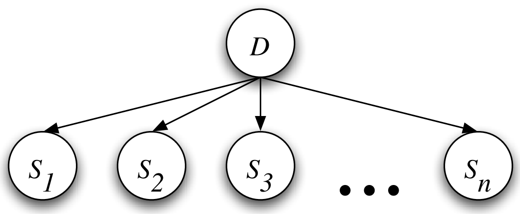
CSE 150. Assignment 2

Out: *Tue Jan 22*
Due: *Tue Jan 29*
Reading: Russell & Norvig, Chapter 14; Korb & Nicholson, Chapter 2.

2.1 Probabilistic reasoning

A patient is known to have contracted a rare disease which comes in two forms, represented by the values of a binary random variable $D \in \{0, 1\}$. Symptoms of the disease are represented by the binary random variables $S_k \in \{0, 1\}$, and knowledge of the disease is summarized by the belief network:



The conditional probability tables (CPTs) for this belief network are as follows. In the absence of evidence, both forms of the disease are equally likely, with prior probabilities: $P(D\!=\!0) = P(D\!=\!1) = \frac{1}{2}$. In the first form of the disease ($D\!=\!0$), the first symptom occurs with probability one,

$$P(S_1\!=\!1|D\!=\!0) = 1,$$

while the k^{th} symptom (with $k\!\geq\!2$) occurs with probability

$$P(S_k\!=\!1|D\!=\!0) = \frac{f(k-1)}{f(k)},$$

where the function $f(k)$ is defined by

$$f(k) = 2^k + (-1)^k.$$

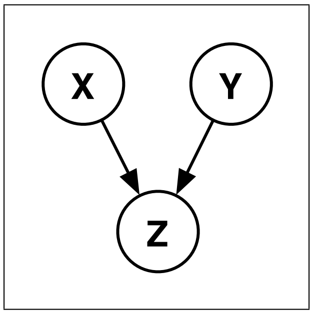
By contrast, in the second form of the disease ($D\!=\!1$), all the symptoms are uniformly likely to be observed, with $P(S_k\!=\!1|D\!=\!1) = \frac{1}{2}$ for all k.

Suppose that on the k^{th} day of the month, a test is done to determine whether the patient is exhibiting the k^{th} symptom, and that each such test returns a positive result. Thus, on the k^{th} day, the doctor observes the patient with symptoms $\{S_1\!=\!1, S_2\!=\!1, \ldots, S_k\!=\!1\}$. Based on the cumulative evidence, the doctor makes a new diagnosis each day by computing the ratio:

$$r_k = \frac{P(D\!=\!1|S_1\!=\!1, S_2\!=\!1, \ldots, S_k\!=\!1)}{P(D\!=\!0|S_1\!=\!1, S_2\!=\!1, \ldots, S_k\!=\!1)}.$$

If this ratio is greater than 1, the doctor diagnoses the patient with the $D\!=\!1$ form of the disease; otherwise, with the $D\!=\!0$ form. Compute the ratio r_k as a function of k. How does the doctor's diagnosis depend on the day of the month? Does the diagnosis become more or less certain as more symptoms are observed? Explain.

2.2 Noisy-OR



Nodes: $X \in \{0,1\}$, $Y \in \{0,1\}$, $Z \in \{0,1\}$

Noisy-OR CPT: $P(Z=1|X,Y) = 1 - (1-p_x)^X\,(1-p_y)^Y$

Parameters: $p_x \in [0,1]$, $p_y \in [0,1]$, $p_y < p_x$

Suppose that the nodes in this network represent binary random variables and that the CPT for $P(Z|X,Y)$ is parameterized by a noisy-OR model, as shown above. Suppose also that

$$0 < P(X=1) < 1,$$
$$0 < P(Y=1) < 1,$$

while the parameters of the noisy-OR model satisfy:

$$0 < p_y < p_x < 1.$$

Consider the following pairs of probabilities. In each case, indicate whether the probability on the left is equal $(=)$, greater than $(>)$, or less than $(<)$ the probability on the right.

(a)	$P(Z=1\|X=0,Y=1)$	☐	$P(Z=1\|X=0,Y=0)$
(b)	$P(Z=1\|X=1,Y=0)$	☐	$P(Z=1\|X=0,Y=1)$
(c)	$P(Z=1\|X=1,Y=0)$	☐	$P(Z=1\|X=1,Y=1)$
(d)	$P(X=1\|Z=1)$	☐	$P(X=1)$
(e)	$P(X=1\|Y=1)$	☐	$P(X=1)$
(f)	$P(X=1\|Z=1)$	☐	$P(X=1\|Y=1,Z=1)$
(g)	$P(X=1)\,P(Y=1)\,P(Z=1)$	☐	$P(X=1,Y=1,Z=1)$

2.3 Conditional independence

For the belief network shown below, indicate whether the following statements of conditional independence are **true (T)** or **false (F)**.

(a) ____________ $P(A, B) = P(A)\,P(B)$

(b) ____________ $P(A|E) = P(A)$

(c) ____________ $P(A, B|C, D, E) = P(A|C, D)\,P(B|D, E)$

(d) ____________ $P(A, B, |C, D, E) = P(A, B|C, D, E, F, G)$

(e) ____________ $P(C, D) = P(C)\,P(D)$

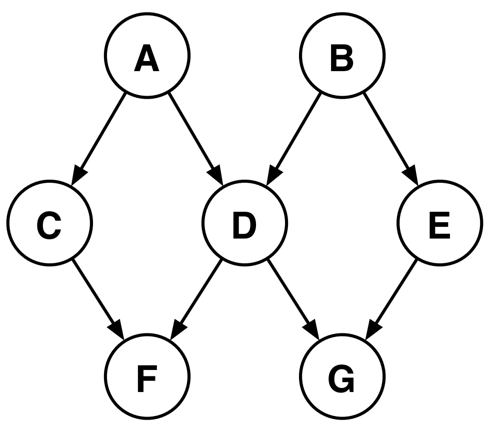
(f) ____________ $P(C|E) = P(C)$

(g) ____________ $P(C, D, E) = P(C)\,P(D|C)\,P(E|D)$

(h) ____________ $P(D|A, B, F, G) = P(D|A, B, F, G, C)$

(i) ____________ $P(C, D|A, F) = P(C|A, F)\,P(D|A, F)$

(j) ____________ $P(G|D) = P(G|A, D)$



2.4 Subsets

For the DAG shown below, consider the following statements of conditional independence. Indicate the largest subset of nodes $\mathcal{S} \subseteq \{A, B, C, D, E, F\}$ for which each statement is true. Note that one possible answer is the empty set $\mathcal{S} = \emptyset$ or $\mathcal{S} = \{\}$ (whichever notation you prefer).

(a) $P(A) = P(A|\mathcal{S})$ ________________

(b) $P(A|C) = P(A|\mathcal{S})$ ________________

(c) $P(A|B,C) = P(A|\mathcal{S})$ ________________

(d) $P(B) = P(B|\mathcal{S})$ ________________

(e) $P(B|A,E) = P(B|\mathcal{S})$ ________________

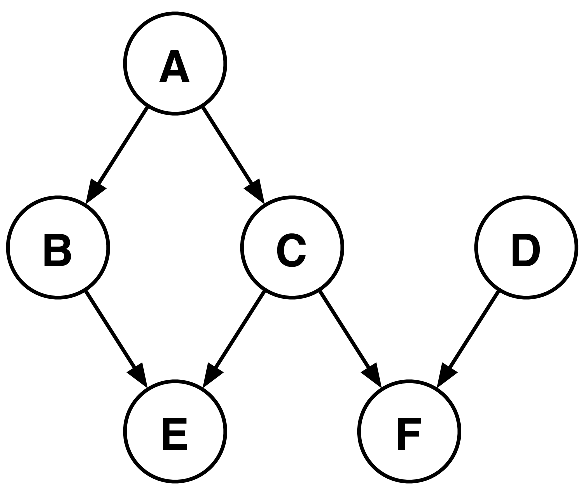
(f) $P(B|A,C,E) = P(B|\mathcal{S})$ ________________

(g) $P(D) = P(D|\mathcal{S})$ ________________

(h) $P(D|F) = P(D|\mathcal{S})$ ________________

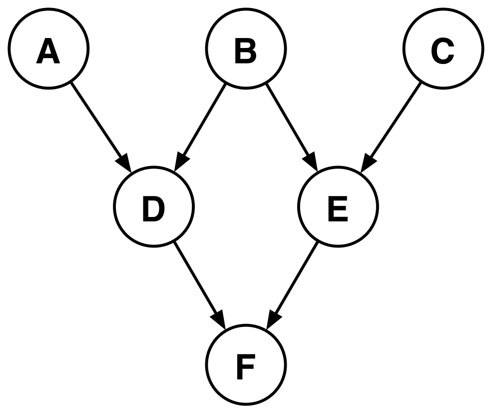
(i) $P(D|C,F) = P(D|\mathcal{S})$ ________________

(j) $P(F) = P(F|\mathcal{S})$ ________________



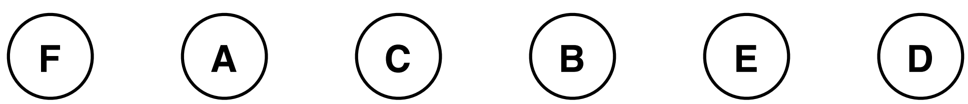
2.5 Node ordering

Consider the following belief network with node ordering $\{A, B, C, D, E, F\}$ and directed acyclic graph (DAG) shown below:



By adding appropriate edges between the nodes shown below, draw the minimal DAG that would be required to represent the same joint distribution for the following alternative node orderings.

(a) $\{F, A, C, B, E, D\}$



(b) $\{D, E, F, A, B, C\}$