Equally-distributed-equivalent from an L-function viewpoint (Working paper)

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February 14, 2026

Abstract

The Atkinson index is a measure of income inequality, first defined in 1970 by invoking the idea of Equally Distributed Equivalent (EDE), i.e. the level of income that, if equally distributed in a hypothetical scenario, gives the same level of welfare as the current income distribution. Originally a concept in economics, the Atkinson index is increasingly applied to studies in health inequality. Our goal is to understand how subgroups are differentially weighted as inequality aversion varies. This is important because it would influence how resources are allocated among different subgroups in society. A deeper understanding on the differential weighting requires adapting mathematical methods not traditionally used in economics. In particular, using functional analytic techniques with an L-function viewpoint, we decompose the Atkinson index into a harmonic-like weighted sum. Our decomposition broaden on the Pigou-Dalton Principle for EDE. Furthermore, we derive a duality principle for EDE, extending the Atkinson index to all welfare metrics other than generic income or health levels. We also explain an application of our decomposition to analyze optimal resource allocation towards achieving maximal welfare under equity considerations.

JEL classification: C65; D63; I14

1 Introduction

Many measures exist to estimate population inequality in both economics and health [3], with the Gini index being the most famous. However, a good measure for inequality analyses should satisfy three important properties [8]: subgroup decomposability, where total inequality is divided into its constituent components; the Pigou-Dalton Principle, where a transfer of a desirable variable (e.g. wealth) from the rich to the poor results in less inequality as long as it does not bring the rich to a worse situation than the poor; and avoids value judgement, for instance by including an explicit parameter that changes the weights placed on various percentiles of the distribution (to allow sensitivity analysis with respect to this parameter for policy decisions). Using this definition of goodness, the Gini index does not qualify as a good measure for two reasons: it does not avoid value judgement; and is only subgroup decomposable if subgroups are explicitly ranked. A detailed analysis (by the same paper [8]) concluded that the Atkinson index [1] stands out as the best inequality measure in health. Additionally, a quantitative analysis of income inequality measures by Shorrocks imply the generalized entropy index [14, Equation 31] is the only one-parameter family that is subgroup decomposable under relatively weak restrictions on homogeneity, and the generalized entropy index can be viewed as a monotonic transformation of the Atkinson index. As such, we focus on the Atkinson index in this paper.

Atkinson defined his index in [1] by invoking the idea of Equally Distributed Equivalent (EDE), reflecting the willingness to trade off aggregate benefits for income to be more equally distributed. This index is defined on the real line as

$$EDE(\epsilon) := \begin{cases} \left(\sum_i H_i^{1-\epsilon} \mu_i \right)^{\frac{1}{1-\epsilon}} & \epsilon \neq 1 \\ \prod_i H_i^{\mu_i} & \epsilon = 1 \end{cases} \tag{1}$$

where H_i is the income level for subgroup i with each H_i distinct and nonzero, μ_i is a weight for subgroup i (with the sum of all μ_i equaling 1), and ϵ is the Atkinson inequality aversion parameter. The weights μ_i are chosen depending on the application. In Atkinson's original formulation, μ_i is simply the proportion of the total population at income level H_i, but μ_i can also be income-dependent, causality-dependent, and so on [13]. Applications of this index in health inequality studies, an emerging body of research [5, 6, 12, 13, 16], replaces the variables H_i with subgroup health levels. For applicability of our results to both economics and health, we will henceforth call H_i the wealth level unless discussing specific examples.

Note that our definition of $EDE(\epsilon)$ is the non-normalized form of the original definition given by Atkinson. This is the perspective taken in health inequality studies, and we find that this form is most suitable for our arguments in this paper. Also, instead of restricting ϵ to be strictly non-negative, we allow use of the entire real line as it has important implications in our analysis (e.g. the duality statement stated at the end of this section).

The goal of this paper is to understand the Atkinson index as weighted sums of the wealth levels H_i. To agree with the original work of Atkinson, such a weighted sum should give greater weight to subgroups with lower wealth at higher inequality aversion ϵ. This implies the weighted sum should be harmonic-like in order to emphasize sensitivity in lower wealth groups. Temporarily suppose $\epsilon > 2$ and abstractly consider the sum

$$AH(\epsilon) := \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

where $f_{i,\epsilon}$ are non-negative functions in ϵ to be determined. We will show the existence of unique $f_{i,\epsilon}$'s minimizing $AH(\epsilon)$ with respect to a technical constraint (Theorem 2.1). Furthermore, this minimum value coincides with the Atkinson index $EDE(\epsilon)$. For $\epsilon < 2$, we will show that $AH(\epsilon)$ can be analytically continued to the entire real line except for a removable singularity at $\epsilon = 1$. This implies the initial supposition of $\epsilon > 2$ can be lifted, and $AH(\epsilon)$ gives a way to decompose $EDE(\epsilon)$ into subgroup components for all values of ϵ. Although our harmonic-like decomposition is related to Shorrock's additive-like decomposition [14],

The decomposition of the Atkinson index into $AH(\epsilon)$ will be proven in Section 2. Following this, we demonstrate how our techniques broaden on the Pigou-Dalton Principle for the Atkinson index, as well as giving greater subgroup-level insights in EDE weighting of subgroups (Section 3). An example of such an insight is the following: subgroups that do not correspond to the highest or lowest wealth may not be monotonically weighted by $f_{i,\epsilon}$ as ϵ varies. In other words, the Atkinson index weighs subgroups in a highly interesting way, especially for subgroups with wealth levels sufficiently close to the one with the lowest wealth, and a precise formulation is given in the discussion leading up to Example 3.4. After an analysis of our decomposition

$AH(\epsilon)$, we illustrate an application in the context of resource allocation (Section 4).

An inference from our decomposition $AH(\epsilon)$ is a surprising connection between opposing metrics in both economics (e.g. income and poverty) and health (e.g. life expectancy and crude mortality rate). More precisely, our decomposition of the Atkinson index $EDE(\epsilon)$ into the harmonic-like sum $AH(\epsilon)$ is transformed into an arithmetic-like sum at $\epsilon \leq 0$. This implies Atkinson's idea of EDE can be carried over to economics or health metrics associated with negative outcomes, such as poverty or mortality rate, with the absolute value of $\epsilon \leq 0$ as the appropriate Atkinson inequality aversion parameter. (As an abuse of notation, we write ϵ to mean its absolute value if $\epsilon \leq 0$.) In particular, a characterization of EDE for metrics associated with negative outcomes should give greater weight to subgroups with higher negative outcomes at higher ϵ, pointing to an arithmetic-like sum as an appropriate subgroup decomposition. Furthermore, a quantitative analysis of this connection between opposing metrics gives rise to a duality between EDE of metrics associated with positive outcomes, versus those associated with negative outcomes. We state this duality for income and poverty here, leaving a more general discussion to Section 5. Our duality principle vastly generalizes a recent observation of Sterck [19] that minimizing overall poverty level (i.e. $\epsilon = 0$) is equivalent to maximizing EDE income at $\epsilon = 2$.

EDE Duality between Income and Poverty (c.f. Duality Principle 5.1)**.** Minimizing EDE poverty at Atkinson inequality aversion parameter ϵ is equivalent to maximizing EDE income at $\epsilon + 2$.

In Section 6, we apply our mathematical methods to analyze potentially avoidable mortality in the setting of Ontario, Canada, before ending with some concluding remarks in Section 7.

2 Functional analysis on Atkinson's index

The definition of Atkinson's index is a little opaque for subgroup-level analyses at first glance. For instance, it does not inform us how the H_i's are differentially weighted at certain Atkinson parameter ϵ. While Shorrocks performed a subgroup decomposition on the Atkinson index into an arithmetic-like sum to obtain subgroup-level information [14, 15], we will decompose $EDE(\epsilon)$ in an alternative way as a harmonic-like sum

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

for a family of functions $f_{i,\epsilon}$ in ϵ and depending on H_i and μ_i. These functions $f_{i,\epsilon}$ are uniquely determined, and an expression can be derived by performing partial differentiation with respect to H_i. However, partial differentiation does not give much insight beyond the work done by Shorrocks in his analysis. In particular, Shorrocks mentioned that his arithmetic-like analog of $f_{i,\epsilon}\mu_i$ cannot be reasonably considered to be weights, as their sum over the subgroups does not equal 1 in general.

It is worth emphasizing that our proposed sum is harmonic-like while Shorrocks used an arithmetic-like sum in his analysis. We believe a harmonic-like sum is more appropriate as the Atkinson index is more sensitive to lower wealth subgroups as ϵ increases, which is reflected in an harmonic-like sum (but not an arithmetic-like sum). However, arithmetic-like sums do come in play in our analysis as they will be essential when considering negative metrics such as poverty level (Section 5).

We now list some basic properties of the Atkinson index $EDE(\epsilon)$ (Equation 1) that is almost immediate from definition. $EDE(\epsilon)$ gives increasingly greater weight to the subgroup with lowest wealth as ϵ increases, and is a strictly decreasing function in ϵ. In particular, $EDE(\epsilon)$ plateaus to the minimum wealth level among the subgroups as ϵ increases, and plateaus to the maximum wealth level among the subgroups as ϵ decreases. Further, by definition of the utility function for Atkinson's index

$$U(x) = \frac{x^{1-\epsilon}}{1-\epsilon}$$

and its concavity, the Atkinson index satisfies the main properties of social welfare measures, such as wealth homogeneity, population homogeneity, and the Pigou-Dalton Principle [1, 14]. These three properties will also be elaborated on in the next Section.

The purpose of this section is to prove the following mathematical results. For convenience in the arguments, these results are formulated as the inverse of the Atkinson index as stated in Equation 1.

Theorem 2.1. *Let $\epsilon > 2$, let $p = \epsilon - 1$, and let q be the number such that $1/p + 1/q = 1$.*

Then there exist a unique collection of non-negative values $f_{i,\epsilon}$ satisfying

$$\sum_i f_{i,\epsilon}^q \mu_i = 1,$$

and such that these $f_{i,\epsilon}$ maximizes the expression

$$F(f_{i,\epsilon}) := \sum_i \frac{f_{i,\epsilon}\mu_i}{H_i}.$$

Furthermore, this maximum value equals $EDE(\epsilon)^{-1}$.

Corollary 2.2. *The values $f_{i,\epsilon}$ in Theorem 2.1 can be explicitly determined:*

$$f_{i,\epsilon} := \left(\sum_j \left(\frac{H_j}{H_i} \right)^{1-\epsilon} \mu_j \right)^{-\left(1+\frac{1}{1-\epsilon}\right)} = \left(\frac{EDE(\epsilon)}{H_i} \right)^{\epsilon-2}.$$

This can be viewed as an analytic function at $\epsilon \neq 1$.

Theorem 2.3. *Let $\epsilon > 2$. Suppose $q > 1$ is a number satisfying the following two conditions.*

- *There exists unique values $f_i > 0$ such that*

  $$\sum_i f_i^q \mu_i = 1$$

  and f_i maximizes the expression

  $$F(a_i) := \sum_i \frac{1}{H_i} a_i \mu_i.$$

- *$F(f_i)$ equals $EDE(\epsilon)^{-1}$, where f_i is specified in the above condition.*

Then q is unique, and equals the value in Theorem 2.1.

Theorem 2.4 (Analytic Continuation)**.** *Let q be as defined in Theorem 2.1. Consider the real-valued function*

$$L(H_i, \epsilon) := \max_{\substack{\sum_i f_{i,\epsilon}^q \mu_i = 1 \\ f_i \geq 0}} \left\{ \sum_i \frac{f_{i,\epsilon}\mu_i}{H_i} \right\}$$

which is well-defined on $\epsilon > 2$ by Theorem 2.3. Then $L(H_i, \epsilon)$ can be analytically continued to the entire real line, except for a removable singularity at $\epsilon = 1$. Furthermore, $L(H_i, \epsilon)$ is a positive function with $L(H_i, \epsilon) = EDE(\epsilon)^{-1}$.

Theorem 2.5 (Functional Equation)**.** *The function $L(H_i, \epsilon)$ defined in Theorem 2.4 satisfies*

$$L(H_i, -\epsilon)^{-1} = L(H_i^{-1}, 2 + \epsilon).$$

In summary, the five mathematical statements above narrate the following story. Theorem 2.1 gives a mathematical justification that a harmonic-like decomposition for the Atkinson index at $\epsilon > 2$ is in fact robust. Corollary 2.2 explicitly defines the components $f_{i,\epsilon}$ of the harmonic-like decomposition. Theorem 2.3 tells us that the technical condition imposed on $f_{i,\epsilon}$ allows us to deform Atkinson weights in a well-defined manner as ϵ varies, and is the only sensible deformation that naturally extends Atkinson weights as ϵ varies. In particular, this Theorem shows our definition of EDE-factors in the next Section makes sense (Definition 3.1). Theorem 2.4 mathematically justifies our harmonic-like decomposition agrees with the Atkinson index at all ϵ (not just $\epsilon > 2$). Finally, Theorem 2.5 exhibits a dual pairing on the harmonic-like decomposition that results in a duality principle that will be explained in Section 5 (Duality Principle 5.1).

Proof of Theorem 2.1. Let X be a countable measure space with discrete probability measure μ, and let F be an injective real-valued positive Lebesgue-measurable function on X. Then the p-norm of F is simply

$$\|F\|_p = \left(\sum_i F_i^p \mu_i \right)^{\frac{1}{p}}.$$

In our case where $F_i = 1/H_i$,

$$\|F\|_p = \left(\sum_i H_i^{-p} \mu_i \right)^{\frac{1}{p}}.$$

By the Riesz-Fréchet Representation Theorem for L^p-spaces [18, Chapter 1], there exist non-negative values f_i such that

$$\|F\|_p = \max_{\substack{\sum_i f_i^q \mu_i \leq 1 \\ f_i \geq 0}} \left\{ \sum_i \frac{1}{H_i} f_i \mu_i \right\},$$

where q is the number such that $1/p + 1/q = 1$. If we can find values f_i for this equality to hold, then we are done as the left-hand side $\|F\|_p$ equals $EDE(\epsilon)^{-1}$ after recalling $p = \epsilon - 1$.

Following this discussion, we need to solve an optimization problem: Find numbers $f_i \geq 0$ that maximizes

$$\sum_i \frac{1}{H_i} f_i \mu_i \tag{2}$$

subject to the condition

$$\sum_i f_i^q \mu_i = l, \qquad 0 \leq l \leq 1. \tag{3}$$

Clearly, f_i cannot be simultaneously zero for all i. Using Lagrange Multipliers, there exist a constant λ_l depending on l such that

$$\frac{1}{H_i} \mu_i = \lambda_l q f_i^{q-1} \mu_i.$$

Hence

$$f_i = \left(\frac{1}{\lambda_l q H_i} \right)^{\frac{1}{q-1}}.$$

Substituting f_i to Equations 2 and 3 gives

$$\sum_i \frac{1}{H_i} f_i \mu_i = \lambda_l^{-\frac{1}{q-1}} \sum_i \left(\frac{1}{q H_i^q} \right)^{\frac{1}{q-1}} \mu_i$$

and

$$\lambda_l^{-\frac{1}{q-1}} = l^{\frac{1}{q-1}} \left(\sum_i \left(\frac{1}{q H_i} \right)^{\frac{q}{q-1}} \mu_i \right)^{-\frac{1}{q}}$$

Since q, H_i, μ_i are all known constants, to maximize Equation 2, we will need to maximize $\lambda_l^{-\frac{1}{q-1}}$, which requires maximizing $l^{\frac{1}{q-1}}$, and this last expression is an increasing function on l as $q - 1 > 0$. Therefore, necessarily $l = 1$ for our optimization problem.

In summary, for each f_i,

$$\begin{aligned} f_i &= \left(\frac{1}{\lambda_1 q H_i}\right)^{\frac{1}{q-1}} \\ &= \left(\frac{1}{qH_i}\right)^{\frac{1}{q-1}} \left(\sum_j \left(\frac{1}{qH_j}\right)^{\frac{q}{q-1}} \mu_j\right)^{-\frac{1}{q}} \\ &= \left(\sum_j \left(\frac{H_i}{H_j}\right)^{\frac{q}{q-1}} \mu_j\right)^{-\frac{1}{q}}. \end{aligned} \tag{4}$$

These f_i's are exactly the $f_{i,\epsilon}$ we seek. □

Proof of Corollary 2.2. This is an algebraic manipulation of Equation 4 to rewrite it into two different ways. □

Proof of Theorem 2.3. By manipulating Equation 4,

$$f_i = \left(\frac{EDE(k)}{H_i}\right)^{-\frac{1}{q-1}}, \qquad k = 2 + \frac{1}{q-1}.$$

Therefore, we can view both f_i and

$$H(q) := \sum_i \frac{1}{H_i} f_i \mu_i$$

as functions in q. To prove the Theorem, it suffices to show that $H(q)$ is monotone decreasing in q. A computation tells us that the derivative with respect to q is

$$\begin{aligned} \frac{d}{dq} H(q) &= \sum_i \frac{1}{H_i} \frac{df_i}{dq} \mu_i \\ &= \frac{1}{(q-1)^2} \sum_i \frac{f_i \mu_i}{H_i} \ln\left(\frac{EDE(k)}{H_i}\right) + \frac{1}{(q-1)^3} \sum_i \frac{f_i \mu_i}{H_i} \frac{EDE'(k)}{EDE(k)} \\ &= -\frac{1}{q-1} \sum_i \frac{f_i \mu_i}{H_i} \ln(f_i) + \frac{1}{(q-1)^3} \sum_i \frac{f_i \mu_i}{H_i} \frac{EDE'(k)}{EDE(k)}. \end{aligned} \tag{5}$$

Here, $EDE'(k)$ is the derivative of $EDE(k)$ with respect to k and not q.

We now need to show that the first and second term of Equation 5 are both negative. The second term is negative as $EDE'(k)$ is the only part of the term that is negative. For the first term, note that

$$\sum_i \frac{f_i\mu_i}{H_i}\ln(f_i) = \ln\left(\prod_i f_i^{\frac{f_i\mu_i}{H_i}}\right)$$

so we are reduced to showing that the product inside the logarithm is at least 1. Using the weighted power mean inequality, more specifically the weighted GM-HM inequality [2, Chapter 3],

$$\prod_i f_i^{\frac{f_i\mu_i}{H_i}} \geq \left(\left(\frac{\sum_i \frac{f_i\mu_i}{H_i} f_i^{-1}}{\sum_i \frac{f_i\mu_i}{H_i}}\right)^{-1}\right)^{\sum_i \frac{f_i\mu_i}{H_i}} . \tag{6}$$

We now make the observation that

$$\sum_i \frac{f_i\mu_i}{H_i} f_i^{-1} = \sum_i \frac{1}{H_i}\cdot 1\cdot \mu_i$$

so by the first condition in the statement of the Theorem,

$$\sum_i \frac{f_i\mu_i}{H_i} f_i^{-1} \leq \sum_i \frac{f_i\mu_i}{H_i}.$$

Hence the fraction in the right hand side of Equation 6 is at most 1, implying

$$\left(\left(\frac{\sum_i \frac{f_i\mu_i}{H_i} f_i^{-1}}{\sum_i \frac{f_i\mu_i}{H_i}}\right)^{-1}\right)^{\sum_i \frac{f_i\mu_i}{H_i}} \geq 1$$

and we are done. $\square$

Proof of Theorem 2.4. Note that both $L(H_i,\epsilon)$ and $EDE(\epsilon)^{-1}$ are analytic functions defined on the interval $(-\infty,1)\cup(1,\infty)$. As $L(H_i,\epsilon) = EDE(\epsilon)^{-1}$ on $(2,\infty)$, the Identity Theorem [7, Chapter 1] implies they must also be equal on $(1,\infty)$.

We now apply the Identity Theorem on $(-\infty,1)$ by showing that $L(H_i,\epsilon)$ and $EDE(\epsilon)^{-1}$ agree on a sequence converging to the limit point 0. In particular, we show that $L(H_i, 1/n)$ equals $EDE(1/n)^{-1}$ for all positive integers $n \geq 2$. By Corollary 2.2,

$$\begin{aligned}
L\left(H_i, \frac{1}{n}\right) &= \sum_i \frac{1}{H_i}\left(\sum_j \left(\frac{H_j}{H_i}\right)^{1-\frac{1}{n}} \mu_j\right)^{-1+\frac{1}{1-\frac{1}{n}}} \mu_i \\
&= \left(\sum_j H_j^{1-\frac{1}{n}} \mu_j\right)^{-\left(1+\frac{n}{n-1}\right)} \left(\sum_i H_i^{1-\frac{1}{n}} \mu_i\right) \\
&= EDE\left(\frac{1}{n}\right)^{-1},
\end{aligned}$$

as desired. □

Proof of Theorem 2.5. If $\epsilon > 0$, a calculation reveals

$$\begin{aligned}
L(H_i^{-1}, 2+\epsilon) &= \sum_j \sum_i H_j \left(\left(\frac{H_j}{H_i}\right)^{-(1+\epsilon)} \mu_i\right)^{-\left(1-\frac{1}{1+\epsilon}\right)} \mu_j \\
&= \left(\sum_j H_j^{1+\epsilon} \mu_j\right)\left(\sum_i H_i^{1+\epsilon} \mu_i\right)^{-\left(1-\frac{1}{1+\epsilon}\right)} \\
&= \left(\sum_i H_i^{1+\epsilon} \mu_i\right)^{\frac{1}{1+\epsilon}} \\
&= EDE(-\epsilon) \\
&= L(H, -\epsilon)^{-1}
\end{aligned}$$

where the last equality is due to Theorem 2.1. Finally, the condition $\epsilon > 0$ can be dropped by the Identity Theorem. □

3 EDE-factors

Due to Section 2, the Atkinson index can be decomposed as a harmonic-like sum

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i\right)^{-1}$$

where $f_{i,\epsilon}$ is defined as

$$f_{i,\epsilon} := \begin{cases} \left(\sum_j \left(\frac{H_j}{H_i} \right)^{1-\epsilon} \mu_j \right)^{-\left(1+\frac{1}{1-\epsilon}\right)} & \epsilon \neq 1 \\ \sum_j \ln \left(\frac{H_j}{H_i} \right) \mu_j & \epsilon = 1. \end{cases}$$

These $f_{i,\epsilon}$ satisfy a technical weighted normality condition:

$$\sum_i f_{i,\epsilon}^q \mu_i = 1, \qquad q = \frac{\epsilon - 1}{\epsilon - 2},$$

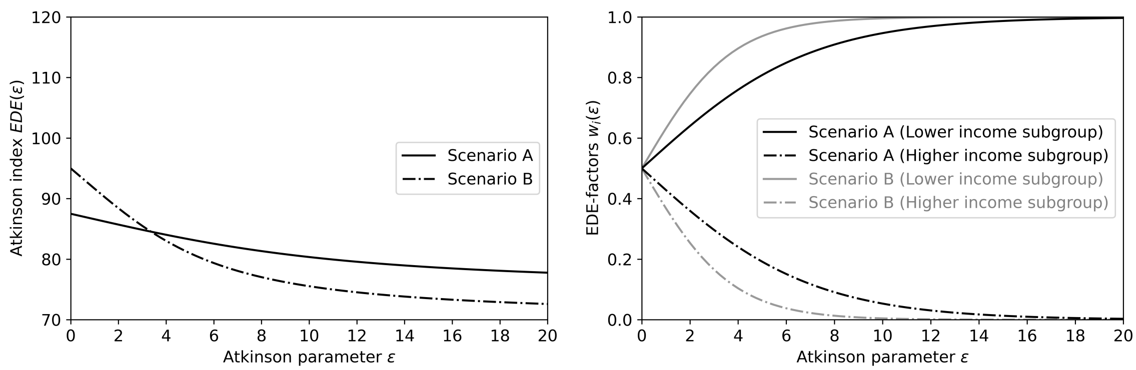
allowing us to extend Atkinson's weights μ_i through the expressions $w_i(\epsilon) := f_{i,\epsilon}^q \mu_i$. In other words, we extend Atkinson's weights via a q-analog of the expression $f_{i,\epsilon}\mu_i$. Such an extension requires agreement with Atkinson's weight without inequality aversion considerations, i.e. $w_i(0) = \mu_i$ or equivalently $f_{i,\epsilon}^q = 1$. As such, we need to shift ϵ by 1 for this equality to hold.

Definition 3.1. The *EDE-factor* for subgroup i can be explicitly defined in two ways:

$$w_i(\epsilon) := \left(\sum_j \left(\frac{H_j}{H_i} \right)^{-\epsilon} \mu_j \right)^{-1} \mu_i = \left(\frac{EDE(\epsilon + 1)}{H_i} \right)^{\epsilon} \mu_i.$$

Before discussing properties of EDE-factors, we give a simple example of EDE-factors on two hypothetical policy scenarios.

Example 3.2. Consider a population split into 2 subgroups with equal Atkinson weights $\mu_1 = \mu_2 = 0.5$ and income levels $H_1 = \$70,000$ (lower income), $H_2 = \$100,000$ (higher income). Also consider two Scenarios: A, where individuals in the lower income subgroup H_1 are given additional $\$5,000$; and B, where individuals in the higher income subgroup H_2 are given additional $\$20,000$. Then the graphs of the Atkinson index (computing EDE) and EDE-factors (computed using Definition 3.1) are graphed below. Our choice of graphing ϵ between 0 and 20 is deliberate as elicitation of ϵ in health or income generally falls in this range [5, 12, 16].



If inequality aversion is not considered, Scenario B results in a higher average income as four-fold more resources are additionally given compared to Scenario A. However, a computation of EDE income shows a more rapid decrease as ϵ increases, with a tipping point at $\epsilon_{tip} \approx 3.37$ and $EDE(\epsilon_{tip}) \approx \$84,500$. This is also evident by the EDE-factors, as Scenario B puts a much higher weight on the lower income subgroup in EDE computations due to a larger income gap between subgroups.

Immediate properties from functional analysis

From our discussion in the previous Section, clearly the EDE-factors $w_i(\epsilon)$ are non-negative and sum to 1 for all ϵ:

$$\sum_i w_i(\epsilon) = 1.$$

At $\epsilon = 0$, this is simply the base case condition that the sum of all μ_i equal 1. These EDE-factors can be seen as a spiritual answer to questions raised in [13, 14] on a method to decompose measures, such as the Atkinson's index, as a simple weighted sum. We now discuss how EDE-factors imply most of the homogeneity and transfer properties in Atkinson's index.

Weatlh homogeneity. By Definition 3.1, the EDE-factors $w_i(\epsilon)$ are not affected by a uniform scaling of wealth levels $H_i \mapsto kH_i$ for some positive constant k. Thus, by Theorem 2.4, this implies wealth homogeneity for Atkinson's index, i.e. $EDE(\epsilon)$ is multiplied by the same constant k under a uniform scaling of wealth levels. To offer another point of view, $EDE(\epsilon)$ is a homogeneous function of degree 1, and Euler's

Theorem (for homogeneous functions) implies the EDE-factors $w_i(\epsilon)$ are homogeneous of degree 0.

Population homogeneity. To show that the EDE-factors satisfy population homogeneity, suppose each subgroup i is replicated n times $(i,1),\ldots,(i,n)$ and each replication is weighted $\omega_1,\ldots,\omega_n$, with the ω_k's summing to 1. Then $H_{(i,1)} = \cdots = H_{(i,n)}$, and each subgroup (i,η) is weighted $\mu_i\omega_\eta$ in Atkinson's index. Therefore, its corresponding EDE-factor is

$$\begin{aligned} w_{(i,\eta)}(\epsilon) &= \left(\sum_\alpha \sum_j \left(\frac{H_{(j,\alpha)}}{H_{(i,\eta)}}\right)^{-\epsilon} \mu_j\omega_\alpha\right)^{-1} \mu_i\omega_\eta \\ &= \left(\sum_j \left(\frac{H_{(j,\alpha)}}{H_{(i,\eta)}}\right)^{-\epsilon} \mu_j \sum_\alpha \omega_\alpha\right)^{-1} \mu_i\omega_\eta \\ &= \omega_\eta \cdot w_i(\epsilon). \end{aligned}$$

As the weights $w_i(\epsilon)$ are q-analogs of the expression $f_{i,\epsilon}\mu_i$,

$$\begin{aligned} \left(\sum_\alpha \sum_i \frac{1}{H_{(i,\alpha)}} f_{(i,\alpha),\epsilon}\mu_i\omega_\alpha\right)^{-1} &= \left(\sum_i \frac{1}{H_i} f_{i,\epsilon}\mu_i \sum_\alpha \omega_\alpha\right)^{-1} \\ &= \left(\sum_i \frac{1}{H_i} f_{i,\epsilon}\mu_i\right)^{-1} \\ &= EDE(\epsilon) \end{aligned}$$

which proves population homogeneity.

Pigou-Dalton Principle. The Pigou-Dalton Principle is a transfer principle that asserts any social welfare function must prefer allocations that are more equitable. Formally, if $H_i > H_j$, then a transfer of $\Delta > 0$ from H_i to H_j, in such a way that

$$H_i - \Delta \geq H_j + \Delta^*, \qquad \Delta^* := \Delta\frac{\mu_i}{\mu_j}$$

must not decrease $EDE(\epsilon)$. This is easily seen to hold for the Atkinson index due to the concavity of the utility function.

We prove that the Pigou-Dalton Principle is a special case of Theorem 2.1 when $\epsilon > 2$, though the Theorem cannot be used to prove the Pigou-Dalton Principle at

$0 < \epsilon < 2$. However, this is sufficient to show that Theorem 2.1 generalizes the Pigou-Dalton Principle for inequality studies using negative metrics (e.g. poverty level); see Section 5 for a discussion on this.

Proposition 3.3. *Theorem 2.1 implies the Pigou-Dalton Principle for Atkinson's index at $\epsilon > 2$.*

Proof. Let $\{H_k^*\}_k$ be the wealth profile such that $H_k^* = H_k$ for $k \neq i, j$, with $H_i^* = H_i - \Delta$ and $H_j^* = H_j + \Delta^*$ where Δ, Δ^* are as defined above. Let $EDE^*(\epsilon)$ be Atkinson's index calculated with the wealth profile $\{H_k^*\}_k$. By Theorem 2.1, there exists $f_{i,\epsilon}, f_{i,\epsilon}^*$ satisfying the conditions of that Theorem such that

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

and

$$EDE^*(\epsilon) = \left(\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i \right)^{-1}.$$

We need to show that $EDE^*(\epsilon) \geq EDE(\epsilon)$. Note that

$$\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i = \Delta \mu_i \left(\frac{f_i^*}{H_i H_i^*} - \frac{f_j^*}{H_j H_j^*} \right) + \sum_i \frac{1}{H_i} f_{i,\epsilon}^* \mu_i.$$

The summation on the right satisfies

$$\sum_i \frac{1}{H_i} f_{i,\epsilon}^* \mu_i \leq \sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i$$

by Theorem 2.1. The term on the left satisfies

$$\begin{aligned} \frac{f_i^*}{H_i H_i^*} - \frac{f_j^*}{H_j H_j^*} &= EDE^*(\epsilon)^{\epsilon-2} \left(\frac{1}{H_i (H_i^*)^{\epsilon-1}} - \frac{1}{H_j (H_i^*)^{\epsilon-1}} \right) \\ &< 0 \end{aligned}$$

where the equality is by definition of f_i^* (Corollary 2.1) and the inequality is because $H_i > H_j$ and $H_i^* > H_j^*$. Therefore

$$\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i \leq \sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i$$

implying $EDE^*(\epsilon) \geq EDE(\epsilon)$. $\square$

A non-monotonic property

As $EDE(\epsilon)$ tends to the minimum wealth level as ϵ increases, the EDE-factors obey the following asymptotic property:

$$\lim_{\epsilon\to\infty} w_i(\epsilon) = \begin{cases} 1 & \text{if } H_i = \min\{H_k\}_k; \\ 0 & \text{otherwise.} \end{cases}$$

However, EDE-factors demonstrate a very interesting property: subgroups that do not correspond to the highest or lowest wealth may not be monotonically weighted as ϵ increases. More precisely, let $w_i(\epsilon)$ correspond to the EDE-factor of such a subgroup. By taking the derivative, one gets

$$\frac{d}{d\epsilon} w_i(\epsilon) = \left(\sum_j \left(\frac{H_j}{H_i}\right)^{-\epsilon} \mu_j\right)^{-2} \mu_i \left(\sum_j (\ln H_j - \ln H_i) \left(\frac{H_j}{H_i}\right)^{-\epsilon} \mu_j\right)$$

This is non-increasing exactly when the right-most sum is non-negative, or equivalently

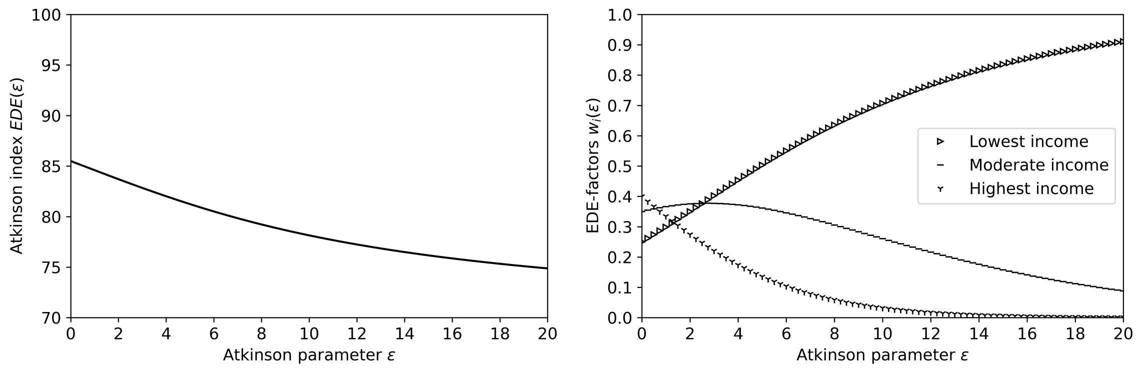
$$\ln H_i \geq \frac{\sum_{j\neq i} H_j^{-\epsilon} \mu_j \ln H_j}{\sum_{j\neq i} H_j^{-\epsilon} \mu_j}.$$

As the right hand side is a decreasing function in ϵ (by applying the Cauchy-Schwarz inequality on its derivative), this implies

$$\ln H_i \geq \sum_{j\neq i} \ln H_j^{\mu_j}.$$

If we assume $H_i > 1$ (with no loss of generality by wealth homogeneity), the above inequality implies its EDE-factor will increase at lower levels of ϵ to a unique maxima before monotonically tending to 0 as long as H_i is less than the relative geometric mean of the other subgroups. In other words, the Atkinson index may increasingly weight subgroups that are close to the lowest wealth at lower levels of ϵ. This fact cannot be seen directly from the original definition of the Atkinson index (Equation 1).

Example 3.4. Consider a population split into 3 subgroups with Atkinson weights $\mu_1 = 0.25$, $\mu_2 = 0.35$, $\mu_3 = 0.4$ and income levels $H_1 = \$70,000$ (lowest income), $H_2 = \$80,000$ (moderate income), $H_3 = \$100,000$ (highest income). Then the graphs of the Atkinson index and EDE-factors are graphed below.



In this example, the moderate income subgroup H_2 is increasingly weighted until $\epsilon_{2,peak} \approx 2.76$, with EDE-factor peaking at $w_2(\epsilon_{2,peak}) \approx 0.377$. This shows that at low levels of inequality aversion, the calculation of $EDE(\epsilon)$ puts emphasis in both H_1 and H_2, and not just the lowest income subgroup H_1.

4 Maximal EDE resource allocation

Consider the problem of reallocating current resources between subgroups to achieve maximal EDE wealth at inequality aversion ϵ. If p_i is the (local) production function of subgroup i and r_i is the amount of resources currently allocated to subgroup i, then the wealth level H_i can be expressed as $H_i = p_i r_i$. Assuming the sum of all current resources equals R, this reallocation problem reduces to the following optimization problem: Maximize

$$EDE(\epsilon) = \left(\sum_i (p_i r_i)^{1-\epsilon} \mu_i \right)^{\frac{1}{1-\epsilon}}$$

subject to the condition

$$\sum_i r_i = R.$$

If inequality aversion is not a consideration ($\epsilon = 0$), this problem has a simple solution: Allocate all resources to the subgroup with the highest value of $p_i \mu_i$, i.e. best weighted production function. However, if equity is a consideration ($\epsilon > 0$), we can solve this problem via Lagrange Multipliers, telling us that $EDE(\epsilon)$ is maximized if $H_i = \widetilde{H_{i,\epsilon}}$, where

$$\widetilde{H_{i,\epsilon}} := R\left(\sum_j \frac{1}{p_j}\left(\frac{p_j\mu_j}{p_i\mu_i}\right)^{\frac{1}{\epsilon}}\right)^{-1}. \tag{7}$$

Equation 7 can be compared to a similar setup described in [13].

EDE-factors offer a quick comparison of arriving at maximal EDE resource allocation given current allocation. Writing $w_{i,\epsilon} = w_i(\epsilon)$, Definition 3.1 implies

$$H_i = d\left(w_{i,\epsilon}^{-1}\mu_i\right)^{\frac{1}{\epsilon}}$$

for an expression d that is constant across all subgroups. As $H_i = p_i r_i$, dividing by p_i and summing across i gives

$$R = d\sum_j \frac{1}{p_j}\left(w_{j,\epsilon}^{-1}\mu_j\right)^{\frac{1}{\epsilon}},$$

and a rearrangement gives

$$H_i = R\left(\sum_j \frac{1}{p_j}\left(\frac{w_{j,\epsilon}^{-1}\mu_j}{w_{i,\epsilon}^{-1}\mu_i}\right)^{\frac{1}{\epsilon}}\right)^{-1} \tag{8}$$

which is very similar to Equation 7 for $\widetilde{H_{i,\epsilon}}$.

Equations 7 and 8 are useful for policy making as it allows us to compare resource allocation as a ratio between subgroups without requiring explicit knowledge on total resources (the R's cancel out under a ratio), allowing for scalability or if total resources are relatively unknown but with known effects. For instance, the expressions $p_i\mu_i$ and $w_{i,\epsilon}^{-1}\mu_i$, in the fractions of Equations 7 and 8 respectively, are related by a factor of $w_{i,\epsilon}p_i$:

$$p_i\mu_i = p_i w_{i,\epsilon}\cdot(w_{i,\epsilon}^{-1}\mu_i).$$

At current resource allocation, this means that, for each subgroup i, the deformation of the weighted production function $p_i w_{i,\epsilon}$, consisting of production function p_i and EDE-factor $w_{i,\epsilon}$, can serve as a measure of current resource allocation compared to the optimal allocation giving rise to the maximal EDE at a given $\epsilon > 0$.

5 EDE calculations for negative metrics

Let M be a metric that varies inversely proportional to wealth level H (e.g. poverty level, mortality rate). Such metrics are important in applications for both economics and health [6]. Note that the EDE-adjusted M as ϵ varies cannot be calculated by directly substituting M into Atkinson's index $EDE(\epsilon)$ (Equation 1) as this would tend to the lowest level of M (i.e. highest level of wealth), contrary to what we expect.

With that said, calculations on negative metrics M can be done through our functional equation (Theorem 2.5). For EDE-adjusted calculation on M, we require tending to the highest level (i.e. lowest level of wealth) as ϵ increases. Furthermore, we would like larger values of subgroup-level M to be emphasized so that the most disadvantageous subgroups bear more magnitude in the computation of an EDE-adjusted M. Therefore, we desire an arithmetic-like sum

$$EDE^{\dagger}(\epsilon) := \sum_i M_i g_{i,\epsilon} \mu_i,$$

where $g_{i,\epsilon}$ are functions depending on M_i and ϵ. This expression must be consistent with the usual arithmetic sum without any considerations on inequality aversion, i.e.

$$EDE^{\dagger}(0) = \sum_i M_i \mu_i.$$

(Equivalently, $g_{i,0} = 1$ for all i.) In addition, for such an $EDE^{\dagger}(\epsilon)$ to be compatible with $EDE(\epsilon)$ for wealth, we require $EDE^{\dagger}(\epsilon)$ to satisfy the same fundamental properties as those outlined in [1, 14]. Therefore, using the fact that the (non-normalized) Atkinson index is a monotonic transformation of Shorrock's work in [14], $EDE^{\dagger}(\epsilon)$ has to equal $EDE(\eta)$ for some η. In particular, the results of Section 2 apply to $EDE^{\dagger}(\epsilon)$ through $EDE(\eta)$.

We show that $\eta = -\epsilon$. Note that the expression $f_{i,\epsilon}$ in Corollary 2.2 satisfies $f_{i,\epsilon} = 1$ for all i precisely when $\epsilon = 2$ (otherwise $EDE(\epsilon) = H_i$ for all i, which is impossible except for the trivial case $i = 1$). Thus $g_{i,\epsilon} = f_{i,2+\epsilon}$ or $g_{i,\epsilon} = f_{i,2-\epsilon}$. Since $g_{j,\epsilon} \to 1$ as $\epsilon \to \infty$ for the subgroup j with highest M_j, necessarily $g_{i,\epsilon} = f_{i,2+\epsilon}$, implying

$$EDE^{\dagger}(\epsilon) = L(M_i^{-1}, 2 + \epsilon).$$

By Theorem 2.5,

$$EDE^{\dagger}(\epsilon) = L(M_i^{-1}, 2+\epsilon) = L(M_i, -\epsilon)^{-1} = EDE(-\epsilon). \tag{9}$$

This equation is a generalization of Sterck's observation [19] that minimizing overall poverty level is equivalent to maximizing EDE wealth at $\epsilon = 2$, for this observation is simply a consequence of substituting $\epsilon = 0$ into Equation 9. In other words, we extend Sterck's pointwise equivalence at $\epsilon = 0$ into a structural equivalence valid for all $\epsilon \geq 0$. This is summarized as the Duality Principle below.

Duality Principle 5.1. Minimizing the EDE of a negative metric M at ϵ is equivalent to maximizing the EDE of its inverse metric at $\epsilon + 2$.

Because of Equation 9, everything discussed in Sections 3 and 4 can be appropriately carried over to negative metrics by replacing ϵ with $-\epsilon$.

EDE-factors for negative metrics

For negative metrics, the EDE-factors are

$$w_i^{\dagger}(\epsilon) := \left(\sum_j \left(\frac{M_i}{M_j} \right)^{-\epsilon} \mu_j \right)^{-1} \mu_i = \left(\frac{M_i}{EDE(-\epsilon+1)} \right)^{\epsilon} \mu_i.$$

The three main properties still hold (wealth homogeneity, population homogeneity, Pigou-Dalton Principle). In fact, our functional-analytic discussion in Section 2 is actually a broadening of the Pigou-Dalton Principle in this case.

Corollary 5.2. *Theorem 2.1 implies the Pigou-Dalton Principle for negative metrics at all $\epsilon \geq 0$.*

Proof. This is immediate by applying Proposition 3.3 to Equation 9. □

The non-monotonic property of EDE-factors works the opposite way for negative metrics: $w_i^{\dagger}(\epsilon)$ is strictly non-increasing as ϵ increases precisely when

$$\ln M_i \leq \sum_{j \neq i} \ln M_j^{\mu_j}.$$

In other words, at lower levels of ϵ the Atkinson index may increasingly weigh a subgroup i with negative metric M_i if M_i greater than the relative geometric mean of (the negative metrics of) the other subgroups.

Resource allocation for negative metrics

Let M be a negative metric. Typically, negative metrics are rates or probabilities (such as poverty level), and resource allocation problems seek to optimally allocate an amount of new resources in order to lower M. This is an important area of research in cost-effectiveness analysis, and while EDE-factors cannot globally solve the issue of resource allocation to minimize M, it can offer a measurement on how far a proposed allocation strategy is from a hypothetical scenario where both M and resources can be traded to achieve the minimal EDE M as ϵ varies.

We outline the modifications required to apply techniques in Section 4. Let M_i and p_i be the respective negative metric and production function allocated to subgroup i. For the amount of resources allocated to subgroups i, we use the formula

$$r_i = \min\{m_i, s_i^2\}$$

where m_i is the maximum capacity of resource the subgroup can accept, and s_i^2 is the (non-negative) amount of resources allocated if below maximum capacity. Let R be the total amount of new resource to be allocated. If M_i is transformed to $M_i^o = M_i - p_i r_i$ after resource reallocation, and $w_i^o(\epsilon) = w_{i,\epsilon}^o$ is the respective EDE-factor after resource allocation, then

$$M_i^o = \left(-R + \sum_j \frac{M_j}{p_j}\right)\left(\sum_j \frac{1}{p_j}\left(\frac{(w_{i,\epsilon}^o)^{-1}\mu_i}{(w_{j,\epsilon}^o)^{-1}\mu_j}\right)^{\frac{1}{\epsilon}}\right)^{-1}.$$

Consider a hypothetical scenario, where the minimal of

$$EDE^o(\epsilon) = \left(\sum_i (M_i - p_i r_i)^{1+\epsilon}\mu_i\right)^{\frac{1}{1+\epsilon}}$$

is attained subject to the condition

$$\sum_i r_i = R.$$

If $\epsilon = 0$, the solution is clear: allocative maximal resources to the subgroups with highest weighted height production functions $p_i\mu_i$. If $\epsilon > 0$, Lagrange Multipliers can be applied to compute the associated resource allocation r_i. If the reallocation satisfies $0 < r_i < m_i$ for all i, then minimality can be achieved when $M_i = \widetilde{M_{i,\epsilon}}$, where

$$\widetilde{M_{i,\epsilon}} := \left(-R + \sum_j \frac{M_j}{p_j}\right)\left(\sum_j \frac{1}{p_j}\left(\frac{p_i\mu_i}{p_j\mu_j}\right)^{\frac{1}{\epsilon}}\right)^{-1}.$$

If $r_i < 0$ or $m_i > 0$ for some i, then we will need to consider cases where $r_i = 0$ or $r_i = m_i$ and separately perform Lagrange Multipliers to obtain the resource allocation minimizing $EDE^o(\epsilon)$. This results in potentially 3^n different cases to consider, where n is the number of subgroups. However, many cases can be immediately discarded by considering the total number of available resources R (e.g. it is usually not feasible to have $r_i = m_i$ for all i unless resources are abundant). A computational example of resource allocation for negative metrics is presented in the following Section.

6 Example: EDE for potentially avoidable mortality

Ontario, Canada, has the only published province-level elicitation of health-based ϵ in North America, and possibly around the world as far as we know, as of writing this paper. Results of the elicitation [5] show that health-based ϵ in Ontario is bimodally distributed, with two classes of nearly equal size (50.7% and 49.3%) having mean values $\epsilon = 0.339$ and $\epsilon = 7.304$ respectively. Additionally, there are no statistically significant socio-economic or demographic differences between these two classes.

We use our tools developed in Sections 2–5 on data from Public Health Ontario [11]. The dataset considered is stratified by 7 geographical regions, with the health outcome as age-standardized mortality rate from avoidable causes in 2023 (latest available year as of writing this paper). In the remainder of this section, "mortality rate" will be an abbreviation of "age-standardized mortality rate from avoidable causes".

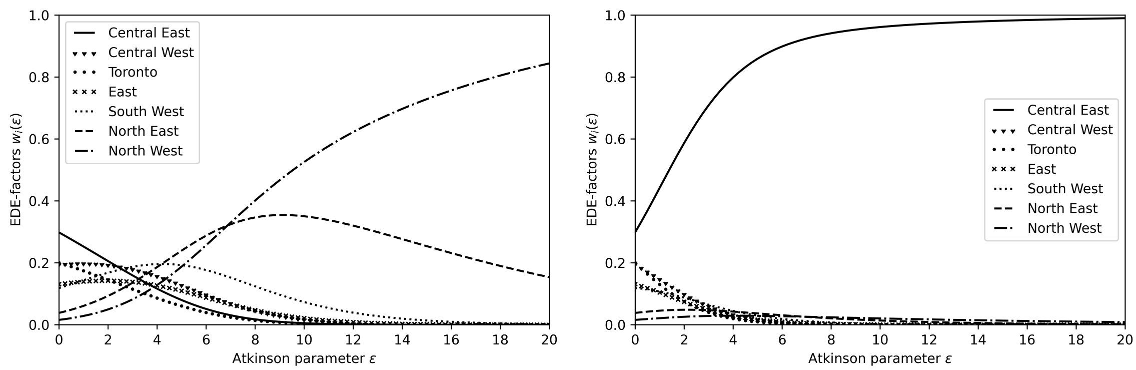
As an illustration, consider a highly effective hypothetical intervention that prevents mortality rate over a one-year period by 80%. Such high effectiveness have been observed for prevention interventions in high-income settings – for instance, in the context of HIV, injectable cabotegravir can be 94% effective [4] with 85% retention and adherence [17], resulting in a plausible real-world effectiveness of approximately 80%. Assume a scale-up of our hypothetical intervention to a coverage level of 90%, which coincides with the measured universal health coverage of Canada in the Global Burden of Disease study [9]. We consider three scenarios: "baseline", where mortality is as listed in [11]; "burden-preference", where our hypothetical intervention is rolled out instantly at 90% coverage with a preference to regions with higher mortality rate; and "uniform", where our hypothetical intervention is rolled out instantly at 90% coverage to all 7 geographical regions. For simplicity, we assume the health production function p_i is the product of intervention efficiency $e = 0.8$, region mortality rate M_i, and the inverse population $1/n_i$:

$$p_i = \frac{eM_i}{n_i}.$$

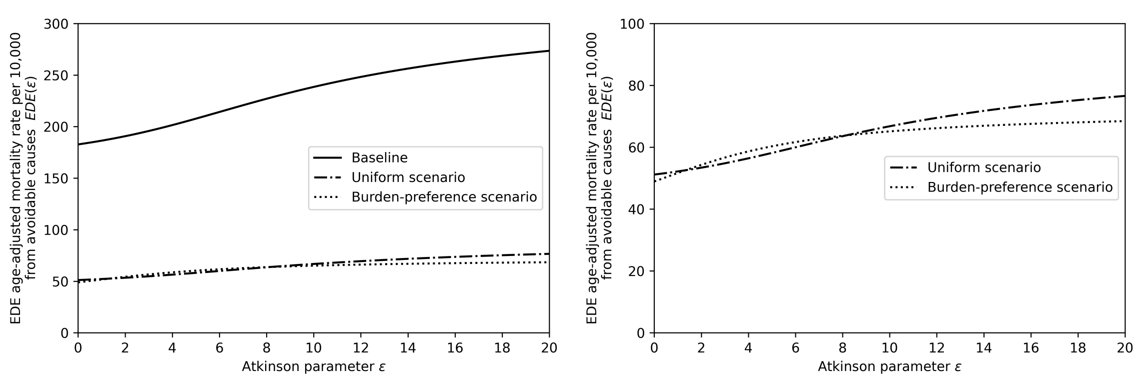
Writing total resources as $R = 0.9n$, where n is the total population of Ontario, this production function allows us to capture the change in death rate by reallocating r_i resources to region i via $M_i - p_i r_i$, with $r_i \leq n_i$ as the upper bound. A summary of the three scenarios is listed in the following table.

Age-standardized mortality rate per 10,000 from avoidable causes, Ontario (2023)				
Region	Baseline scenario	Uniform scenario	Burden-preference scenario	Population
North West	331.1	92.7	66.2	224,481
North East	290.8	81.4	58.2	548,020
South West	222.1	62.2	44.4	1,696,228
East	194.3	54.4	38.9	1,879,714
Central West	185.6	52.0	37.1	2,841,315
Toronto	158.6	44.4	31.7	2,875,513
Central East	154.8	43.3	72.5	4,277,775

By Equation 9, the EDE mortality rate at ϵ can be calculated by substituting $-\epsilon$ into the Atkinson index (Equation 1). Also let μ_i be the original weights considered by Atkinson (i.e. μ_i is the proportion of population in subgroup i). The weighted mortality rate among the three scenarios are: 182.8 (baseline); 51.2 (uniform); 48.9 (burden-preference). A plot of EDE-factors for baseline and uniform scenarios (left), and burden-preference scenario (right), is given below. Note that the EDE-factors for baseline and uniform scenarios are the same due to homogeneity (Definition 3.1).

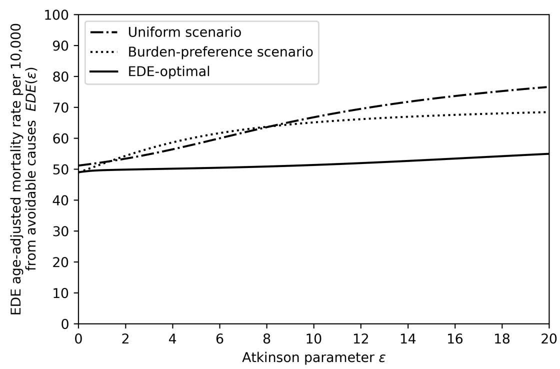


One immediately notices two phenomena. Firstly, a disproportionate amount of weight is placed on the Central East region for the burden-preference scenario at $\epsilon < 3$. This implies a faster increase in EDE mortality rate for the burden-preference scenario compared to the uniform scenario at low ϵ, and a tipping point should exist where EDE mortality rate for the burden-preference scenario rises above that for the uniform scenario as ϵ increases. Secondly, EDE-factors for the burden-preference scenario is heavily weighted by the Central East region at $\epsilon > 6$, compared to two dominant regions (North West and North East) for the uniform scenario. This implies EDE mortality for the burden-preference scenario has a much flatter slope at $\epsilon > 6$ compared to the uniform scenario, suggesting another tipping point in this interval. Indeed, this is the case as a plot of EDE mortality shows.



The first tipping point is at $\epsilon_1 \approx 1.328$ with $EDE(\epsilon_1) \approx 52.6$, and the second tipping point is at $\epsilon_2 \approx 8.149$ with $EDE(\epsilon_2) \approx 63.8$. Considering the bimodally distributed ϵ in Ontario, the population with mean $\epsilon = 0.339$ would prefer burden-preference to uniform scenarios (EDE mortality rate: 49.9 for burden-preference; 51.5 for uniform), while the population with mean $\epsilon = 7.304$ would prefer uniform to burden-preference scenarios (EDE mortality rate: 63.1 for burden-preference; 62.4 for uniform).

Following the discussion on resource allocation for negative metrics in Section 5, let us briefly compare the three scenarios considered above with "EDE-optimal", where resources are allocated to maximize EDE mortality rate at ϵ.



At $\epsilon = 0$, the burden-preference scenario achieves EDE-optimal. At $\epsilon = 0.339$, the burden-preference scenario is still closer to EDE-optimal than the uniform scenario, and to achieve EDE-optimal we should allocate maximal resources to the non-Toronto regions and allocate remaining resources to Toronto. At $\epsilon = 7.304$, the uniform scenario is closer to EDE-optimal than the burden-preference scenario, and to achieve EDE-optimal we should allocate maximal resources to all but Toronto and Central East, and distribute remaining resources in the ratio of 58.585% to 41.415% among Central East and Toronto, such that 16.5% of the Central East population and 23.4% of the Toronto population receives the hypothetical intervention.

7 Concluding remarks

This paper demonstrated a new decomposition of the Atkinson index by way of EDE-factors (Equation 3.1). Although many kinds of wealth inequality measures exist [3], we chose to focus on the Atkinson index for applicability in both health and economics.

In health, studies have shown the Atkinson index may be the most appropriate index for inequality analyses [8] as it allows for many different interpretations of subgroup decomposability, satisfies the Pigou-Dalton Principle, and avoid value judgement. In economics, Shorrocks [14, 15] gave mathematical justification for the generalized entropy index to be the family of inequality measures for our purposes, and this index can be viewed as a monotonic transformation of the Atkinson index.

The technical aspect of our paper contributes a novel way to decompose the Atkinson index as we used an approach via a L-function viewpoint borrowed from number theory. Our main objectives were to seek a general principle behind Sterck's observed duality between income at $\epsilon = 2$ and poverty at $\epsilon = 0$, as well as a broadening of the Pigou-Dalton Principle. An L-function viewpoint is essential to obtain our duality principle as it is mathematically expressed via a functional equation (Theorem 2.5). This duality is also hinted at in current working papers on decomposition of measures [10, 19]. As for a broadening of the Pigou-Dalton Principle, our result (Proposition 5.2; Corollary 5.2) does not require any reference on the direction of wealth allocation, and is also used to define the EDE-factors.

The empirical aspect of our paper applies EDE-factors to resource allocation. We additionally considered a case study of age-adjusted mortality rate from avoidable causes in Ontario, Canada to illustrate our mathematical methods. This case study is interesting as elicitation of health-based ϵ in [4] tells us that the Ontario population is bimodally distributed, with one population class strongly inequality neutral while the other class has high inequality aversion. Our case study, on resource allocation of a hypothetical intervention, shows that the two different classes may support opposing public health policies.

We believe the framework developed in this paper may be generalized to more classes of wealth inequality measures satisfying the three conditions listed at the start of this paper (subgroup decomposability; Pigou-Dalton principle; avoids value judgement). In particular, for inequality metrics that avoid value judgement by introducing an explicit parameter ϵ, a functional equation (dependent on ϵ) for a number-theoretic-like L-function arising from subgroup decomposability should imply a duality principle, while an analog of the Riesz-Fréchet Representation Theorem should broaden on the Pigou-Dalton Principle.

8 Acknowledgements

The author would like to thank R. Scott Braithwaite for his countless insights in inequality measures, and Nao Yamamoto for the numerous helpful criticisms in a preliminary draft of this work.

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