The Satake isomorphism (2017-08-01)

The goal is to understand the Satake isomorphism, following the article by Cartier in the Corvallis volumes. We have the usual setup:

- F is a (nonarchimedean) local field.
- G is a connected reductive group over F.
- A is a maximal split in G.
- M is the centralizer of A in G.
- $N(A)$ is the normalizer of A in G.
- $W = N(A)/M$ is the Weyl group.
- P is a Borel subgroup, so $P = MN$ with N the unipotent radical.
- K is a "special" maximal compact of G, so it satisfies things like the Iwasawa decomposition $G = PK$ ($= MNK$).
- $X^*(H) = \mathrm{Hom}_{k\text{-alg}}(H, \mathbb{G}_m)$ is the character group of $H \subset G$.
- $X_*(H) = \mathrm{Hom}_{\mathbb{Z}}(X^*(H), \mathbb{Z})$ is the cocharacter group of $H \subset G$.

We write ${}^\circ M = M \cap K$. Here is another characterization. Define $\mathrm{ord}_M : M \to X_*(M)$ by $\mathrm{ord}_M(m)(f) = \mathrm{ord}_F f(m)$. Then ${}^\circ M$ is the kernel of ord_M, i.e.

$$1 \to {}^\circ M \longrightarrow M \xrightarrow{\mathrm{ord}_M} \Lambda =: \mathrm{ord}_M(M) \longrightarrow 1.$$

$\boxed{\text{What we want}}$ To understand the Hecke algebra $\mathcal{H}(G,K)$.

↳ Element of $\mathcal{H}(G,K)$ are compactly supported functions $f : G \to \mathbb{C}$ that are K-biinvariant, with pointwise addition and convolution as multiplication: $f * g(m) = \int_{..} f(x)\, g(x^{-1}m)\, dm$.

Define $\mathcal{H}(M, {}^\circ M)$ similarly. Notice that a $\mathbb{C}$-basis for $\mathcal{H}(M, {}^\circ M)$ is the set $\{ch_\lambda : \lambda \in \Lambda\}$, where ch_λ is the characteristic function of $\mathrm{ord}_M^{-1}(\lambda)$ in M.

<u>Easy to check</u>: $ch_\lambda * ch_{\lambda'} = ch_{\lambda+\lambda'}$ if we choose dm so that $\int_{{}^\circ M} dm = 1$.

↳ The idea is the usual $\int_M \cdots = \int_{{}^\circ M} \int_{{}^\circ M \backslash M} \cdots$.

<u>Corollary</u>: $\mathcal{H}(M, {}^\circ M) = \mathbb{C}[\Lambda]$, where $\mathbb{C}[\Lambda]$ is the algebra generated by $\{e^\lambda : \lambda \in \Lambda\}$.

Let now $\mathfrak{n} = \mathrm{Lie}(N)$, and define, for $m \in M$, $\delta(m) := |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F$. Note that

$$\int_N f(mnm^{-1})\,dn = \delta(m)^{-1} \int_N f(n)\,dn. \quad \text{—— } (*)$$

<u>Def</u>: The <u>Satake map</u> $S : \mathcal{H}(G, K) \longrightarrow \mathcal{H}(M, {}^\circ M) = \mathbb{C}[\Lambda]$ is defined by

$$Sf(m) = \delta(m)^{\frac{1}{2}} \int_N f(mn)\,dn \overset{(*)}{=} \delta(m)^{-\frac{1}{2}} \int_N f(nm)\,dn.$$

Clearly Sf is ${}^\circ M$-biinvariant and compactly supported.

<u>Thm (Satake)</u>: S is an algebra isomorphism onto $\mathbb{C}[\Lambda]^W$, so $\mathcal{H}(G,K)$ is abelian.

↳ What is the action of W on Λ? Notice that we have the diagram

$$\begin{tikzcd}
1 \arrow[r] & {}^\circ A \arrow[r] \arrow[d, hook] & A \arrow[r, two heads, "\mathrm{ord}_A"] \arrow[d, hook] & X_*(A) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & {}^\circ M \arrow[r] & M \arrow[r, "\mathrm{ord}_M"] & X_*(M) \arrow[r] & 1
\end{tikzcd}$$

with $X_*(A) \subset \Lambda \subset X_*(M)$. Thus $N(A)$ acts on $M, {}^\circ M, A, {}^\circ A$ via conjugation, and W acts on $X_*(M)$ with M as an invariant subspace.

Over there A is already k-split so $M = A$.

<u>Remark</u>. In Buzzard-Gee the required isomorphism is $\mathcal{H}(G, K) = \mathbb{C}[X_*(T_d)]^W$

Remark. The Satake isomorphism is the p-adic version of the Harish-Chandra isomorphism $Z(\mathfrak{g}) \to (S\mathfrak{h})^W$ for G a complex semisimple Lie group.

We now sketch the proof of the above theorem.

Step 1. Check that S is a homomorphism by observing that it is actually the composition of three algebra maps

$$\mathcal{H}(G,K) \xrightarrow{\alpha} \mathcal{H}(P) \xrightarrow{\beta} \mathcal{H}(M) \xrightarrow{\gamma} \mathcal{H}(M)$$

where α is just restriction

β is given by $\beta u(m) = \int_N u(mn)\, dn$

γ is given by $\gamma f(m) = f(m)\, \delta(m)^{\frac{1}{2}}$.

Step 2. Check that $S(\mathcal{H}(G,K))$ is contained in $\mathbb{C}[\Lambda]^W$.

↳ Use fact: $Sf(m) = \Delta(m)\, \delta(m)^{-\frac{1}{2}} \int_{G/A} f(gmg^{-1})\, dm$.

Another property of K is that $W = (N(A) \cap K)/{}^{\circ}M$. Hence we need to show that $Sf(xmx^{-1}) = Sf(m)$ for $m \in M$ and $x \in N(A) \cap K$. In fact it suffices to show it for element $m \in M$ such that $m \mapsto \det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)$ is (polynomial) nonzero.

$\boxed{\text{Invariance for } D(m)}$ In fact $D(xmx^{-1}) = D(m)$ for $x \in N(A)$. This follows because

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{n}^- \text{ so}$$

$$D(m)^2 = |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)|_F^2 \cdot |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F^{-1}$$

$$= |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)_F|\, |\det(\mathrm{Ad}_{\mathfrak{n}}(m^{-1}) - 1|_F = |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)|_F\, |\det(\mathrm{Ad}_{\mathfrak{n}^-}(m) - 1|_F$$

Thus $D(x) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{m}}(m) - 1)|_F^{\frac{1}{2}}$, so $D(xmx^{-1}) = D(m)$.

$\boxed{\text{Invariance for integral}}$ Note that $N(A) \cap K$ acts by inner automorphisms on G and A; so leaves invariant the measure on G/A. Let $m \in M$ be regular, $x \in N(A) \cap K$, and $f \in \mathcal{H}(G,K)$. Thus $f(xgx^{-1}) = f(g)$ for any $g \in G$, and

$$\int_{G/A} f(g(xmx^{-1})g^{-1})\, d\bar{g} = \int_{G/A} f((x^{-1}gx)m(x^{-1}gx)^{-1})\, d\bar{g} = \int_{G/A} f(gmg^{-1})\, d\bar{g}.$$

Step 3. Check that $S(\mathcal{H}(G,K)) \xrightarrow{\cong} \mathbb{C}[\Lambda]^W$.

↳ The idea is to find another basis of $\mathcal{H}(G,K)$ using the Cartan decomposition and show that the image of the basis is "upper triangular" with respect to the original basis ch_λ of $\mathbb{C}[\Lambda]$.

Here is an application. Let us try to determine all unitary algebra homomorphisms $\mathcal{H}(G,K) \to \mathbb{C}$. We do this by looking at the ones for $\mathcal{H}(M, {}^\circ M)$ and then passing over to $\mathcal{H}(G,K)$ via the Satake isomorphism. Since $\int_{{}^\circ M} dm = 1$ by assumption, the map $f \longmapsto \int_M f(m)\chi(m)\, dm$ is a unitary homomorphism for all unramified $\chi: M \to \mathbb{C}^\times$, and all such unitary maps arises in this way.

Corollary: Any unitary $\mathcal{H}(G,K) \to \mathbb{C}^\times$ is of the form $\omega_\chi(f) = \int_M Sf(m)\chi(m)\, dm$ for an unramified χ. Moreover, $\omega_\chi = \omega_{\chi'}$ iff $\chi' = w \cdot \chi$ for some $w \in W$.

<u>Definition</u>: A <u>spherical function</u> of G with repect to K is a function

$\Gamma : G \to \mathbb{C}$ that is K-invariant, with $\Gamma(1) = 1$, and such that

for any $f \in \mathcal{H}(G,K)$ there is a constant $\lambda(f)$ with $f * \Gamma = \Gamma * f = \lambda(f)\,\Gamma$.

Our goal now is to translate our results in terms of this language. Let χ be an unramified character of M, and define

$$\Phi_{K,\chi}(\underbrace{mnk}^{G=MNK}) = \chi(m)\,\delta^{\frac{1}{2}}(m) \quad \text{for } m \in M,\ n \in N,\ k \in K.$$

$$\Gamma_\chi(g) = \int_K \Phi_{K,\chi}(kg)\,dk, \quad \text{for } g \in G.$$

<u>Thm</u>: (a) The spherical functions are the functions Γ_χ.

(b) $\Gamma_\chi = \Gamma_{\chi'} \iff \chi = w\chi'$ (some $w \in W$).

↳ Computation for (b) in the notes.

↳ Spherical functions comes up in the Hecke theory for irreducible admissible representations and automorphic forms for GL_n, say.

(Godement's notes as a reference).

The last condition for spherical function is that it should be like an eigenfunction for the Hecke operator on $L^2_0(G_k \backslash G_{\mathbb{A}}, \omega)$.

1. Constructing quaternion and dihedral extensions by class field theory.

This problem has to do with constructing degree 8 quaternion and dihedral extensions using class field theory.

1. Suppose H is a subgroup of finite index in a group G. The transfer homomorphism

$$\mathrm{Ver}_G^H : G^{ab} \to H^{ab}$$

between the maximal abelian quotients of G and H is defined in the following way. Let T be a set of representatives for the right cosets of H in G, so that $H\backslash G = \{Ht : t \in T\}$. If $g \in G$ and $t \in T$, then $tg = h_{g,t}t'$ for some $t' \in T$ and $h_{g,t} \in H$. Define

$$\mathrm{Ver}_G^H(\overline{g}) = \overline{h} \quad \text{when} \quad h = \prod_{t \in T} h_{g,t}$$

where $\overline{g}$ (resp. $\overline{h}$) is the image of g in G^{ab} (resp. the image of h in H^{ab}). Show that if H is cyclic of order 8 and G is a dihedral (resp. quaternion) group of order 8, then Ver_G^H is trivial if G is dihedral, and otherwise Ver_G^H is the unique non-trivial homomorphism which has kernel the image of H in G^{ab}.

2. Let L/K be a finite extension of global fields. Define $C_K = J_K/K^*$ to be the idele class group of K. Let K^{ab} be the maximal abelian extension of K in some algebraic closure containining L. Two basic properties of the Artin map $\Psi_K : C_K \to \mathrm{Gal}(K^{ab}/K)$ are that the two following two diagrams commute:

(1.1)
$$\begin{tikzcd} C_L \arrow[r, "\Psi_L"] \arrow[d, "Norm_{L/K}"'] & \mathrm{Gal}(L^{ab}/L) \arrow[d, "\mathrm{res}_{L^{ab}/K^{ab}}"] \\ C_K \arrow[r, "\Psi_L"] & \mathrm{Gal}(K^{ab}/K) \end{tikzcd}$$

(1.2)
$$\begin{tikzcd} C_K \arrow[r, "\Psi_L"] \arrow[d, "i_{K/L}"'] & \mathrm{Gal}(K^{ab}/K) \arrow[d, "\mathrm{Ver}_{L/K}"] \\ C_L \arrow[r, "\Psi_K"] & \mathrm{Gal}(L^{ab}/L) \end{tikzcd}$$

in which $\mathrm{res}_{L^{ab}/K^{ab}}$ is induced by restriction, $i_{K/L}$ is induced by the inclusion of K into L and $\mathrm{Ver}_{L/K}$ is the transfer map.

Use this to show that all dihedral and quaternion extensions of K arise from the following construction. Let L/K be a quadratic separable extension, and let $\epsilon_L : C_K \to \{\pm 1\}$ be the unique surjective homomorphism corresponding to L via class field theory. Write $\mathrm{Gal}(L/K) = \{e, \sigma\}$, with σ of order 2. Let $\mu_4 = \{\pm 1, \pm\sqrt{-1}\}$ be the group of fourth roots of unity in $\mathbb{C}^*$. A surjective homomorphism $\chi : C_L \to \mu_4$ is of dihedral (resp. quaternion) type if:

a. $\chi^\sigma = \chi^{-1}$ when $\chi^\sigma : C_L \to \mu_4$ is defined by $\chi^\sigma(j) = \chi(\sigma(j))$ for $j \in C_L$

b. The restriction $\chi|_{C_K}$ of χ to C_K via the map $C_K \to C_L$ induced by including K into L is trivial (in the dihedral case) or the character ϵ_L (in the quaternion case).

Let N be the extension of L which corresponds to the kernel of χ via class field theory over L. Show that N/K is a dihedral (resp. quaternion) extension of degree 8 if χ is of dihedral (resp. quaternion) type, and that all such extensions arise from this construction as L ranges over the quadratic Galois extensions of K. Which pairs (L, χ) give rise to the same N?

3. The character $\chi : C_L = J_L/L^* \to \mu_4$ then has local components $\chi_v : L_v^* \to \mu_4$ for each place v of L defined by $\chi_v(j_v) = \chi(\iota_v(j_v))$ when $\iota_v : L_v^* \to C_L$ results from the inclusion of L_v into J_L at the place v followed by the projection $J_L \to C_L/L^*$.

a. Suppose K is a number field and that K and L have class number 1. Show that there are exact sequences

$$1 \to O_L^* \to \prod_v O_v^* \to C_L \to 1 \quad \text{and} \quad 1 \to O_K^* \to \prod_w O_w^* \to C_K \to 1 \tag{1.3}$$

where v and w range over all places of L and K, respectively, including the archimedean places. Conclude from this that to specify a finite order continuous homomorphism $\chi : C_L \to \mathbf{C}^*$ it is necessary and sufficient to specify continuous local characters $\chi_v' : O_v^* \to \mathbf{C}^*$ which are trivial for almost all v such that $\prod_v \chi_v'$ vanishes on O_L^*.

b. With the notations of problem (3a), what conditions on the restrictions χ_v' are equivalent to χ being of dihedral or quaternion type? (Note that by the same reasoning, the character $\epsilon : C_K \to \{\pm 1\}$ is determined by its restrictions to the multiplicative groups O_w^* of all places w of K, and that each such O_w^* embeds naturally into the product of the O_v^* associated to v over w in L.)

c. Suppose $K = \mathbf{Q}$ and $L = \mathbf{Q}(\sqrt{5})$. Show that there is a quaternion character $\chi : C_L \to \mu_4$ such that the $\chi_v' = \chi_v | O_v^*$ have the following properties. The character χ_v' is trivial unless v is the unique place v_5 over 5 or one of the two first degree places v_{41} and v_{41}' over 41. The order of χ_v' is 2 if $v = v_5$ and 4 if $v = v_{41}$ or $v = v_{41}'$. Finally, when we use the natural inclusion $K = \mathbf{Q} \to L$ to identify both $O_{v_{41}}$ and $O_{v_{41}'}$ with $\mathbf{Z}_{41}$, the characters $\chi_{v_{41}}'$ and $\chi_{v_{41}'}'$ are inverses of each other when we view them both as characters of $\mathbf{Z}_{41}^*$.

Need to add condition that

if v real then $\chi_v' : \mathbb{R}^\times \to \mathbb{C}^\times$ is of order 1 or 2,

if v complex then $\chi_v' : \mathbb{C}^\times \to \mathbb{C}^\times$ is trivial.

Either that, or delete the "finite order" condition.

1. Computation.

$\boxed{G = D_8}$

$$G = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

$$G^{ab} = \{1, \bar{r}, \bar{s}, \overline{sr}\} = \mathbb{Z}/2 \times \mathbb{Z}/2$$

$$H = \{1, r, r^2, r^3\}$$

$$T = \{1, s\}$$

$$H \backslash G = \{H, Hs\}$$

$$\begin{array}{lll@{\qquad\qquad}lll}
1 \cdot 1 & = & 1 \cdot 1 & s \cdot 1 & = & 1 \cdot s \\
1 \cdot r & = & r \cdot 1 & s \cdot r & = & r^{-1} \cdot s \\
1 \cdot s & = & 1 \cdot s & s \cdot s & = & 1 \cdot s \\
1 \cdot sr & = & r^{-1} \cdot s & s \cdot sr & = & r \cdot 1
\end{array}$$

$\Rightarrow$ Ver_G^H is trivial.

$\boxed{G = Q_8}$

$$G = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$G^{ab} = \{1, \bar{i}, \bar{j}, \bar{k}\} = \mathbb{Z}/2 \times \mathbb{Z}/2$$

$$H = \{\pm 1, \pm i\} \quad \text{(the other choices are symmetrical)}$$

$$T = \{1, j\}$$

$$H \backslash G = \{H, Hj\}$$

$$\begin{array}{lll@{\qquad\qquad}lll}
1 \cdot 1 & = & 1 \cdot 1 & j \cdot 1 & = & 1 \cdot j \\
1 \cdot i & = & i \cdot 1 & j \cdot i & = & -i \cdot j \\
1 \cdot j & = & 1 \cdot j & j \cdot j & = & -1 \cdot 1 \\
1 \cdot k & = & i \cdot j & j \cdot k & = & i \cdot 1
\end{array}$$

$\Rightarrow$ Ver_G^H is nontrivial with image $\{\pm 1\}$ and kernel $\mathrm{im}(H \hookrightarrow G \twoheadrightarrow G^{ab})$.

2. Let $\chi : C_L \to \mu_4$ be as in the problem. We know that N/K is a degree 8 extension. If one can show that $\mathrm{Gal}(L/K)$ acts nontrivially on $\mathrm{Gal}(N/L)$, then the classification of groups of order 8 will imply that N/K is either of dihedral or quaternionic type.

Note that, by definition of the σ-action on C_L $\left(\sigma(x_v)_v := (\sigma x_{\sigma^{-1}v})_v\right)$ one has a commutative diagram

$$\begin{tikzcd}
C_L \arrow[r, two heads] \arrow[d, "\sigma"'] \arrow[dd, bend right=60, "\chi^{-1}"', "\text{(condition a)}"'] & C_L/\ker\chi \arrow[r, "\mathrm{Art}_L", "\cong"'] & \mathrm{Gal}(N/L) \arrow[d, "\sigma(-)\sigma^{-1}"] \\
C_L \arrow[r, two heads] \arrow[d, "\chi"'] & C_L/\ker\chi \arrow[r, "\mathrm{Art}_L", "\cong"'] \arrow[dl, "\cong"] & \mathrm{Gal}(N/L) \\
\mu_4 & &
\end{tikzcd}$$

Since $\chi \neq \chi^{-1}$, this means $\mathrm{Gal}(L/K)$ acts nontrivially on $\mathrm{Gal}(N/L)$, as desired. Accordingly, $\mathrm{Gal}(N^{ab}/K) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ by problem 1. Let us now consider

$$\begin{tikzcd}
C_K \arrow[r, two heads] \arrow[d] & C_K/N_{N/K}(C_N) \arrow[r, "\mathrm{Art}_K", "\cong"'] & \mathrm{Gal}(N^{ab}/K) \arrow[d, "\mathrm{Ver}_{L/K}"] \\
C_L \arrow[r, two heads] \arrow[d] & C_L/\ker\chi \arrow[r, "\mathrm{Art}_L", "\cong"'] \arrow[dl, "\cong"] & \mathrm{Gal}(N/L) \\
\mu_4 & &
\end{tikzcd}$$

We now use condition b.

- If χ is of dihedral type, then the leftmost composition is trivial, implying $Ver_{L/K}$ is trivial and L/K is dihedral by problem 1.
- Similarly if χ is of quaternionic type then L/K is quaternionic.

Conversely suppose we are given N/K of dihedral or quaternionic type of degree 8. Then one simply let L/K be any quadratic extension in N, and consider $\chi : C_L \twoheadrightarrow C_L / N_{N/L}(C_N) \cong \mu_4$ (which satisfies conditions a and b).

Now suppose (L, χ) and (L', χ') give rise to the same N.

- If N/K is dihedral, then there is only one subgroup of order 4, so necessarily $L = L'$. We then need $\ker \chi = \ker \chi'$ to induce the same N by the existence theorem.
- If N/K is quaternionic, we have two cases. If $L = L'$ then one again require $\ker \chi = \ker \chi'$. Now suppose $L \neq L'$. Then LL' is of degree 4 over K, and $Gal(N/LL')$ corresponds to the unique subgroup $\{\pm 1\}$ of Q_8 of order 2. Since $Gal(N/LL')$ is of index two in both $Gal(N/L)$ and $Gal(N/L')$, the requirement is that $\ker \chi$ and $\ker \chi'$ contains a common index two subgroup (after letting L and L' be contained in the same algebraic closure).

3. (a) As L and K has class number 1,

$$1 = Cl(L) = \mathbb{A}_L^\times / L^\times \prod_v \mathcal{O}_v^\times \Rightarrow \mathbb{A}_L^\times = L^\times \prod_v \mathcal{O}_v^\times .$$

Thus by isomorphism theorems

$$C_L = \mathbb{A}_L^\times / L^\times = \prod_v \mathcal{O}_v^\times / L^\times \cap \prod_v \mathcal{O}_v^\times = \prod_v \mathcal{O}_v^\times / \mathcal{O}_L^\times .$$

Therefore there is an exact sequence

$$1 \to \mathcal{O}_L^\times \to \prod_v \mathcal{O}_v^\times \to C_L \to 1 .$$

If we have local continuous homomorphisms $\chi_v' : \mathcal{O}_v^\times \to \mathbb{C}^\times$ with almost all χ_v' trivial and $\prod_v \chi_v'$ vanishing on $\mathcal{O}_L^\times$, then we get a homomorphism $\chi : C_L \to \mathbb{C}^\times$ that is still continuous by the topology of $\mathbb{A}_L^\times$. Conversely, given $\chi : C_L \to \mathbb{C}^\times$ a continuous homomorphism, one gets $\chi_v' : \mathcal{O}_v^\times \to \mathbb{C}^\times$ with $\prod_v \chi_v^\times$ vanishing on $\mathcal{O}_L^\times$ (by above, as L has class number one). Almost all of the χ_v' are trivial: by picking a neighborhood U of $\mathbb{C}^\times$ containing $\{1\}$ as its unique subgroup, and considering an open subset $\prod_{\substack{i \in I \\ \text{finite}}} U_i \prod_{\substack{i \notin I \\ \text{infinite}}} \mathcal{O}_v^\times$ of $\chi^{-1}(U)$ containing the identity, one sees that the group $\prod_{i \in I} \{1\} \prod_{i \notin I} \mathcal{O}_v^\times$ must map to $\{1\}$ under χ.

Let us also show that χ is of finite order iff $\chi|_{(C_L)_1}$ is trivial, where

$(C_L)_1$ is the neutral component of $|\ |$. The 'only if' direction is clear. Now suppose $\chi|_{(C_L)_1}$ is trivial. Then by class field theory, and the fact that χ factors through $(C_L)_1$, one has

$$\begin{tikzcd}
C_L/(C_L)_1 \arrow[rr, "\chi"] \arrow[dr, "\mathrm{Art}"', "\wr"] & & \mathbb{C}^\times \\
& \mathrm{Gal}(L^{ab}/L) \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Just as before, as $\mathbb{C}^\times$ has no small subgroups, η is trivial on some $\mathrm{Gal}(L^{ab}/H)$, with H/L finite (abelian) Galois. Hence η factors through a finite index subgroup, implying χ is of finite order. ⌋

(b) $\boxed{\text{Dihedral case}}$ We require $(\chi_v')^\sigma = (\chi_v')^{-1}$, and, for w a place of K,

$$\chi_w' : O_w^\times \hookrightarrow \prod_{v|w} O_v^\times \xrightarrow{\chi} \mathbb{C}^\times$$

is trivial for all w. Also, almost all $O_v^\times \xrightarrow{\chi_v'} \mathbb{C}^\times$ need to be trivial, (and $\prod_v \chi_v'|_{O_L^\times}=1$) with the nontrivial ones of order dividing 4, and at least one of order 4.

$\boxed{\text{Quaternion case}}$ We require $(\chi_v')^\sigma = (\chi_v')^{-1}$, and almost all χ_v' trivial (and $\prod_v \chi_v'|_{O_L^\times}=1$) with the nontrivial ones of order dividing 4, and at least one of order 4.

The condition on χ_w' for w a place of K is as follow:

- ↳ if w is unramified in L then χ_w' is trivial,
- ↳ if w is ramified in L then χ_w' has order two.

These conditions follow from local class field theory.

(c) L/K is ramified only at the prime 5. Since we want χ_v' to be trivial at all archimedean places, the condition $\prod \chi_v'|_{\mathcal{O}_L^\times} = 1$ can be ignored (as χ factors through the finite ideles, and $L^\times \cap \left(\prod_{v\nmid\infty} \mathcal{O}_v^\times \times \begin{pmatrix}\text{neutral}\\ \text{component}\end{pmatrix}\right) = \{1\}$).

For the finite places, define

$$\chi_v' = \begin{cases} \text{trivial} & \text{if } v \nmid 5 \text{ and } v \nmid 41 \\ \mathcal{O}_v^\times \cong \mu_4 \times (1+\mathfrak{p}) \to \mathbb{C}^\times,\ (\zeta_4, -) \mapsto -1 & \text{if } v = v_5 \\ \mathcal{O}_v^\times \cong \mu_{40} \times (1+\mathfrak{p}') \to \mathbb{C}^\times,\ (\zeta_{40}, -) \mapsto \zeta_4^{\pm} & \text{if } v = v_{41} \text{ or } v_{41}'. \end{cases}$$

This certainly satisfies $(\chi_v')^\sigma = (\chi_v')^{-1}$ and $\chi = \prod_v \chi_v'$ of order 4. We just need to check that its restriction to K agrees with

$$\varepsilon_K : C_K \cong \prod \mathbb{Z}_p^\times \times \mathbb{R}_{>0}^\times \longrightarrow \mathbb{Z}_5^\times \to \{\pm 1\}$$
$$u \longmapsto \left(\frac{u \bmod 5}{5}\right)$$

But this is clear, for

$\chi_{41} : \mathbb{Z}_{41}^\times \hookrightarrow \mathcal{O}_{v_{41}}^\times \times \mathcal{O}_{v_{41}'}^\times \to \mathbb{C}^\times$ is trivial

$\chi_5 : \mathbb{Z}_5^\times \hookrightarrow \mathcal{O}_{v_5}^\times \to \mathbb{C}^\times$ agrees with the Legendre symbol ε_K

$\chi_p : \mathbb{Z}_p^\times \hookrightarrow \prod_{v|p} \mathcal{O}_v^\times \to \mathbb{C}^\times$ is certainly trivial for $p \nmid 5, 41$.