_4)$$

N = total number of solutions $= C(17+4-1, 3)$

$= 1140$

$N(P_1) = C(16,3) = 560$

$N(P_2) = C(15,3) = 455$

$N(P_3) = C(14,3) = 364$

$N(P_4) = C(11,3) = 165$

$N(P_1P_2) = C(11,3) = 165$

$N(P_1P_3) = C(10,3) = 120$

$N(P_1P_4) = C(7,3) = 35$

$N(P_2P_3) = C(9,3) = 84$

$N(P_2P_4) = C(6,3) = 20$

$N(P_3P_4) = C(5,3) = 10$

$N(P_1P_2P_3) = C(5,3) = 10$

$N(P_1P_2P_4) = N(P_1P_3P_4) = N(P_2P_3P_4) = N(P_1P_2P_3P_4) = 0$

$\Rightarrow N(P_1' \cdot P_2' \cdot P_3' \cdot P_4') = 1140 - 560 - 455 - 364 - 165$
$+165 + 120 + 35 + 84 + 20 + 10 - 10 = 20$

Section 9.1

#6.

a). not reflexive, symmetric, not antisymmetric, not transitive

b). reflexive, symmetric, not antisymmetric, transitive

c). reflexive, symmetric, not antisymmetric, transitive.

d). not reflexive, not symmetric, antisymmetric, not transitive

e). reflexive, symmetric, not antisymmetric, not transitive

f). not reflexive, symmetric, not antisymmetric, not transitive

g). not reflexive, not symmetric, antisymmetric, transitive

h). not reflexive, symmetric, not antisymmetric, not transitive

#12.

(a). $R = \{ \}$ on $A = \{1, 2, 3\}$

(b). $R = \{(1,2), (2,1), (1,3)\}$ on $A = \{1, 2, 3\}$

#50.

(a). For any element $a \in A$

R is reflexive $\Rightarrow$ $(a,a) \in R$

S is reflexive $\Rightarrow$ $(a,a) \in S$

$R \cup S$ contain all elements in either R or S, which means that $\{a,a\}$ is in $R \cup S$, by defination, $R \cup S$ is also reflexive.

(b). For any element $a \in A$

R is reflexive $\Rightarrow$ $(a,a) \in R$

S is reflexive $\Rightarrow$ $(a,a) \in S$

$R \cap S$ containg all element in both R and S, which means that

$(a,a) \in R \cap S$ for any elemet a.

Thus, $R \cap S$ is also reflexive.

c). For any element $a \in A$

R is reflexive $\Rightarrow (a,a) \in R$

S is reflexive $\Rightarrow (a,a) \in S$

Since (a,a) is in both R and S.

$R \oplus S$ will not contain any element (a,a)

So $R \oplus S$ is irreflexive.

d). For any element $a \in A$

R is reflexive $\Rightarrow (a,a) \in R$

S is reflexive $\Rightarrow (a,a) \in S$

Since (a,a) is in both R and S.

$R-S$ will not contain any element (a,a)

So $R-S$ is irreflexive.

d). For any element $a \in A$

R is reflexive $\Rightarrow (a,a) \in R$

S is reflexive $\Rightarrow (a,a) \in S$

Since $(a,a) \in R$ and $(a,a) \in S$. by

defination of composite,

$(a,a) \in S \circ R$

Thus, $S \circ R$ is also reflexive.

Section 9.5

#2.

a). It is a equivalence relation

b). It is a equivalence relation

c). It lacks trasitivity because (a, b) may have the same mother and (b, c) may have the same father, but (a, c) doesn't share a common parent.

d). It lacks transitivity. Because if (a, b) met each other and (b, c) met each other it is possible tha a, c haven't met.

e). it lacks transitivity. If a, b share a common language and b, c share a common language but it is possible that a, c don't share a common language.

#48.

a). $\{a,b\} \to \{\{a,a\}, \{a,b\}, \{b,a\}, \{b,b\}\}$

$\{c,d\} \to \{\{c,c\}, \{c,d\}, \{d,c\}, \{d,d\}\}$

$\{e,f,g\} \to \{\{e,e\}, \{e,f\}, \{e,g\}, \{f,e\}, \{f,f\}, \{f,g\}, \{g,e\}, \{g,f\}, \{g,g\}\}$

The union of these subset:

$\{\{a,a\}, \{a,b\}, \{b,a\}, \{b,b\}, \{c,c\}, \{c,d\}, \{d,c\}, \{d,d\}$
$\{e,e\}, \{e,f\}, \{e,g\}, \{f,e\}, \{f,f\}, \{f,g\}, \{g,e\}, \{g,f\}, \{g,g\}\}$

b). $\{a\} \to \{\{a,a\}\}$

$\{b\} \to \{\{b,b\}\}$

$\{c,d\} \to \{\{c,c\}, \{c,d\}, \{d,c\}, \{d,d\}\}$

$\{e,f\} \to \{\{e,e\}, \{e,f\}, \{f,e\}, \{f,f\}\}$

$\{g\} \to \{\{g,g\}\}$

The union of these subset:

$\{\{a,a\}, \{b,b\}, \{c,c\}, \{c,d\}, \{d,c\}, \{d,d\}$
$\{e,e\}, \{e,f\}, \{f,e\}, \{f,f\}, \{g,g\}\}$

c). $\{a,b,c,d\} \to \{\{a,a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,a\}$
$\{b,b\}, \{b,c\}, \{b,d\}, \{c,a\}, \{c,b\}, \{c,c\},$
$\{c,d\}, \{d,a\}, \{d,b\}, \{d,c\}, \{d,d\}\}.$

$\{e,f,g\} \to \{\{e,e\}, \{e,f\}, \{e,g\}, \{f,e\}\{f,f\}, \{f,g\},$
$\{g,e\}, \{g,f\}, \{g,g\}\},$

The union of these subset:

$\{\{a,a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,a\}, \{b,b\}, \{b,c\}, \{b,d\}, \{c,a\},$
$\{c,b\}, \{c,c\}, \{c,d\}, \{d,a\}, \{d,b\}, \{d,c\}, \{d,d\}\}. \{e,e\}, \{e,f\},$
$, \{e,g\}, \{f,e\}\{f,f\}, \{f,g\}, \{g,e\}, \{g,f\}, \{g,g\}\}$

d). $\{a,c,e,g\} \to \{\{a,a\}, \{a,c\}, \{a,e\}, \{a,g\}, \{c,a\},$
$\{c,c\}, \{c,e\}, \{c,g\}, \{e,a\}, \{e,c\},$
$\{e,e\}, \{e,g\}, \{g,a\} \{g,c\}, \{g,e\}, \{g,g\}\}$

$\{b,d\} \to \{\{b,b\}, \{b,d\}, \{d,b\}, \{d,d\}\}.$

$\{f\} \to \{\{f,f\}\}.$

The union of these subset:

$\{\{a,a\}, \{a,c\}, \{a,e\}, \{a,g\}, \{c,a\}, \{c,c\}, \{c,e\}, \{c,g\}, \{e,a\},$
$\{e,c\}, \{e,e\}, \{e,g\}, \{g,a\} \{g,c\}, \{g,e\}, \{g,g\}, \{b,b\}, \{b,d\},$
$\{d,b\}, \{d,d\}, \{f,f\}\}.$

Section 9.6

#4.

a). It is not a poset because the lack of antisymmetry.

b). It is not a poset because the lack of reflexivity.

c). It is a poset.

d). It is not a poset because the lack of reflexivity.

#24.

$\{a,b,c,d\} \rightarrow \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}$