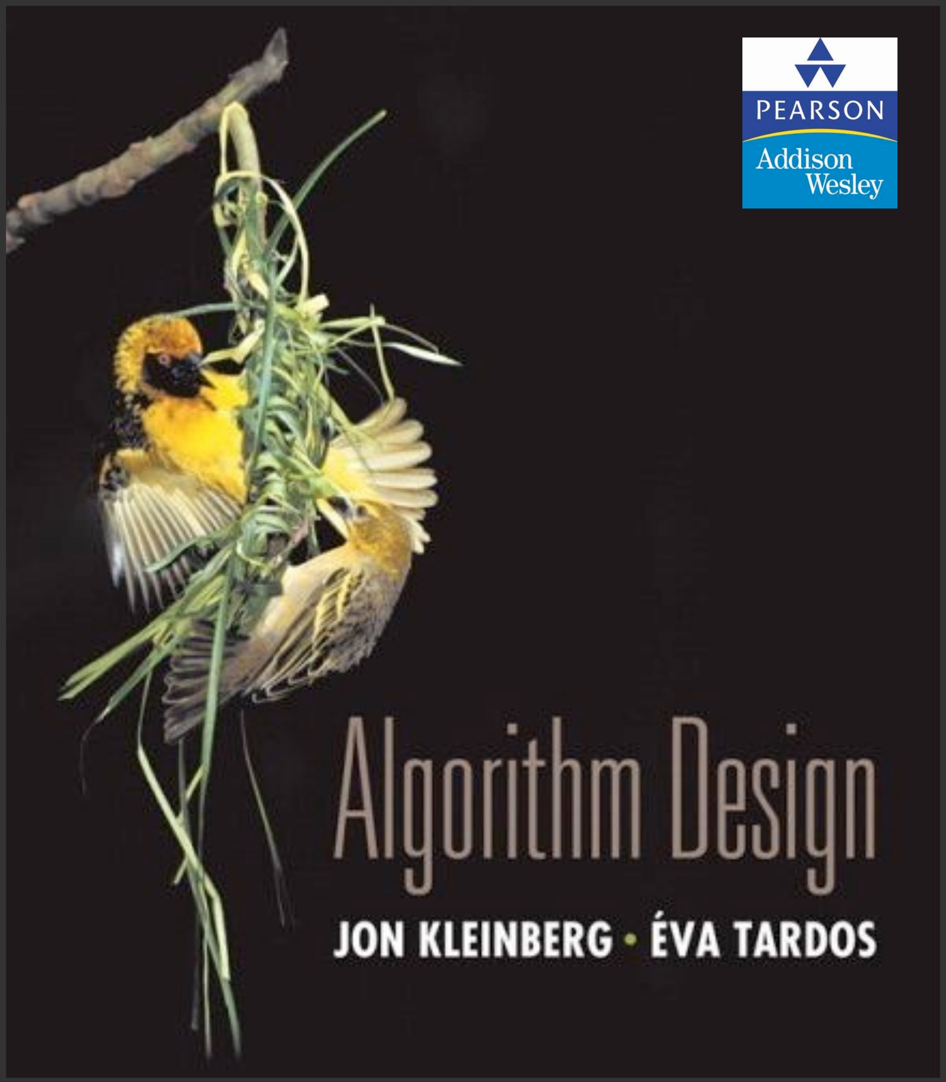


PRIORITY QUEUES

- *binary heaps*
- *d-ary heaps*
- *binomial heaps*
- *Fibonacci heaps*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 7/25/17 11:07 AM

Priority queue data type

A min-oriented priority queue supports the following core operations:

- MAKE-HEAP(): create an empty heap.
- INSERT(H, x): insert an element x into the heap.
- EXTRACT-MIN(H): remove and return an element with the smallest key.
- DECREASE-KEY(H, x, k): decrease the key of element x to k.

The following operations are also useful:

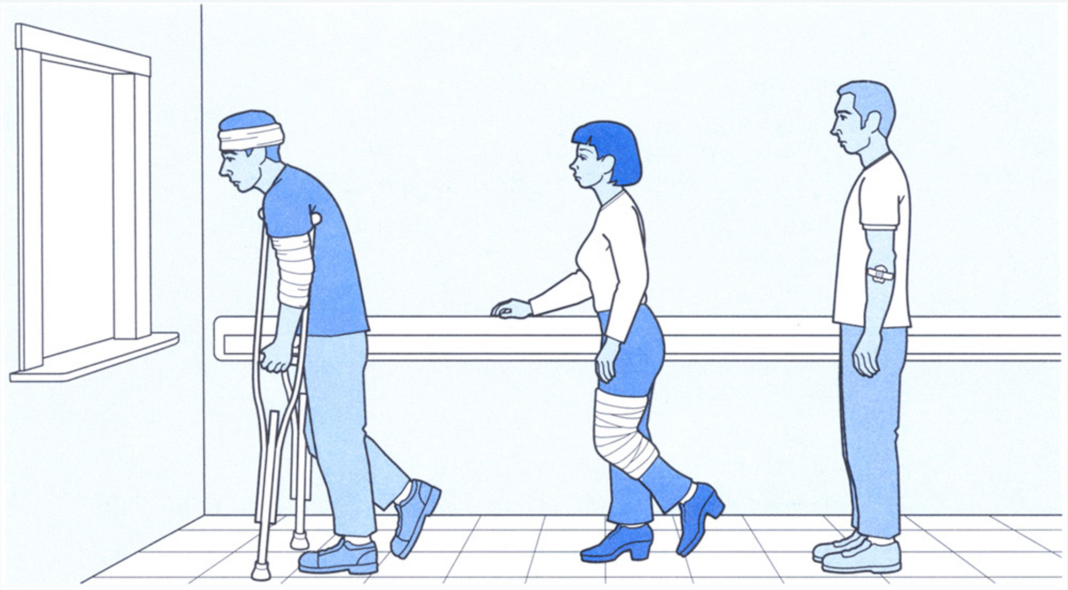
- IS-EMPTY(H): is the heap empty?
- FIND-MIN(H): return an element with smallest key.
- DELETE(H, x): delete element x from the heap.
- MELD(H_1, H_2): replace heaps H_1 and H_2 with their union.

Note. Each element contains a key (duplicate keys are permitted) from a totally-ordered universe.

Priority queue applications

Applications.

- A* search.
- Heapsort.
- Online median.
- Huffman encoding.
- Prim's MST algorithm.
- Discrete event-driven simulation.
- Network bandwidth management.
- Dijkstra's shortest-paths algorithm.
- ...



http://younginc.site11.com/source/5895/fos0092.html

SECTION 2.4

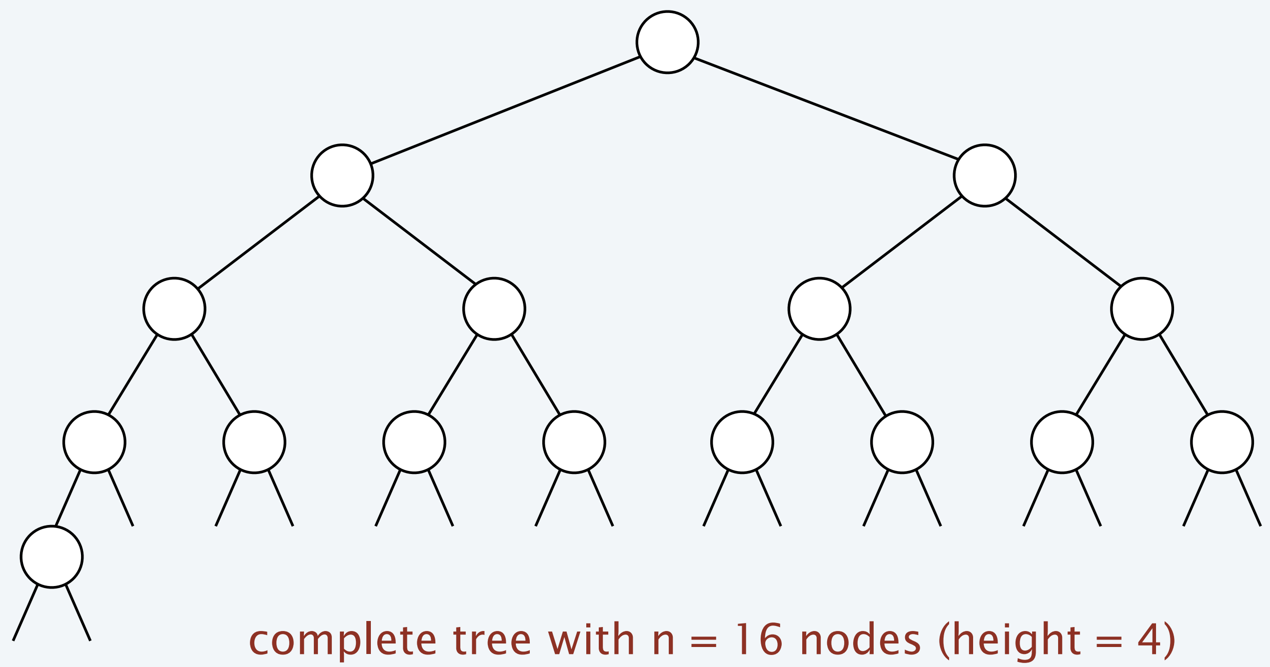
PRIORITY QUEUES

- *binary heaps*
- *d-ary heaps*
- *binomial heaps*
- *Fibonacci heaps*

Complete binary tree

Binary tree. Empty or node with links to two disjoint binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete tree with n = 16 nodes (height = 4)

Property. Height of complete binary tree with n nodes is $\lfloor \log_2 n \rfloor$.

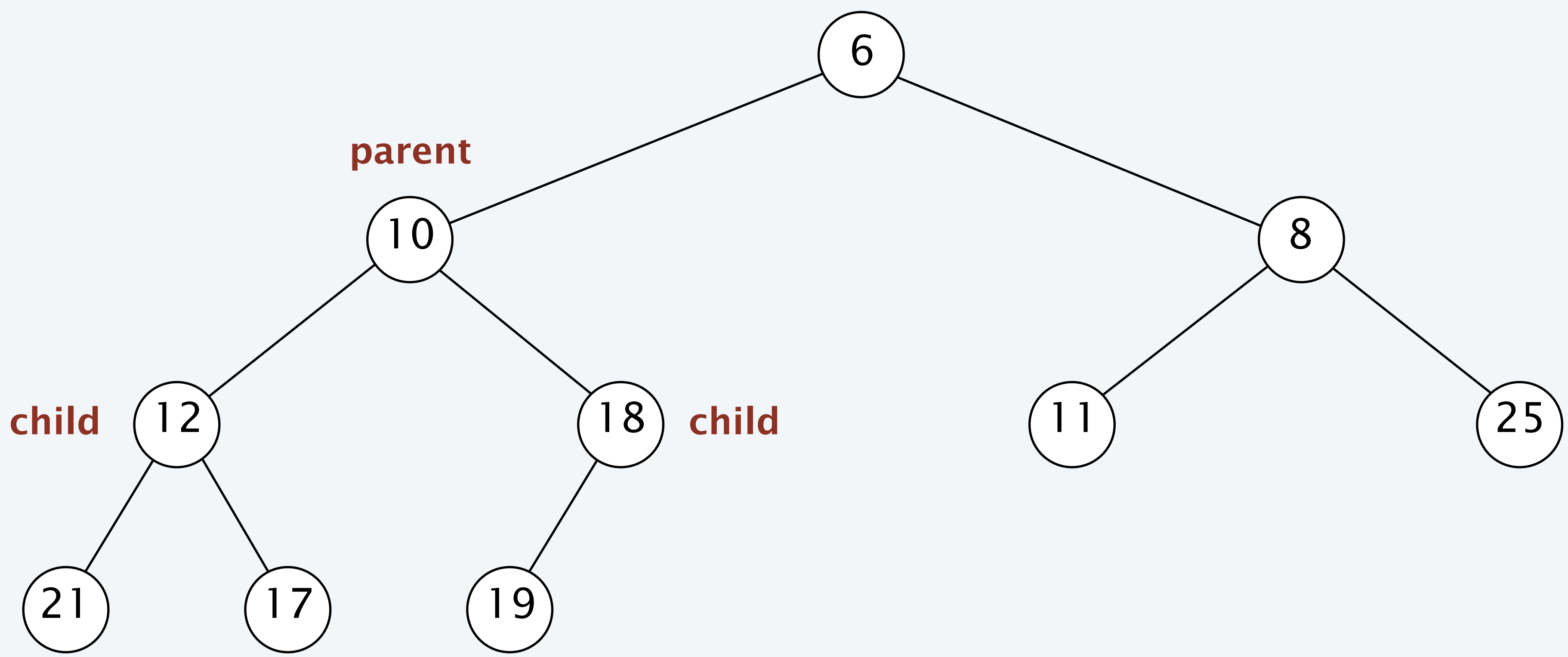
Pf. Height increases (by 1) only when n is a power of 2. ▪

A complete binary tree in nature

Binary heap

Binary heap. Heap-ordered complete binary tree.

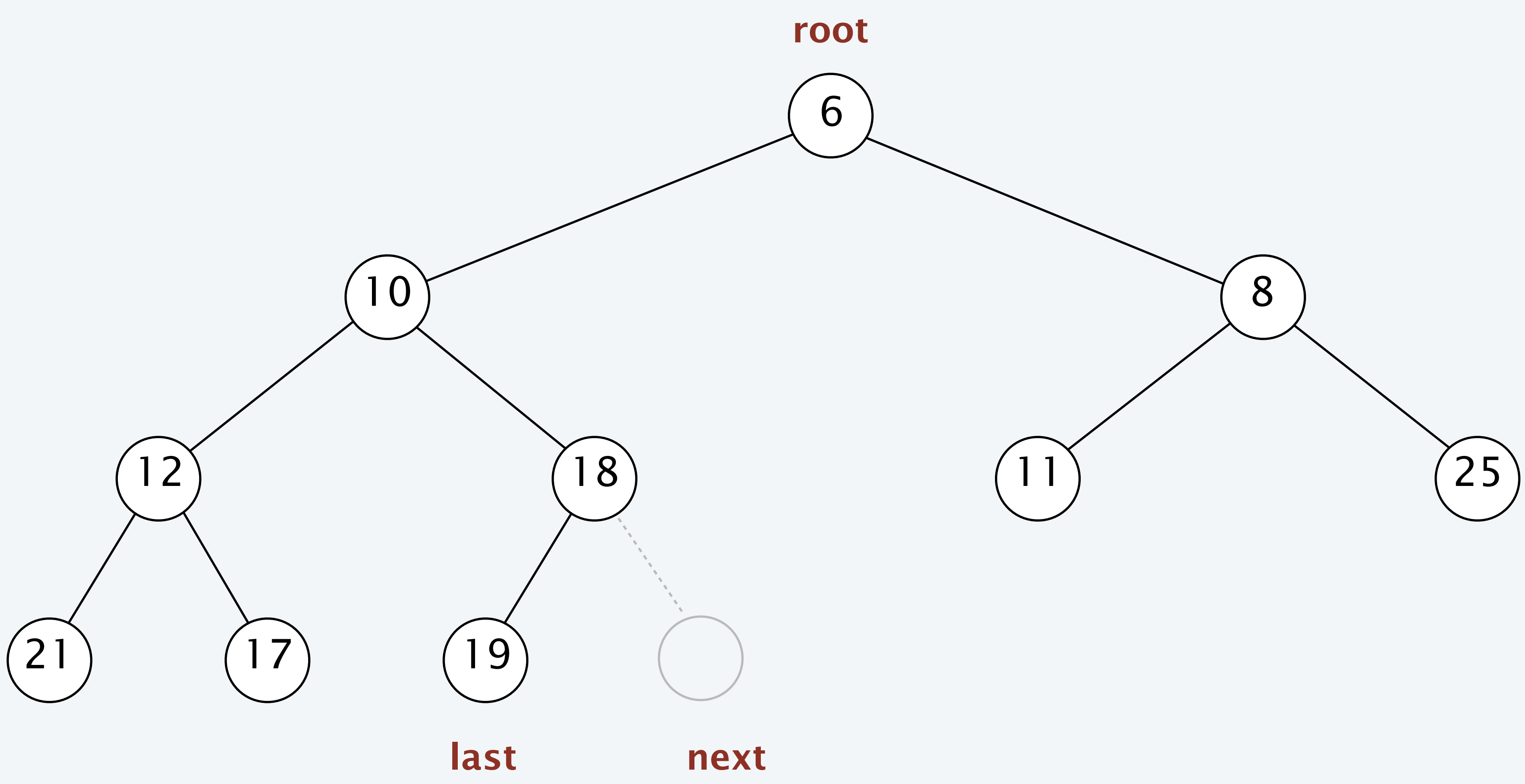
Heap-ordered tree. For each child, the key in child ≥ key in parent.



Explicit binary heap

Pointer representation. Each node has a pointer to parent and two children.

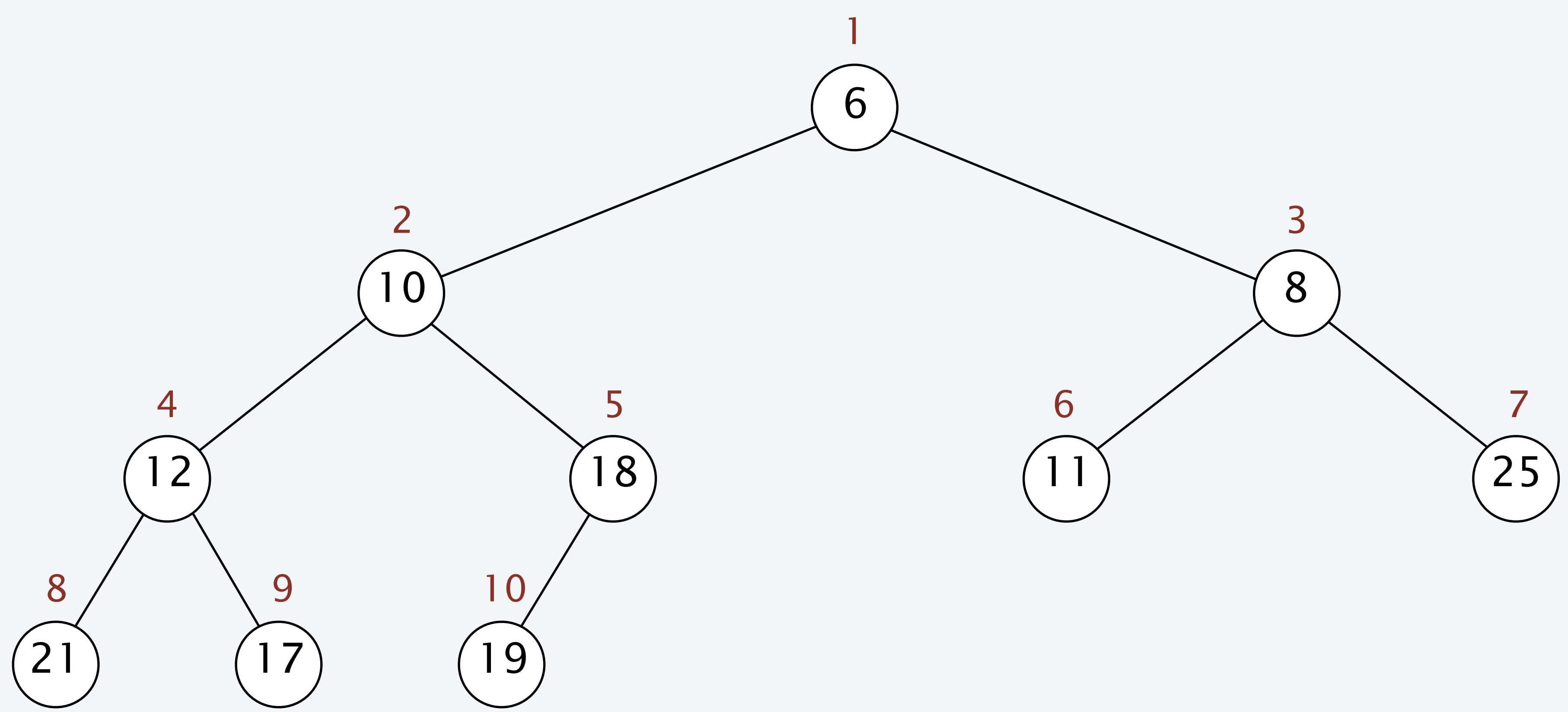
- Maintain number of elements n.
- Maintain pointer to root node.
- Can find pointer to last node or next node in $O(\log n)$ time.



Implicit binary heap

Array representation. Indices start at 1.

- Take nodes in level order.
- Parent of node at k is at $\lfloor k / 2 \rfloor$.
- Children of node at k are at $2k$ and $2k + 1$.



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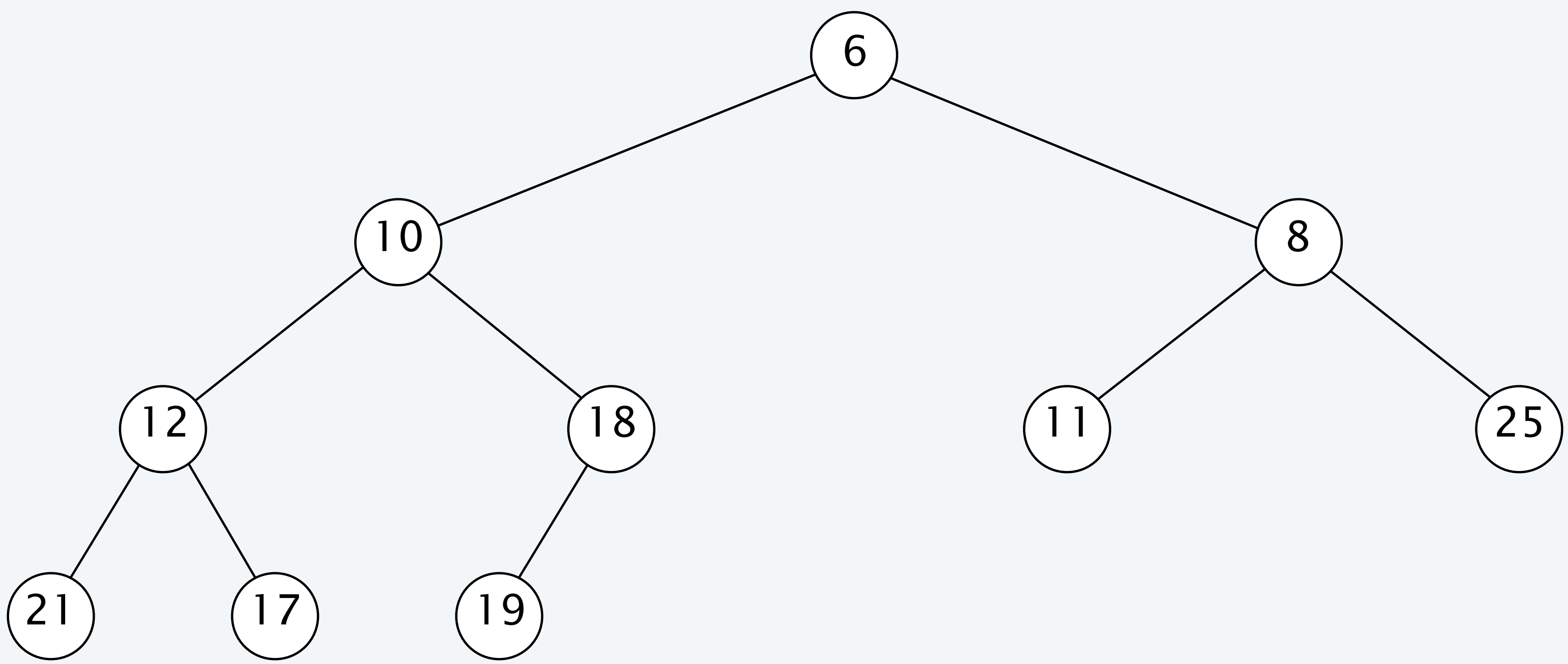
6	10	8	12	18	11	25	21	17	19						

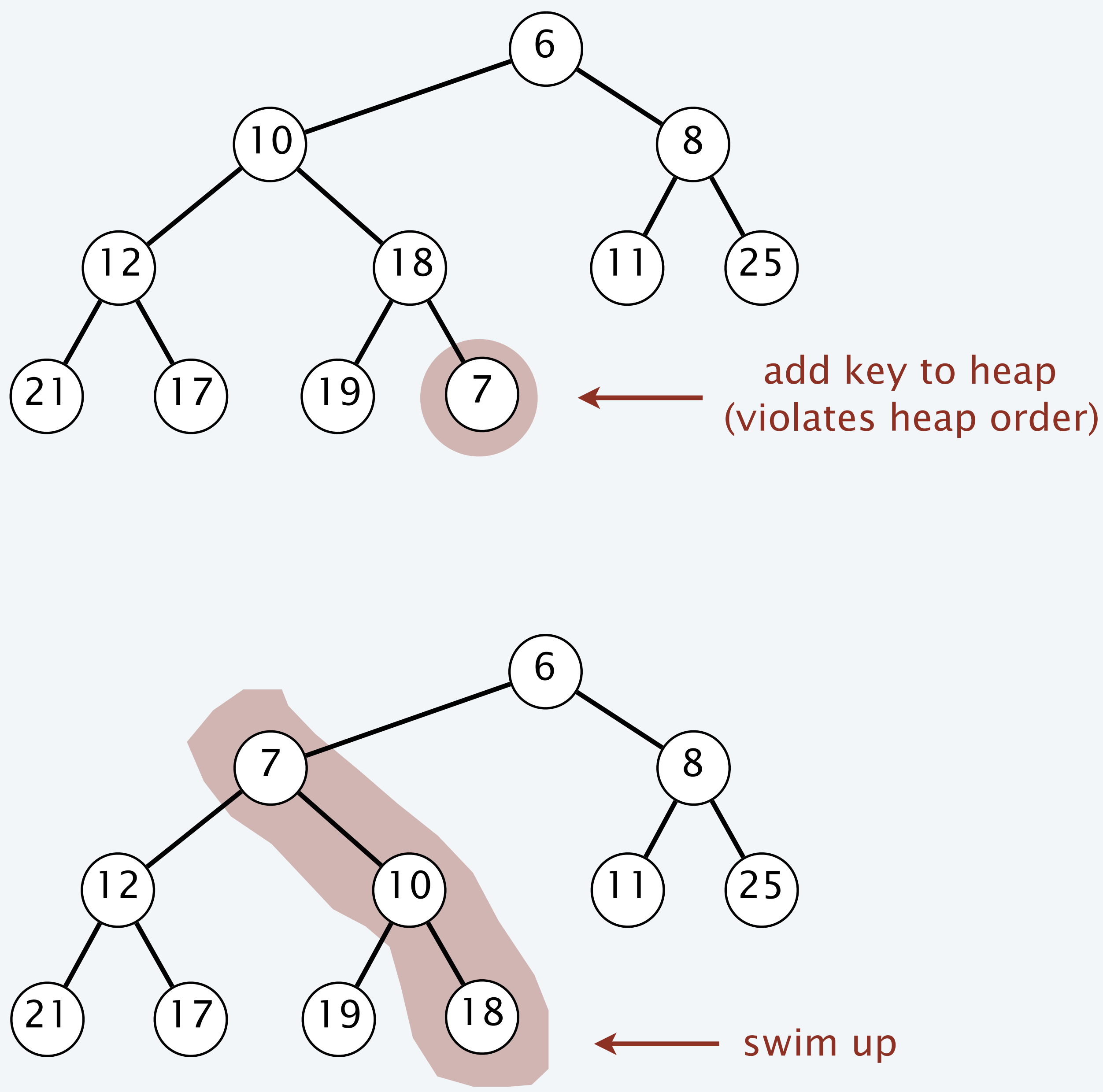
Binary heap demo

heap ordered



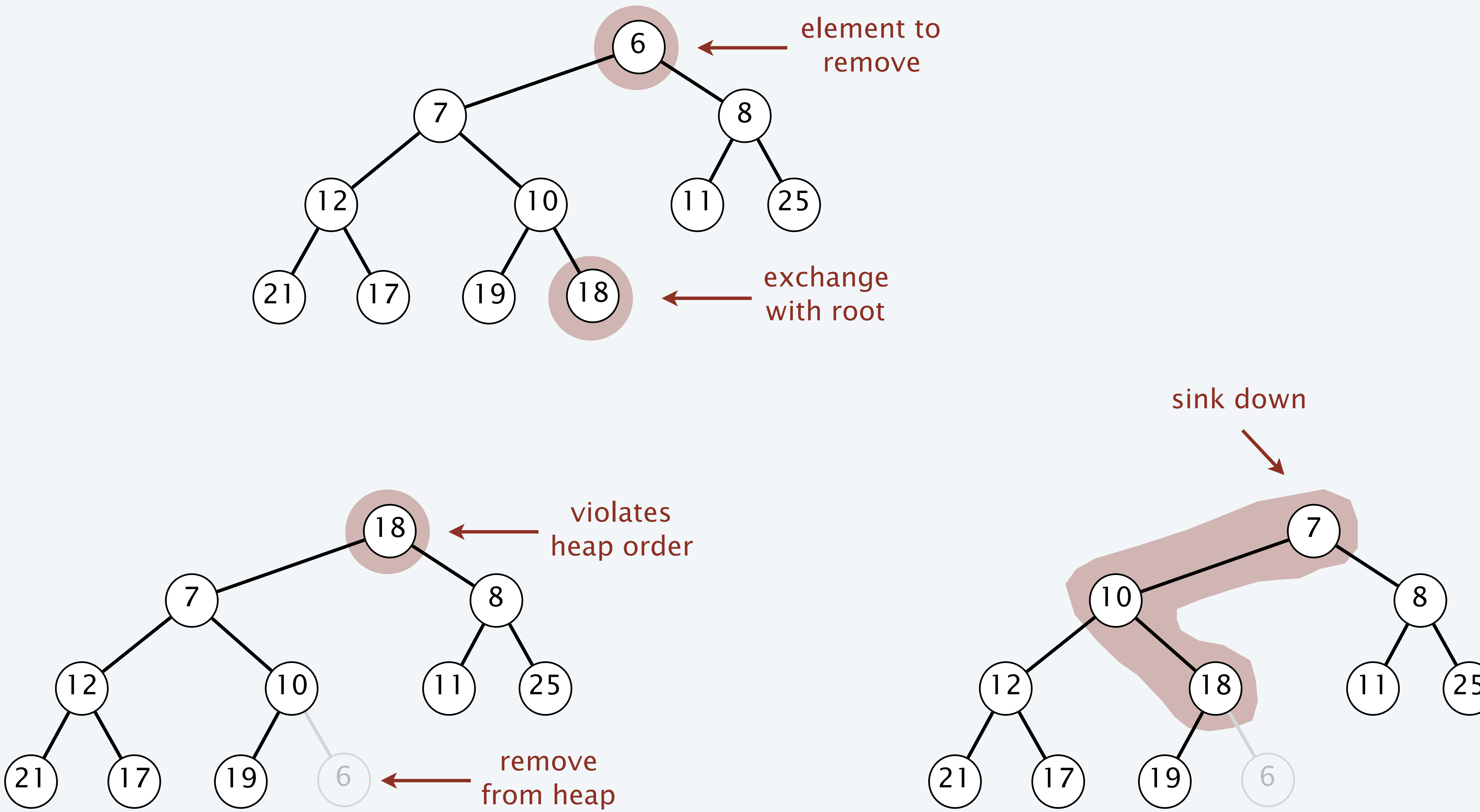
Binary heap: insert

Insert. Add element in new node at end; repeatedly exchange new element with element in its parent until heap order is restored.



Binary heap: extract the minimum

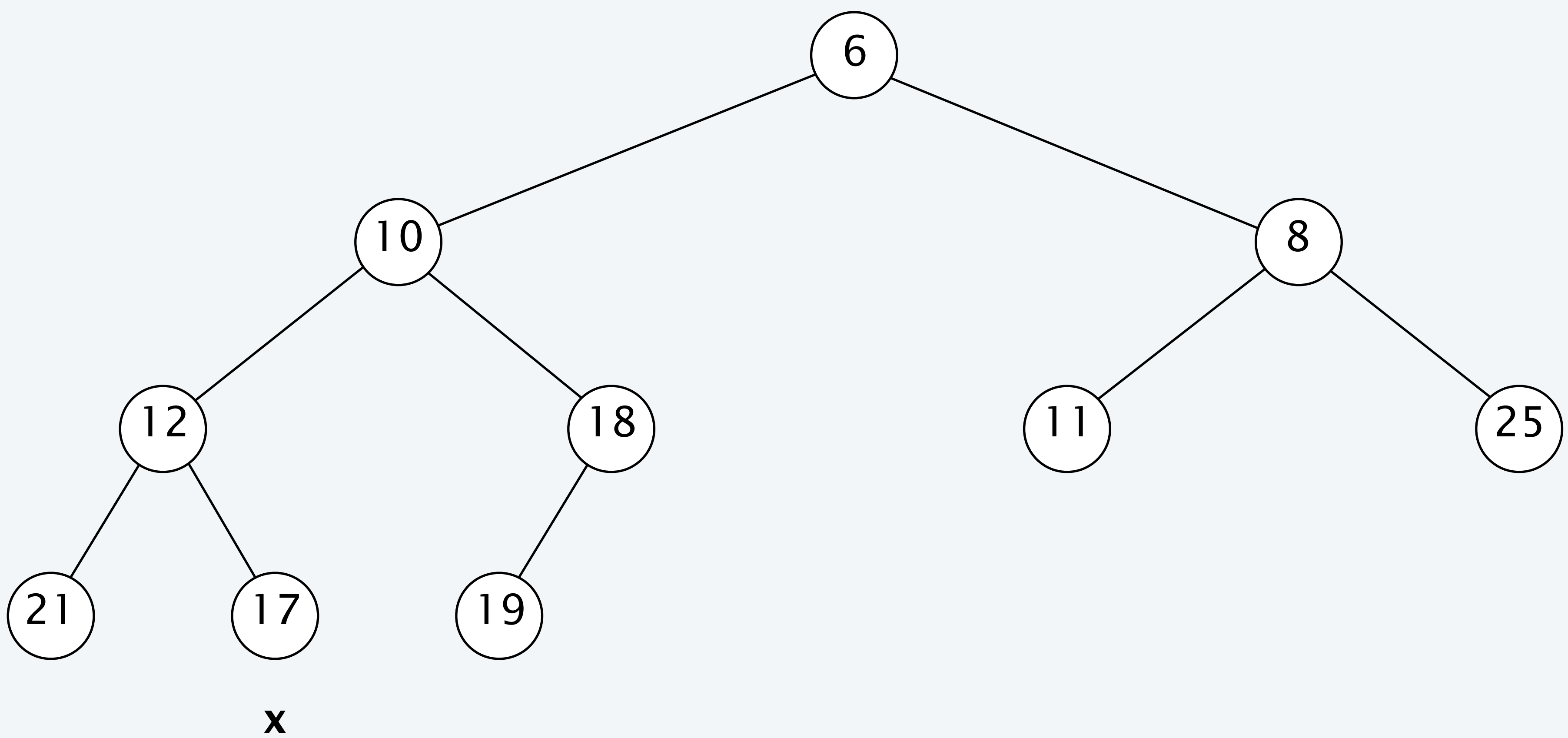
Extract min. Exchange element in root node with last node; repeatedly exchange element in root with its smaller child until heap order is restored.



Binary heap: decrease key

Decrease key. Given a handle to node, repeatedly exchange element with its parent until heap order is restored.

decrease key of node x to 11



Binary heap: analysis

Theorem. In an implicit binary heap, any sequence of m INSERT, EXTRACT-MIN, and DECREASE-KEY operations with n INSERT operations takes $O(m \log n)$ time.

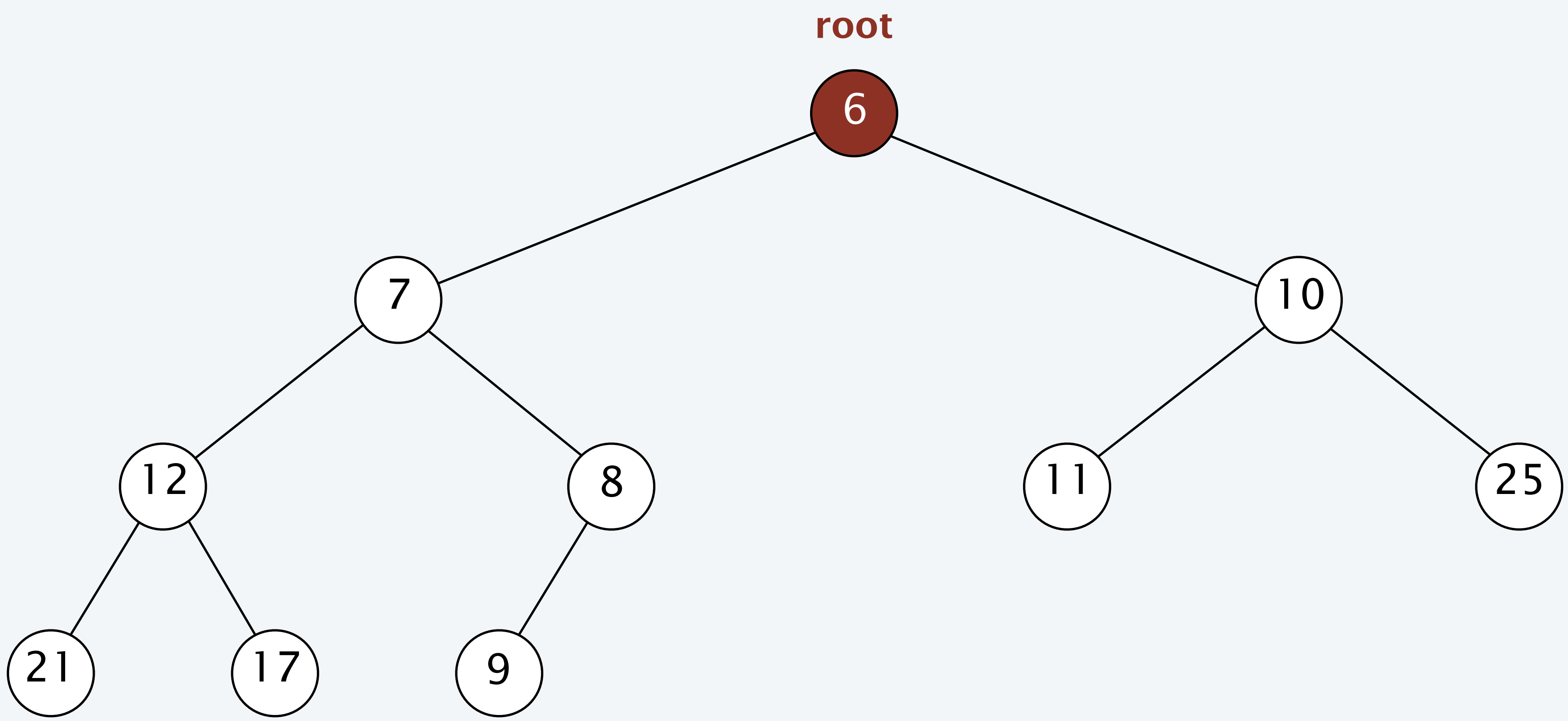
Pf.

- Each heap op touches nodes only on a path from the root to a leaf; the height of the tree is at most $\log_2 n$.
- The total cost of expanding and contracting the arrays is $O(n)$. ▪

Theorem. In an explicit binary heap with n nodes, the operations INSERT, DECREASE-KEY, and EXTRACT-MIN take $O(\log n)$ time in the worst case.

Binary heap: find-min

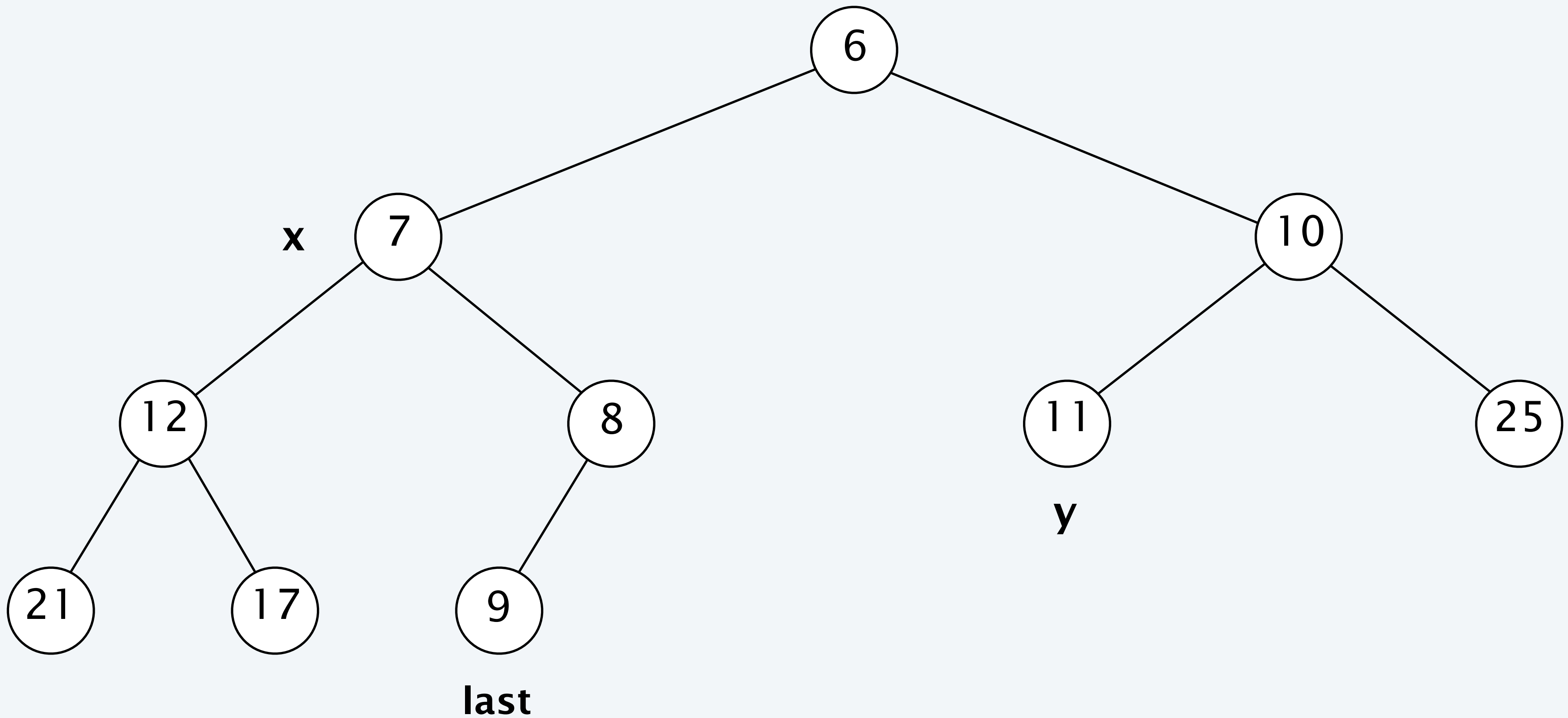
Find the minimum. Return element in the root node.



Binary heap: delete

Delete. Given a handle to a node, exchange element in node with last node; either swim down or sink up the node until heap order is restored.

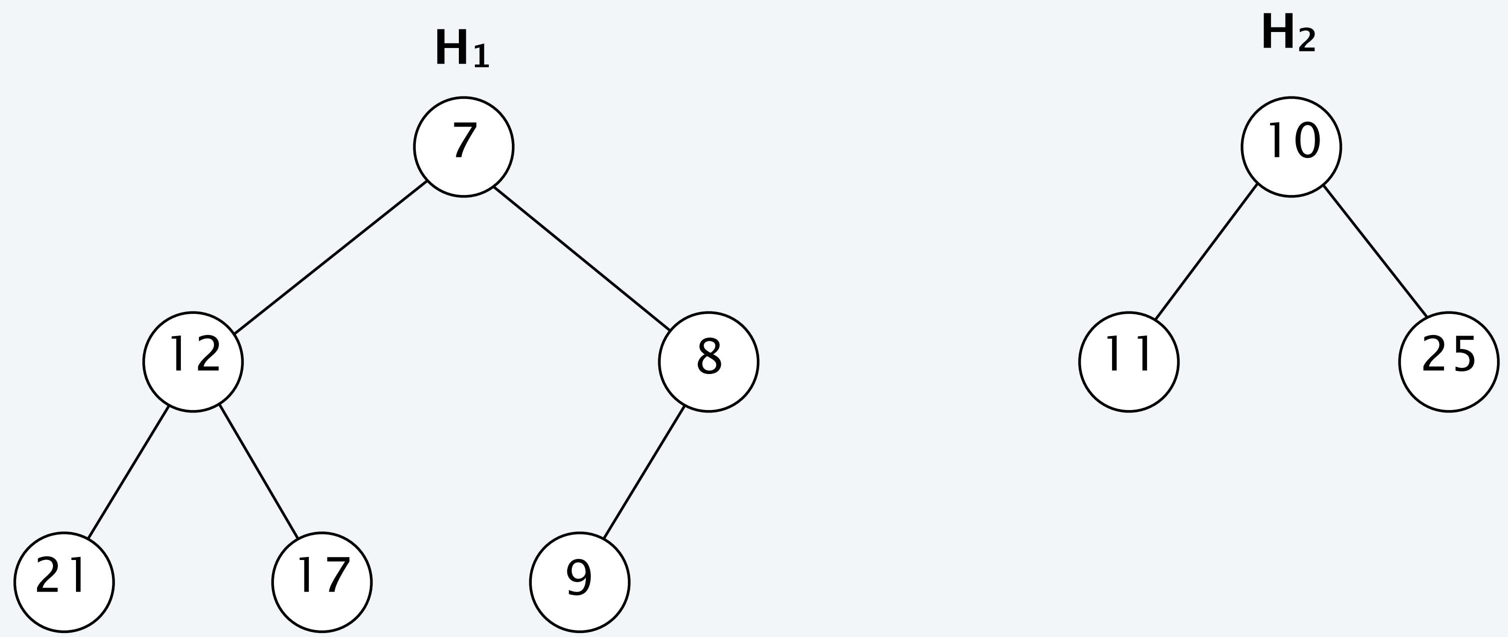
delete node x or y



Binary heap: meld

Meld. Given two binary heaps H_1 and H_2, merge into a single binary heap.

Observation. No easy solution: $\Omega(n)$ time apparently required.

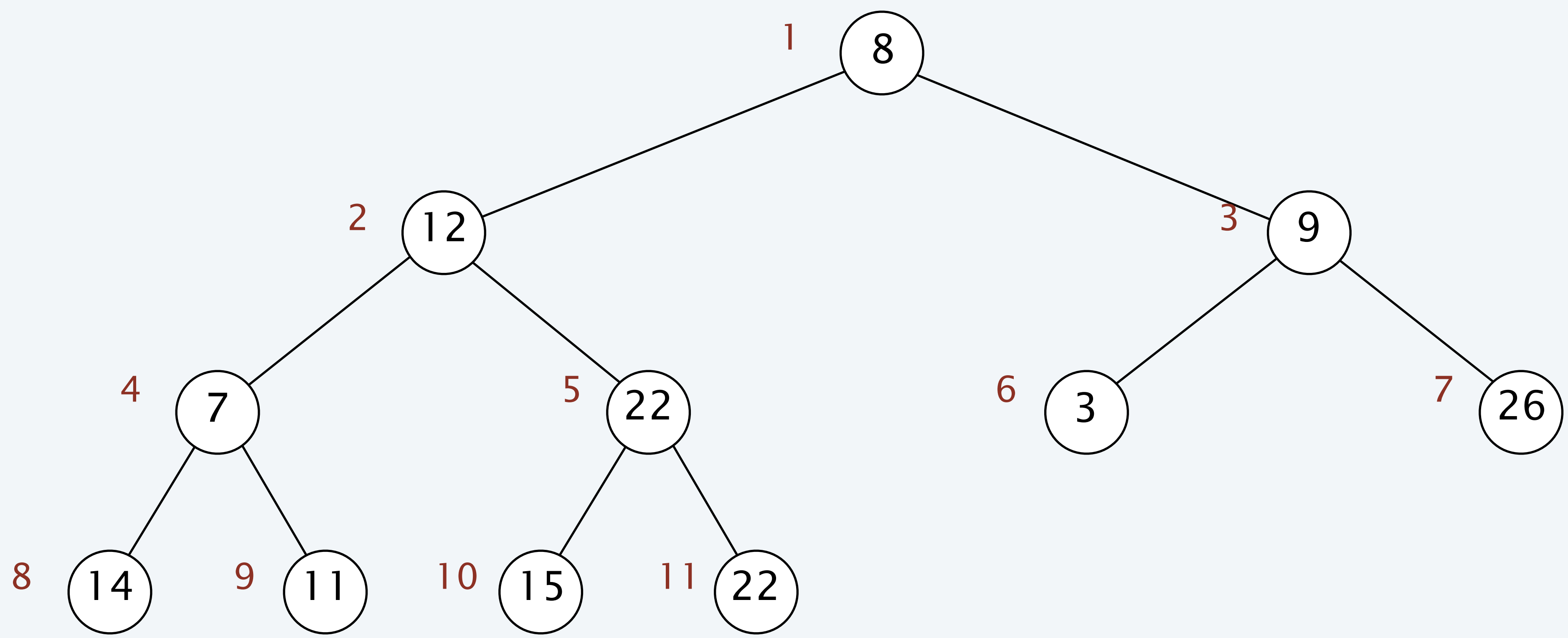


Binary heap: heapify

Heapify. Given n elements, construct a binary heap containing them.

Observation. Can do in $O(n \log n)$ time by inserting each element.

Bottom-up method. For $i = n$ to 1, repeatedly exchange the element in node i with its smaller child until subtree rooted at i is heap-ordered.



8	12	9	7	22	3	26	14	11	15	22
1	2	3	4	5	6	7	8	9	10	11

Binary heap: heapify

Theorem. Given n elements, can construct a binary heap containing those n elements in $O(n)$ time.

Pf.

- There are at most $\lceil n / 2^{h+1} \rceil$ nodes of height h.
- The amount of work to sink a node is proportional to its height h.
- Thus, the total work is bounded by:

$$\sum_{h=0}^{\lfloor \log_2 n \rfloor} \lceil n / 2^{h+1} \rceil \, h \;\leq\; \sum_{h=0}^{\lfloor \log_2 n \rfloor} n \, h / 2^h \;\leq\; 2n \quad \blacksquare$$

$$\sum_{i=1}^{k} \frac{i}{2^i} = 2 - \frac{k}{2^k} - \frac{1}{2^{k-1}} \leq 2$$

Corollary. Given two binary heaps H_1 and H_2 containing n elements in total, can implement MELD in $O(n)$ time.

Priority queues performance cost summary

operation	linked list	binary heap
MAKE-HEAP	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$
MELD	$O(1)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$

Priority queues performance cost summary

Q. Reanalyze so that EXTRACT-MIN and DELETE take $O(1)$ amortized time?

operation	linked list	binary heap	binary heap †
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(1)$ †
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$	$O(1)$ †
MELD	$O(1)$	$O(n)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$	$O(1)$

† amortized

SECTION 2.4

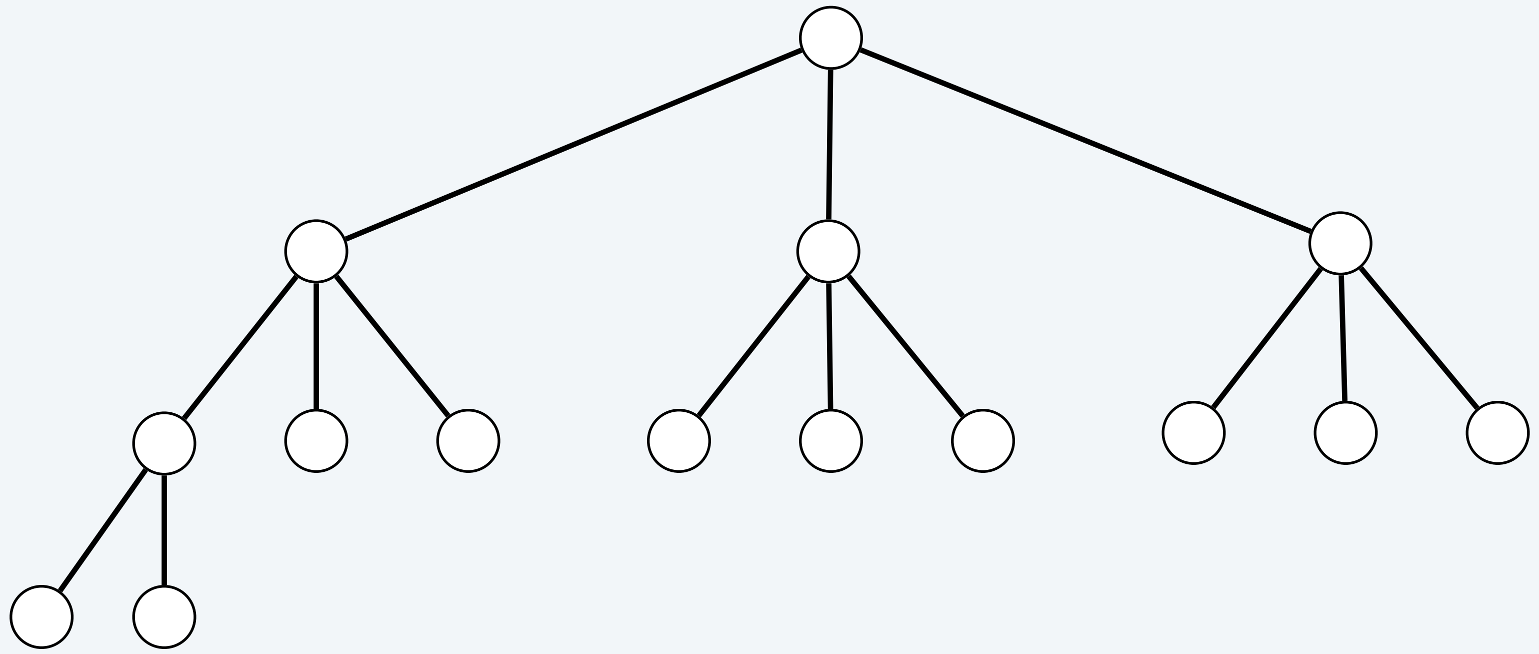
PRIORITY QUEUES

- binary heaps
- *d*-ary heaps
- binomial heaps
- Fibonacci heaps

Complete d-ary tree

d-ary tree. Empty or node with links to d disjoint d-ary trees.

Complete tree. Perfectly balanced, except for bottom level.

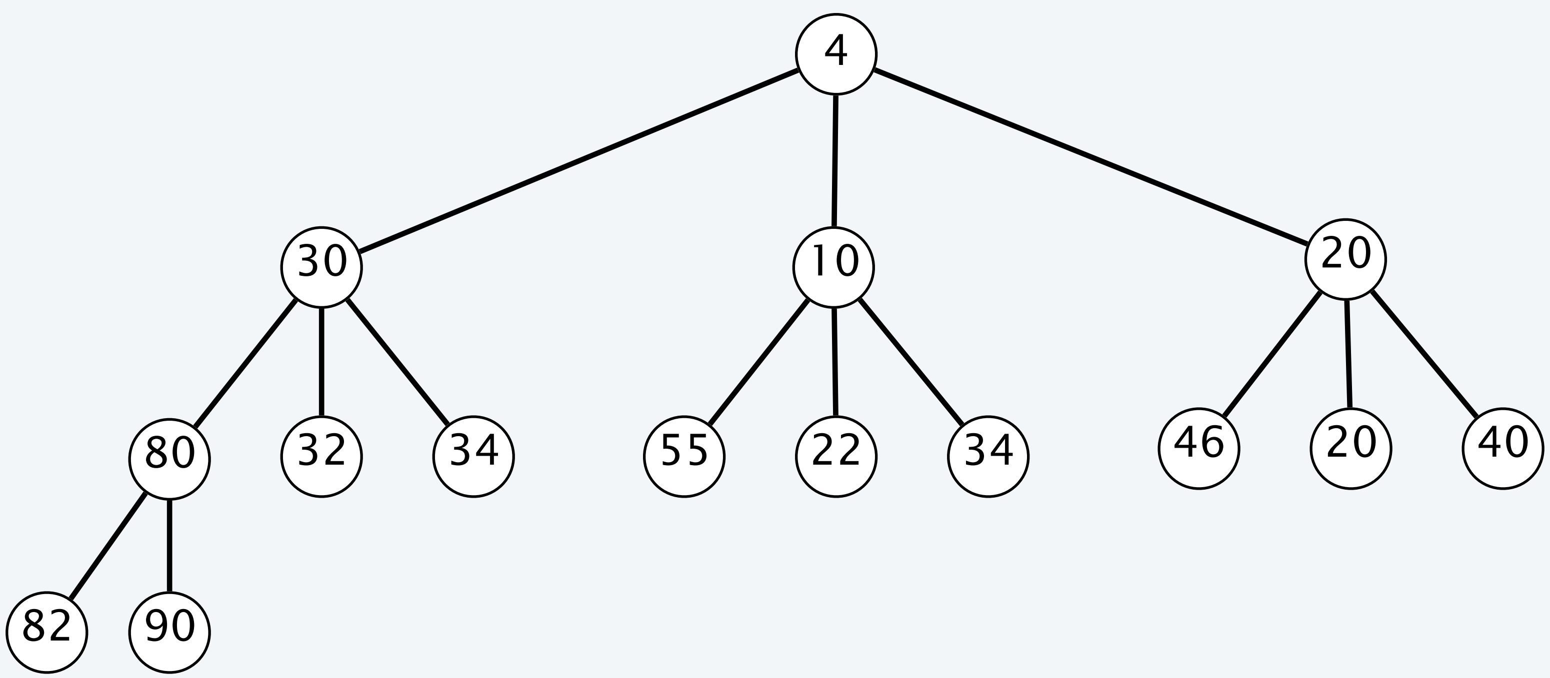


Fact. The height of a complete d-ary tree with n nodes is $\leq \lceil \log_d n \rceil$.

d-ary heap

d-ary heap. Heap-ordered complete d-ary tree.

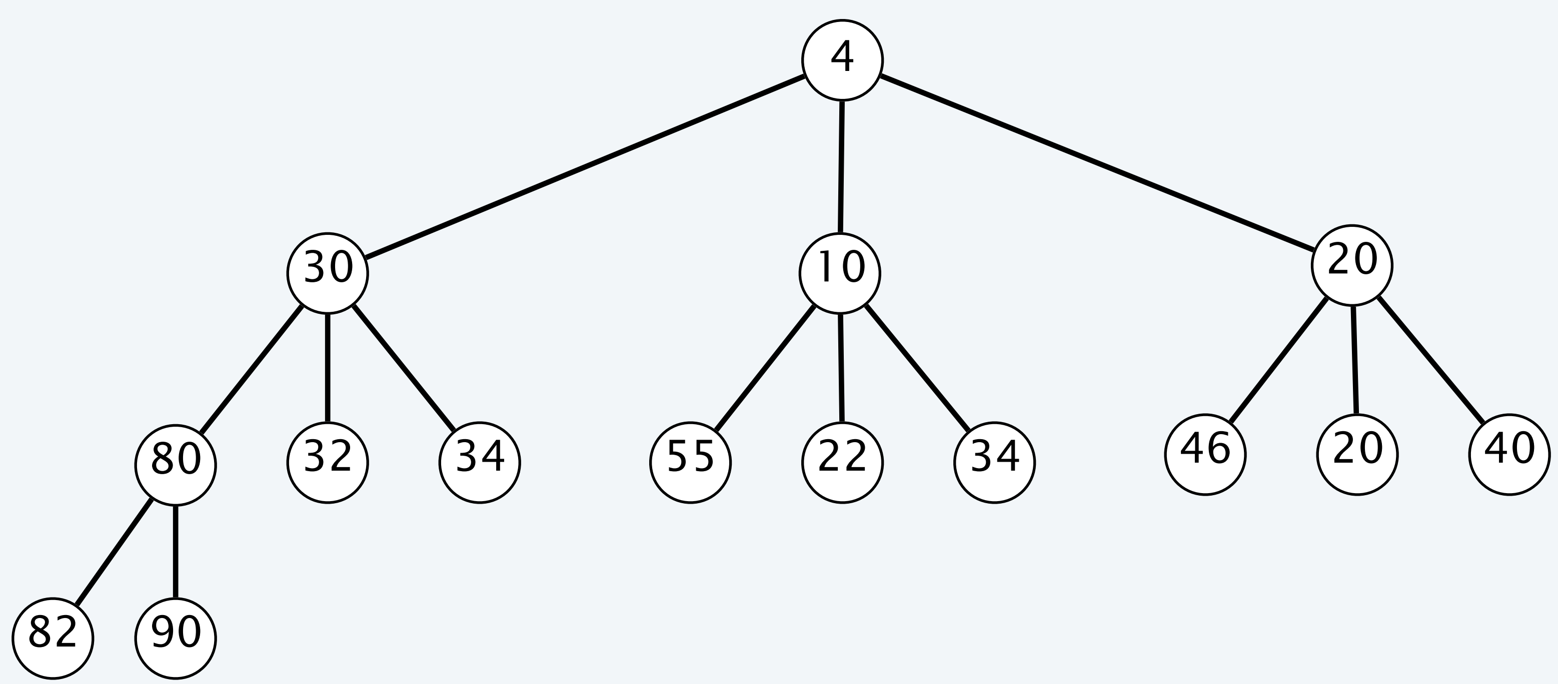
Heap-ordered tree. For each child, the key in child ≥ key in parent.



d-ary heap: insert

Insert. Add node at end; repeatedly exchange element in child with element in parent until heap order is restored.

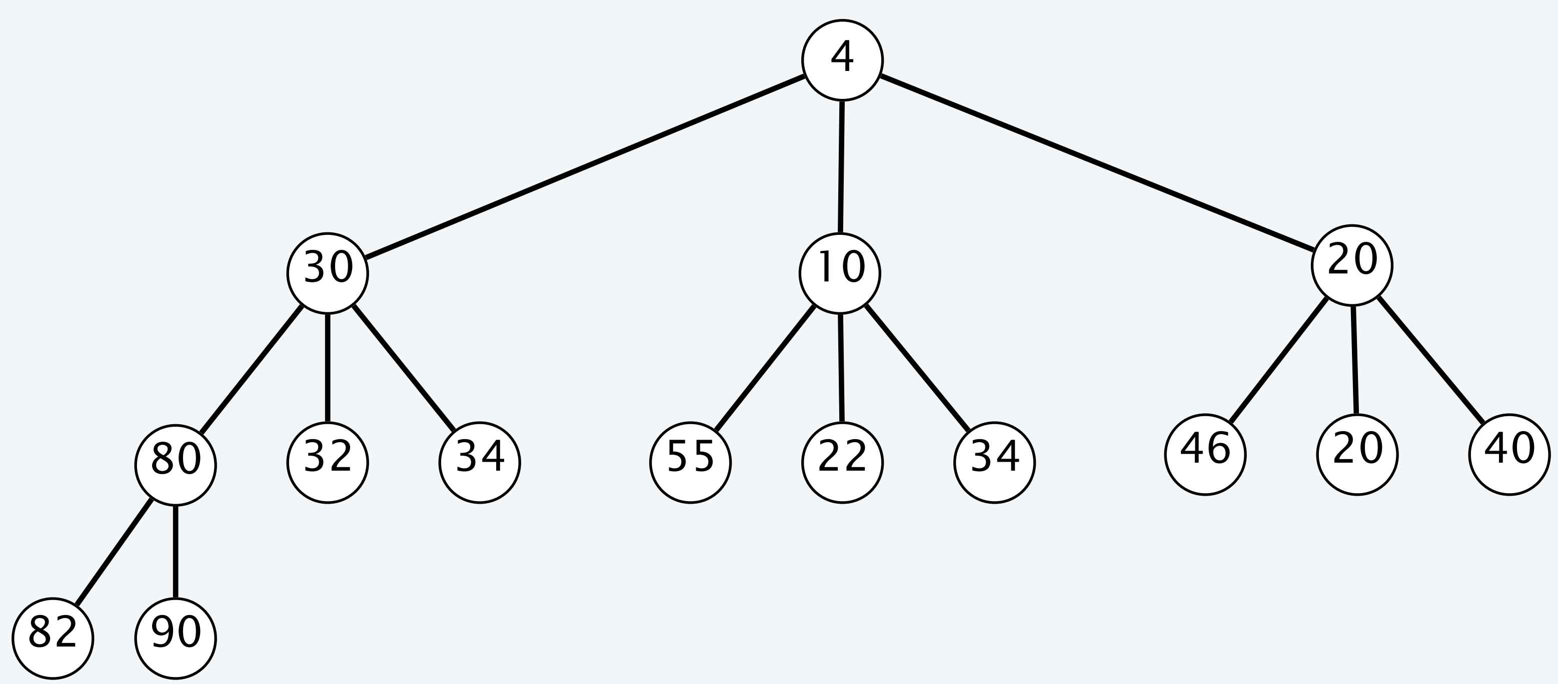
Running time. Proportional to height = $O(\log_d n)$.



d-ary heap: extract the minimum

Extract min. Exchange root node with last node; repeatedly exchange element in parent with element in largest child until heap order is restored.

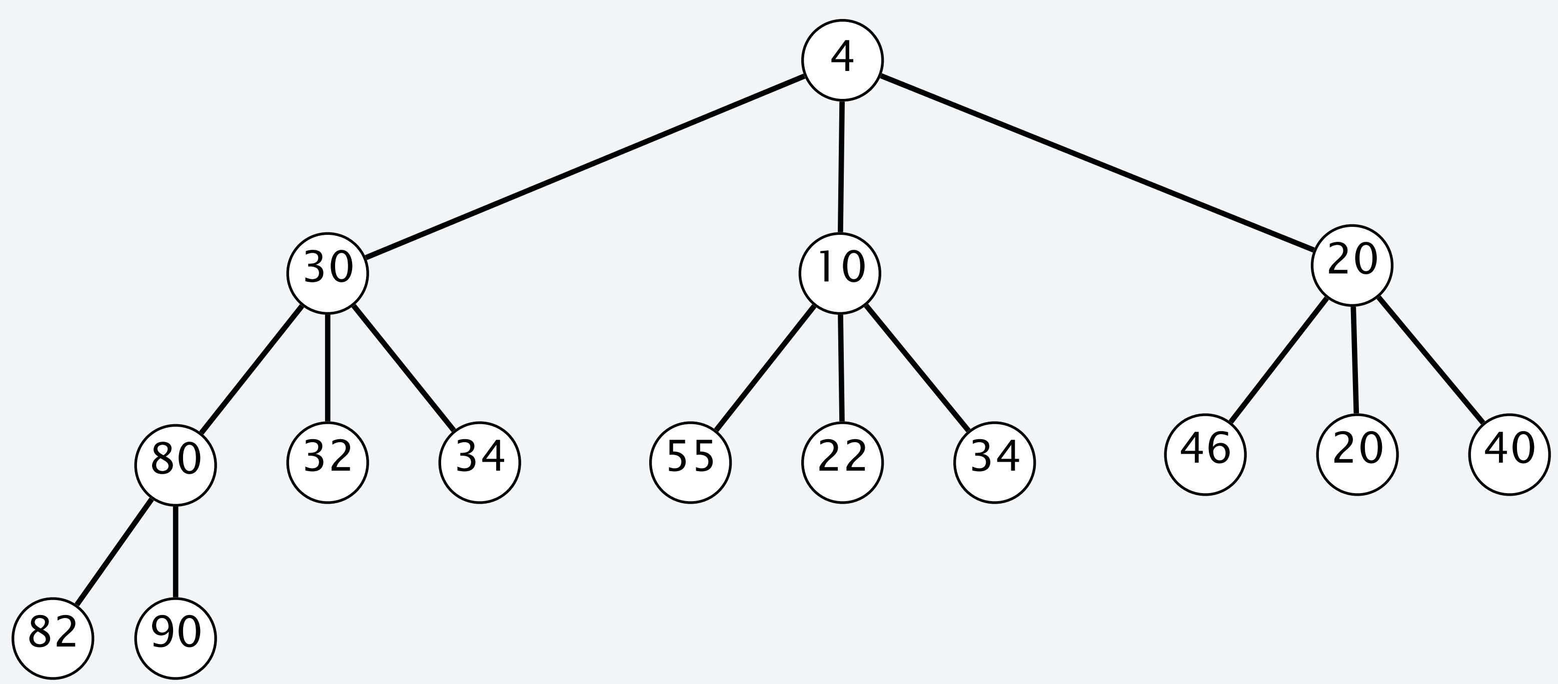
Running time. Proportional to $d \times$ height $= O(d \log_d n)$.



d-ary heap: decrease key

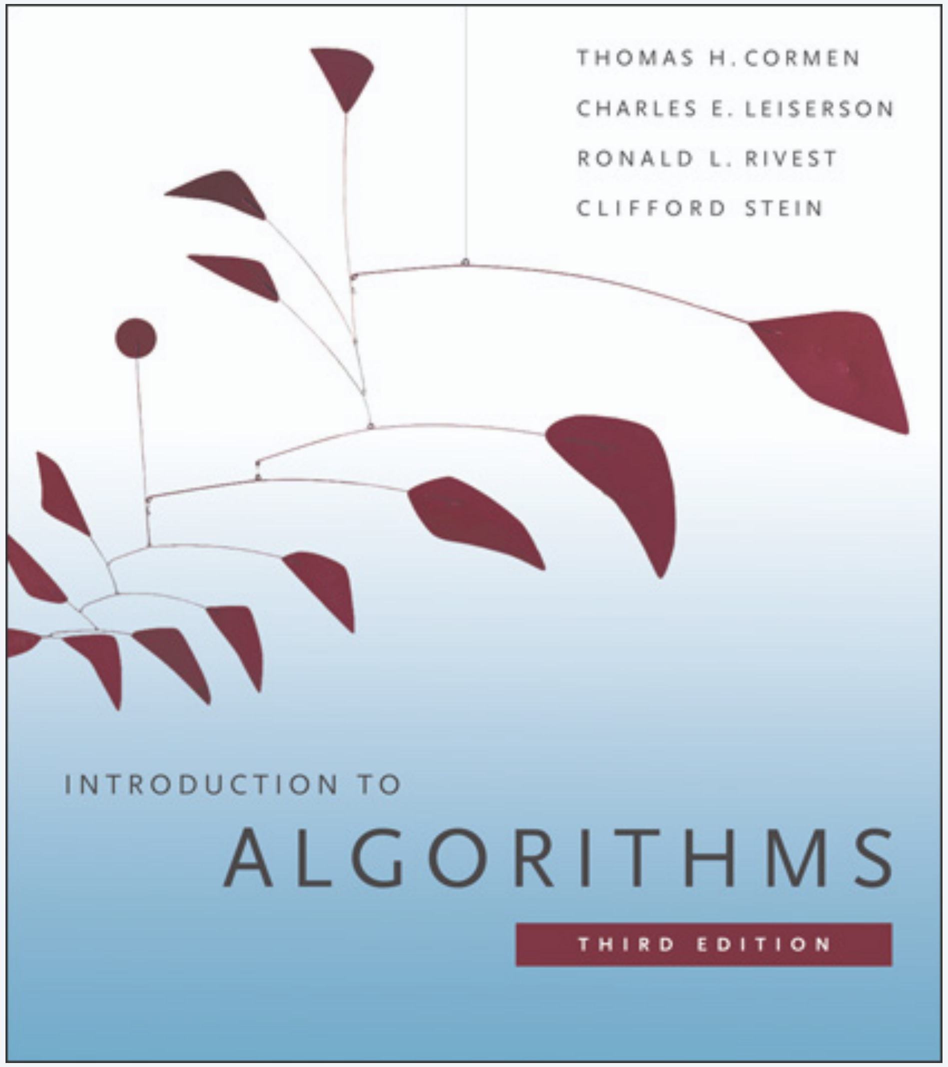
Decrease key. Given a handle to an element x, repeatedly exchange it with its parent until heap order is restored.

Running time. Proportional to height = $O(\log_d n)$.



Priority queues performance cost summary

operation	linked list	binary heap	d-ary heap
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log_d n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(d \log_d n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log_d n)$
DELETE	$O(1)$	$O(\log n)$	$O(d \log_d n)$
MELD	$O(1)$	$O(n)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$	$O(1)$



CHAPTER 6 (2^{ND} EDITION)

PRIORITY QUEUES

- *binary heaps*
- *d-ary heaps*
- *binomial heaps*
- *Fibonacci heaps*

Priority queues performance cost summary

O(1) average cost

operation	linked list	binary heap	d-ary heap
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log_d n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(d \log_d n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log_d n)$
DELETE	$O(1)$	$O(\log n)$	$O(d \log_d n)$
MELD	$O(1)$	$O(n)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$	$O(1)$

Goal. $O(\log n)$ INSERT, DECREASE-KEY, EXTRACT-MIN, and MELD.

also achieve O(1) insert on average

mergeable heap

Binomial heaps

Programming Techniques
S.L. Graham, R.L. Rivest Editors

A Data Structure for Manipulating Priority Queues

Jean Vuillemin
Université de Paris-Sud

A data structure is described which can be used for representing a collection of priority queues. The primitive operations are insertion, deletion, union, update, and search for an item of earliest priority.

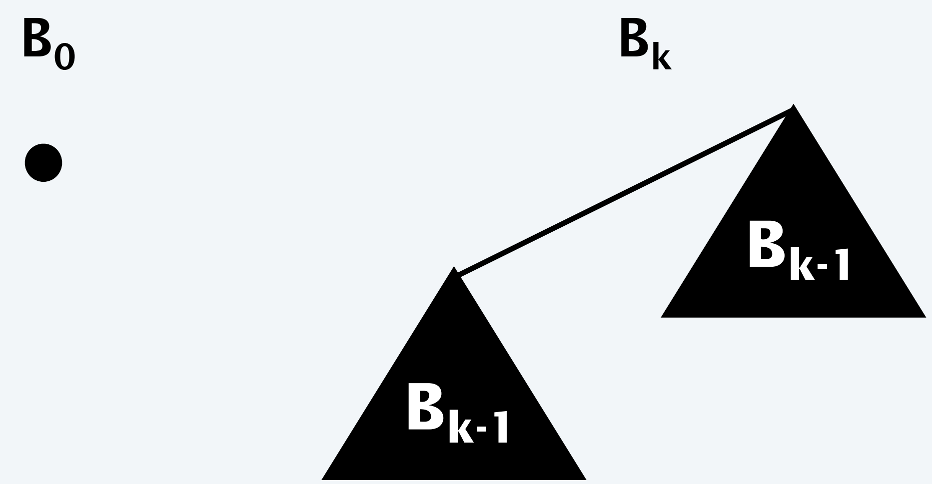
Key Words and Phrases: data structures, implementation of set operations, priority queues, mergeable heaps, binary trees

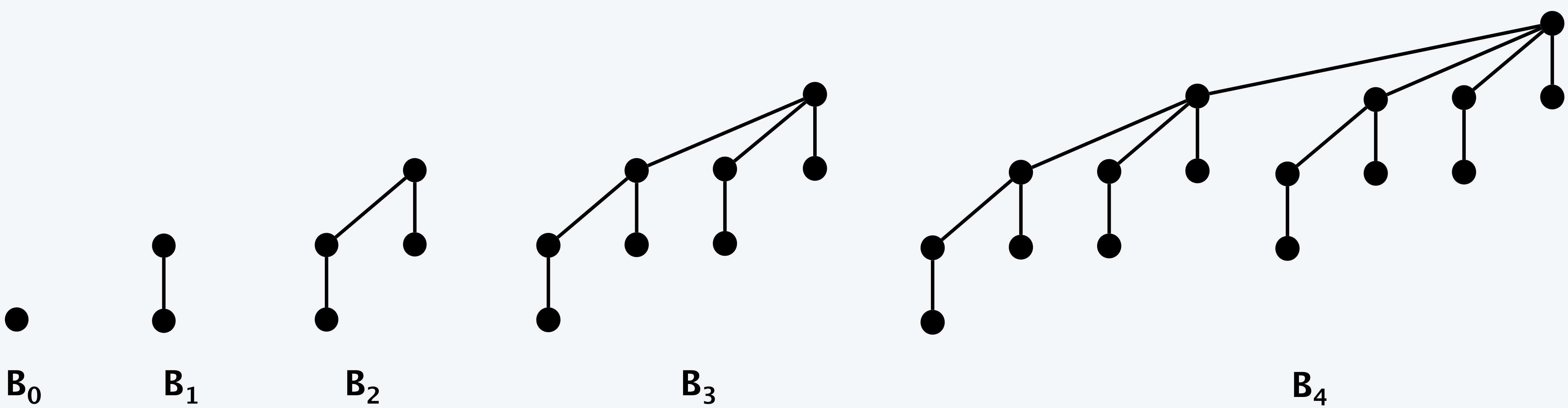
CR Categories: 4.34, 5.24, 5.25, 5.32, 8.1

Binomial tree

Def. A binomial tree of order k is defined recursively:

- Order 0: single node.
- Order k: one binomial tree of order $k-1$ linked to another of order $k-1$.



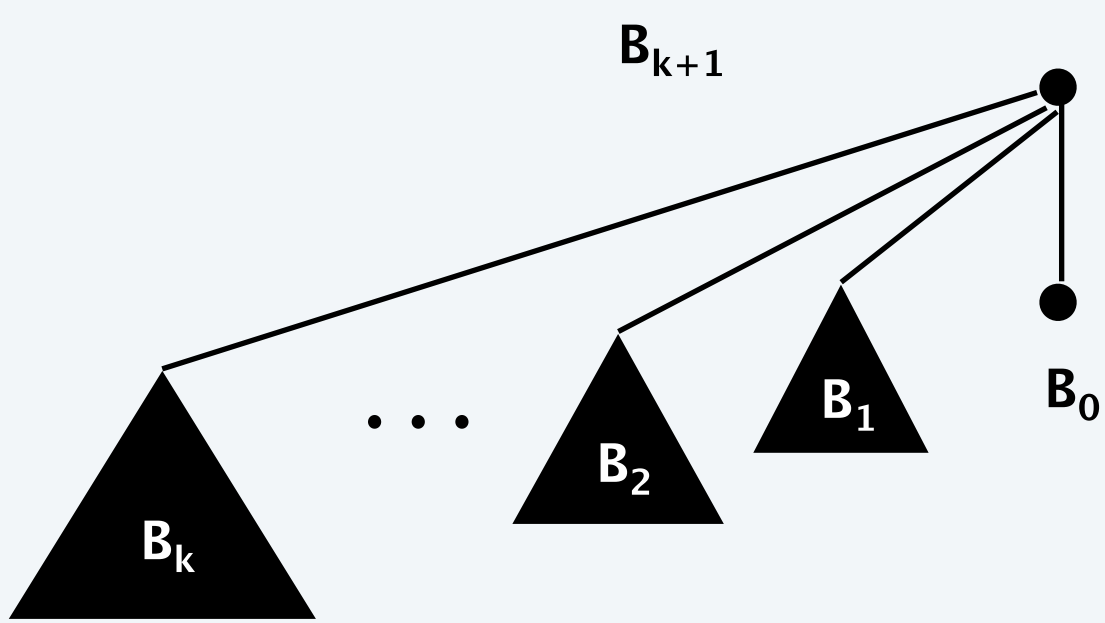


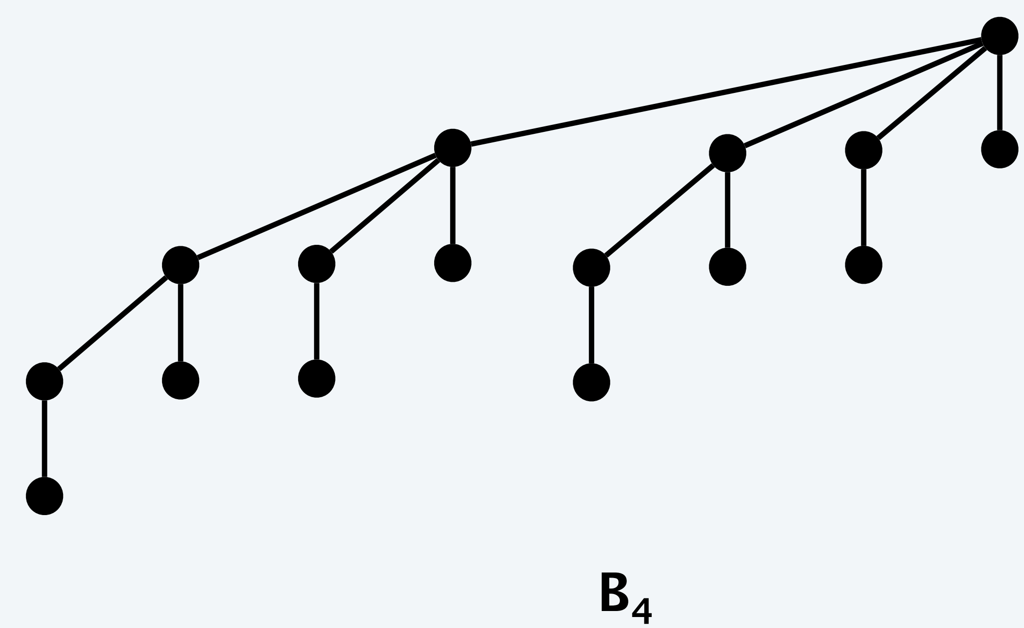
Binomial tree properties

Properties. Given an order k binomial tree B_k,

- Its height is k.
- It has 2^k nodes.
- It has $\binom{k}{i}$ nodes at depth i.
- The degree of its root is k.
- Deleting its root yields k binomial trees $B_{k-1}, \ldots, B_0$.

Pf. [by induction on k]

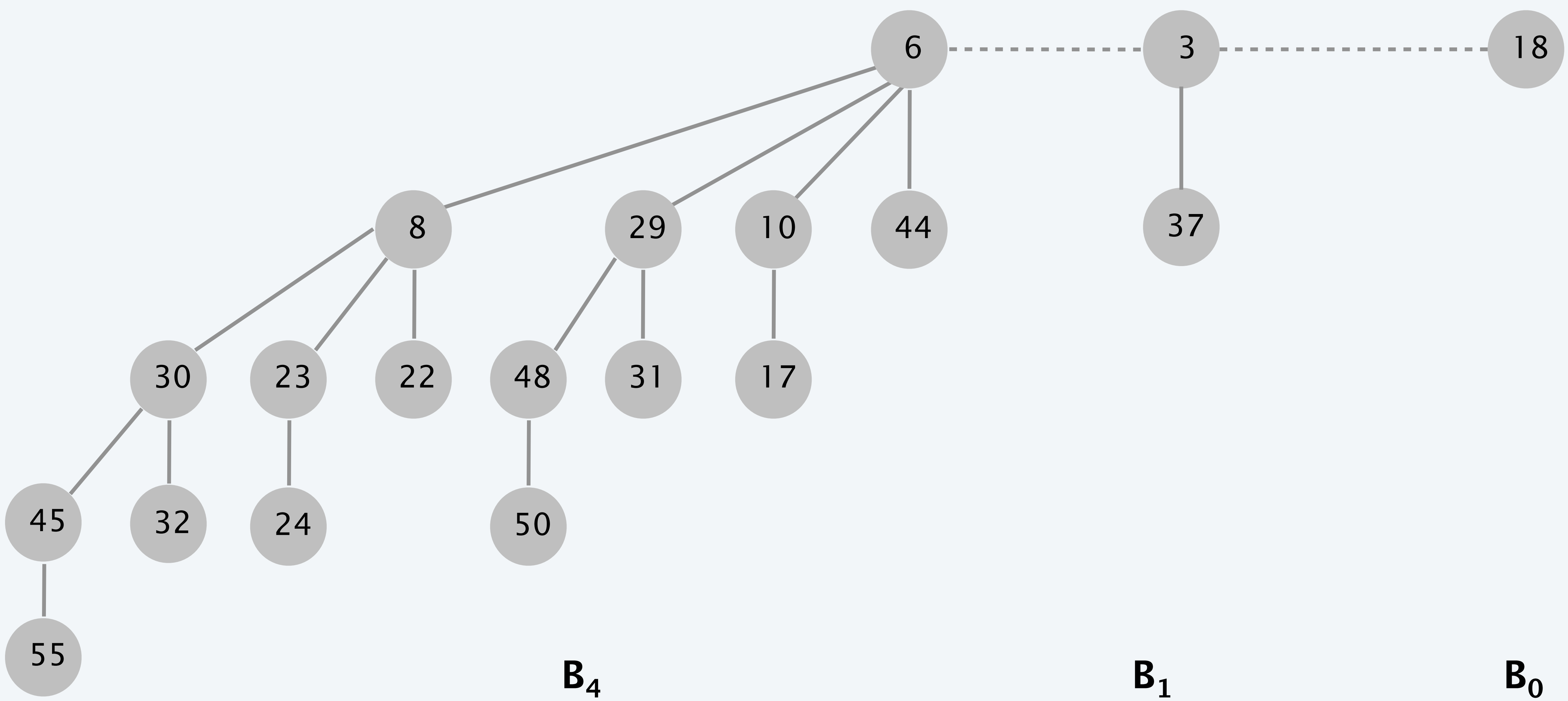




Binomial heap

Def. A binomial heap is a sequence of binomial trees such that:

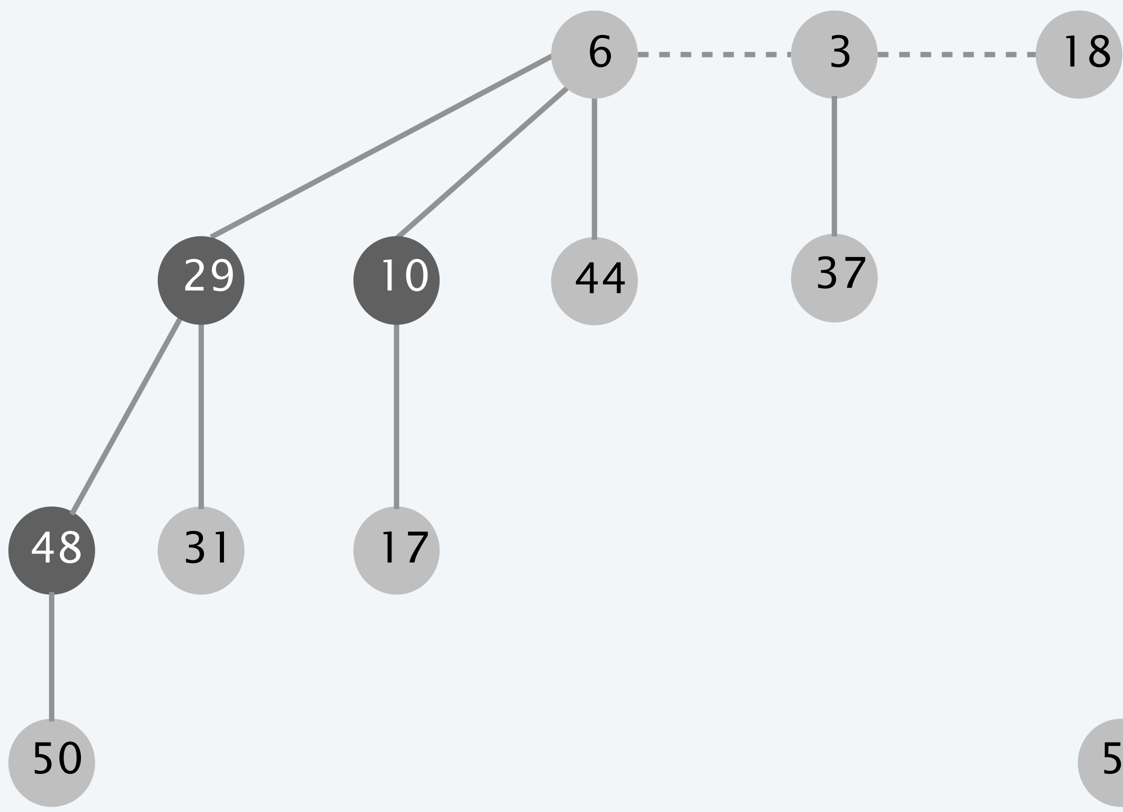
- Each tree is heap-ordered.
- There is either 0 or 1 binomial tree of order k.



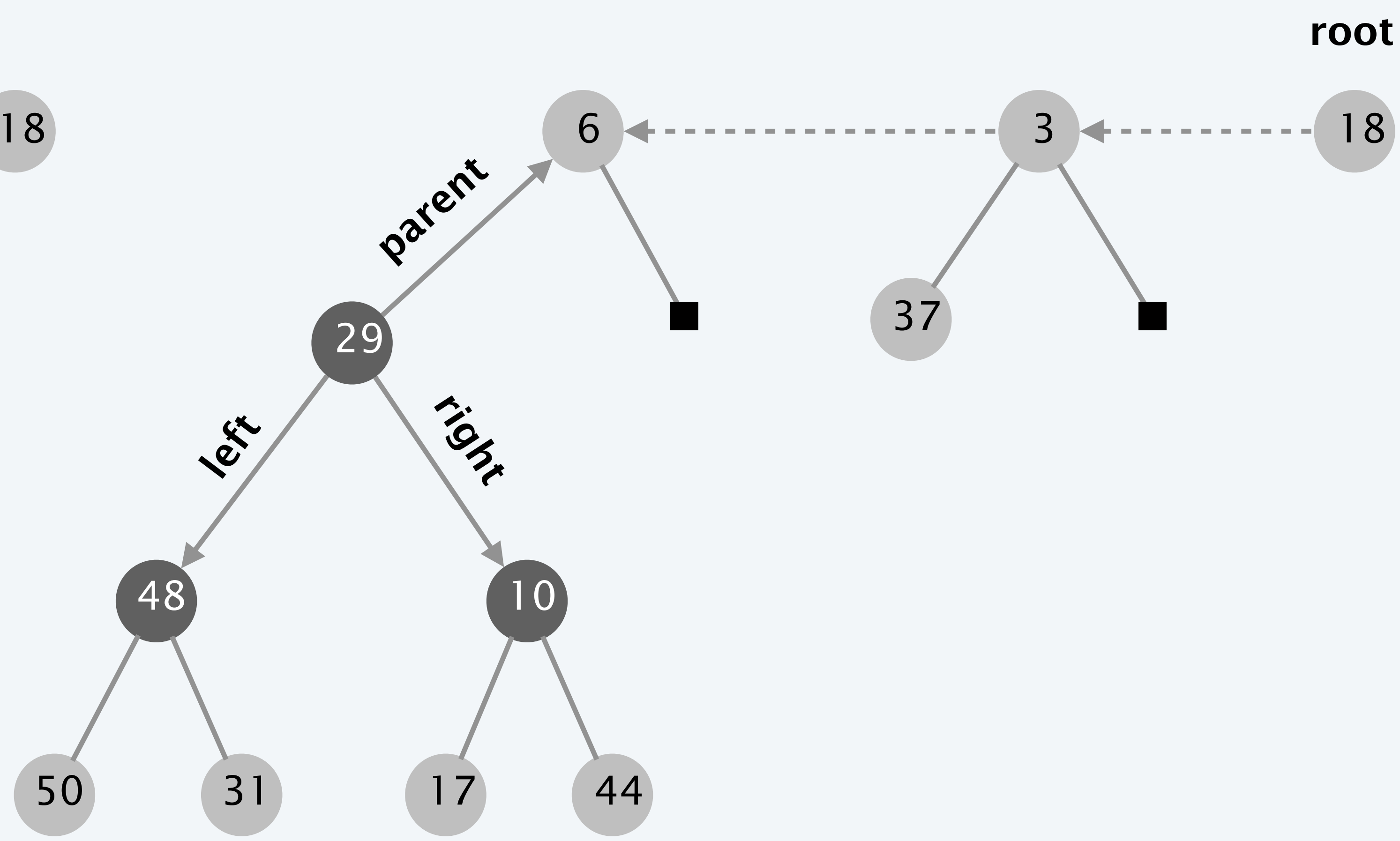
Binomial heap representation

Binomial trees. Represent trees using left-child, right-sibling pointers.

Roots of trees. Connect with singly-linked list, with degrees decreasing from left to right.



binomial heap

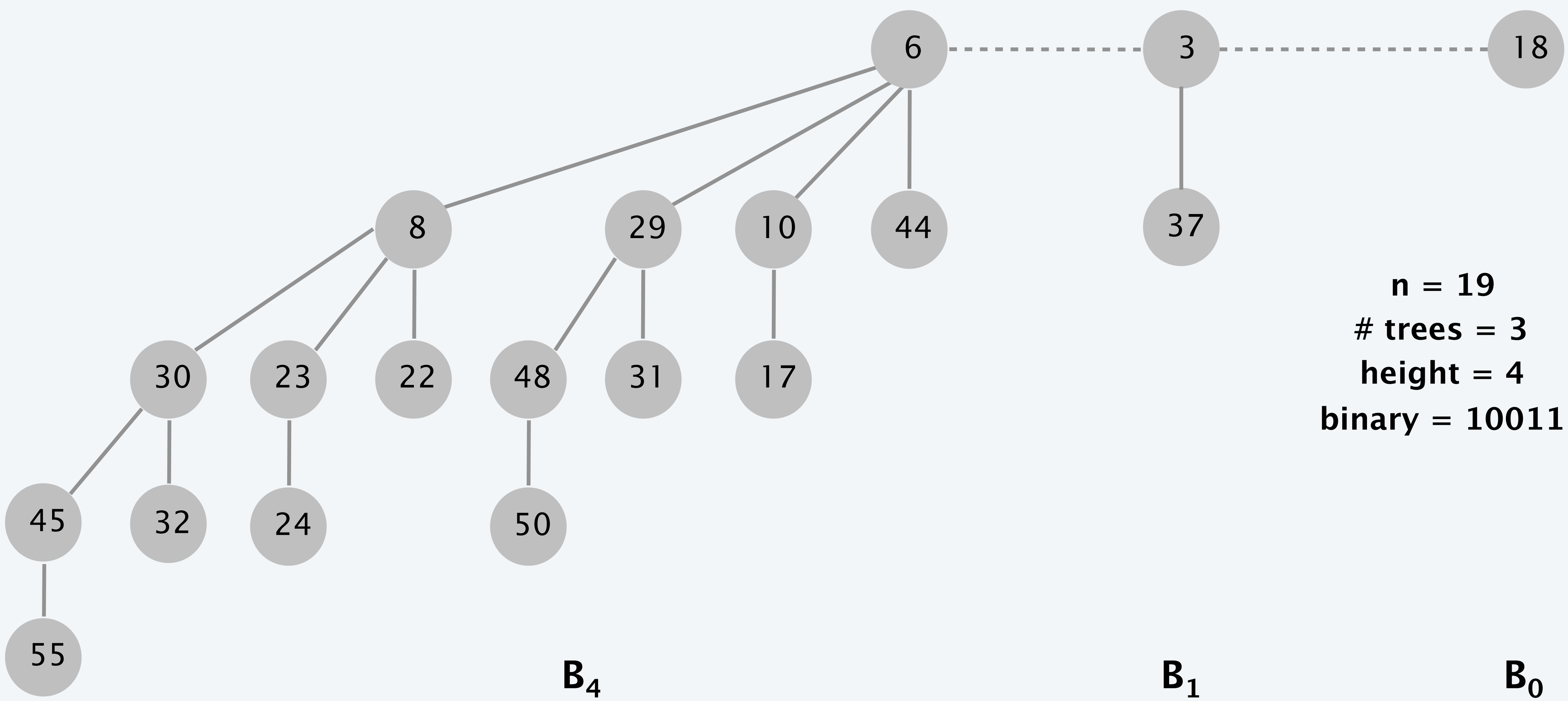


leftist power-of-2 heap representation

Binomial heap properties

Properties. Given a binomial heap with n nodes:

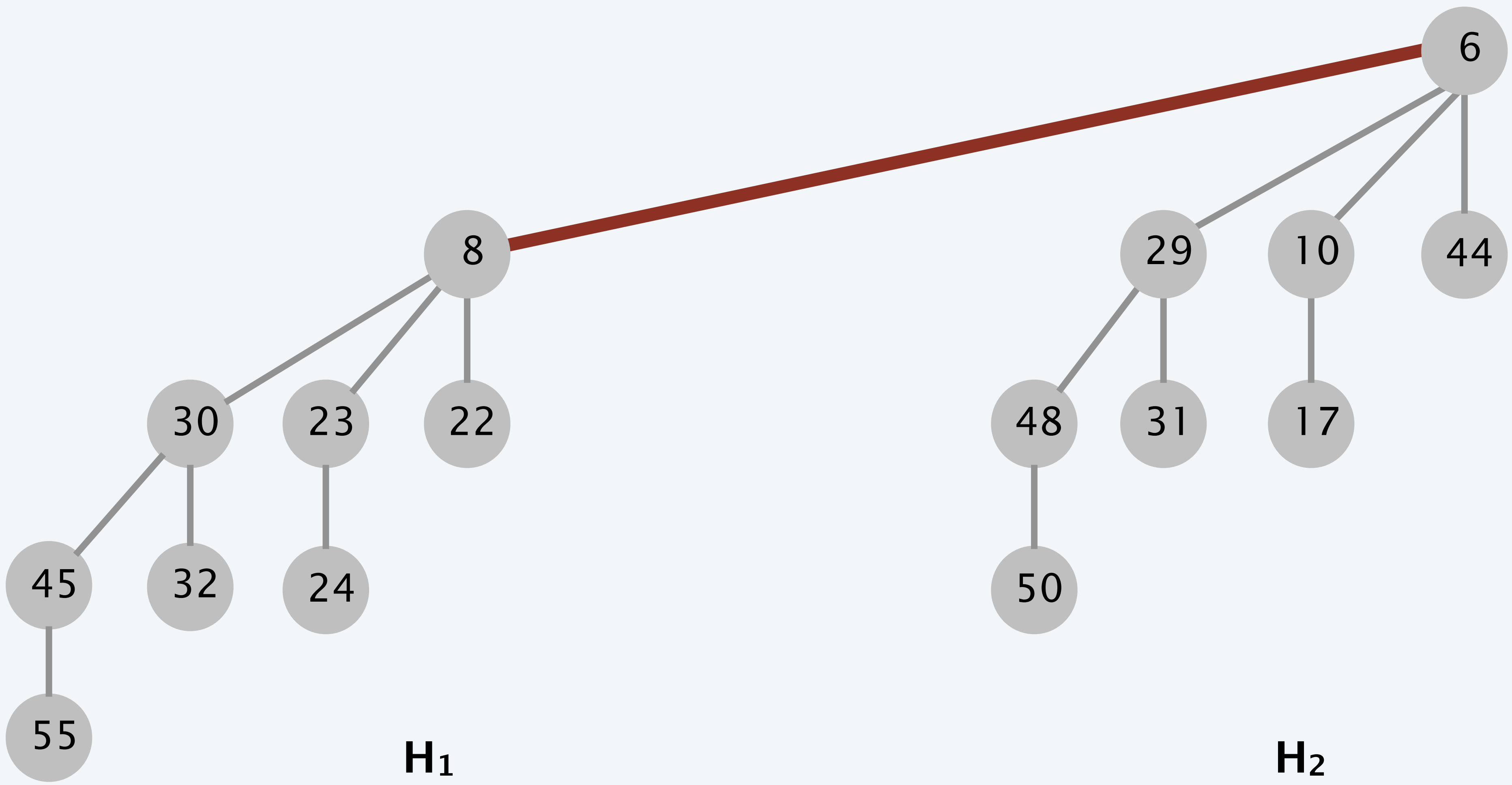
- The node containing the min element is a root of B_0, B_1, …, or B_k.
- It contains the binomial tree B_i iff $b_i = 1$, where $b_k \cdot b_2 b_1 b_0$ is binary representation of n.
- It has $\le \lfloor \log_2 n \rfloor + 1$ binomial trees.
- Its height $\le \lfloor \log_2 n \rfloor$.

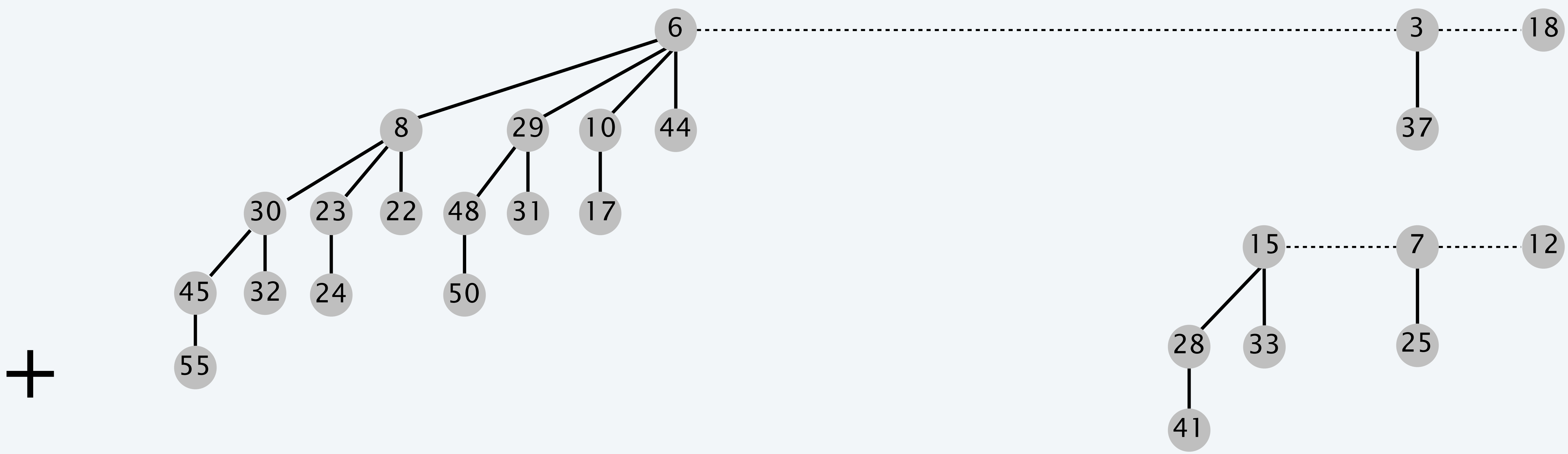


Binomial heap: meld

Meld operation. Given two binomial heaps H_1 and H_2, (destructively) replace with a binomial heap H that is the union of the two.

Warmup. Easy if H_1 and H_2 are both binomial trees of order k.
- Connect roots of H_1 and H_2.
- Choose node with smaller key to be root of H.





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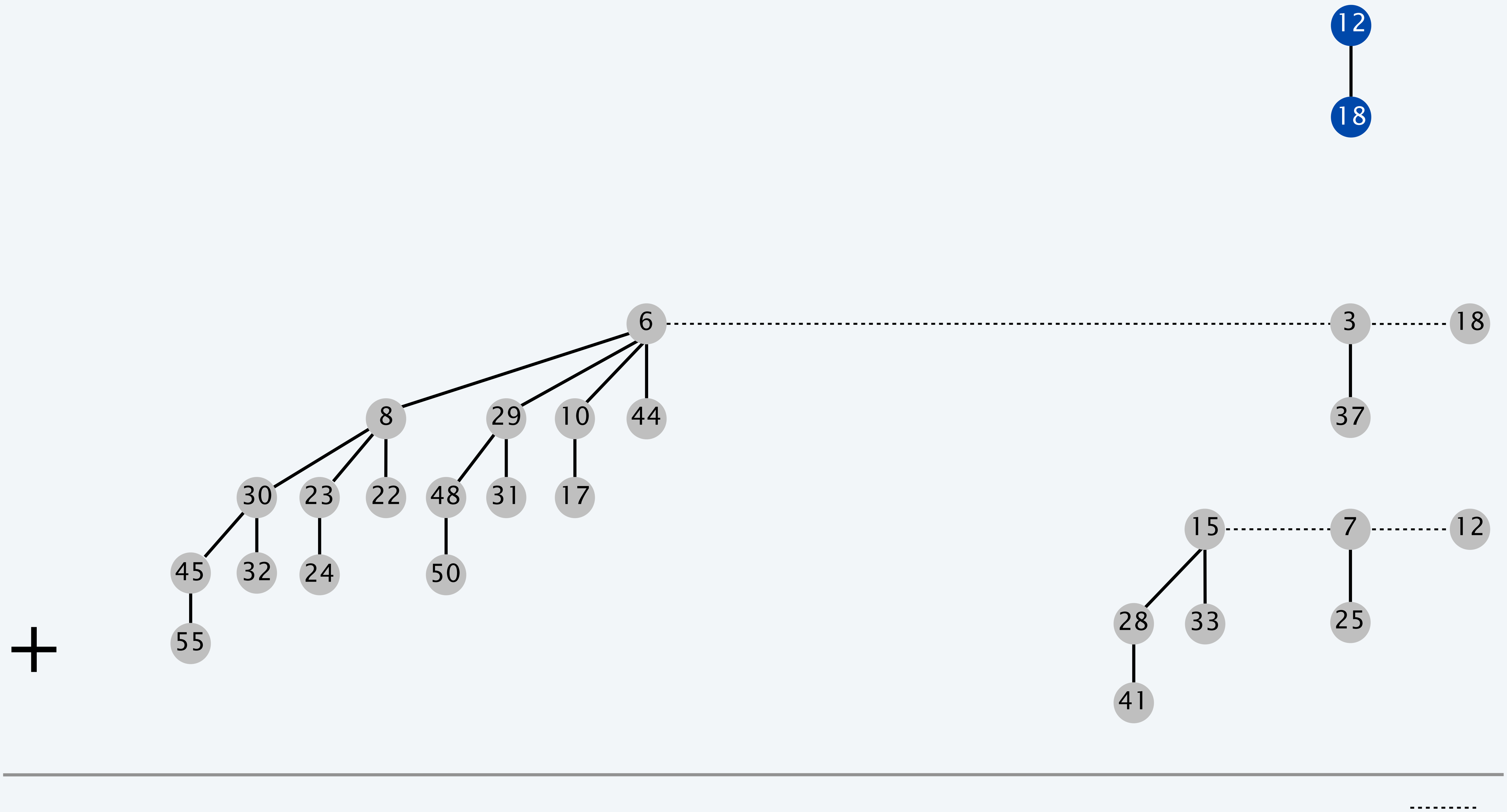
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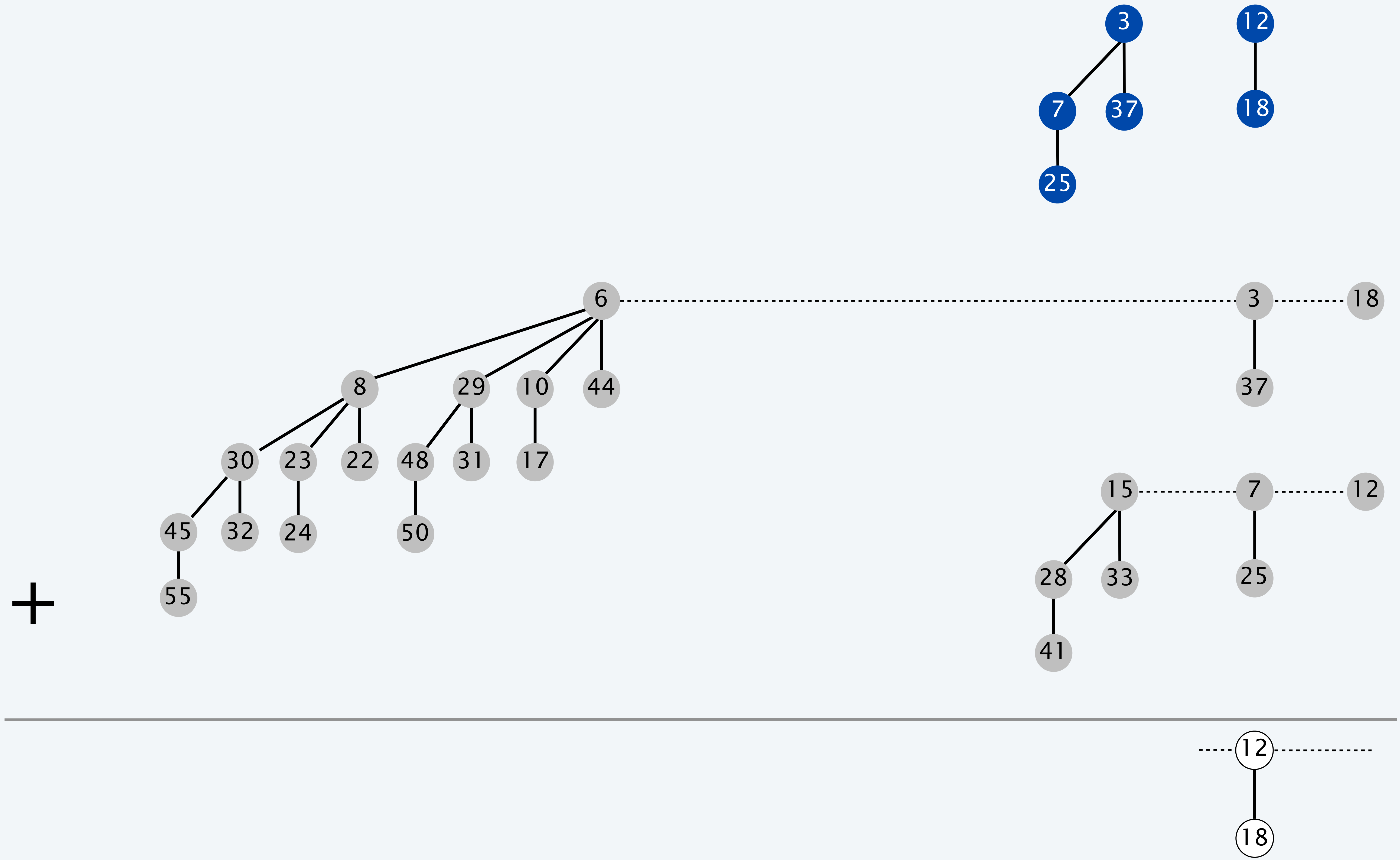
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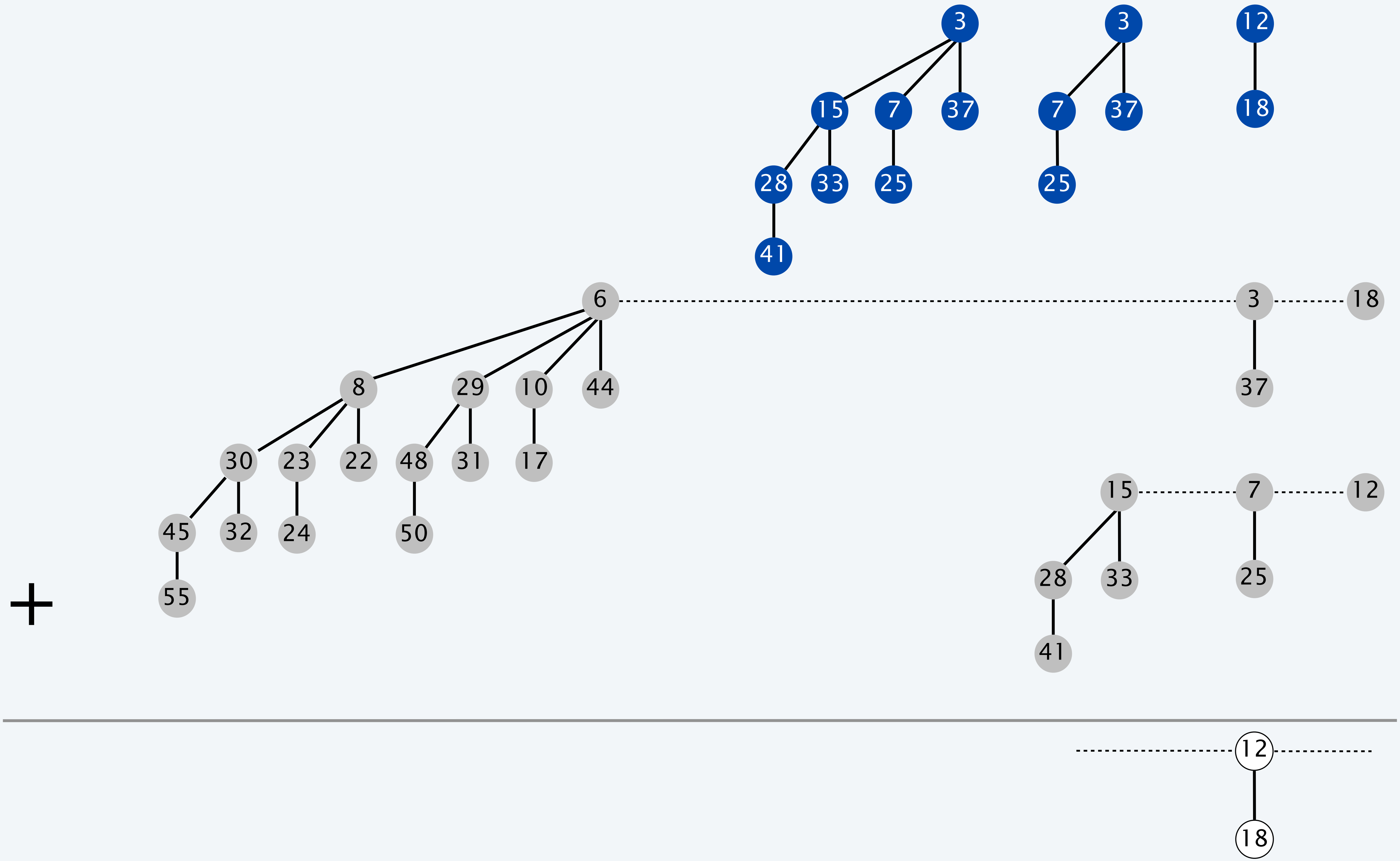

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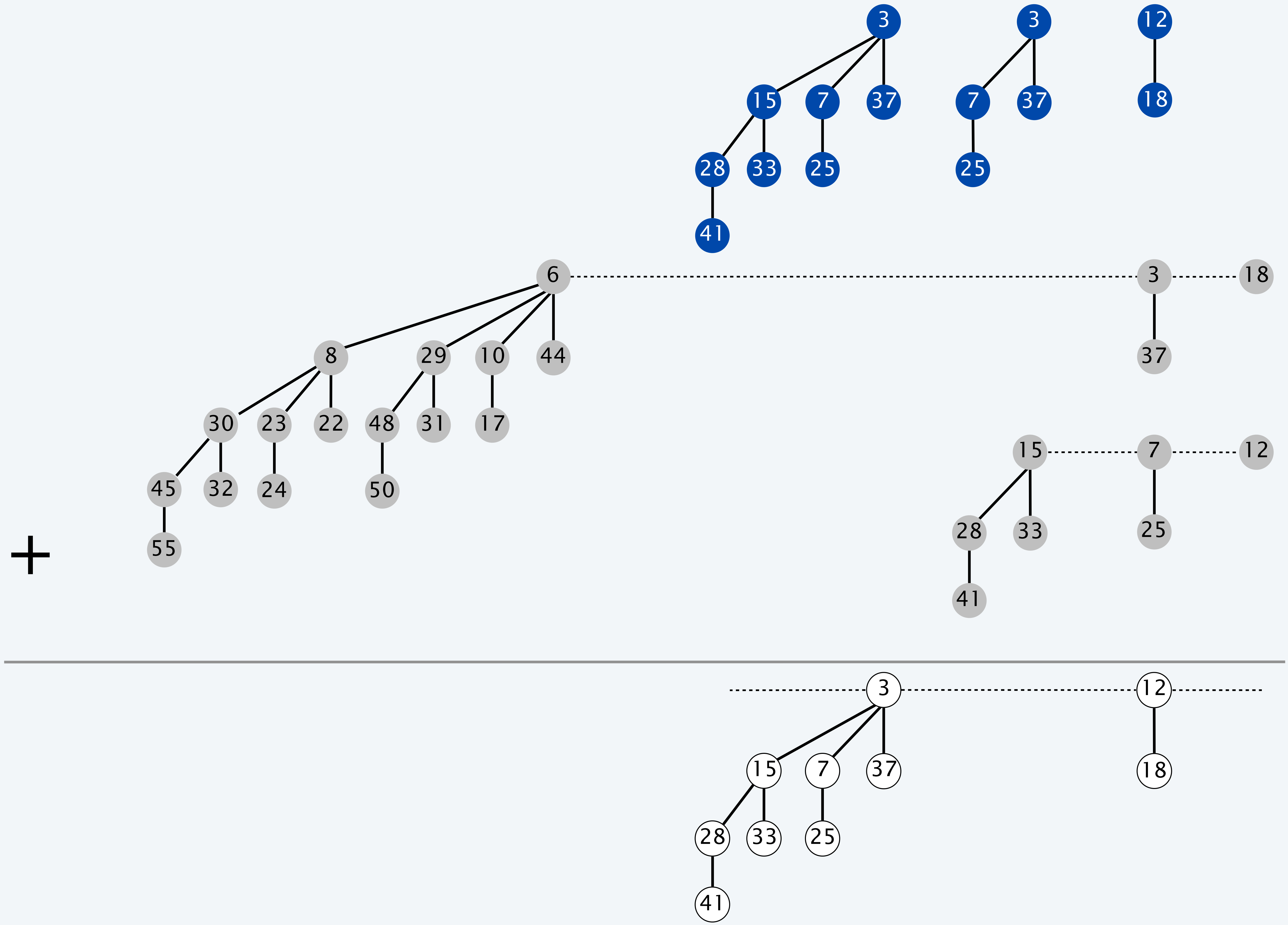
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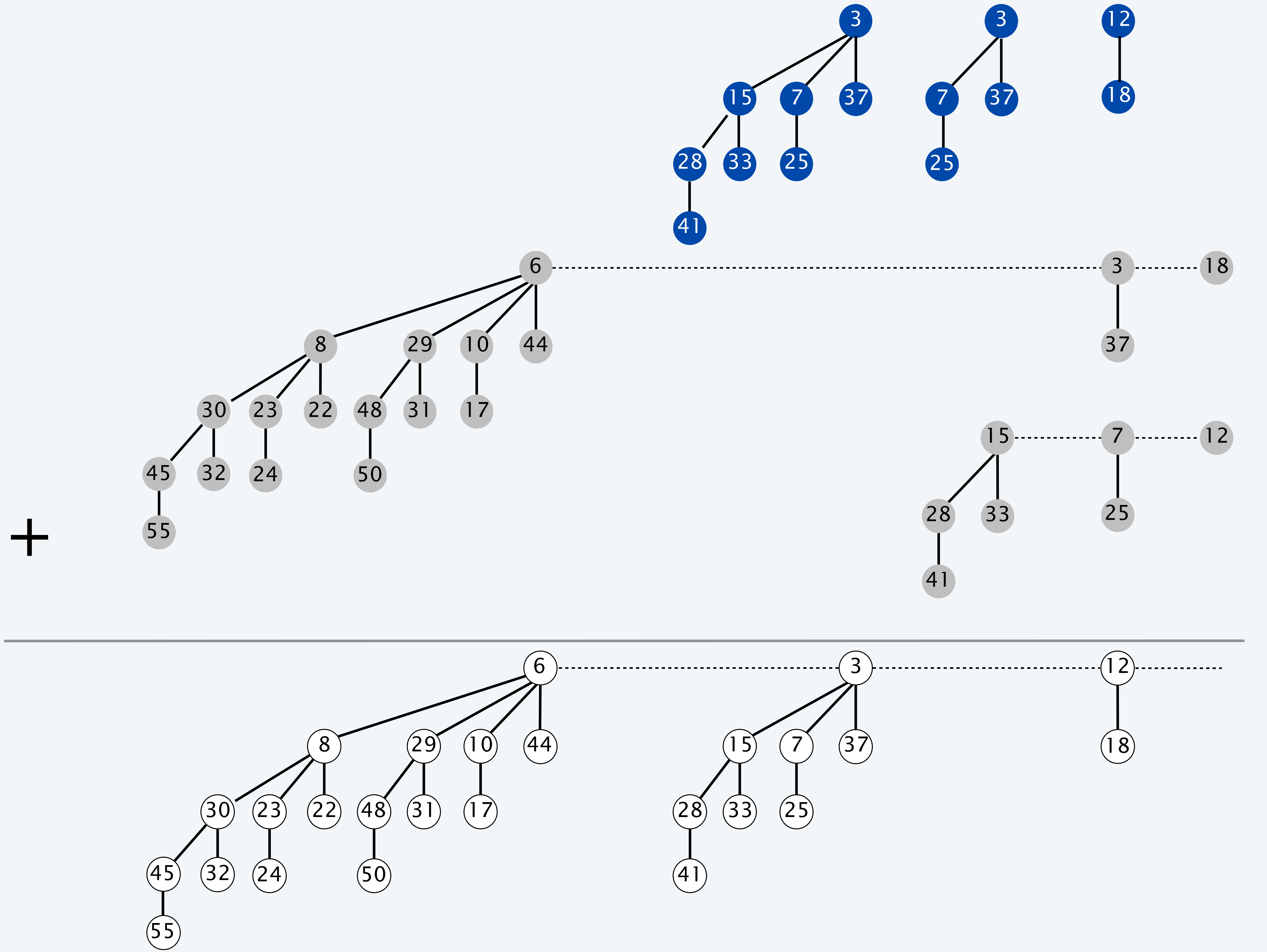

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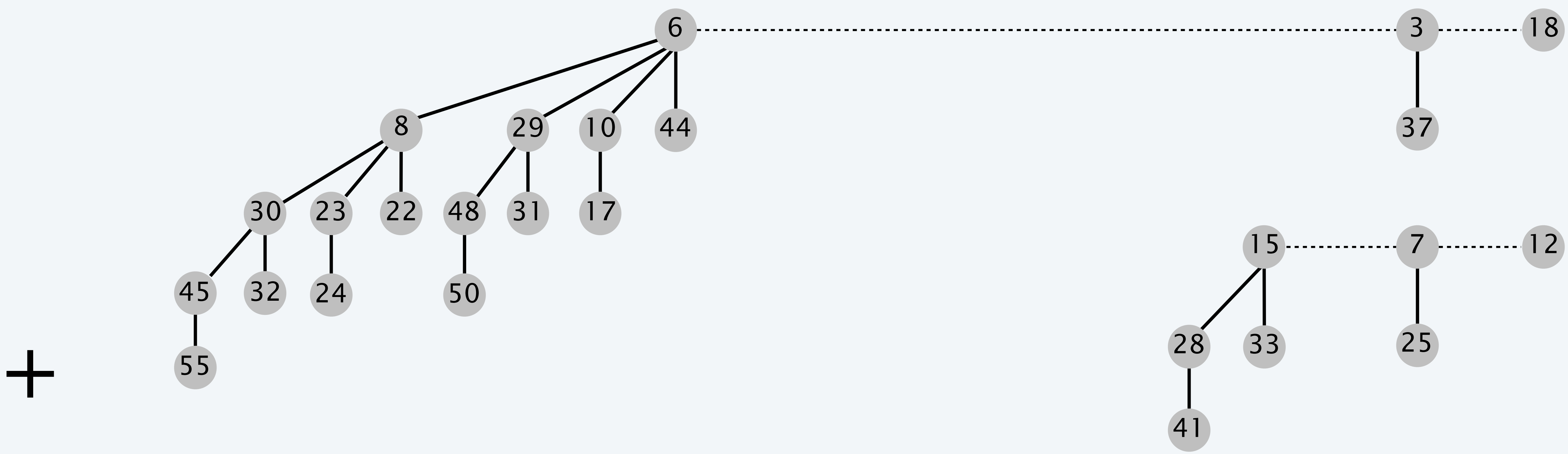


19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

Binomial heap: meld

Meld operation. Given two binomial heaps H_1 and H_2, (destructively) replace with a binomial heap H that is the union of the two.

Solution. Analogous to binary addition.

Running time. $O(\log n)$.
Pf. Proportional to number of trees in root lists $\leq 2\,(\lfloor \log_2 n \rfloor + 1)$. ▪

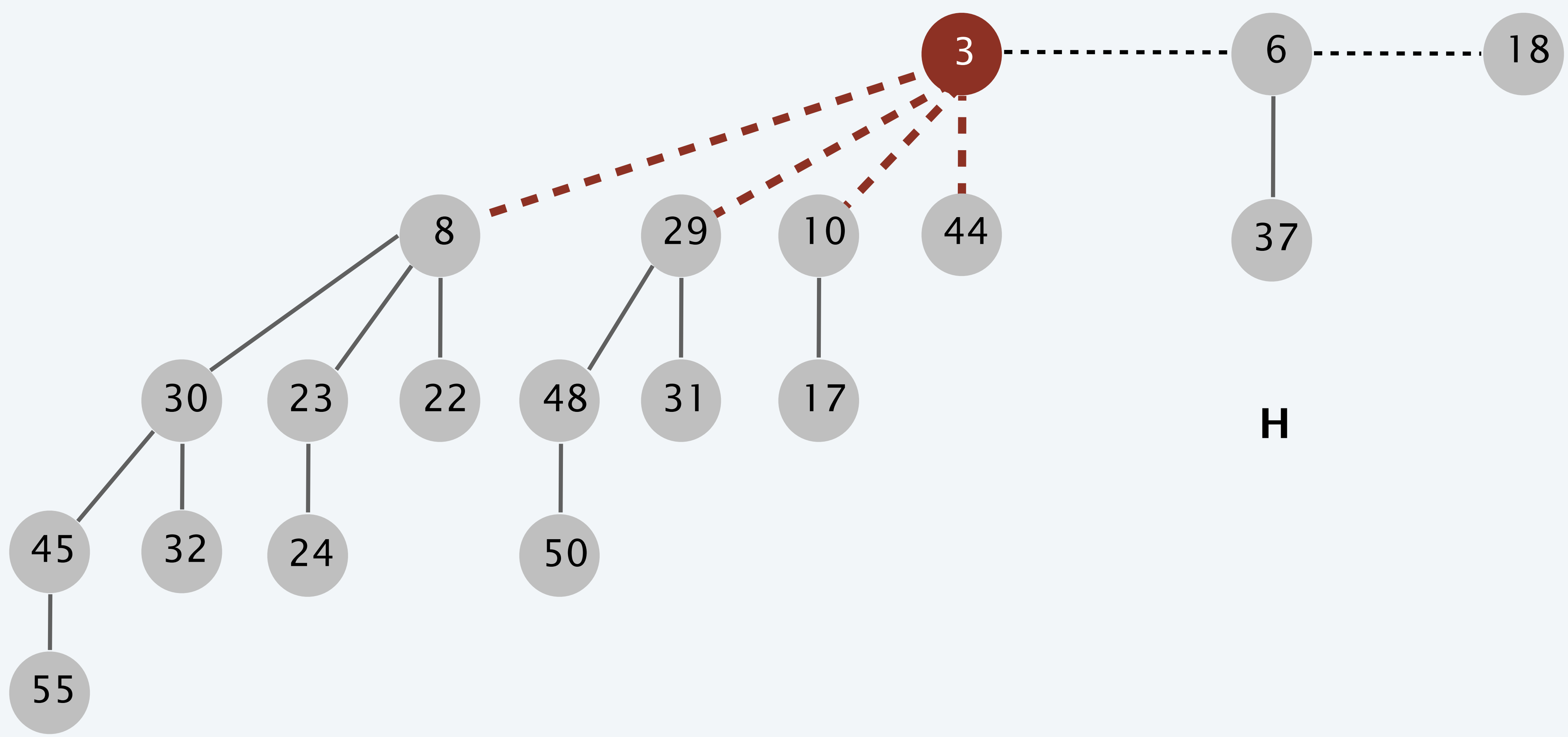
19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

Binomial heap: extract the minimum

Extract-min. Delete the node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete.

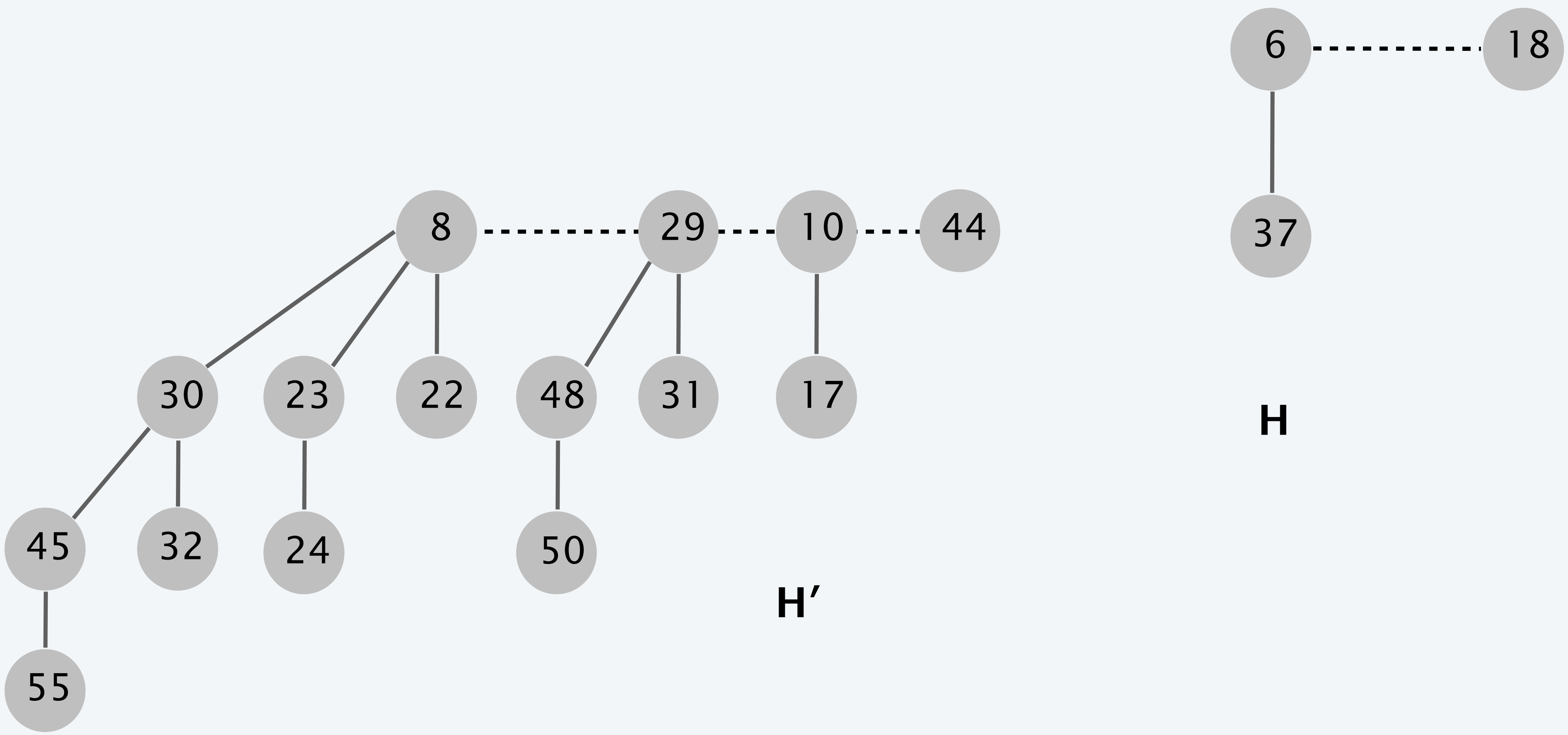


Binomial heap: extract the minimum

Extract-min. Delete the node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete.
- $H' \leftarrow$ broken binomial trees.
- $H \leftarrow$ MELD(H', H).

Running time. $O(\log n)$.

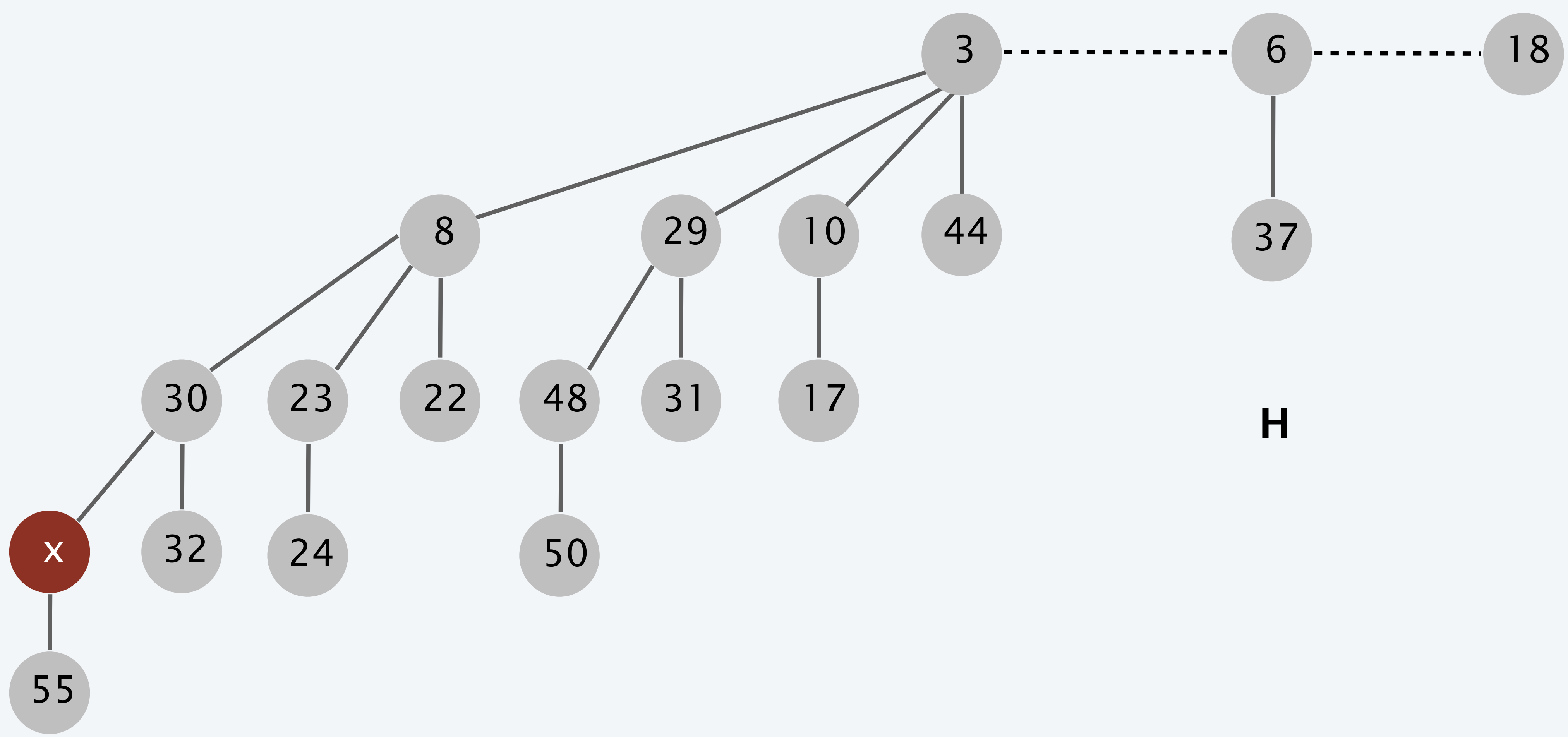


Binomial heap: decrease key

Decrease key. Given a handle to an element x in H, decrease its key to k.

- Suppose x is in binomial tree B_k.
- Repeatedly exchange x with its parent until heap order is restored.

Running time. $O(\log n)$.

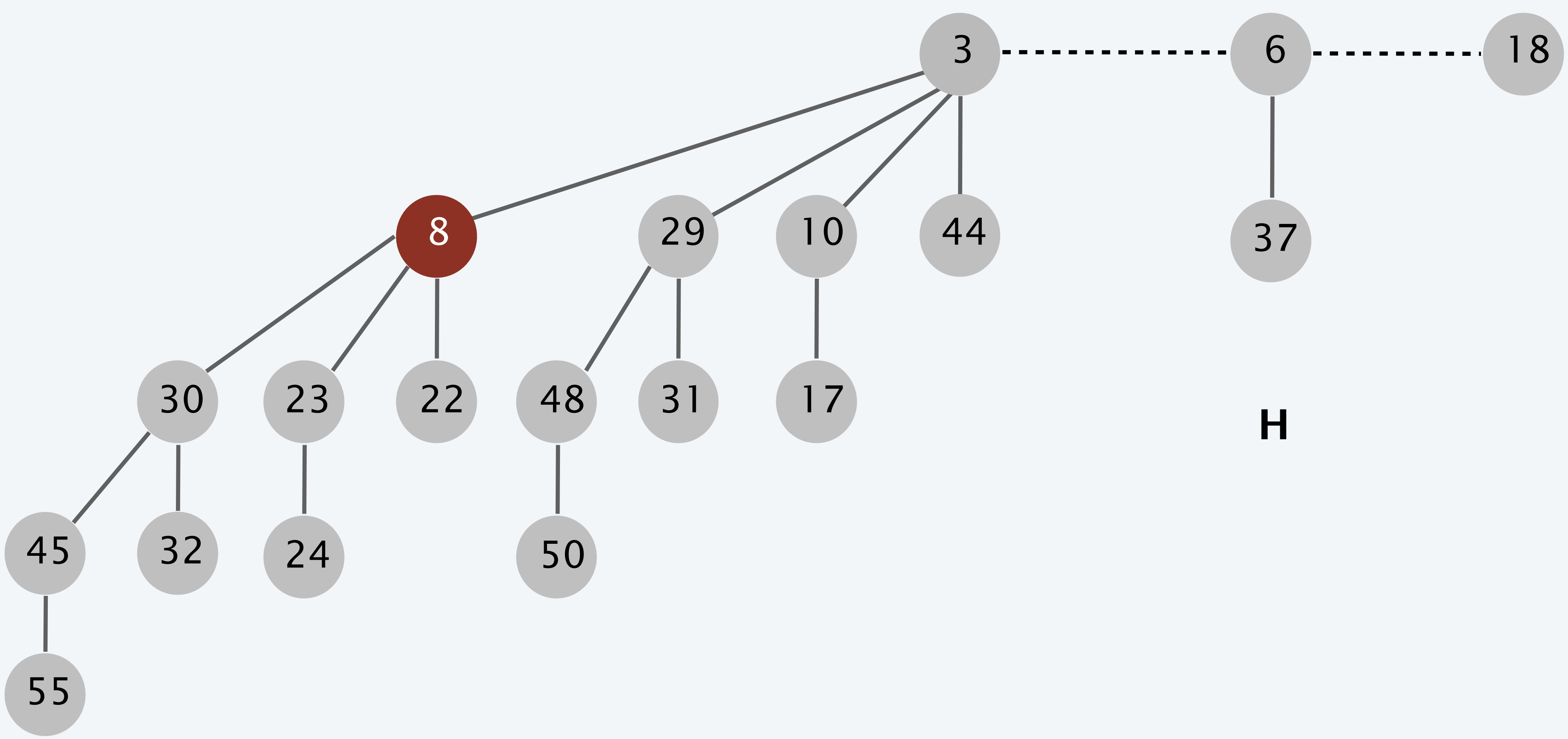


Binomial heap: delete

Delete. Given a handle to an element x in a binomial heap, delete it.

- DECREASE-KEY$(H, x, -\infty)$.
- DELETE-MIN(H).

Running time. $O(\log n)$.

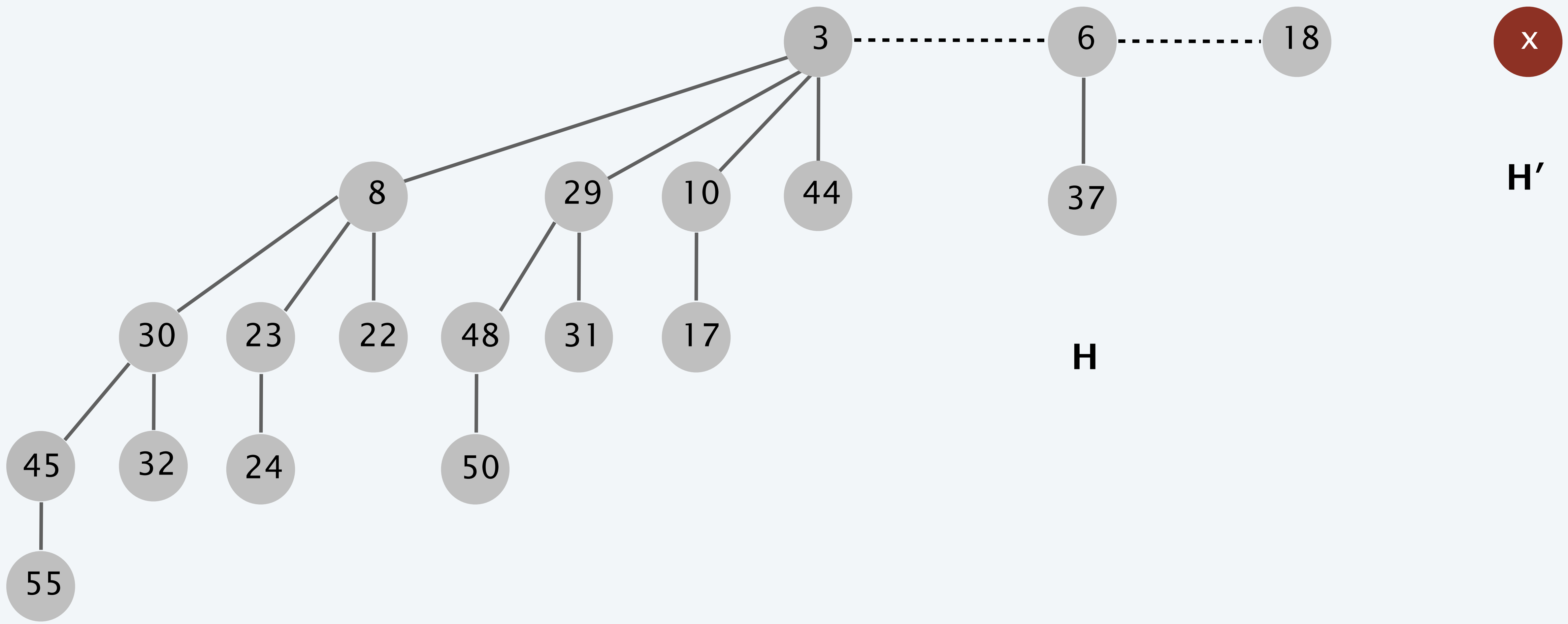


Binomial heap: insert

Insert. Given a binomial heap H, insert an element x.

- $H' \leftarrow$ MAKE-HEAP( ).
- $H' \leftarrow$ INSERT(H', x).
- $H \leftarrow$ MELD(H', H).

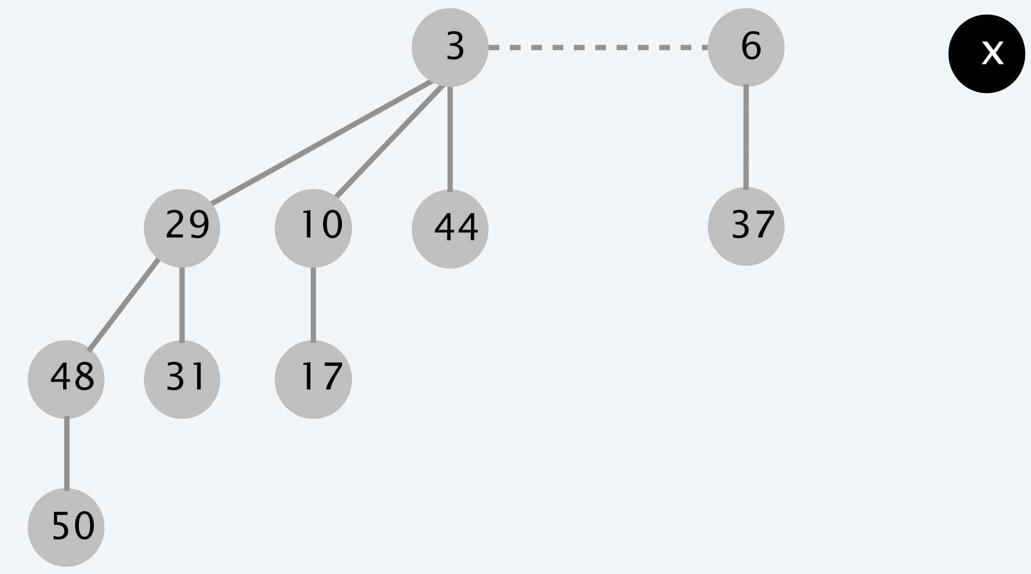
Running time. $O(\log n)$.



Binomial heap: sequence of insertions

Insert. How much work to insert a new node x?

- If $n =$ ....0, then only 1 credit.
- If $n =$ ....01, then only 2 credits.
- If $n =$ ....011, then only 3 credits.
- If $n =$ ....0111, then only 4 credits.



Observation. Inserting one element can take $\Omega(\log n)$ time.

if n = 11...111

Theorem. Starting from an empty binomial heap, a sequence of n consecutive INSERT operations takes $O(n)$ time.

Pf. $(n / 2)(1) + (n / 4)(2) + (n / 8)(3) + \ldots \leq 2n$. ▪

$$\sum_{i=1}^{k} \frac{i}{2^i} = 2 - \frac{k}{2^k} - \frac{1}{2^{k-1}}$$
$$\leq 2$$

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = trees(H_i) = \#$ trees in binomial heap H_i.

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i.

Case 1. [INSERT]

- Actual cost c_i = number of trees merged $+ 1$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) = 1 -$ number of trees merged.
- Amortized cost $= \hat{c}_i = c_i + \Phi(H_i) - \Phi(H_{i-1}) = 2$.

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = \mathit{trees}(H_i) = \#$ trees in binomial heap H_i.

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i.

Case 2. [DECREASE-KEY]

- Actual cost $c_i = O(\log n)$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) = 0$.
- Amortized cost $= \hat{c}_i = c_i = O(\log n)$.

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = trees(H_i) = \#$ trees in binomial heap H_i.

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i.

Case 3. [EXTRACT-MIN or DELETE]

- Actual cost $c_i = O(\log n)$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) \leq \Phi(H_i) \leq \lfloor \log_2 n \rfloor$.
- Amortized cost $= \hat{c}_i = c_i + \Phi(H_i) - \Phi(H_{i-1}) = O(\log n)$. ▪

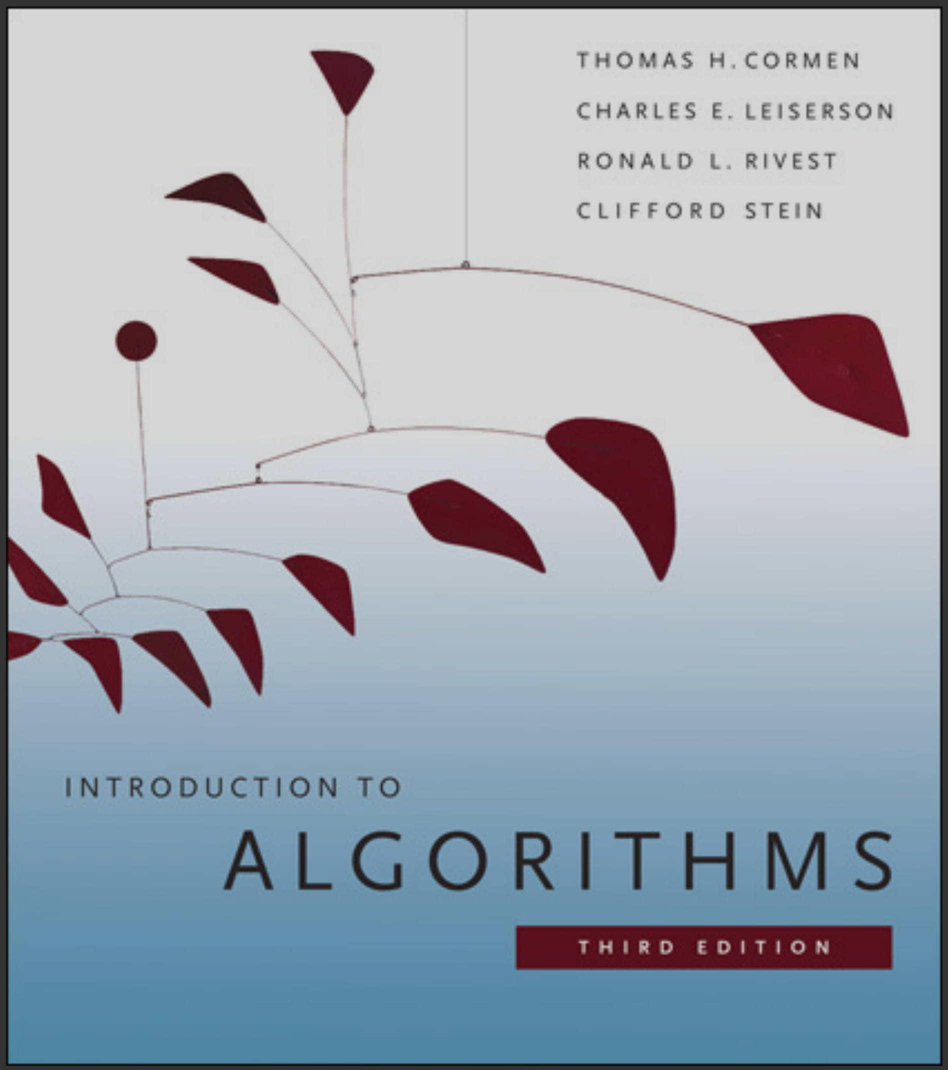
Priority queues performance cost summary

operation	linked list	binary heap	binomial heap	binomial heap
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$ †
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
MELD	$O(1)$	$O(n)$	$O(\log n)$	$O(1)$ † homework
FIND-MIN	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$

† amortized

Hopeless challenge. $O(1)$ INSERT, DECREASE-KEY and EXTRACT-MIN. Why?

Challenge. $O(1)$ INSERT and DECREASE-KEY, $O(\log n)$ EXTRACT-MIN.



Fibonacci Heaps

- *preliminaries*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 7/25/17 11:07 AM

Priority queues performance cost summary

operation	linked list	binary heap	binomial heap	Fibonacci heap †
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$	$O(1)$
IS-EMPTY	$O(1)$	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$
DELETE	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
MELD	$O(1)$	$O(n)$	$O(\log n)$	$O(1)$
FIND-MIN	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$

† amortized

Ahead. $O(1)$ INSERT and DECREASE-KEY, $O(\log n)$ EXTRACT-MIN.

Fibonacci heaps

Theorem. [Fredman-Tarjan 1986] Starting from an empty Fibonacci heap, any sequence of m INSERT, EXTRACT-MIN, and DECREASE-KEY operations involving n INSERT operations takes $O(m + n \log n)$ time.

this statement is a bit weaker than the actual theorem

Fibonacci Heaps and Their Uses in Improved Network Optimization Algorithms

MICHAEL L. FREDMAN

University of California, San Diego, La Jolla, California

AND

ROBERT ENDRE TARJAN

AT&T Bell Laboratories, Murray Hill, New Jersey

Abstract. In this paper we develop a new data structure for implementing heaps (priority queues). Our structure, *Fibonacci heaps* (abbreviated *F-heaps*), extends the binomial queues proposed by Vuillemin and studied further by Brown. F-heaps support arbitrary deletion from an n-item heap in $O(\log n)$ amortized time and all other standard heap operations in $O(1)$ amortized time. Using F-heaps we are able to obtain improved running times for several network optimization algorithms. In particular, we obtain the following worst-case bounds, where n is the number of vertices and m the number of edges in the problem graph:

(1) $O(n \log n + m)$ for the single-source shortest path problem with nonnegative edge lengths, improved from $O(m \log_{(m/n+2)} n)$;
(2) $O(n^2 \log n + nm)$ for the all-pairs shortest path problem, improved from $O(nm \log_{(m/n+2)} n)$;
(3) $O(n^2 \log n + nm)$ for the assignment problem (weighted bipartite matching), improved from $O(nm \log_{(m/n+2)} n)$;
(4) $O(m\beta(m, n))$ for the minimum spanning tree problem, improved from $O(m \log\log_{(m/n+2)} n)$, where $\beta(m, n) = \min \{i \mid \log^{(i)} n \le m/n\}$. Note that $\beta(m, n) \le \log^* n$ if $m \ge n$.

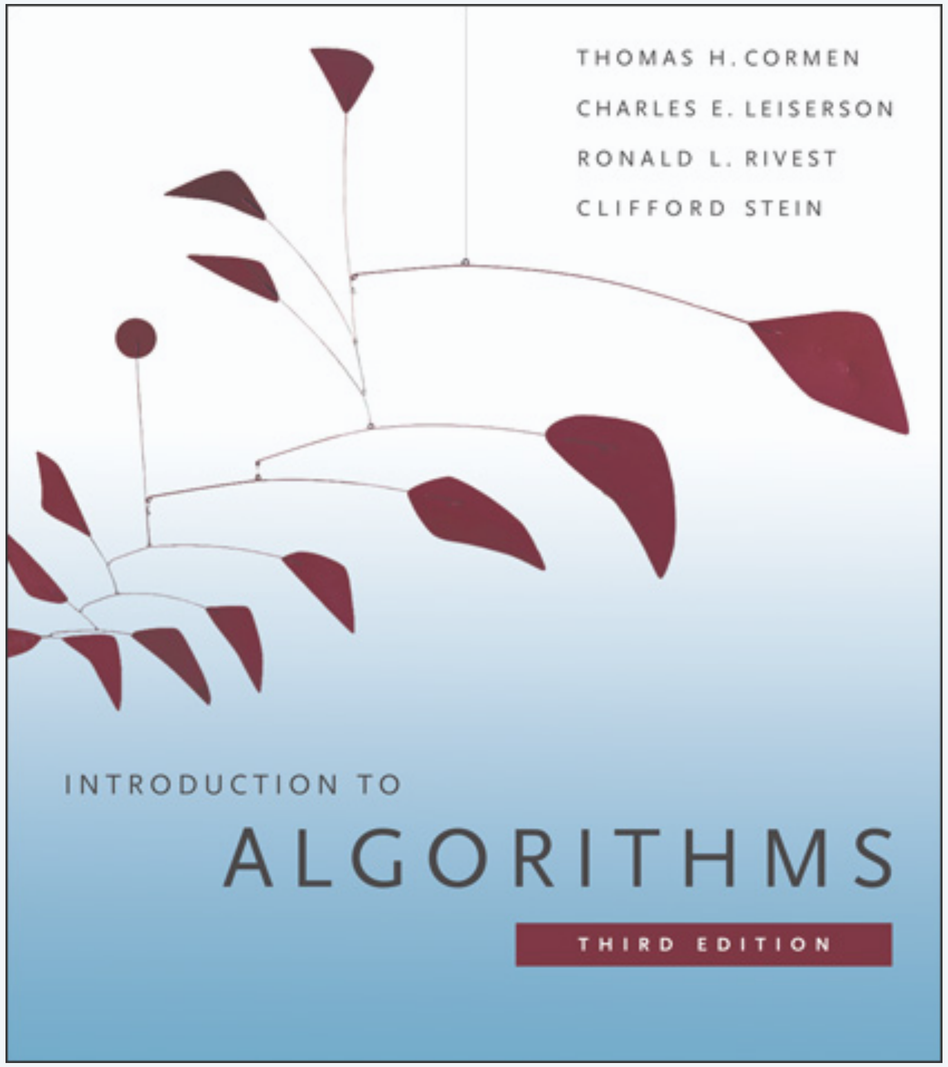
Of these results, the improved bound for minimum spanning trees is the most striking, although all the results give asymptotic improvements for graphs of appropriate densities.

Fibonacci heaps

Theorem. [Fredman–Tarjan 1986] Starting from an empty Fibonacci heap, any sequence of m INSERT, EXTRACT-MIN, and DECREASE-KEY operations involving n INSERT operations takes $O(m + n \log n)$ time.

History.

- Ingenious data structure and application of amortized analysis.
- Original motivation: improve Dijkstra's shortest path algorithm from $O(m \log n)$ to $O(m + n \log n)$.
- Also improved best-known bounds for all-pairs shortest paths, assignment problem, minimum spanning trees.



SECTION 19.1

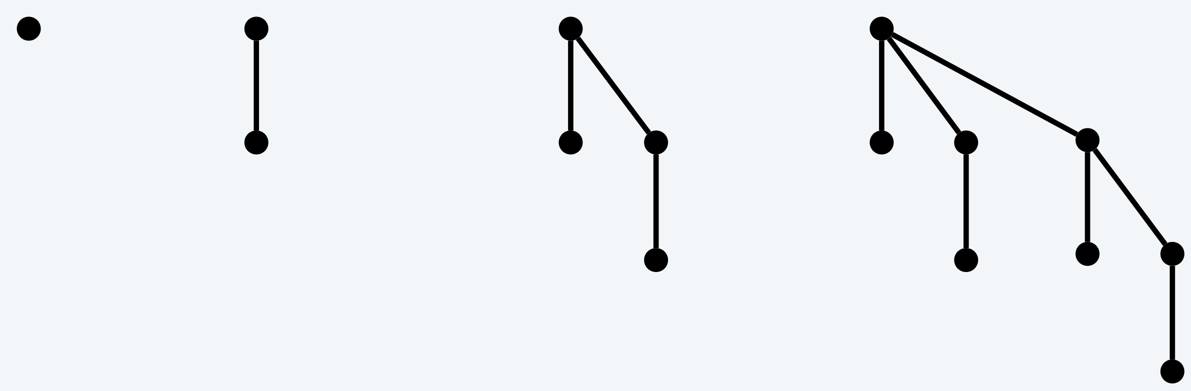
FIBONACCI HEAPS

- *structure*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Fibonacci heaps

Basic idea.

- Similar to binomial heaps, but less rigid structure.
- Binomial heap: eagerly consolidate trees after each INSERT; implement DECREASE-KEY by repeatedly exchanging node with its parent.



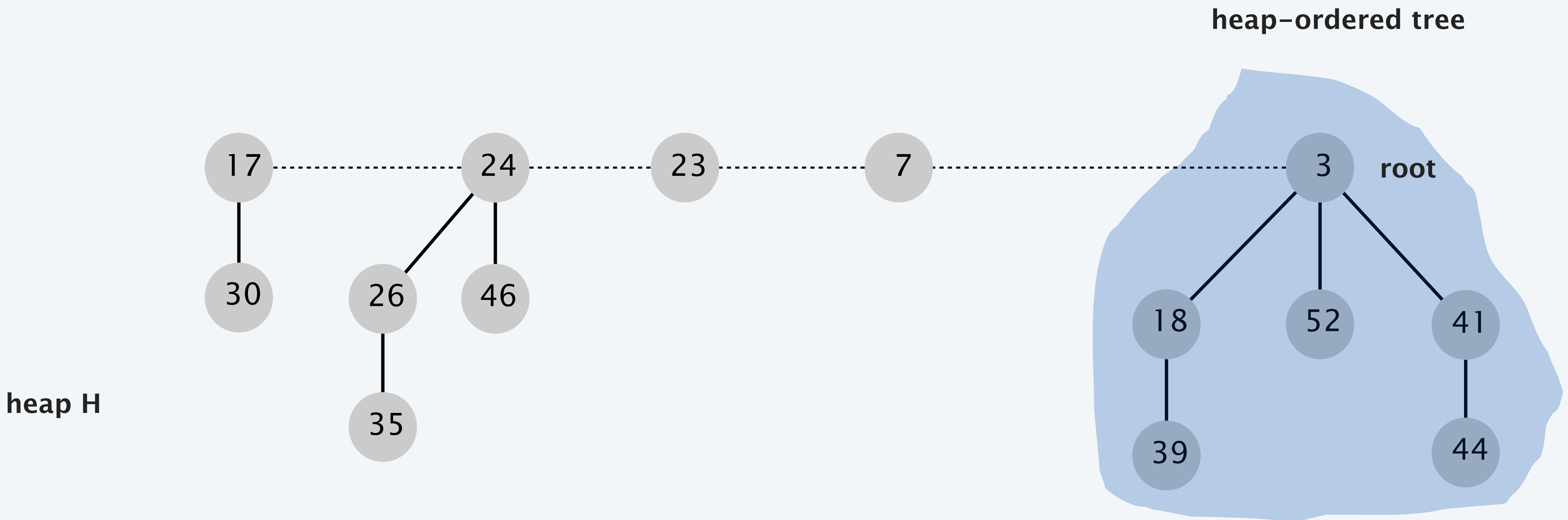
- Fibonacci heap: lazily defer consolidation until next EXTRACT-MIN; implement DECREASE-KEY by cutting off node and splicing into root list.

Remark. Height of Fibonacci heap is $\Theta(n)$ in worst case, but it doesn't use sink or swim operations.

Fibonacci heap: structure

- Set of heap-ordered trees.

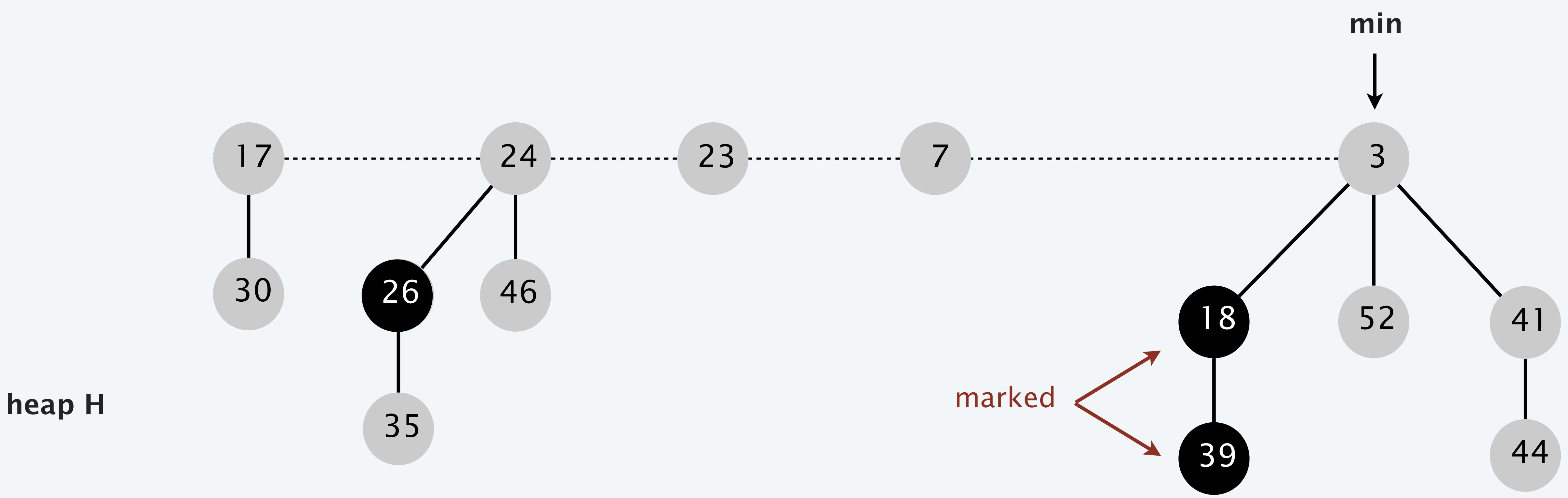
each child no smaller than its parent



Fibonacci heap: structure

- Set of heap-ordered trees.
- Set of marked nodes.

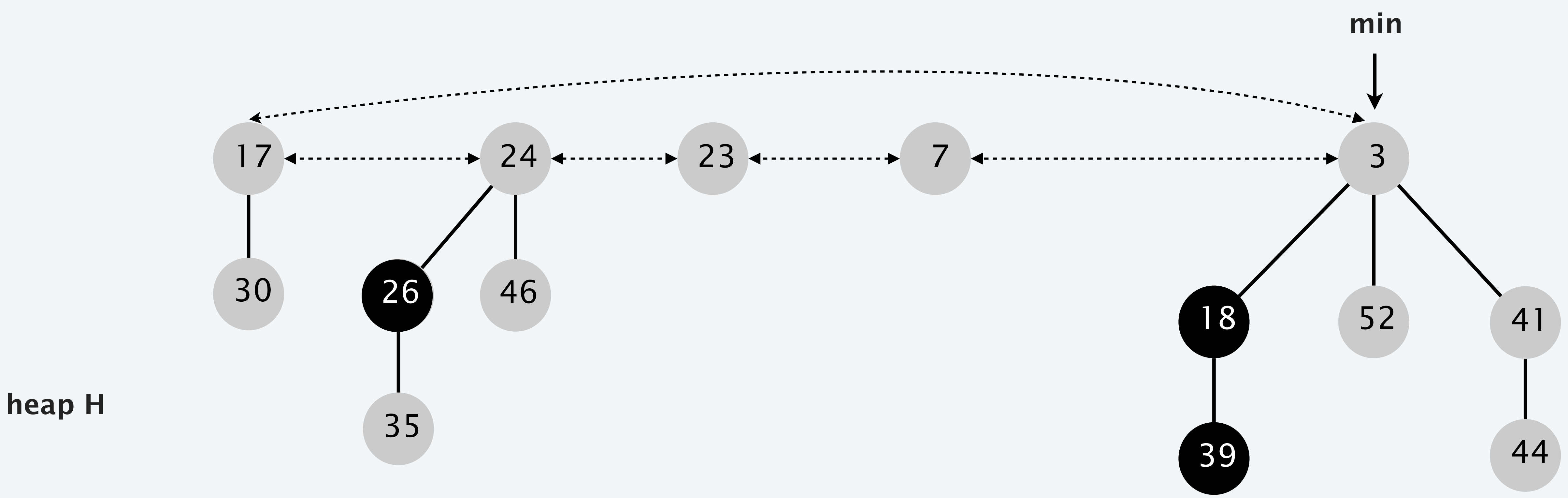
used to keep trees bushy
(stay tuned)



Fibonacci heap: structure

Heap representation.

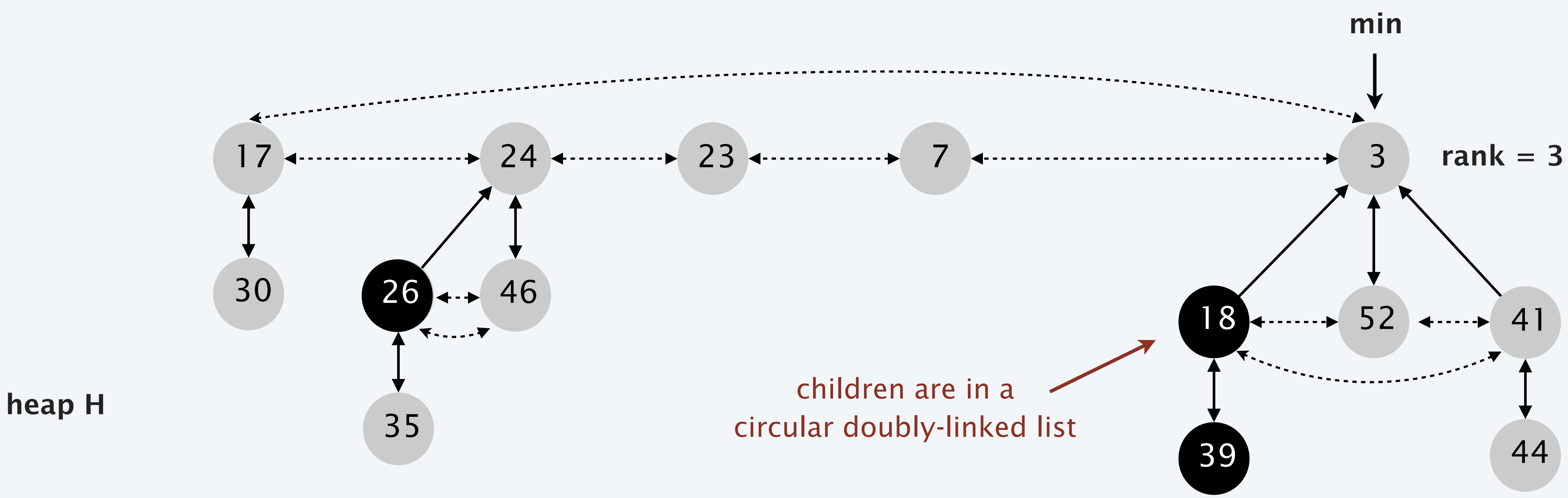
- Store a pointer to the minimum node.
- Maintain tree roots in a circular, doubly-linked list.



Fibonacci heap: representation

Node representation. Each node stores:

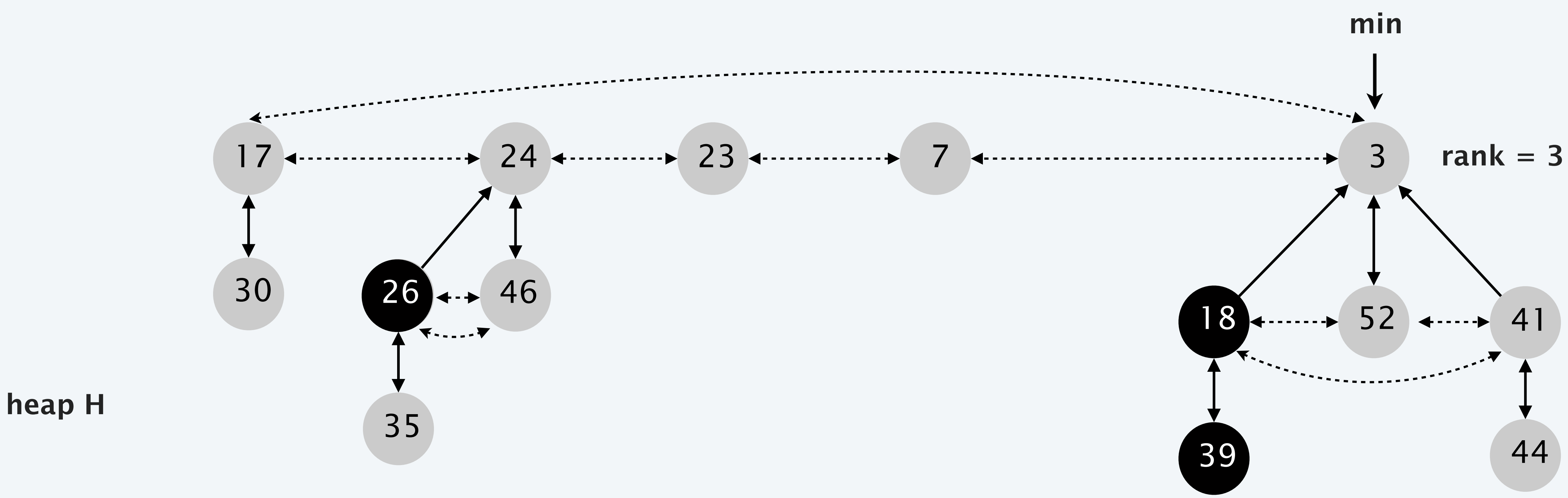
- A pointer to its parent.
- A pointer to any of its children.
- A pointer to its left and right siblings.
- Its rank = number of children.
- Whether it is marked.



Fibonacci heap: representation

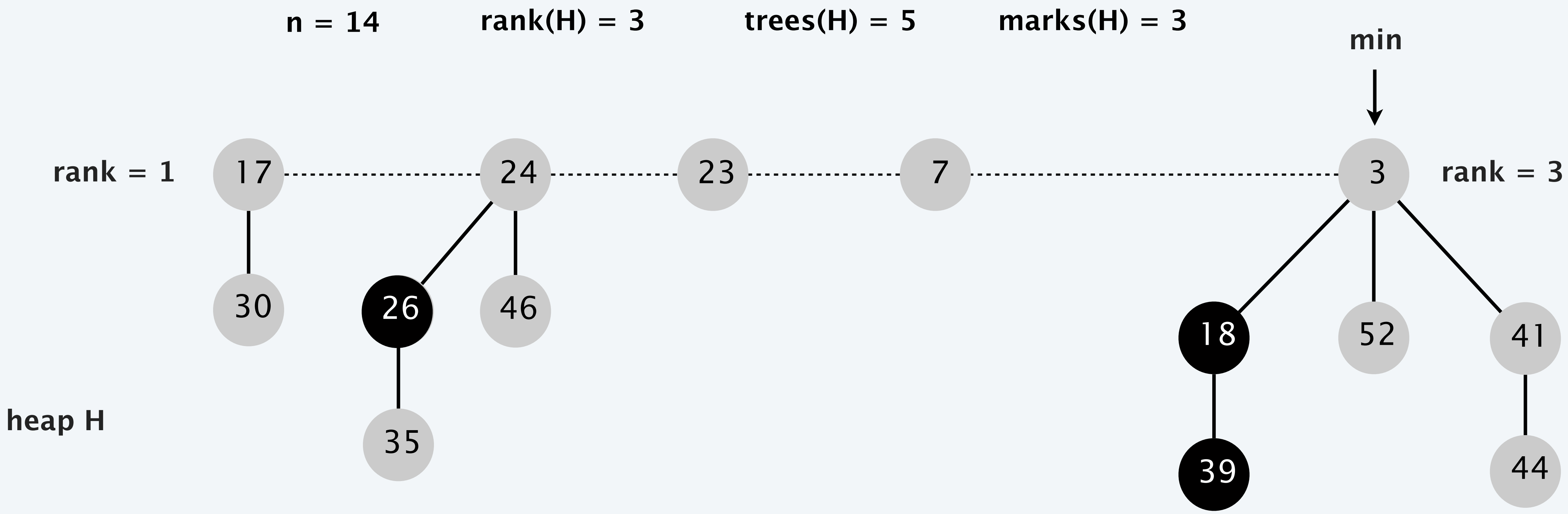
Operations we can do in constant time:

- Determine rank of a node.
- Find the minimum element.
- Merge two root lists together.
- Add or remove a node from the root list.
- Remove a subtree and merge into root list.
- Link the root of a one tree to root of another tree.



Fibonacci heap: notation

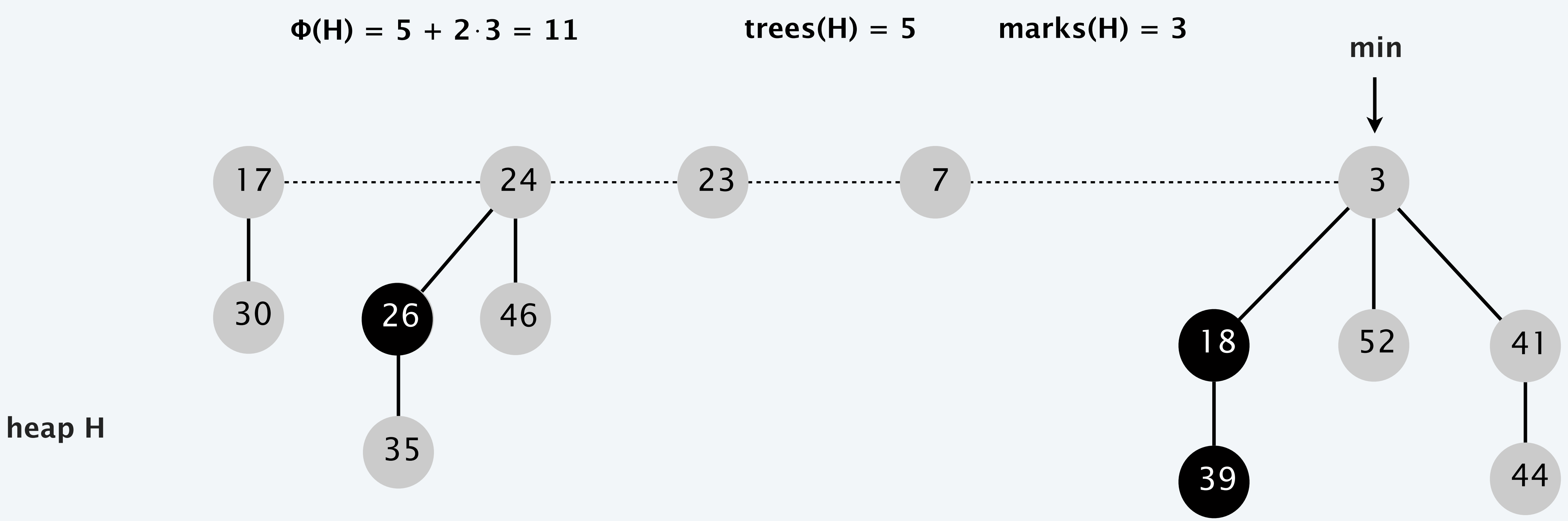
notation	meaning
n	number of nodes
$rank(x)$	number of children of node x
$rank(H)$	max rank of any node in heap H
$trees(H)$	number of trees in heap H
$marks(H)$	number of marked nodes in heap H

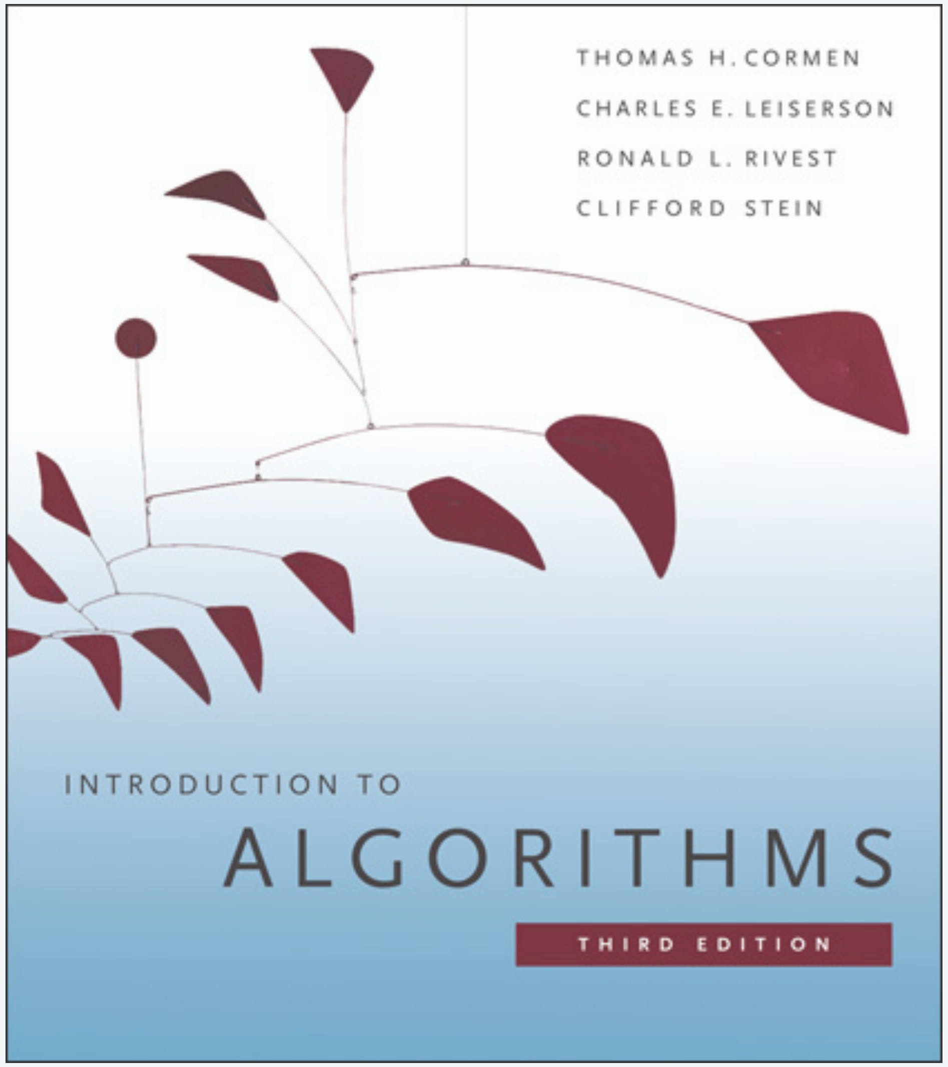


Fibonacci heap: potential function

Potential function.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$





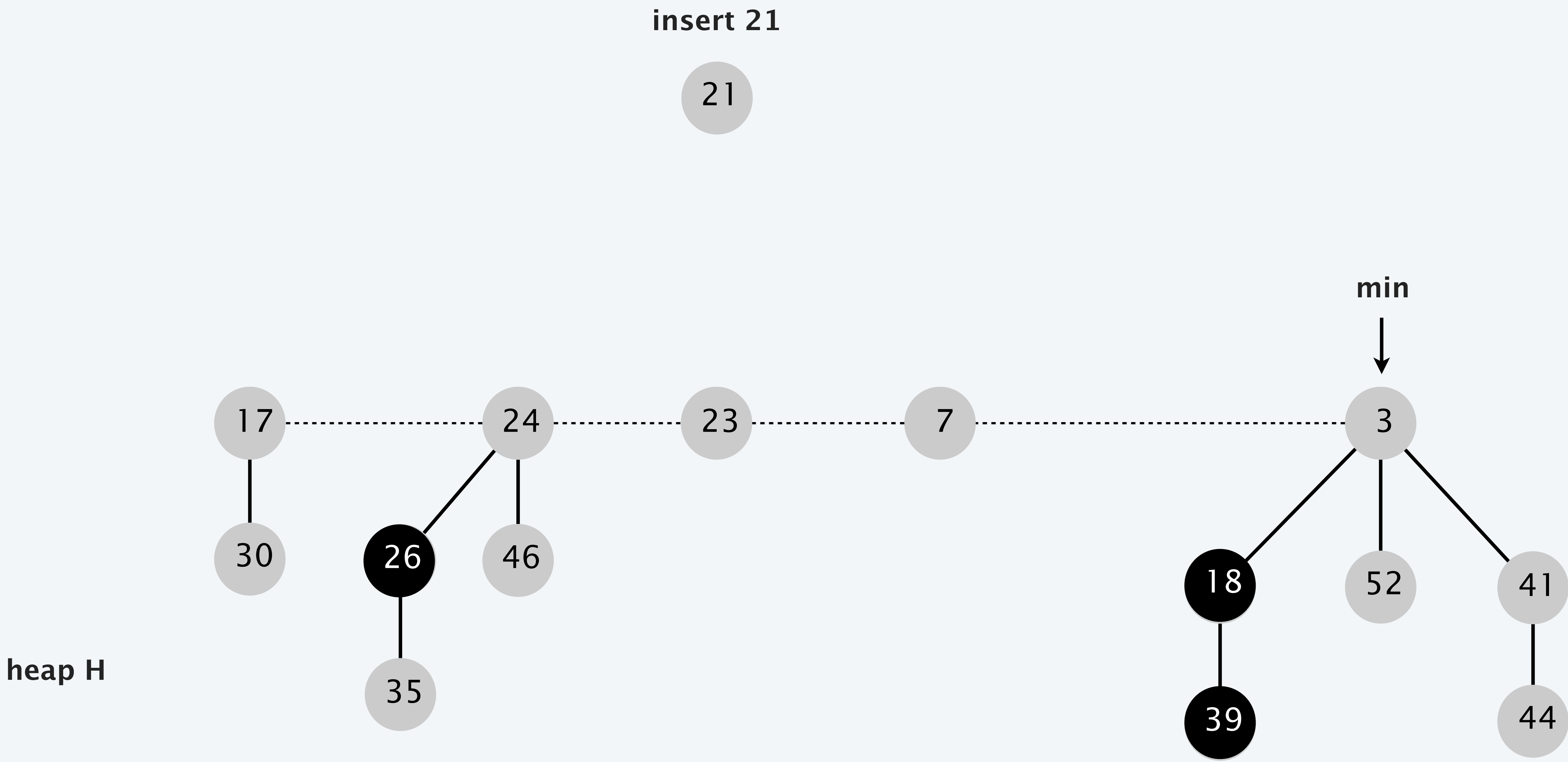
SECTION 19.2

FIBONACCI HEAPS

- *preliminaries*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Fibonacci heap: insert

- Create a new singleton tree.
- Add to root list; update min pointer (if necessary).



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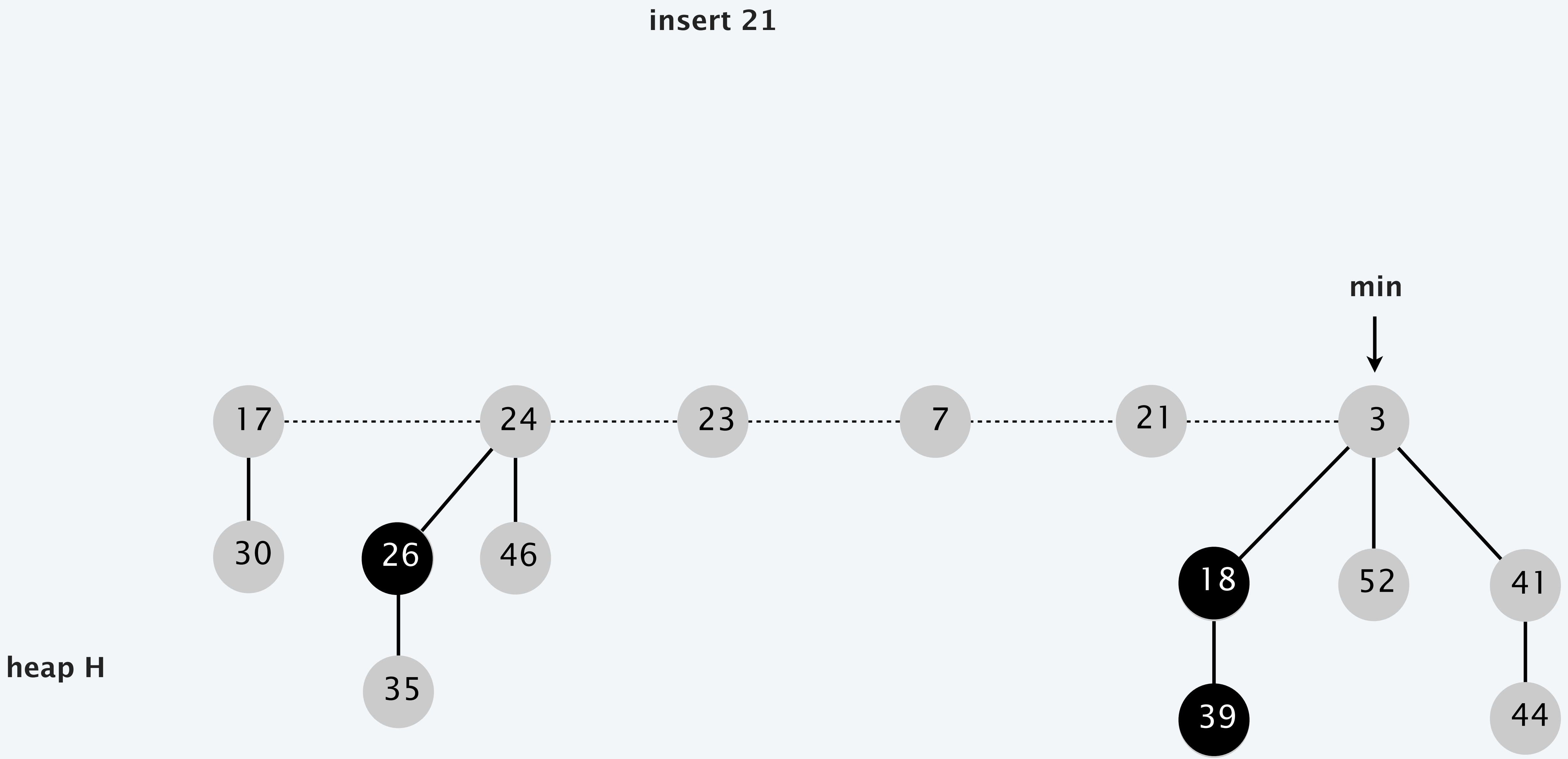


Fibonacci heap: insert analysis

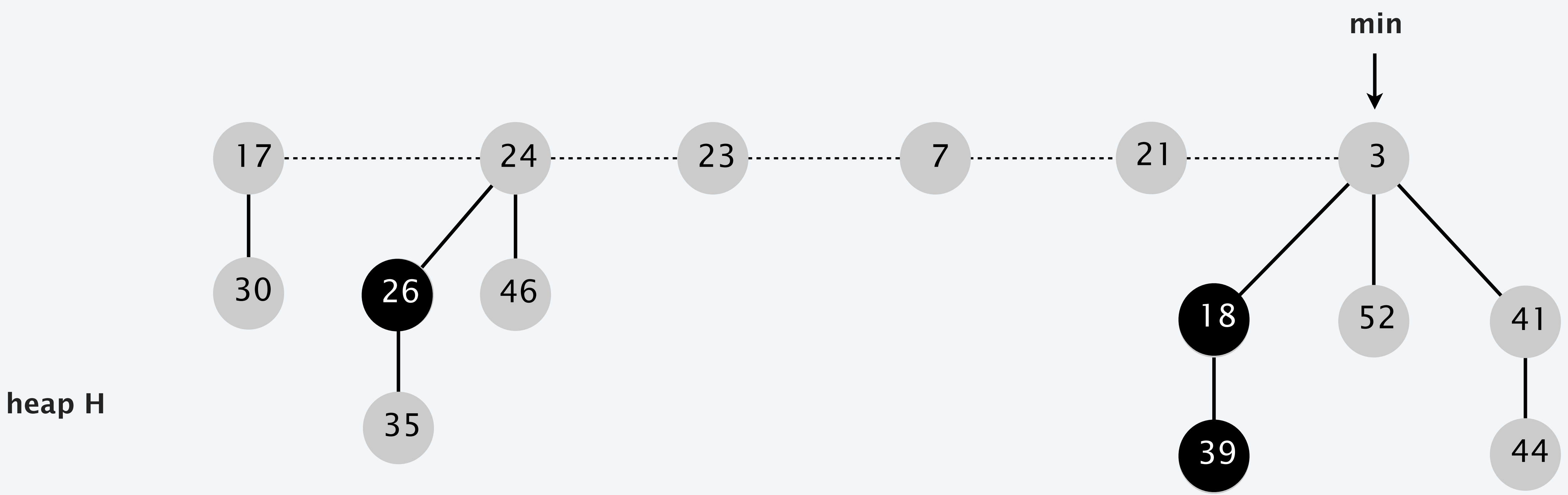
Actual cost. $c_i = O(1)$.

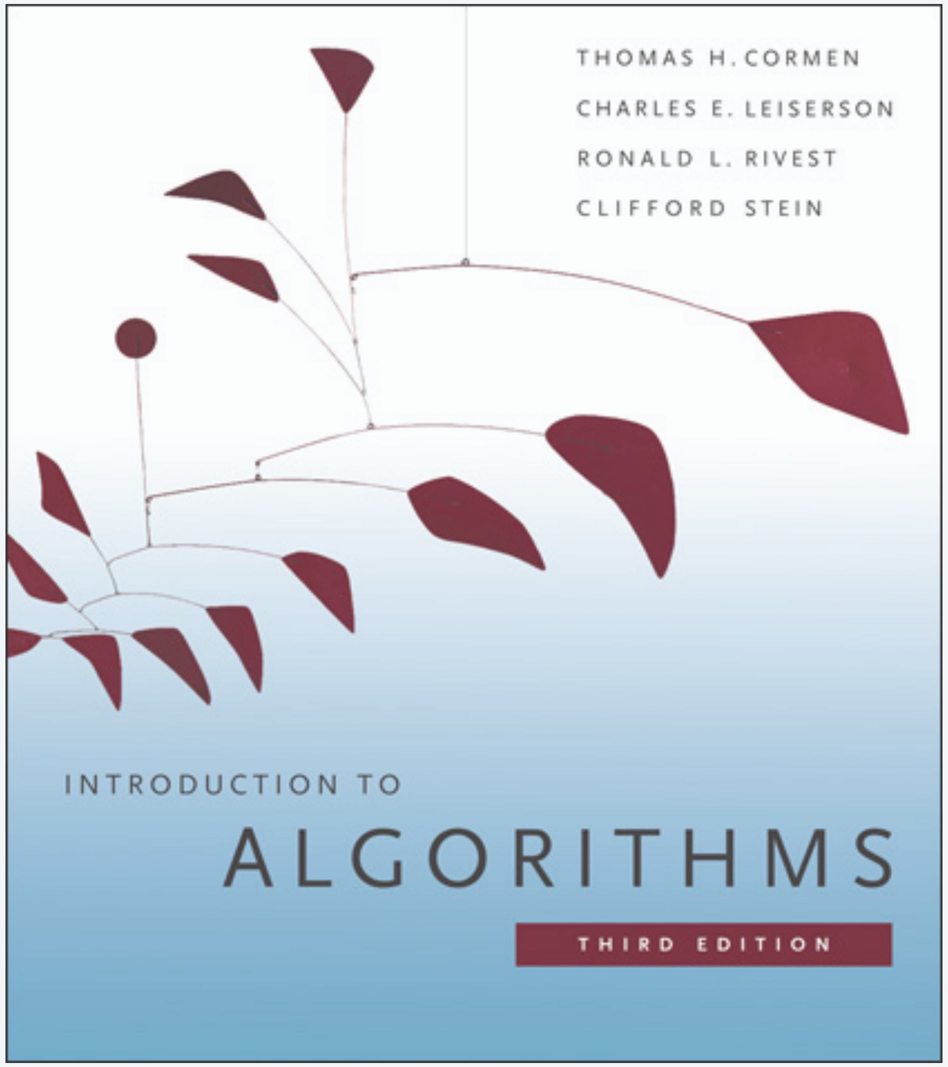
Change in potential. $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) = +1$.

one more tree;
no change in marks

Amortized cost. $\hat{c}_i = c_i + \Delta\Phi = O(1)$.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$





SECTION 19.2

FIBONACCI HEAPS

- *preliminaries*
- *insert*
- *extract the minimum*
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- *bounding the rank*
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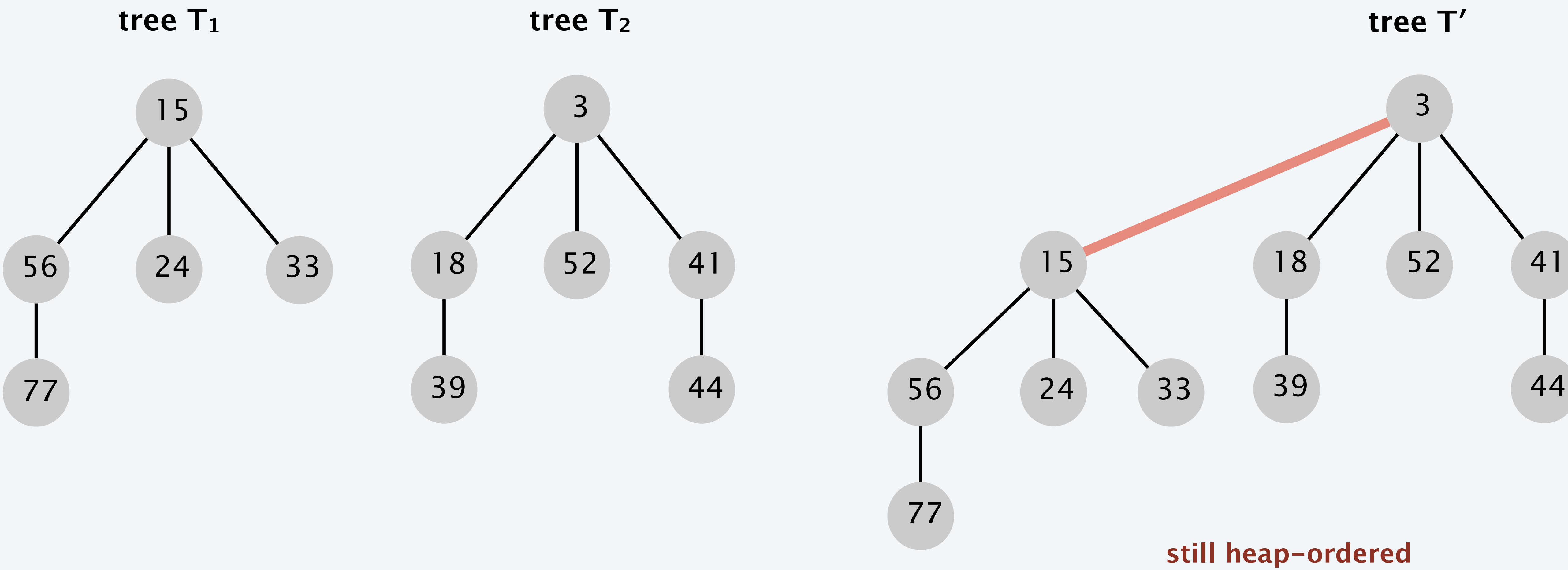
Linking operation

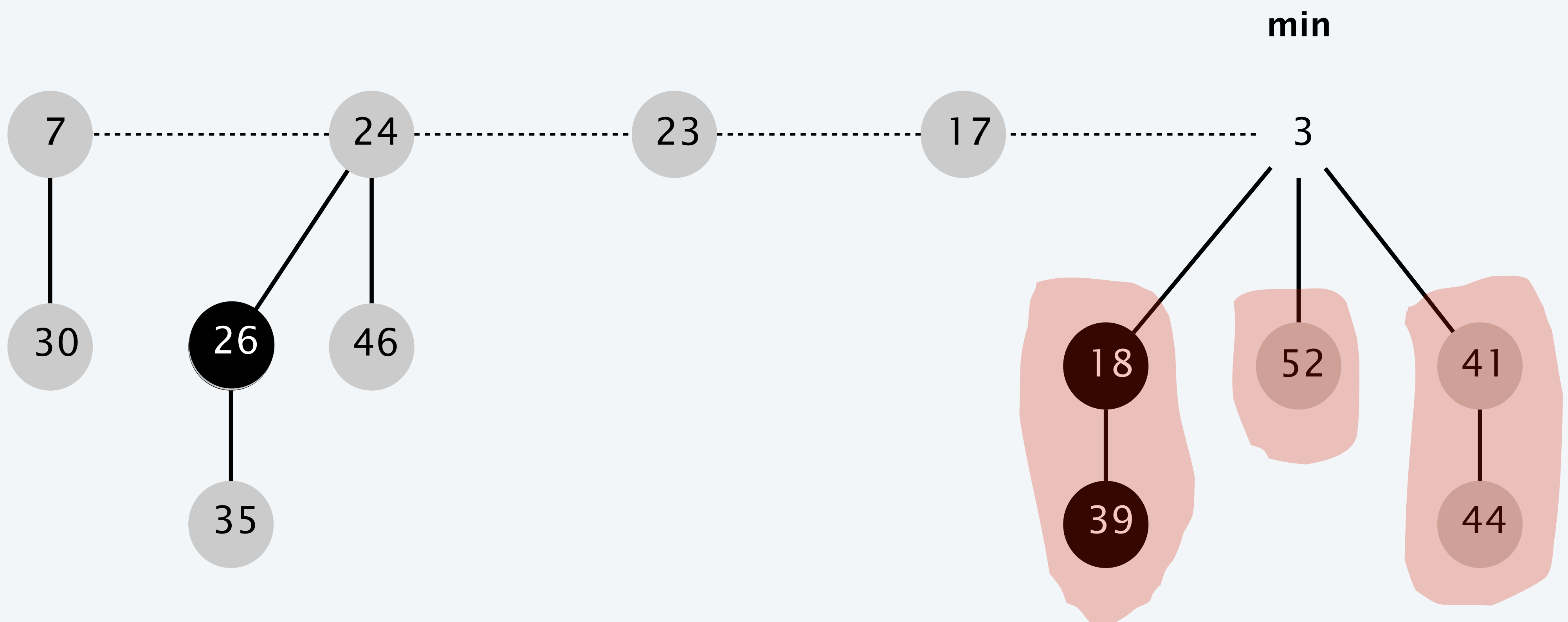
Useful primitive. Combine two trees T_1 and T_2 of rank k.

- Make larger root be a child of smaller root.
- Resulting tree T' has rank $k + 1$.



Fibonacci heap: extract the minimum

- Delete min; meld its children into root list; update min.
- Consolidate trees so that no two roots have same rank.



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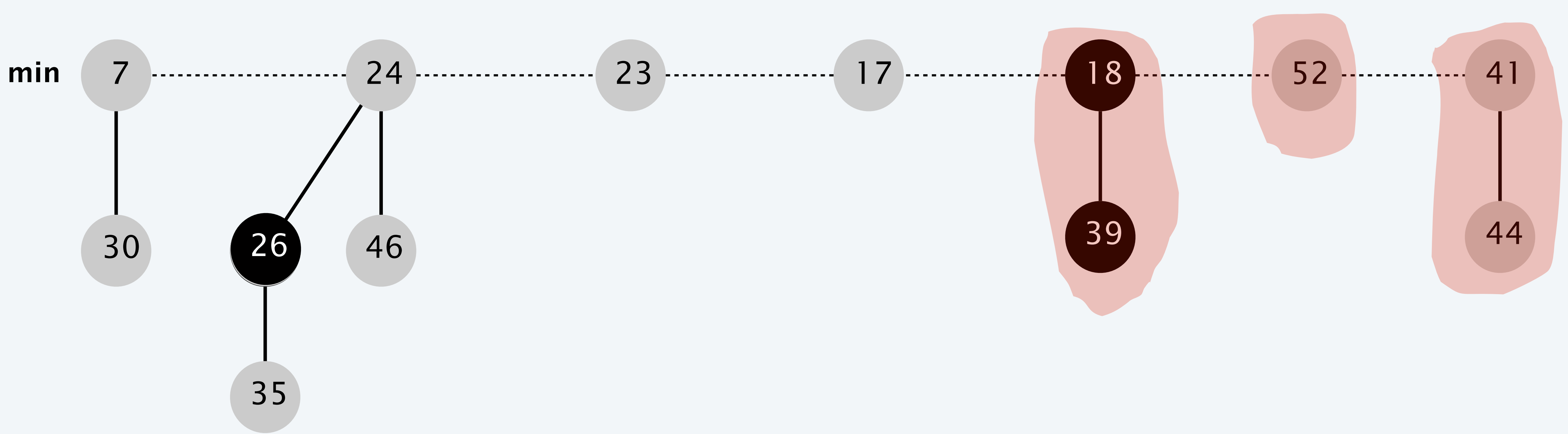

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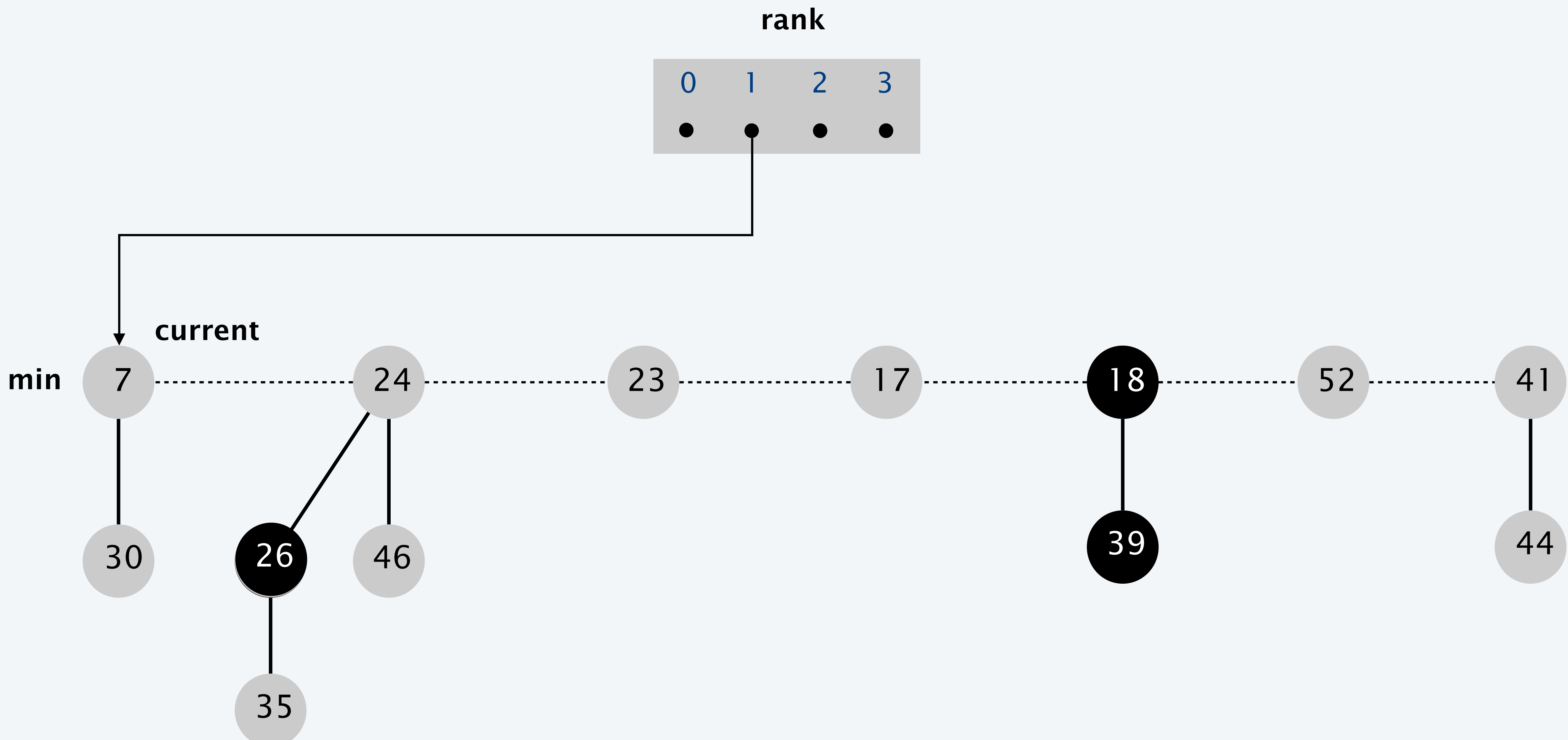

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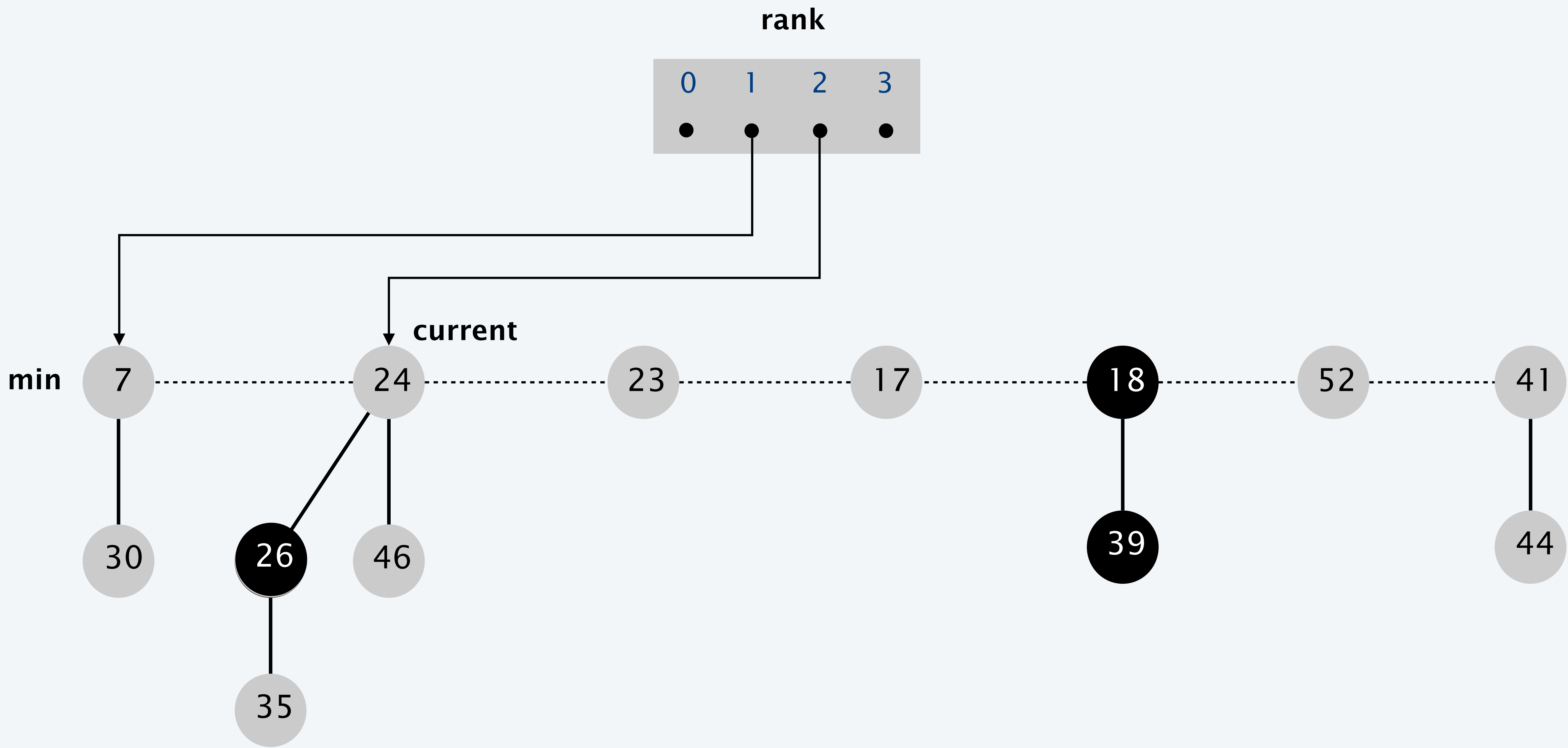

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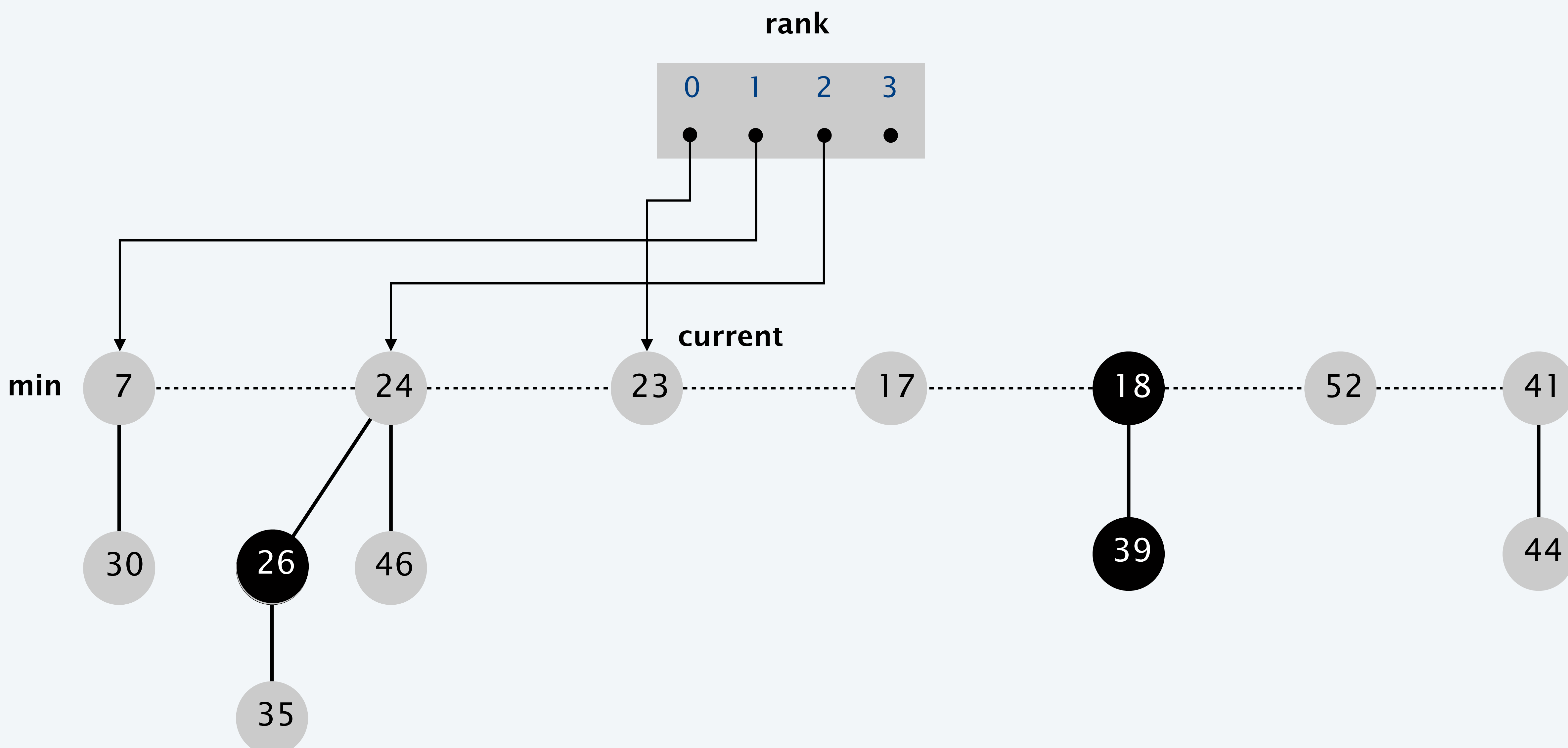

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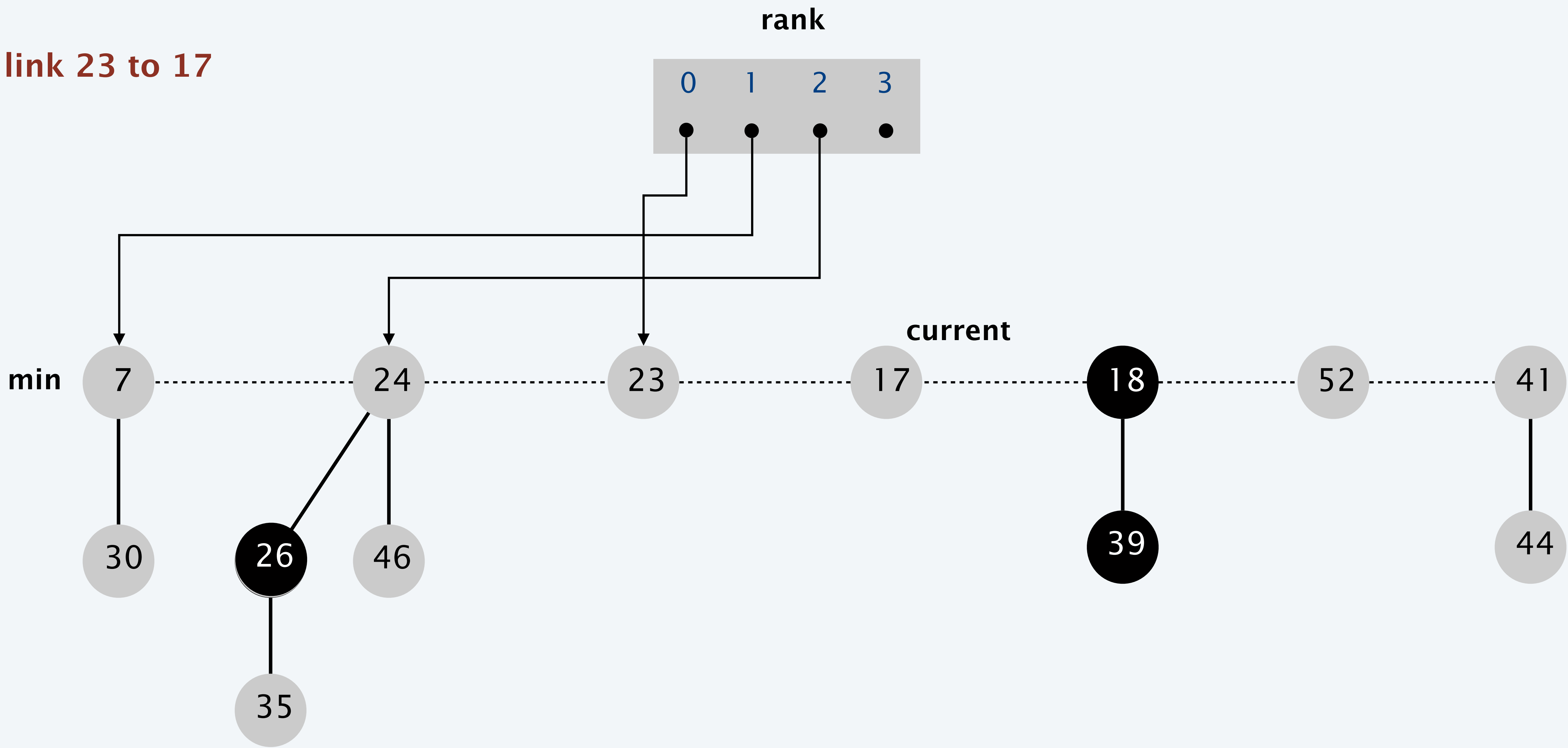

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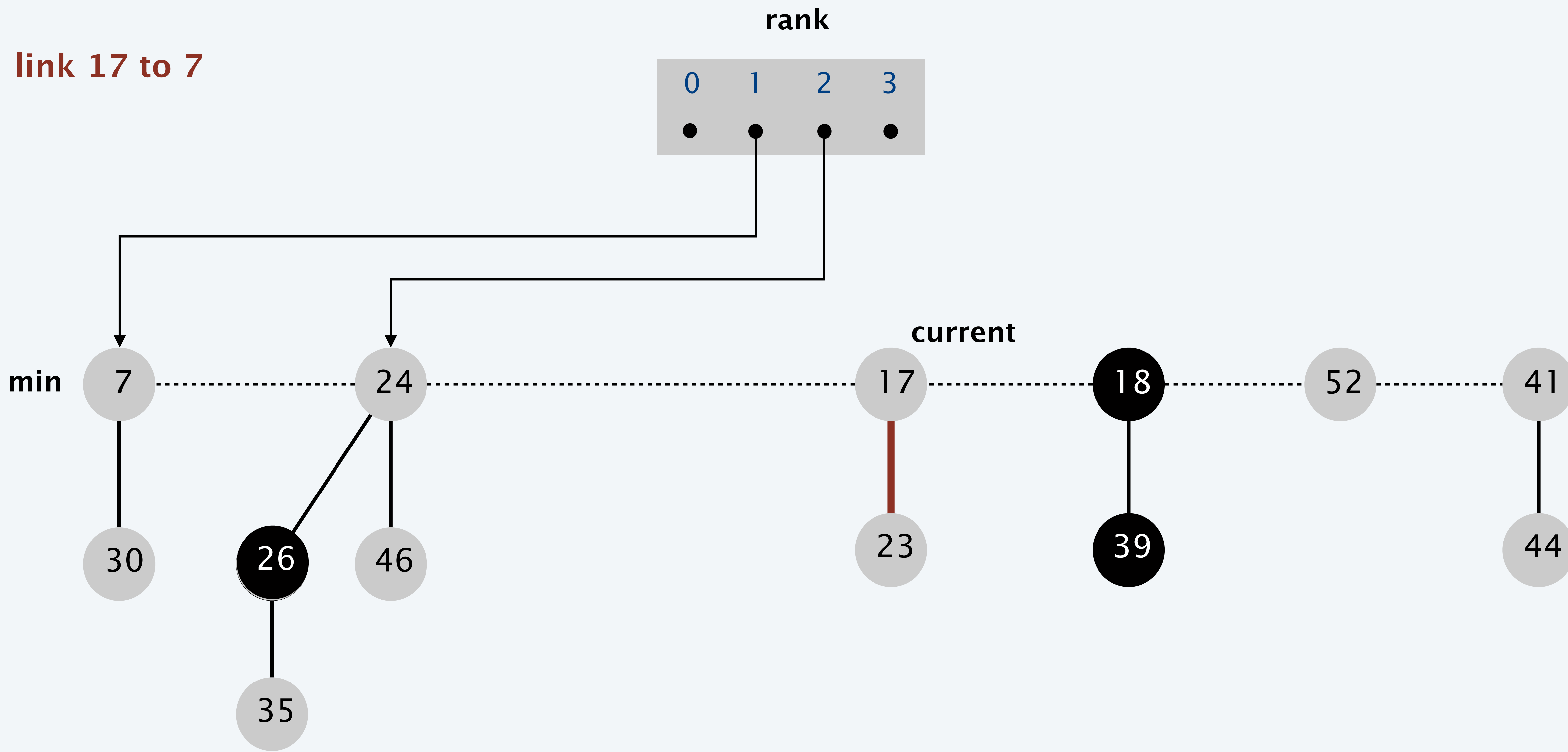

Fibonacci heap: extract the minimum

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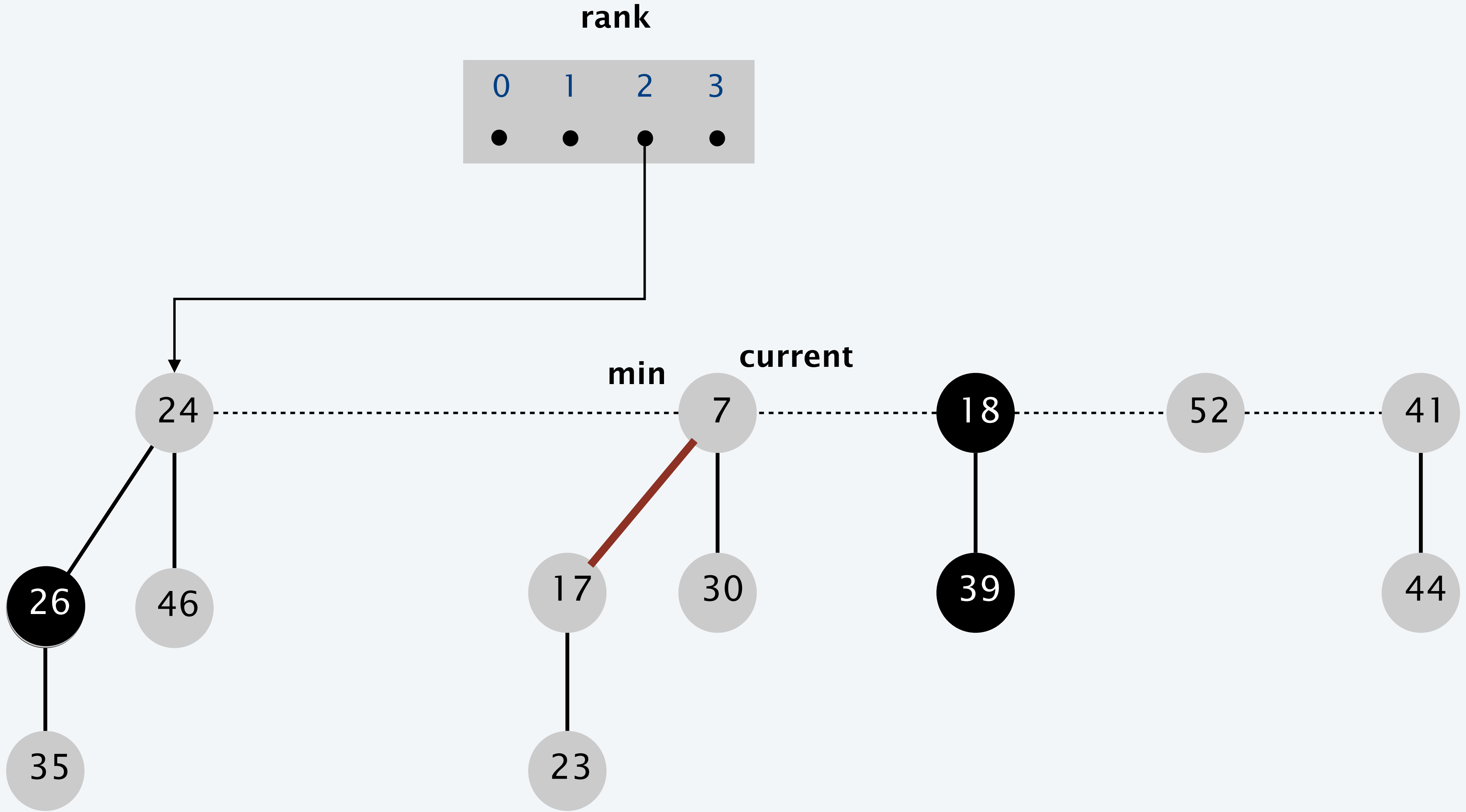
link 24 to 7



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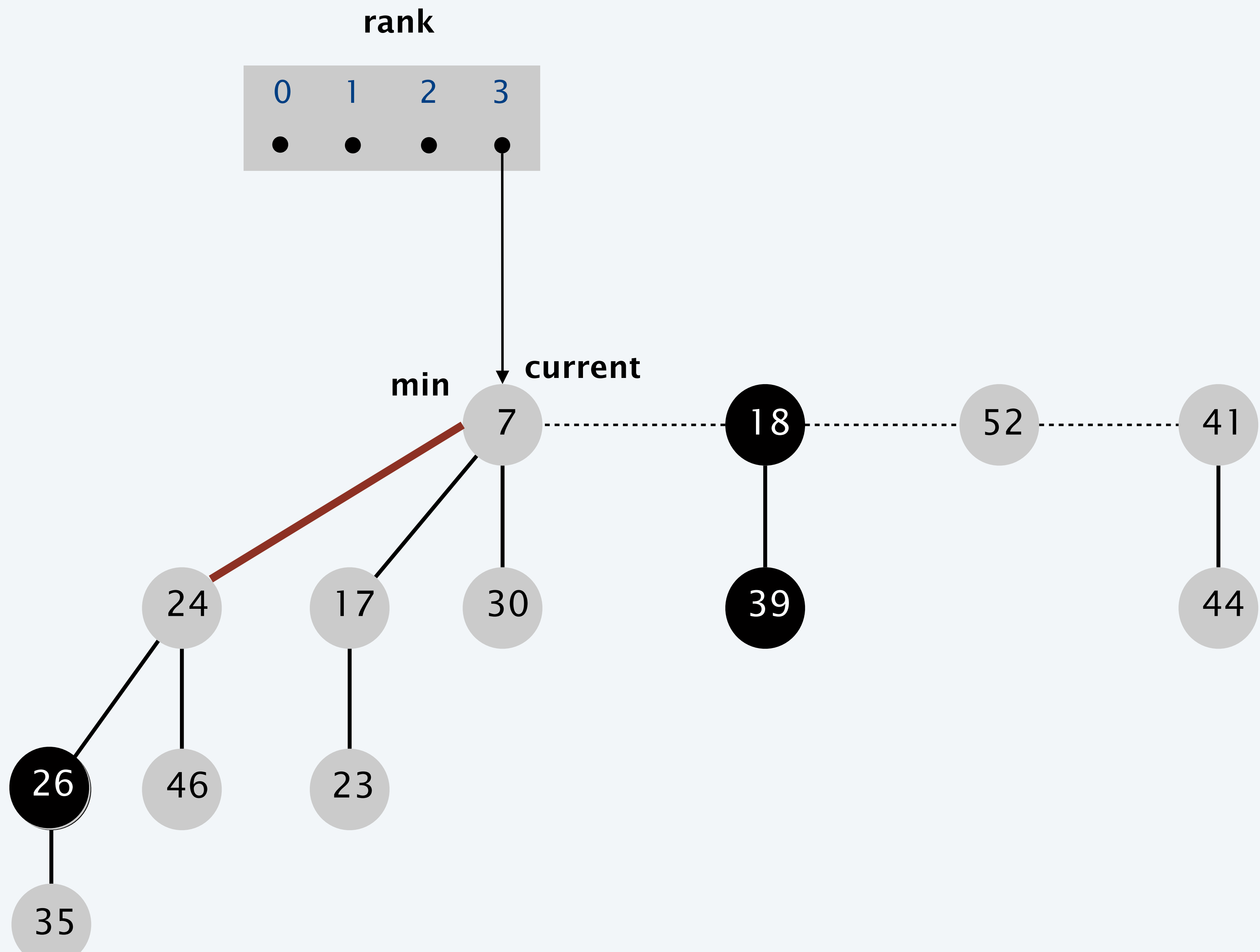

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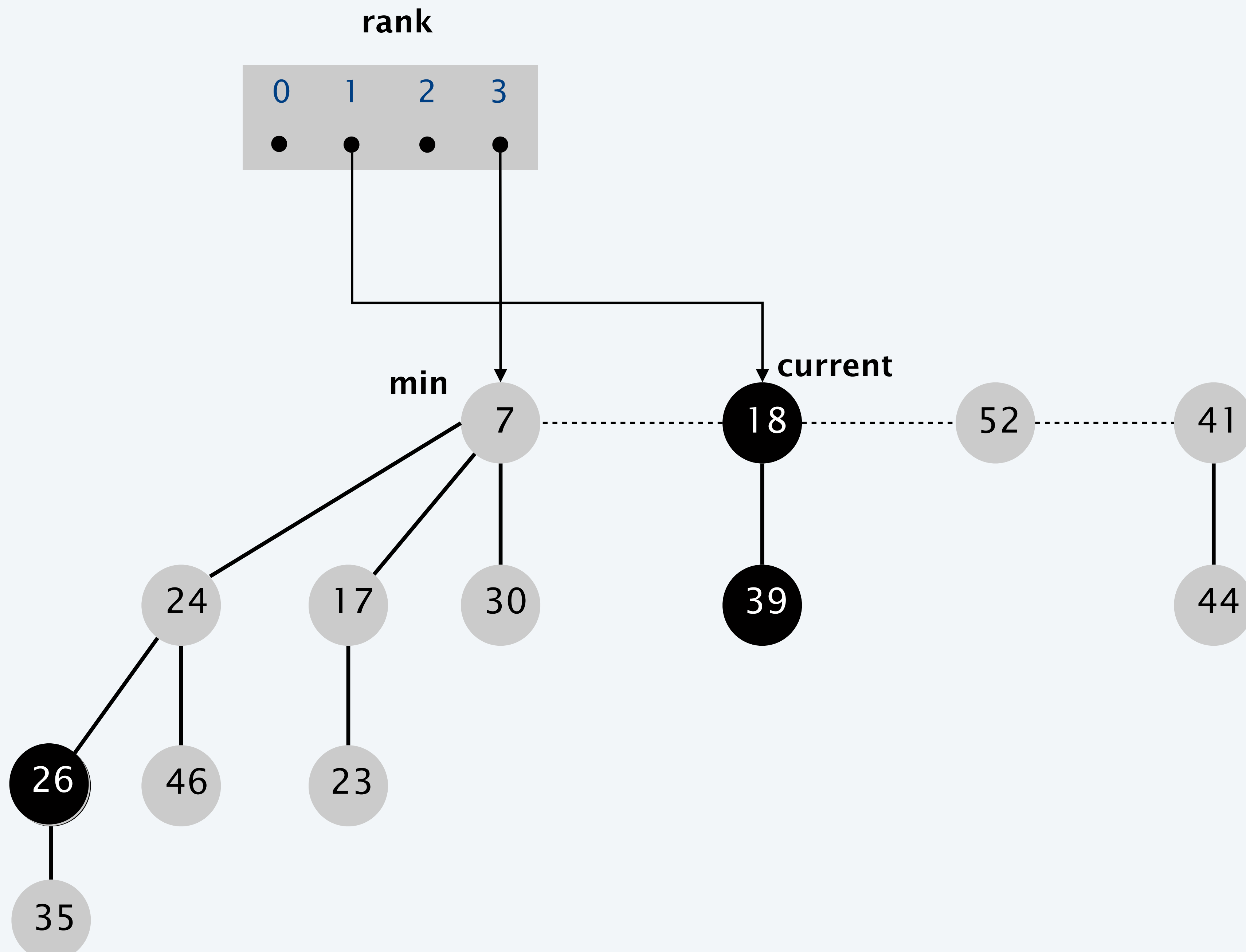

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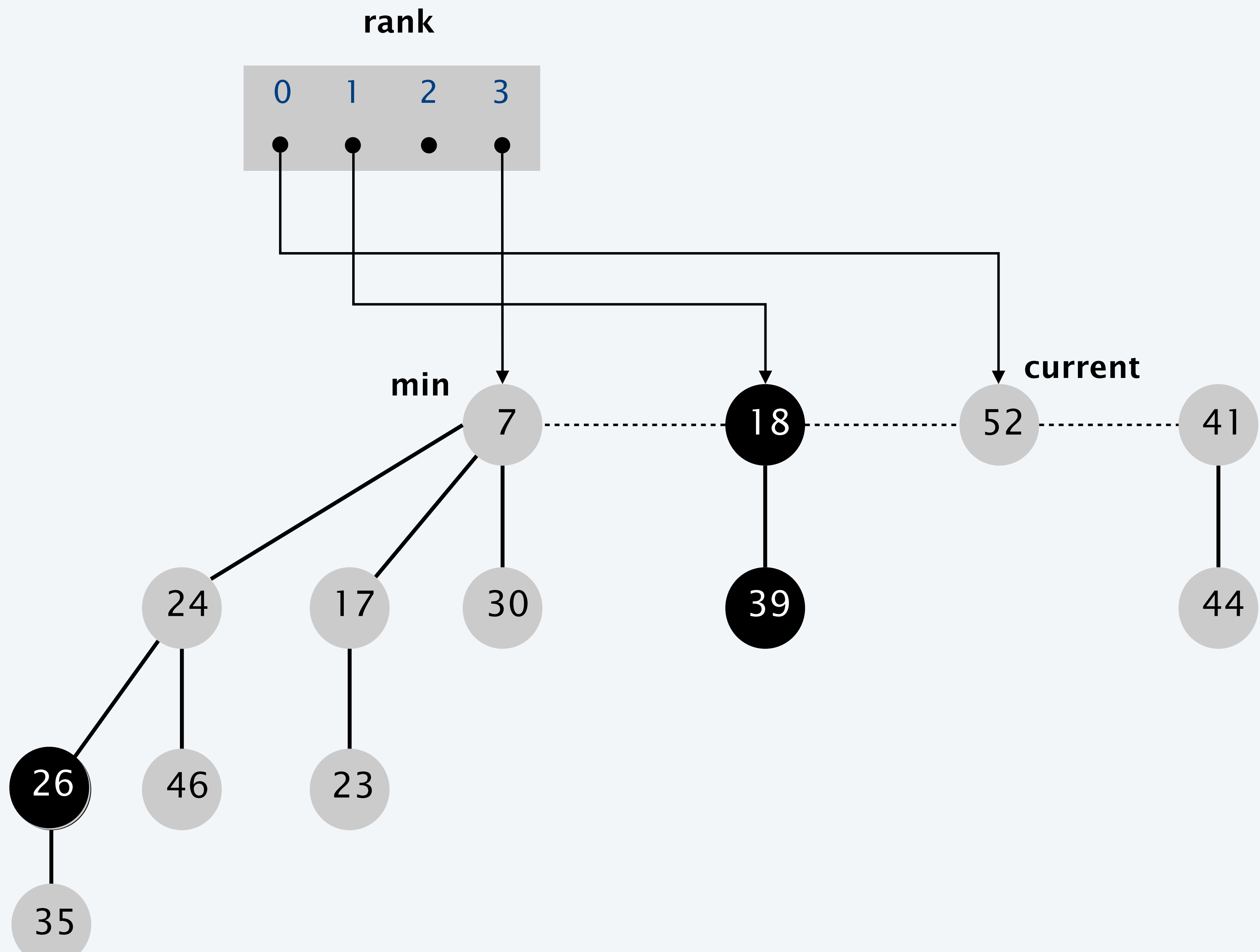

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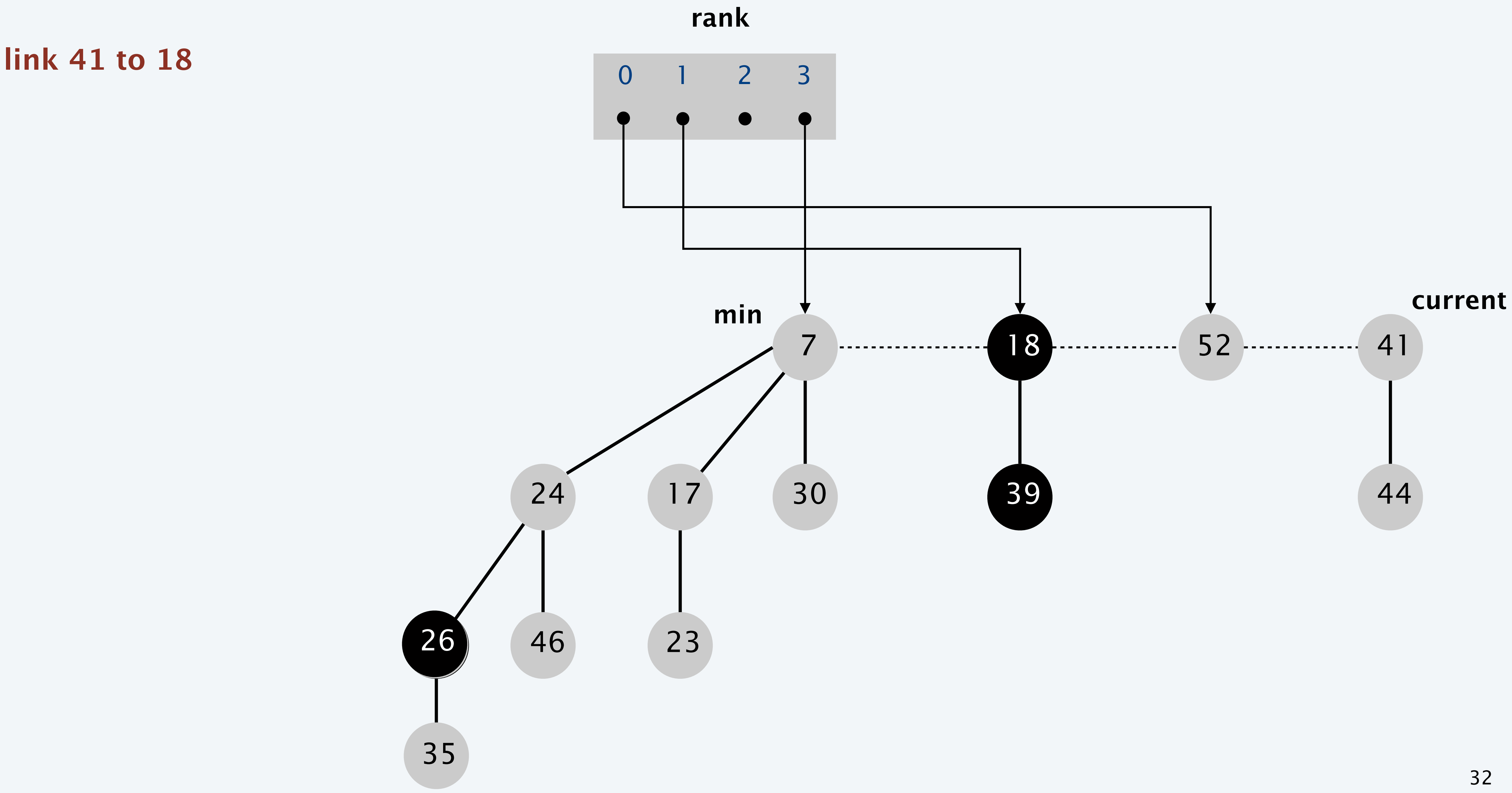

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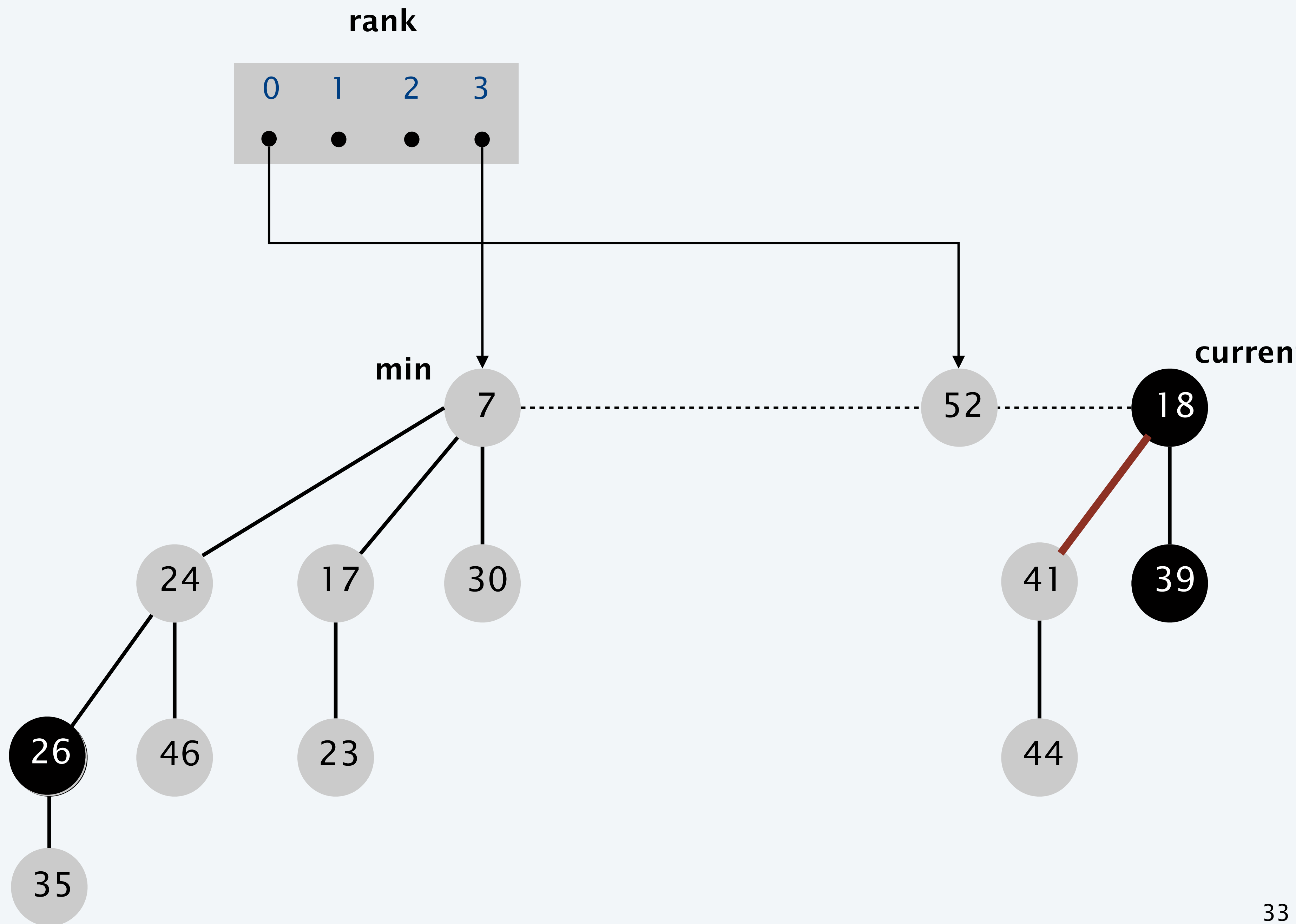

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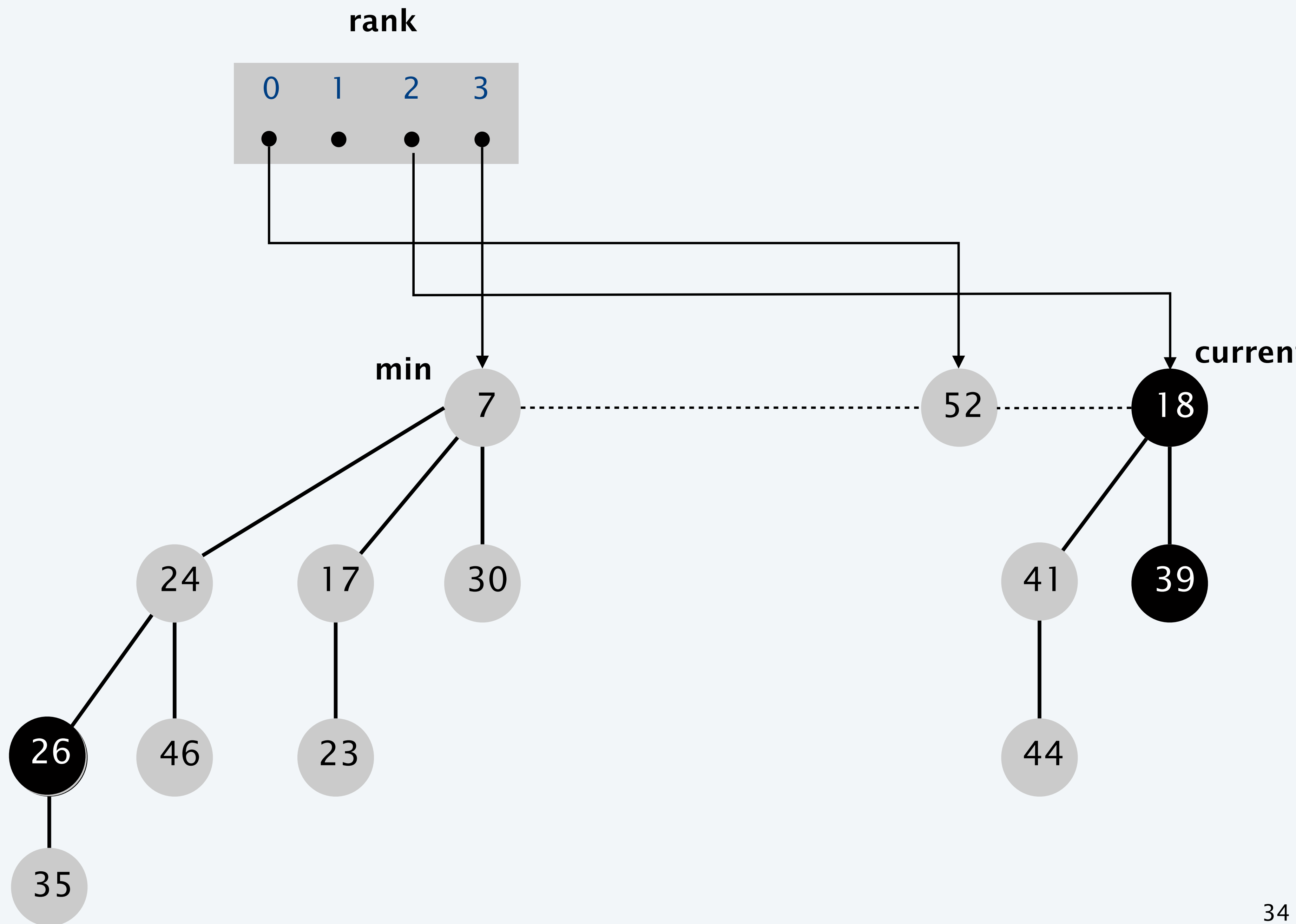
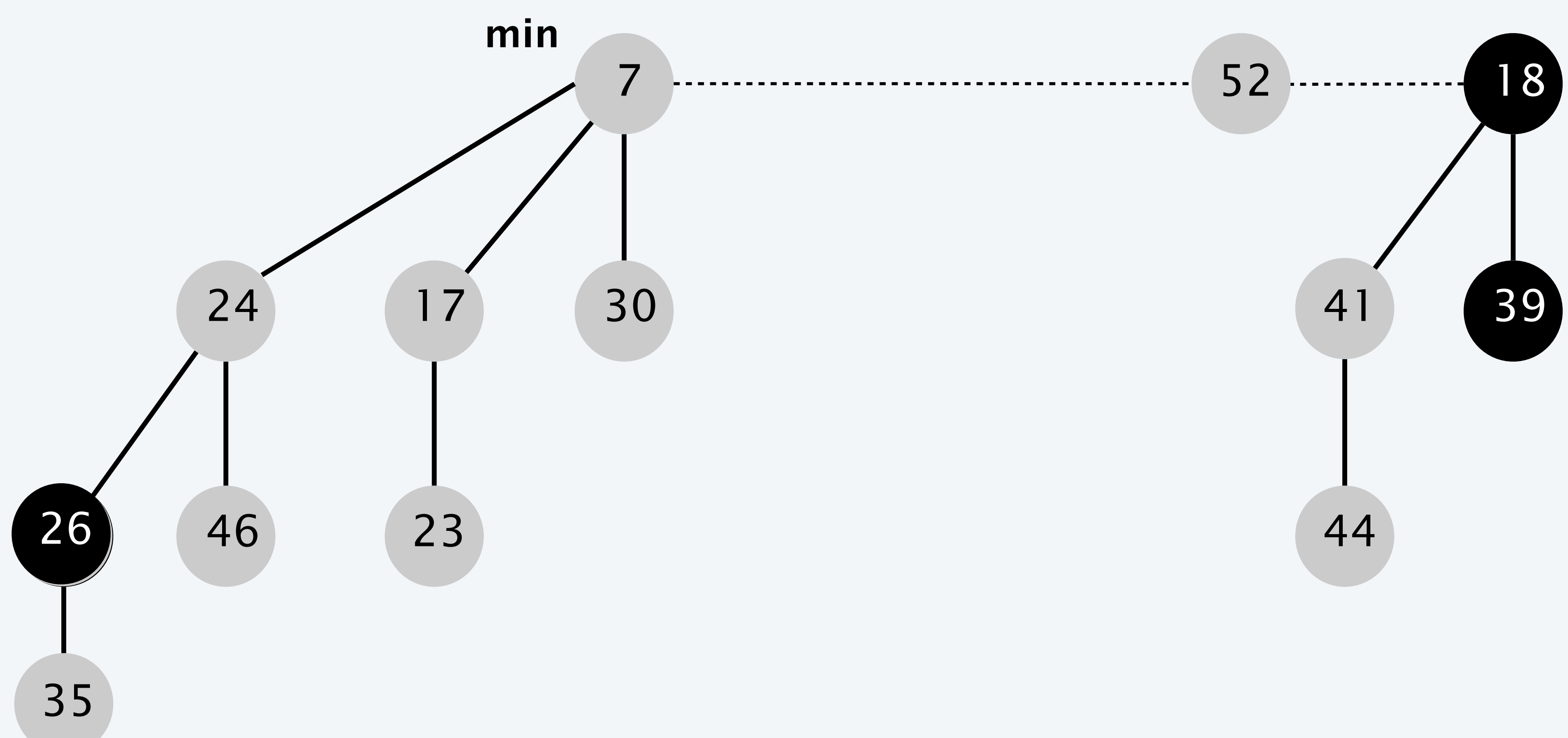


Fibonacci heap: extract the minimum

- Delete min; meld its children into root list; update min.
- Consolidate trees so that no two roots have same rank.

stop (no two trees have same rank)



Fibonacci heap: extract the minimum analysis

Actual cost. $c_i = O(rank(H)) + O(trees(H))$.

- $O(rank(H))$ to meld min's children into root list. ← $\leq$ rank(H) children
- $O(rank(H)) + O(trees(H))$ to update min. ← $\leq$ rank(H) + trees(H) – 1 root nodes
- $O(rank(H)) + O(trees(H))$ to consolidate trees. ← number of roots decreases by 1 after each linking operation

Change in potential. $\Delta\Phi \leq rank(H') + 1 - trees(H)$.

- No new nodes become marked.
- $trees(H') \leq rank(H') + 1$. ← no two trees have same rank after consolidation

Amortized cost. $O(\log n)$.

- $\hat{c}_i = c_i + \Delta\Phi = O(rank(H)) + O(rank(H'))$.
- The rank of a Fibonacci heap with n elements is $O(\log n)$.

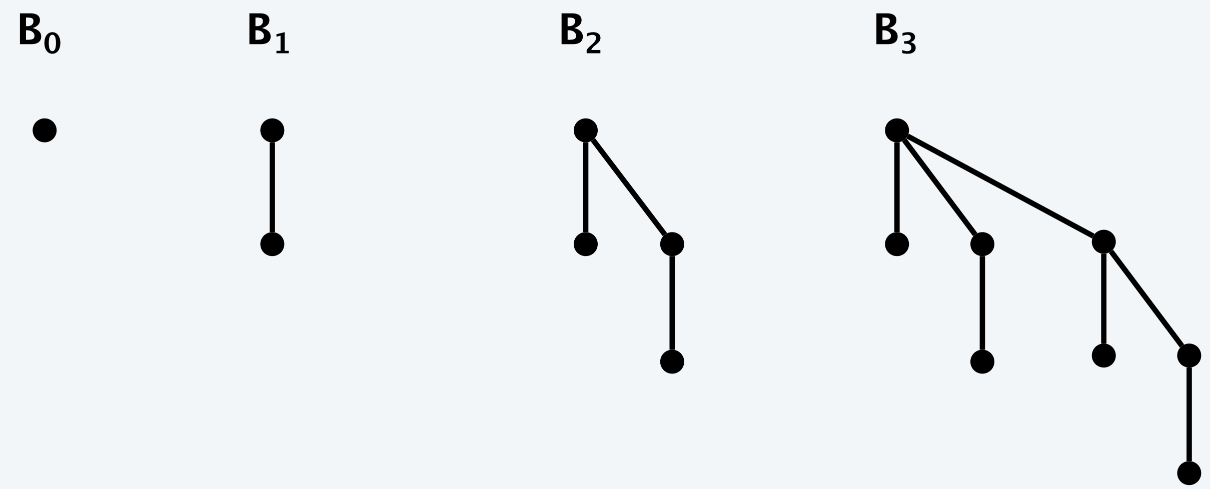
Fibonacci lemma
(stay tuned)

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Fibonacci heap vs. binomial heaps

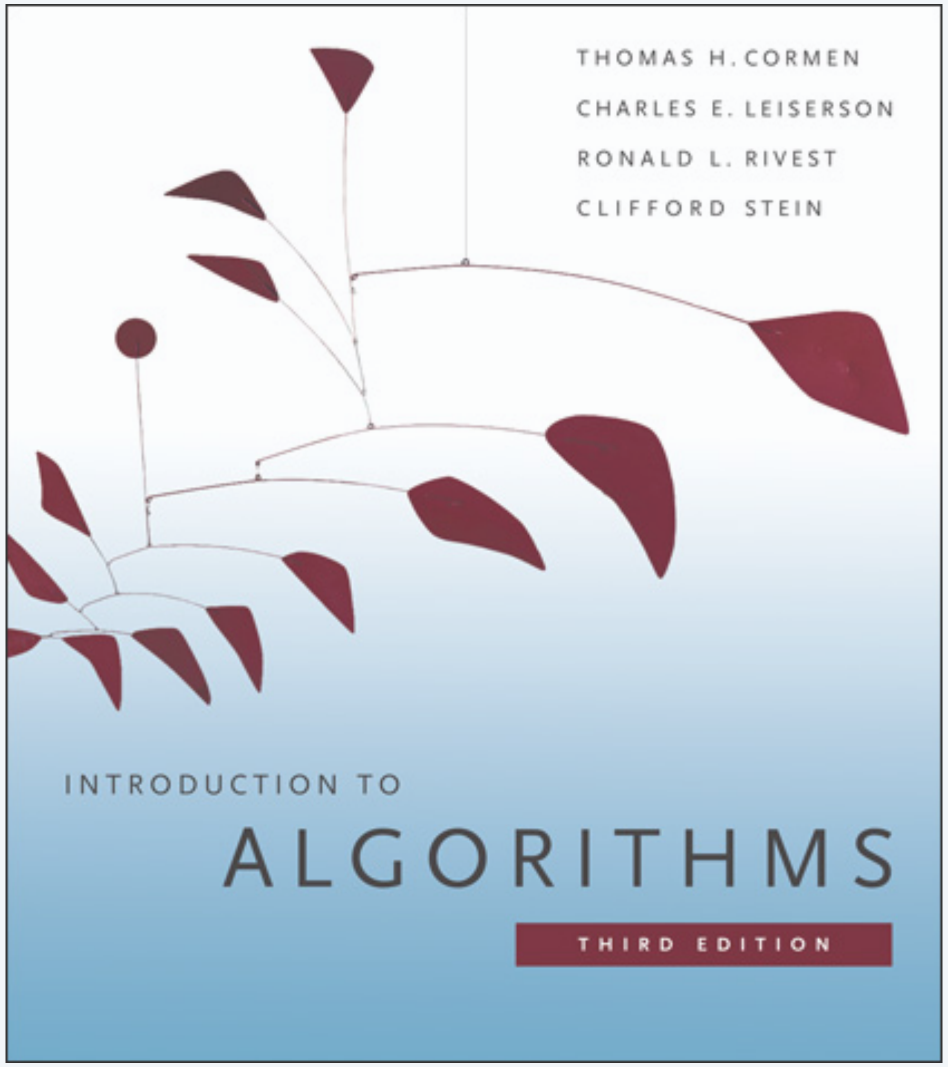
Observation. If only INSERT and EXTRACT-MIN operations, then all trees are binomial trees.

we link only trees of equal rank



Binomial heap property. This implies $rank(H) \leq \log_2 n$.

Fibonacci heap property. Our DECREASE-KEY implementation will not preserve this property, but we will implement it in such a way that $rank(H) \leq \log_\phi n$.



SECTION 19.3

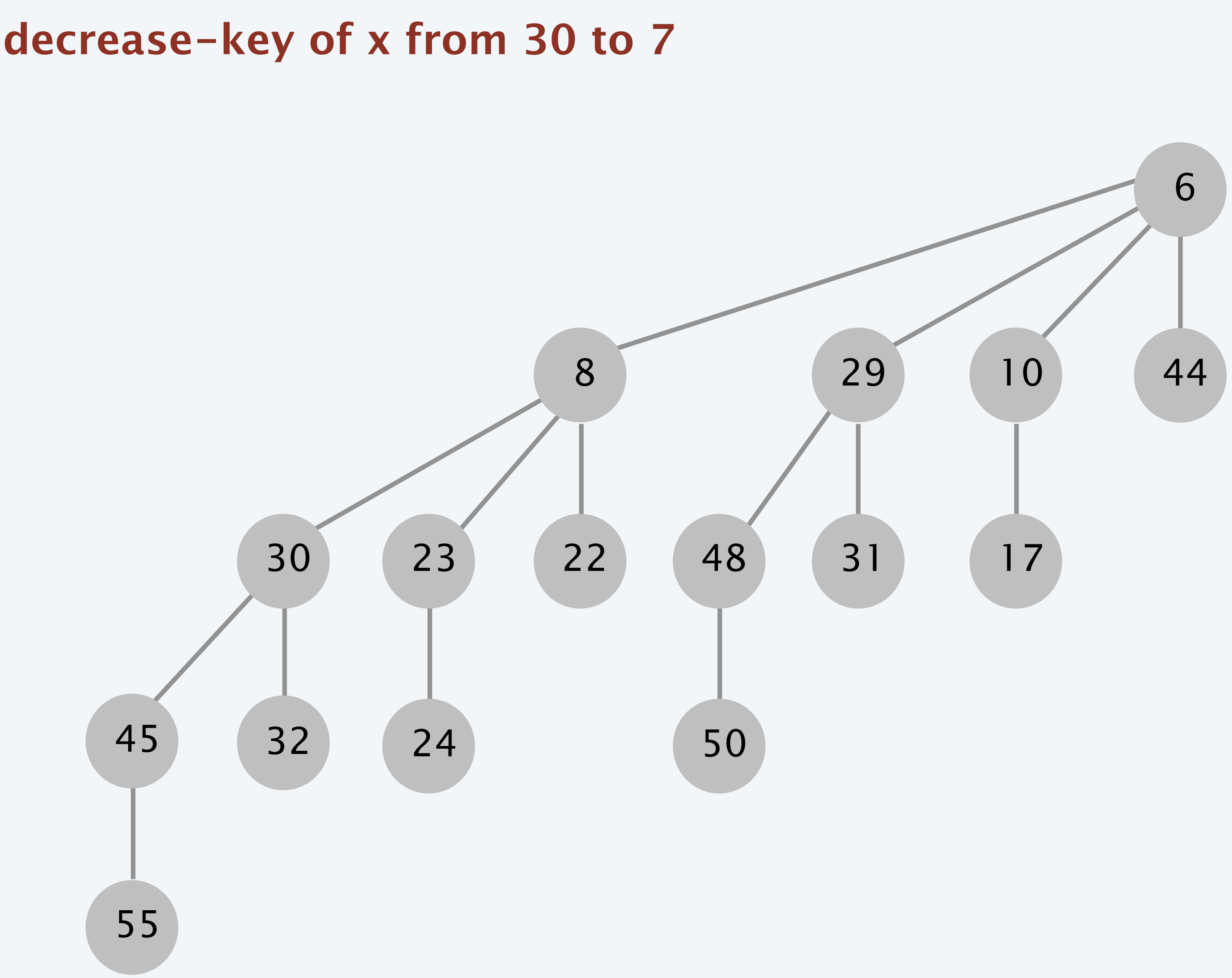
FIBONACCI HEAPS

- *preliminaries*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Fibonacci heap: decrease key

Intuition for deceasing the key of node x.

- If heap-order is not violated, decrease the key of x.
- Otherwise, cut tree rooted at x and meld into root list.

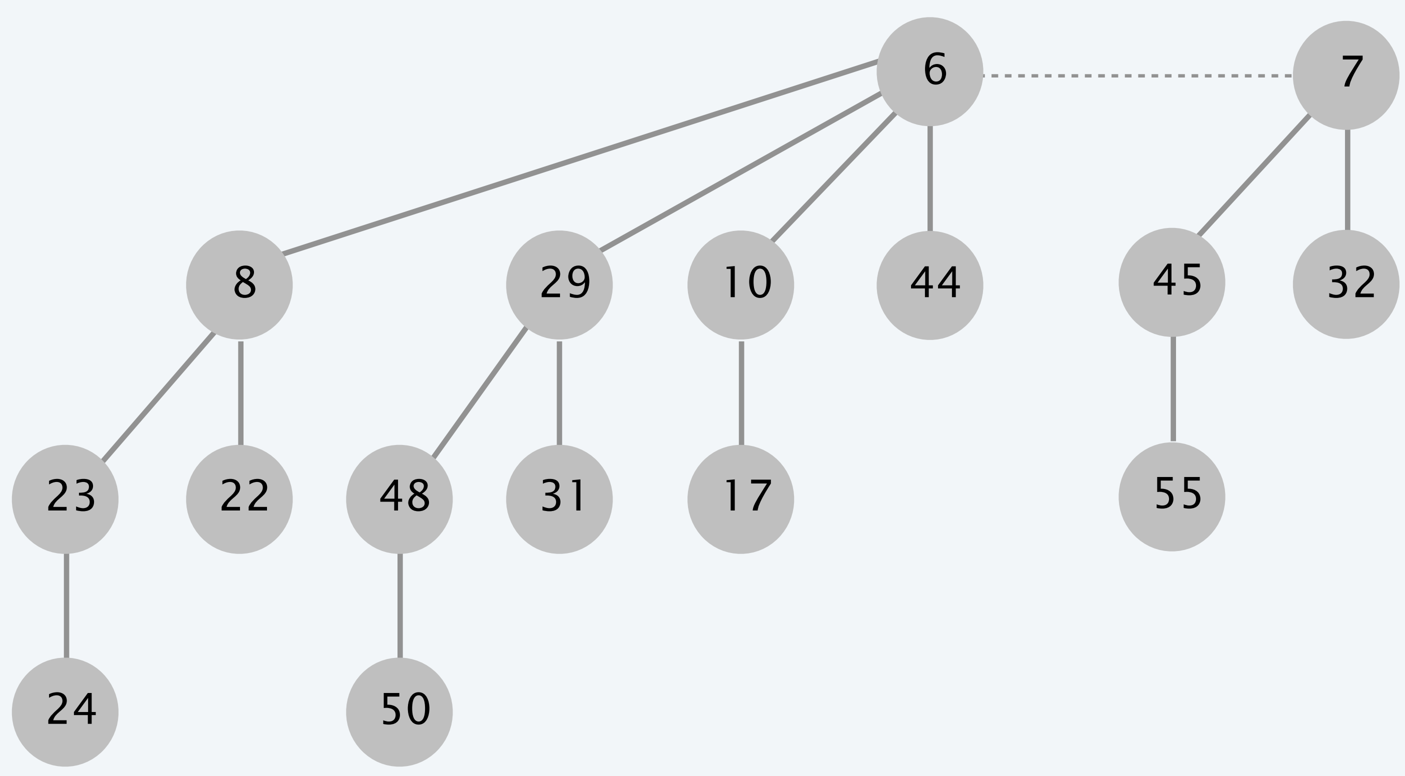


Fibonacci heap: decrease key

Intuition for deceasing the key of node x.

- If heap-order is not violated, decrease the key of x.
- Otherwise, cut tree rooted at x and meld into root list.

decrease-key of x from 23 to 5

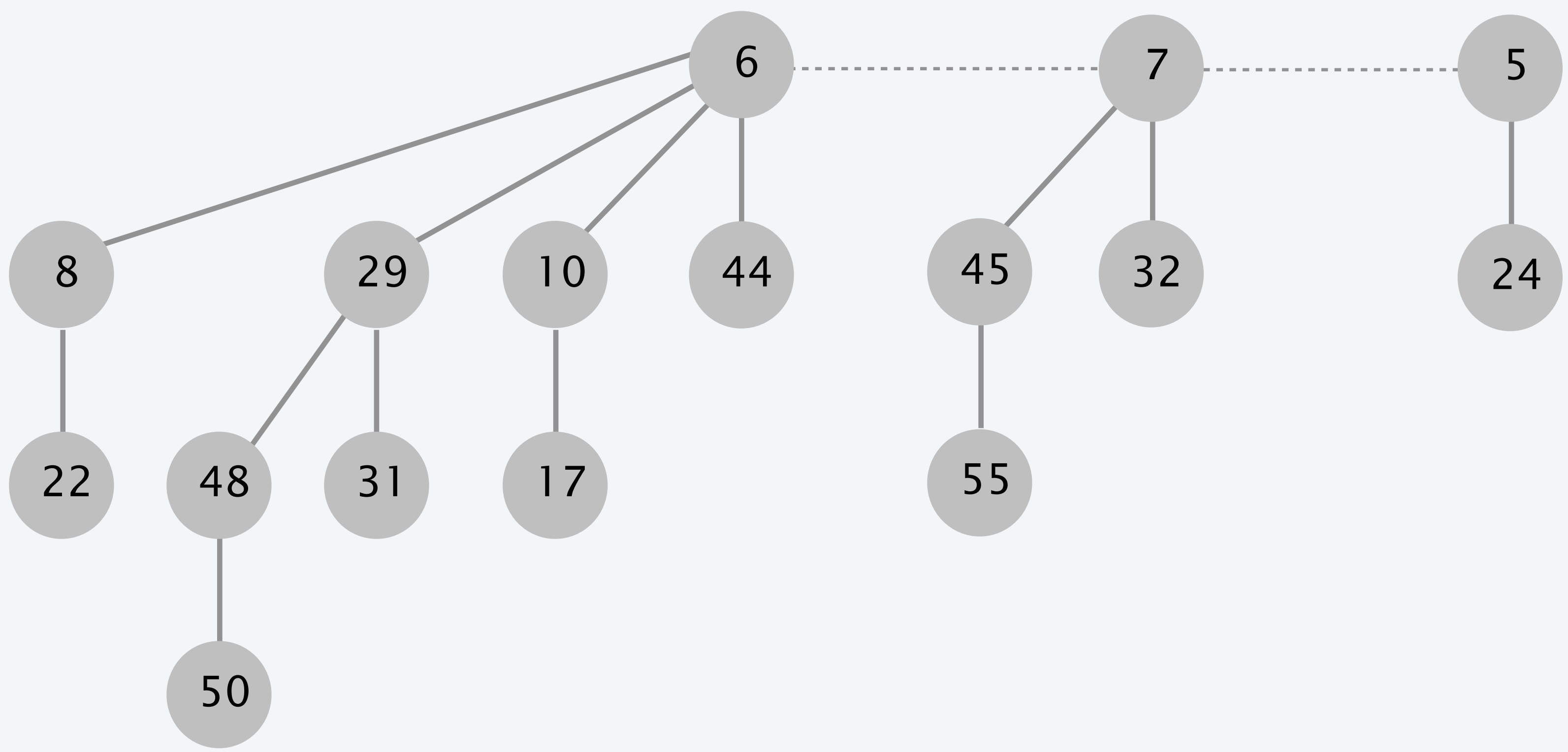


Fibonacci heap: decrease key

Intuition for deceasing the key of node x.

- If heap-order is not violated, decrease the key of x.
- Otherwise, cut tree rooted at x and meld into root list.

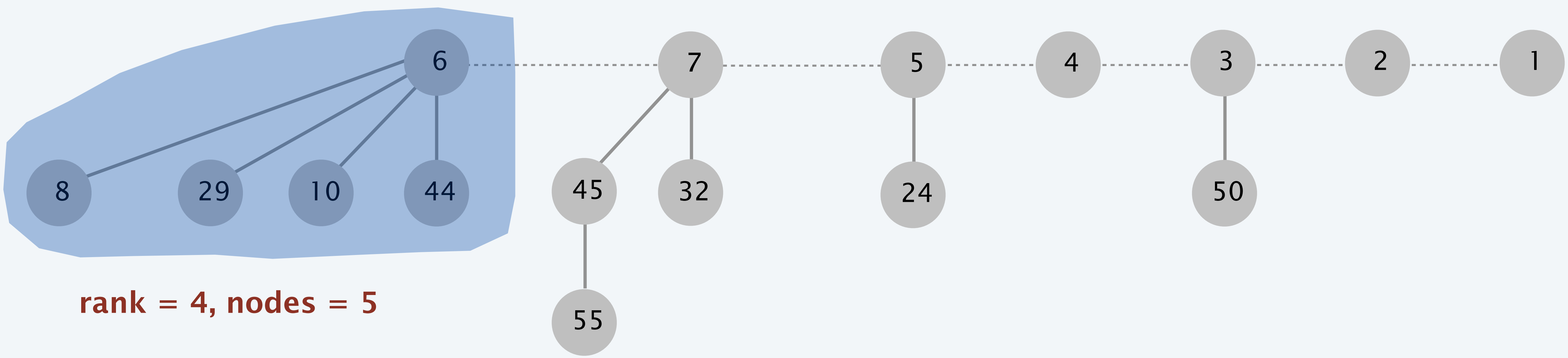
decrease-key of 22 to 4
decrease-key of 48 to 3
decrease-key of 31 to 2
decrease-key of 17 to 1



Fibonacci heap: decrease key

Intuition for deceasing the key of node x.

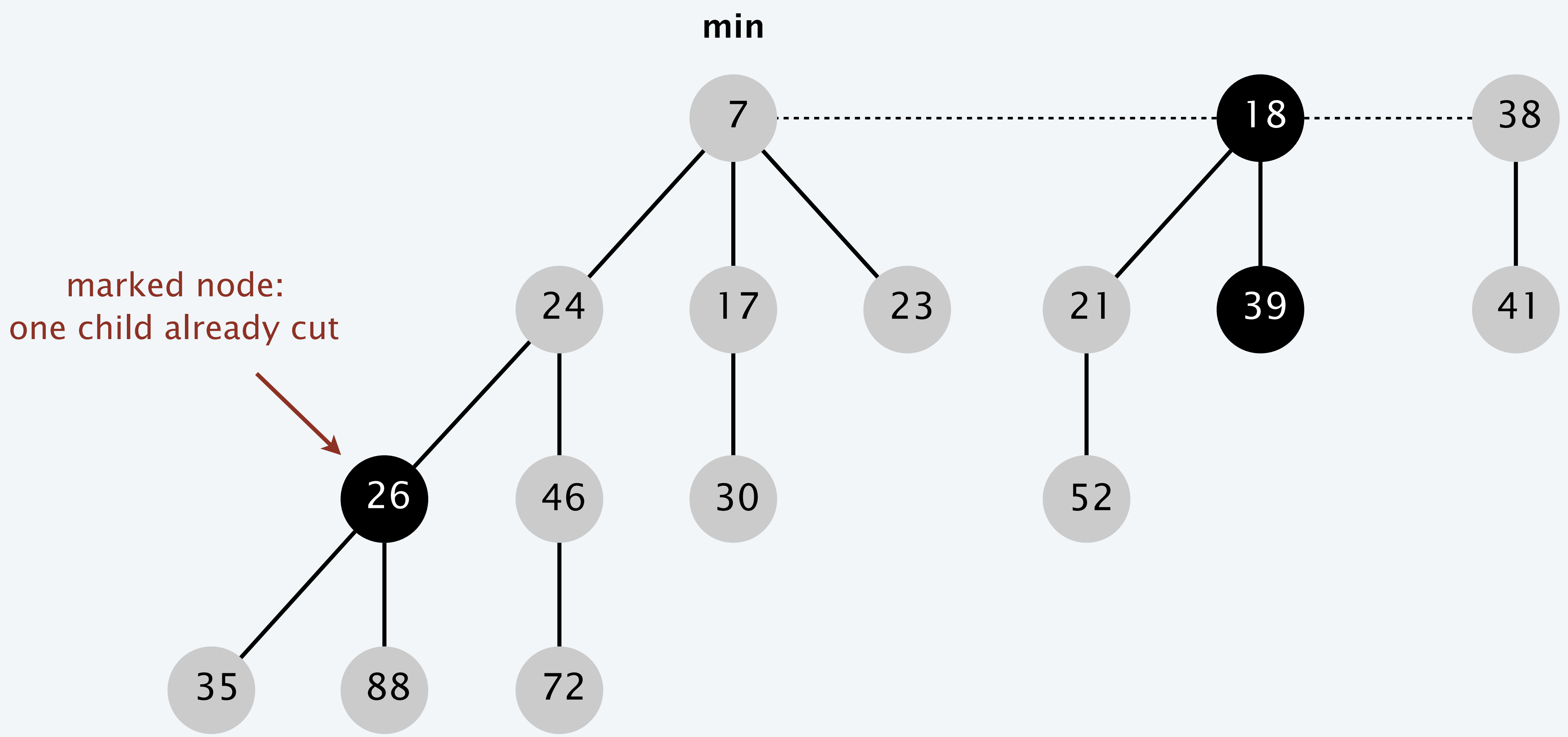
- If heap-order is not violated, decrease the key of x.
- Otherwise, cut tree rooted at x and meld into root list.
- Problem: number of nodes not exponential in rank.



Fibonacci heap: decrease key

Intuition for deceasing the key of node x.

- If heap-order is not violated, decrease the key of x.
- Otherwise, cut tree rooted at x and meld into root list.
- Solution: as soon as a node has its second child cut, cut it off also and meld into root list (and unmark it).

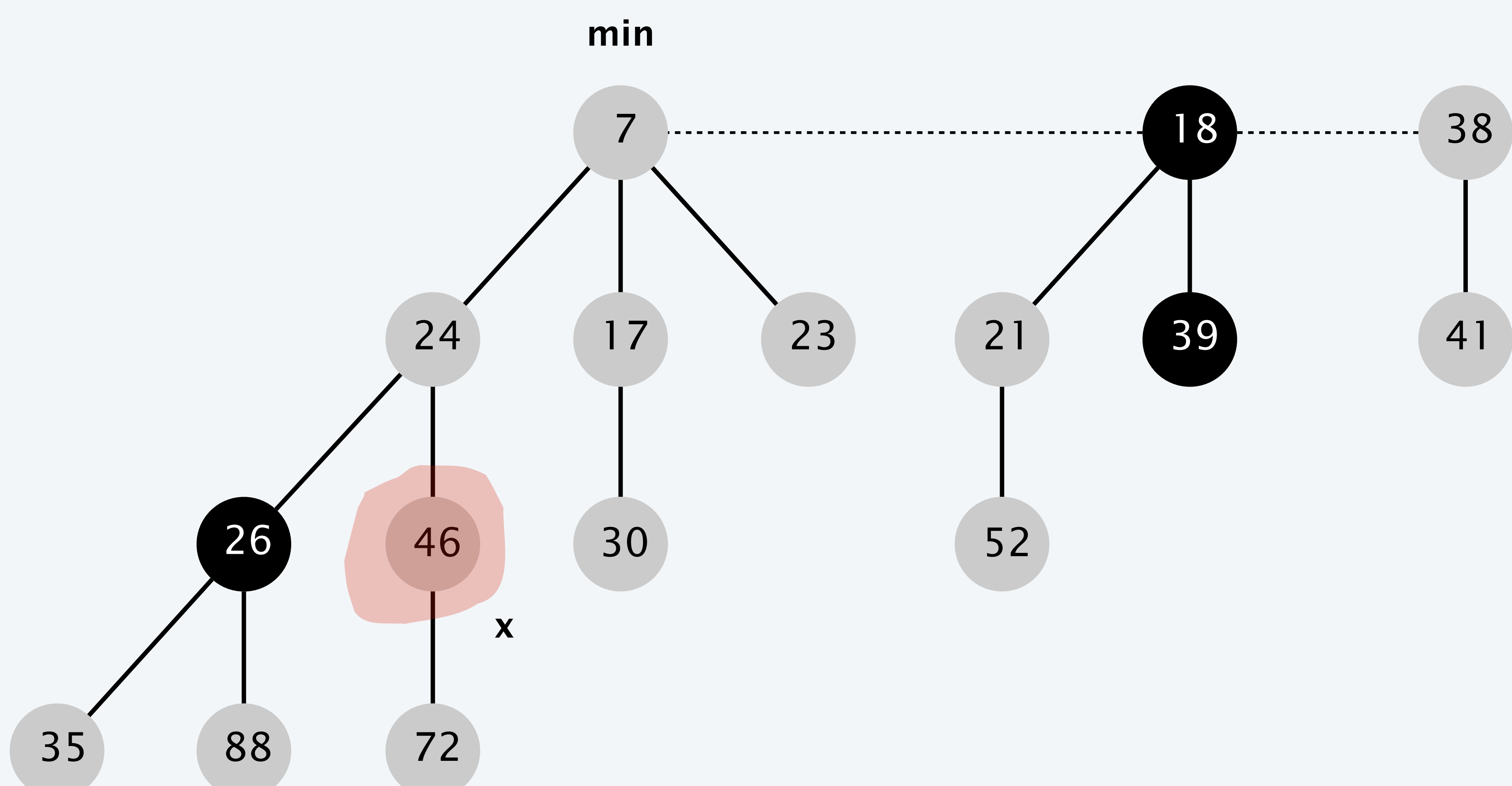


Fibonacci heap: decrease key

Case 1. [heap order not violated]

- Decrease key of x.
- Change heap min pointer (if necessary).

decrease-key of x from 46 to 29

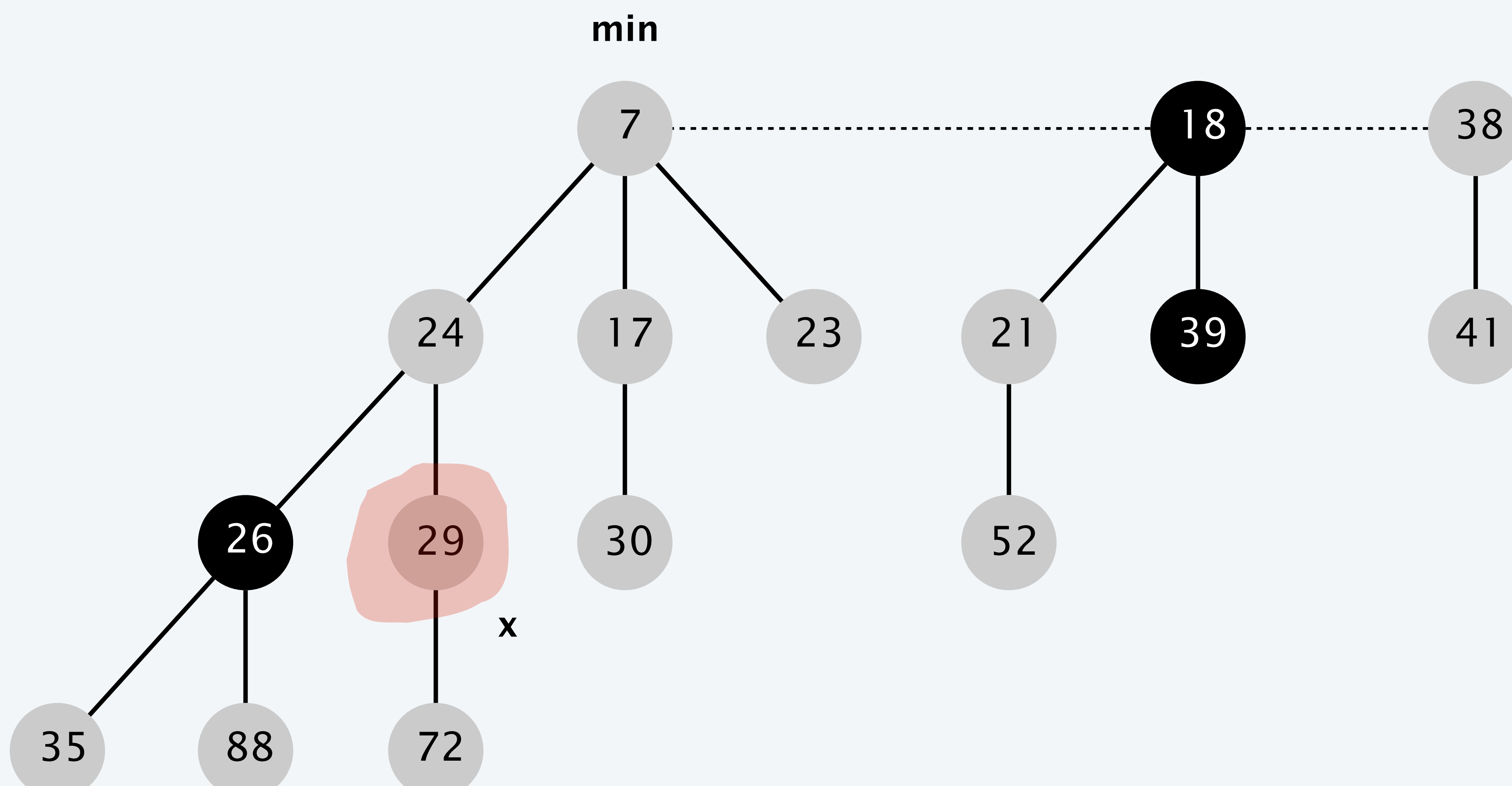


Fibonacci heap: decrease key

Case 1. [heap order not violated]

- Decrease key of x.
- Change heap min pointer (if necessary).

decrease-key of x from 46 to 29

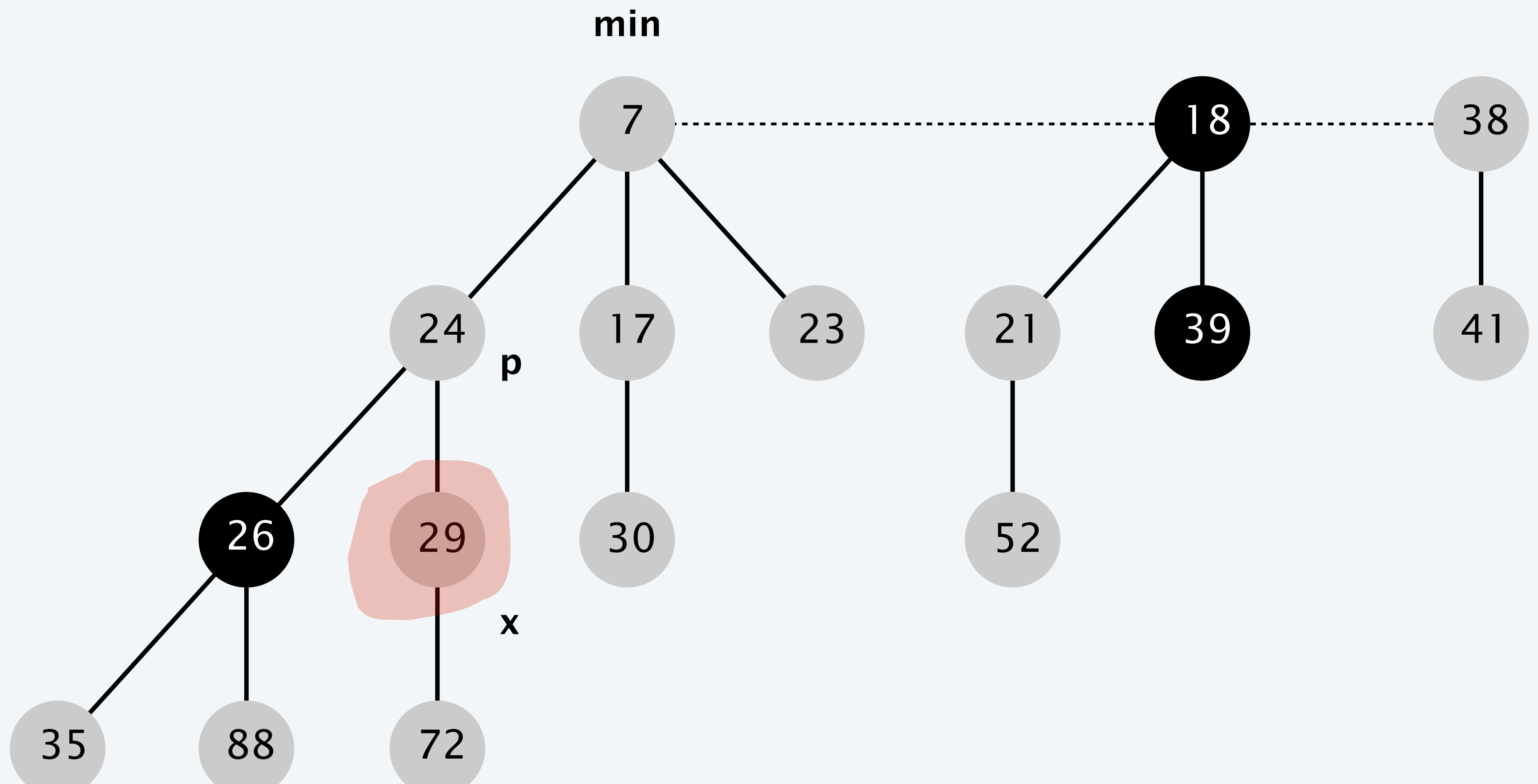


Fibonacci heap: decrease key

Case 2a. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 29 to 15



Fibonacci heap: decrease key

Case 2a. [heap order violated]

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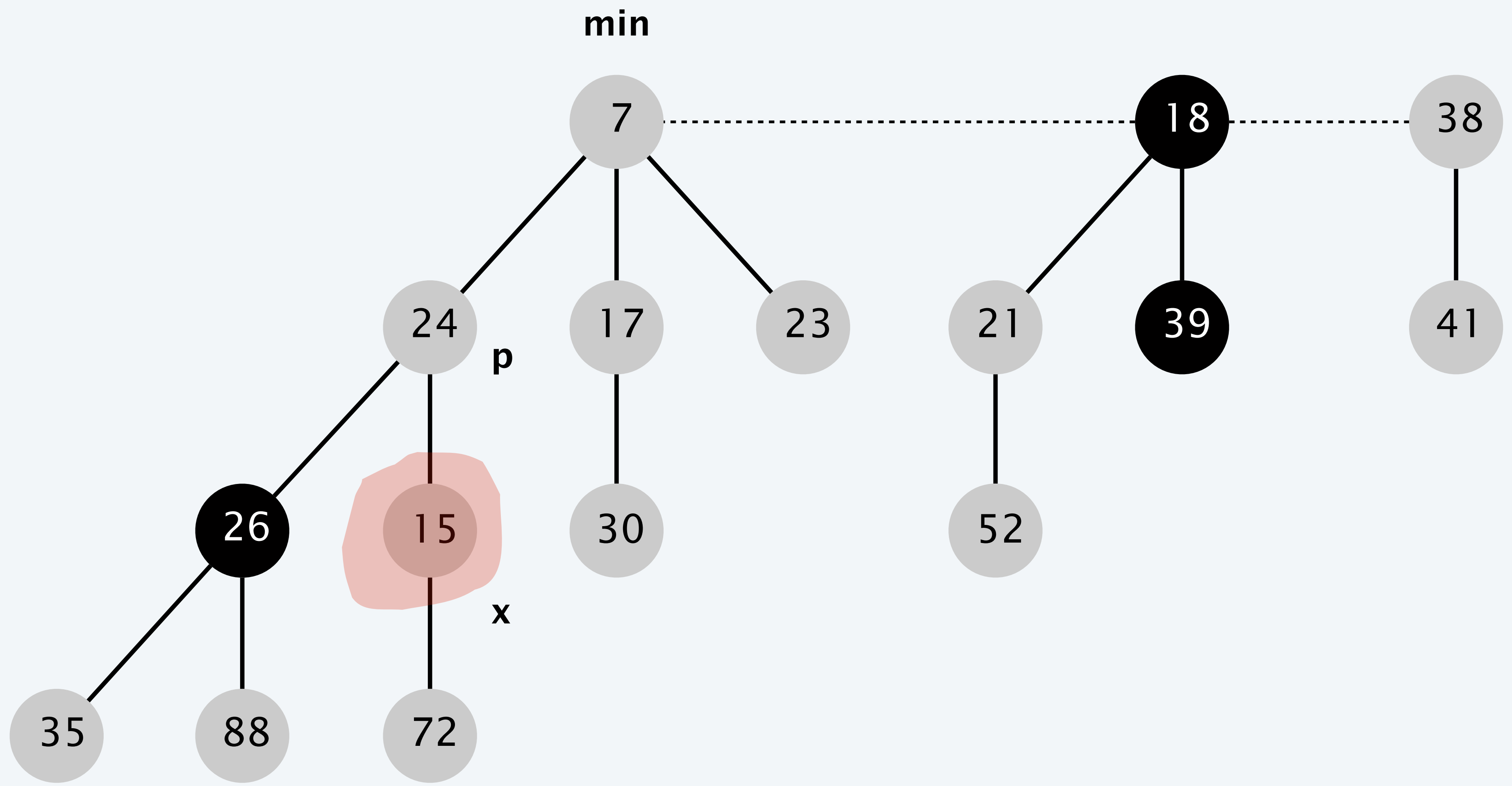
decrease-key of x from 29 to 15



Fibonacci heap: decrease key

Case 2a. [heap order violated]

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- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
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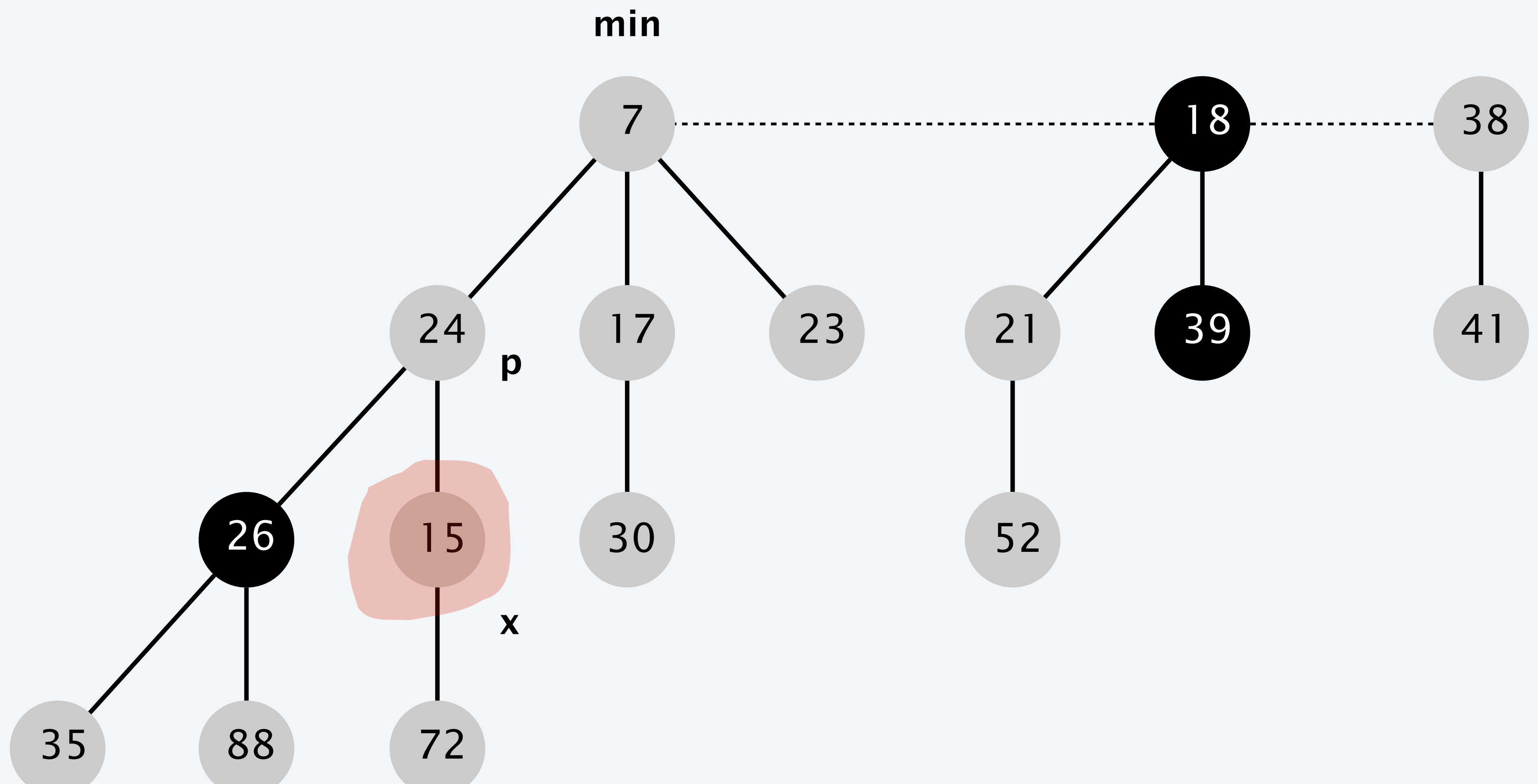
decrease-key of x from 29 to 15



Fibonacci heap: decrease key

Case 2a. [heap order violated]

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- Cut tree rooted at x, meld into root list, and unmark.
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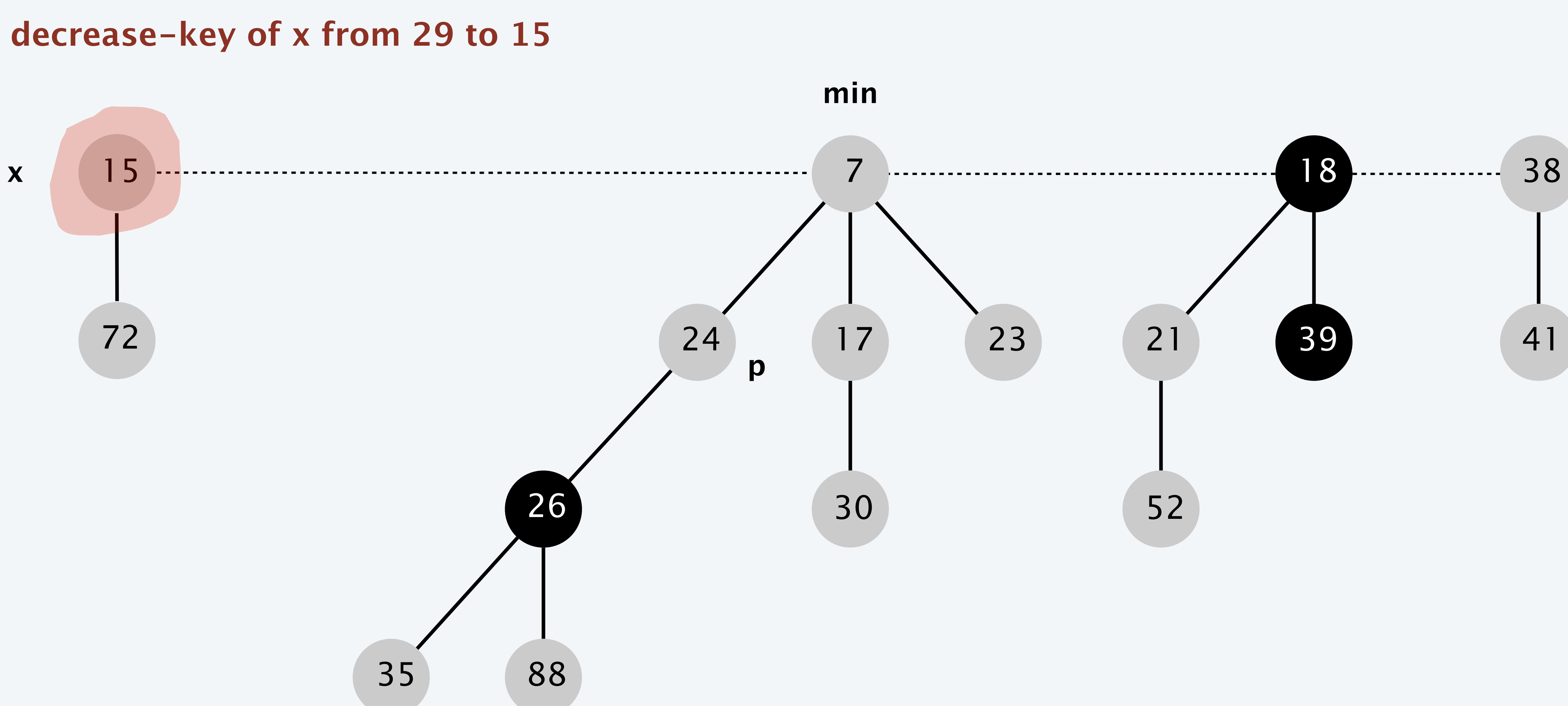
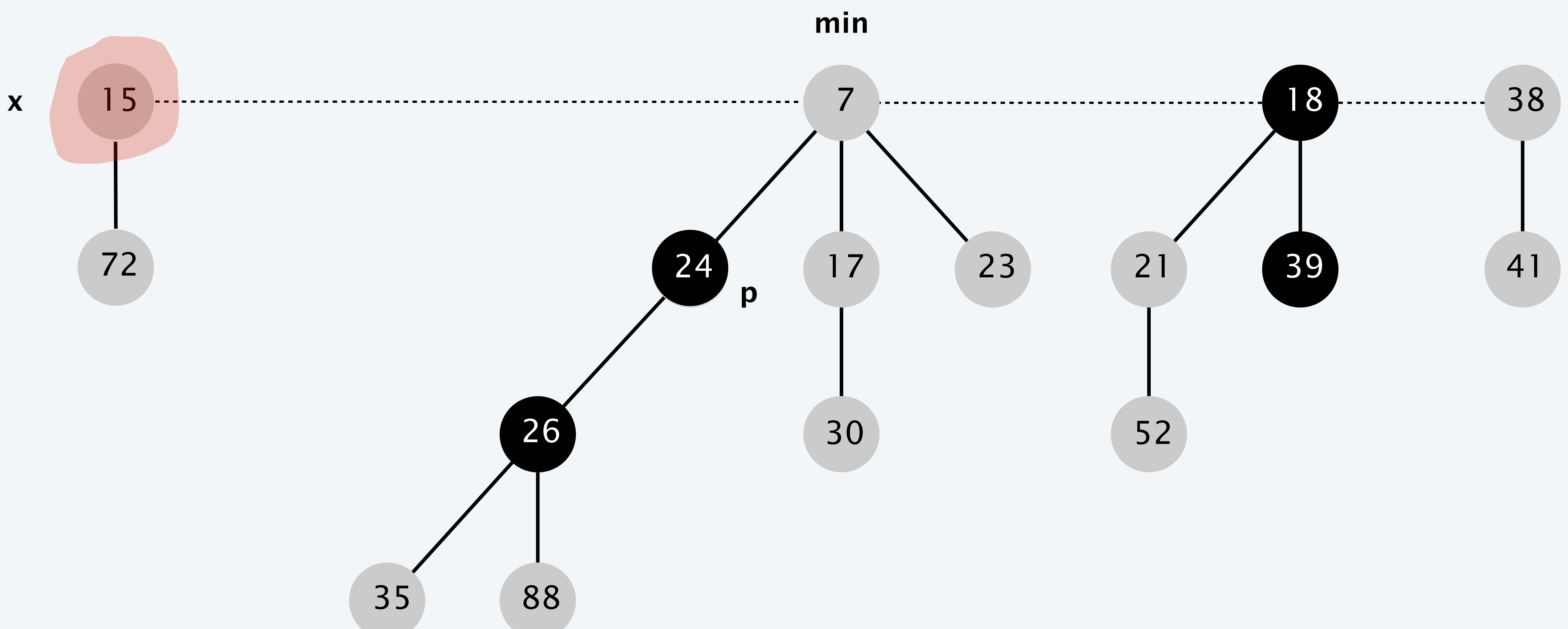


Fibonacci heap: decrease key

Case 2a. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 29 to 15

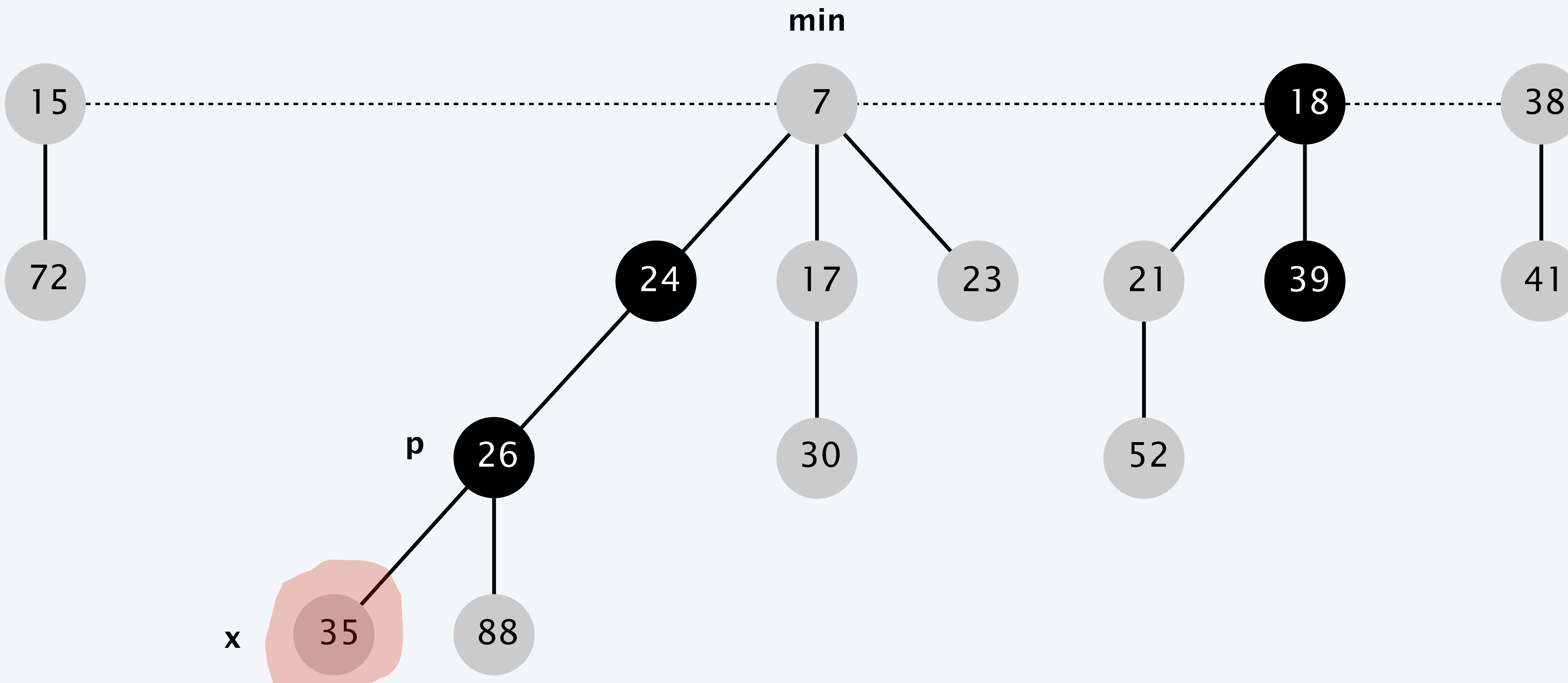


Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 35 to 5

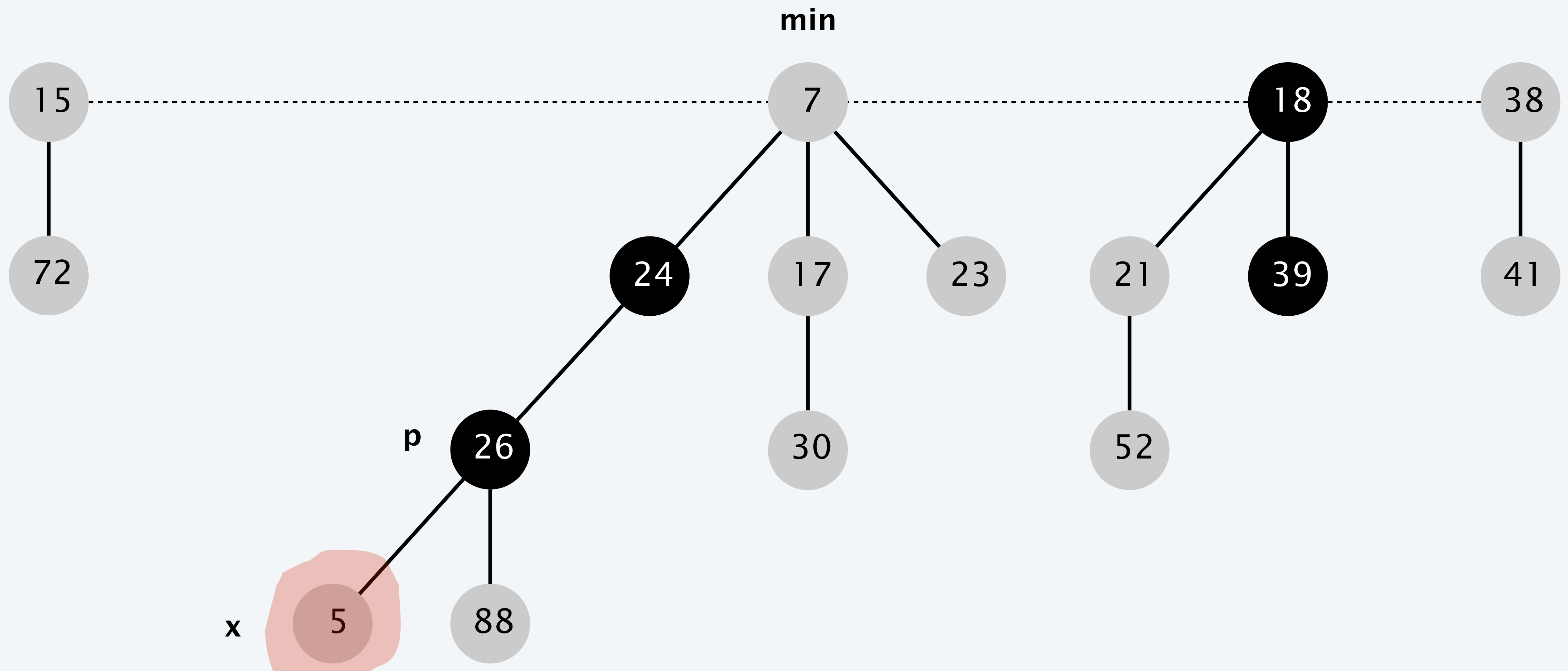


Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 35 to 5



Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
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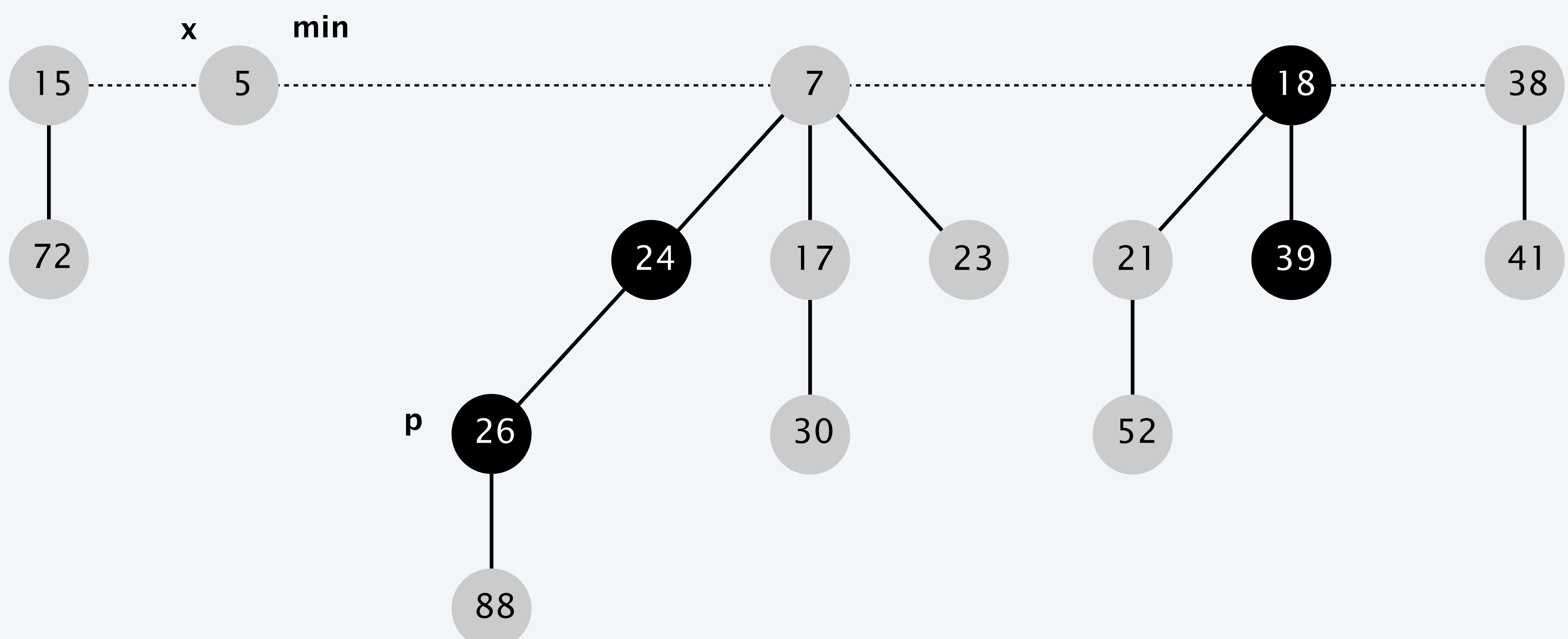
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 35 to 5

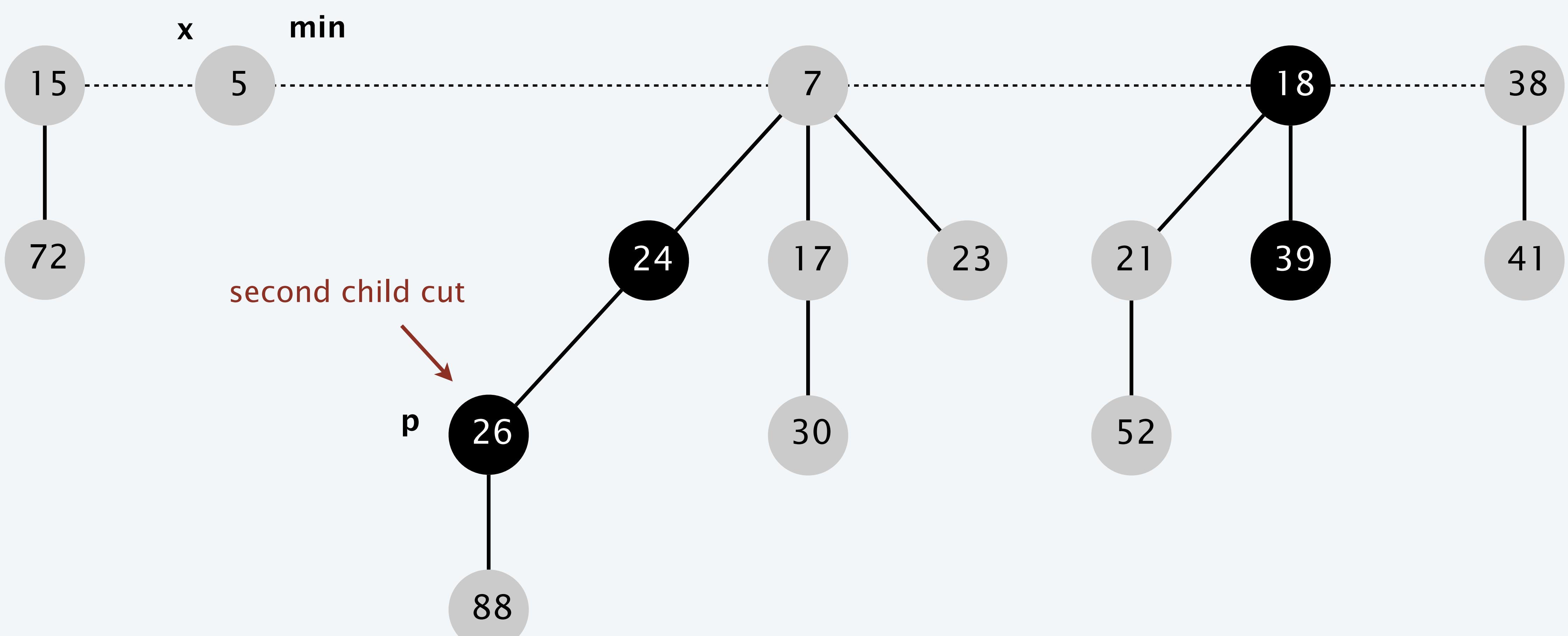


Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
- If parent p of x is unmarked (hasn't yet lost a child), mark it;
  Otherwise, cut p, meld into root list, and unmark
  (and do so recursively for all ancestors that lose a second child).

decrease-key of x from 35 to 5



Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
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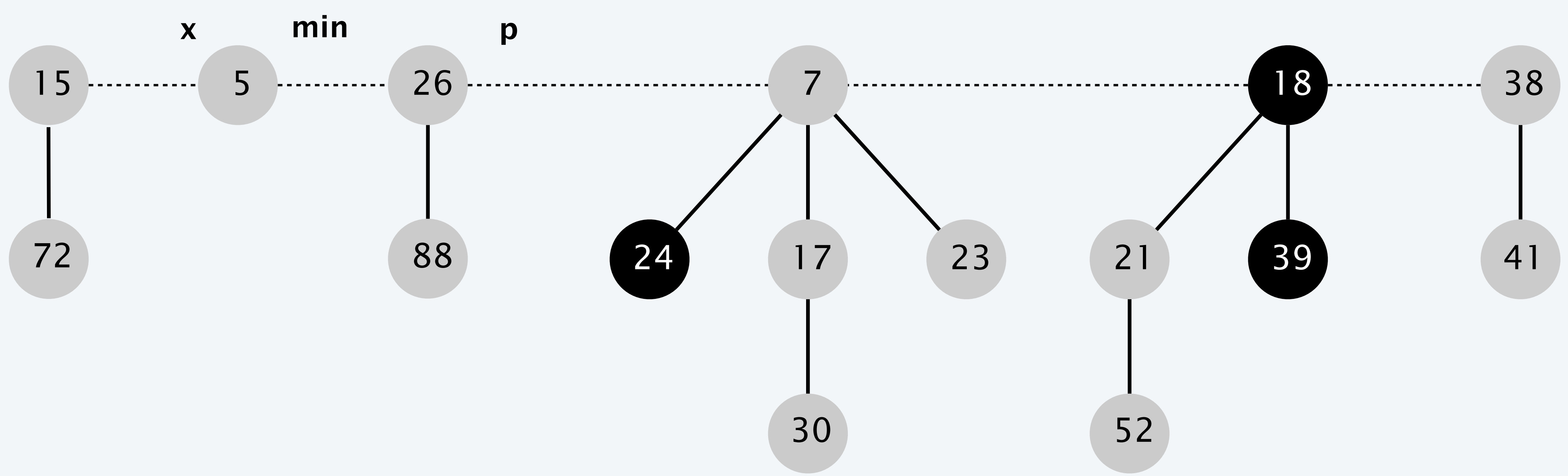
decrease-key of x from 35 to 5



Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
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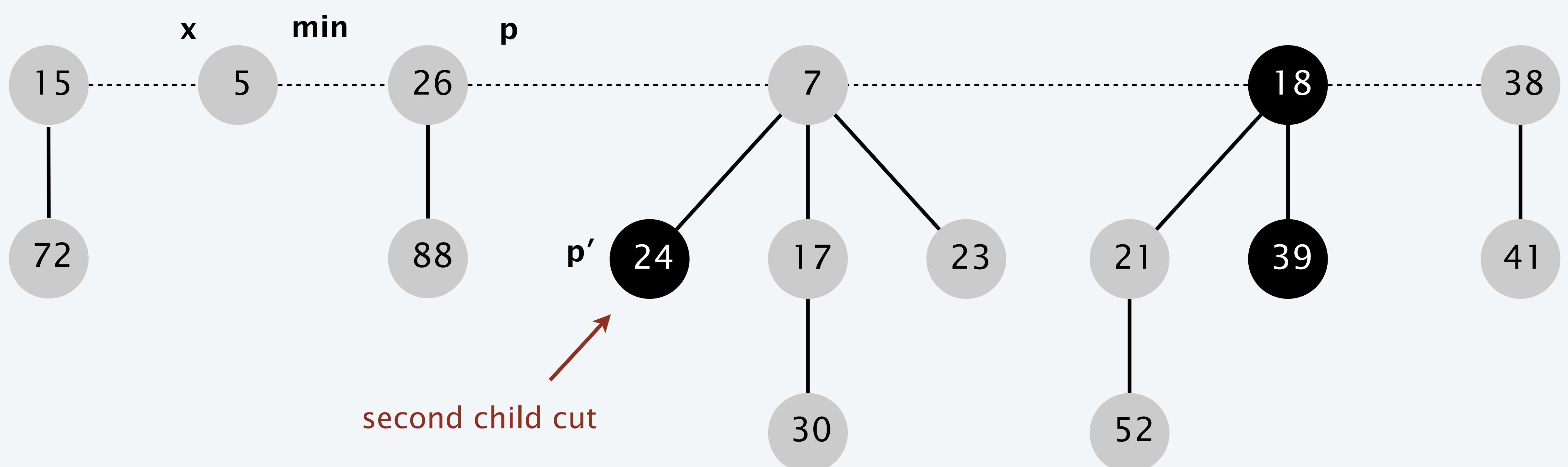
decrease-key of x from 35 to 5



Fibonacci heap: decrease key

Case 2b. [heap order violated]

- Decrease key of x.
- Cut tree rooted at x, meld into root list, and unmark.
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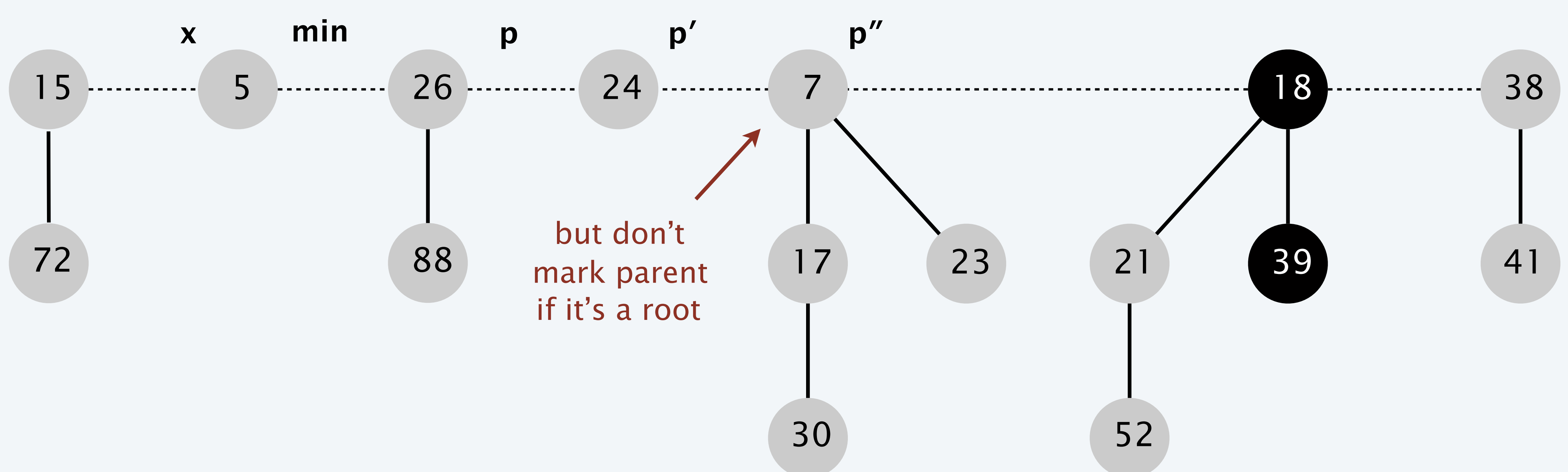
decrease-key of x from 35 to 5



Fibonacci heap: decrease key analysis

Actual cost. $c_i = O(c)$, where c is the number of cuts.

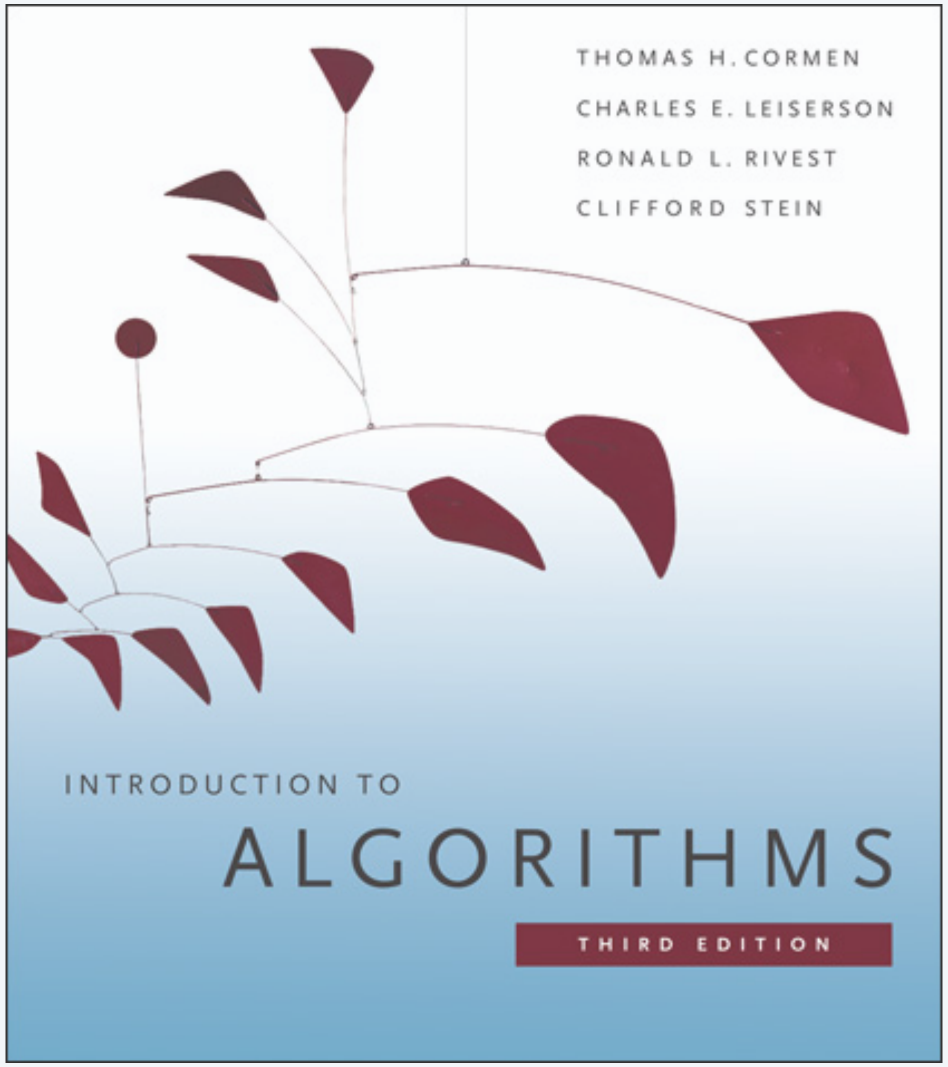
- $O(1)$ time for changing the key.
- $O(1)$ time for each of c cuts, plus melding into root list.

Change in potential. $\Delta\Phi = O(1) - c$.

- $trees(H') = trees(H) + c$.
- $marks(H') \le marks(H) - c + 2$. ← each cut (except first) unmarks a node; last cut may or may not mark a node
- $\Delta\Phi \le c + 2 \cdot (-c + 2) = 4 - c$.

Amortized cost. $\hat{c}_i = c_i + \Delta\Phi = O(1)$.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$



SECTION 19.4

FIBONACCI HEAPS

- *preliminaries*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Analysis summary

Insert. $O(1)$.

Delete-min. $O(rank(H))$ amortized.

Decrease-key. $O(1)$ amortized.

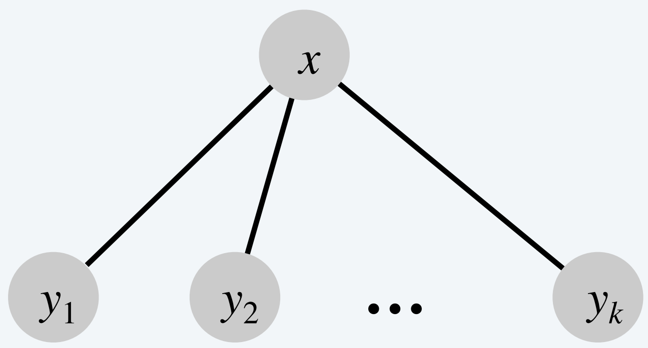
Fibonacci lemma. Let H be a Fibonacci heap with n elements.
Then, $rank(H) = O(\log n)$.

number of nodes is
exponential in rank

Bounding the rank

Lemma 1. Fix a point in time. Let x be a node of rank k, and let $y_1, \ldots, y_k$ denote its current children in the order in which they were linked to x. Then:

$$rank(y_i) \;\geq\; \begin{cases} 0 & \text{if } i = 1 \\ i-2 & \text{if } i \geq 2 \end{cases}$$



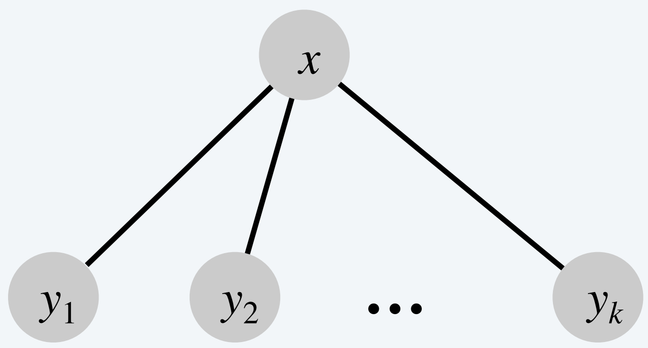
Pf.

- When y_i was linked into x, x had at least $i - 1$ children $y_1, \ldots, y_{i-1}$.
- Since only trees of equal rank are linked, at that time $rank(y_i) \;=\; rank(x) \;\geq\; i - 1$.
- Since then, y_i has lost at most one child (or y_i would have been cut).
- Thus, right now $rank(y_i) \;\geq\; i - 2$. ▪

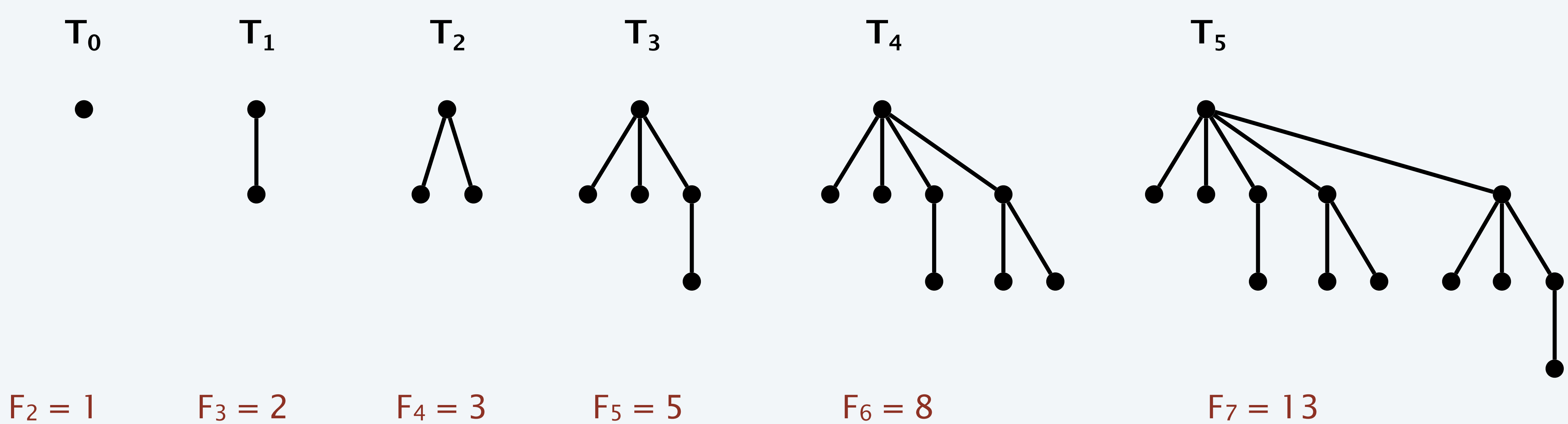
Bounding the rank

Lemma 1. Fix a point in time. Let x be a node of rank k, and let $y_1, \ldots, y_k$ denote its current children in the order in which they were linked to x. Then:

$$rank(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i-2 & \text{if } i \geq 2 \end{cases}$$



Def. Let T_k be smallest possible tree of rank k satisfying property.



Bounding the rank

Lemma 1. Fix a point in time. Let x be a node of rank k, and let $y_1, \ldots, y_k$ denote its current children in the order in which they were linked to x. Then:

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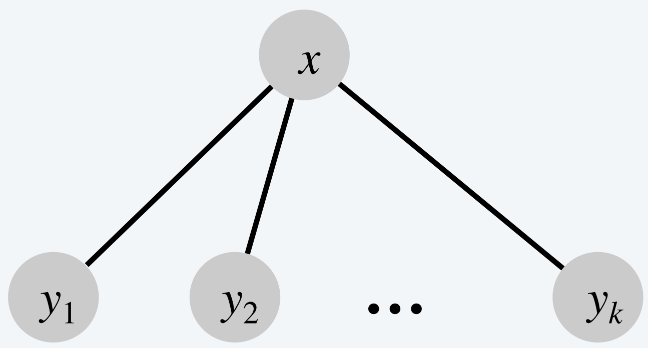
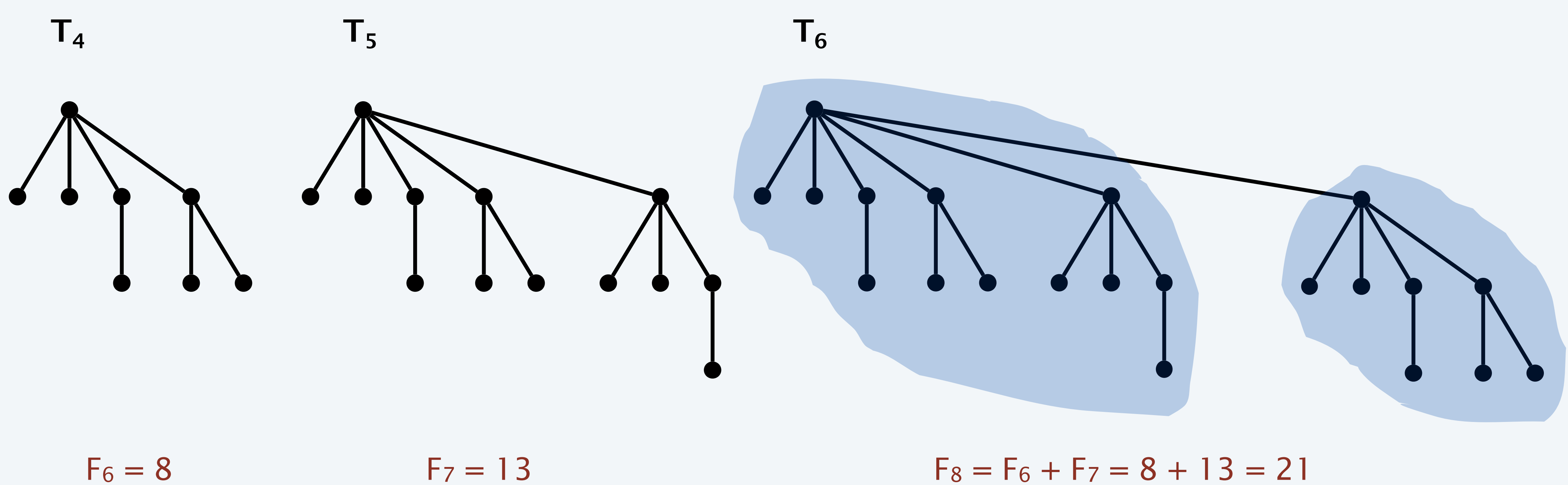


Def. Let T_k be smallest possible tree of rank k satisfying property.



Bounding the rank

Lemma 2. Let s_k be minimum number of elements in any Fibonacci heap of rank k. Then $s_k \ge F_{k+2}$, where F_k is the k^{th} Fibonacci number.

Pf. [by strong induction on k]

- Base cases: $s_0 = 1$ and $s_1 = 2$.
- Inductive hypothesis: assume $s_i \ge F_{i+2}$ for $i = 0, \ldots, k - 1$.
- As in Lemma 1, let let $y_1, \ldots, y_k$ denote its current children in the order in which they were linked to x.

$$
\begin{aligned}
s_k &\ge 1 + 1 + (s_0 + s_1 + \ldots + s_{k-2}) && \text{(Lemma 1)} \\
&\ge (1 + F_1) + F_2 + F_3 + \ldots + F_k && \text{(inductive hypothesis)} \\
&= F_{k+2}. \quad \blacksquare && \text{(Fibonacci fact 1)}
\end{aligned}
$$

Bounding the rank

Fibonacci lemma. Let H be a Fibonacci heap with n elements. Then, $rank(H) \leq \log_\phi n$, where ϕ is the golden ratio $= (1 + \sqrt{5}) / 2 \approx 1.618$.

Pf.

- Let H is a Fibonacci heap with n elements and rank k.
- Then $n \underset{\text{Lemma 2}}{\geq} F_{k+2} \underset{\text{Fibonacci Fact 2}}{\geq} \phi^k$.
- Taking logs, we obtain $rank(H) = k \leq \log_\phi n$. ▪

Fibonacci fact 1

Def. The Fibonacci sequence is: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Fibonacci fact 1. For all integers $k \geq 0$, $F_{k+2} = 1 + F_0 + F_1 + \ldots + F_k$.

Pf. [by induction on k]

- Base case: $F_2 = 1 + F_0 = 2$.
- Inductive hypothesis: assume $F_{k+1} = 1 + F_0 + F_1 + \ldots + F_{k-1}$.

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} && \text{(definition)} \\ &= F_k + (1 + F_0 + F_1 + \ldots + F_{k-1}) && \text{(inductive hypothesis)} \\ &= 1 + F_0 + F_1 + \ldots + F_{k-1} + F_k. \quad \blacksquare && \text{(algebra)} \end{aligned}$$

Fibonacci fact 2

Def. The Fibonacci sequence is: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Fibonacci fact 2. $F_{k+2} \geq \phi^k$, where $\phi = (1 + \sqrt{5}) / 2 \approx 1.618$.

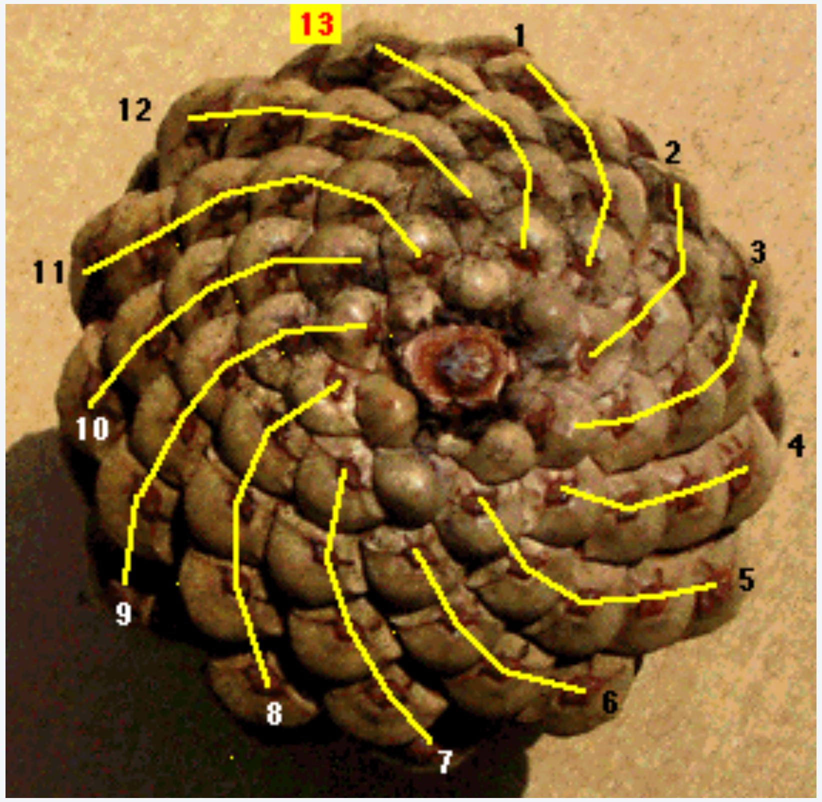
Pf. [by induction on k]

- Base cases: $F_2 = 1 \geq 1$, $F_3 = 2 \geq \phi$.
- Inductive hypotheses: assume $F_k \geq \phi^k$ and $F_{k+1} \geq \phi^{k+1}$

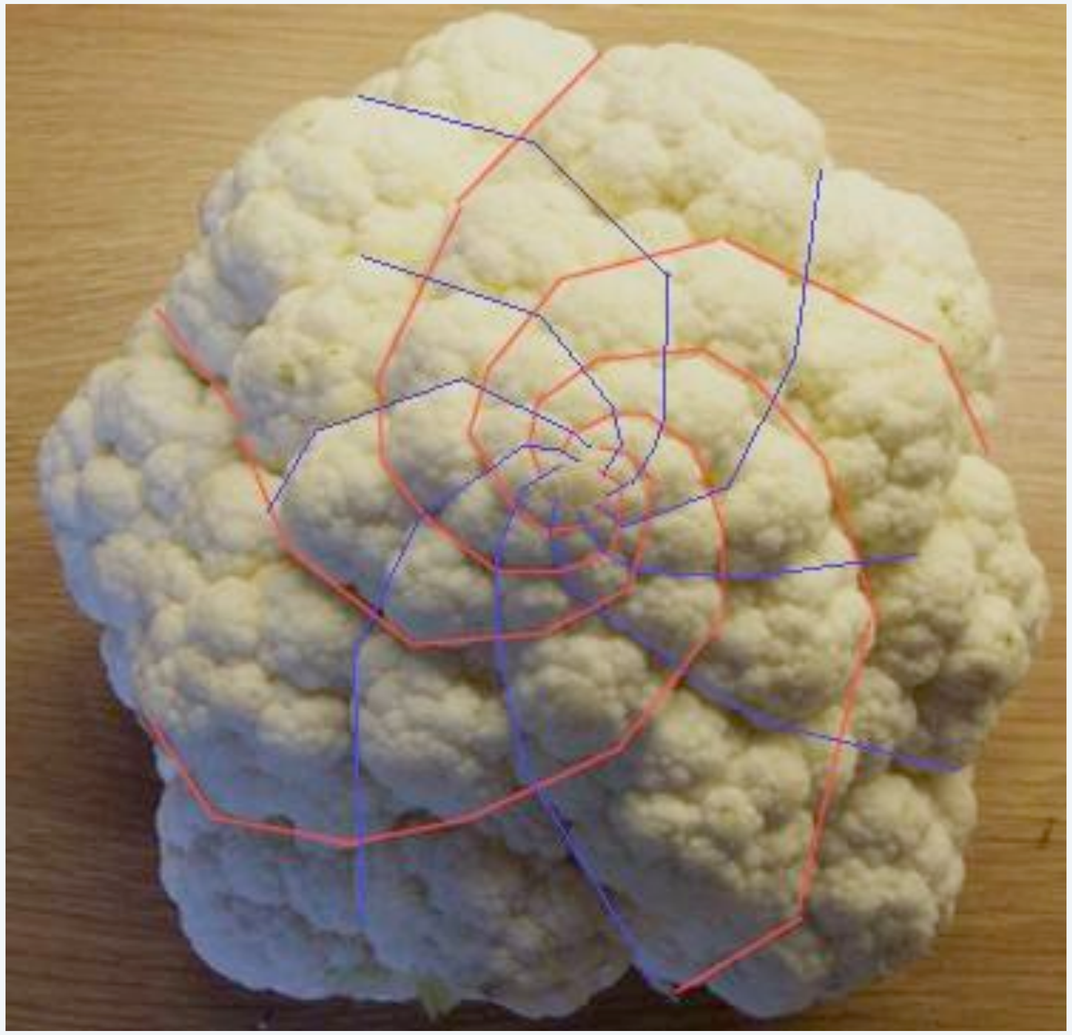
$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} && \text{(definition)} \\ &\geq \phi^{k-1} + \phi^{k-2} && \text{(inductive hypothesis)} \\ &= \phi^{k-2}(1 + \phi) && \text{(algebra)} \\ &= \phi^{k-2}\,\phi^2 && (\phi^2 = \phi + 1) \\ &= \phi^k. \quad \blacksquare && \text{(algebra)} \end{aligned}$$

Fibonacci numbers and nature

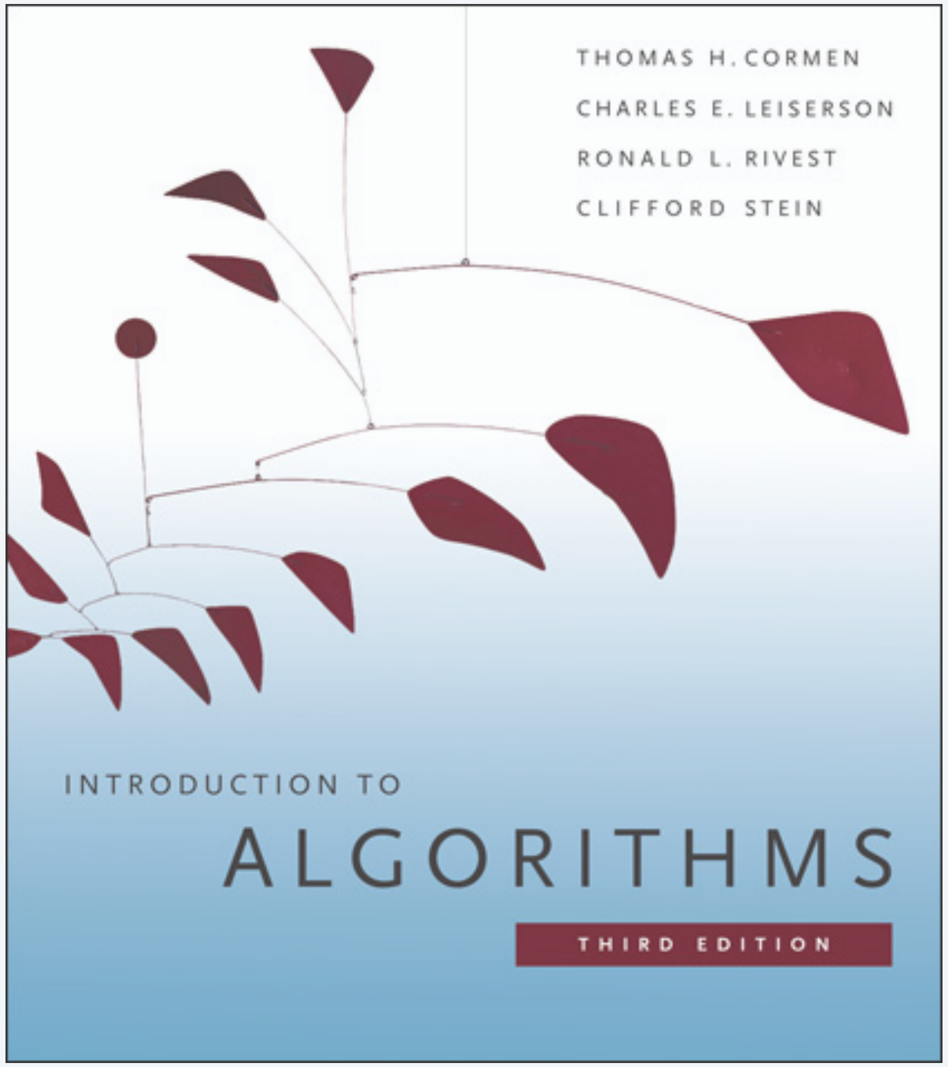
Fibonacci numbers arise both in nature and algorithms.



pinecone



cauliflower



SECTION 19.2, 19.3

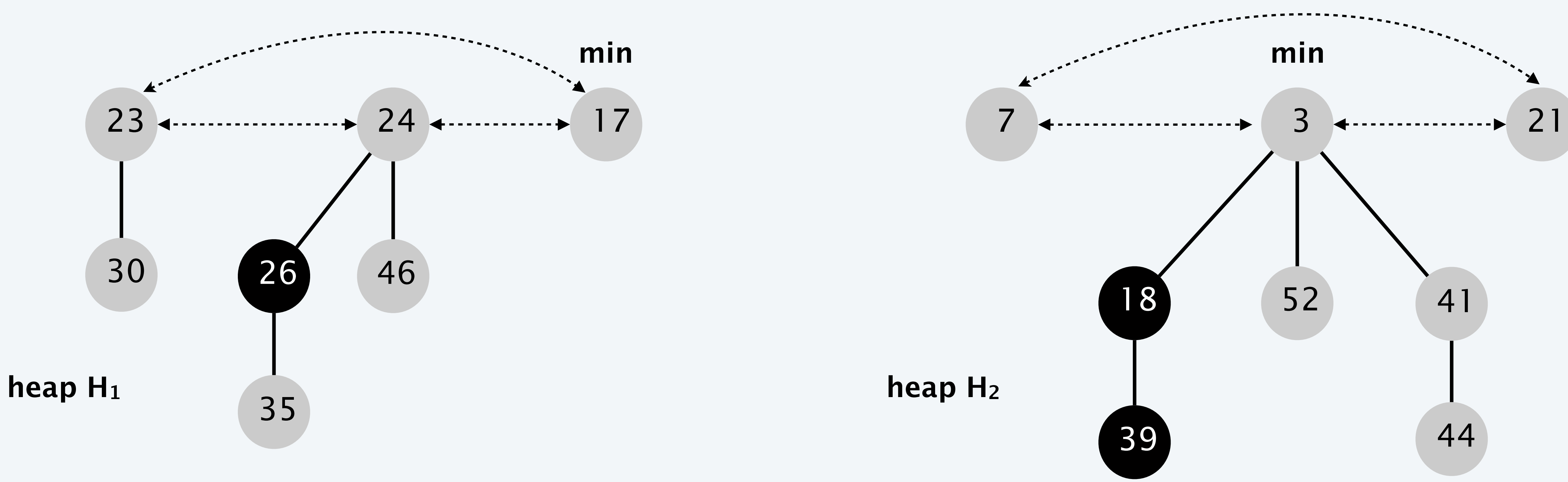
FIBONACCI HEAPS

- *preliminaries*
- *insert*
- *extract the minimum*
- *decrease key*
- *bounding the rank*
- *meld and delete*

Fibonacci heap: meld

Meld. Combine two Fibonacci heaps (destroying old heaps).

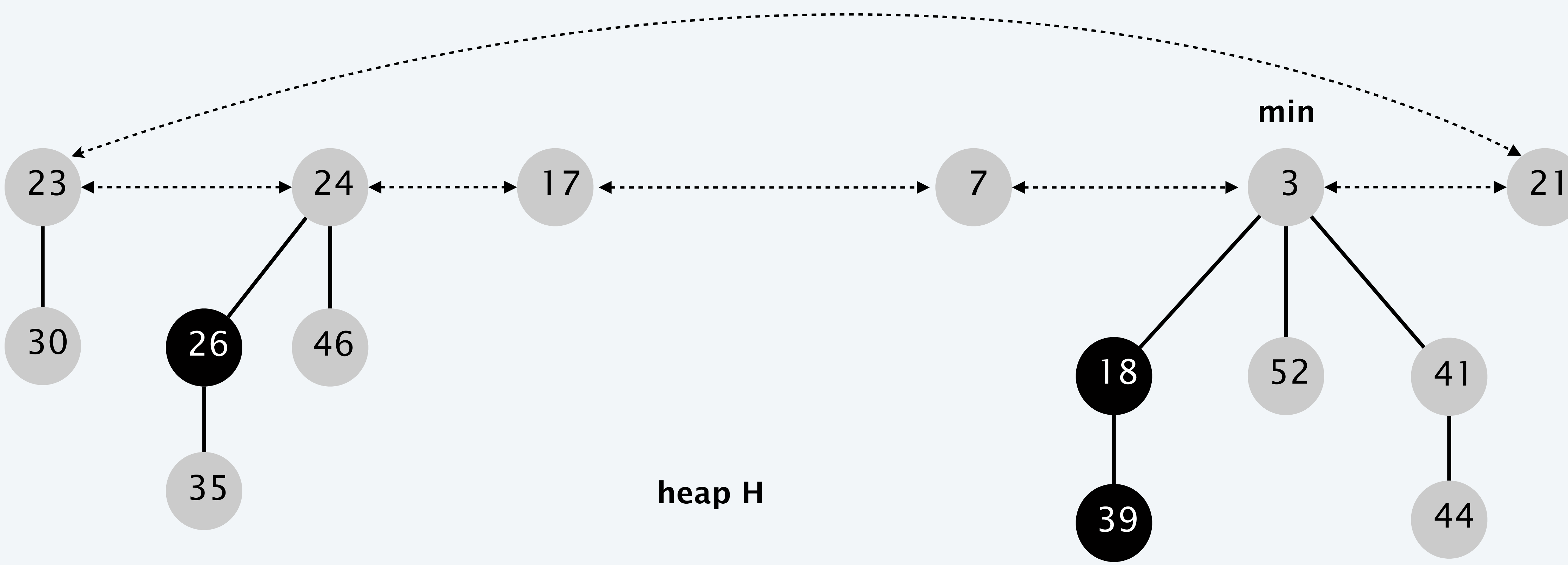
Recall. Root lists are circular, doubly-linked lists.



Fibonacci heap: meld

Meld. Combine two Fibonacci heaps (destroying old heaps).

Recall. Root lists are circular, doubly-linked lists.



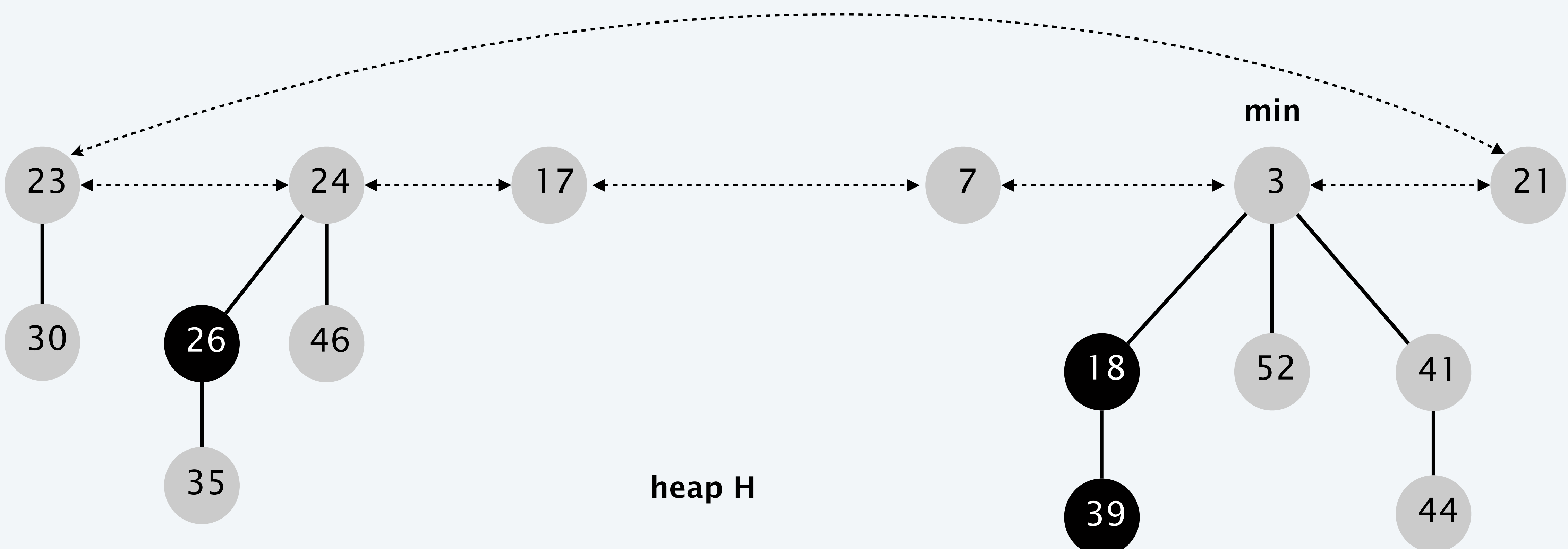
Fibonacci heap: meld analysis

Actual cost. $c_i = O(1)$.

Change in potential. $\Delta\Phi = 0$.

Amortized cost. $\hat{c}_i = c_i + \Delta\Phi = O(1)$.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$



Fibonacci heap: delete

Delete. Given a handle to an element x, delete it from heap H.

- DECREASE-KEY($H, x, -\infty$).
- EXTRACT-MIN(H).

Amortized cost. $\hat{c}_i = O(rank(H))$.

- $O(1)$ amortized for DECREASE-KEY.
- $O(rank(H))$ amortized for EXTRACT-MIN.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Priority queues performance cost summary

operation	linked list	binary heap	binomial heap	Fibonacci heap †
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$	$O(1)$
IS-EMPTY	$O(1)$	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$
DELETE	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
MELD	$O(1)$	$O(n)$	$O(\log n)$	$O(1)$
FIND-MIN	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$

† amortized

Accomplished. $O(1)$ INSERT and DECREASE-KEY, $O(\log n)$ EXTRACT-MIN.

PRIORITY QUEUES

- *binary heaps*
- *d-ary heaps*
- *binomial heaps*
- *Fibonacci heaps*
- *advanced topics*

Heaps of heaps

- b-heaps.
- Fat heaps.
- 2–3 heaps.
- Leaf heaps.
- Thin heaps.
- Skew heaps.
- Splay heaps.
- Weak heaps.
- Leftist heaps.
- Quake heaps.
- Pairing heaps.
- Violation heaps.
- Run-relaxed heaps.
- Rank-pairing heaps.
- Skew-pairing heaps.
- Rank-relaxed heaps.
- Lazy Fibonacci heaps.

Brodal queues

Q. Can we achieve same running time as for Fibonacci heap but with worst-case bounds per operation (instead of amortized)?

Theory. [Brodal 1996] Yes.

Worst-Case Efficient Priority Queues*

Gerth Stølting Brodal†

Abstract

An implementation of priority queues is presented that supports the operations MAKEQUEUE, FINDMIN, INSERT, MELD and DECREASEKEY in worst case time $O(1)$ and DELETEMIN and DELETE in worst case time $O(\log n)$. The space requirement is linear. The data structure presented is the first achieving this worst case performance.

Practice. Ever implemented? Constants are high (and requires RAM model).

Strict Fibonacci heaps

Q. Can we achieve same running time as for Fibonacci heap but with worst-case bounds per operation (instead of amortized) in pointer model?

Theory. [Brodal–Lagogiannis–Tarjan 2012] Yes.

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ABSTRACT

We present the first pointer-based heap implementation with time bounds matching those of Fibonacci heaps in the worst case. We support make-heap, insert, find-min, meld and decrease-key in worst-case $O(1)$ time, and delete and delete-min in worst-case $O(\lg n)$ time, where n is the size of the heap. The data structure uses linear space.

A previous, very complicated, solution achieving the same time bounds in the RAM model made essential use of arrays and extensive use of redundant counter schemes to maintain balance. Our solution uses neither. Our key simplification is to discard the structure of the smaller heap when doing a meld. We use the pigeonhole principle in place of the redundant counter mechanism.

Fibonacci heaps: practice

Q. Are Fibonacci heaps useful in practice?

A. They are part of LEDA and Boost C++ libraries.

(but other heaps seem to perform better in practice)

Pairing heaps

Pairing heap. A self-adjusting heap-ordered general tree.

The Pairing Heap: A New Form of Self-Adjusting Heap

Michael L. Fredman[1,4], Robert Sedgewick[2,5], Daniel D. Sleator[3], and Robert E. Tarjan[2,3,6]

Abstract. Recently, Fredman and Tarjan invented a new, especially efficient form of heap (priority queue) called the *Fibonacci heap*. Although theoretically efficient, Fibonacci heaps are complicated to implement and not as fast in practice as other kinds of heaps. In this paper we describe a new form of heap, called the *pairing heap*, intended to be competitive with the Fibonacci heap in theory and easy to implement and fast in practice. We provide a partial complexity analysis of pairing heaps. Complete analysis remains an open problem.

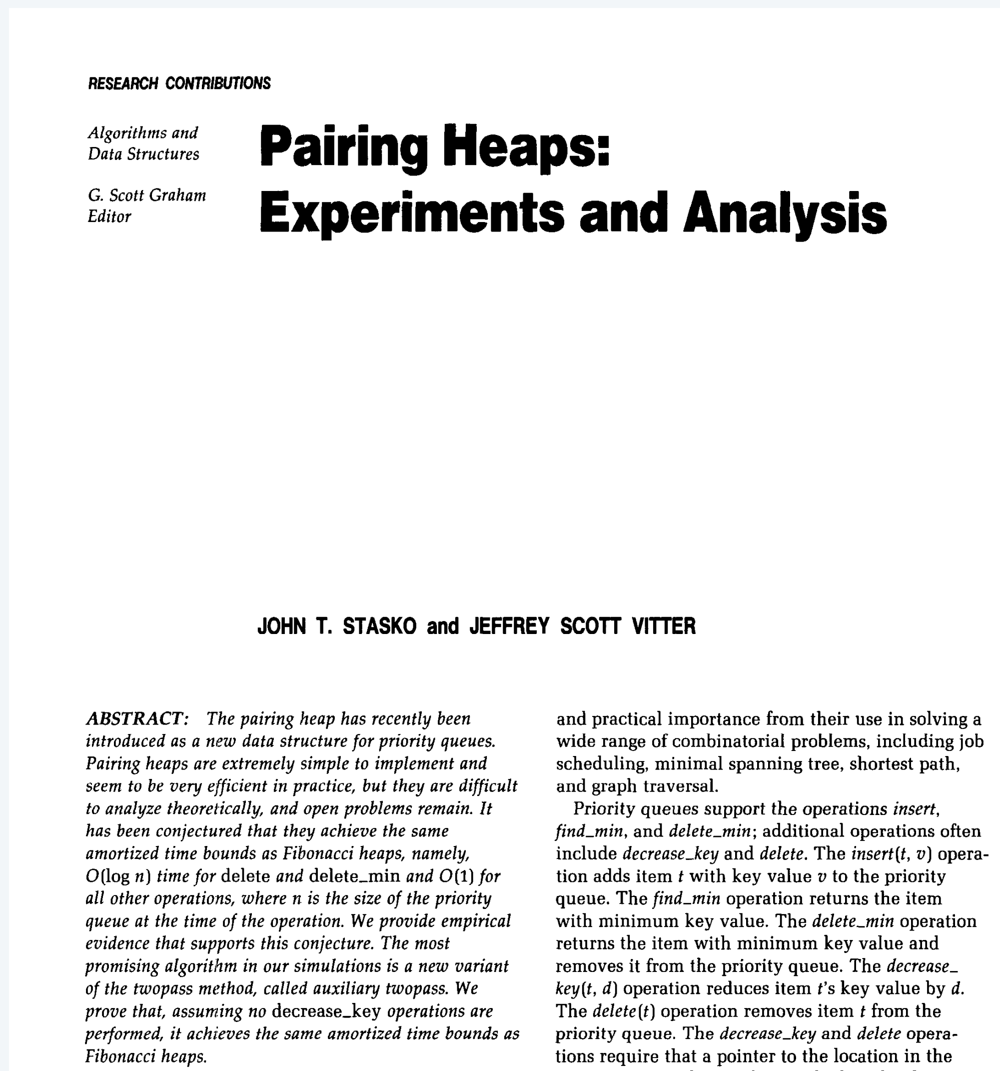
Theory. Same amortized running times as Fibonacci heaps for all operations except DECREASE-KEY.

- $O(\log n)$ amortized. [Fredman et al. 1986]
- $\Omega(\log \log n)$ lower bound on amortized cost. [Fredman 1999]
- $2^{\sqrt{O(\log \log n)}}$ amortized. [Pettie 2005]

Pairing heaps

Pairing heap. A self-adjusting heap-ordered general tree.

Practice. As fast as (or faster than) the binary heap on some problems.
Included in GNU C++ library and LEDA.



RESEARCH CONTRIBUTIONS

Algorithms and Data Structures

G. Scott Graham Editor

Pairing Heaps: Experiments and Analysis

JOHN T. STASKO and JEFFREY SCOTT VITTER

ABSTRACT: The pairing heap has recently been introduced as a new data structure for priority queues. Pairing heaps are extremely simple to implement and seem to be very efficient in practice, but they are difficult to analyze theoretically, and open problems remain. It has been conjectured that they achieve the same amortized time bounds as Fibonacci heaps, namely, $O(\log n)$ time for delete *and* delete_min *and $O(1)$ for all other operations, where n is the size of the priority queue at the time of the operation. We provide empirical evidence that supports this conjecture. The most promising algorithm in our simulations is a new variant of the twopass method, called auxiliary twopass. We prove that, assuming no* decrease_key *operations are performed, it achieves the same amortized time bounds as Fibonacci heaps.*

and practical importance from their use in solving a wide range of combinatorial problems, including job scheduling, minimal spanning tree, shortest path, and graph traversal.

Priority queues support the operations *insert*, *find_min*, and *delete_min*; additional operations often include *decrease_key* and *delete*. The *insert*(t, v) operation adds item t with key value v to the priority queue. The *find_min* operation returns the item with minimum key value. The *delete_min* operation returns the item with minimum key value and removes it from the priority queue. The *decrease_key*(t, d) operation reduces item t's key value by d. The *delete*(t) operation removes item t from the priority queue. The *decrease_key* and *delete* operations require that a pointer to the location in the

Priority queues performance cost summary

operation	linked list	binary heap	binomial heap	pairing heap †	Fibonacci heap †	Brodal queue
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$
IS-EMPTY	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)$	$O(1)$	$O(1)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$	$2^{\sqrt{O(\log \log n)}}$	$O(1)$	$O(1)$
DELETE	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
MELD	$O(1)$	$O(n)$	$O(\log n)$	$O(1)$	$O(1)$	$O(1)$
FIND-MIN	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$	$O(1)$	$O(1)$

† amortized

Priority queues with integer priorities

Assumption. Keys are integers between 0 and C.

Theorem. [Thorup 2004] There exists a priority queue that supports INSERT, FIND-MIN, and DECREASE-KEY in constant time and EXTRACT-MIN and DELETE-KEY in either $O(\log \log n)$ or $O(\log \log C)$ time.

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Integer priority queues with decrease key in constant time and the single source shortest paths problem

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Abstract

We consider Fibonacci heap style integer priority queues supporting find-min, insert, and decrease key operations in constant time. We present a deterministic linear space solution that with n integer keys supports delete in $O(\log \log n)$ time. If the integers are in the range $[0, N)$, we can also support delete in $O(\log \log N)$ time.

Priority queues with integer priorities

Assumption. Keys are integers between 0 and C.

Theorem. [Thorup 2004] There exists a priority queue that supports INSERT, FIND-MIN, and DECREASE-KEY in constant time and EXTRACT-MIN and DELETE-KEY in either $O(\log \log n)$ or $O(\log \log C)$ time.

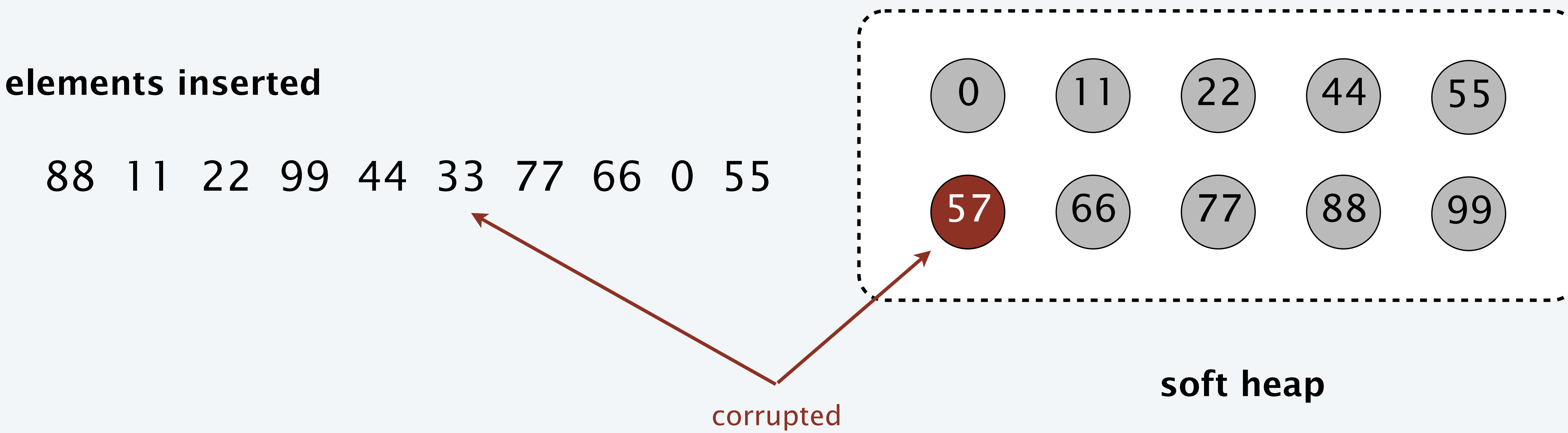
Corollary 1. Can implement Dijkstra's algorithm in either $O(m \log \log n)$ or $O(m \log \log C)$ time.

Corollary 2. Can sort n integers in $O(n \log \log n)$ time.

Computational model. Word RAM.

Soft heaps

Goal. Break information-theoretic lower bound by allowing priority queue to corrupt 10% of the keys (by increasing them).



Soft heaps

Goal. Break information-theoretic lower bound by allowing priority queue to corrupt 10% of the keys (by increasing them).

Representation.

- Set of binomial trees (with some subtrees missing).
- Each node may store several elements.
- Each node stores a value that is an upper bound on the original keys.
- Binomial trees are heap-ordered with respect to these values.

Soft heaps

Goal. Break information-theoretic lower bound by allowing priority queue to corrupt 10% of the keys (by increasing them).

Theorem. [Chazelle 2000] Starting from an empty soft heap, any sequence of n INSERT, MIN, EXTRACT-MIN, MELD, and DELETE operations takes $O(n)$ time and at most 10% of its elements are corrupted at any given time.

The Soft Heap: An Approximate Priority Queue with Optimal Error Rate

BERNARD CHAZELLE

Princeton University, Princeton, New Jersey, and NEC Research Institute

Abstract. A simple variant of a priority queue, called a *soft heap*, is introduced. The data structure supports the usual operations: insert, delete, meld, and findmin. Its novelty is to beat the logarithmic bound on the complexity of a heap in a comparison-based model. To break this information-theoretic barrier, the entropy of the data structure is reduced by artificially raising the values of certain keys. Given any mixed sequence of n operations, a soft heap with error rate ε (for any $0 < \varepsilon \leq 1/2$) ensures that, at any time, at most εn of its items have their keys raised. The amortized complexity of each operation is constant, except for insert, which takes $O(\log 1/\varepsilon)$ time. The soft heap is optimal for any value of ε in a comparison-based model. The data structure is purely pointer-based. No arrays are used and no numeric assumptions are made on the keys. The main idea behind the soft heap is to move items across the data structure not individually, as is customary, but in groups, in a data-structuring equivalent of "car pooling." Keys must be raised as a result, in order to preserve the heap ordering of the data structure. The soft heap can be used to compute exact or approximate medians and percentiles optimally. It is also useful for approximate sorting and for computing minimum spanning trees of general graphs.

Soft heaps

Goal. Break information-theoretic lower bound by allowing priority queue to corrupt 10% of the keys (by increasing them).

Q. Brilliant. But how could it possibly be useful?

Ex. Linear-time deterministic selection. To find k^{th} smallest element:

- Insert the n elements into soft heap.
- Extract the minimum element $n / 2$ times.
- The largest element deleted $\geq 4n / 10$ elements and $\leq 6n / 10$ elements.
- Can remove $\geq 4n / 10$ of elements and recur.
- $T(n) \leq T(3n / 5) + O(n) \Rightarrow T(n) = O(n)$. ▪

Soft heaps

Theorem. [Chazelle 2000] There exists an $O(m\,\alpha(m, n))$ time deterministic algorithm to compute an MST in a graph with n nodes and m edges.

Algorithm. Borůvka + nongreedy + divide-and-conquer + soft heap + …

A Minimum Spanning Tree Algorithm with Inverse-Ackermann Type Complexity

BERNARD CHAZELLE

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Abstract. A deterministic algorithm for computing a minimum spanning tree of a connected graph is presented. Its running time is $O(m\alpha(m, n))$, where α is the classical functional inverse of Ackermann's function and n (respectively, m) is the number of vertices (respectively, edges). The algorithm is comparison-based: it uses pointers, not arrays, and it makes no numeric assumptions on the edge costs.