Advanced Data Structures & Algorithm Analysis

丁尧相
浙江大学

2024.5.20

Randomized Algorithms

- Hiring problem: probabilistic analysis
- Randomized hiring algorithm
- Randomized quicksort
- Min-cut and max-cut
- Take-home messages

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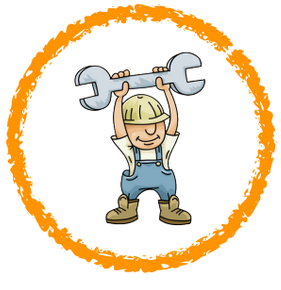

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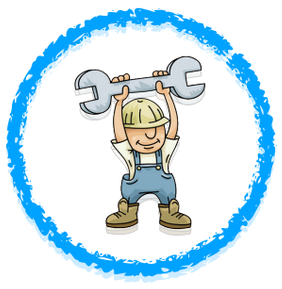

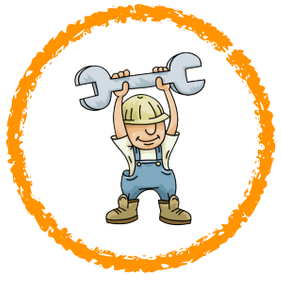

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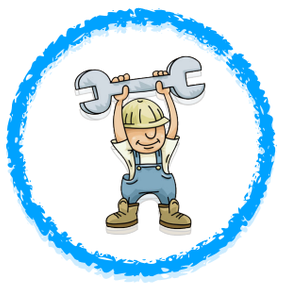

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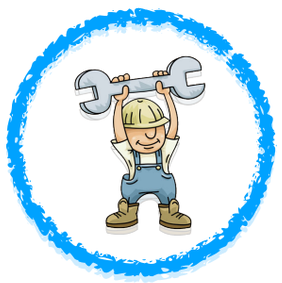

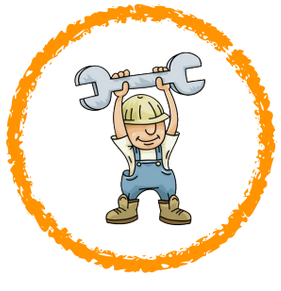

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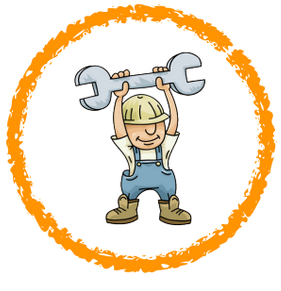

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Hire if find better candidate and spend a big cost.

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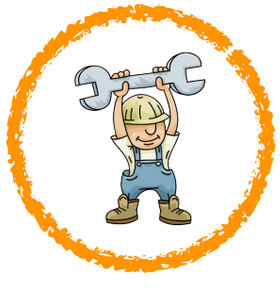

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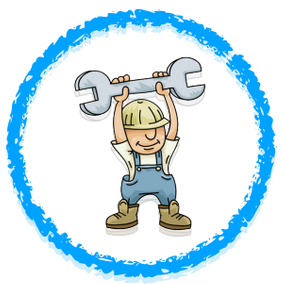

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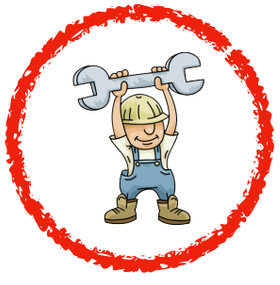

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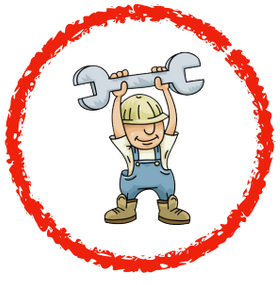

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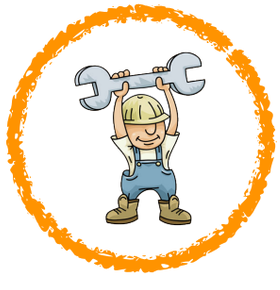

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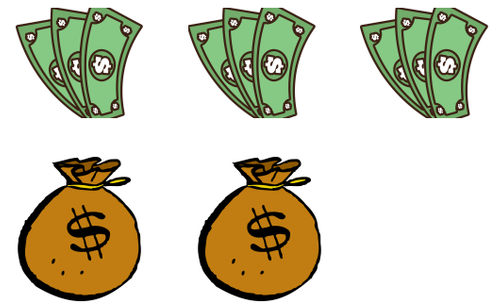

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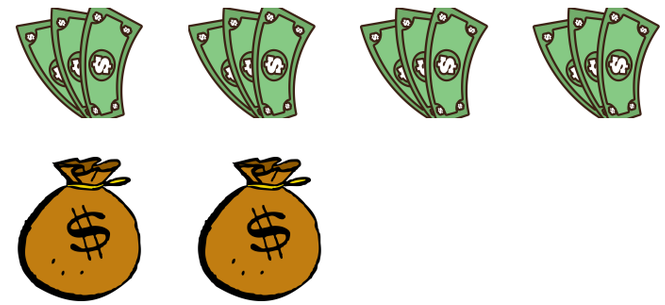

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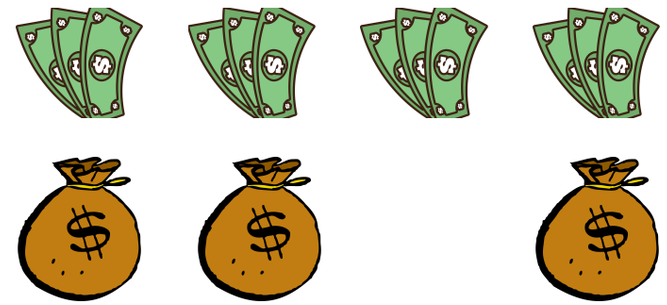

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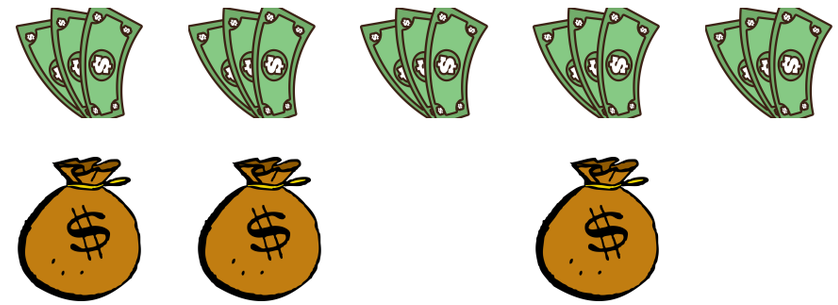

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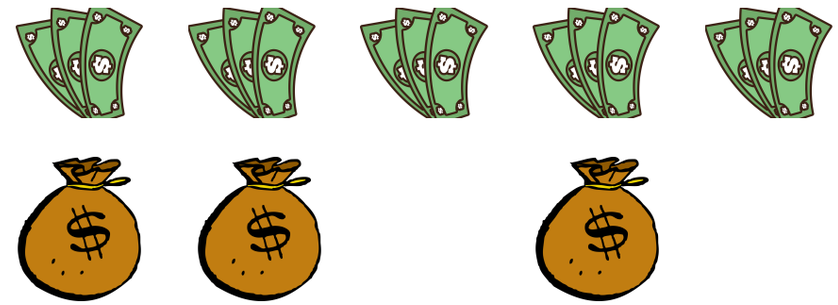

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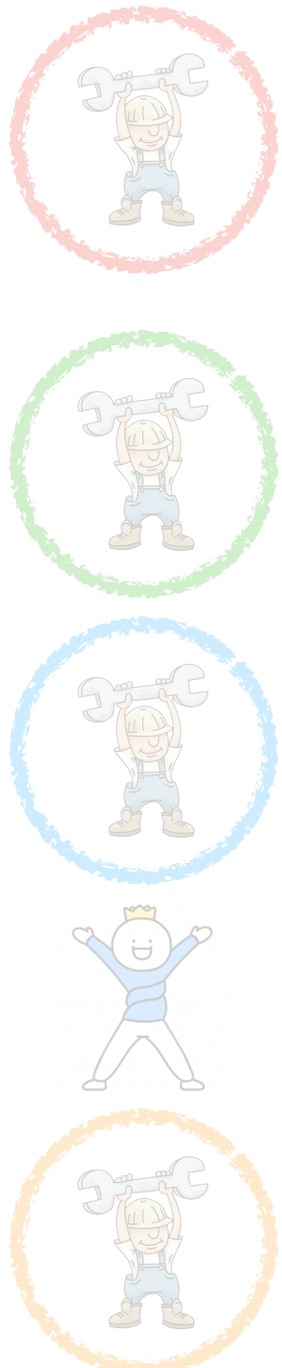

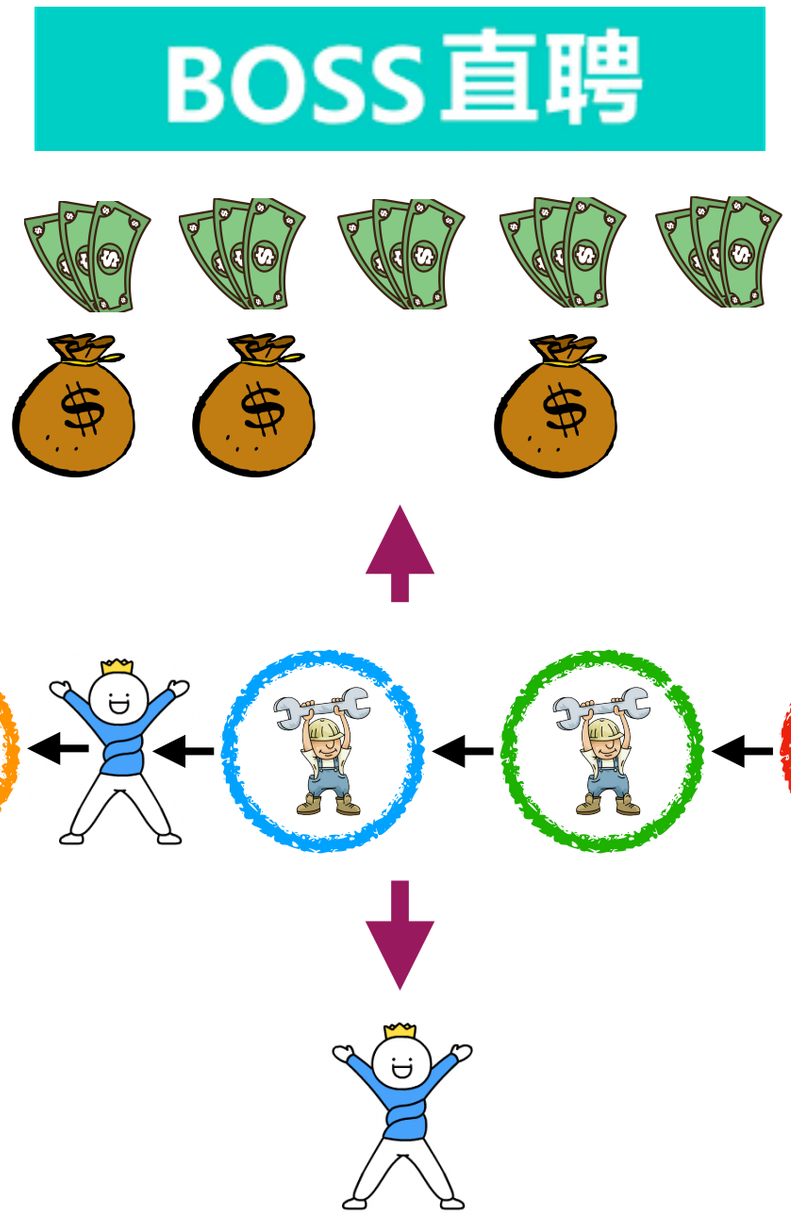

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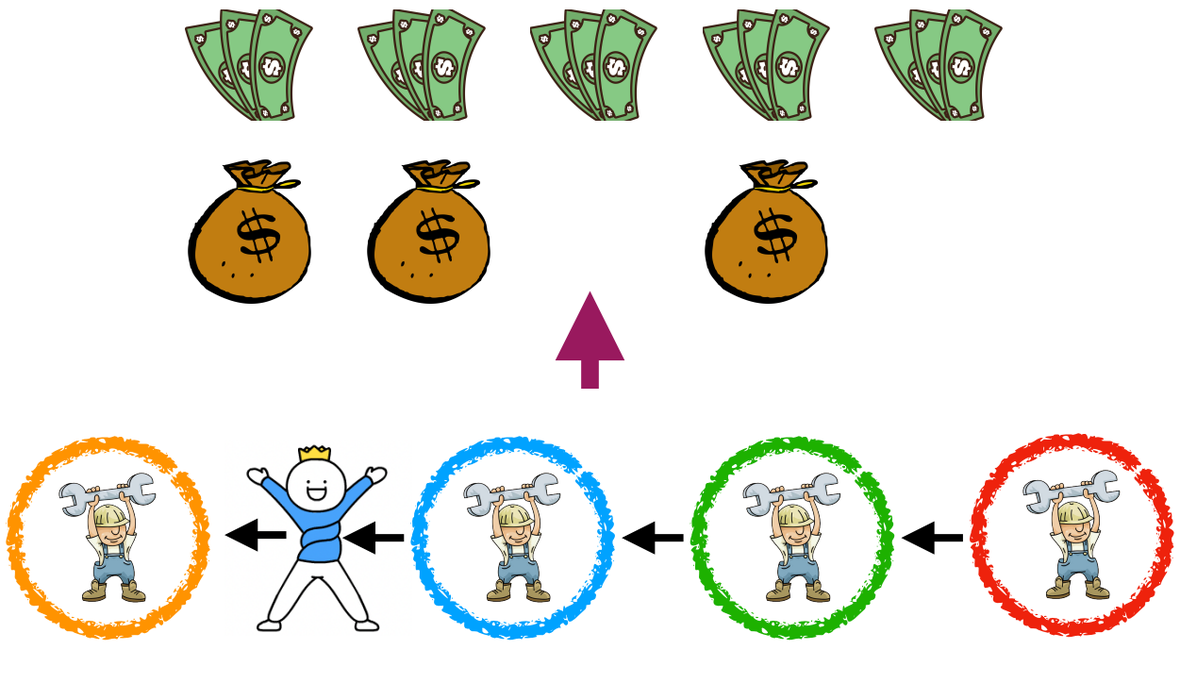
Cost Analysis



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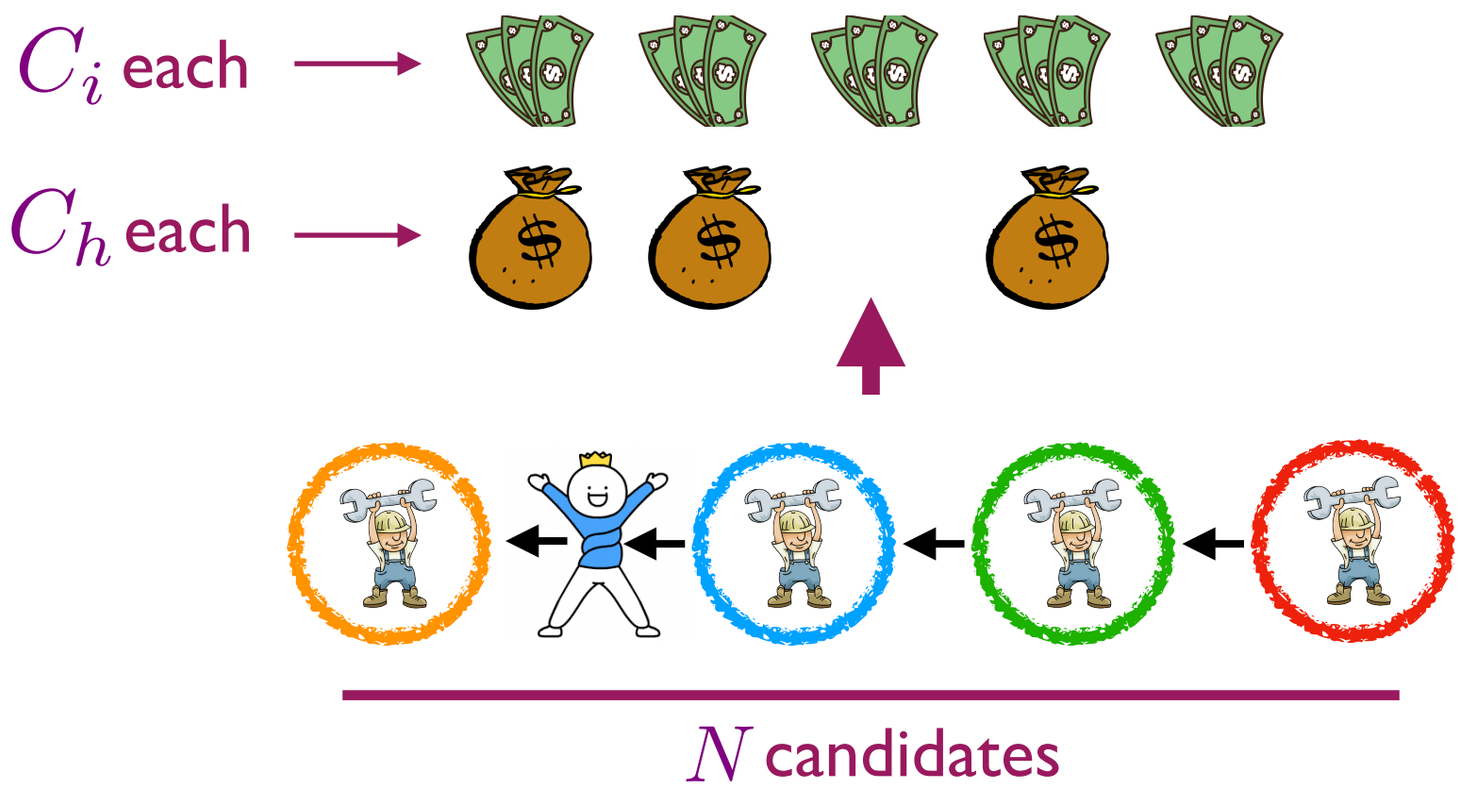

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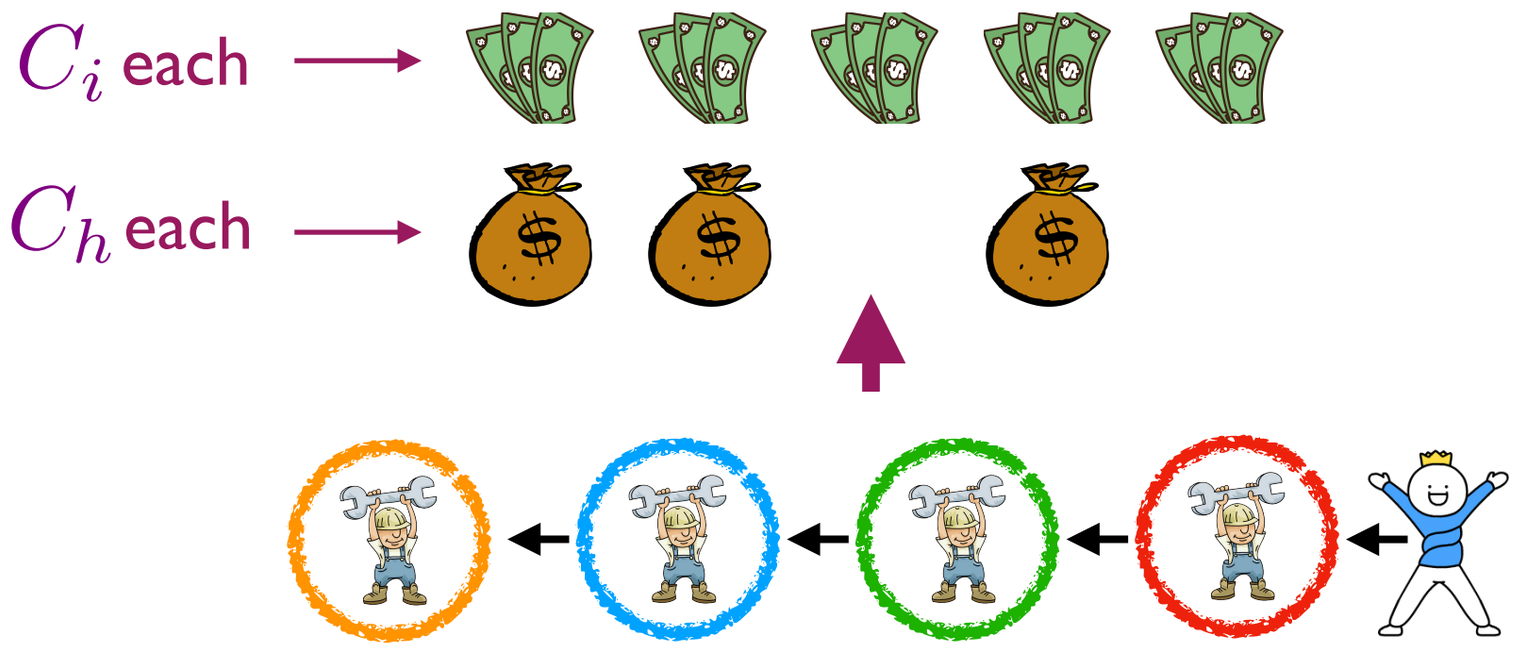

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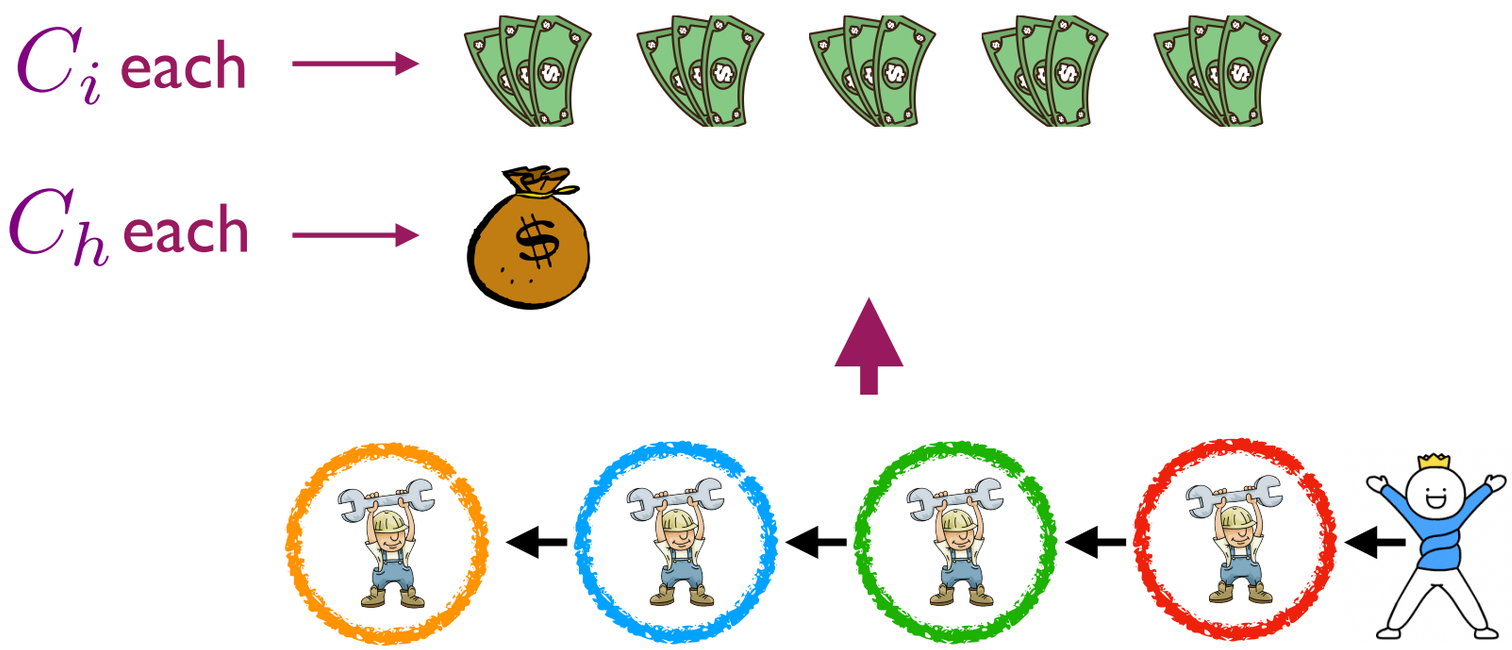

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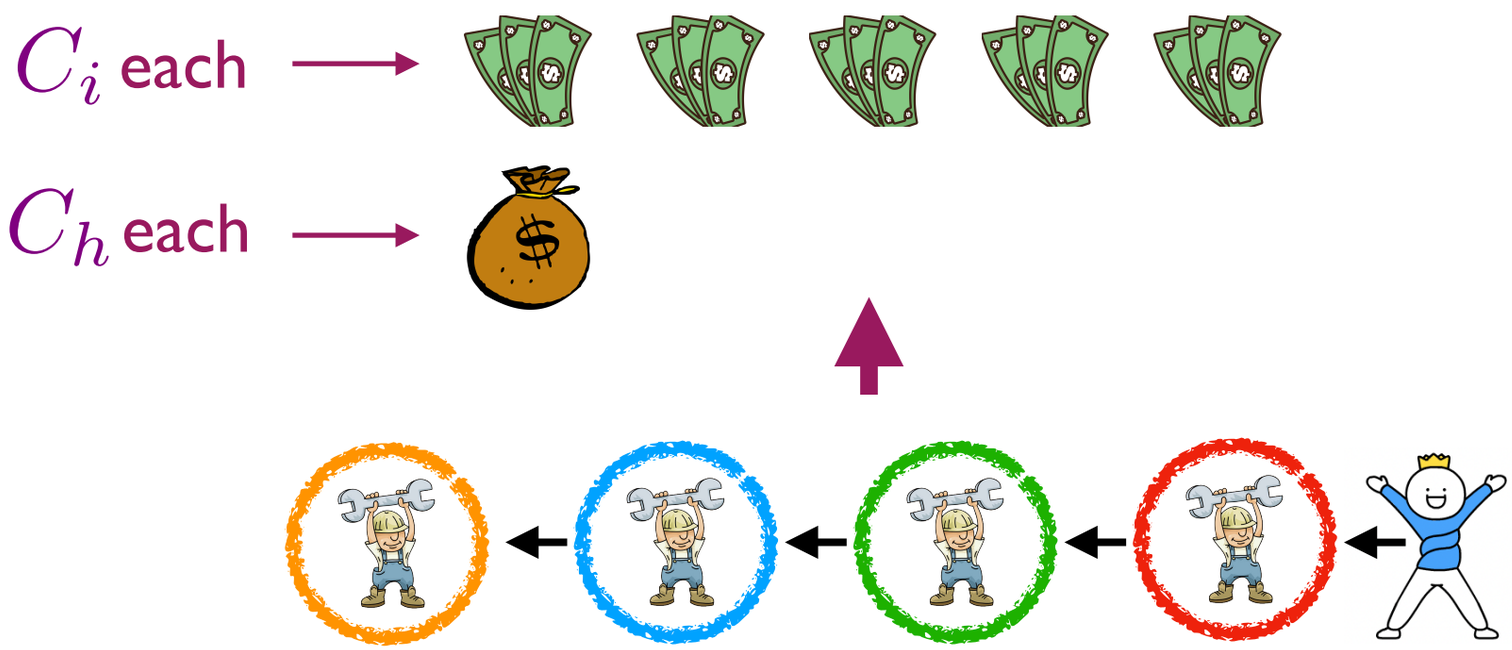

- Time complexity is always $O(N)$.
- More meaningful to consider the total cost on C_h.
- Related to finding the maximum of an array: winner stays.
  But need to minimize the number of winner changes.

Cost Analysis

```
int Hiring ( EventType C[ ], int N )
{   /* candidate 0 is a least-qualified dummy candidate */
    int Best = 0;
    int BestQ = the quality of candidate 0;
    for ( i=1; i<=N; i++ ) {
        Qi = interview( i ); /* Ci */
        if ( Qi > BestQ ) {
            BestQ = Qi;
            Best = i;
            hire( i );  /* Ch */
        }
    }
    return Best;
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```
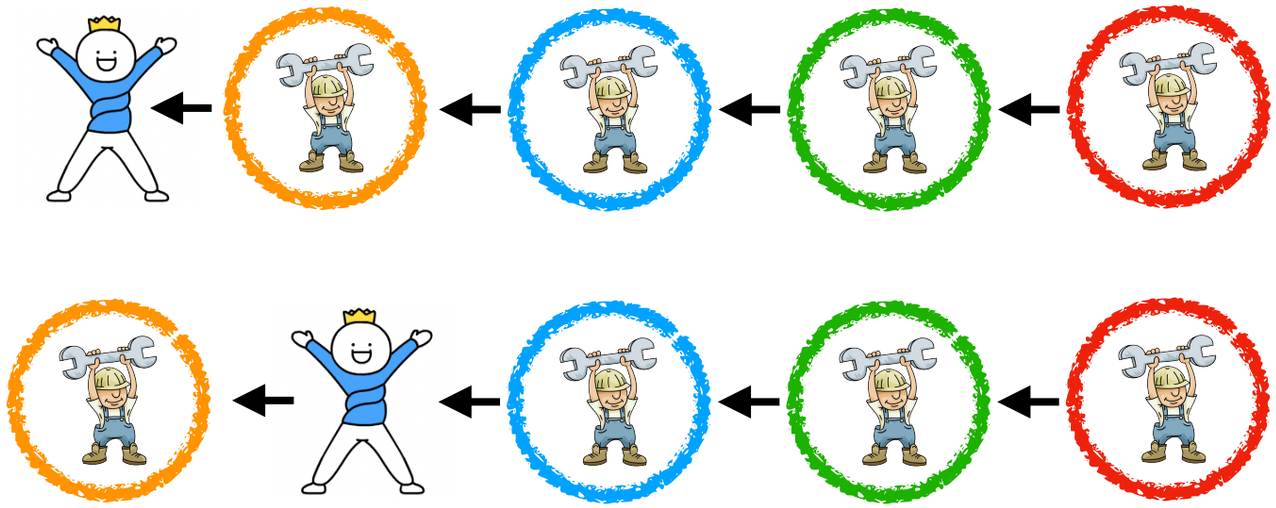

Unfortunately, the cost is not determined by the algorithm, but the order of input sequence.
Worst case: fully reverse order with NC_h cost.
What is the average cost over all possible orders?

Probabilistic Analysis



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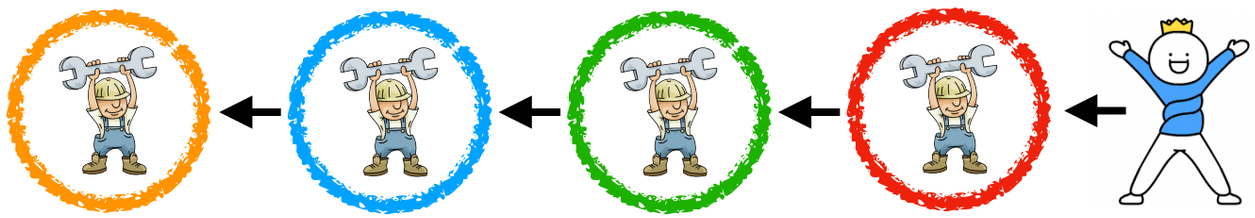
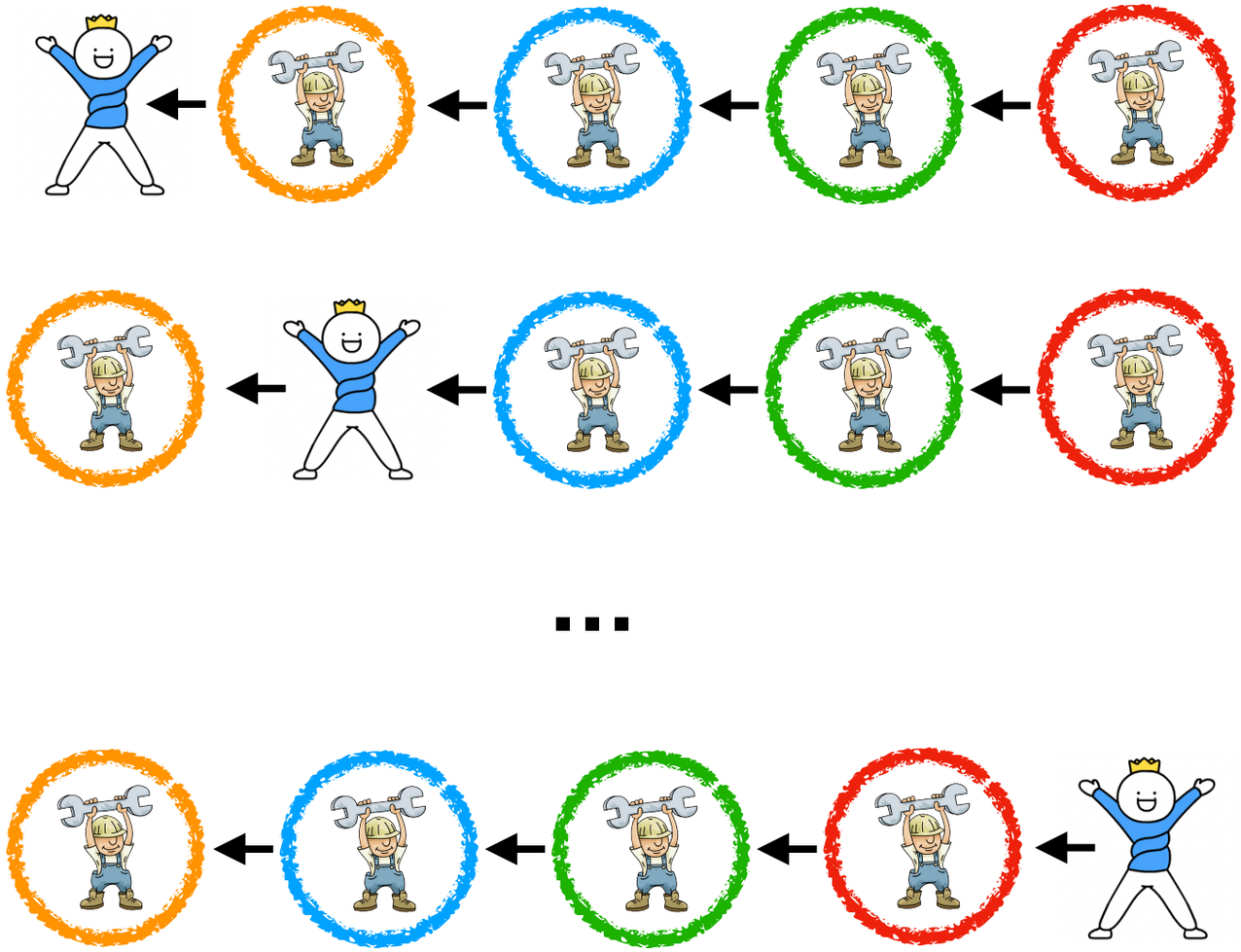


$N!$ possible orders

Probabilistic Analysis



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uniformly draw one from
$N!$ possible orders
randomly as the input.

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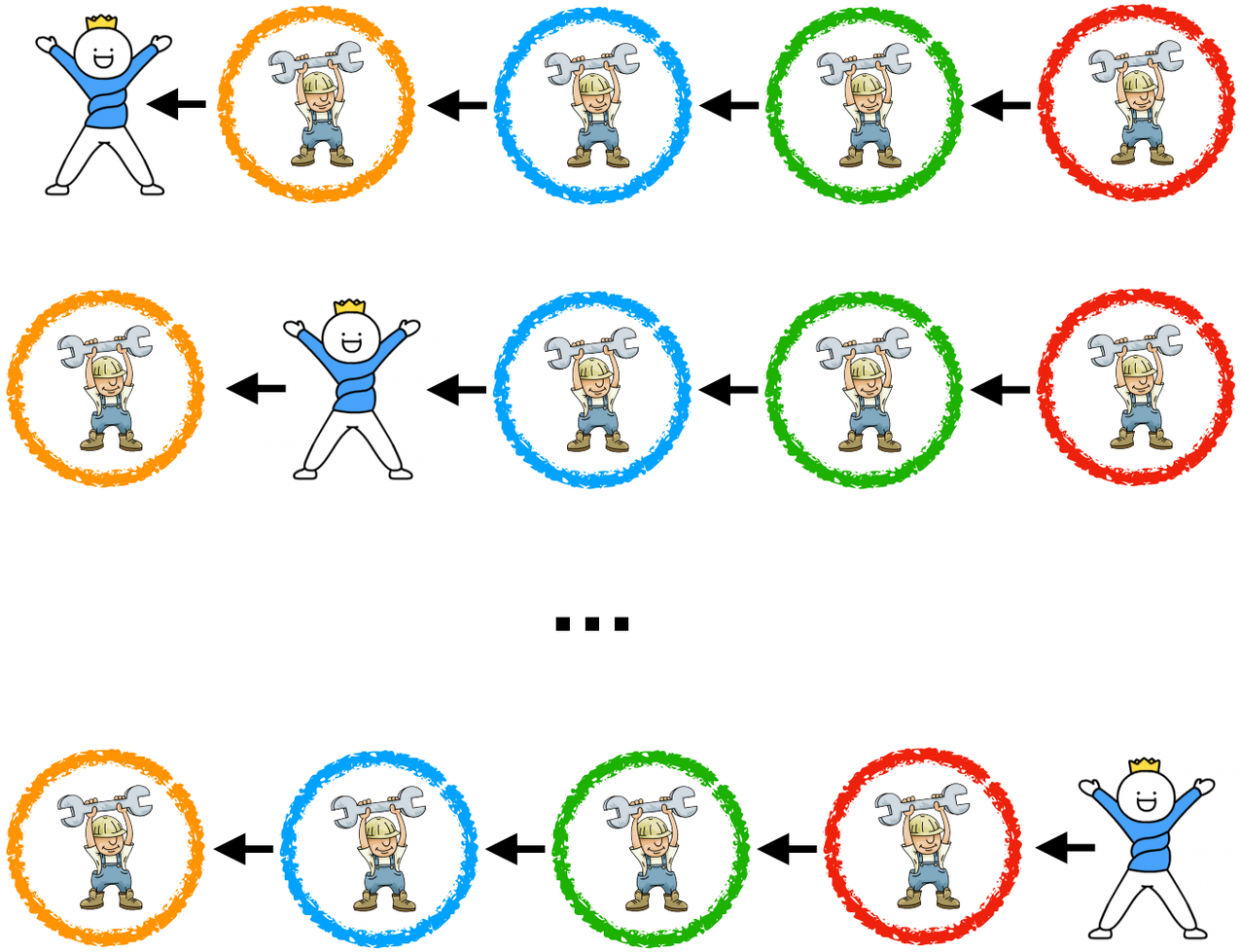

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- The "average" cost is the cost in expectation over random sampling.
- Assumption: all inputs can appear with equal chance.

Expectation and Indicator Random Variable

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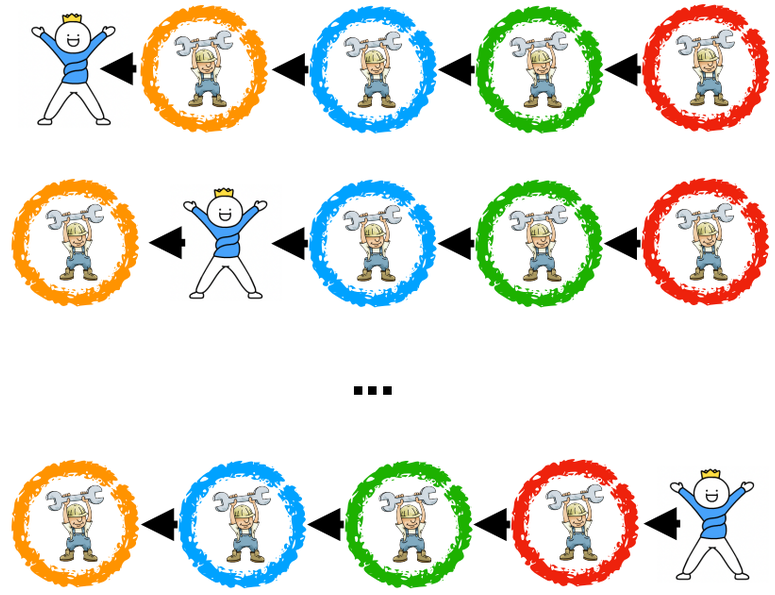
Indicator random variable measures whether a specific random event happens.

Prob. Analysis of Hiring Problem

- Assume uniform distribution over all $N!$ orders. Calculate cost in expectation.



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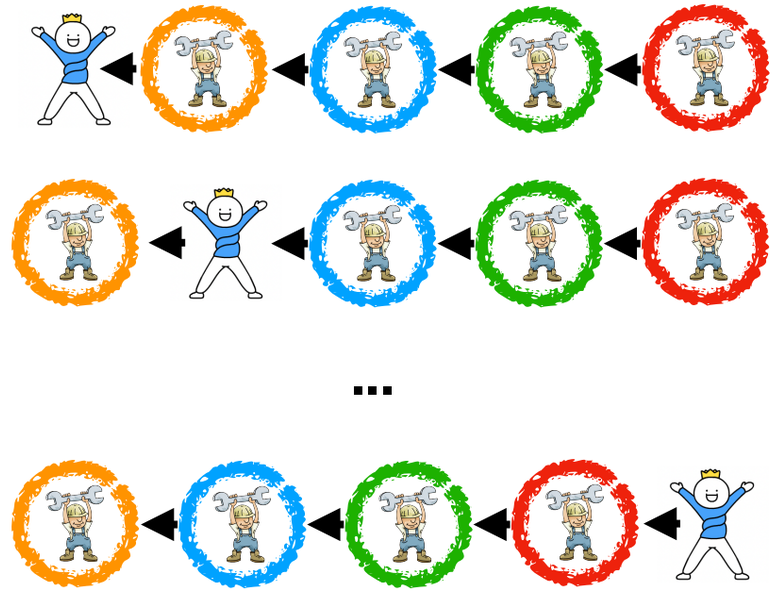

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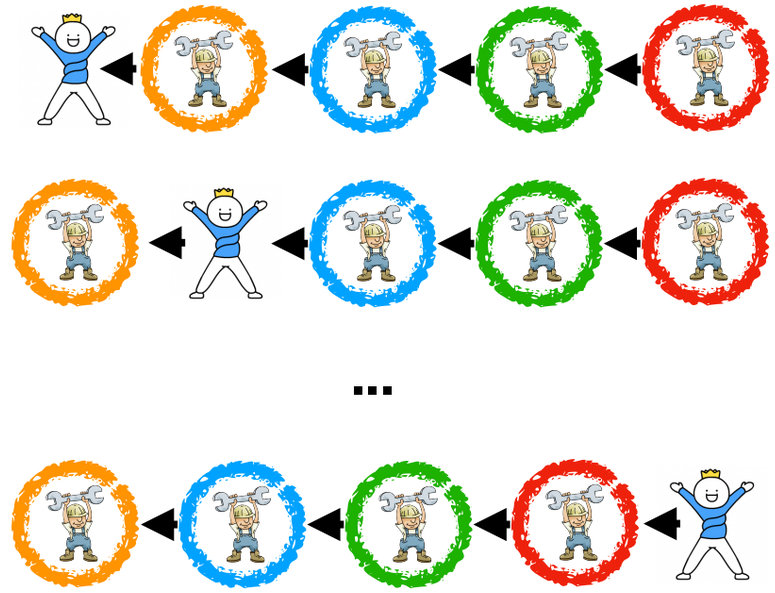

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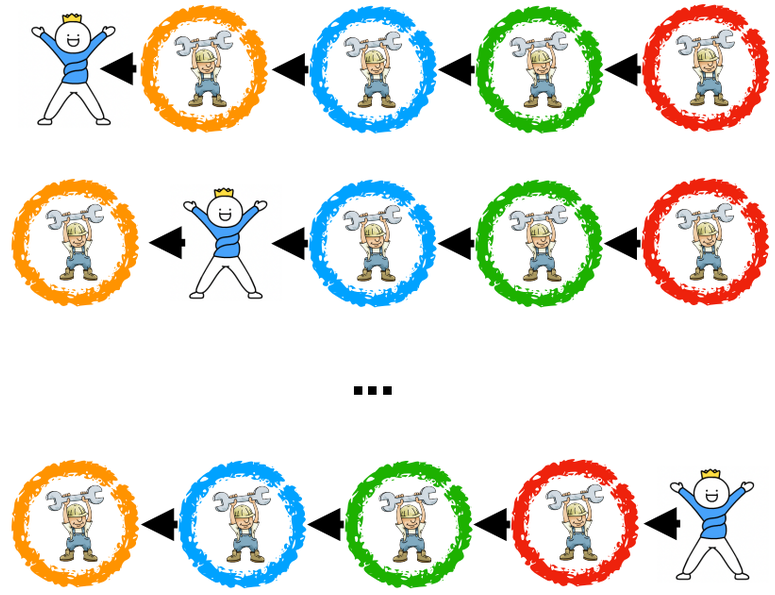

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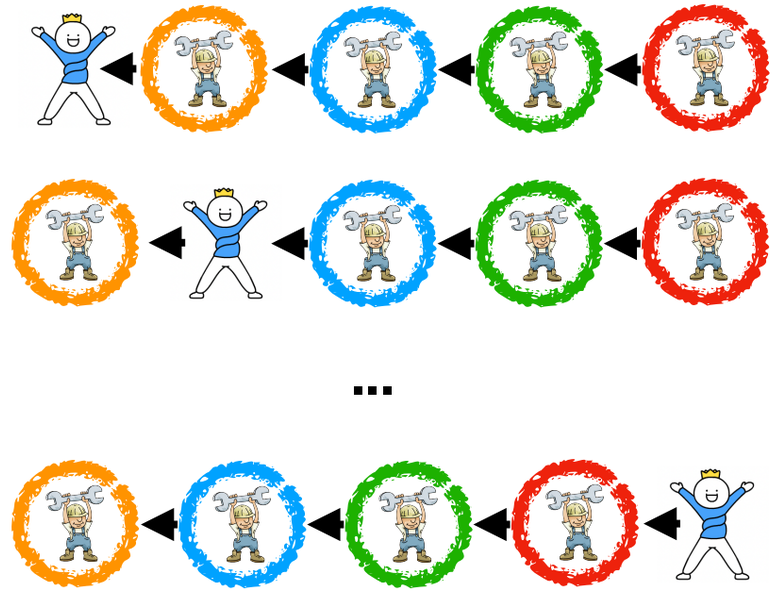

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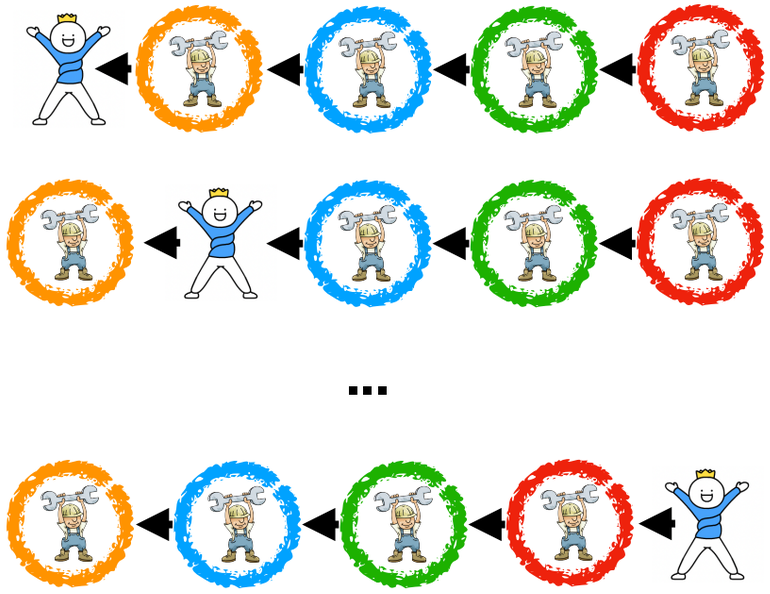

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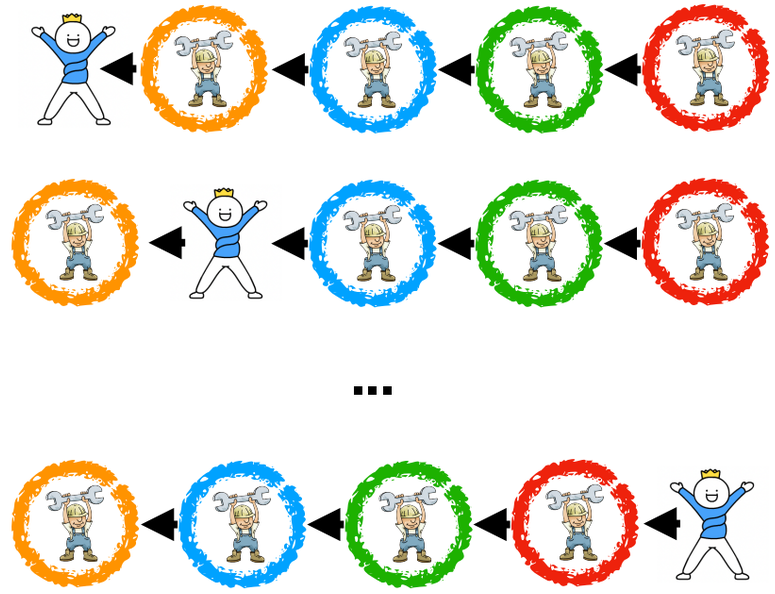

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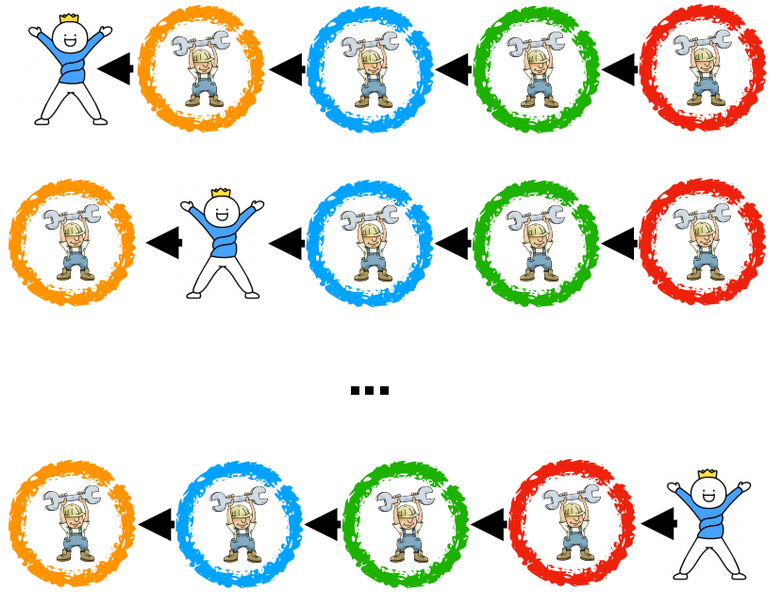
The i-the candidate is the best over the first i candidate, while the best has equal chance to appear in any place.

Prob. Analysis of Hiring Problem

- Assume uniform distribution over all $N!$ orders. Calculate cost in expectation.



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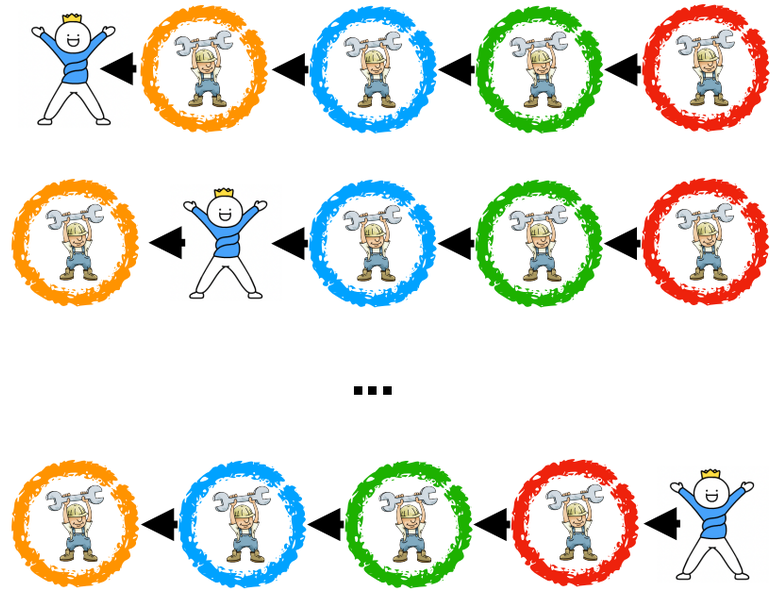

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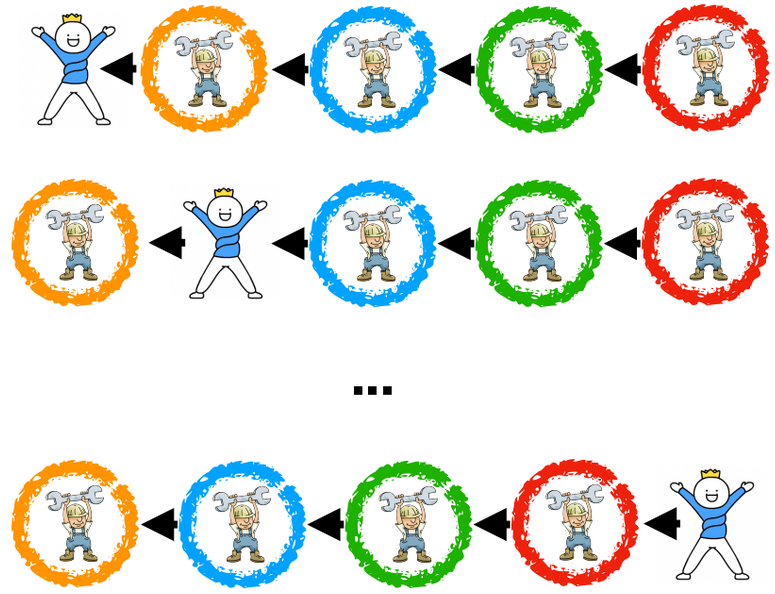

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Can we utilize randomness in algorithm design?

Randomized Algorithms

- Hiring problem: probabilistic analysis
- Randomized hiring algorithm
- Randomized quicksort
- Min-cut and max-cut
- Take-home messages

Randomized Hiring Algorithm

```
int RandomizedHiring ( EventType C[ ], int N )
{   /* candidate 0 is a least-qualified dummy candidate */
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    int BestQ = the quality of candidate 0;

    randomly permute the list of candidates;

    for ( i=1; i<=N; i++ ) {
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the only change in the algorithm

Obviously, the average cost remains the same to the original algorithm.
While even for the worst input case, we can expect the average cost!
In many problems, randomization is a strong technique to reduce the worst-case complexity.

Radomized Permutation Algorithm

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void PermuteBySorting ( ElemType A[ ], int N )
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   for ( i=1; i<=N; i++ )
      A[i].P = 1 + rand()%(N^3);
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   Sort A, using P as the sort keys;
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Claim: **PermuteBySorting produces a *uniform random permutation* of the input, assuming all priorities are distinct.**

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int OnlineHiring ( EventType C[ ], int N, int k )
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        Qi = interview( i );
        if ( Qi > BestQ )   BestQ = Qi;
    }
    for ( i=k+1; i<=N; i++ ) {
        Qi = interview( i );
        if ( Qi > BestQ ) {
            Best = i;
            break;
        }
    }
    return Best;
}
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☞ What is the **probability** we hire the best qualified candidate for a given k?

☞ What is the **best value of** k to maximize above probability?

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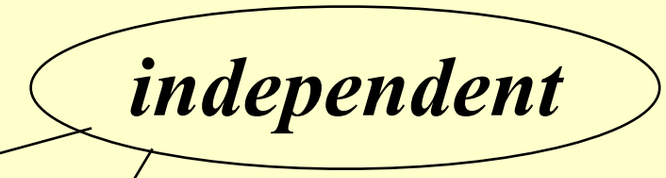

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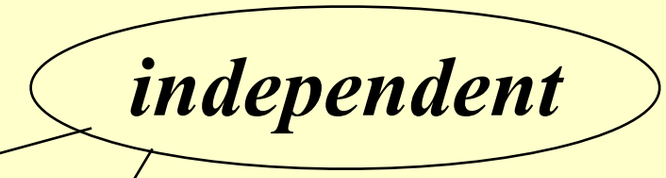

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Discussion 18: Prove $\int_k^N \frac{1}{x} dx \le \sum_{i=k}^{N-1} \frac{1}{i} \le \int_{k-1}^{N-1} \frac{1}{x} dx$

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Discussion 19: What is the maximum value of $f(k) = \frac{k}{N} \ln\left(\frac{N}{k}\right)$? And what is the best k?

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$$\frac{d}{dk}\left[\frac{k}{N} \ln\left(\frac{N}{k}\right)\right] = \frac{1}{N}(\ln N - \ln k - 1) = 0$$

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☞ **Succeed in hiring the best-qualified applicant with probability at least**

$$\Pr[S] = \sum_{i=k+1}^{N} \Pr[S_i] = \sum_{i=k+1}^{N} \frac{k}{N(i-1)} = \frac{k}{N} \sum_{i=k}^{N-1} \frac{1}{i}$$

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☞ **Succeed in hiring the best-qualified applicant with probability at least $1/e$**

Randomization in Algorithm Design

Randomized algorithms has steps where randomness are injected, resulted in two types of algorithms:

- Monte-Carlo algorithm:
  - Deterministic running time
  - Return correct result with probability

- Las Vegas algorithm:
  - Randomized running time
  - Always return the correct result

Relaxation is necessary for dealing with hard computation problems.

Randomized Algorithms

- Hiring problem: probabilistic analysis
- Randomized hiring algorithm
- Randomized quicksort
- Min-cut and max-cut
- Take-home messages

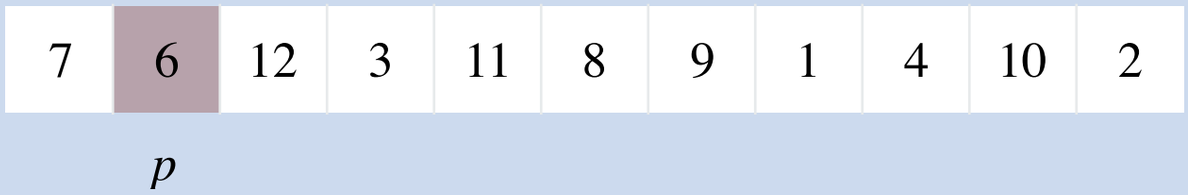
3-WAY PARTITIONING

Goal. Given an array A and pivot element p, partition array so that:

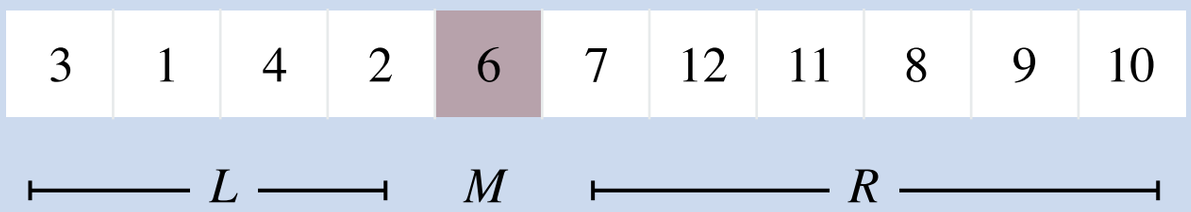
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Challenge. $O(n)$ time and $O(1)$ space.

the array A



the partitioned array A



Randomized quicksort

- Pick a random pivot element $p \in A$.
- 3-way partition the array into L, M, and R.
- Recursively sort both L and R.

the array A	7	6	12	3	11	8	9	1	4	10	2
		p									

partition A	3	1	4	2	6	7	12	11	8	9	10
sort L	1	2	3	4	6	7	12	11	8	9	10
sort R	1	2	3	4	6	7	8	9	10	11	12
the sorted array A	1	2	3	4	6	7	8	9	10	11	12

Randomized quicksort

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RANDOMIZED-QUICKSORT(A)

IF (array A has zero or one element)

  RETURN.

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY(A, p). ← $\Theta(n)$

RANDOMIZED-QUICKSORT(L). ← $T(i)$

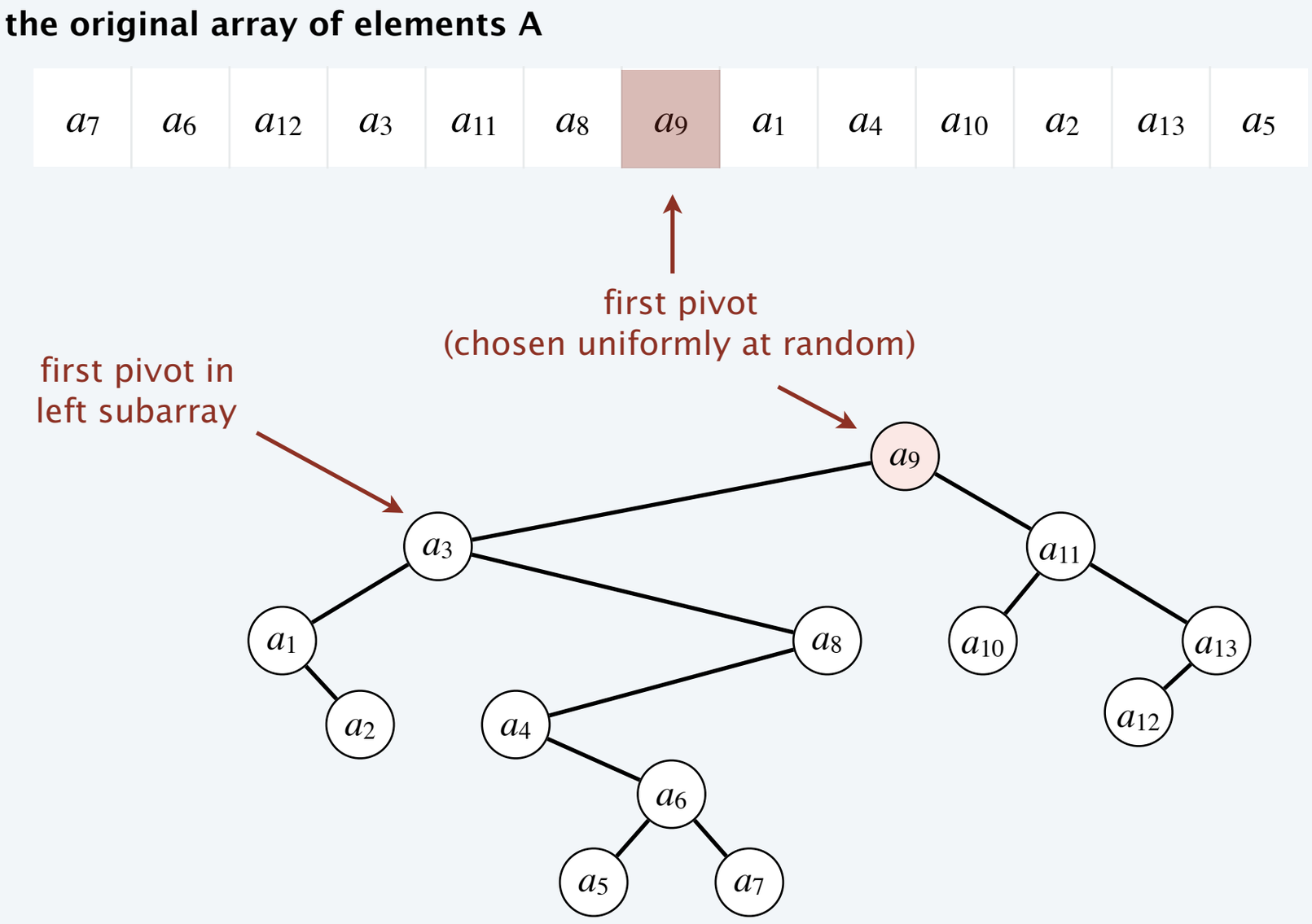
RANDOMIZED-QUICKSORT(R). ← $T(n - i - 1)$

new analysis required
(i is a random variable—depends on p)

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements $a_1 < a_2 < \cdots < a_n$ is $O(n \log n)$.

Pf. Consider BST representation of pivot elements.



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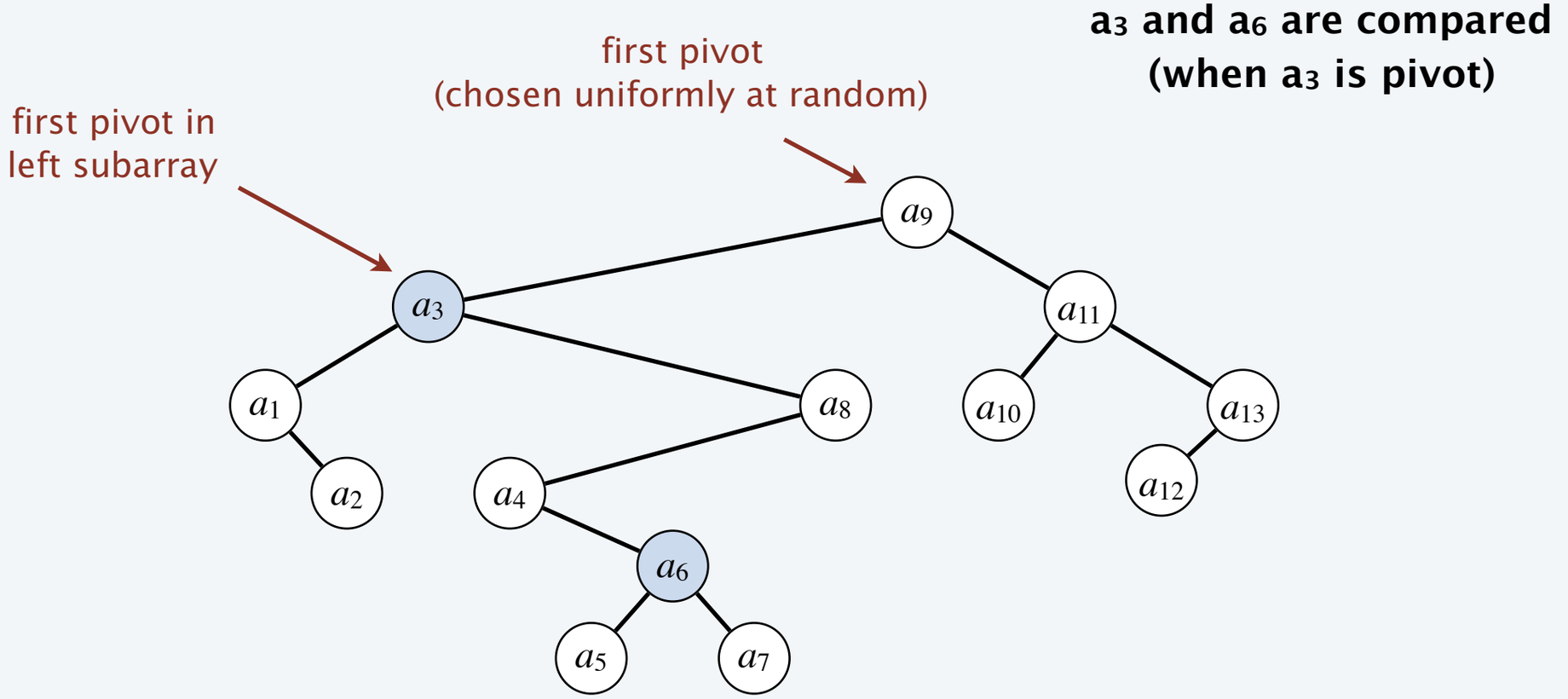
- a_i and a_j are compared once iff one is an ancestor of the other.



Analysis of randomized quicksort

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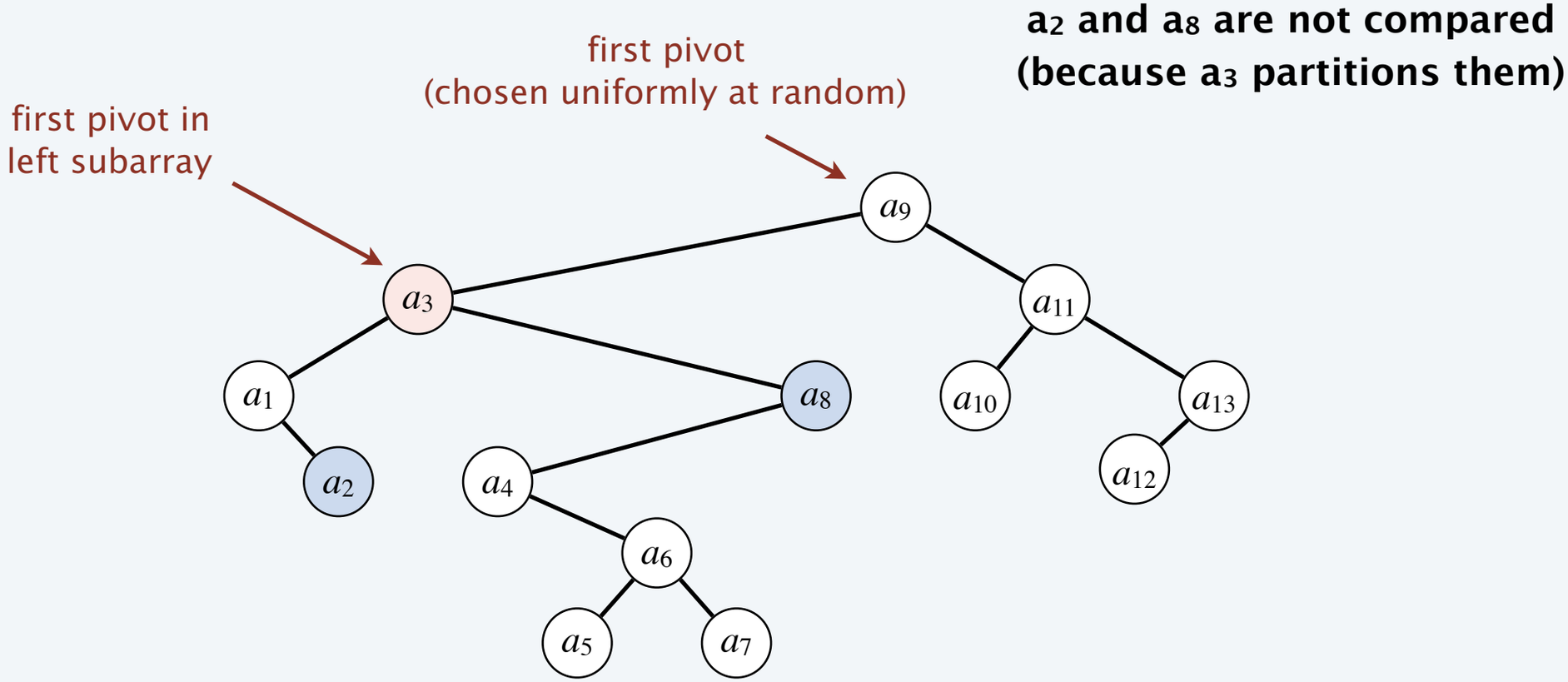


Divide-and-conquer: quiz 2

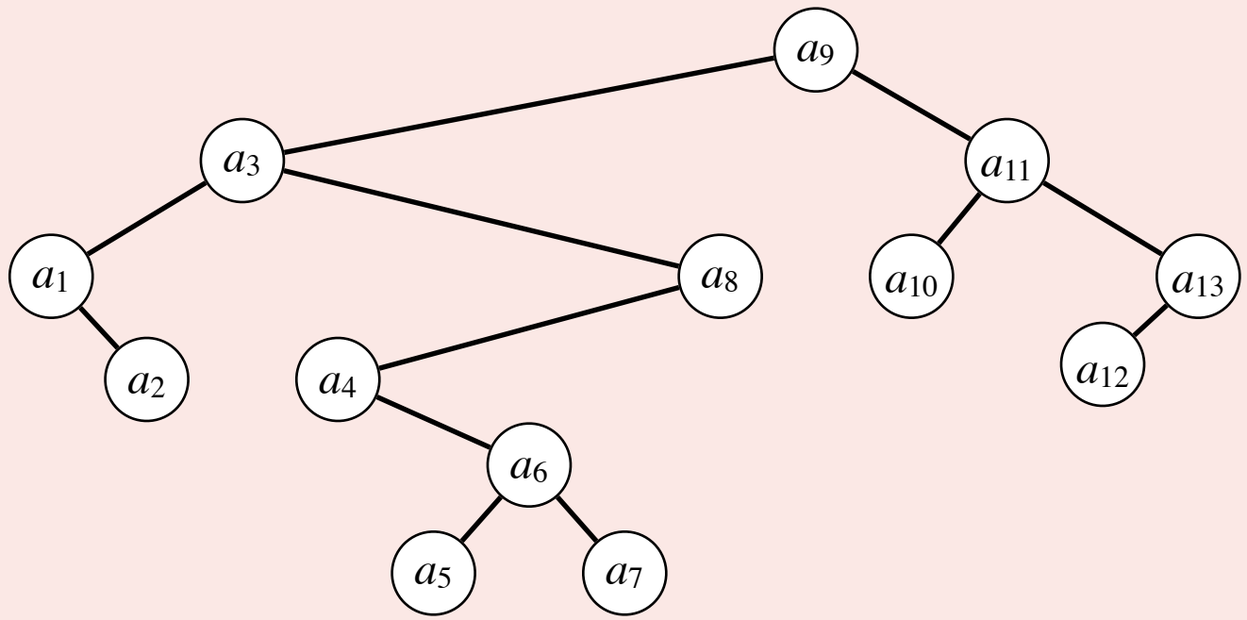
Given an array of $n \geq 8$ distinct elements $a_1 < a_2 < \cdots < a_n$, what is the probability that a_7 and a_8 are compared during randomized quicksort?

A. 0

B. $1 / n$

C. $2 / n$

D. 1



Divide-and-conquer: quiz 3

Given an array of $n \geq 2$ distinct elements $a_1 < a_2 < \cdots < a_n$, what is the probability that a_1 and a_n are compared during randomized quicksort?

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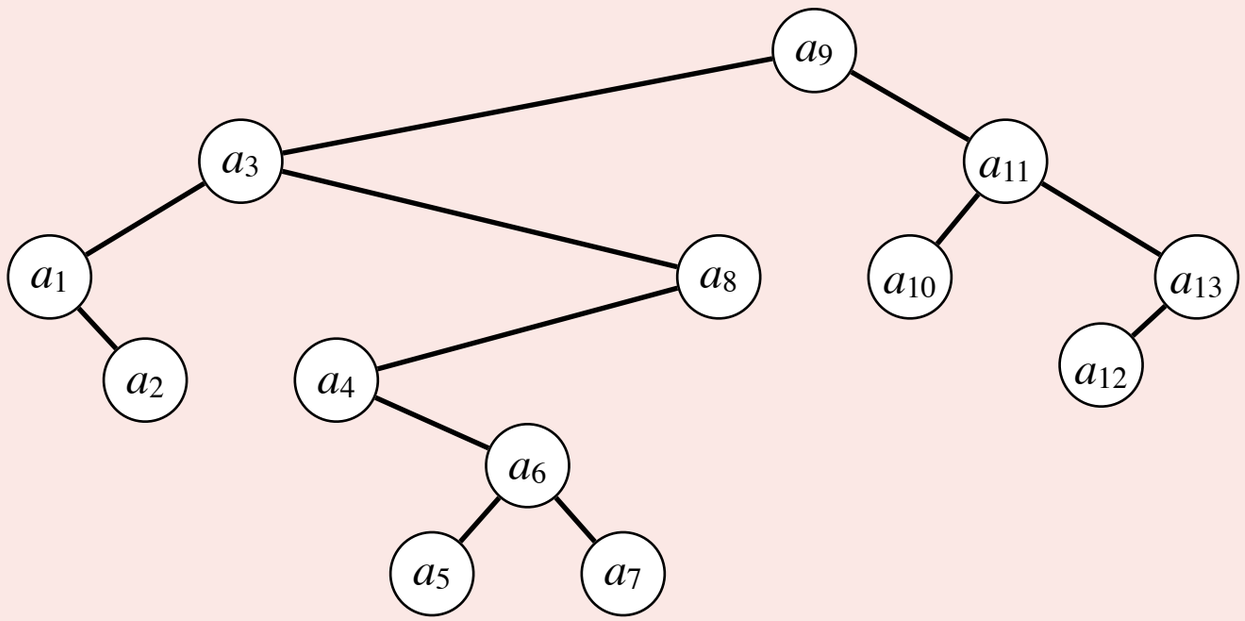

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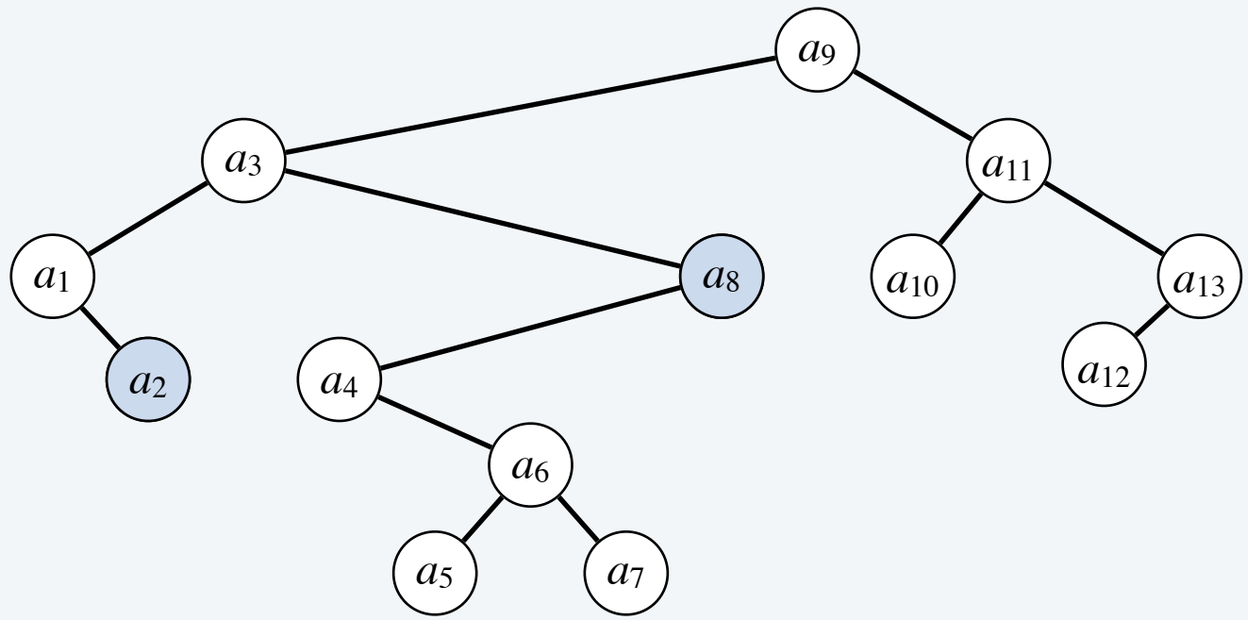
- a_i and a_j are compared once iff one is an ancestor of the other.
- **Pr** [a_i and a_j are compared] $= 2 \ / \ (j - i + 1)$, where $i < j$.

Pr[a_2 and a_8 compared] = 2/7
compared iff either a_2 or a_8 is chosen as pivot before any of { a_3, a_4, a_5, a_6, a_7}



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- **Pr** [a_i and a_j are compared] $= 2 \; / \; (j - i + 1)$, where $i < j$.

- Expected number of compares $= \displaystyle\sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n} \sum_{j=2}^{n-i+1} \frac{1}{j}$

all pairs i and j

$$\leq 2n \sum_{j=1}^{n} \frac{1}{j}$$

$$\leq 2n \, (\ln n + 1) \quad \blacksquare$$

harmonic sum

Remark. Number of compares only decreases if equal elements.

〖Example〗 Quicksort

〖Example〗 Quicksort

🕮 Deterministic Quicksort

〖Example〗 Quicksort

📖 **Deterministic Quicksort**

- ☞ $\Theta(N^2)$ worst-case running time
- ☞ $\Theta(N \log N)$ average case running time, ***assuming every input permutation is equally likely***

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☝ **How about choosing the pivot *uniformly at random*?**

〖Example〗 Quicksort

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Central splitter := the pivot that divides the set so that each side contains at least $n/4$

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Central splitter := the pivot that divides the set so that each side contains **at least $n/4$**

Modified Quicksort := always select a central splitter before recursions

Claim: The expected number of iterations needed until we find a central splitter is at most 2.

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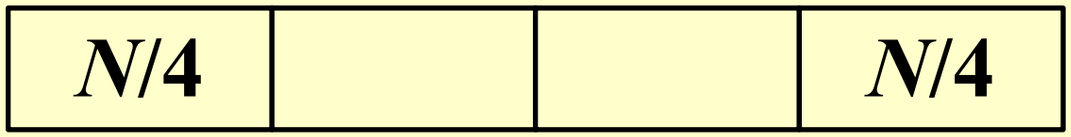

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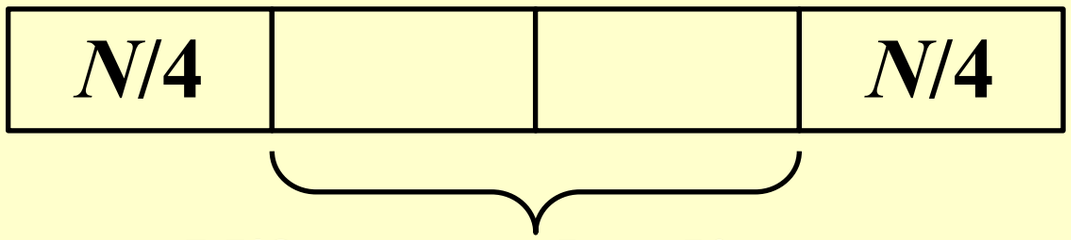

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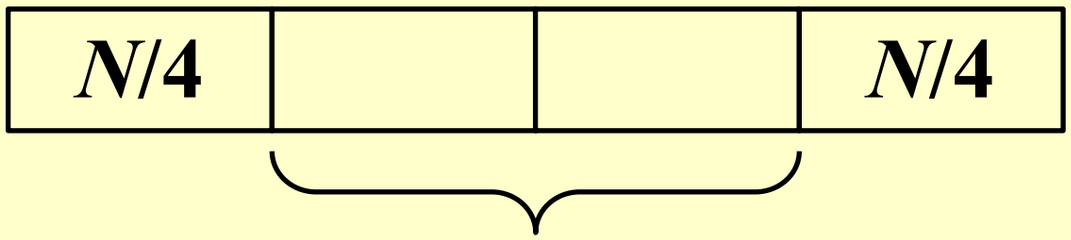

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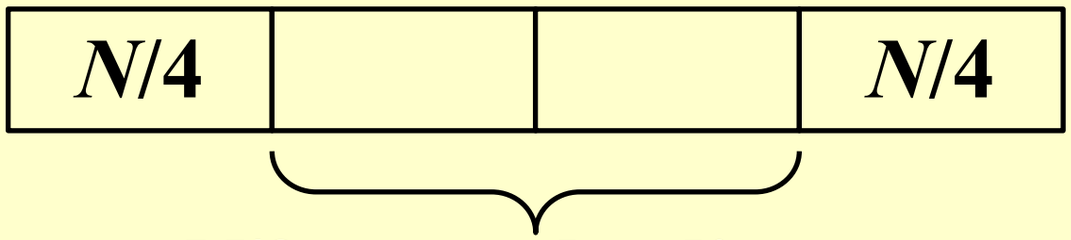

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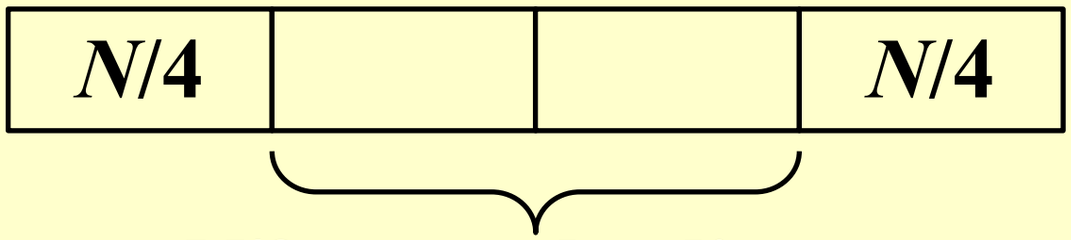

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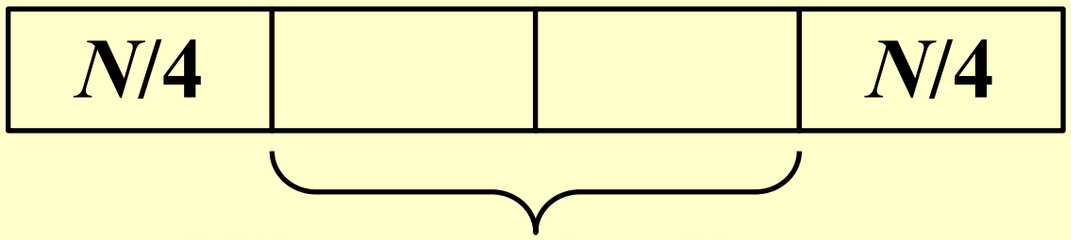

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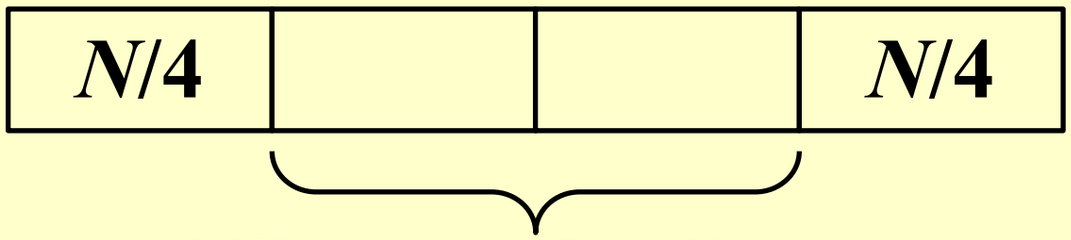

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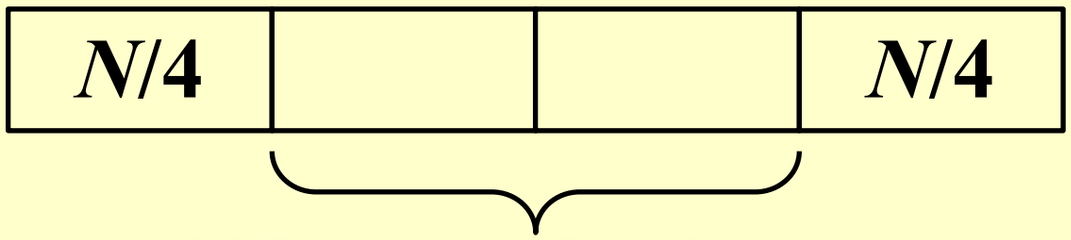

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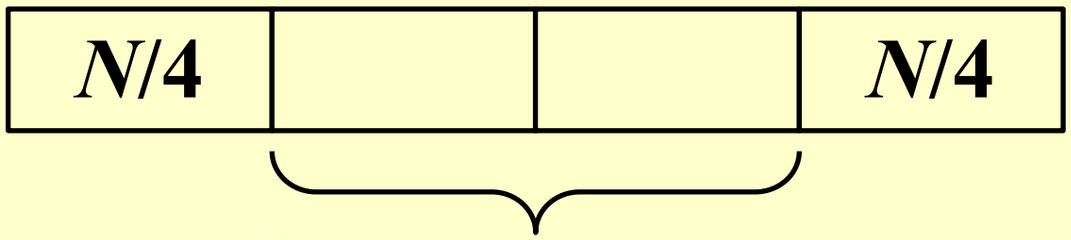

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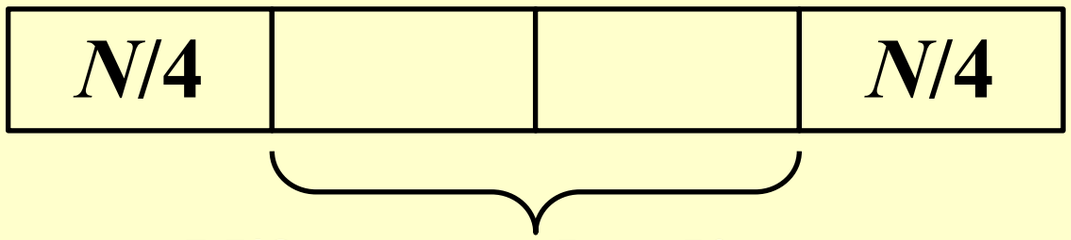

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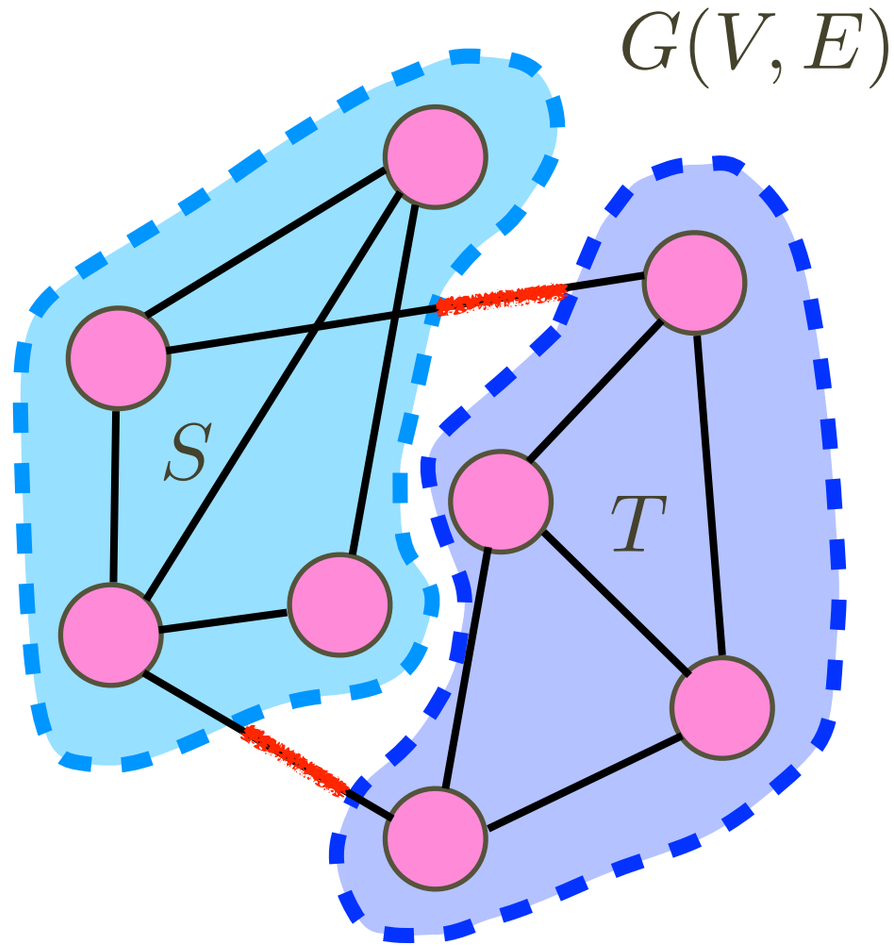
$\Big\}\ O(N \log N)$

Randomized Algorithms

- Hiring problem: probabilistic analysis
- Randomized hiring algorithm
- Randomized quicksort
- Min-cut and max-cut
- Take-home messages

Min-Cut

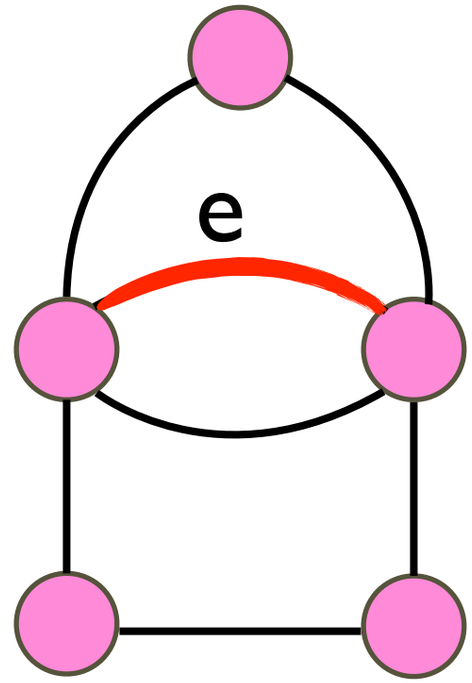
- Partition V into two parts: S and T
- Minimize the cut $E(S,T)$
- deterministic algorithm:
  - max-flow min-cut
  - best known upper bound: $O(mn + n^2 \log n)$



$$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$$

Contraction

- multigraph $G(V, E)$
- multigraph: allow parallel edges
- for an edge e, $\text{contract}(e)$ merges the two endpoints.



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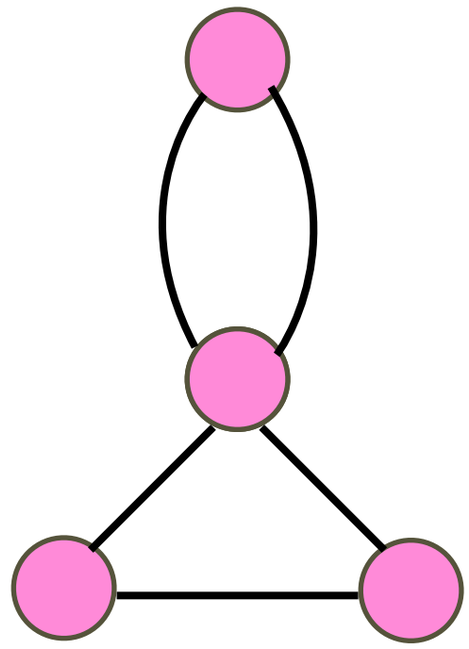

Karger's min-cut Algorithm

MinCut (multigraph $G(V,E)$)

while $|V|>2$ do

  choose a uniform $e \in E$;

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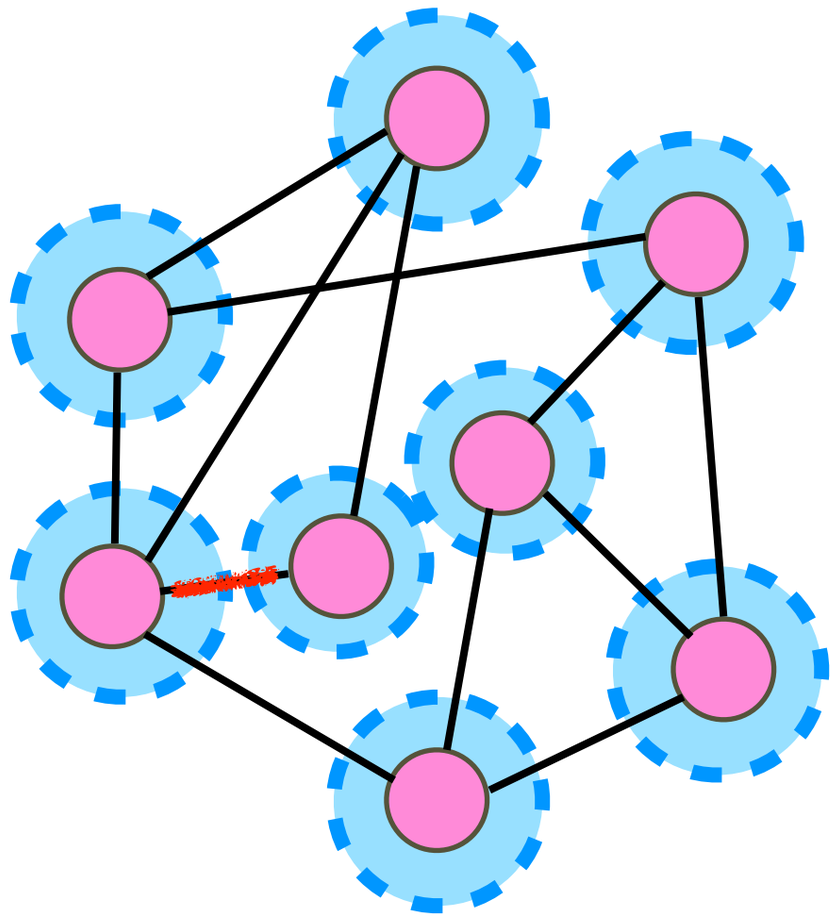
return remaining edges;



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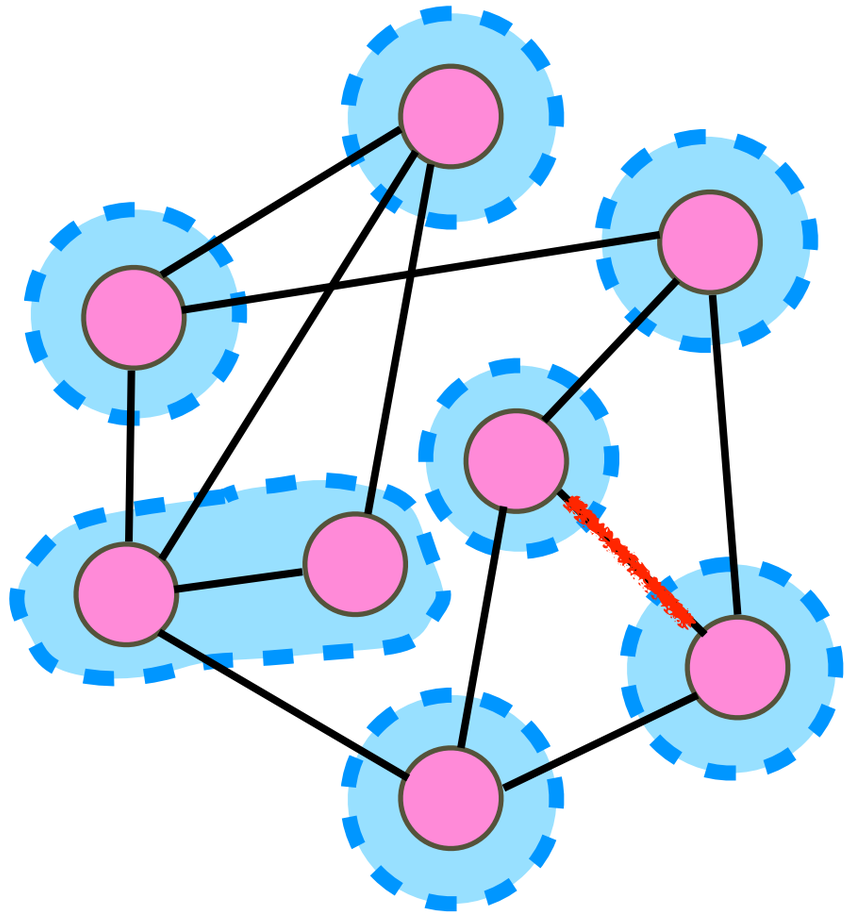

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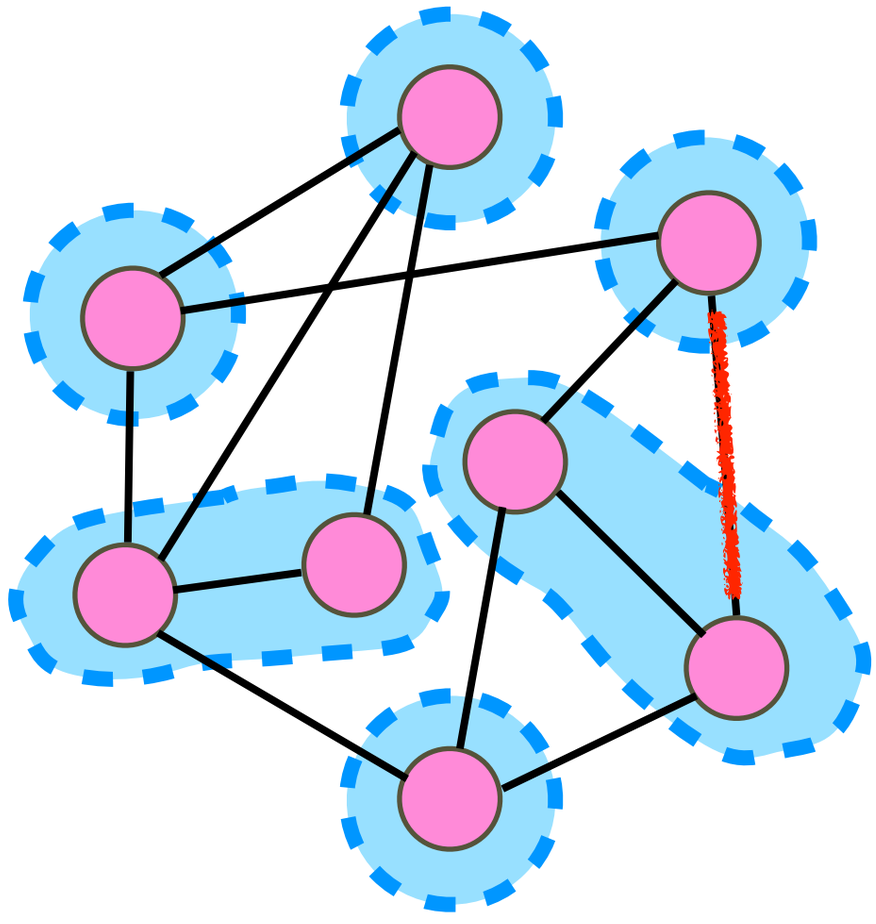

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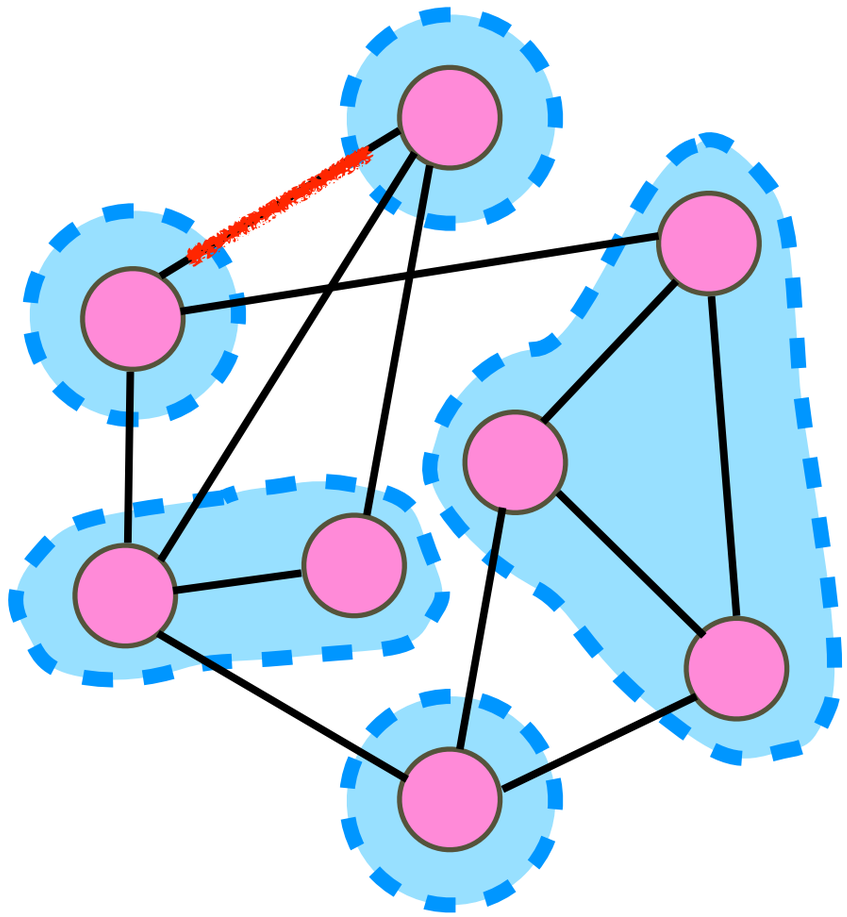

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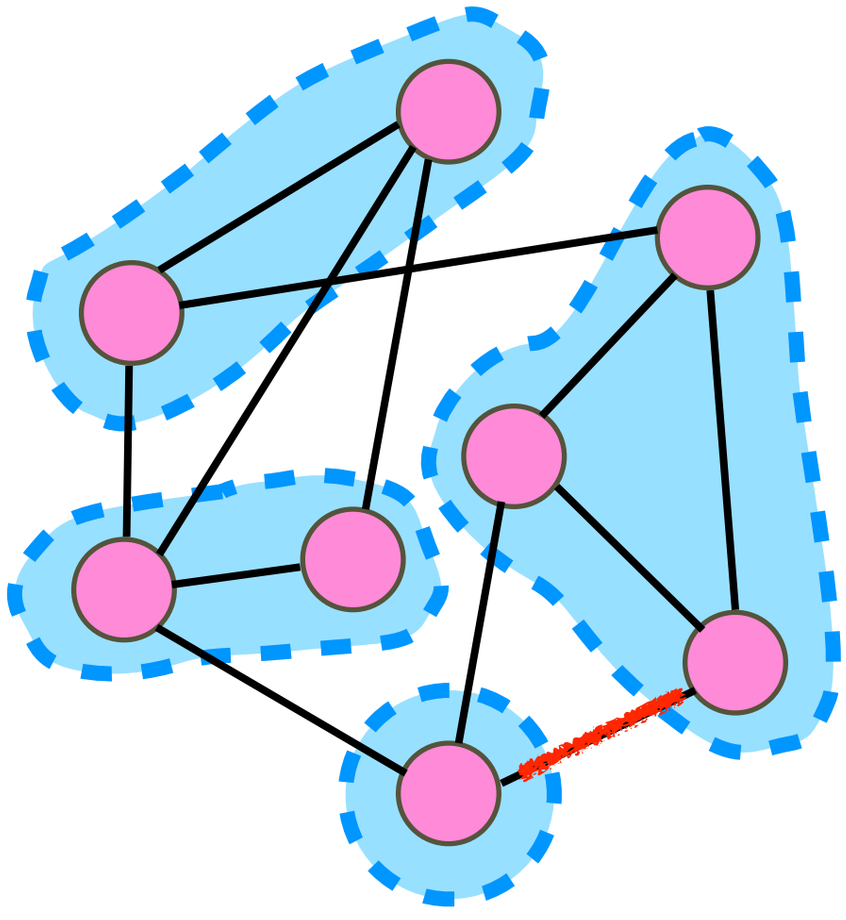

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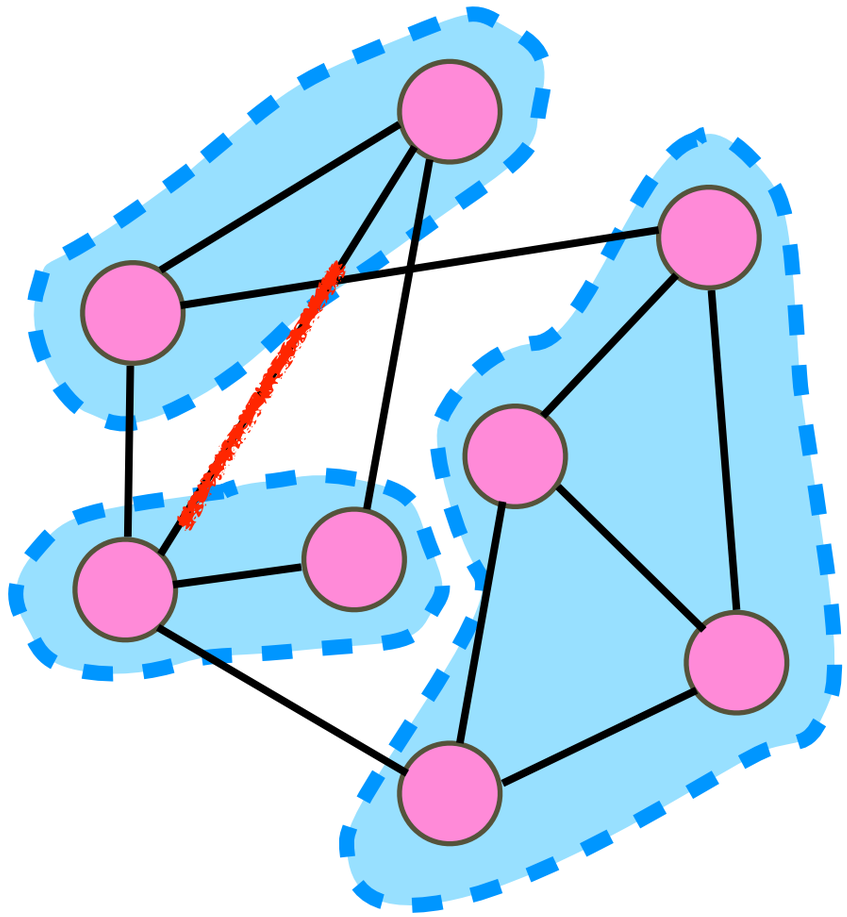

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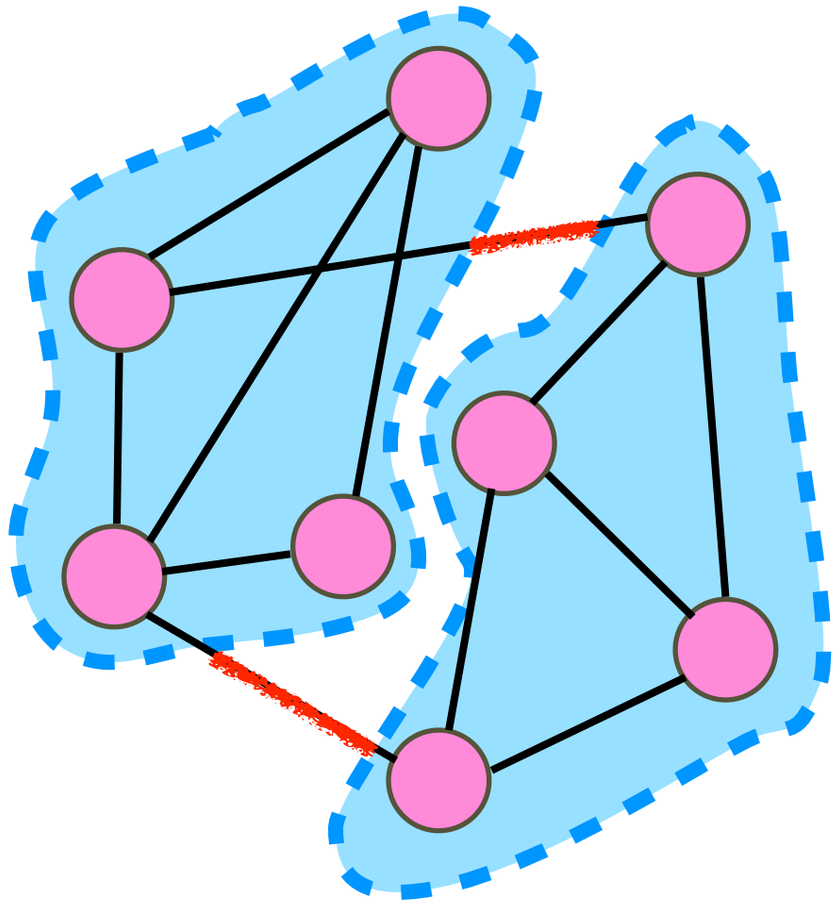

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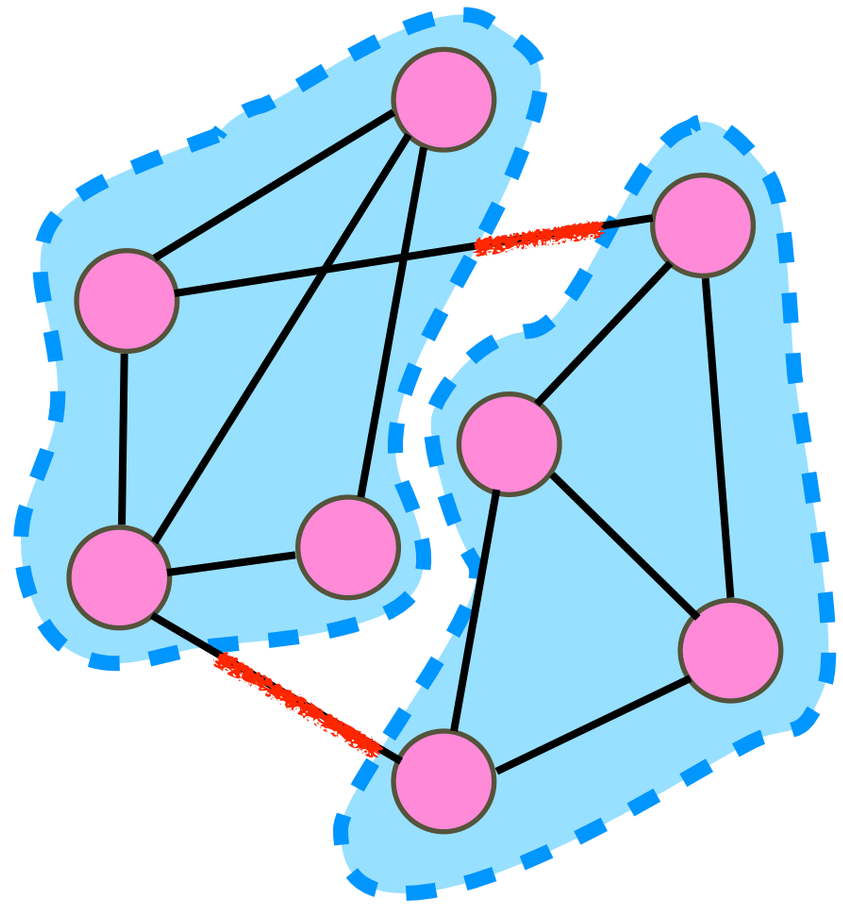

edges returned

Each contract operation takes O(n) time.

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Pr[a min-cut is returned] $\geq \frac{2}{n(n-1)}$

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repeat independently
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$$\Pr[\text{ fail to finally return a min-cut }]$$

$$= \Pr[\text{ fail to construct a min-cut in one trial }]^{n(n-1)/2}$$

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)/2} < \frac{1}{\mathrm{e}}$$

MinCut (multigraph $G(V,E)$)

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suppose $e_1, e_2, \ldots, e_{n-2}$

are contracted edges

initially: $G_1 = G$

i-th round:

$$G_i = \mathrm{contract}(G_{i-1}, e_{i-1})$$

```
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$\Rightarrow$ C is a min-cut in G_i

C: a min-cut of G

$$\Pr[C \text{ is returned}] \geq \Pr[e_1, e_2, \ldots, e_{n-2} \notin C]$$

chain rule:

$$= \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

suppose $e_1, e_2, \ldots, e_{n-2}$ are contracted edges

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$e_{i-1} \notin C$
⇒ C is a min-cut in G_i

C: a min-cut of G

$$\Pr[C \text{ is returned}] \geq \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

$e_1, e_2, \ldots, e_{i-1} \notin C$ ⇒ C is a min-cut in G_i

C is a min-cut in $G(V, E)$
⇒ $|E| \geq \frac{1}{2}|C||V|$

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min-degree of $G \geq |C|$

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$$\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{(n-i+1)}\right)$$

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min-degree of $G \geq |C|$

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$$\Pr[C \text{ is returned}] \geq \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

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C is a min-cut in $G(V, E)$
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Proof:

min-degree of $G \geq |C|$

$$= \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{2}{n(n-1)}$$

MinCut (multigraph $G(V,E)$)

while $|V|>2$ do

  choose a uniform $e \in E$;

  contract(e);

return remaining edges;

Theorem (Karger 1993)**:**

For any min-cut C,

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repeat *independently* for $O(n^2 \log n)$ times

returns a min-cut with probability $1\text{-}O(1/n)$

total running time: $O(n^4 \log n)$

Number of Min-Cuts

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Corollary

The number of distinct min-cuts
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Number of Min-Cuts

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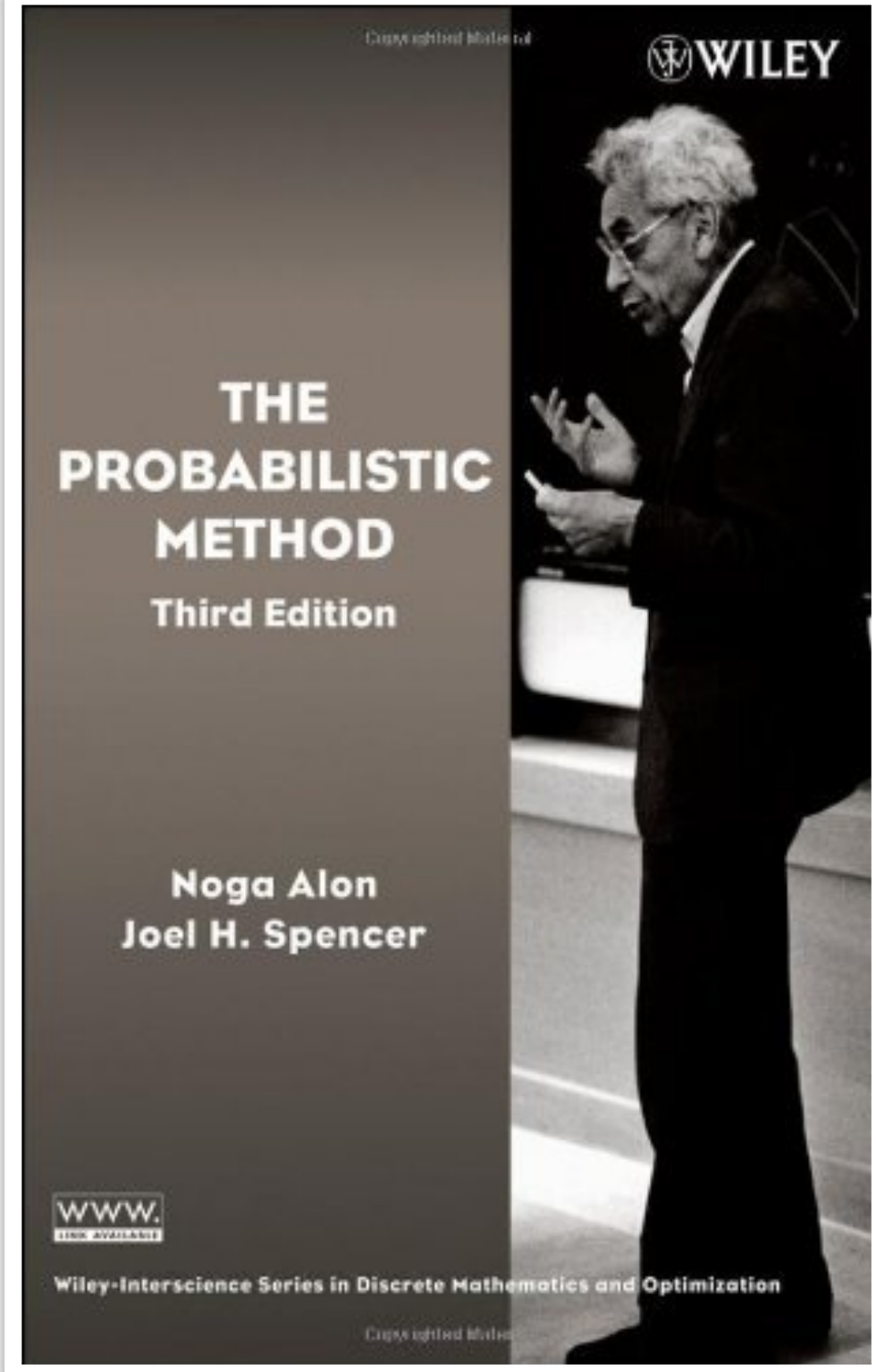
Corollary

The number of distinct min-cuts
in a graph of n vertices is at most $n(n\text{-}1)/2$.

No randomness in the corollary at all.
An example of the probabilistic method.


WILEY
THE PROBABILISTIC METHOD
Third Edition
Noga Alon
Joel H. Spencer
WWW.
Wiley-Interscience Series in Discrete Mathematics and Optimization

An Observation

MinCut (multigraph $G(V,E)$)

while $|V|> t$ do

choose a uniform $e \in E$;

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C: a min-cut of G

$$\Pr[e_1, \ldots, e_{n-t} \notin C] = \prod_{i=1}^{n-t} \Pr[e_i \notin C \mid e_1, \ldots, e_{i-1} \notin C]$$

$$\geq \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)}$$

only getting bad when t is small

Fast Min-Cut

```
Contract ( G, t )
while |V|> t do
  choose a uniform e ∈E ;
  contract(e);
return remaining edges;
```

```
FastCut ( G )
if |V|≤6 then return a min-cut by brute force;
else:  (t to be fixed later)
  G1= Contract(G,t);
  G2= Contract(G,t);          (independently)
    return min{FastCut(G1), FastCut(G2)};
```

FastCut (G)

if $|V| \le 6$ then return a min-cut by brute force;

else: (t to be fixed later)

G_1= Contract(G,t);

G_2= Contract(G,t); (independently)

return min{FastCut(G_1), FastCut(G_2)};

C: a min-cut in G

A: no edge in C is contracted during Contract(G,t)

$$\Pr[A] = \prod_{i=1}^{n-t} \Pr[e_i \notin C \mid e_1, \ldots, e_{i-1} \notin C]$$

$$\geq \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} \quad \geq \frac{t(t-1)}{n(n-1)} \quad \geq \left(\frac{t-1}{n-1}\right)^2$$

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if $|V| \le 6$ then return a min-cut by brute force;

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G_1= Contract(G,t);

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C: a min-cut in G set $t = \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$

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$$\Pr[A] \ge \left(\frac{t-1}{n-1}\right)^2 \ge \frac{1}{2}$$

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$$\Pr[A] \ge \left(\frac{t-1}{n-1}\right)^2 \ge \frac{1}{2}$$

$$p(n) = \min_{G:|V|=n} \Pr[\,\text{FastCut}(G) \text{ returns a mincut in } G\,]$$

succeeds

$$\ge 1 - (1 - \Pr[A]\Pr[\text{FastCut}(G_1) \text{ succeeds} \mid A])^2$$

$$\ge 1 - \left(1 - \left(\frac{t-1}{n-1}\right)^2 p\,(t)\right)^2 \ge p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) - \frac{1}{4} p\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)^2$$

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by induction: $p(n) = \Omega\left(\frac{1}{\log n}\right)$

running time: $T(n) = 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) + O(n^2)$

by induction: $T(n) = O(n^2 \log n)$

FastCut (G)

if $|V| \leq 6$ then return a min-cut by brute force;

else: set $t = \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$

G_1= Contract(G,t);

G_2= Contract(G,t);

(independently)

return min{FastCut(G_1), FastCut(G_2)};

Theorem (Karger-Stein 1996)**:**

FastCut runs in time $O(n^2 \log n)$ and returns a min-cut with probability $\Omega(1/\log n)$.

repeat *independently* for $O((\log n)^2)$ times

total running time: $O(n^2 \log^3 n)$

returns a min-cut with probability $1\text{-}O(1/n)$

Take-Home Messages

- Probabilistic analysis:
  - Considering probability distributions over inputs for average analysis.
- Randomized algorithm:
  - Injection of randomness in running steps.
  - Relaxation by allowing uncertainty in correctness or running time.

Thanks for your attention!
Discussions?

Reference

Introduction to algorithms, 4th edition, Section 5.

Algorithm design: Chap. 13.

Slides from Kevin Wayne: https://www.cs.princeton.edu/~wayne/kleinberg-tardos/

Yitong Yin's lecture:

https://tcs.nju.edu.cn/wiki/index.php?title=%E9%AB%98%E7%BA%A7%E7%AE%97%E6%B3%95_(Fall_2020)/Min-Cut_and_Max-Cut

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