Advanced Data Structures and Algorithm Analysis

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浙江大学

Spring & Summer 2024

Lecture 1

2024-2-26

Outline:
Balanced Binary Search Trees (1)

- Binary search trees
- AVL trees
- Splay trees
- Amortized analysis
- Take-home messages

Acknowledgements:

This lecture is adapted from the slides designed by Prof. Yue Chen and the ZJU ADS course group.

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Data Structures

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  - queries:
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  - modifying operations:
    - insert, delete…
- A proper data structure effectively speeds up the set operations.

in terms of the size of the DS

Binary Search Trees (BSTs)

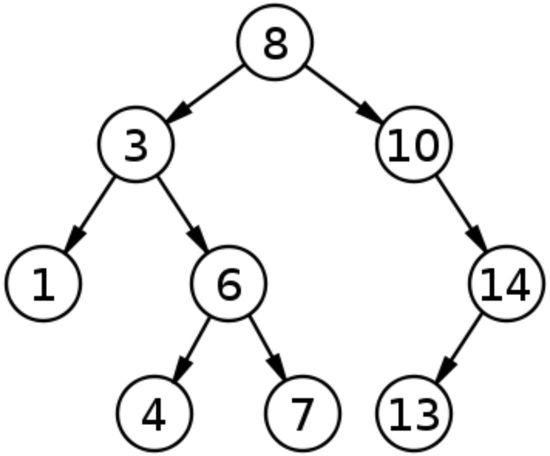

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Figure courtesy: https://en.wikipedia.org/wiki/Binary_search_tree

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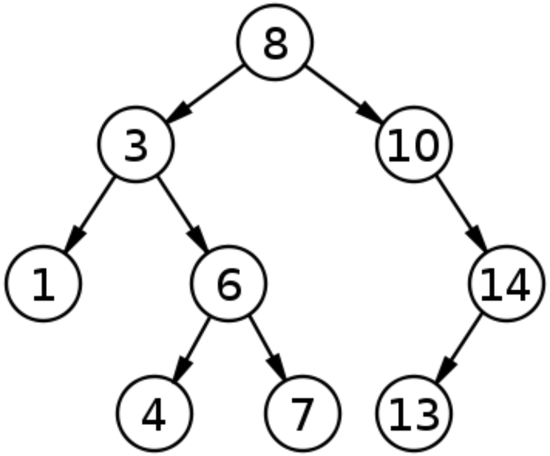

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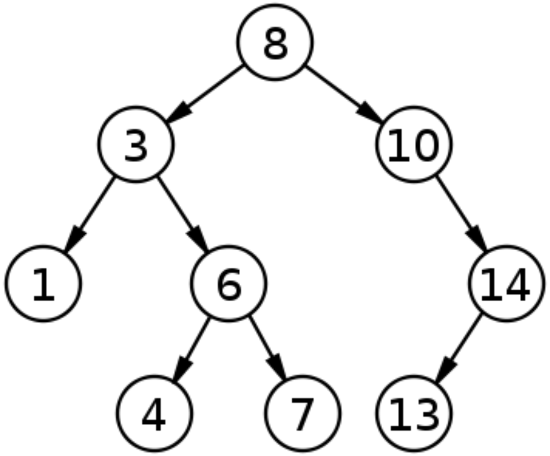

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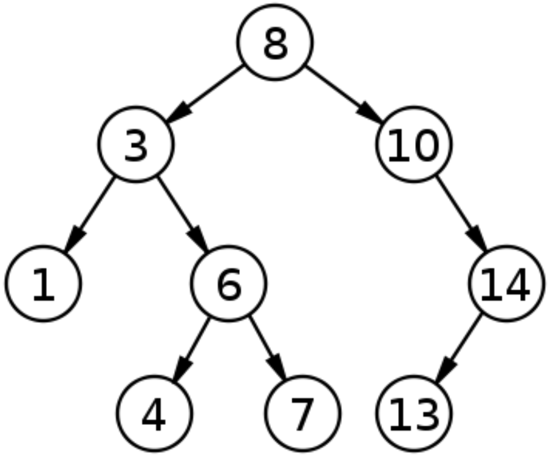

- Every node has at most two children.
- The left child is smaller, and the right child is larger.
- The tree operations (search, insert, delete, minimum, maximum, successor, predecessor…) have time costs closely related to tree depth.
- Balancing is to reduce tree depth in order to reduce time costs.

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Balanced BSTs

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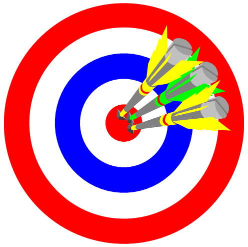

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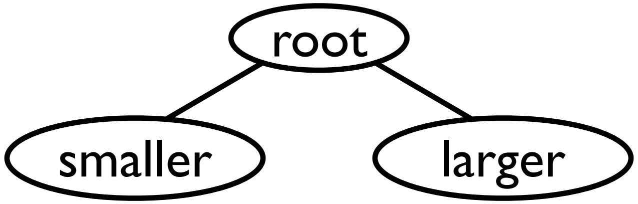
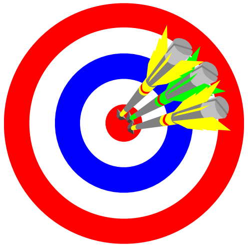


Balanced BSTs



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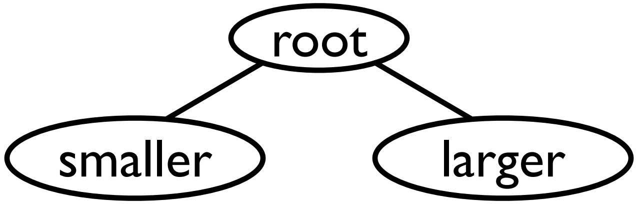
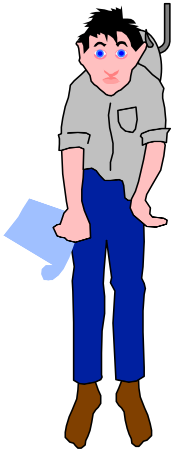


root

smaller

larger



Problem : Although T_p = O(height), but the height can be as bad as O(N).

〖Example〗 2 binary search trees obtained for the months of the year

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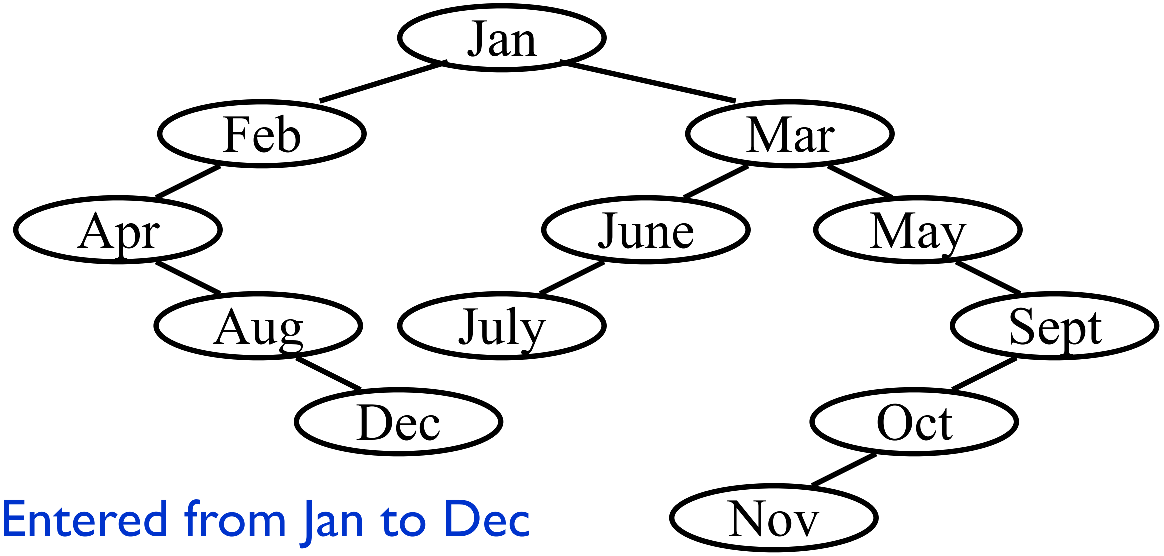

Entered from Jan to Dec

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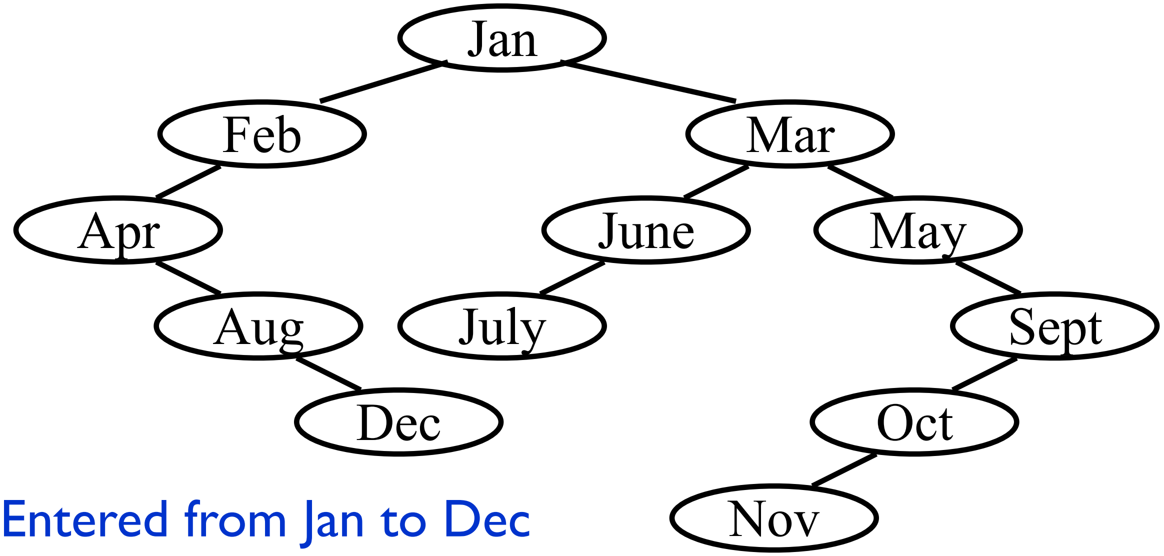

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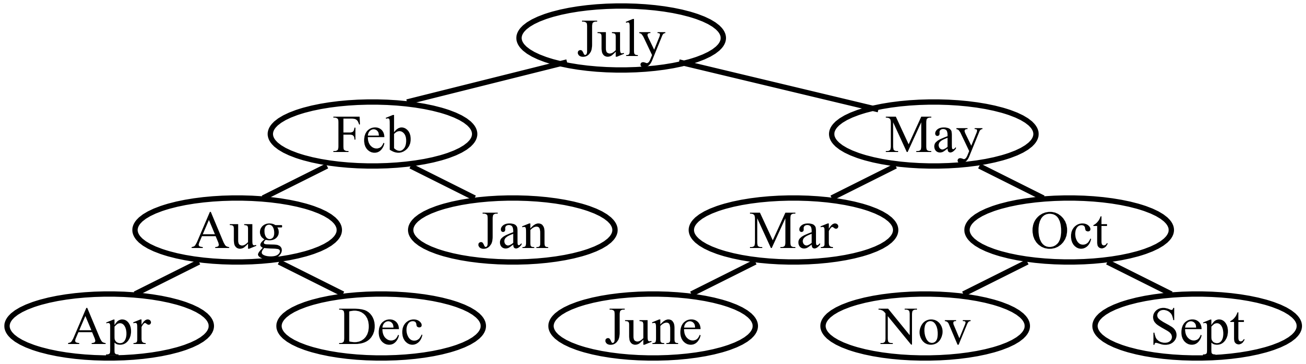

A balanced tree

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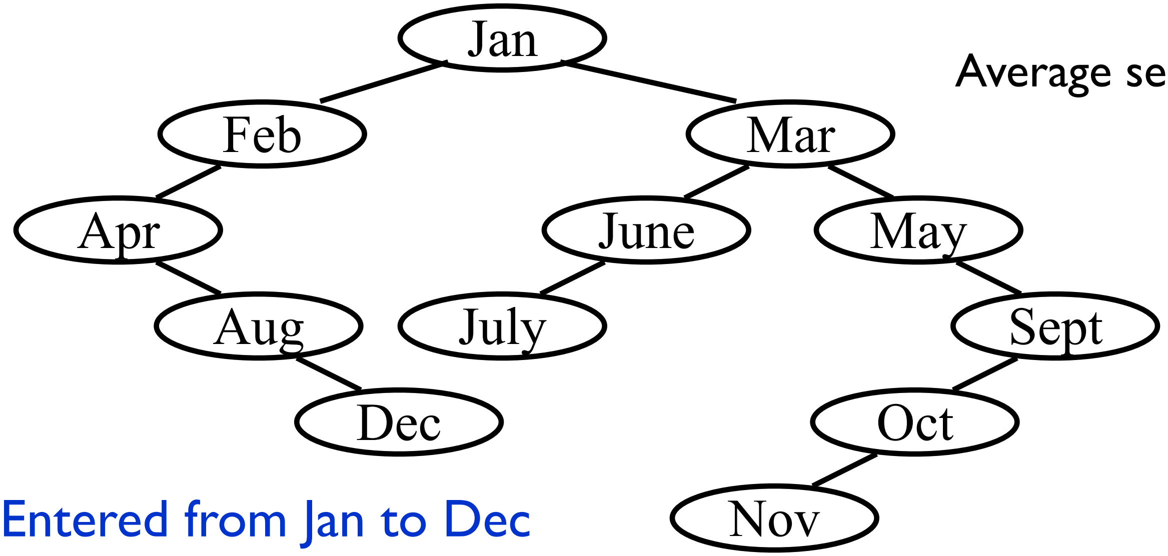
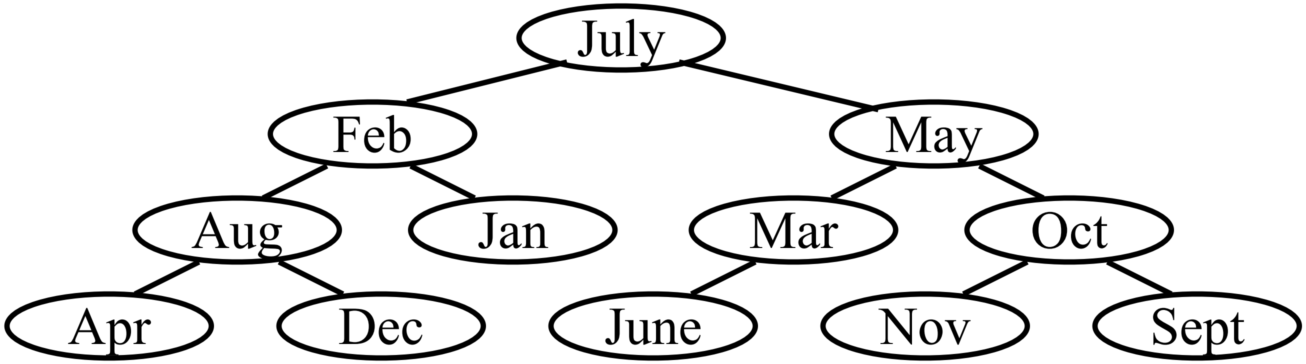


Average search time = 3.5

Entered from Jan to Dec



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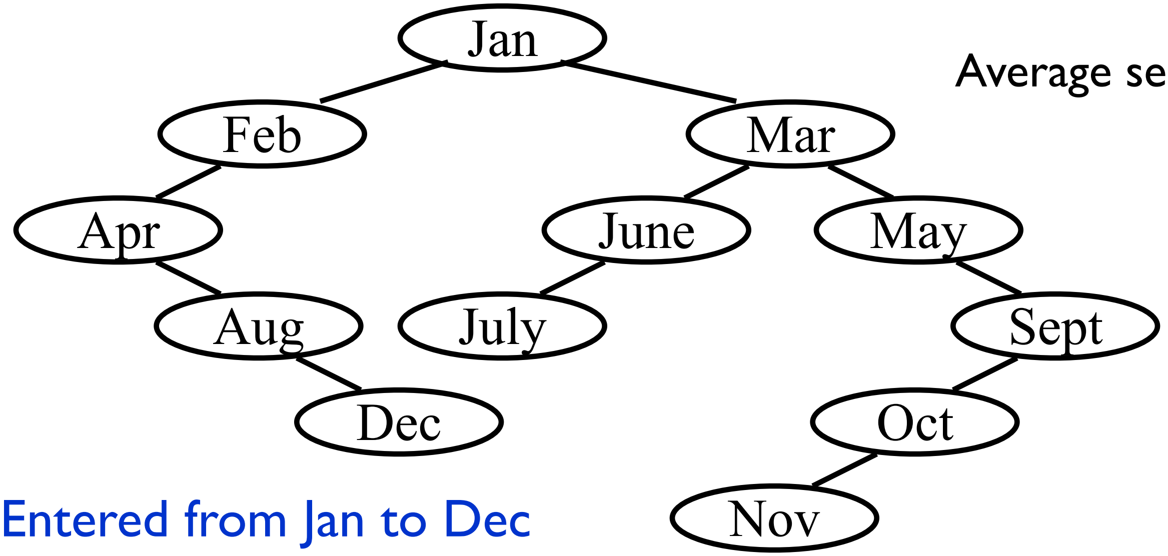

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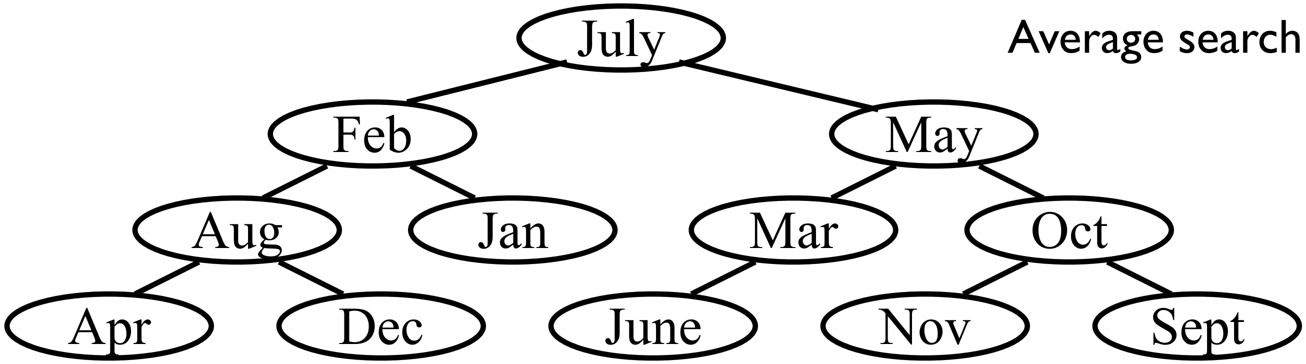

Average search time = 3.1

A balanced tree

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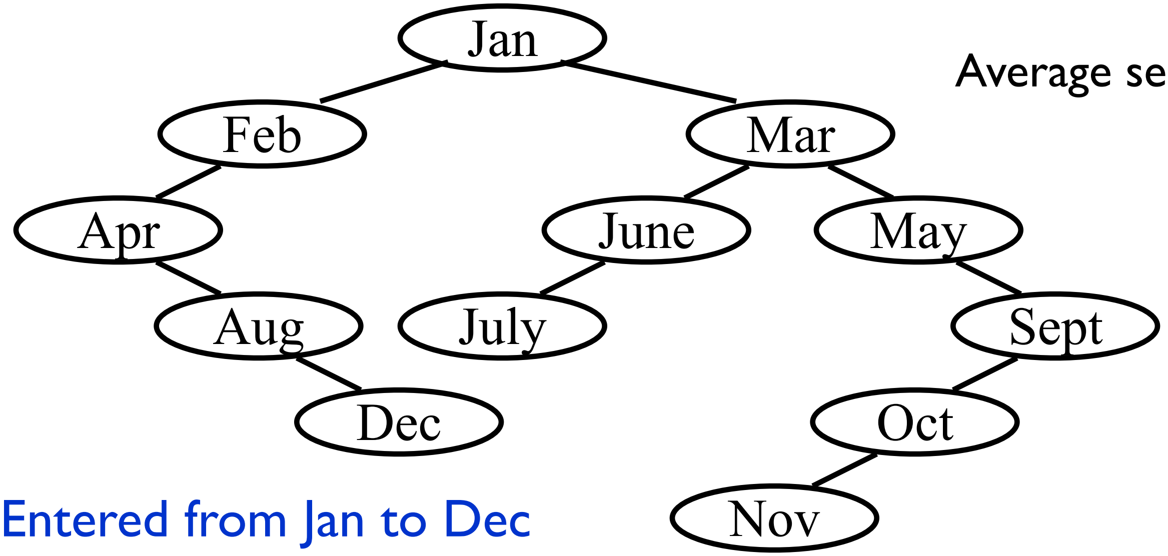
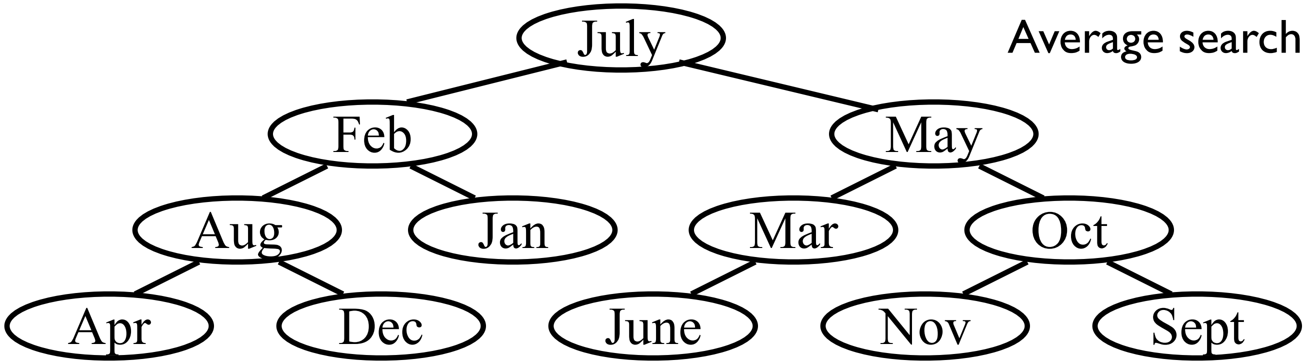


Average search time = 3.5

Average search time of the skew tree = 6.5

Entered from Jan to Dec



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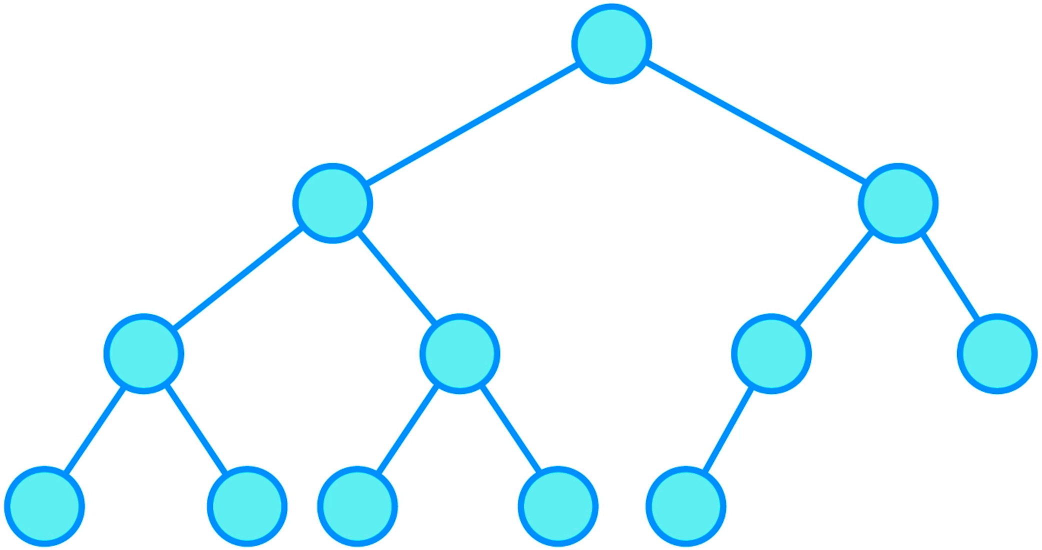
Why Not Use Complete BST?



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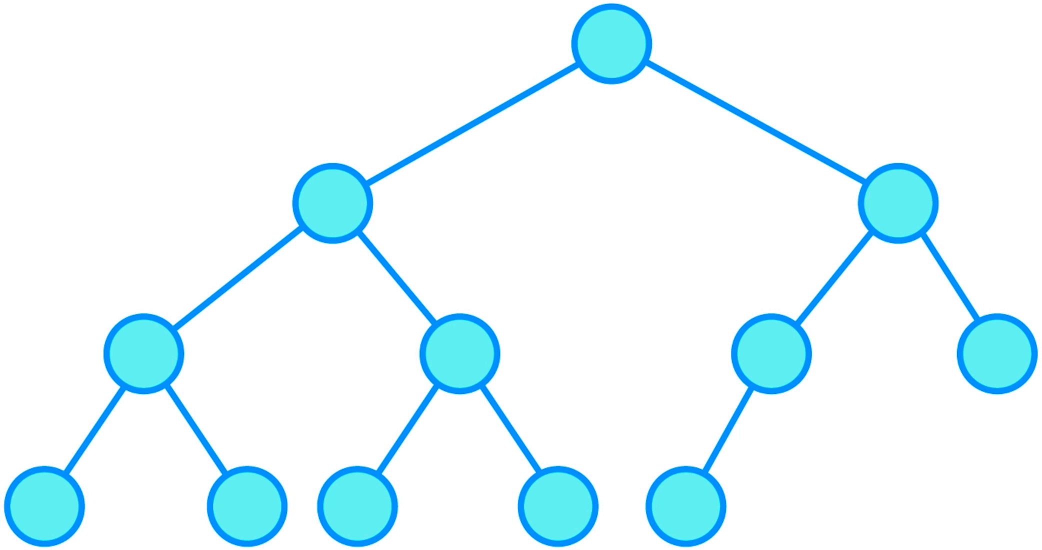

The constraint is too strong.
BST needs to preserve instance order,
every operation involves global tuning of the structure.
We should relax the constraint.

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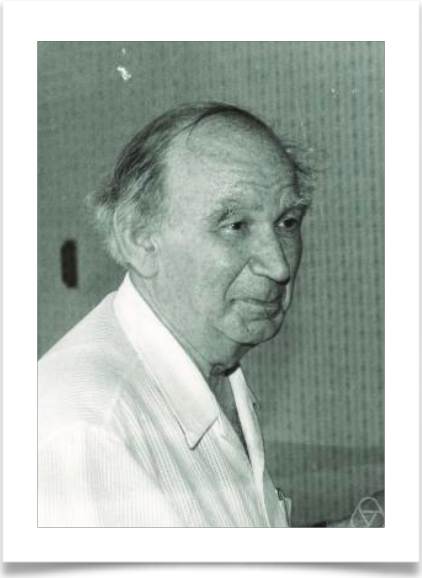
Adelson-Velskii-Landis (AVL) Trees (1962)



- Self-balanced trees which dynamically modifies tree structure to keep the tree balanced during operations.

Figure courtesy: https://www.chessprogramming.org/Georgy_Adelson-Velsky
https://en.wikipedia.org/wiki/Evgenii_Landis

Adelson-Velskii-Landis (AVL) Trees (1962)

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AVL Trees

【Definition】An empty binary tree is height-balanced. If T is a nonempty binary tree with T_L and T_R as its left and right subtrees, then T is height-balanced iff

(1) T_L and T_R are height balanced, and

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AVL Trees

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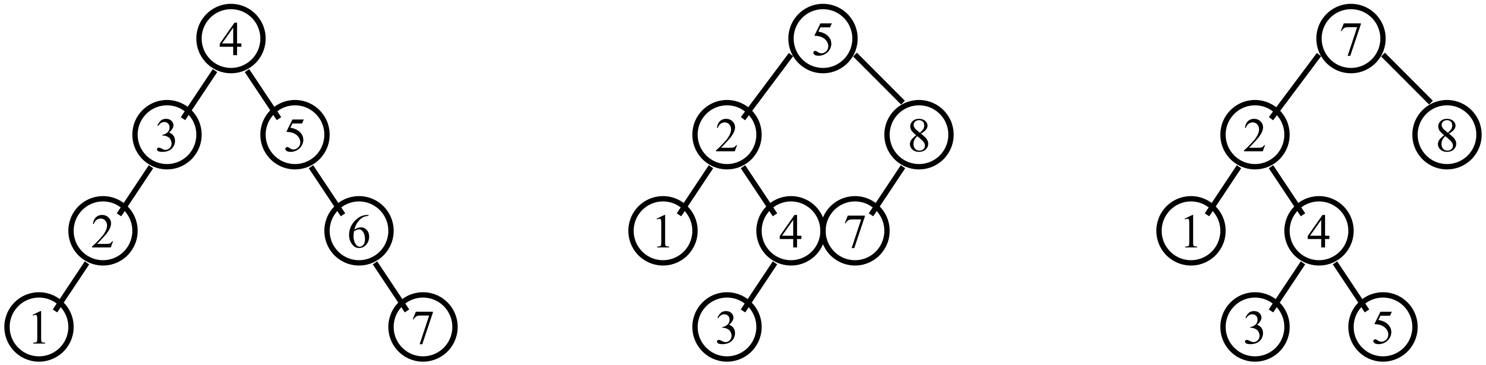

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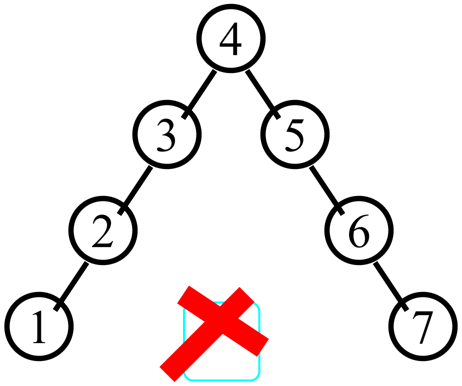

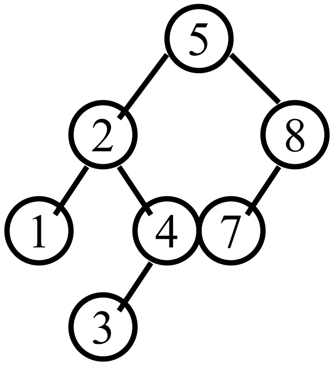

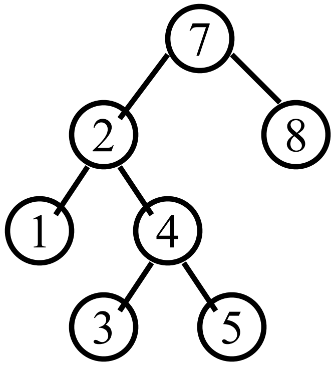

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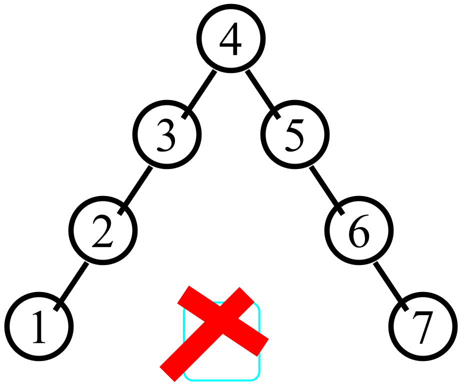

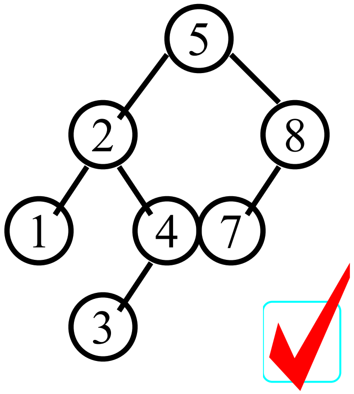

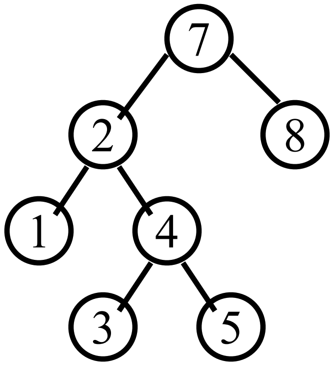

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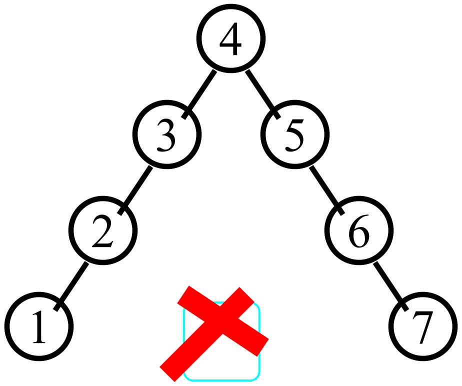

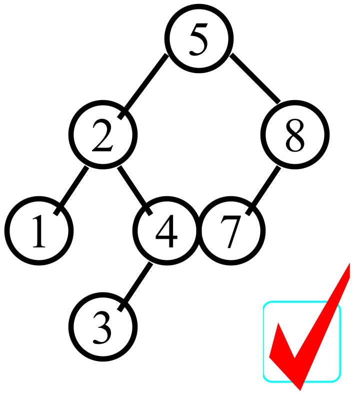

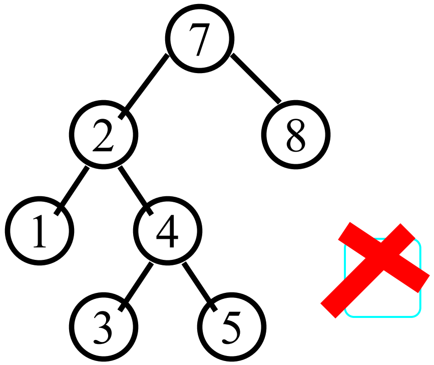

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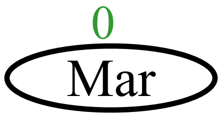

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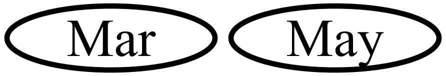

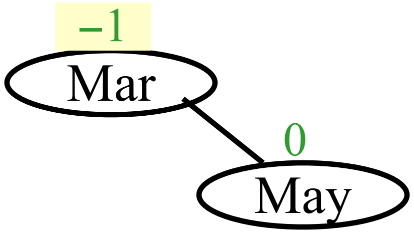

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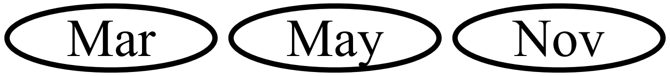

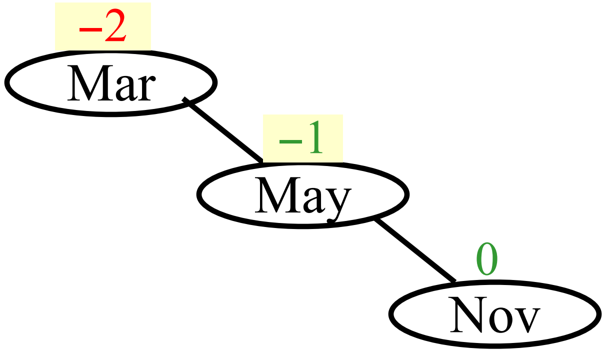

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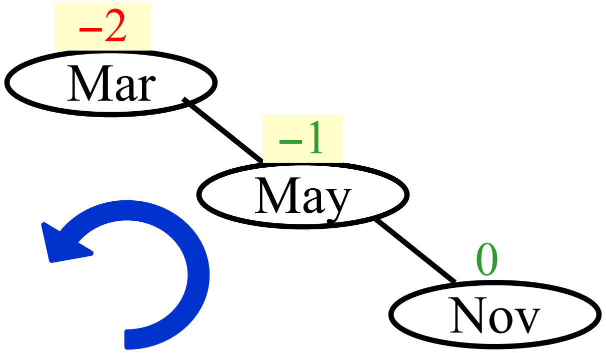

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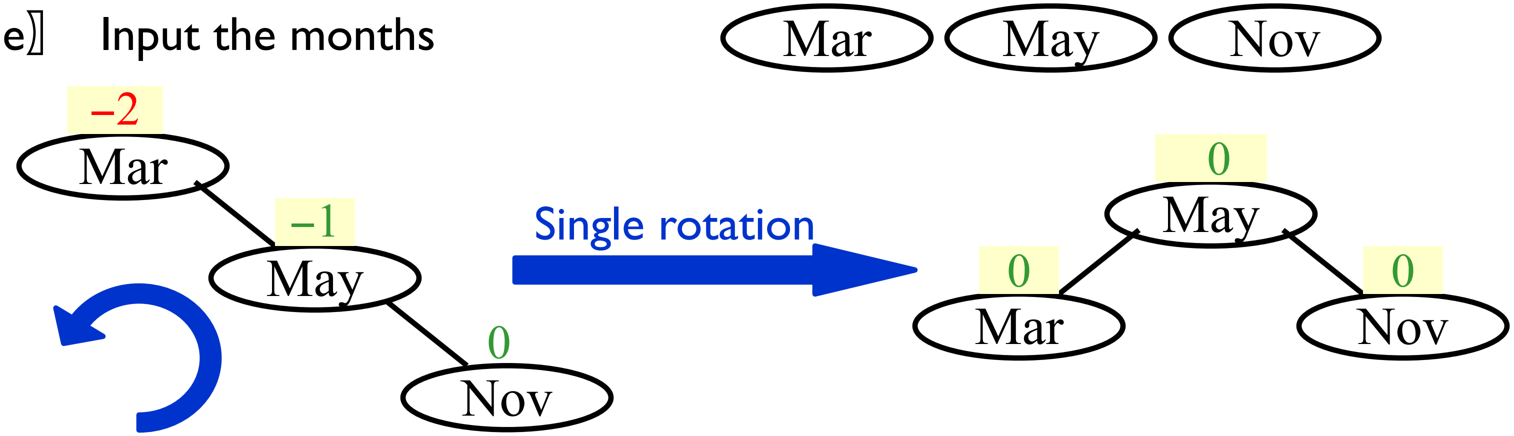

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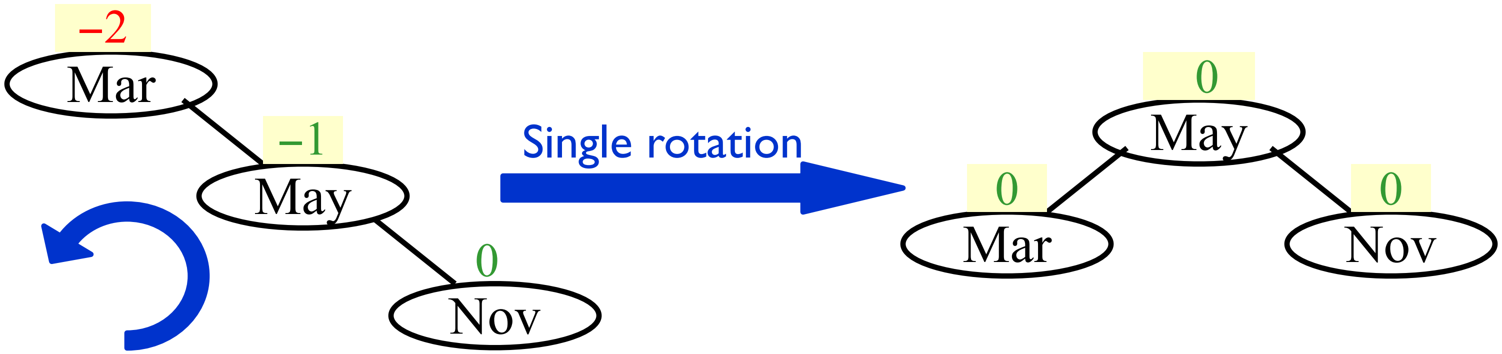

👁 The trouble maker Nov is in the right subtree's right subtree of the trouble finder Mar. Hence it is called an RR rotation.

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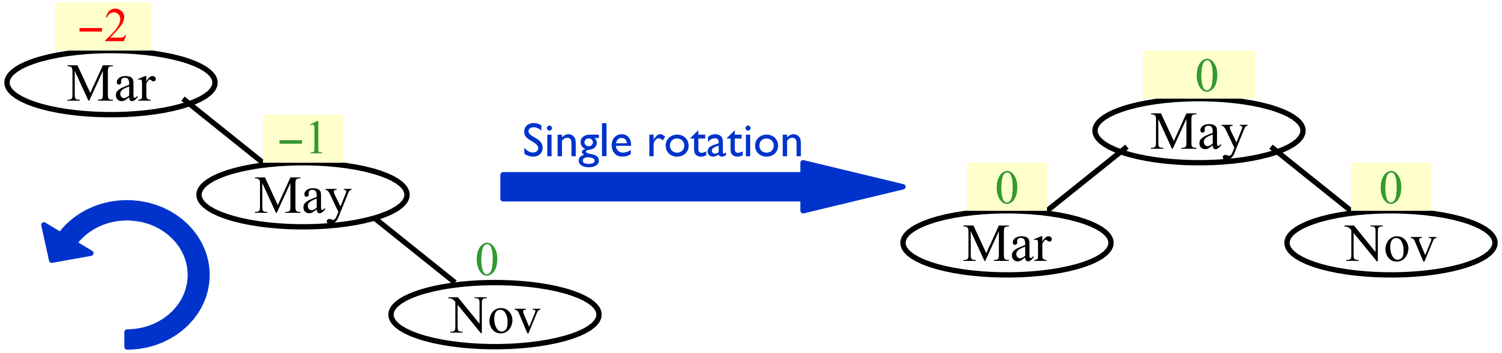

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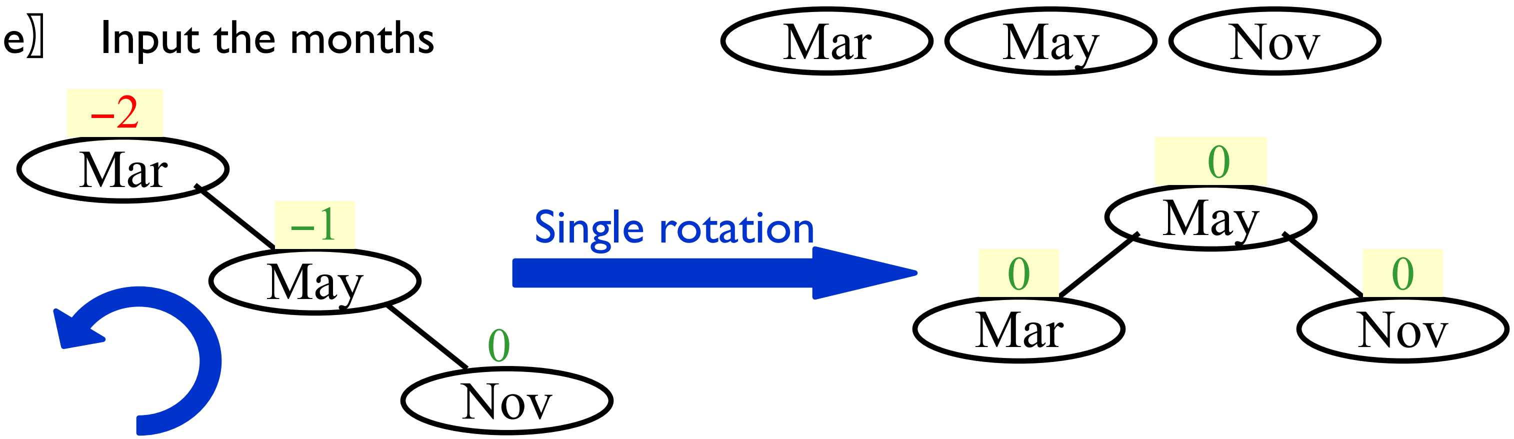

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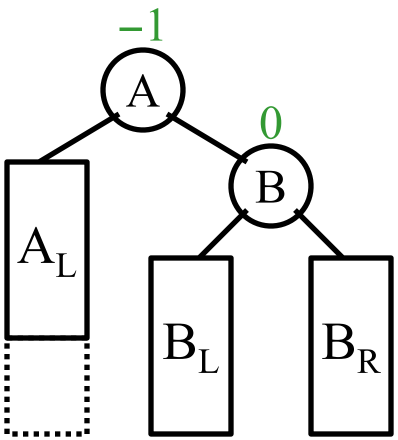

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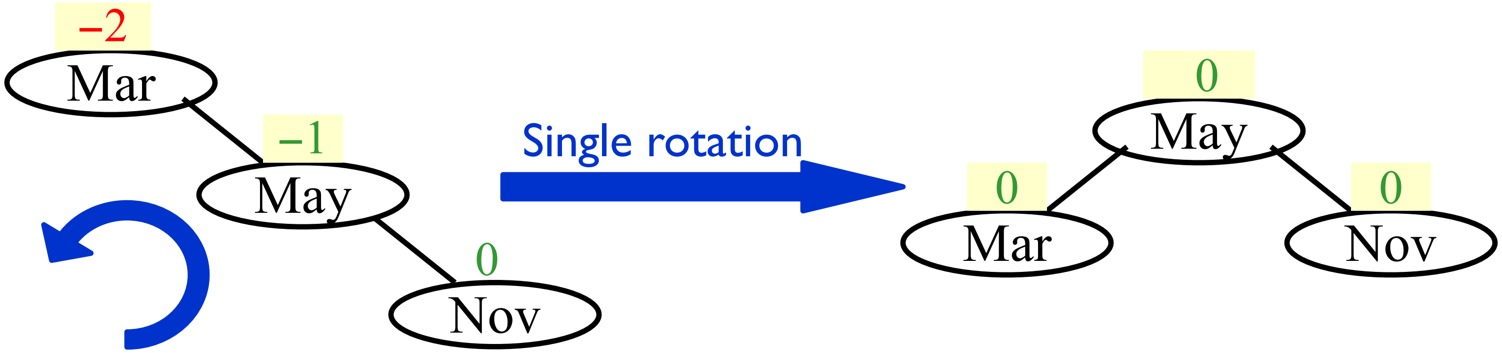

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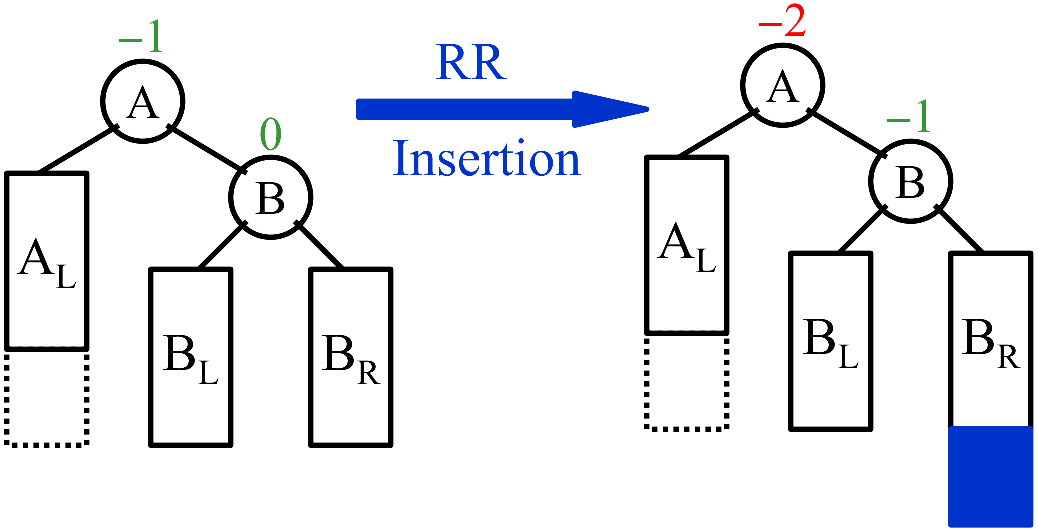

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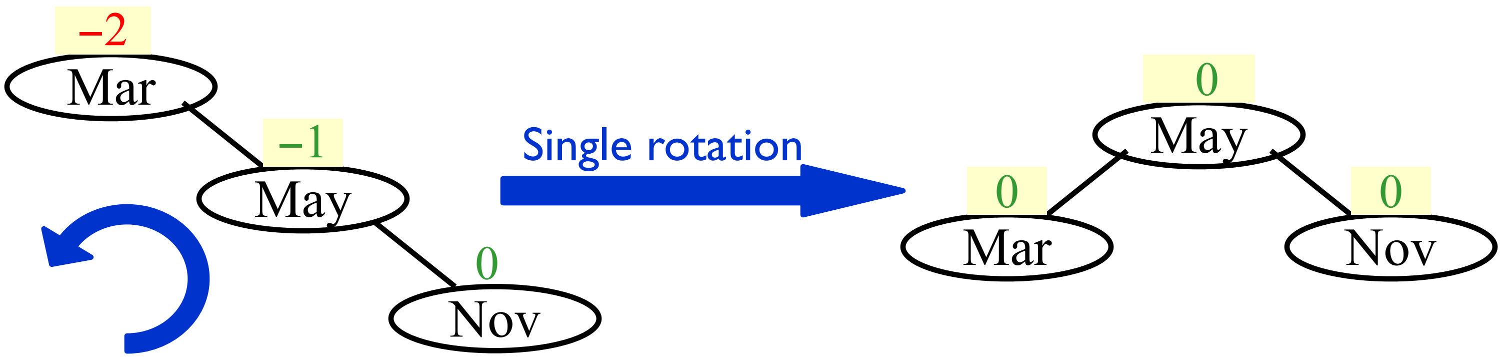

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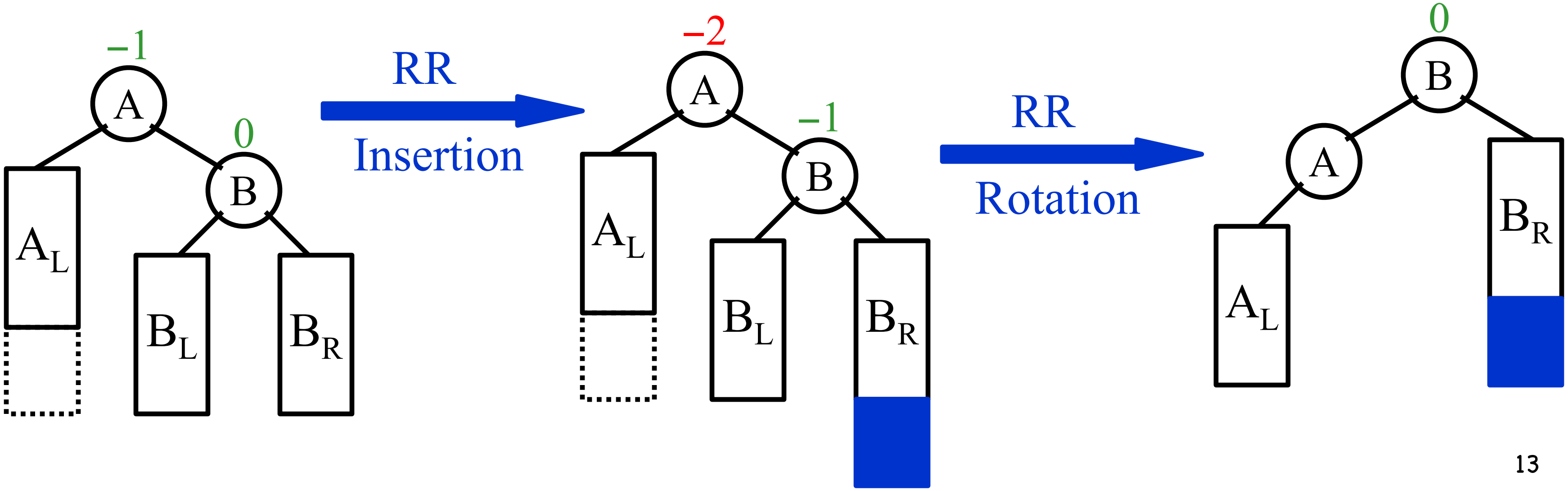

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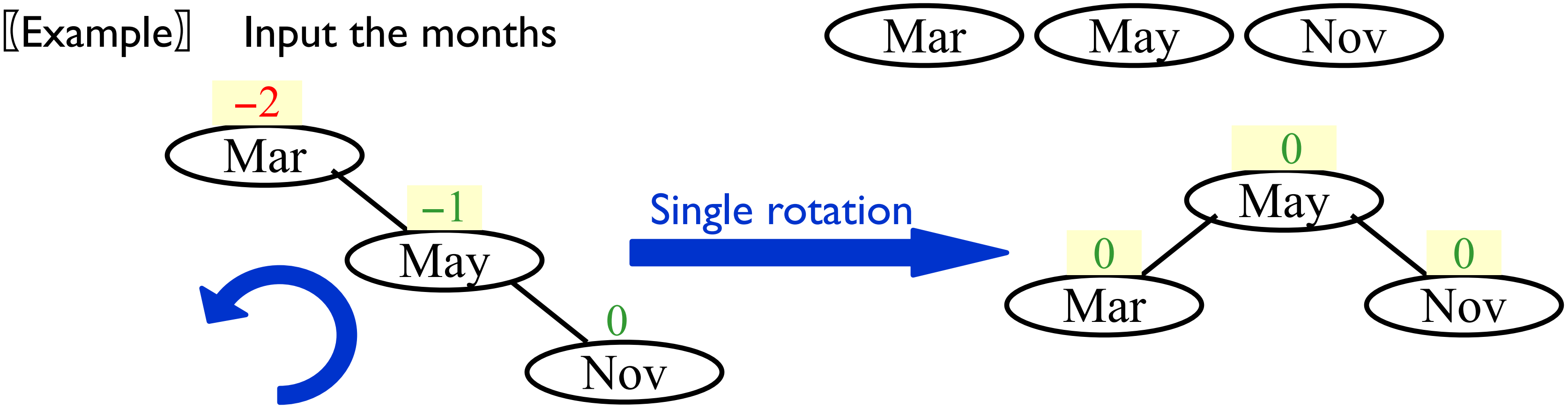

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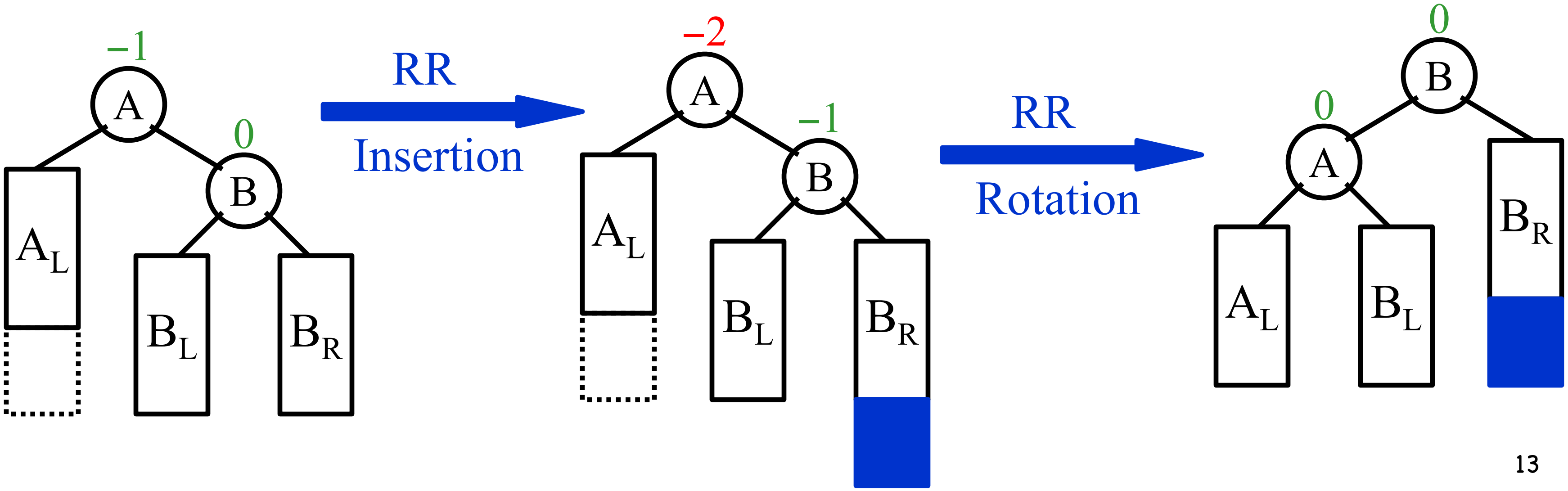

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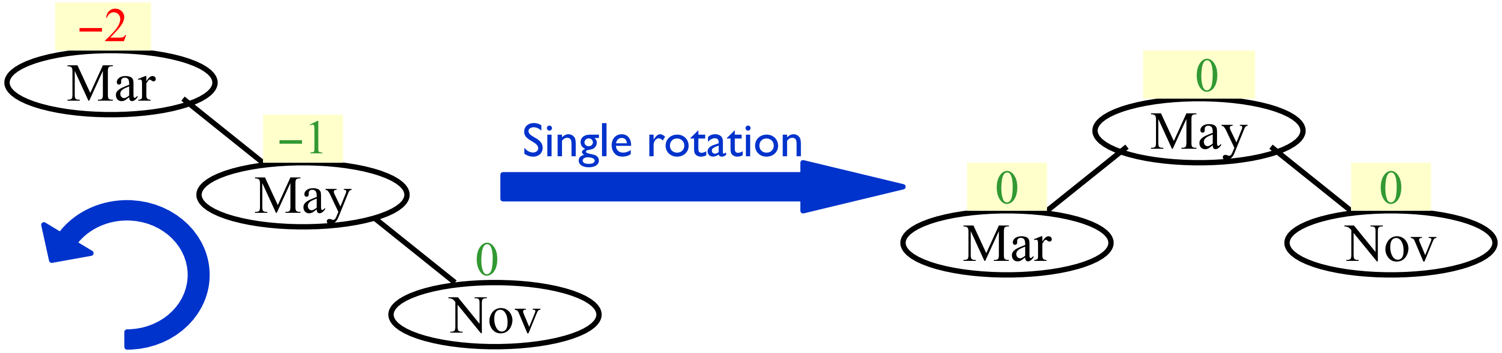
Single rotation

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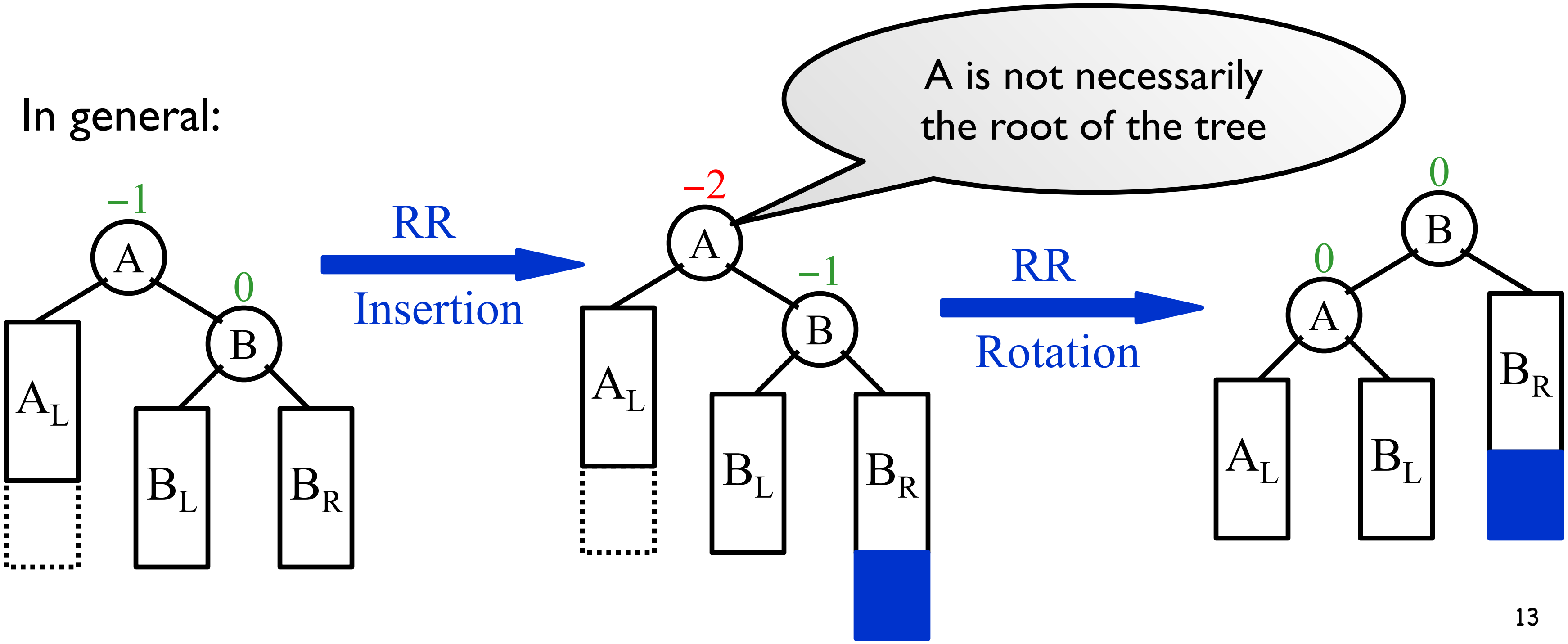
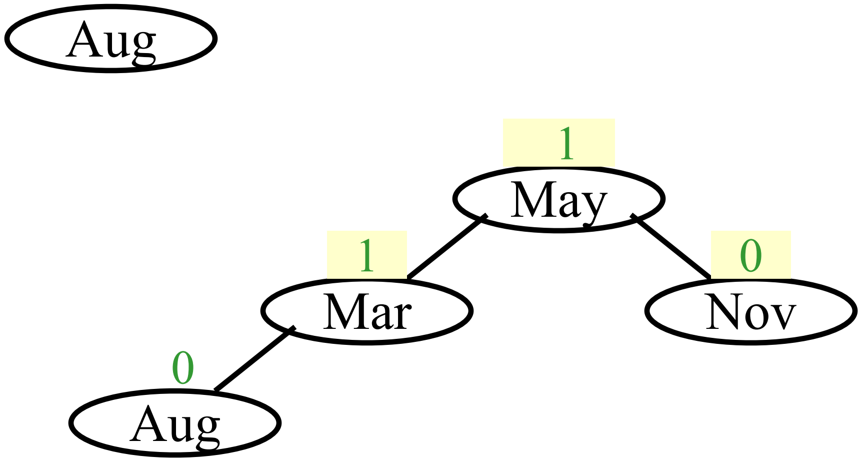
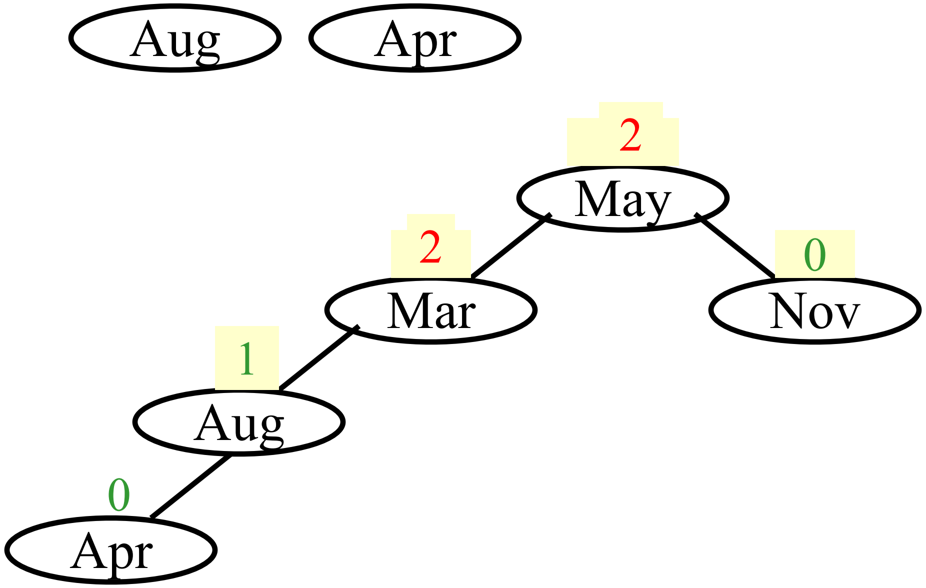
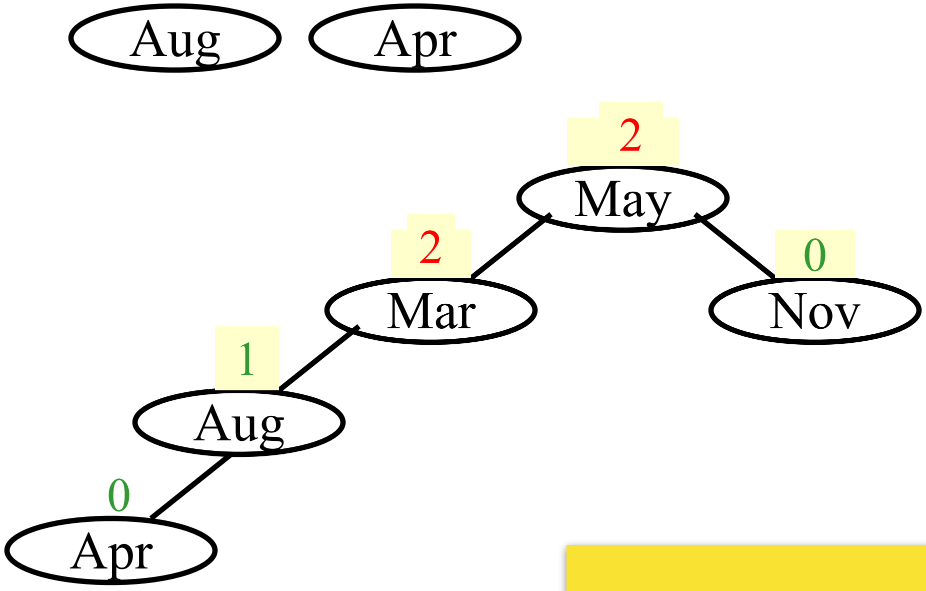
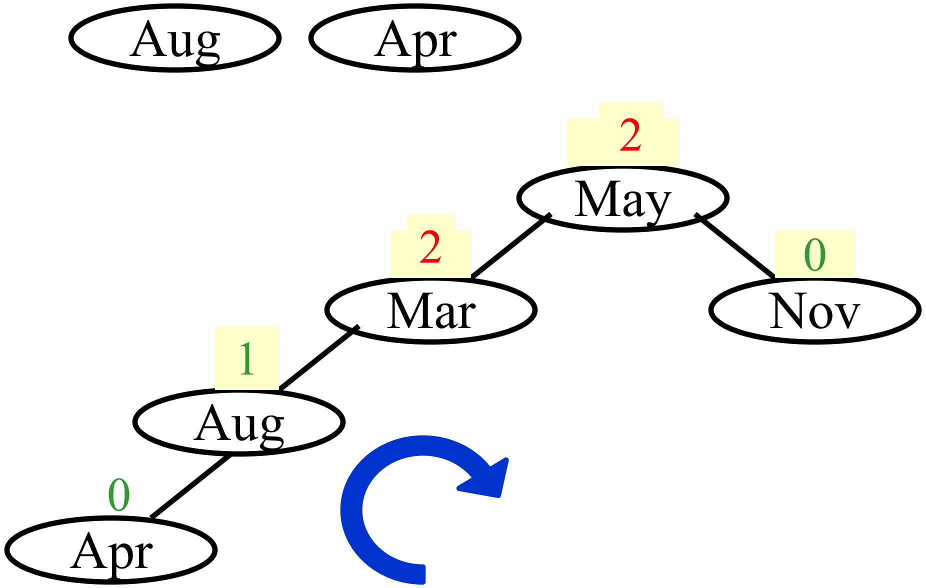

A is not necessarily the root of the tree

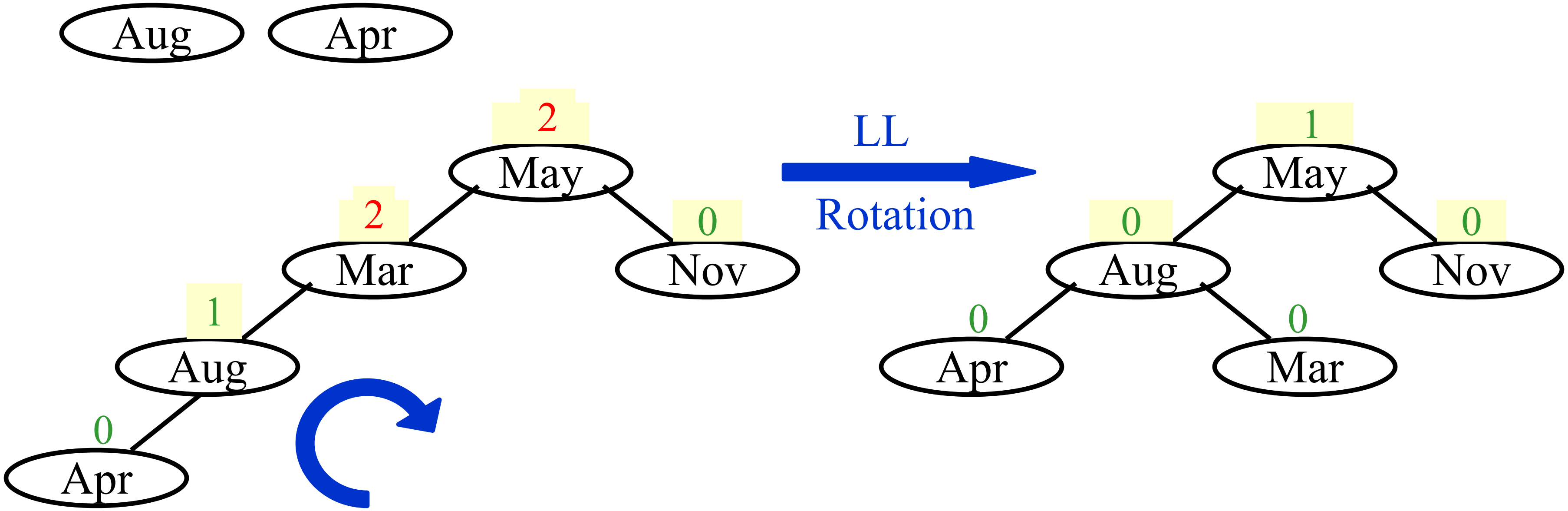

Aug
1
May
1
Mar
0
Nov
0
Aug

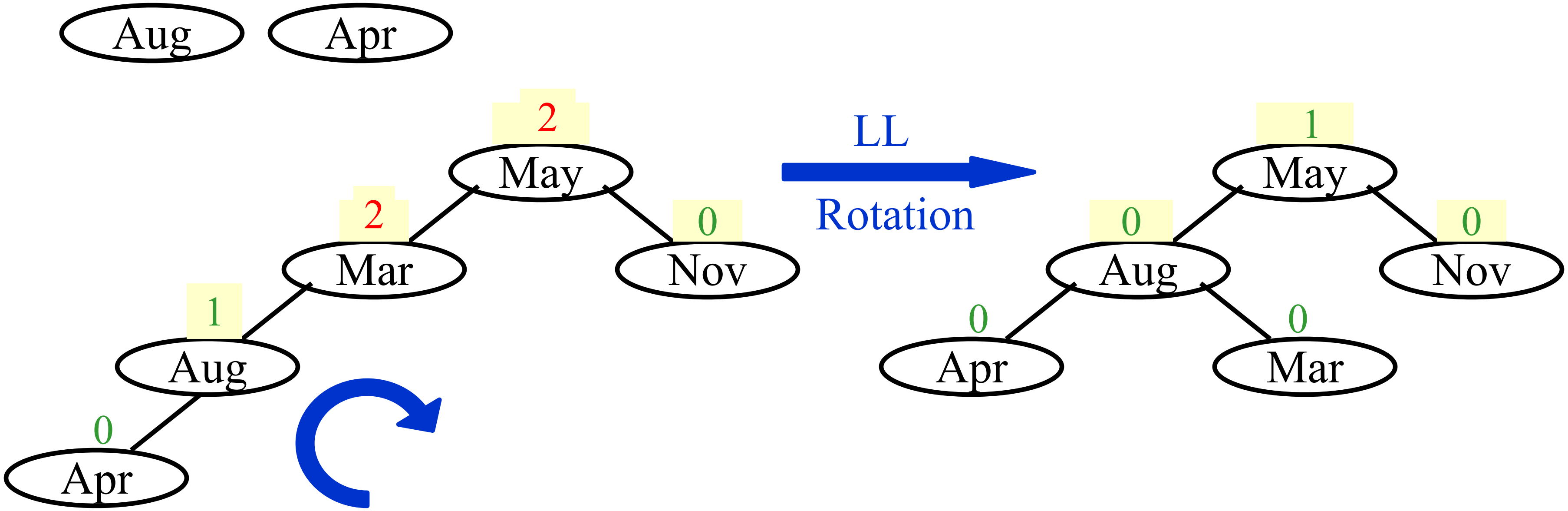

Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr



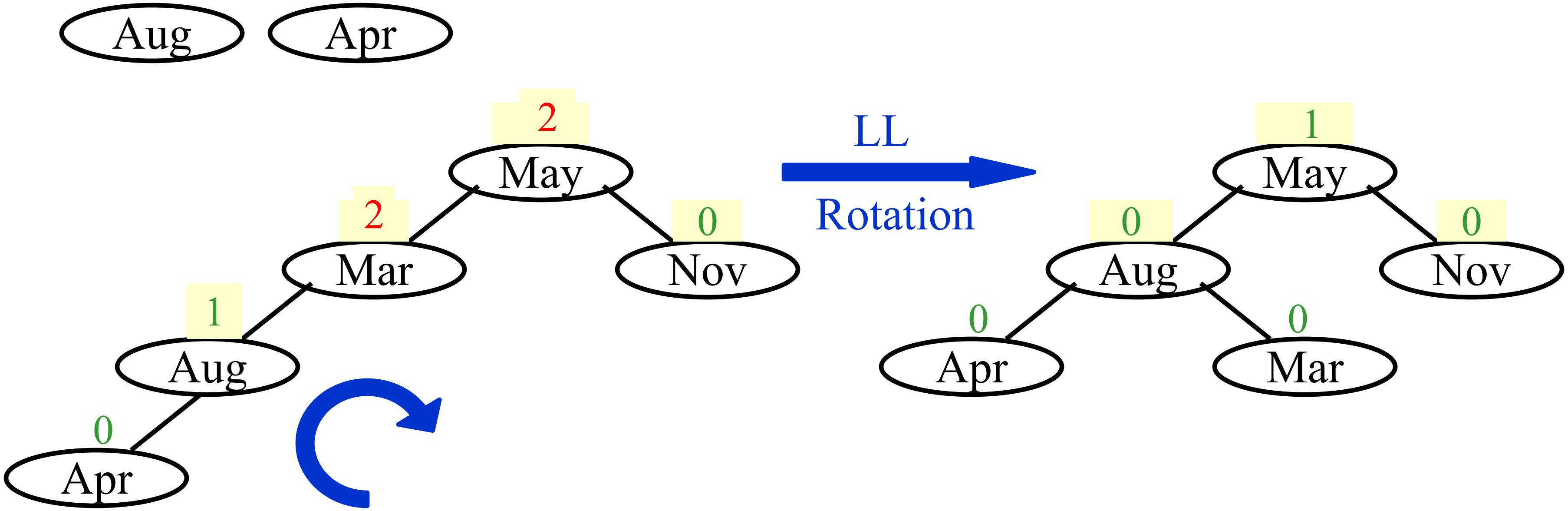
What can we do now?


Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr


Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr
LL
Rotation
1
May
0
Aug
0
Nov
0
Apr
0
Mar


Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr
LL
Rotation
1
May
0
Aug
0
Nov
0
Apr
0
Mar

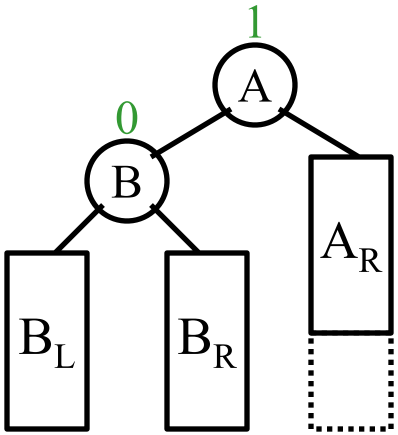
In general:

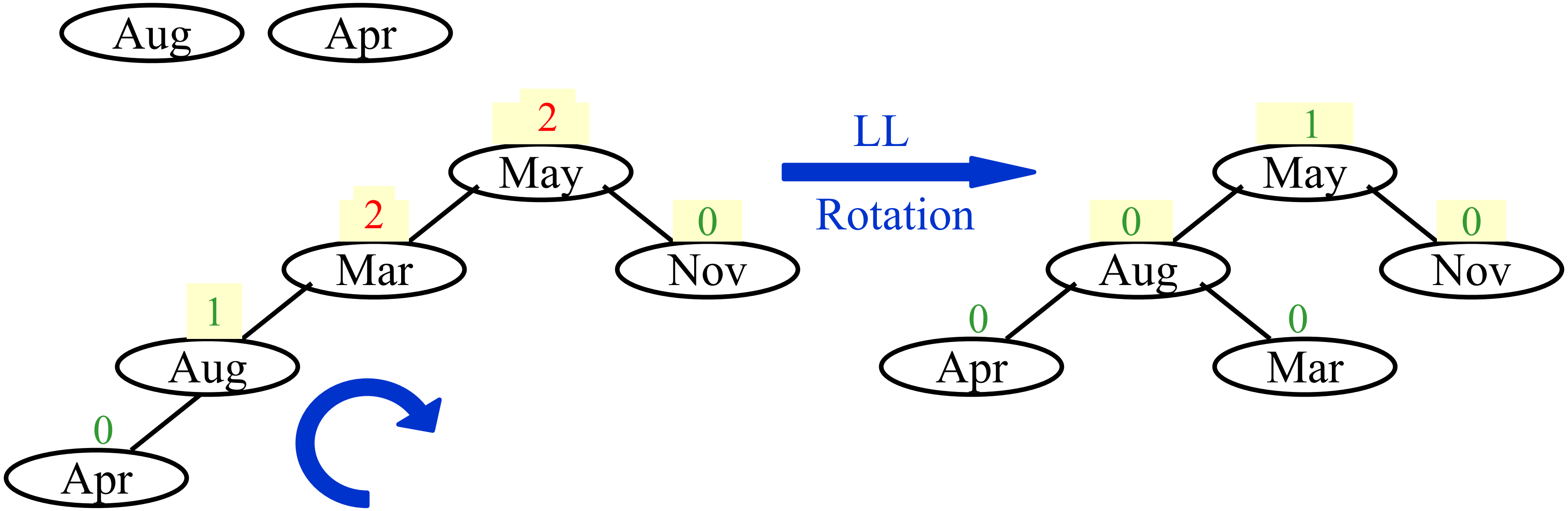


LL

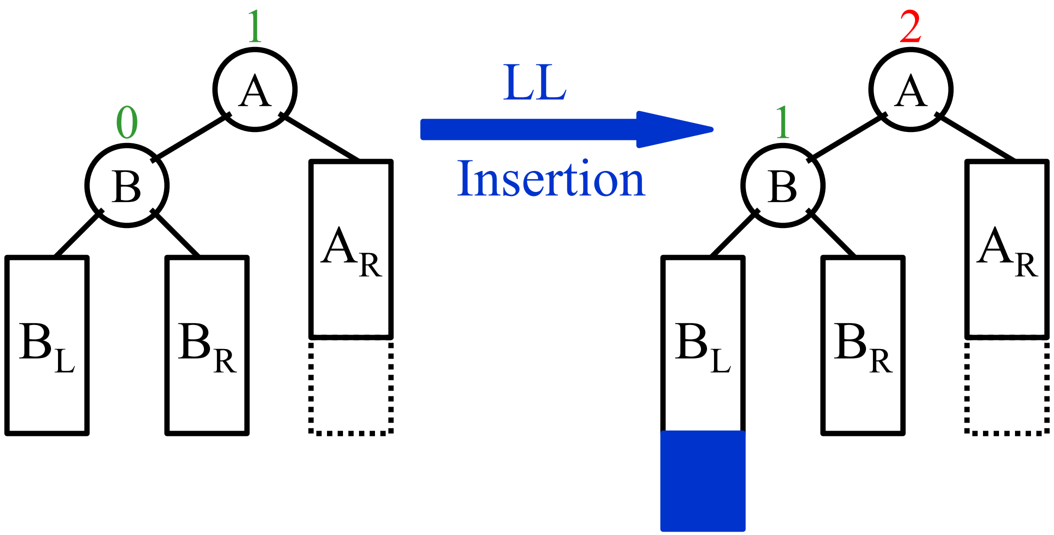
Rotation

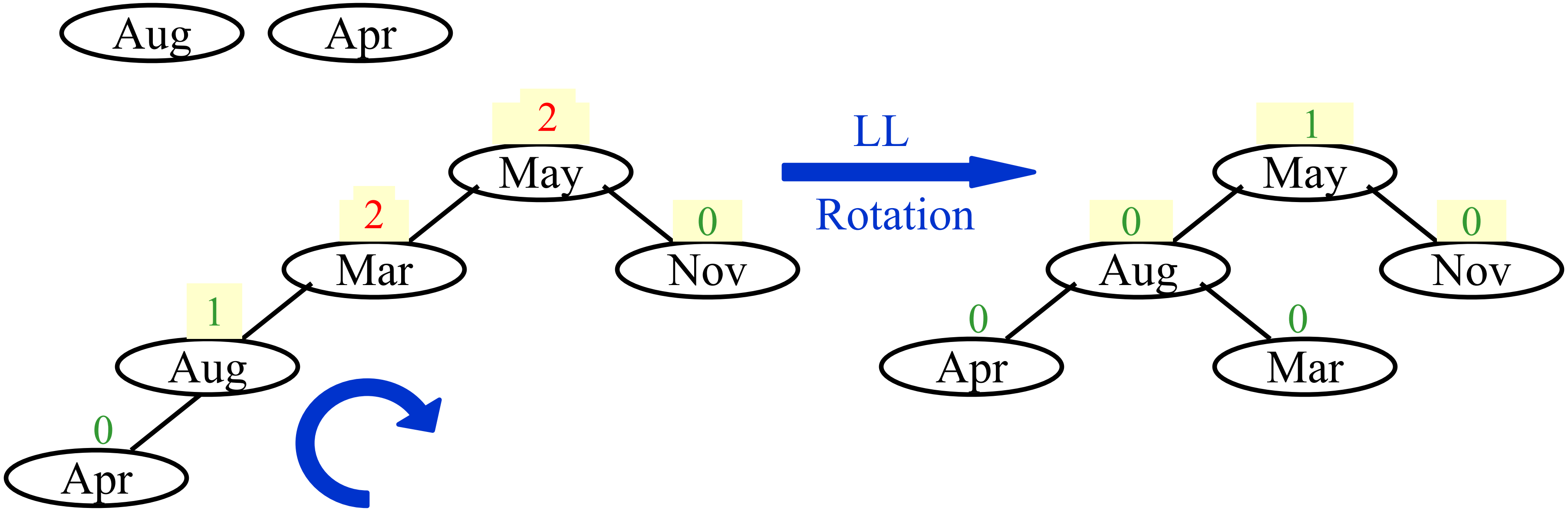
In general:



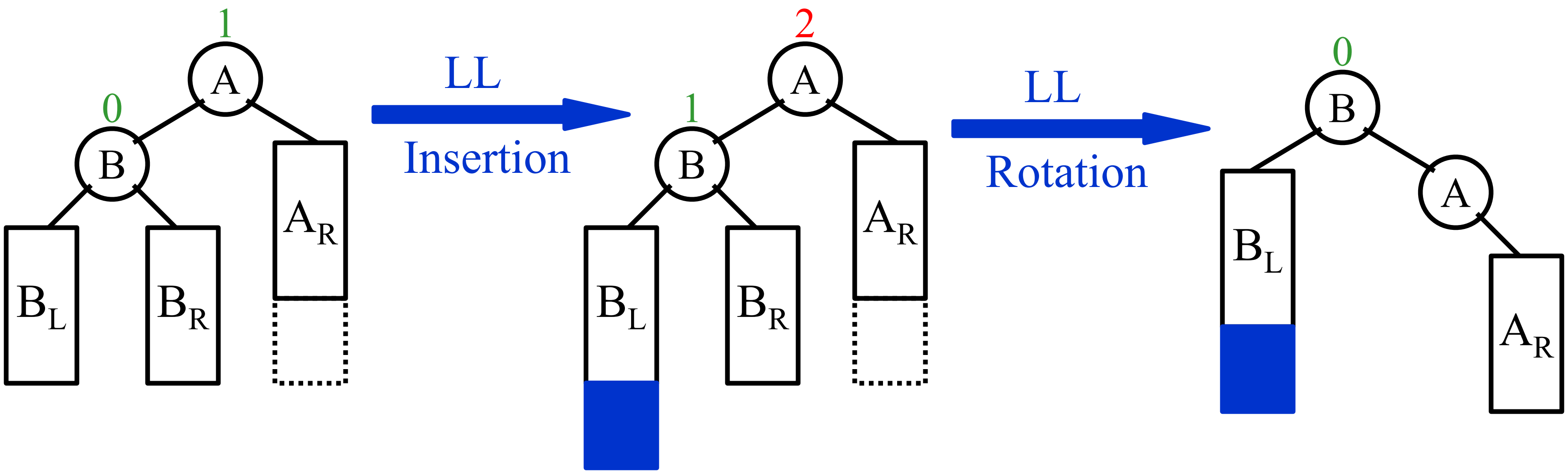

Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr
LL
Rotation
1
May
0
Aug
0
Nov
0
Apr
0
Mar

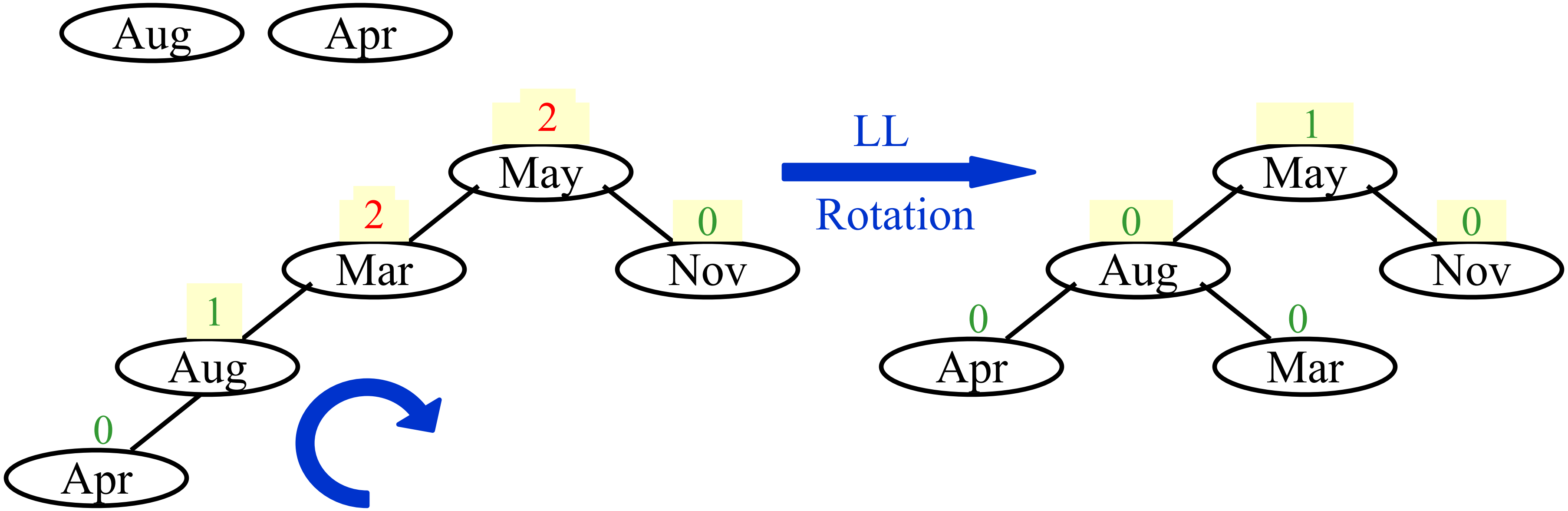
In general:


1
A
0
B
B_L
B_R
A_R
LL
Insertion
2
A
1
B
B_L
B_R
A_R

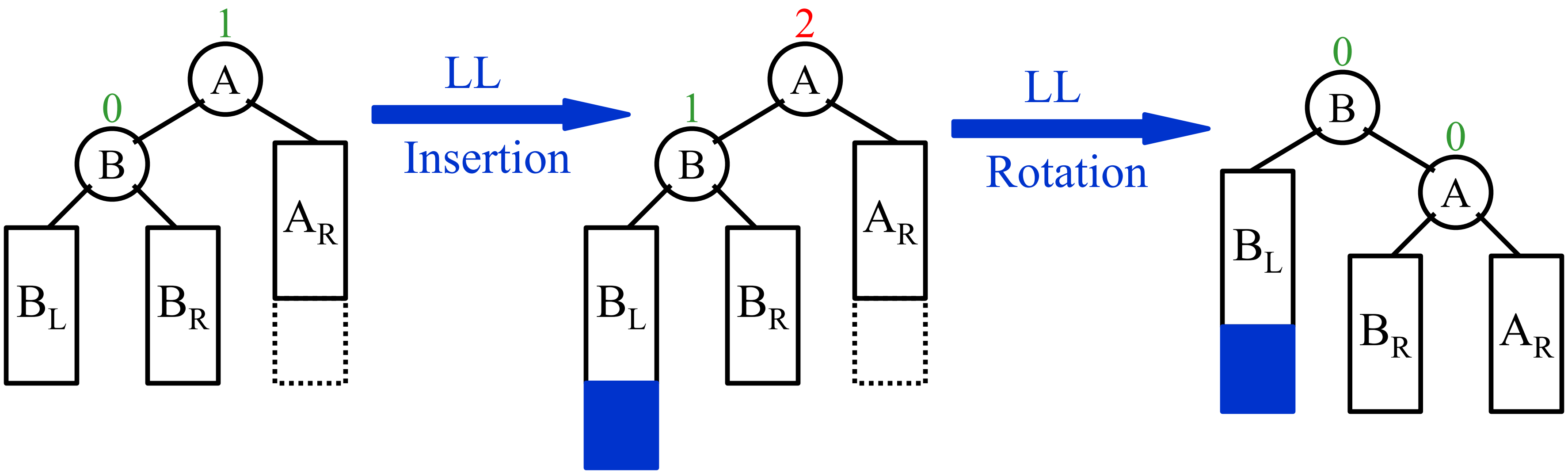

Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr
LL
Rotation
1
May
0
Aug
0
Nov
0
Apr
0
Mar

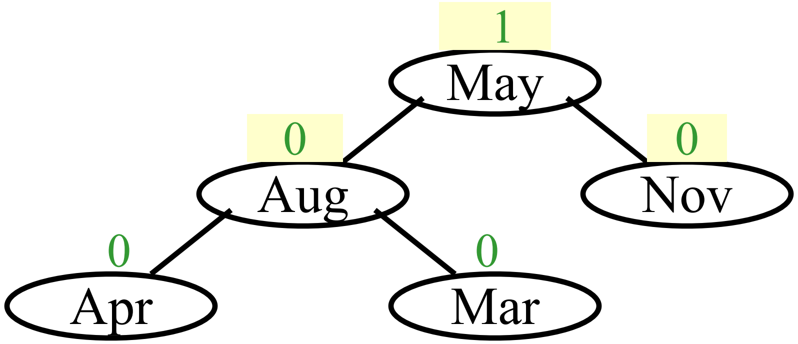
In general:

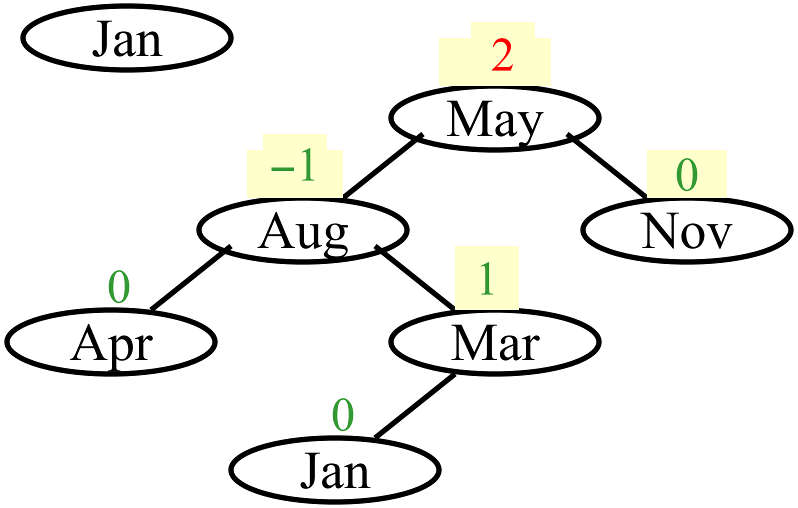

1
A
0
B
B_L
B_R
A_R
LL
Insertion
2
A
1
B
B_L
B_R
A_R
LL
Rotation
0
B
A
B_L
A_R

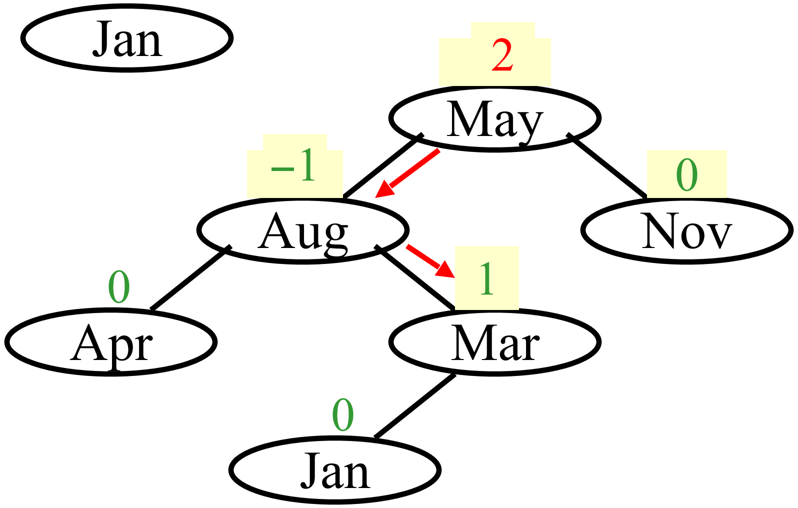

Aug
Apr
2
May
2
Mar
0
Nov
1
Aug
0
Apr
LL
Rotation
1
May
0
Aug
0
Nov
0
Apr
0
Mar

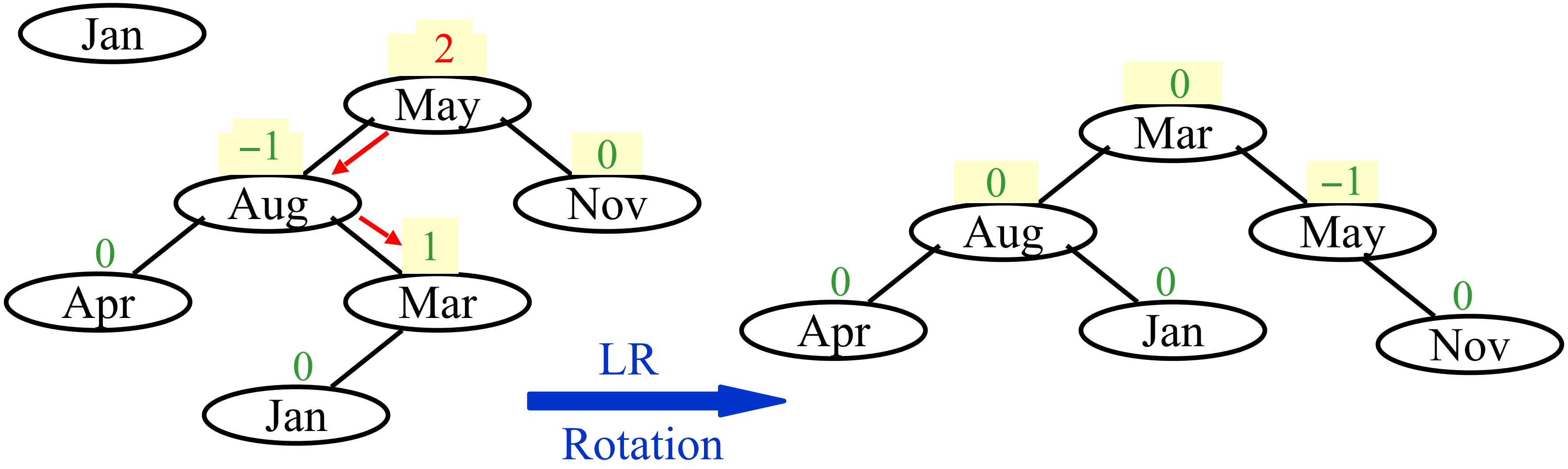
In general:

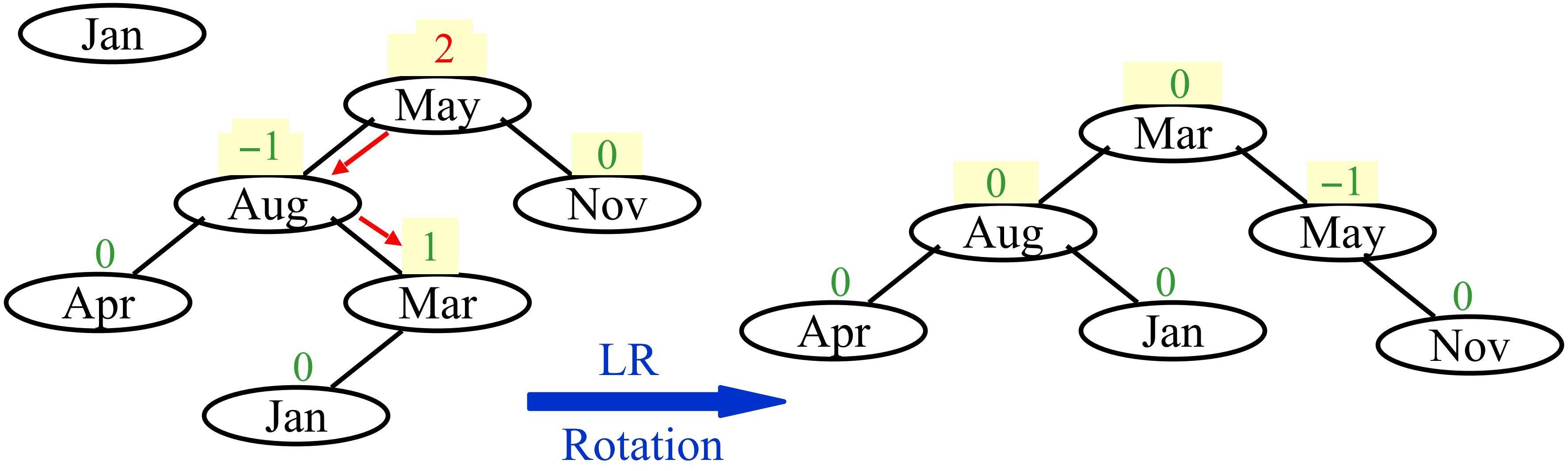

1
A
0
B
B_L
B_R
A_R
LL
Insertion
2
A
1
B
B_L
B_R
A_R
LL
Rotation
0
B
0
A
B_L
B_R
A_R


1
May
0
Aug
0
Nov
0
Apr
0
Mar

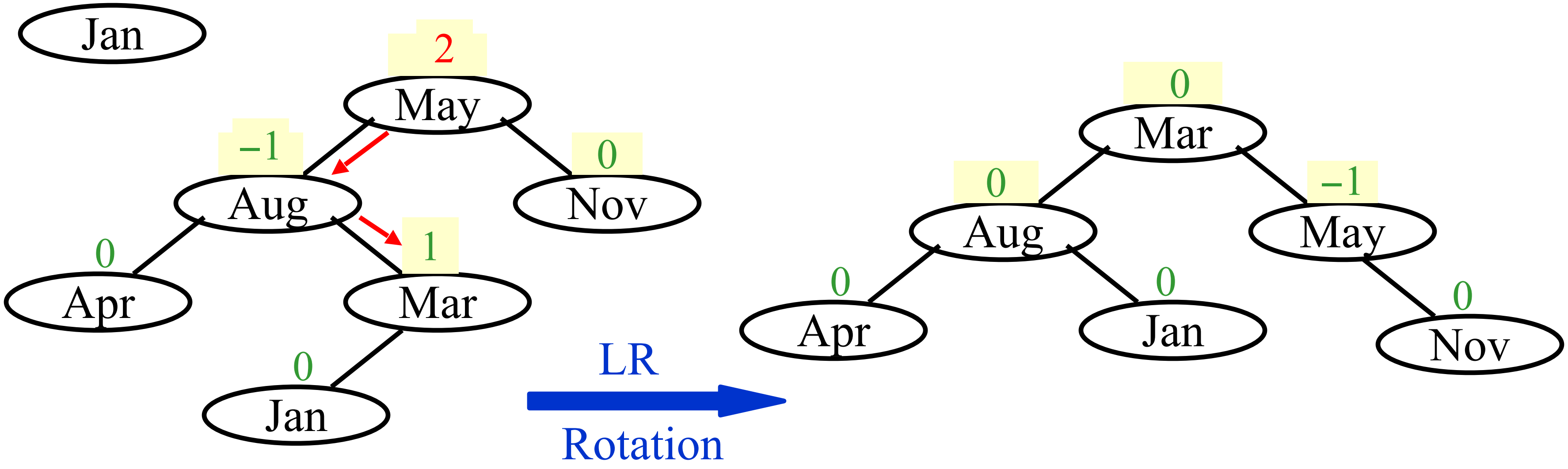

Jan
2
May
-1
Aug
0
Nov
0
Apr
1
Mar
0
Jan


Jan
2
May
−1
Aug
0
Nov
0
Apr
1
Mar
0
Jan

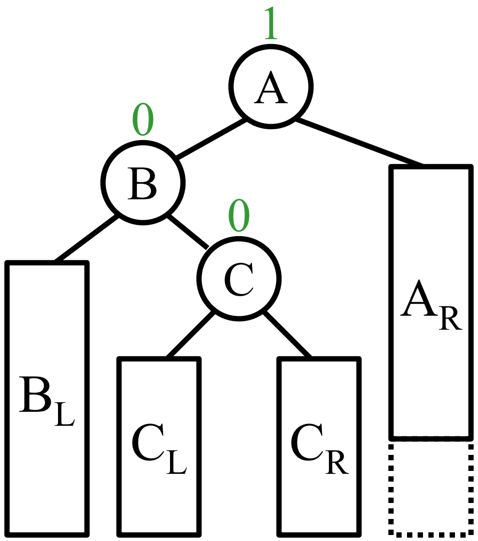

Jan
2
May
–1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
–1
May
0
Apr
0
Jan
0
Nov

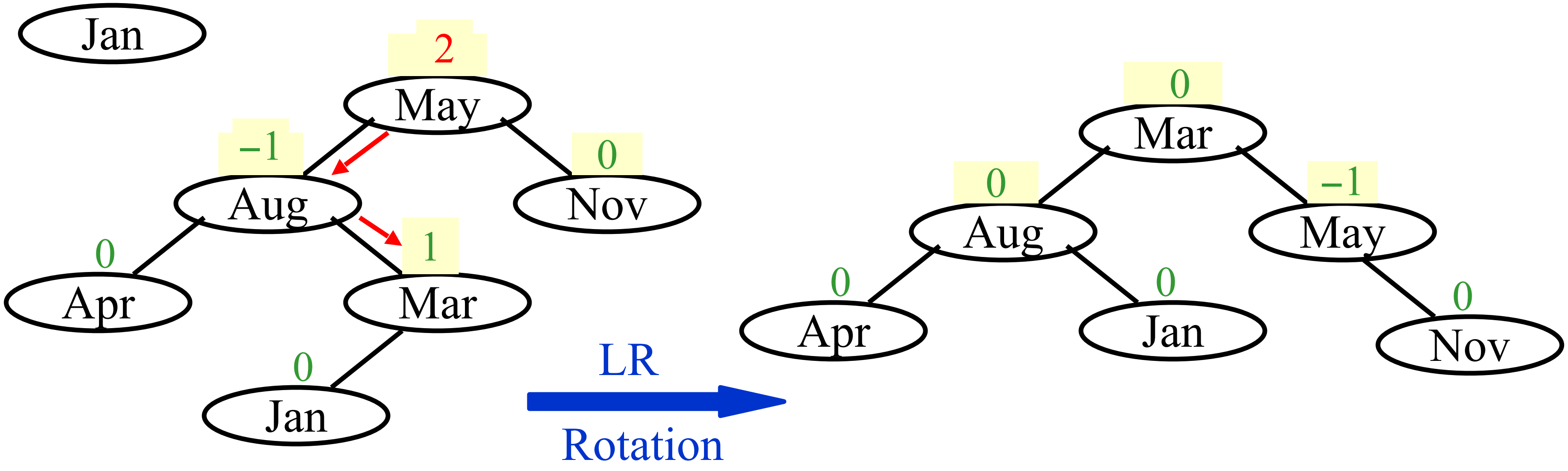

Jan
2
May
−1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
−1
May
0
Apr
0
Jan
0
Nov

In general:

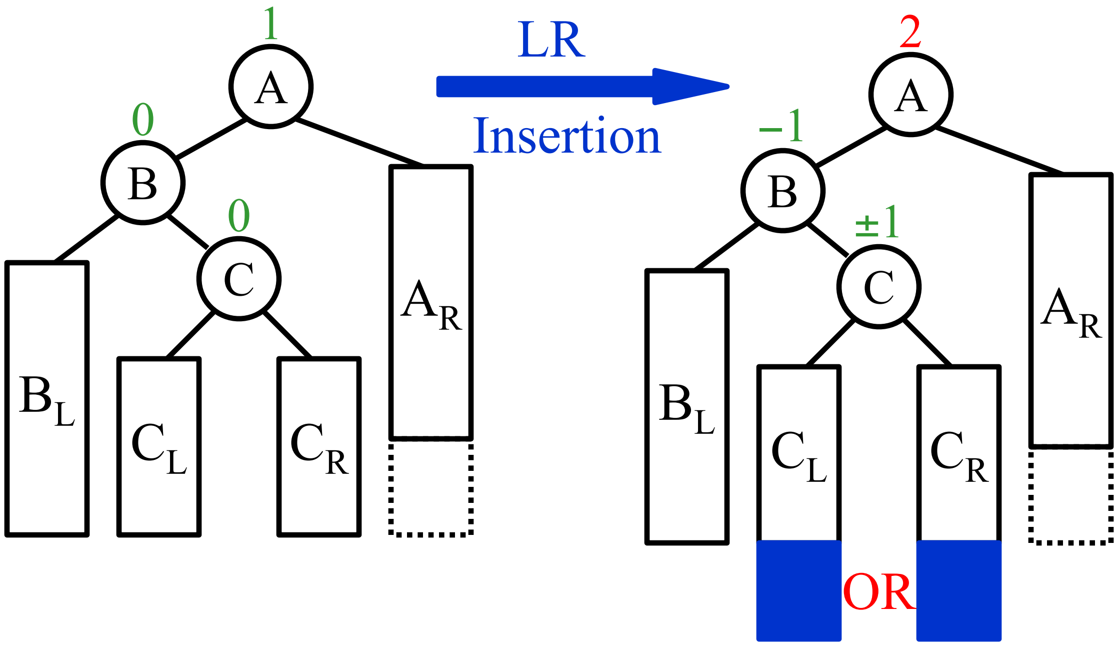

Jan
2
May
–1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
–1
May
0
Apr
0
Jan
0
Nov

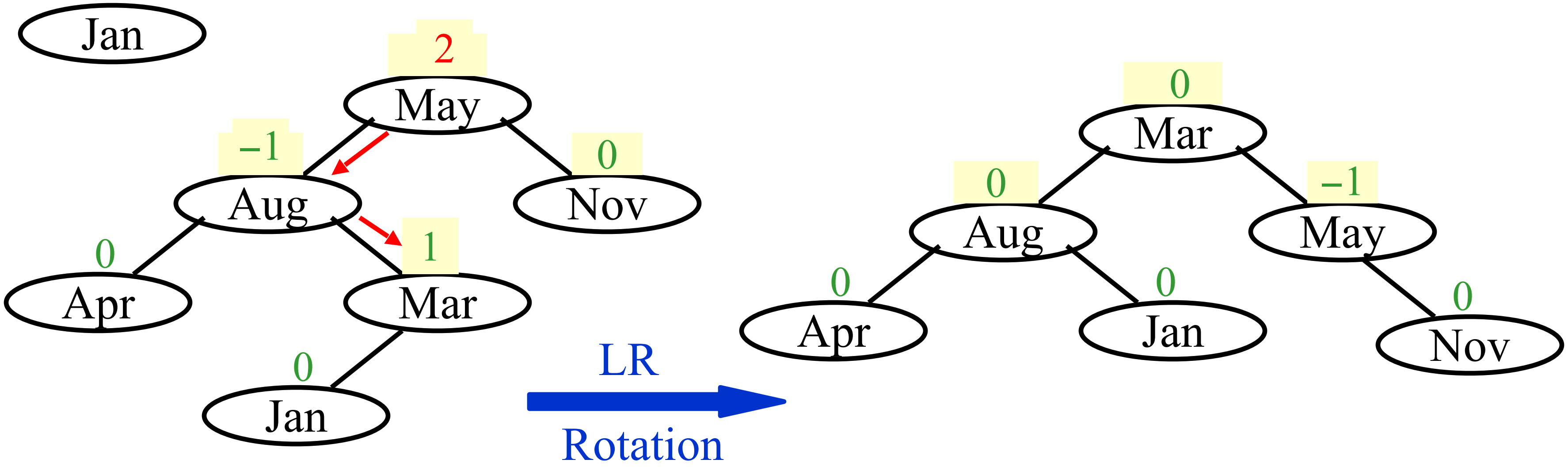
In general:


1
A
0
B
0
C
B_L
C_L
C_R
A_R


Jan
2
May
−1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
−1
May
0
Apr
0
Jan
0
Nov

In general:

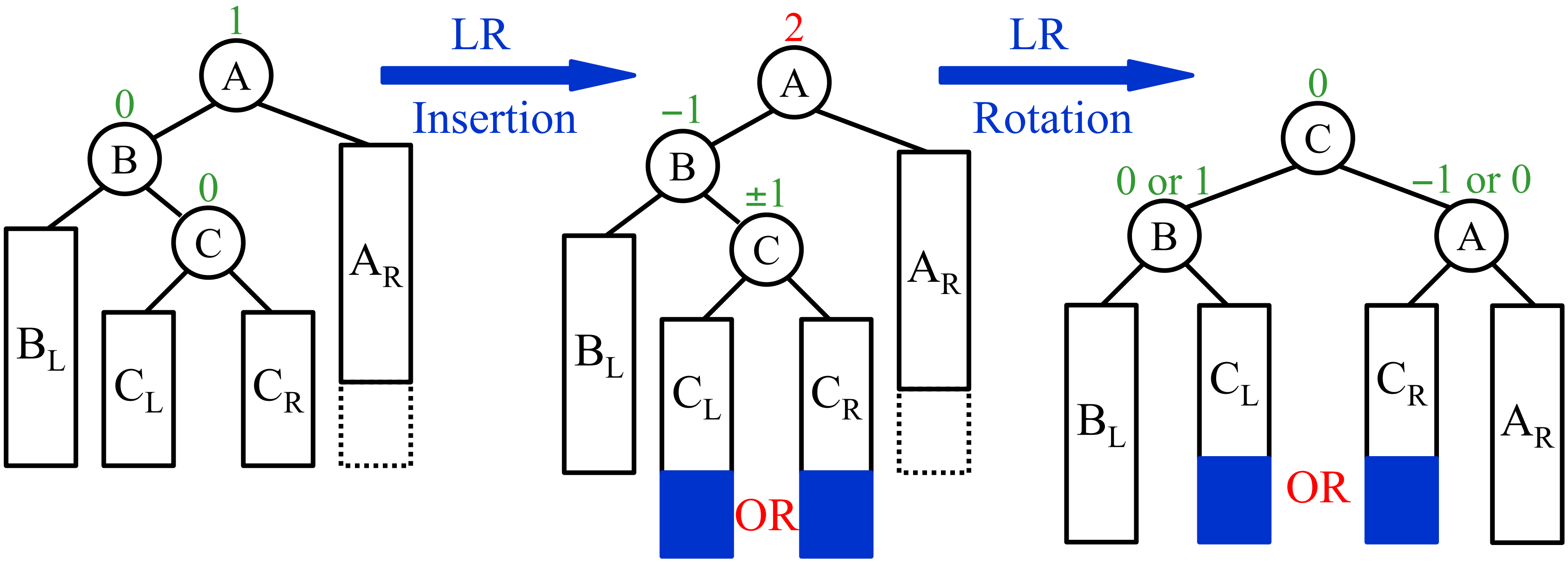

1
A
0
B
0
C
B_L
C_L
C_R
A_R
LR
Insertion
2
A
−1
B
±1
C
B_L
C_L
C_R
A_R
OR



LR

Rotation

In general:

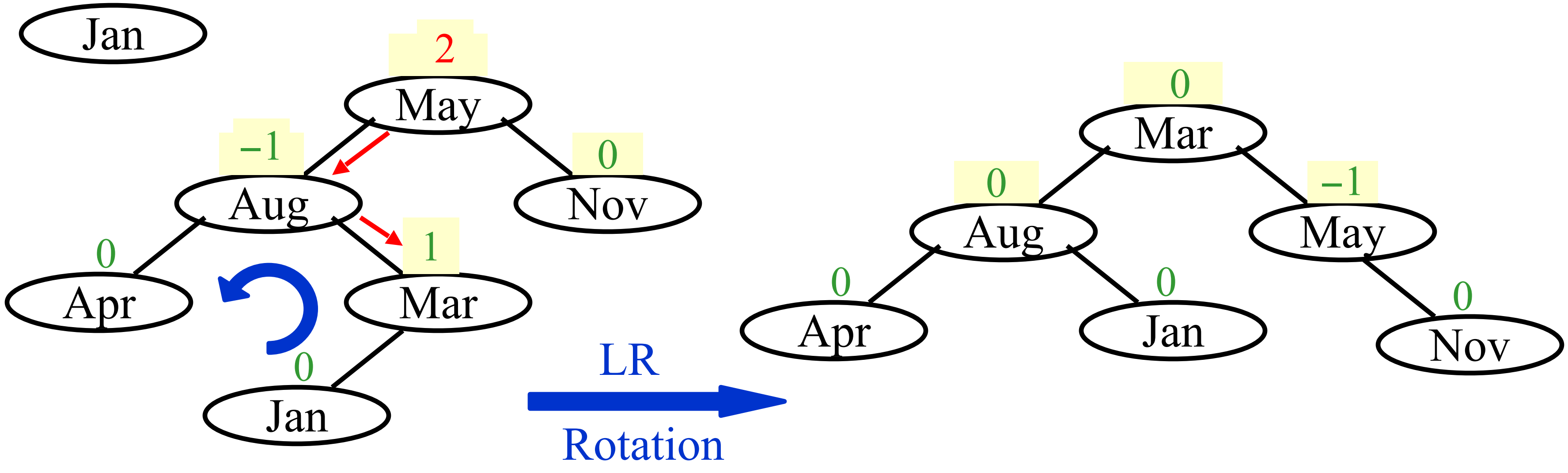


LR

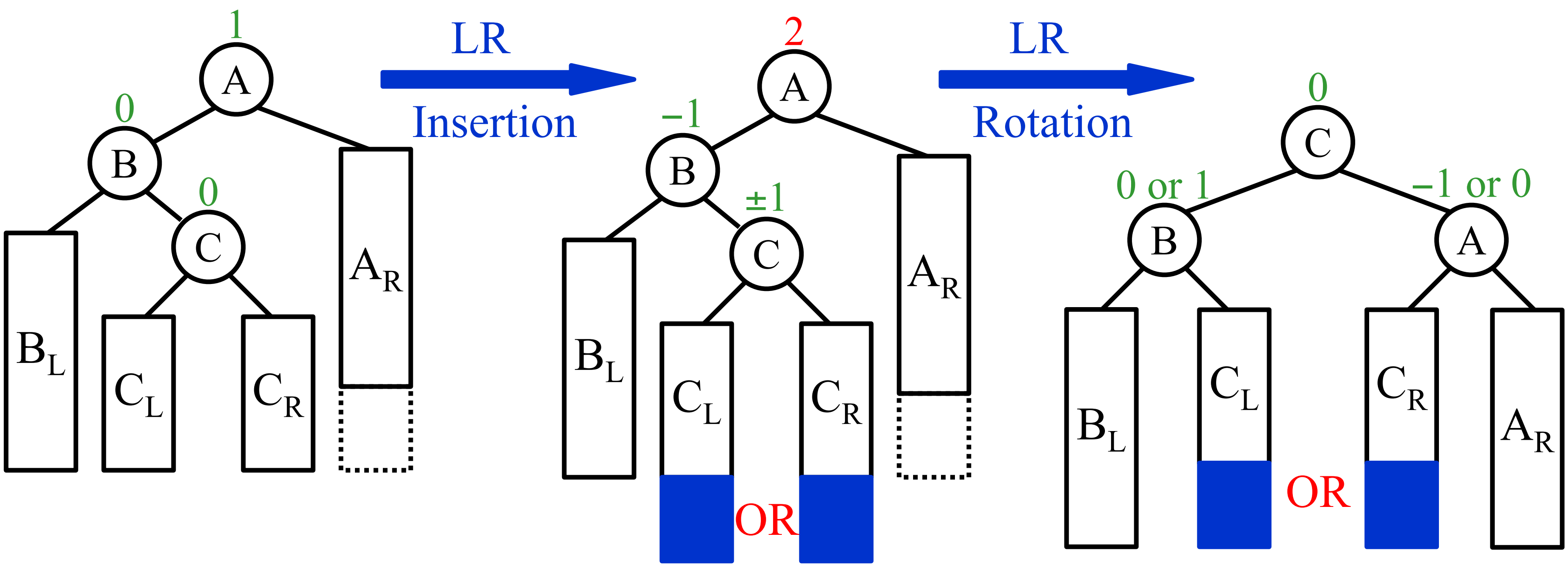
Insertion

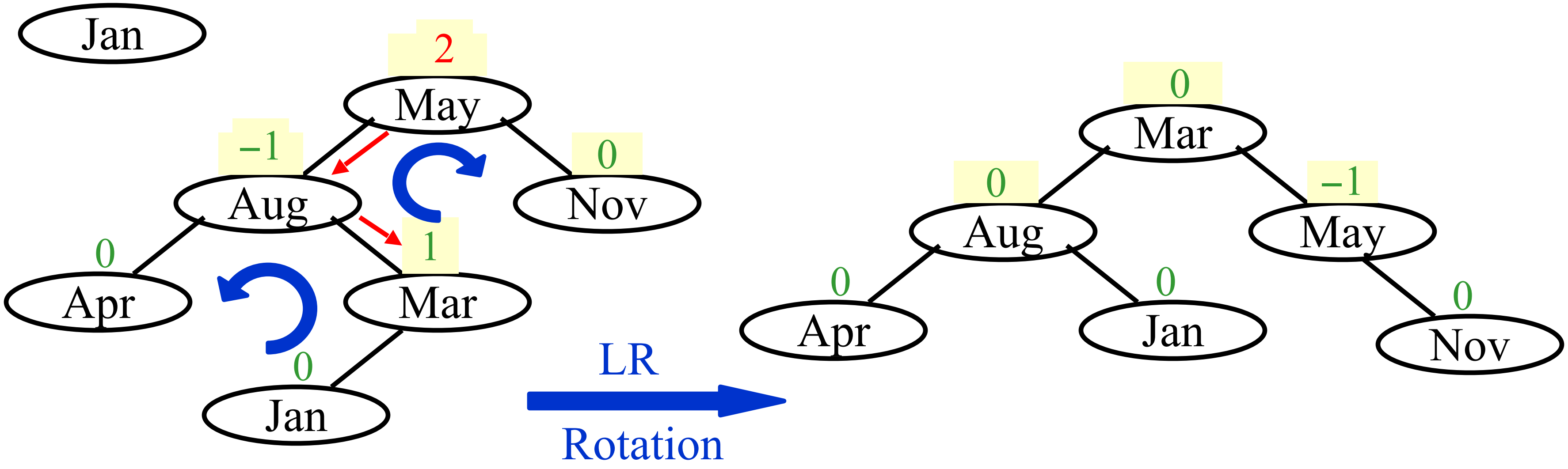
LR

Rotation

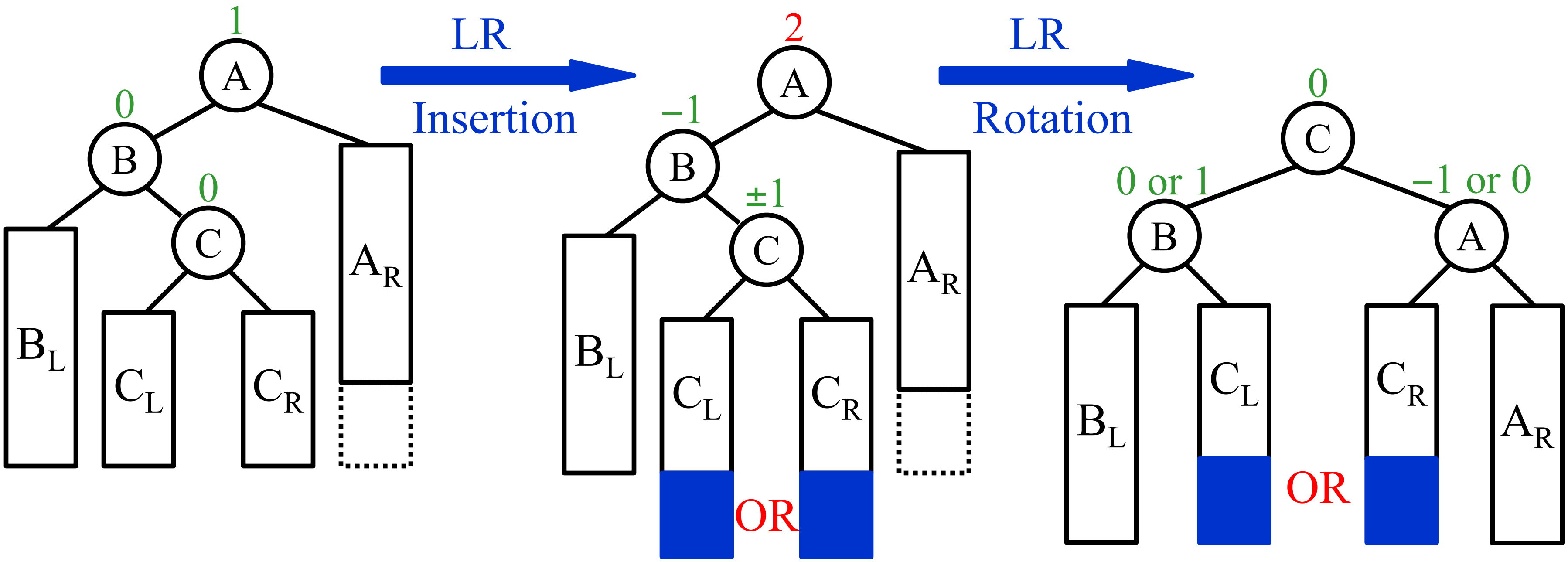

Jan
2
May
−1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
−1
May
0
Apr
0
Jan
0
Nov

In general:

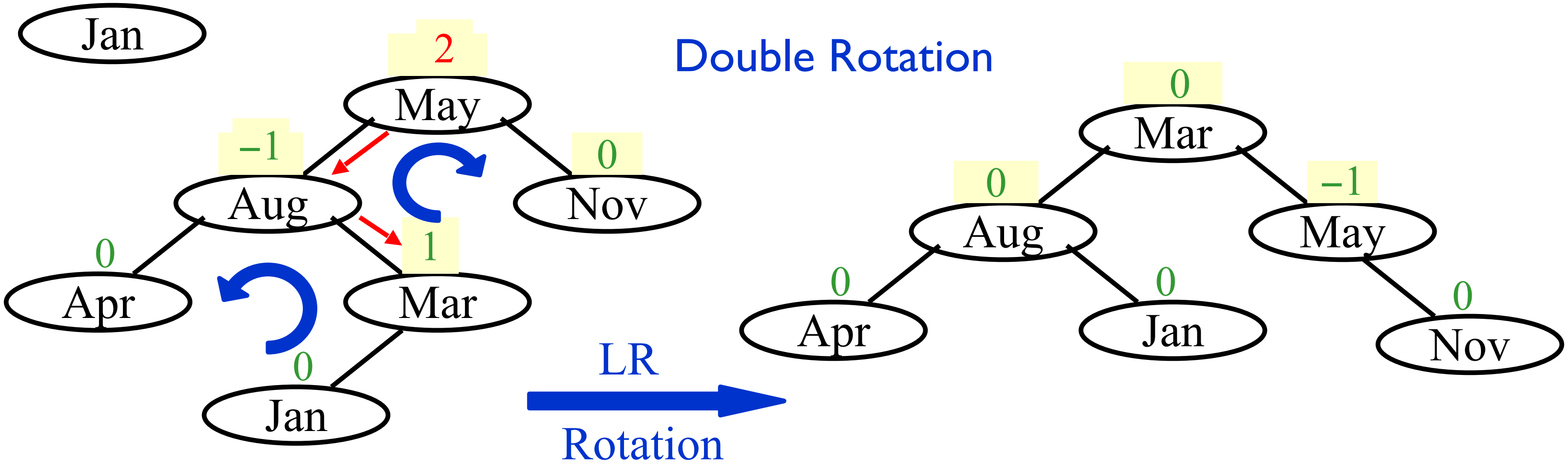

1
A
0
B
0
C
B_L
C_L
C_R
A_R
LR
Insertion
2
A
−1
B
±1
C
B_L
C_L
C_R
A_R
OR
LR
Rotation
0
C
0 or 1
B
−1 or 0
A
B_L
C_L
C_R
A_R
OR


Jan
2
May
−1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
LR
Rotation
0
Mar
0
Aug
−1
May
0
Apr
0
Jan
0
Nov

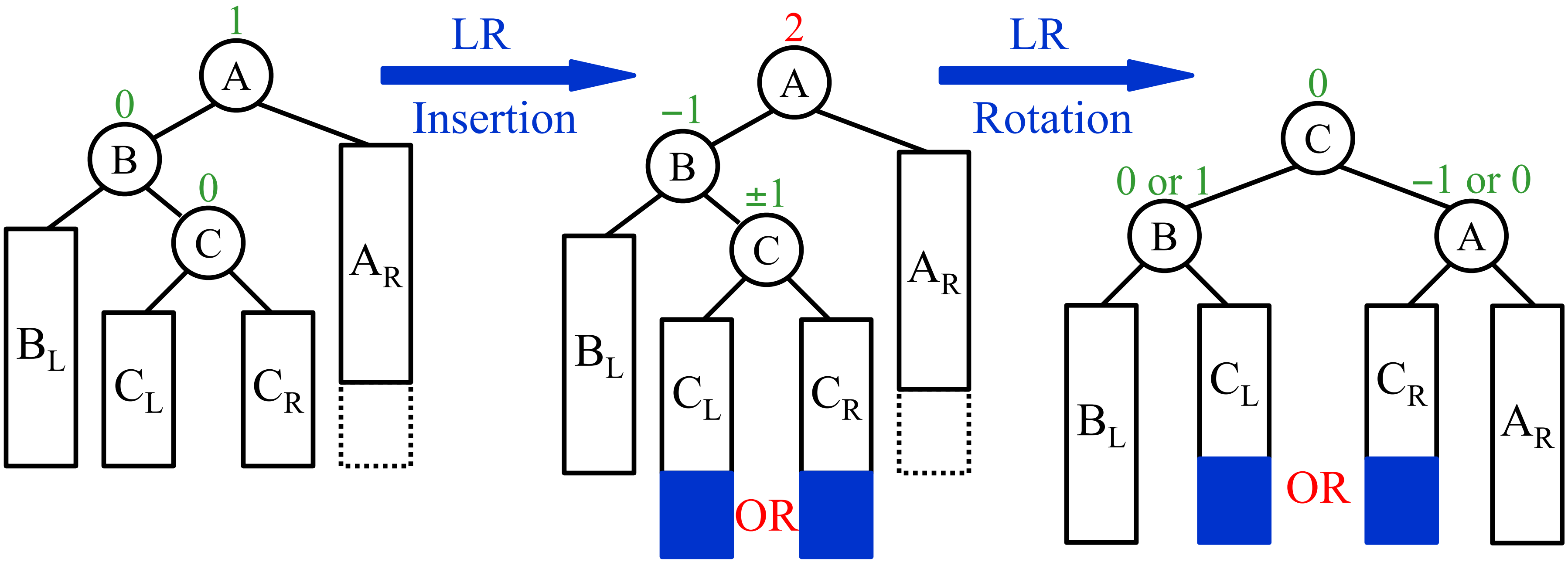
In general:

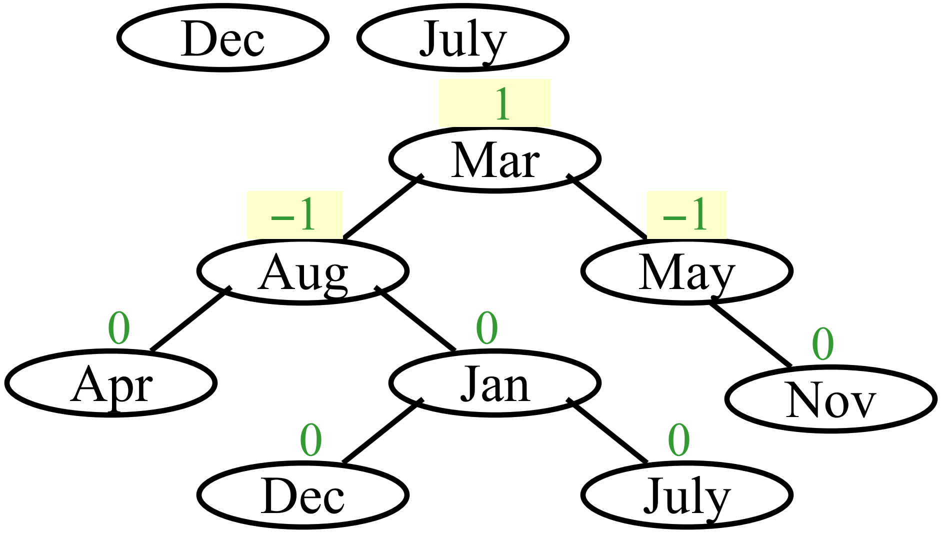

1
A
0
B
0
C
B_L
C_L
C_R
A_R
LR
Insertion
2
A
−1
B
±1
C
B_L
C_L
C_R
A_R
OR
LR
Rotation
0
C
0 or 1
B
−1 or 0
A
B_L
C_L
C_R
A_R
OR

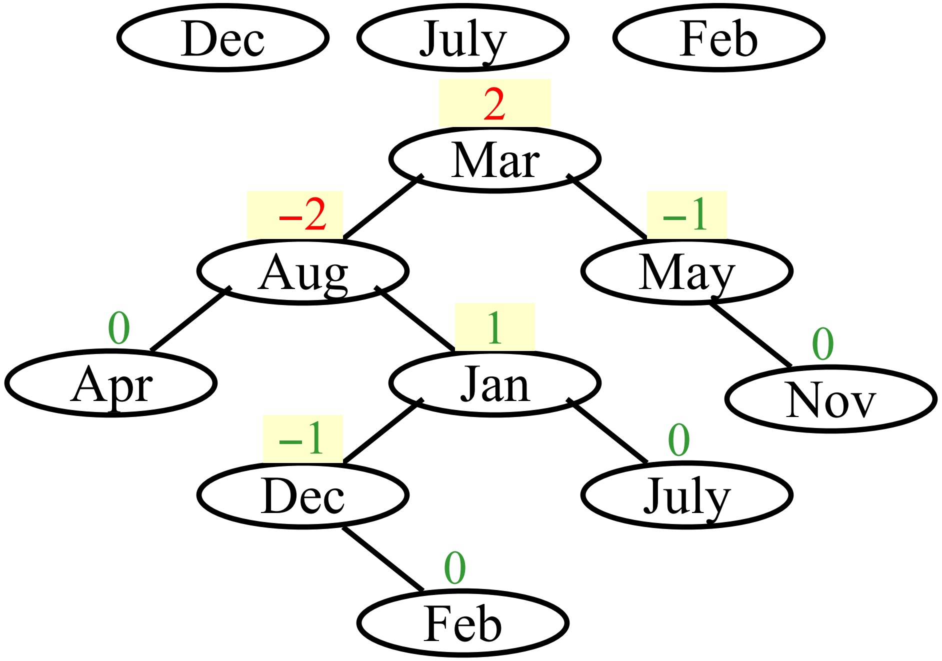
Double Rotation

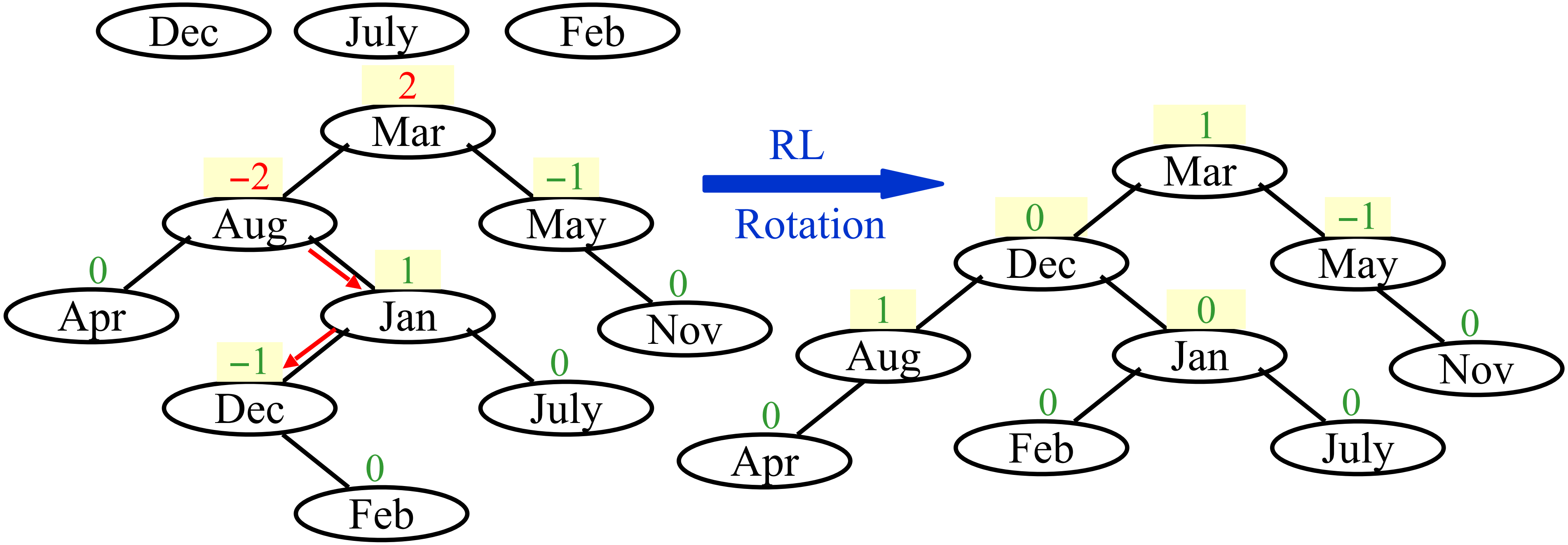


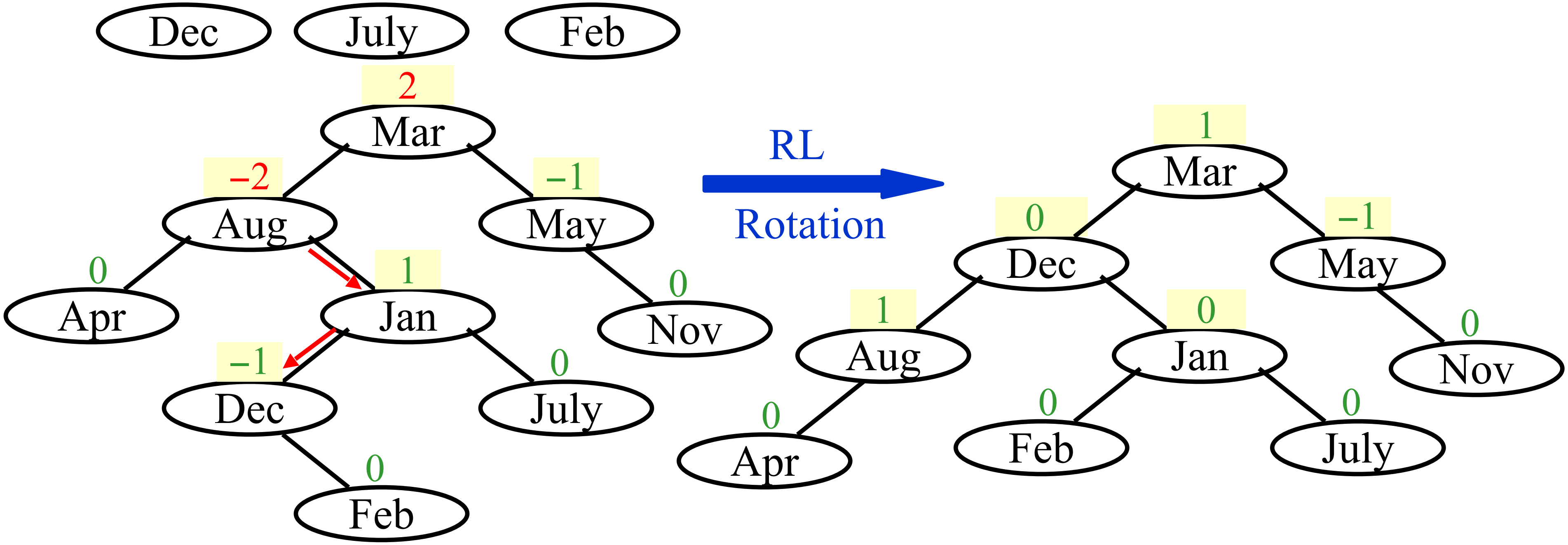
In general:



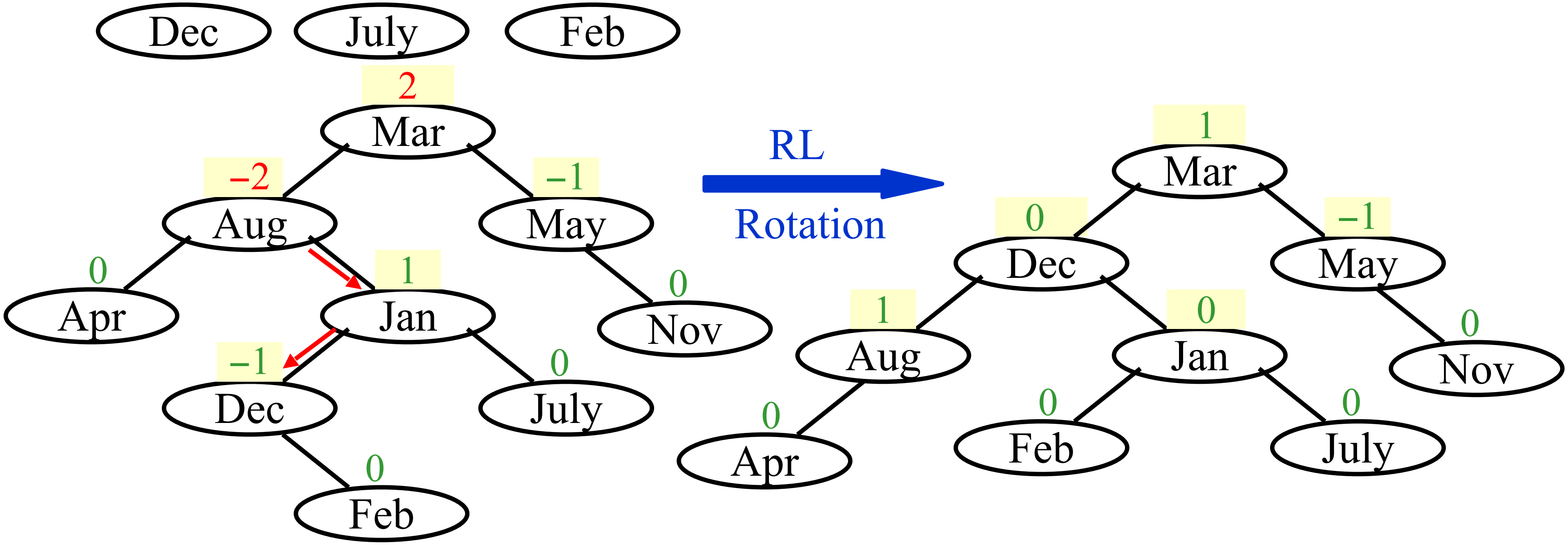

Dec
July
1
Mar
−1
Aug
−1
May
0
Apr
0
Jan
0
Nov
0
Dec
0
July


Dec
July
Feb
2
Mar
–2
Aug
–1
May
0
Apr
1
Jan
0
Nov
–1
Dec
0
July
0
Feb

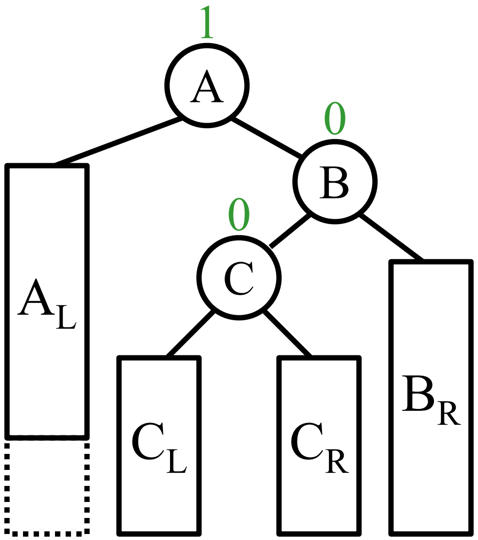

Dec
July
Feb
2
Mar
−2
Aug
−1
May
0
Apr
1
Jan
0
Nov
−1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
−1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

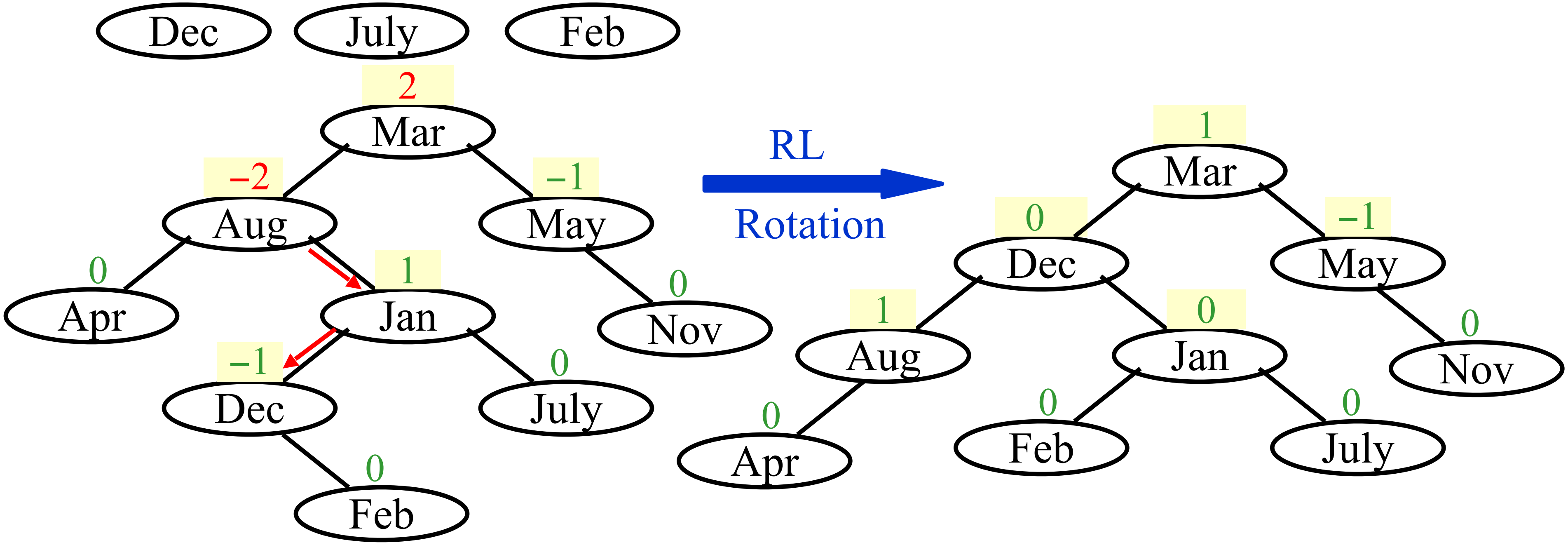

Dec
July
Feb
2
Mar
–2
Aug
–1
May
0
Apr
1
Jan
0
Nov
–1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
–1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

In general:


Dec
July
Feb
2
Mar
−2
Aug
−1
May
0
Apr
1
Jan
0
Nov
−1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
−1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

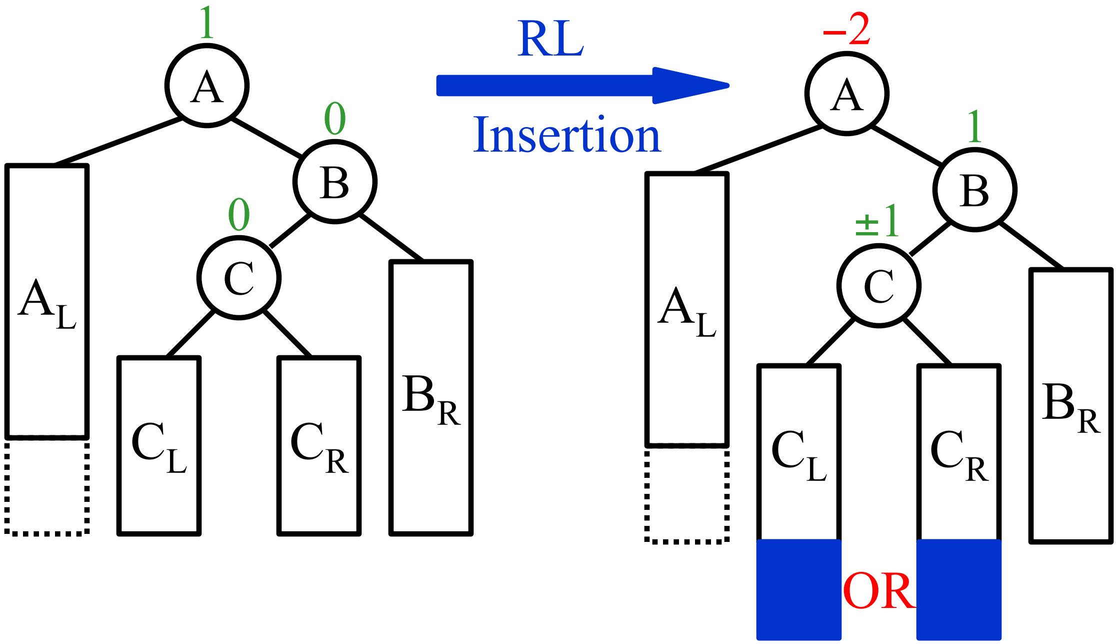
In general:


1
A
0
B
0
C
A_L
C_L
C_R
B_R

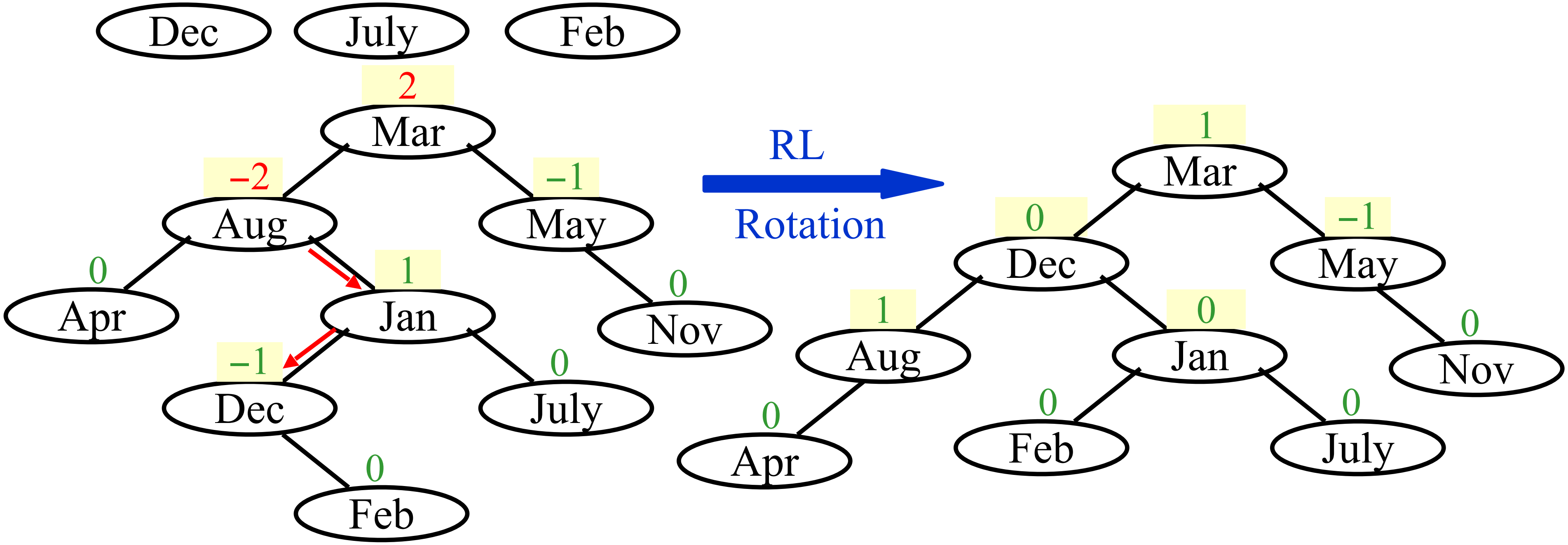


RL Rotation

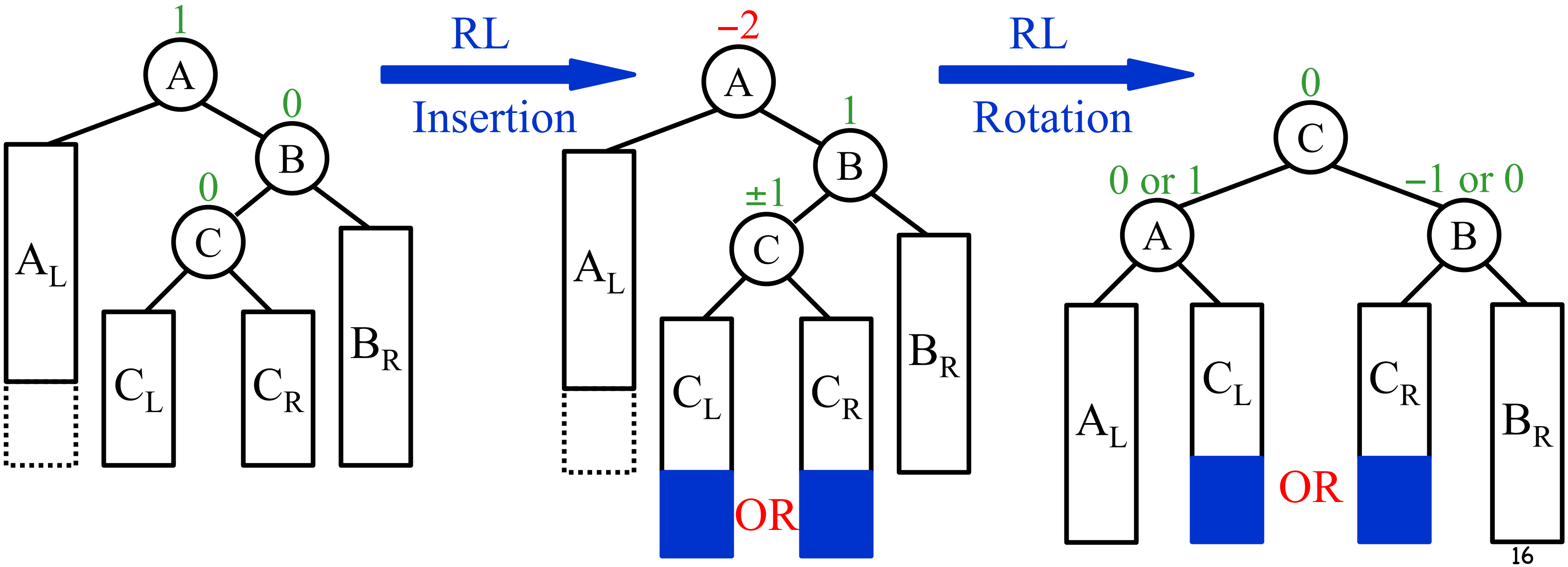
In general:

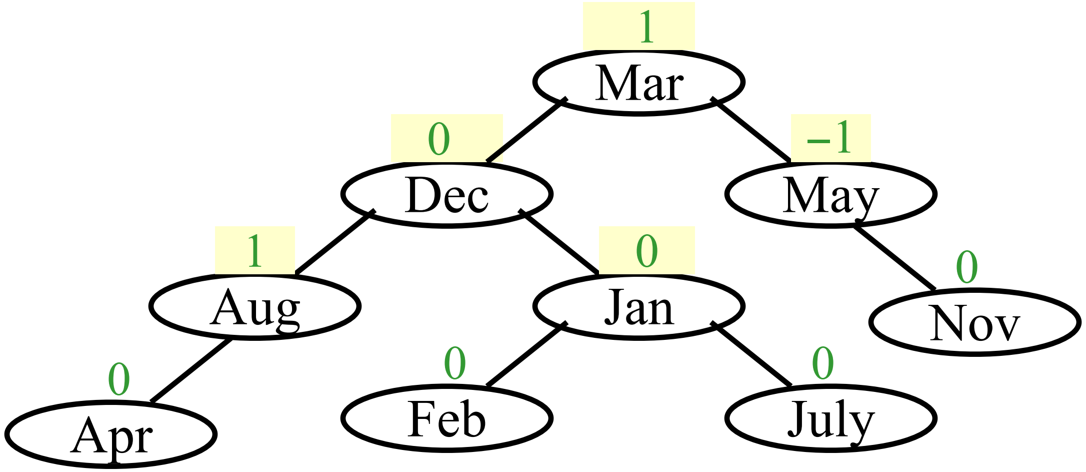


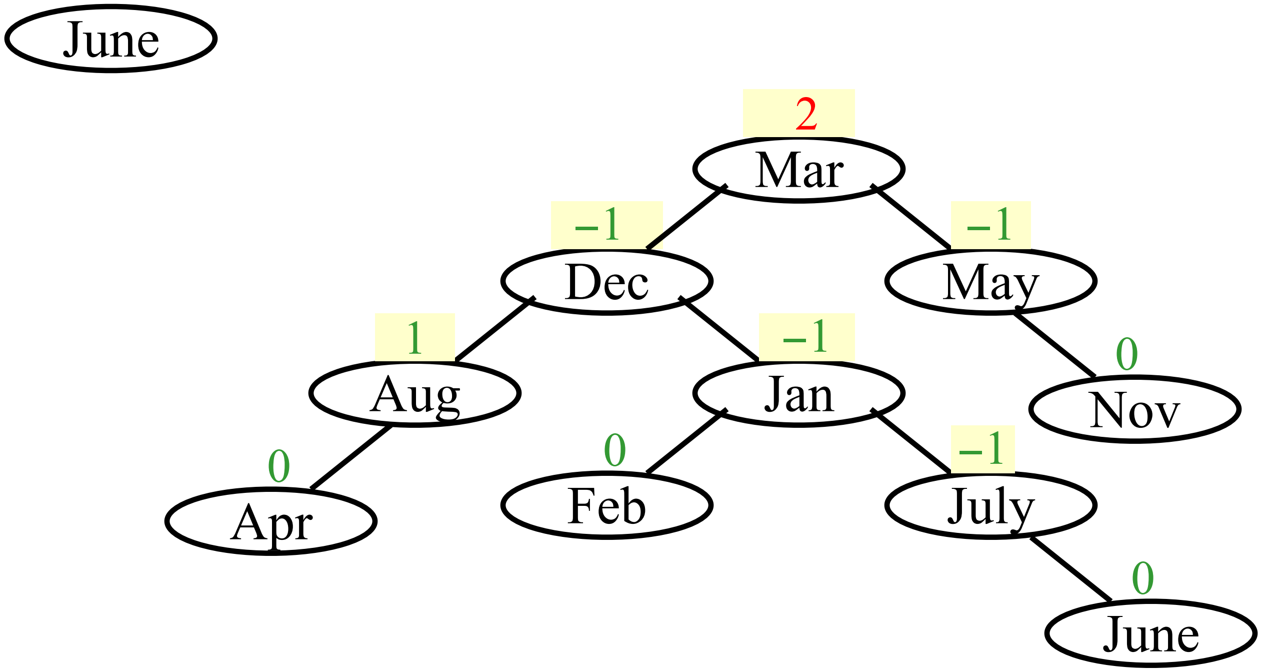
RL Insertion

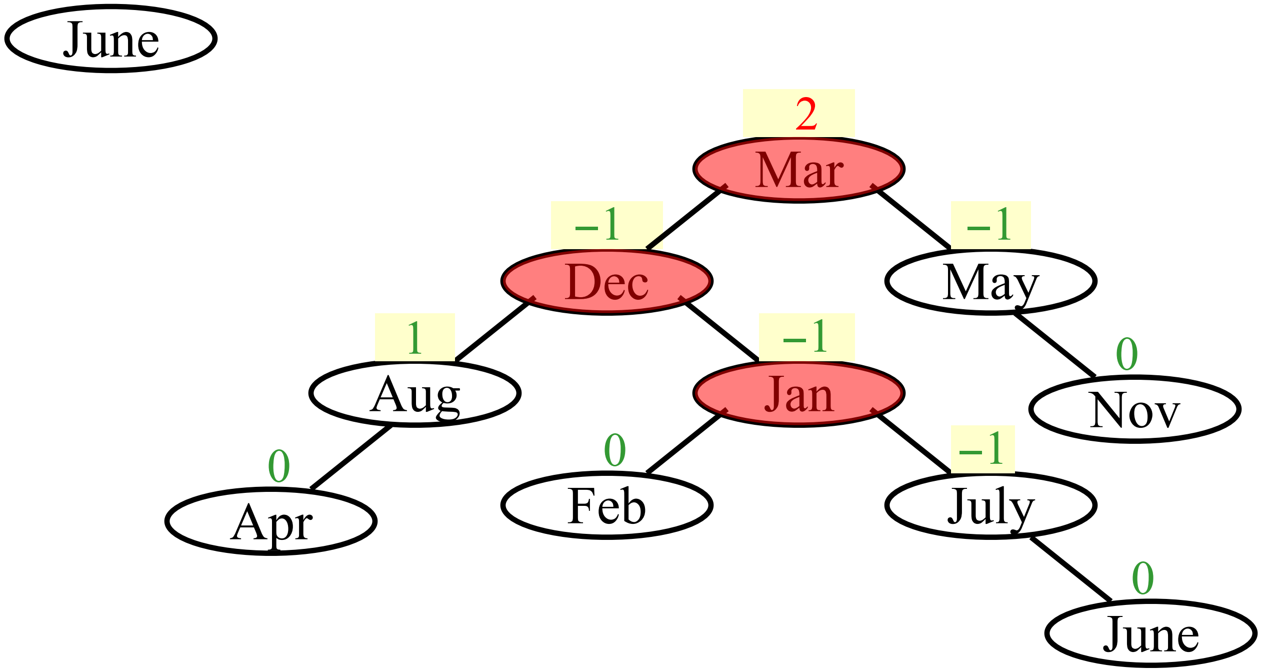

Dec
July
Feb
2
Mar
–2
Aug
–1
May
0
Apr
1
Jan
0
Nov
–1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
–1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

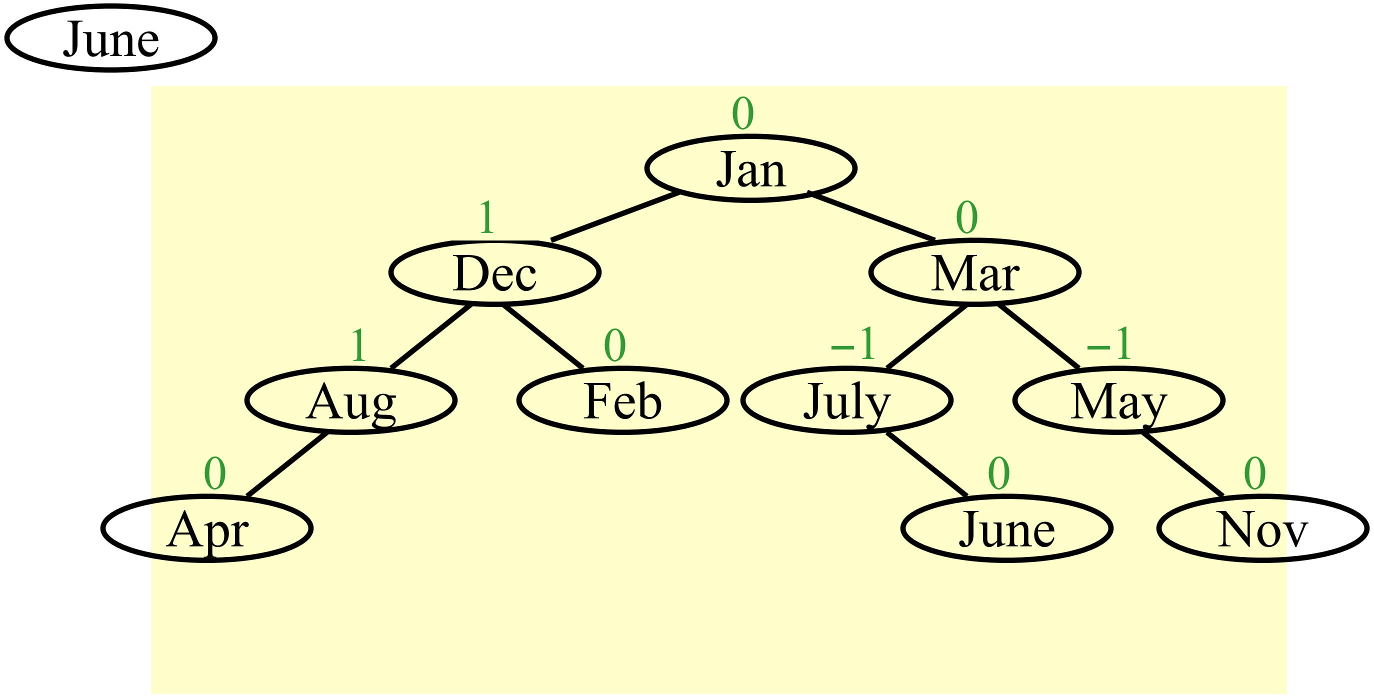
In general:

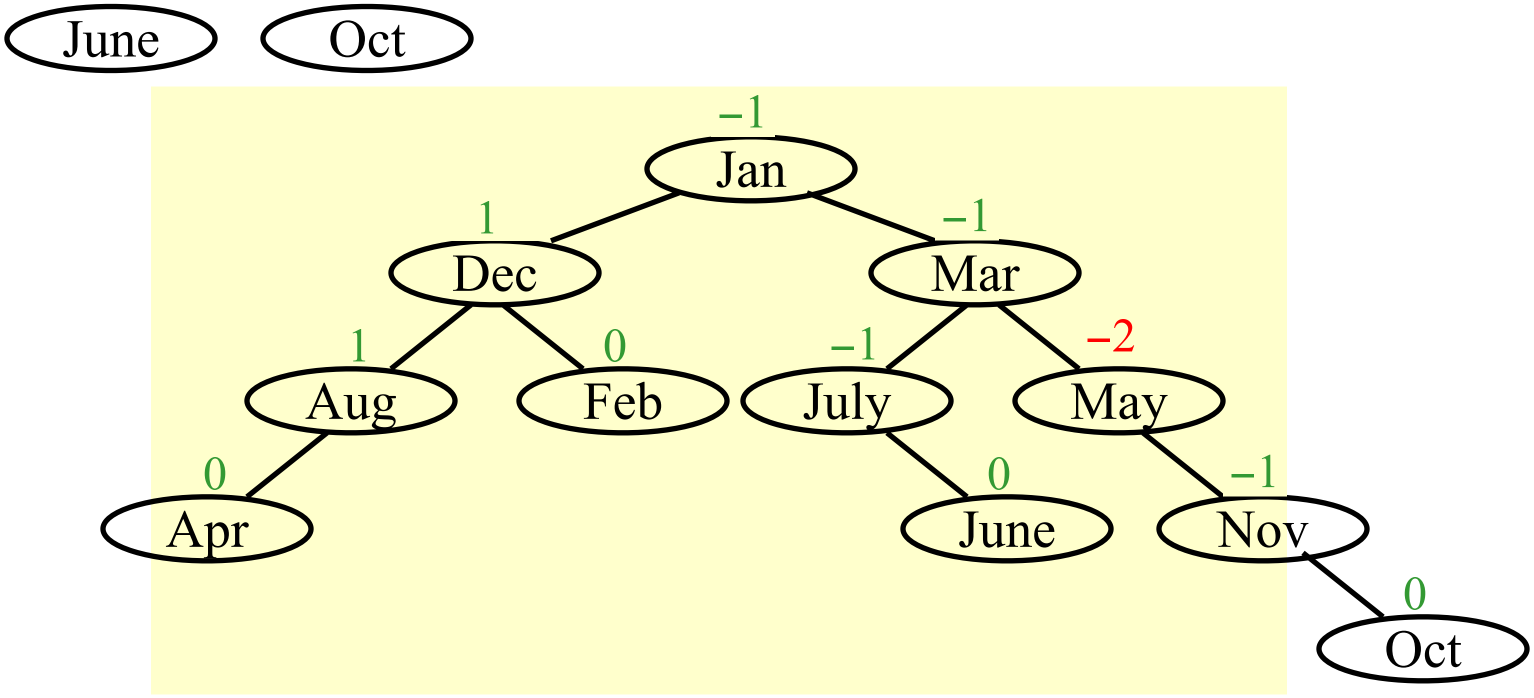

1
A
0
B
0
C
A_L
C_L
C_R
B_R
RL
Insertion
–2
A
1
B
±1
C
A_L
C_L
C_R
B_R
OR
RL
Rotation
0
C
0 or 1
A
–1 or 0
B
A_L
C_L
C_R
B_R
OR

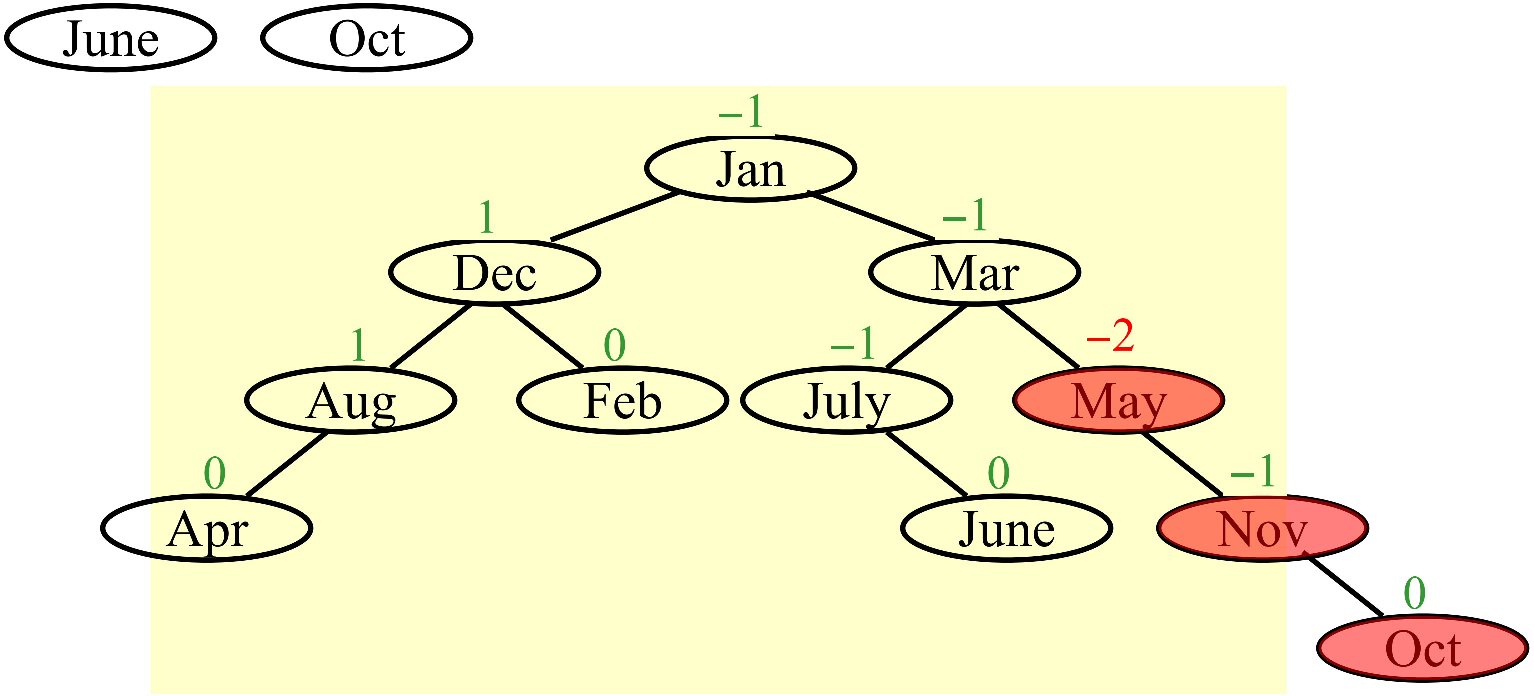

1
Mar
0
Dec
–1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

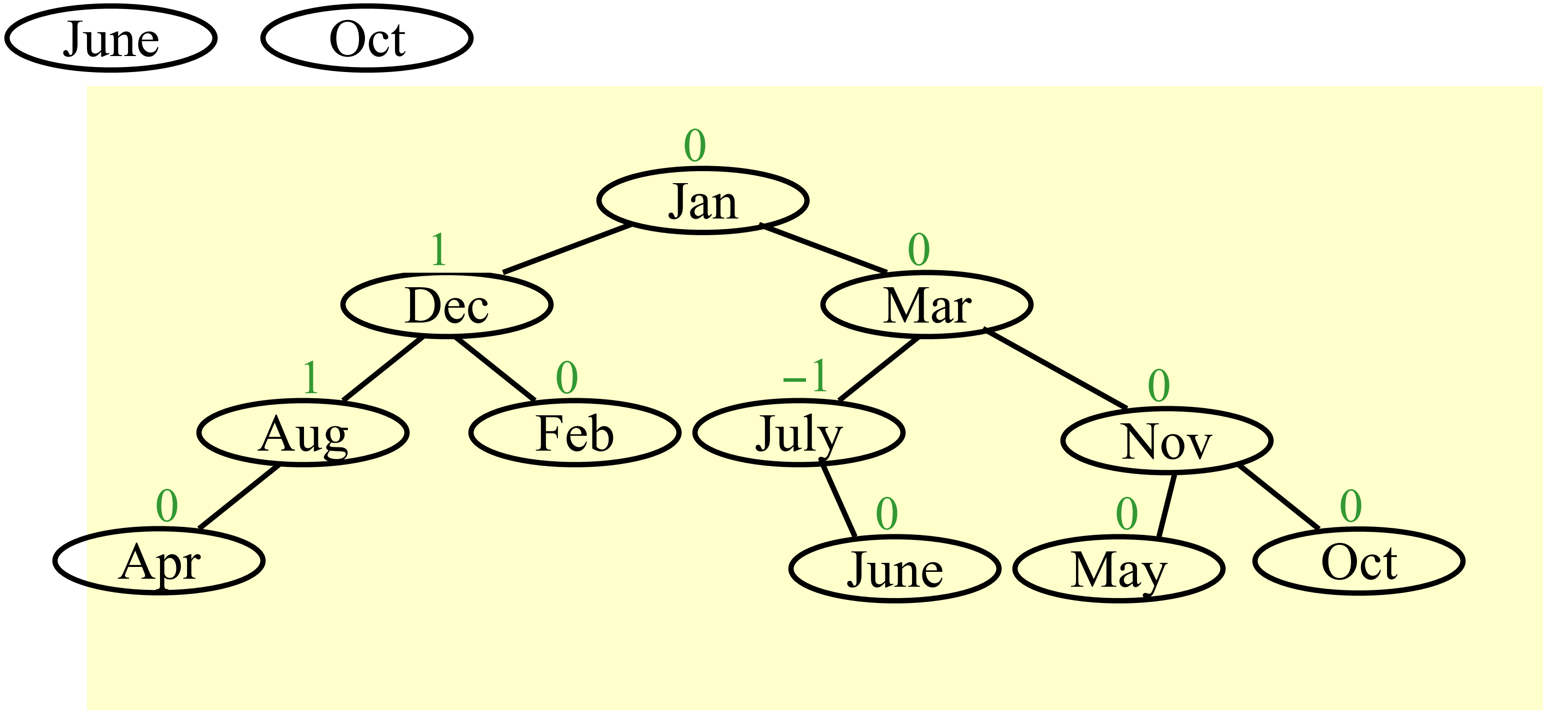

June
2
Mar
–1
Dec
–1
May
1
Aug
–1
Jan
0
Nov
0
Apr
0
Feb
–1
July
0
June

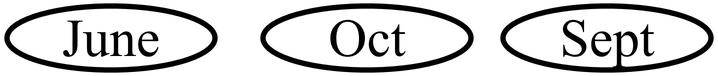

June
2
Mar
−1
Dec
−1
May
1
Aug
−1
Jan
0
Nov
0
Apr
0
Feb
−1
July
0
June

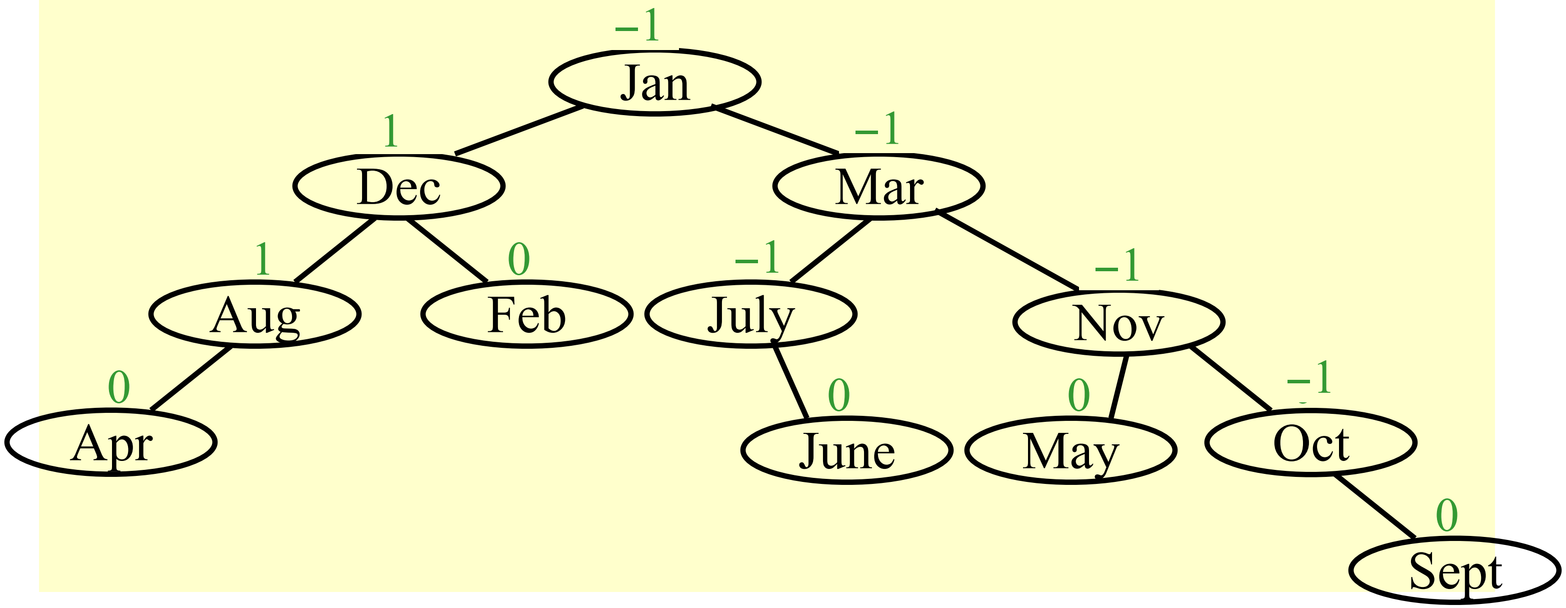

June
0
Jan
1
Dec
0
Mar
1
Aug
0
Feb
–1
July
–1
May
0
Apr
0
June
0
Nov

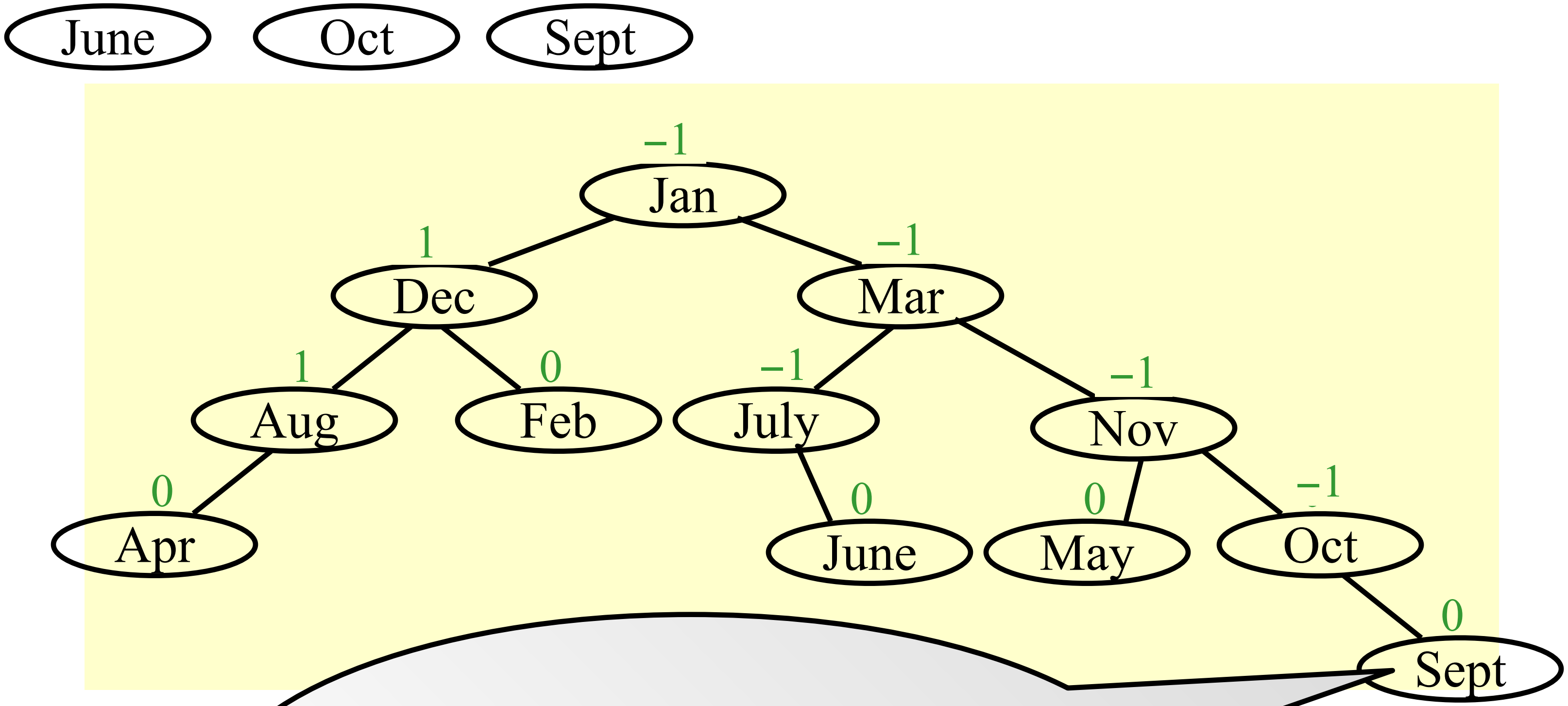

June
Oct
–1
Jan
1
Dec
–1
Mar
1
Aug
0
Feb
–1
July
–2
May
0
Apr
0
June
–1
Nov
0
Oct


June
Oct
–1
Jan
1
Dec
–1
Mar
1
Aug
0
Feb
–1
July
–2
May
0
Apr
0
June
–1
Nov
0
Oct

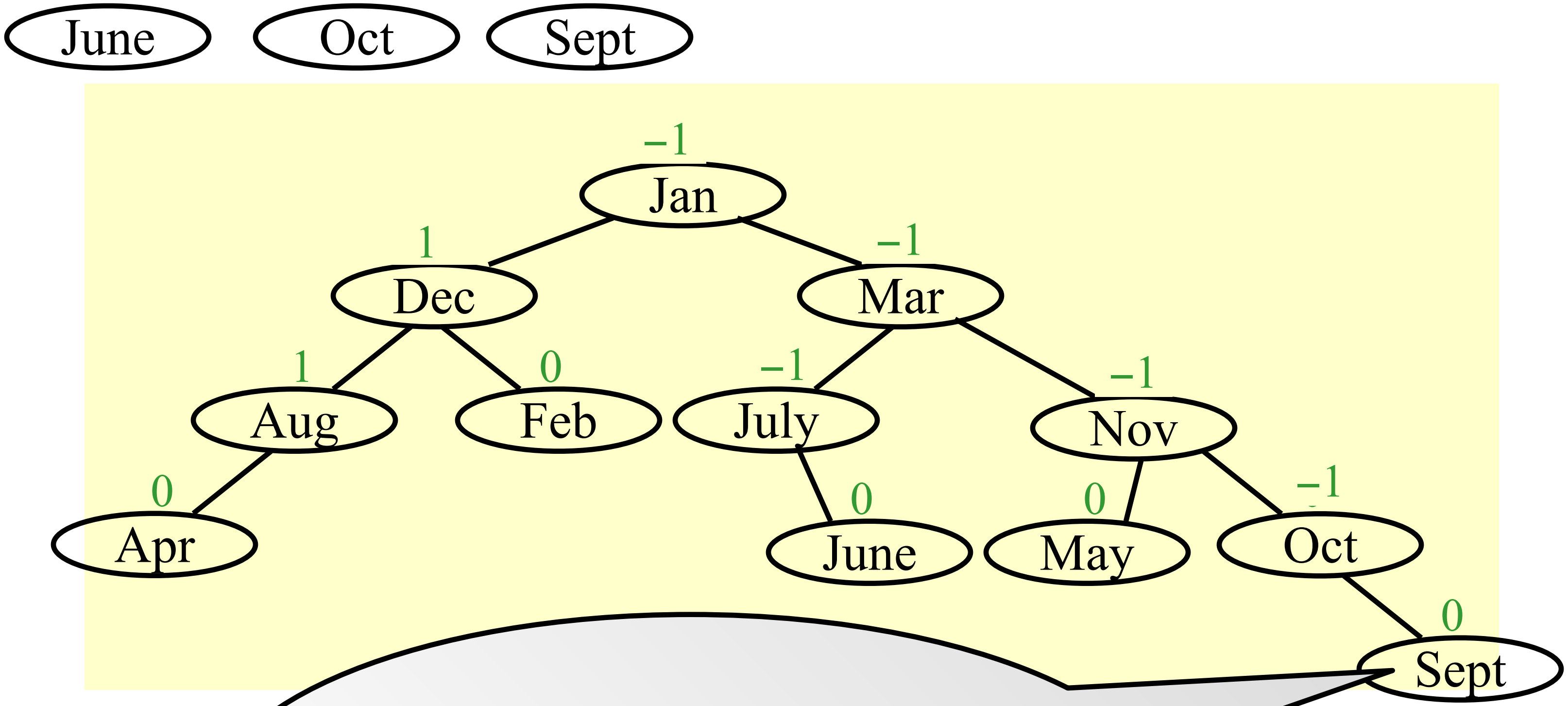

June
Oct
0
Jan
1
Dec
0
Mar
1
Aug
0
Feb
−1
July
0
Nov
0
Apr
0
June
0
May
0
Oct


June
Oct
Sept


−1
Jan
1
Dec
−1
Mar
1
Aug
0
Feb
−1
July
−1
Nov
0
Apr
0
June
0
May
−1
Oct
0
Sept



Note: Several bf's
might be changed even if
we don't need to reconstruct
the tree.



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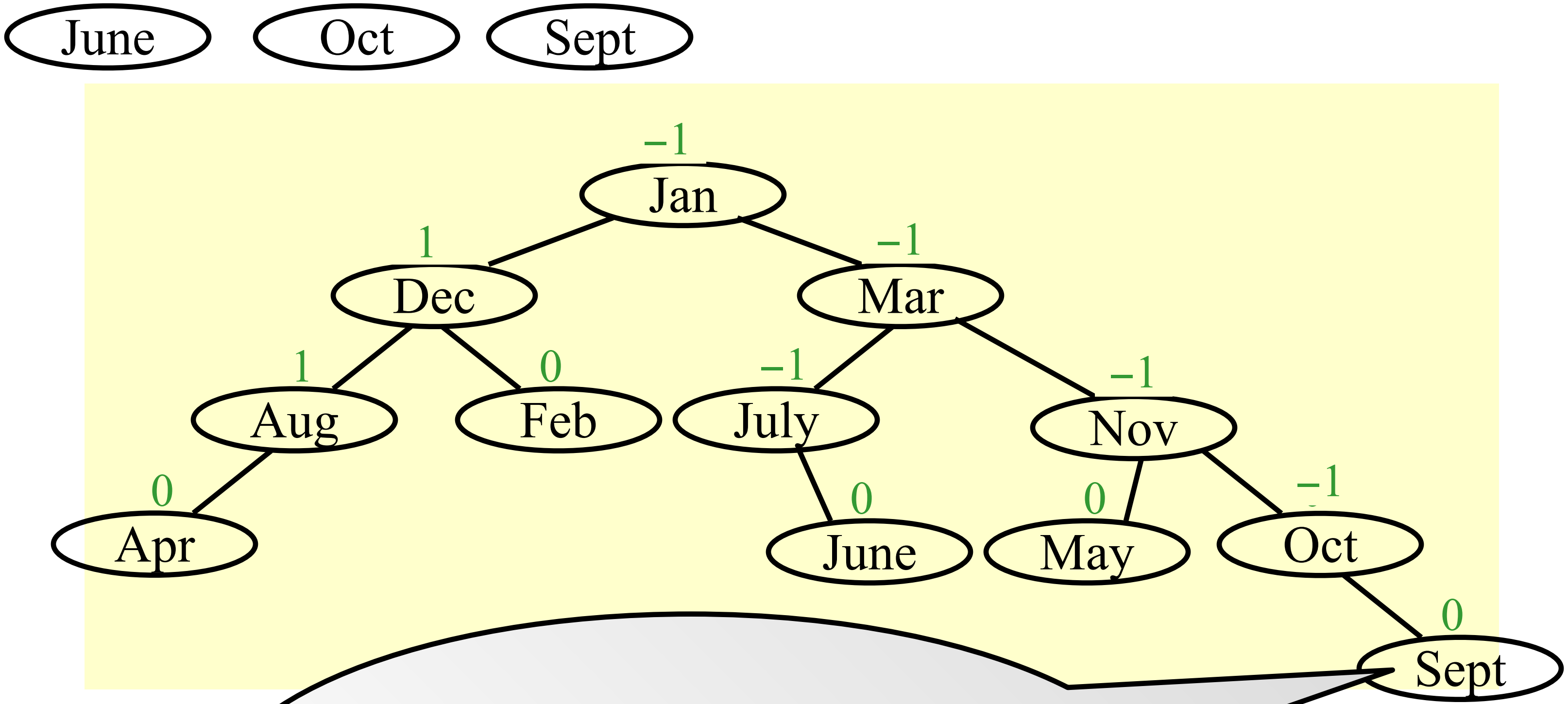
Another option is to keep a *height* field for each node.



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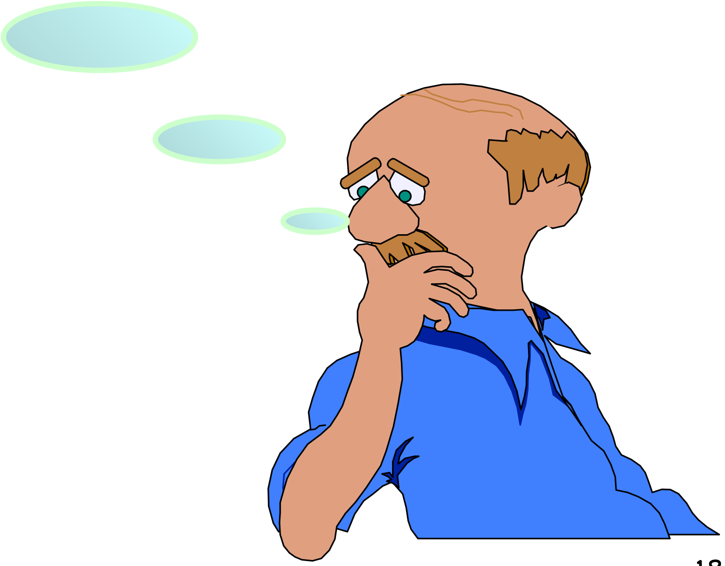
Another option is to keep a *height* field for each node.

Read the declaration and functions in [Weiss] Figures 4.42 – 4.48

One last question:
Obviously we have T_p = O(h)
where h is the height of the tree.
But h = ?



Let n_h be the minimum number of nodes in a height-balanced tree of height h. What does the tree look like?

The worst case for AVL tree of height h.

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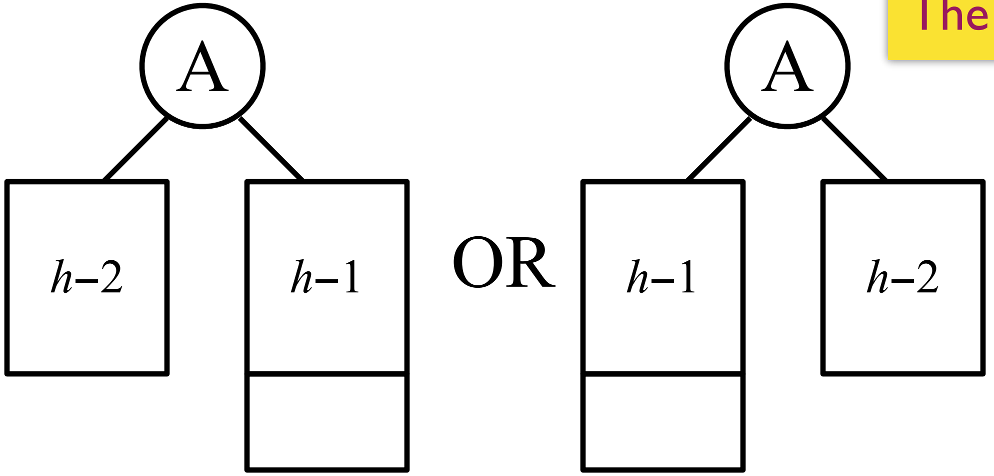

$$\Rightarrow \quad n_h = n_{h-1} + n_{h-2} + 1$$

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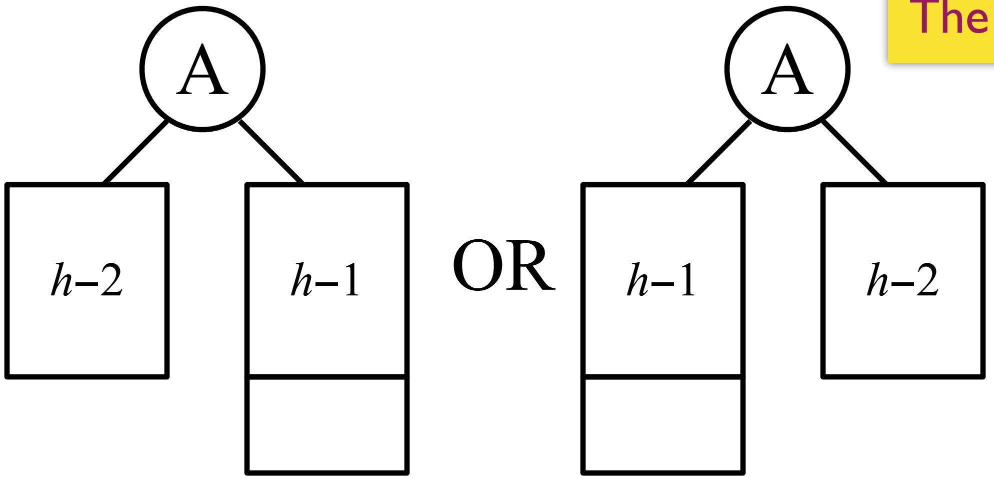

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Fibonacci numbers:

$$F_0 = 0,\ F_1 = 1,\ F_i = F_{i-1} + F_{i-2} \quad \text{for } \ i > 1$$

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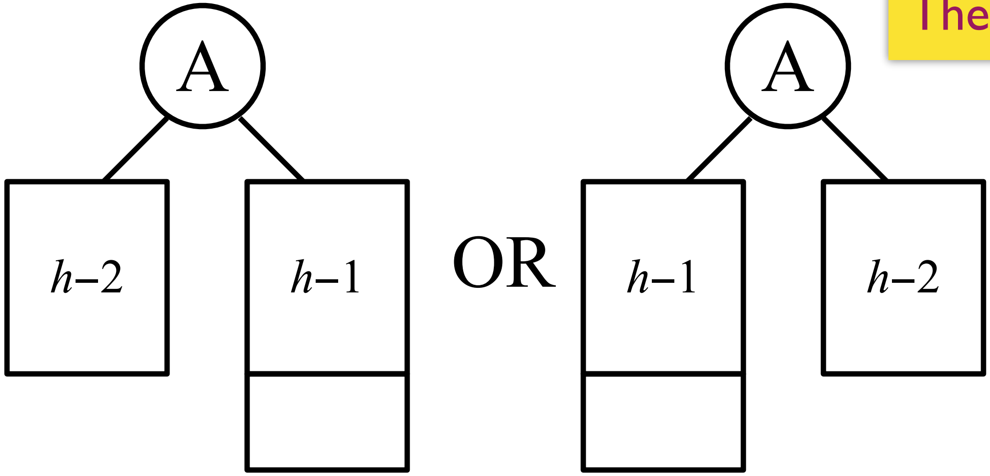

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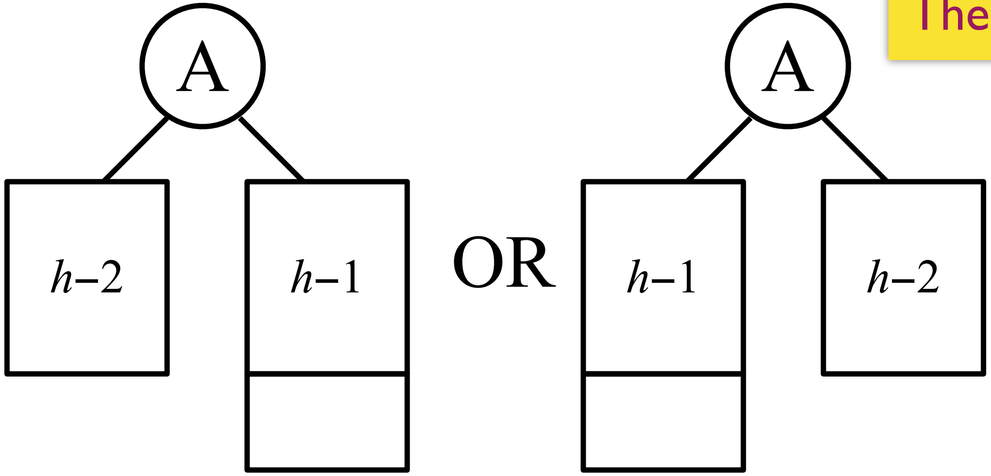

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Fibonacci number theory gives that

$$F_i \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i$$

3

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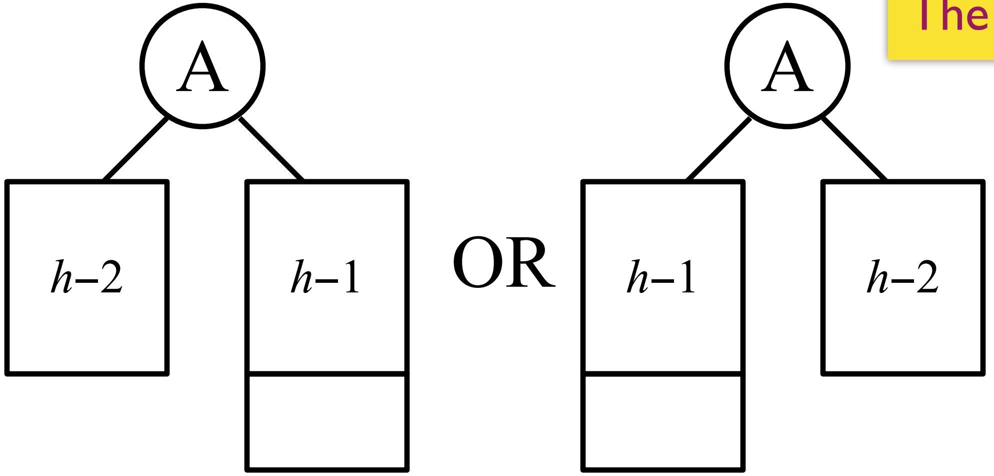

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$$\Rightarrow \quad n_h \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+3} - 1 \qquad \Rightarrow \quad h = O(\ln n)$$

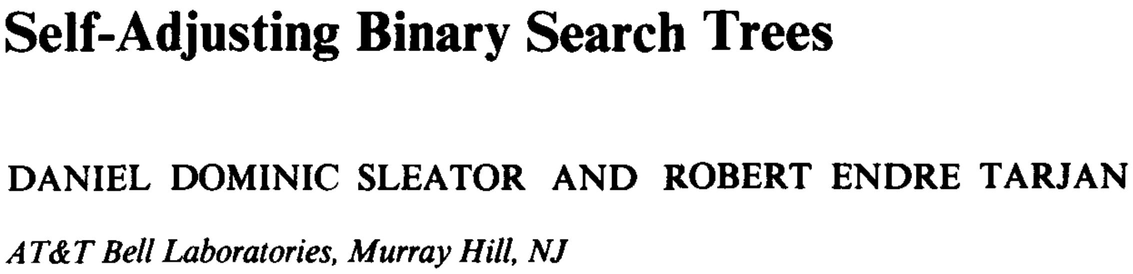
Outline: Balanced Binary Search Trees (1)

- Binary search trees
- AVL trees
- Splay trees
- Amortized analysis
- Take-home messages

Splay Trees (1985)

Daniel Sleator

Robert Tarjan



Self-Adjusting Binary Search Trees

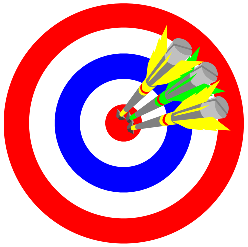
DANIEL DOMINIC SLEATOR AND ROBERT ENDRE TARJAN

AT&T Bell Laboratories, Murray Hill, NJ

Figure courtesy: https://csd.cmu.edu/people/faculty/daniel-sleator
https://en.wikipedia.org/wiki/Robert_Tarjan

Splay Trees

Splay Trees

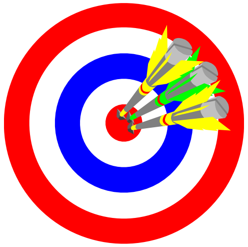


Target : Any M consecutive tree operations starting from an empty tree take at most $O(M \log N)$ time.

Splay Trees

Target : Any *M* consecutive tree operations starting from an empty tree take at most O(*M* log *N*) time.

Splay Trees



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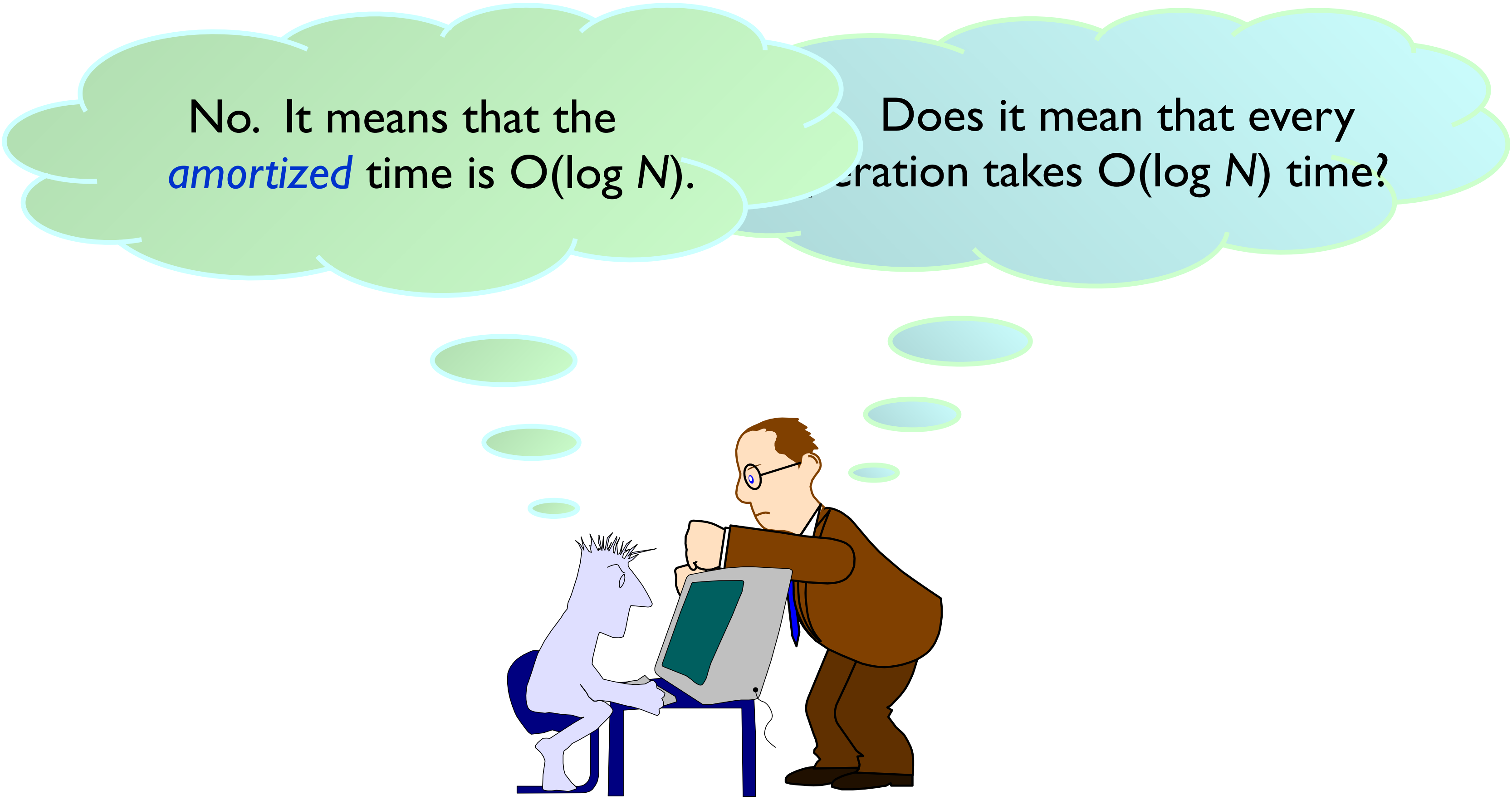


Does it mean that every eration takes O(log N) time?

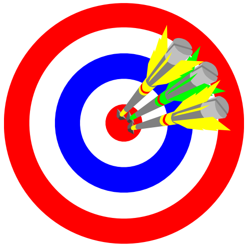
No. It means that the *amortized* time is O(log N).

Splay Trees

Target : Any M consecutive tree operations starting from an empty tree take at most O(M log N) time.

So a single operation might still take O(N) time? Then what's the point?
No. It means that the *amortized* time is O(log N).

Splay Trees



Target : Any *M* consecutive tree operations starting from an empty tree take at most O(*M* log *N*) time.

The bound is weaker.
But the effect is the same:
There are no bad input sequences.

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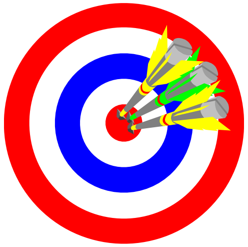
Splay Trees

Target : Any *M* consecutive tree operations starting from an empty tree take at most O(*M* log *N*) time.

The bound is weaker.
But the effect is the sam
There are no bad input seque

But if one node takes O(*N*) time to access, we can keep accessing it for *M* times, can't we?

Splay Trees



Target : Any *M* consecutive tree operations starting from an empty tree take at most O(*M* log *N*) time.

Surely we can – that only means that whenever a node is accessed, it must be moved. Otherwise visiting a bad node repeatedly leads to bad performance

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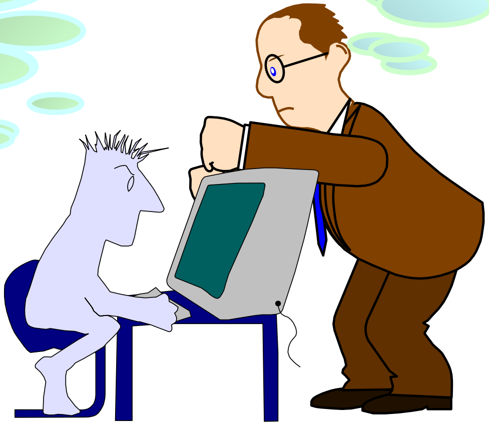

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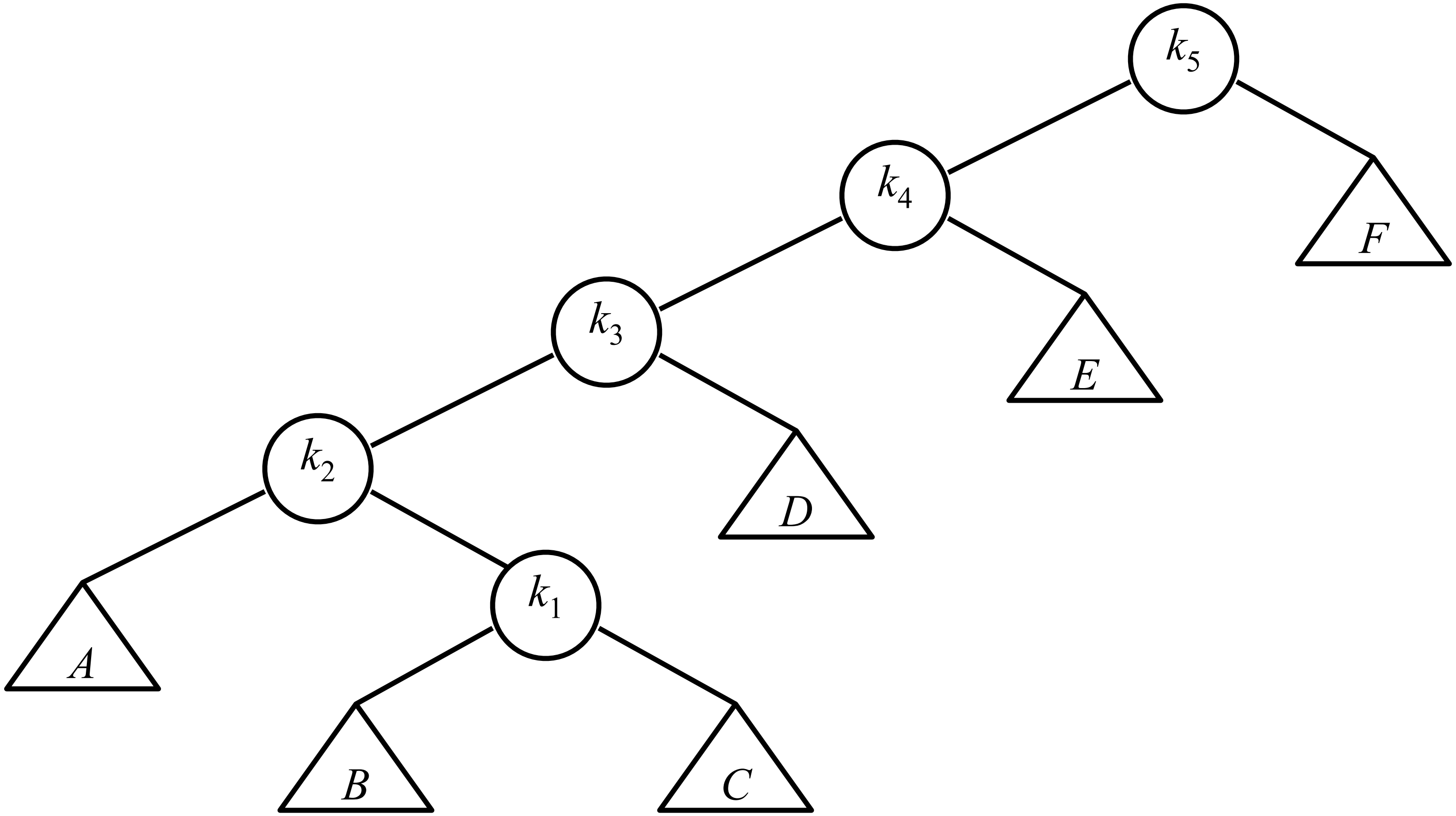
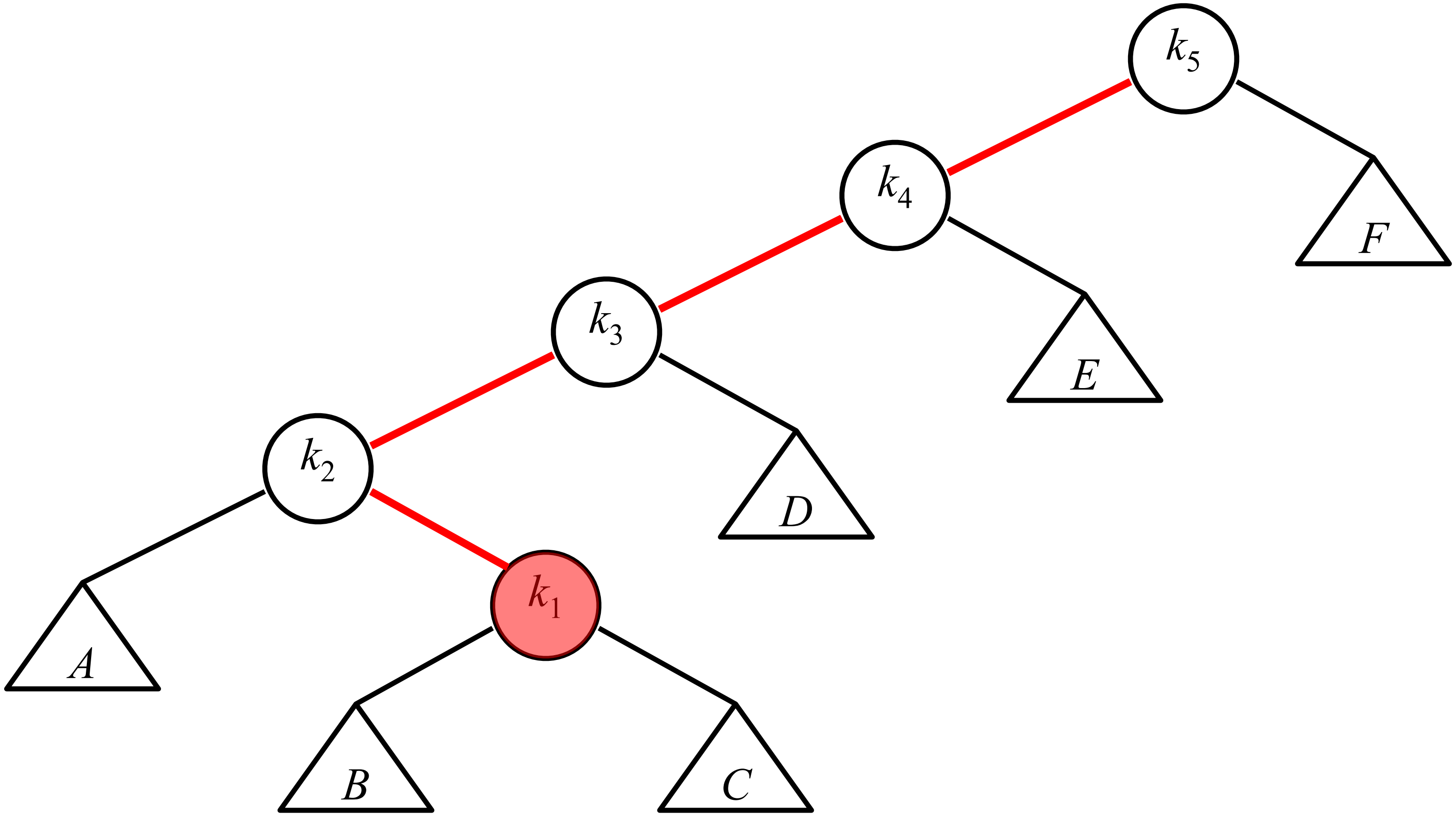
Idea : After a node is accessed, it is pushed to the root by a series of AVL tree rotations.


k_5
k_4
k_3
k_2
k_1
A
B
C
D
E
F


k_5
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F
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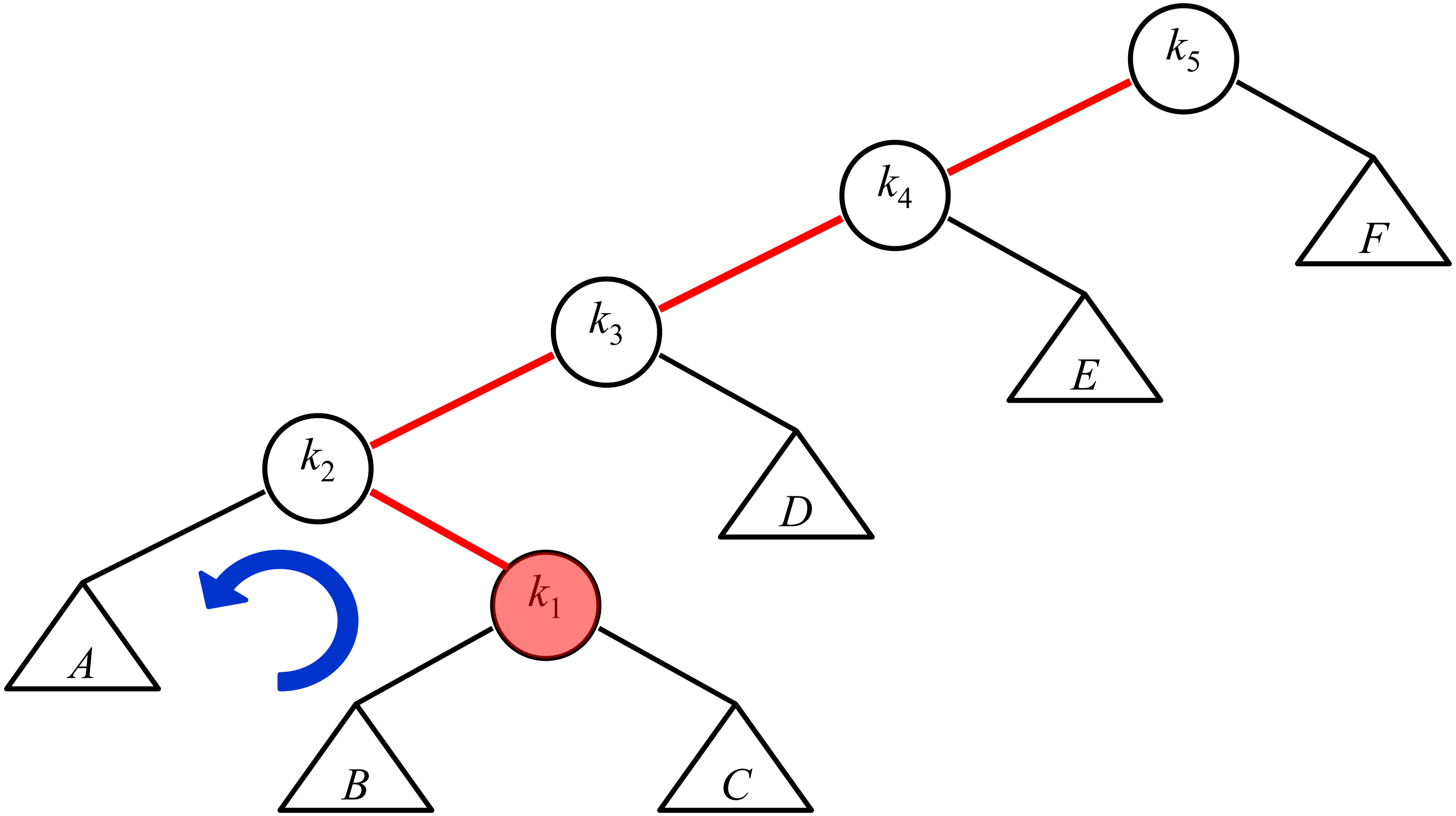
k_5
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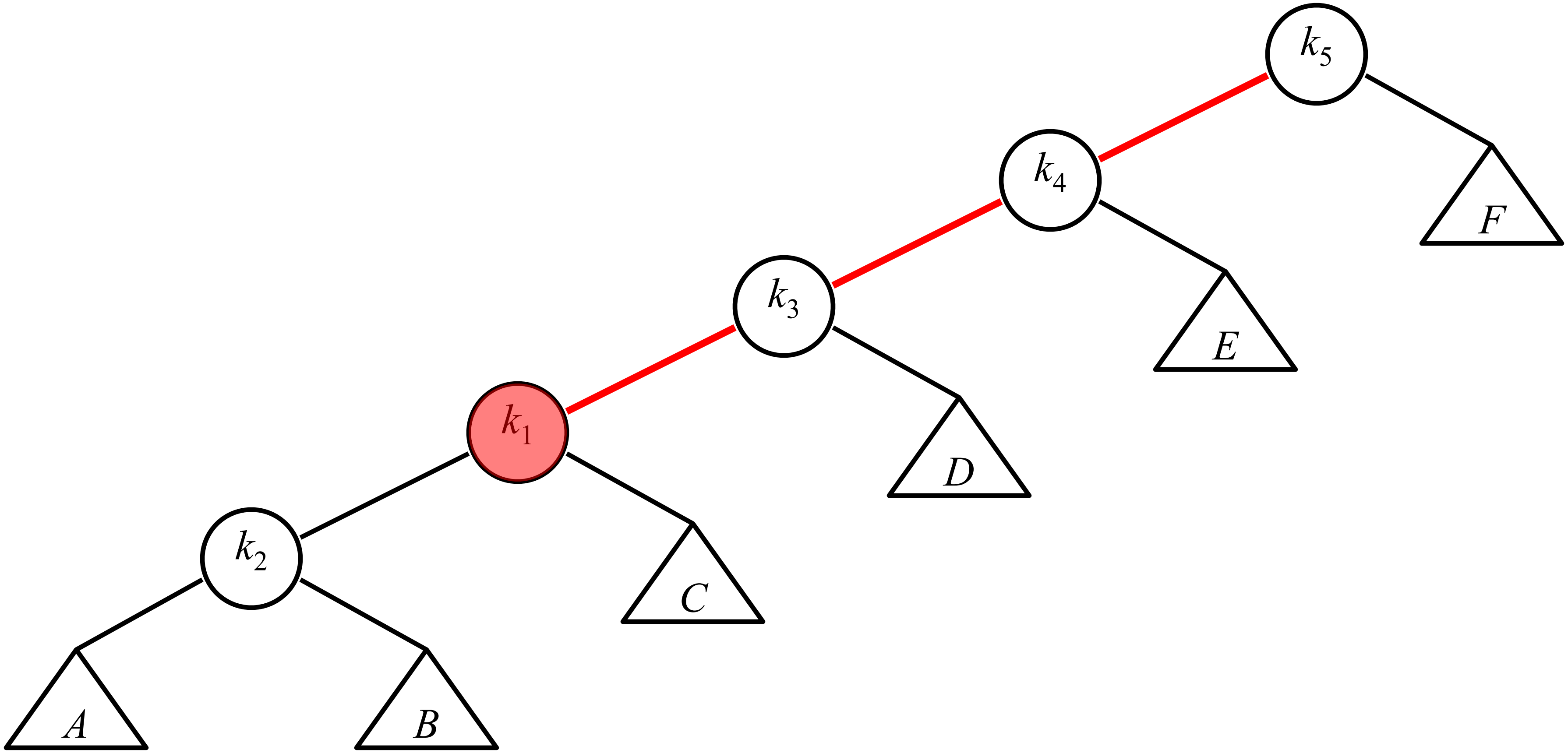
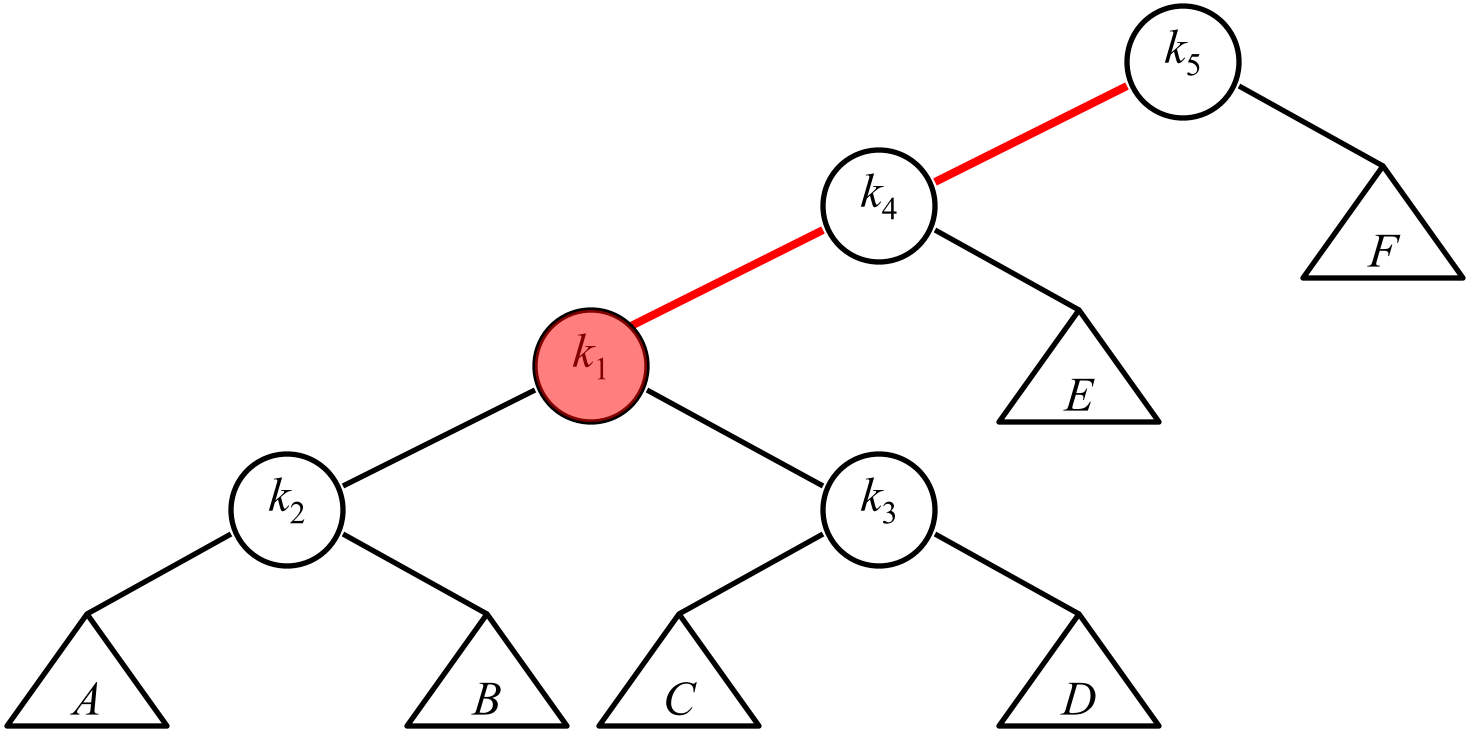

k_5
k_4
F
k_3
E
k_1
D
k_2
C
A
B


k_5
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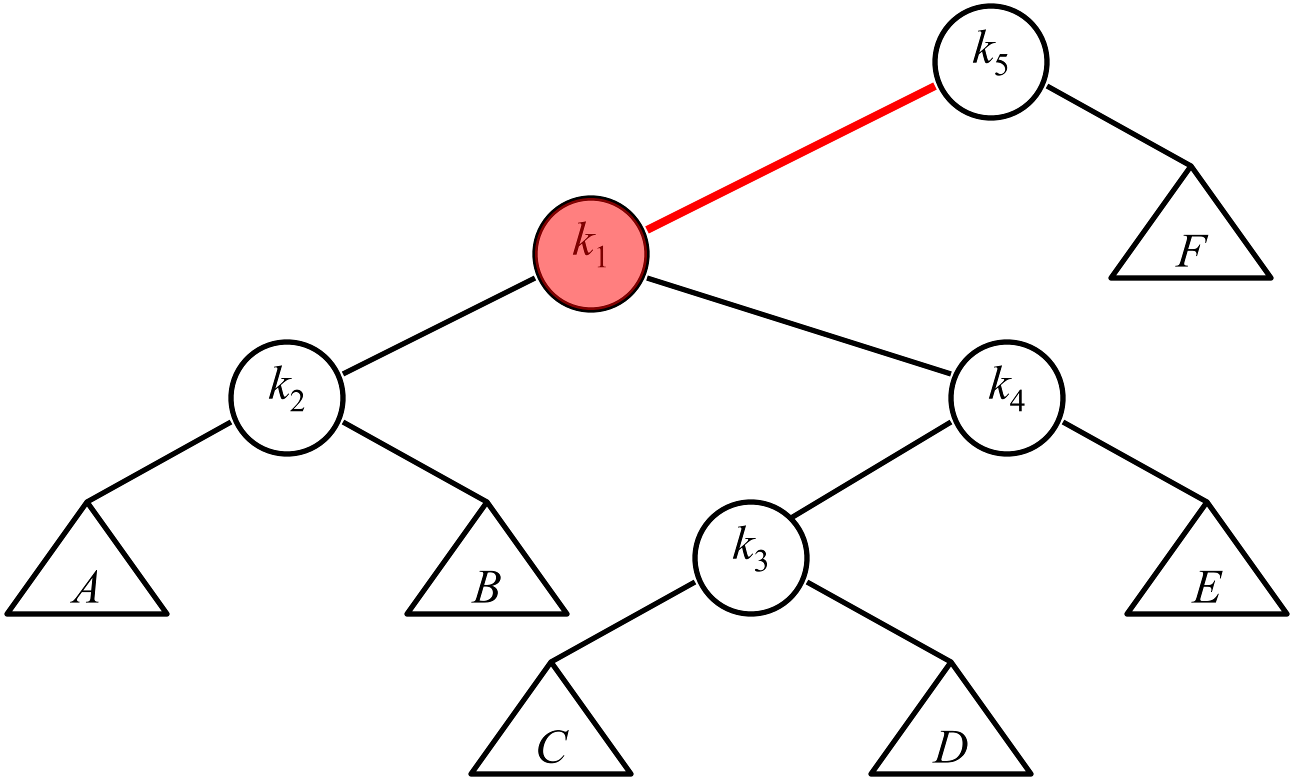
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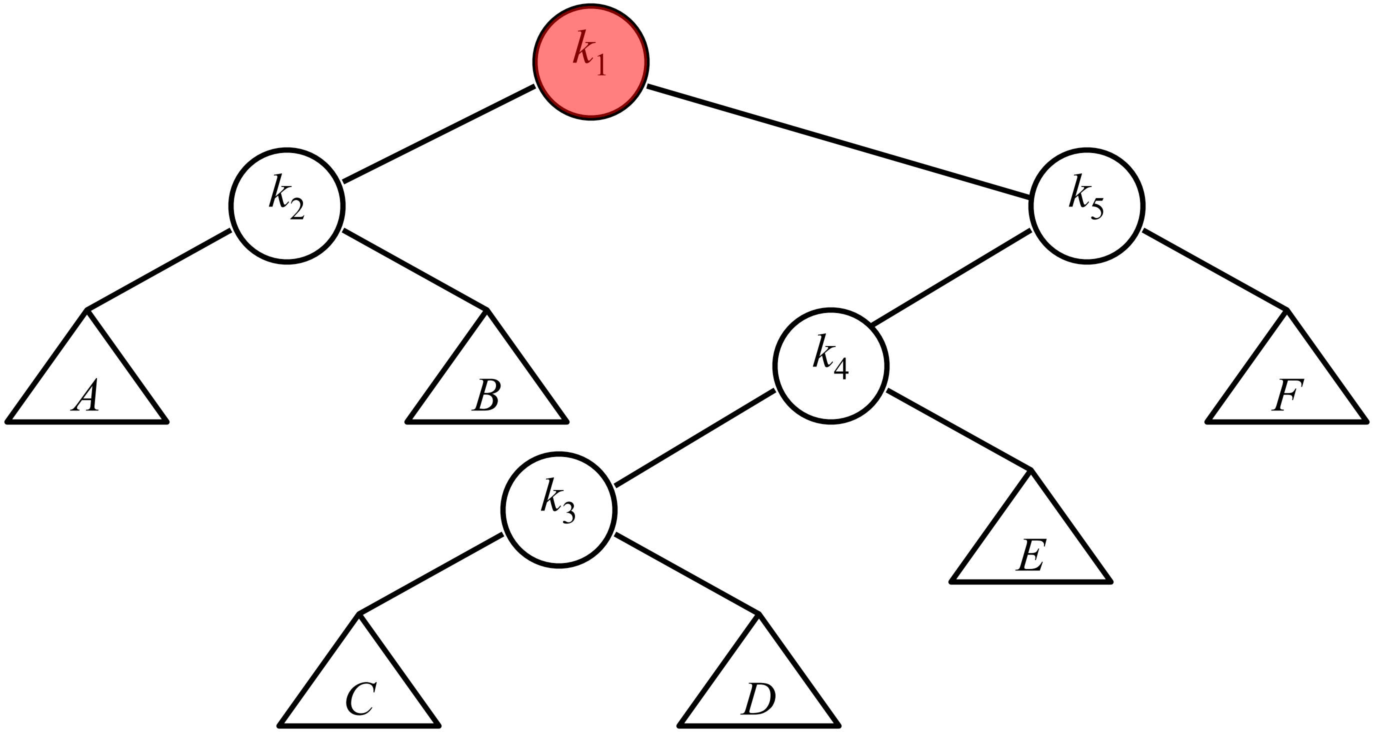
k_1
k_2
k_5
k_4
k_3
A
B
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D
E
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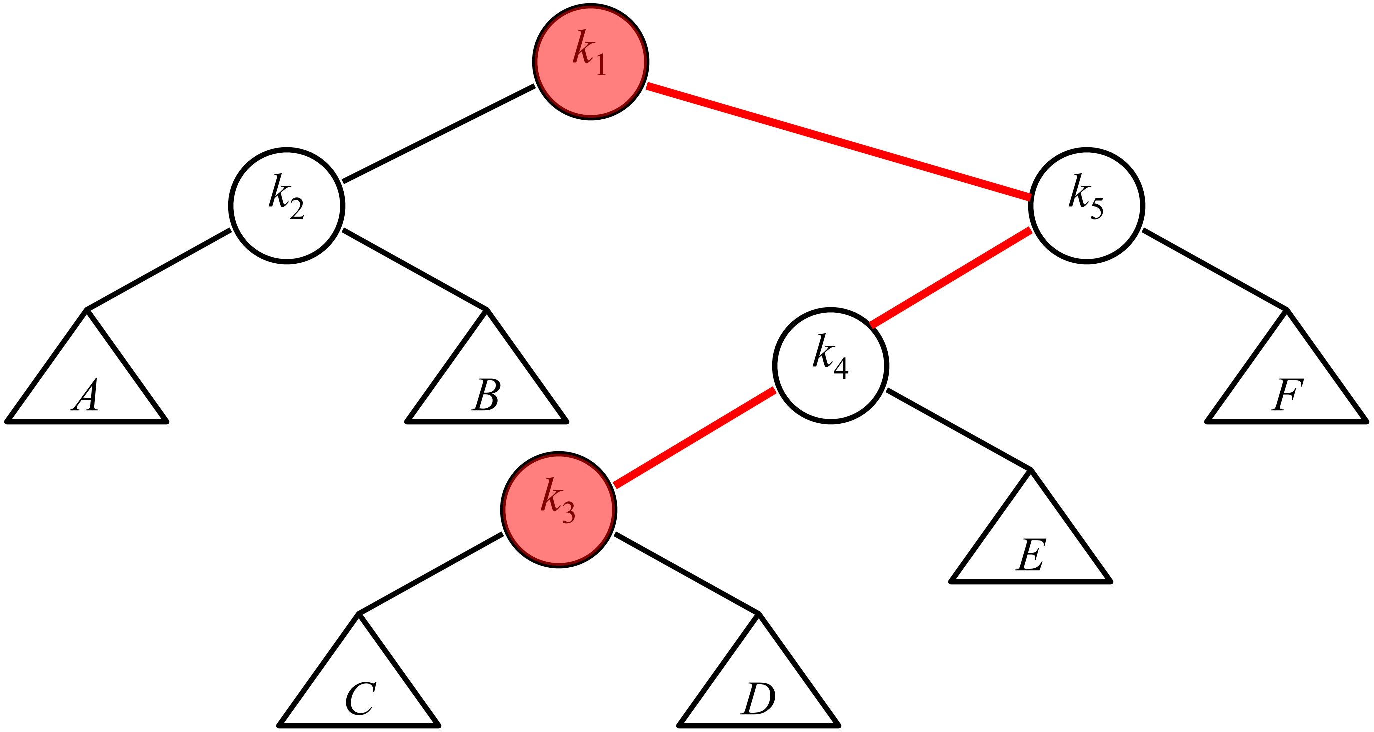
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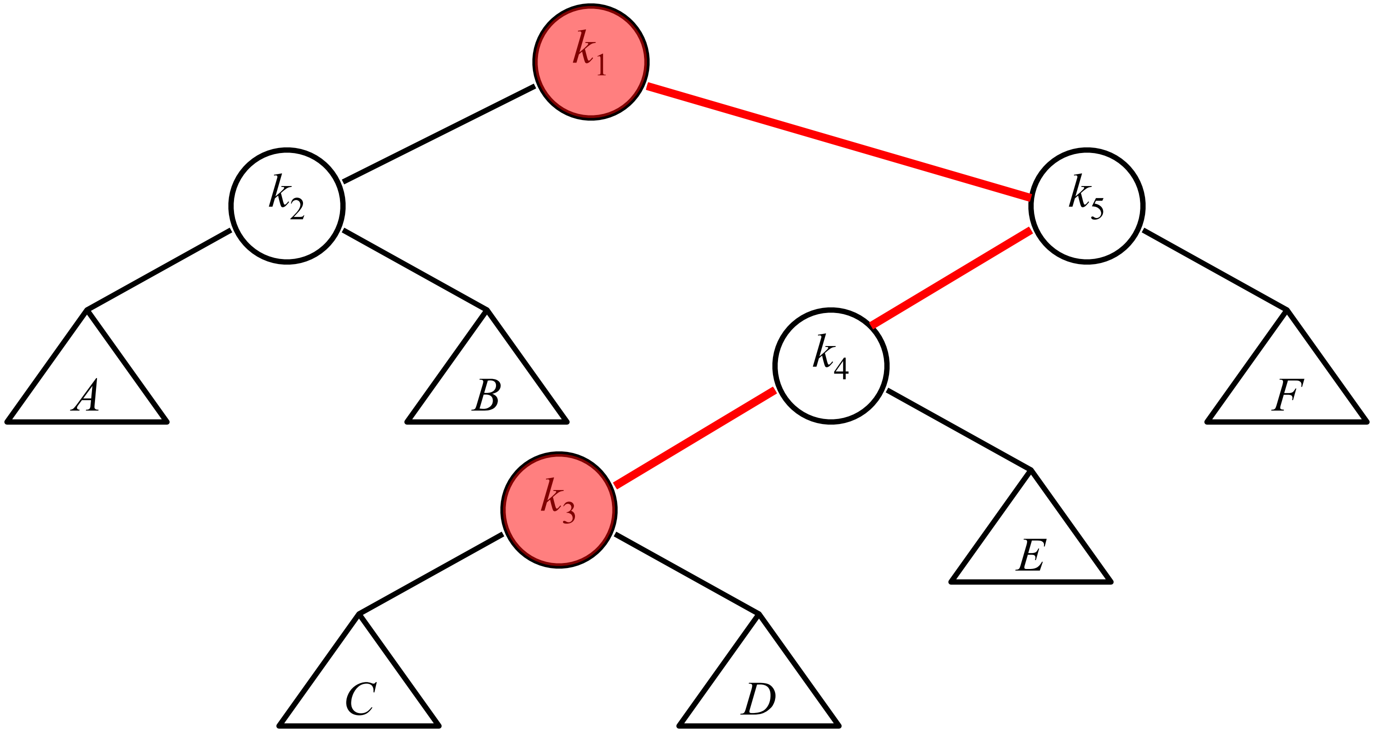

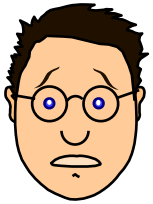

Does NOT work!

The rotation pushes other nodes deeper

An even worse case:

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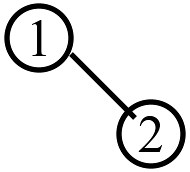

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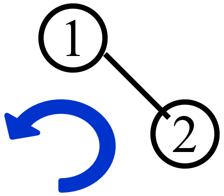

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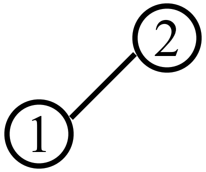

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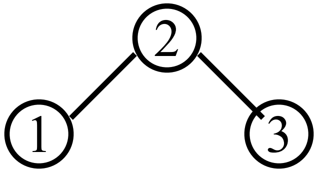

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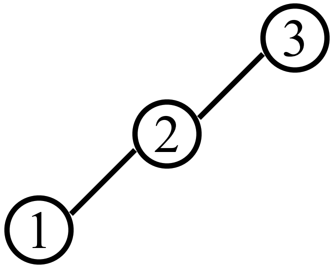
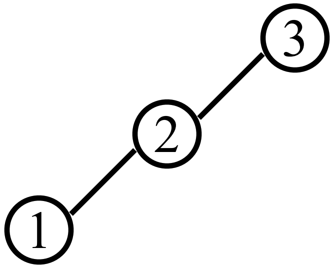


An even worse case:

Insert: 1, 2, 3, … *N*



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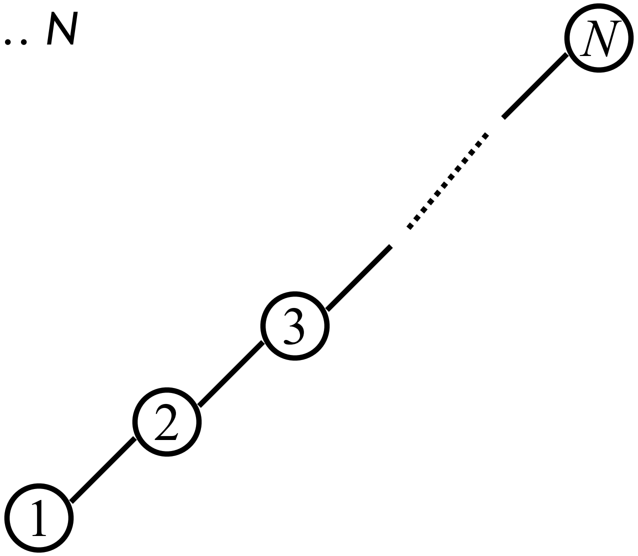

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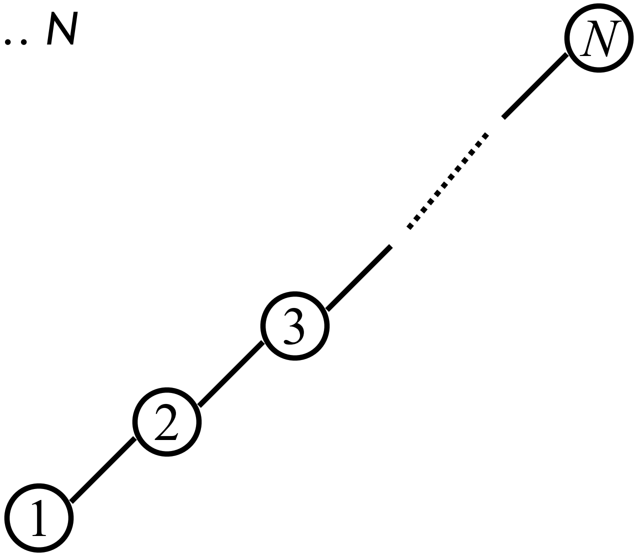
Find: 1



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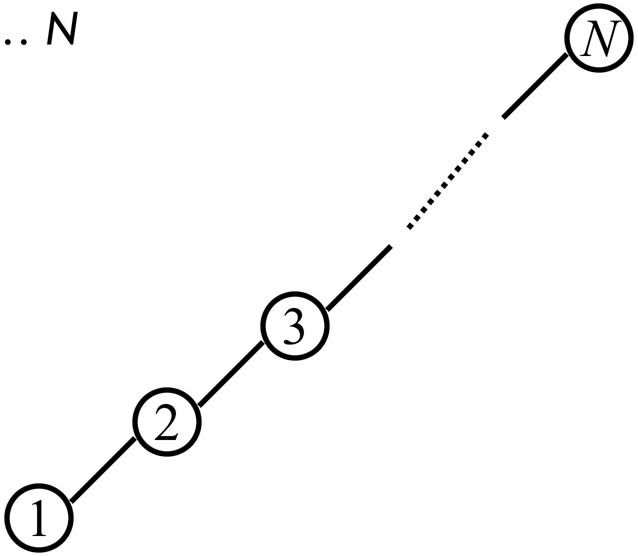

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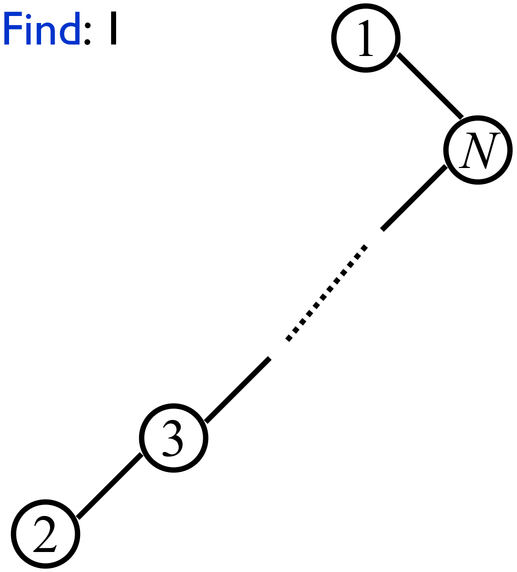

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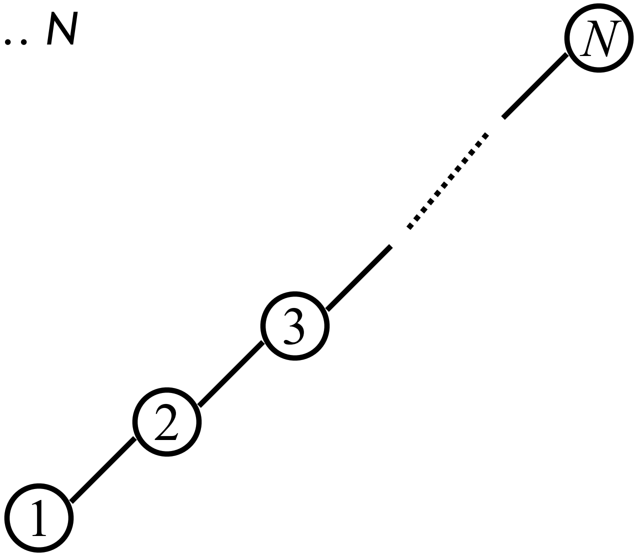

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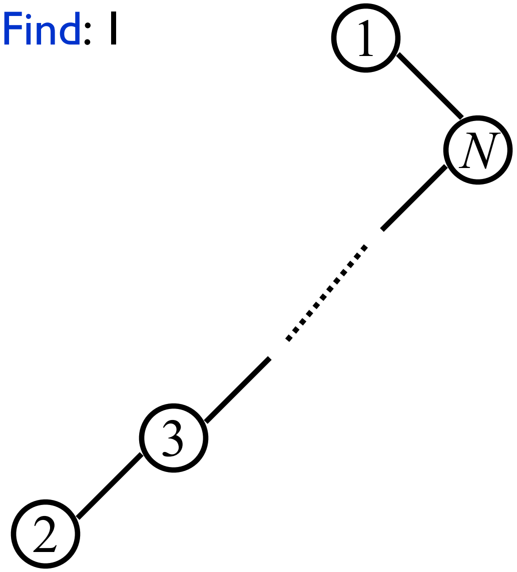

Find: 2

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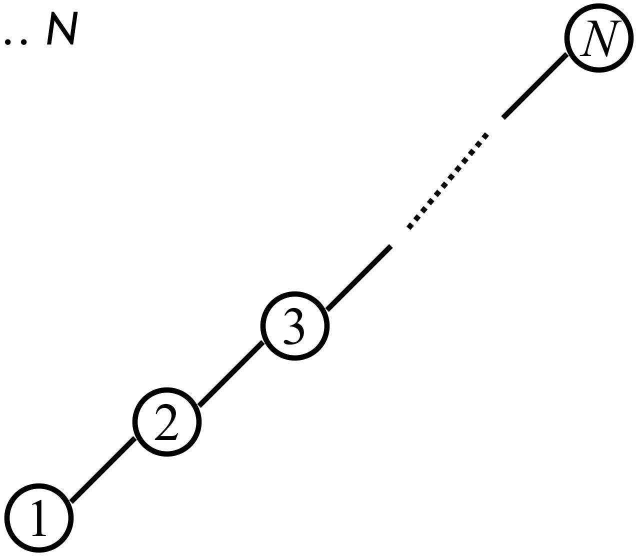
Insert: 1, 2, 3, … *N*



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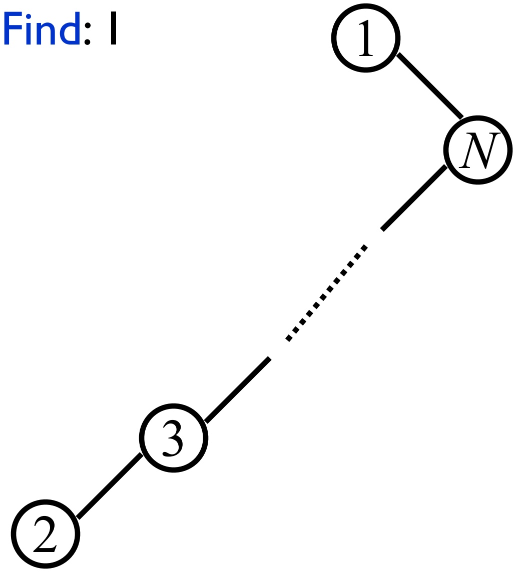
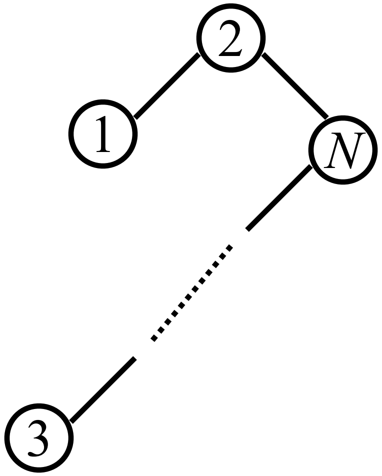


Find: 2



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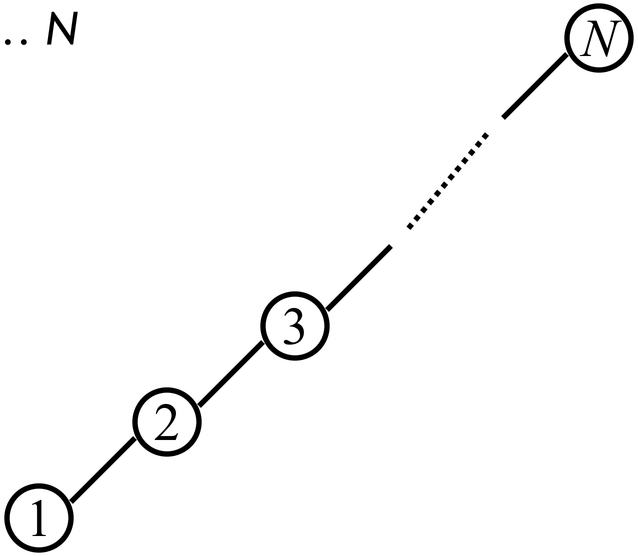

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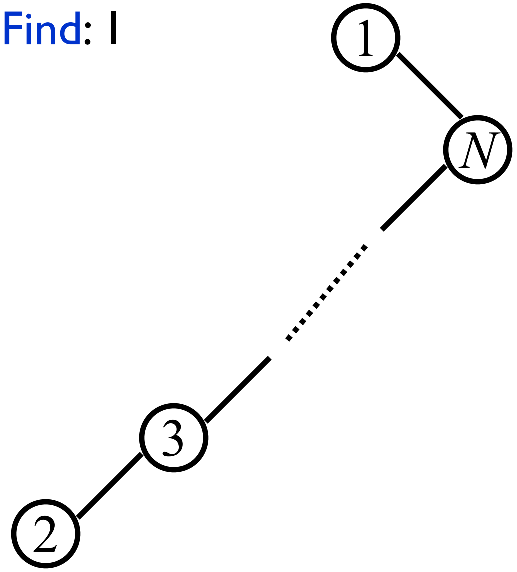
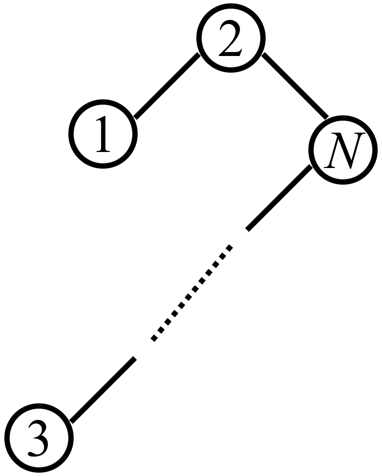


Find: 2



…… Find: *N*

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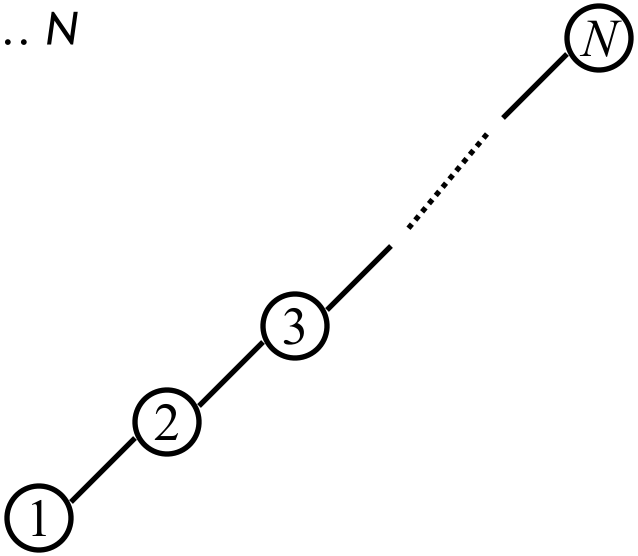

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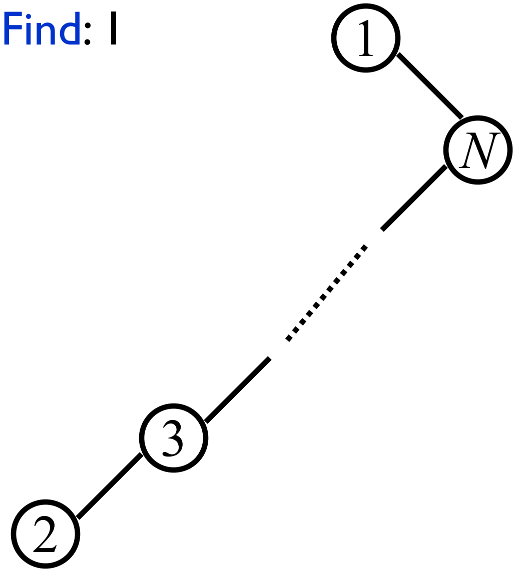
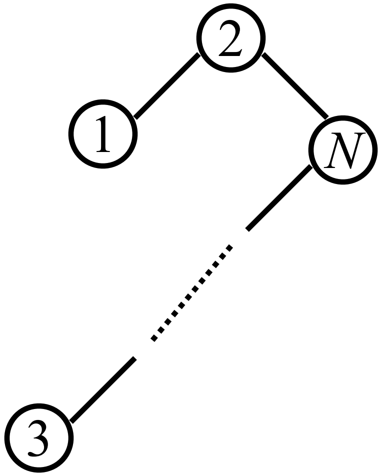


Find: 2



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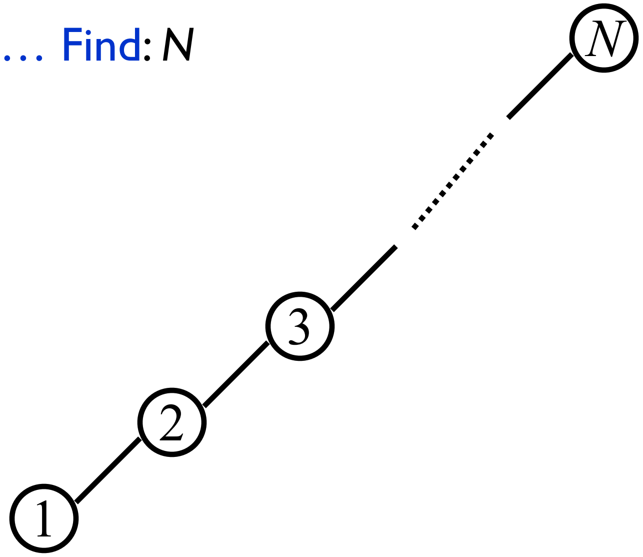

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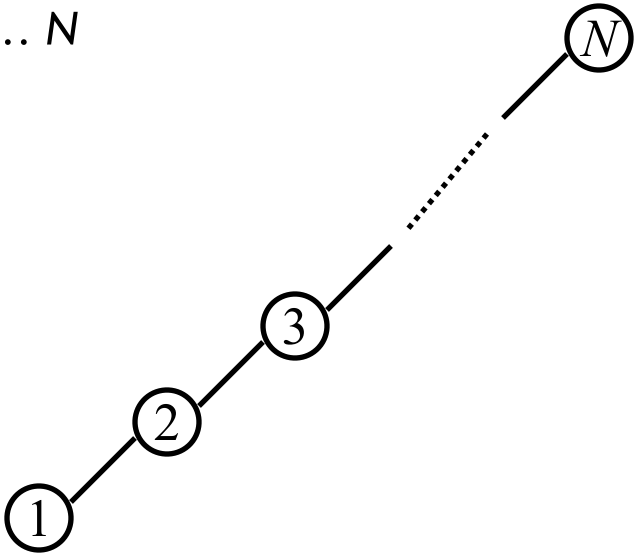

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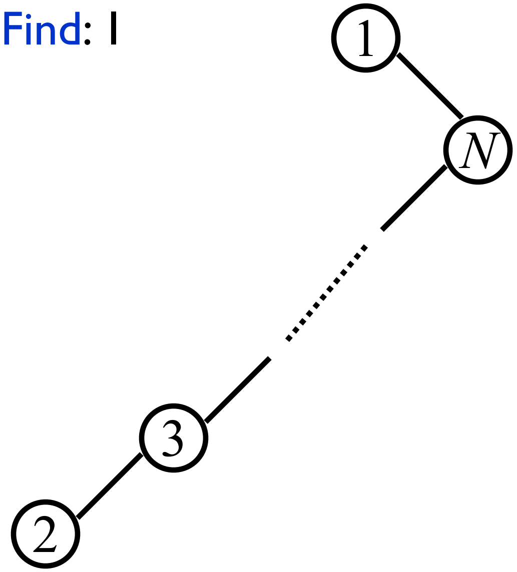
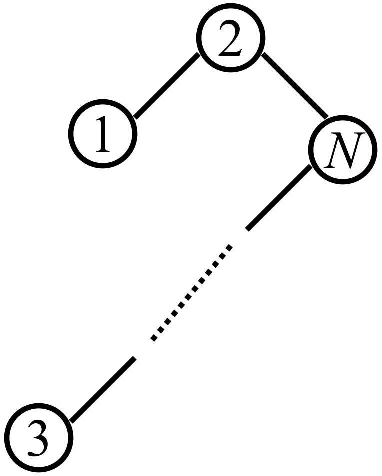


Find: 2



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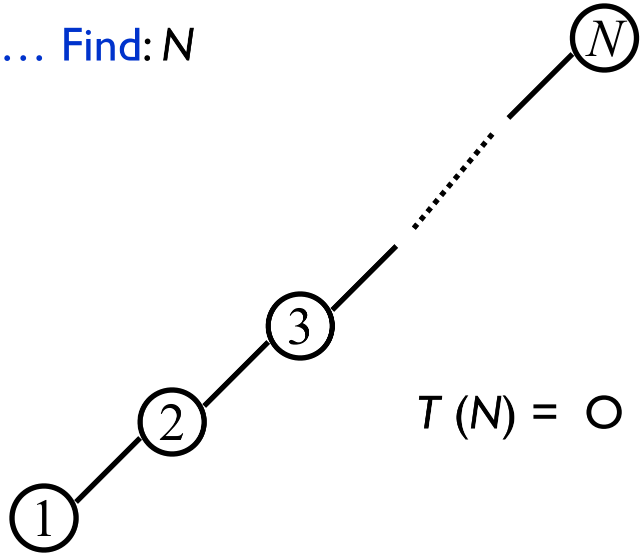

$T(N) = O(N^2)$

Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :

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Zig Case 1: *P* is the root

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Case 2: *P* is not the root

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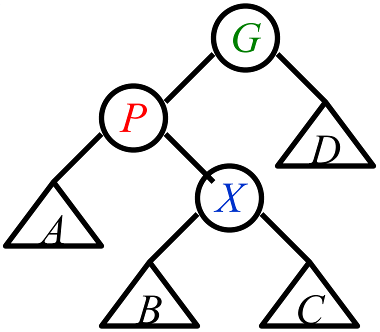
Zig Case 1: *P* is the root ⟶ Rotate *X* and *P*

Case 2: *P* is not the root

Zig-zag

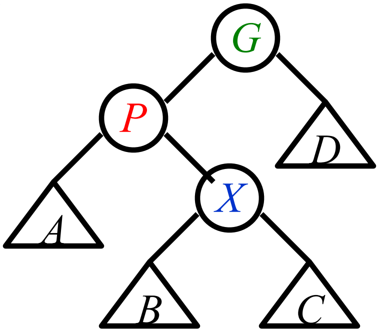


Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :

Zig Case 1: *P* is the root → Rotate *X* and *P*

Case 2: *P* is not the root

Zig-zag



Double rotation →

Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :

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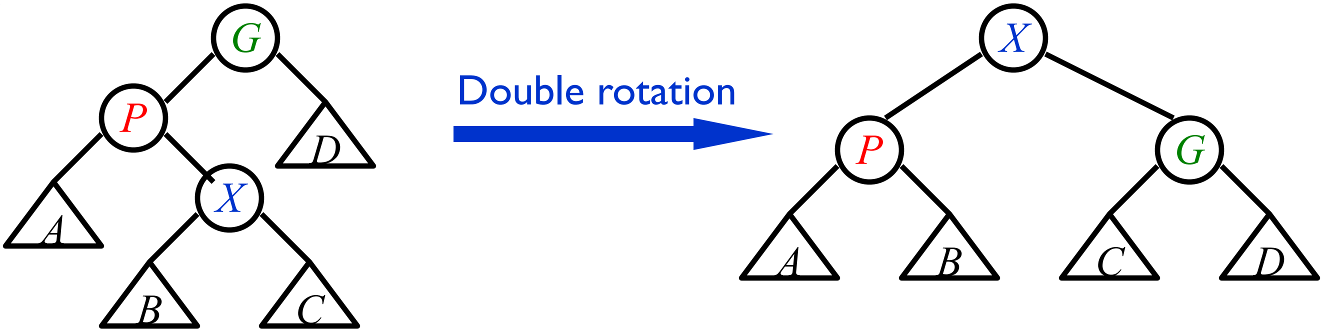
Zig-zag



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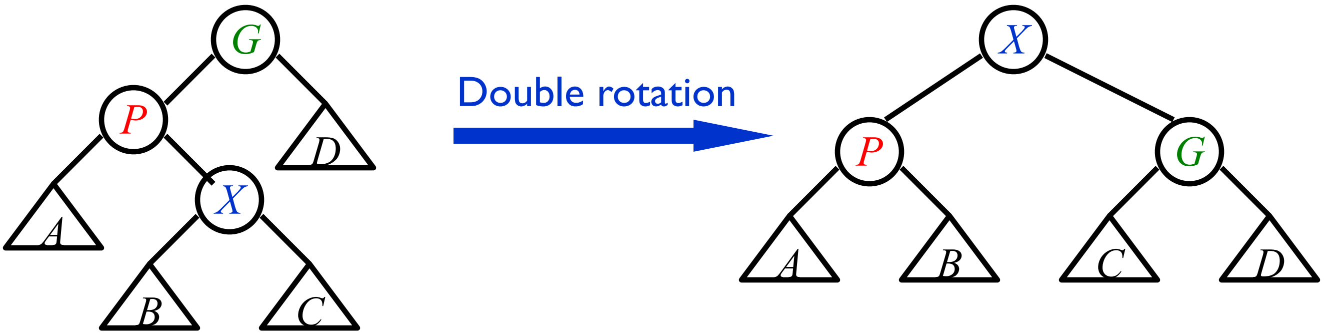
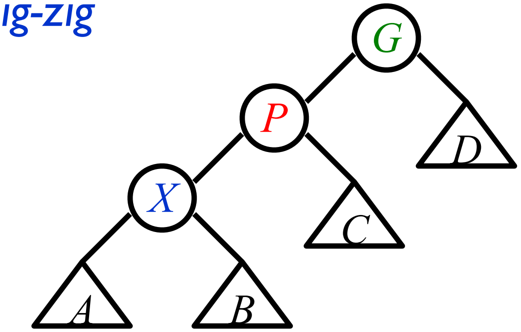


Double rotation

Zig-zig



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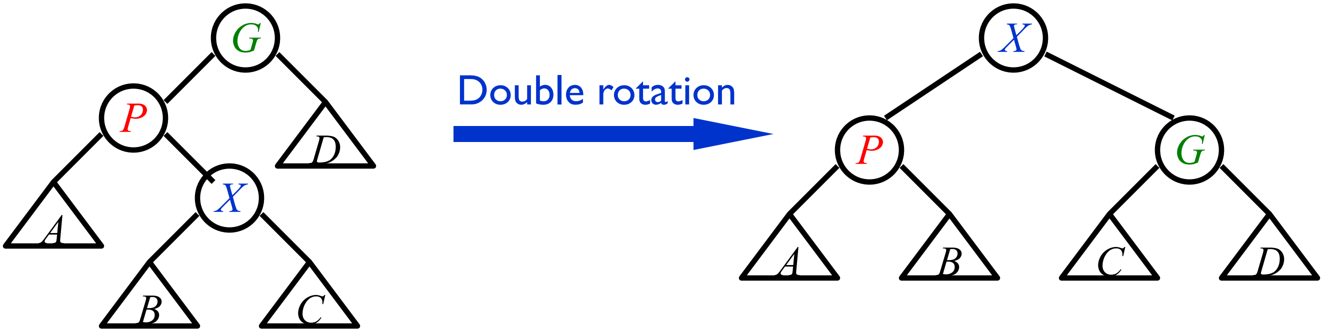

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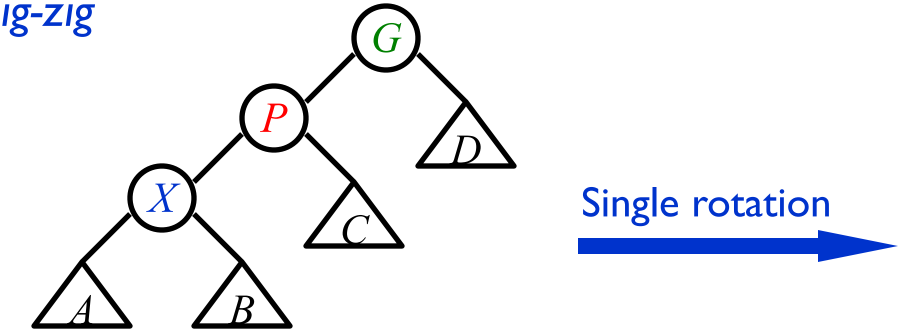

Single rotation

Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :

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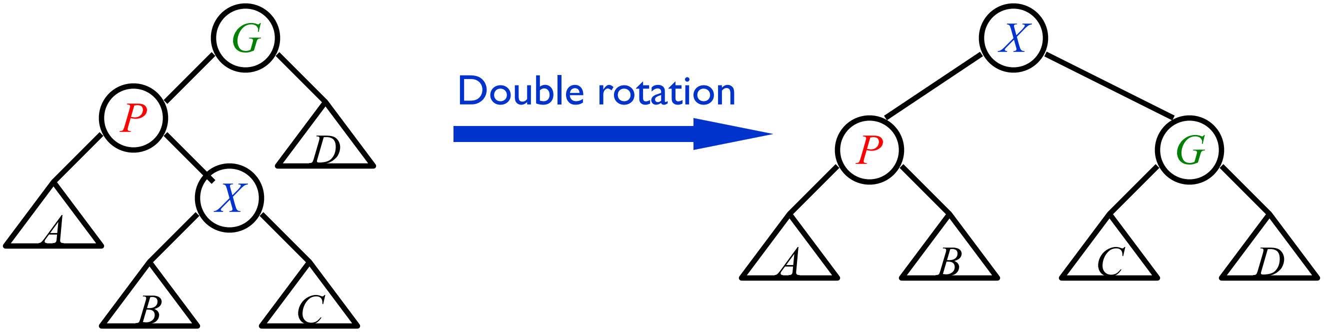
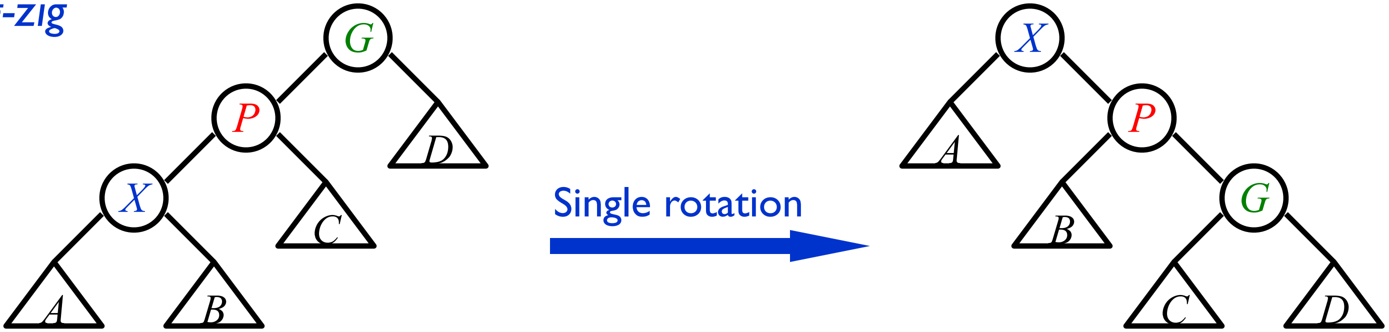
Zig-zag



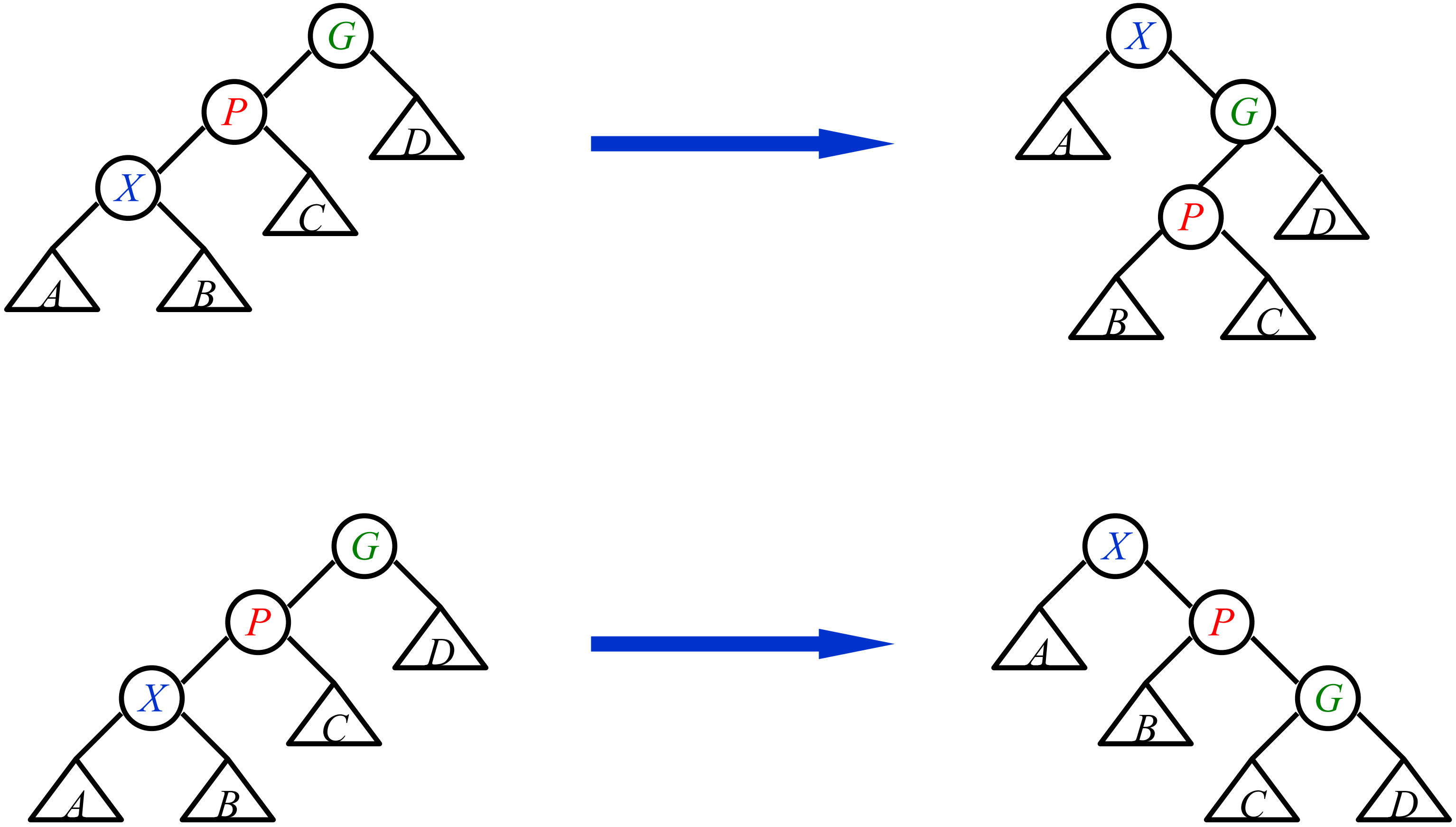
Double rotation

Zig-zig

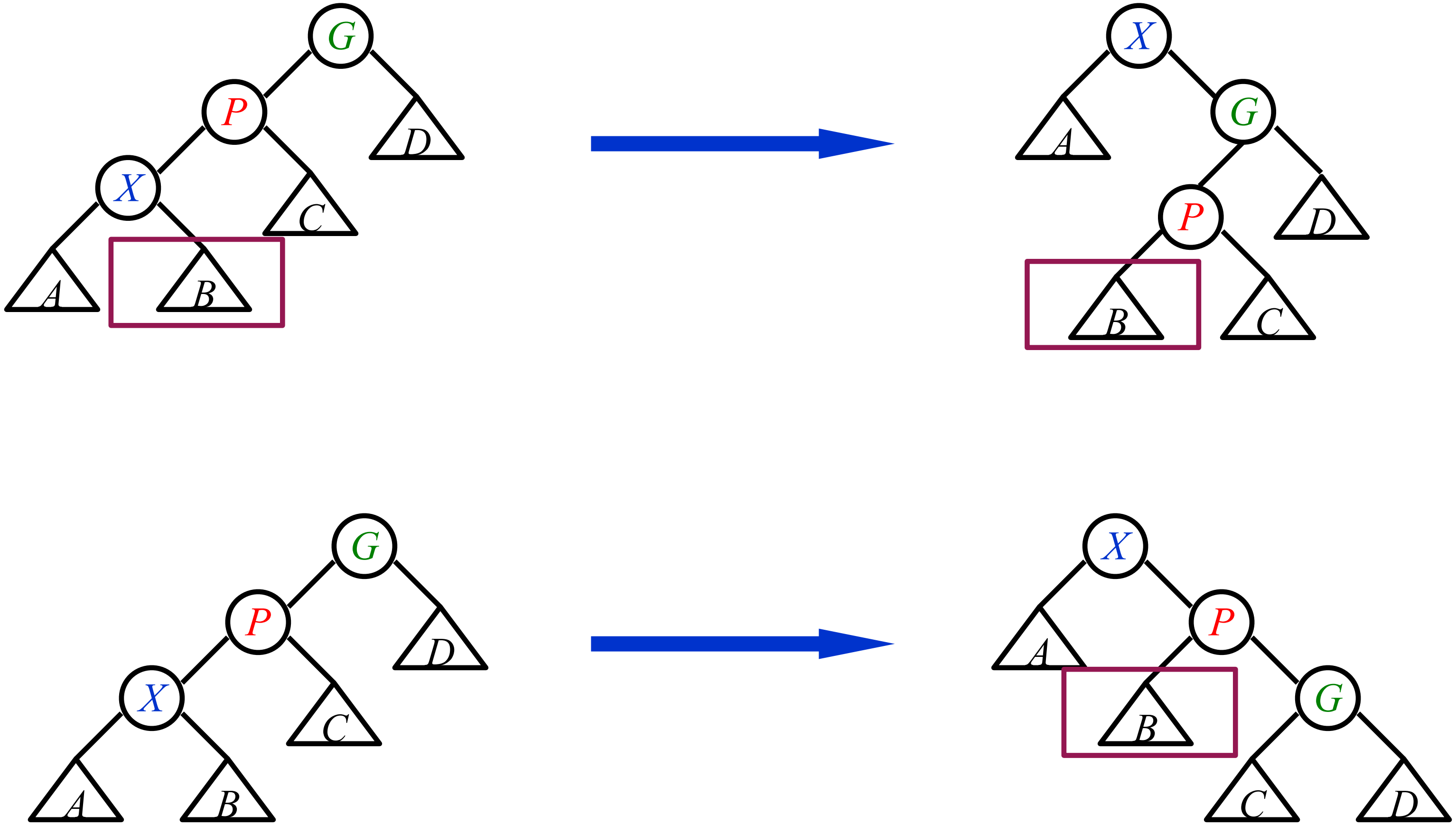


Single rotation

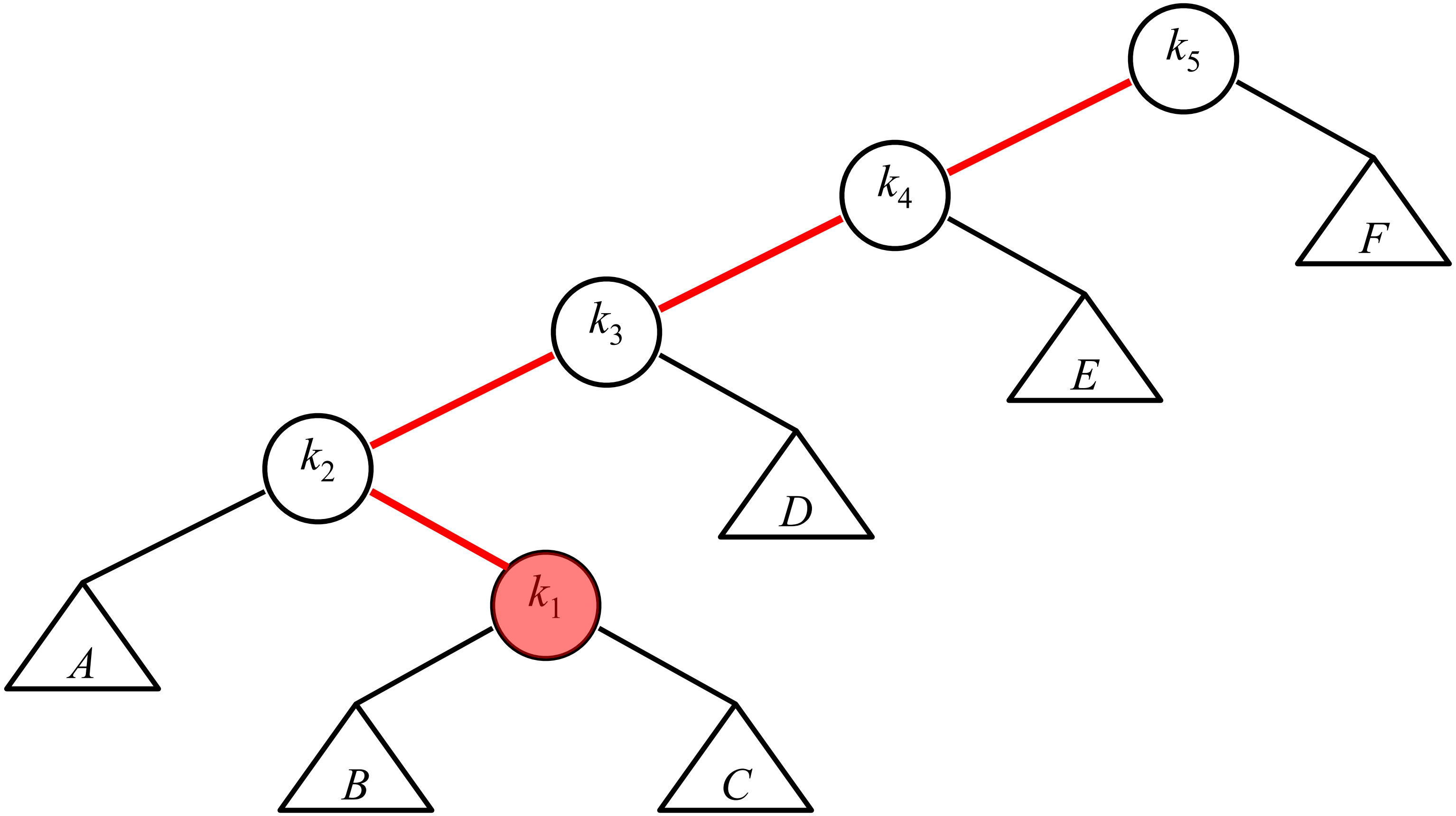
Compare the Zig-zig case:

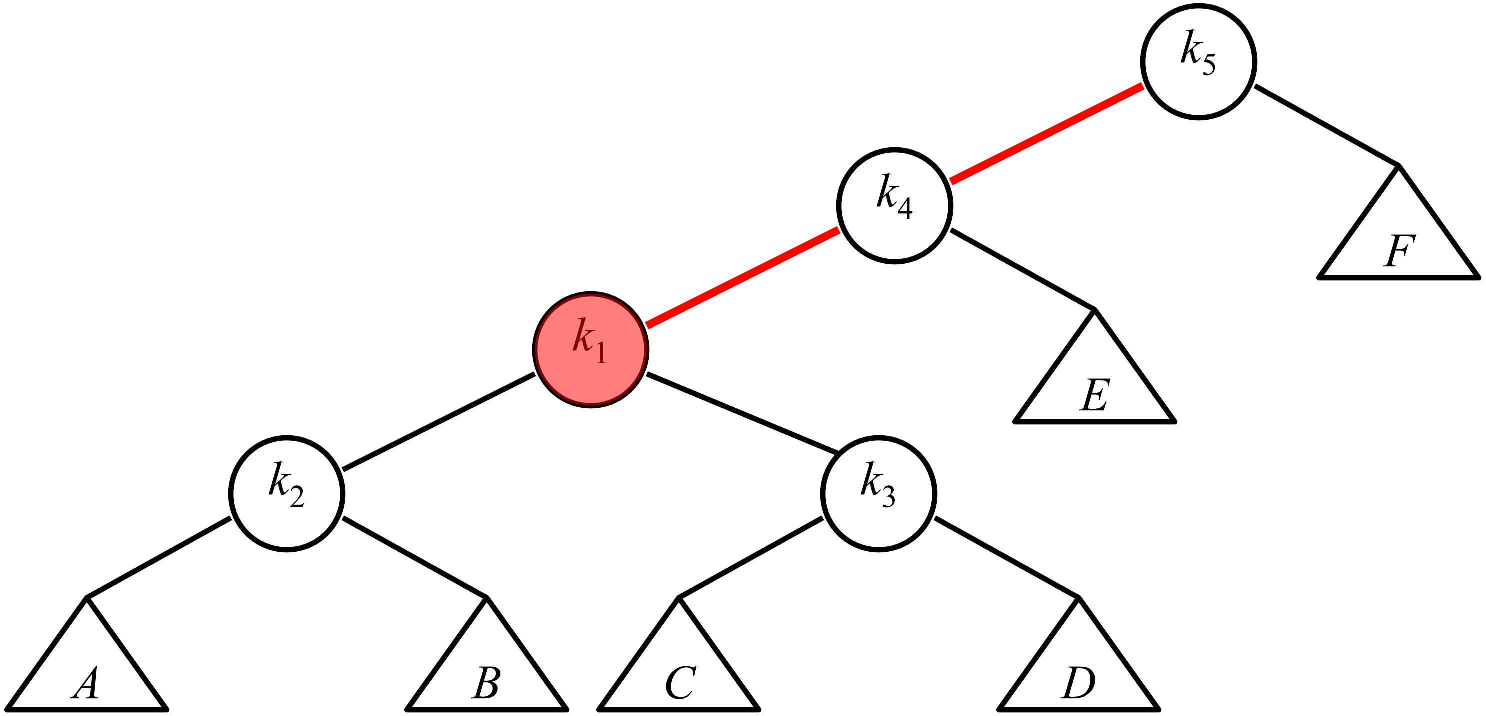


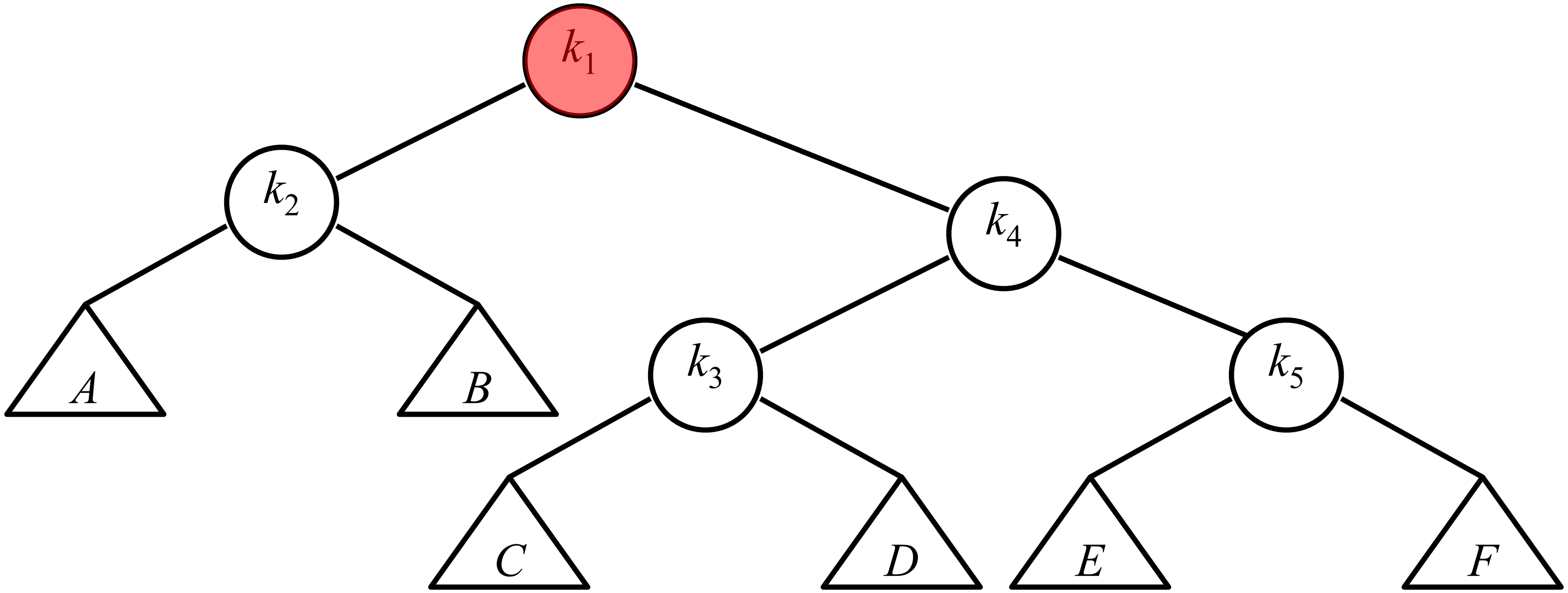
Compare the Zig-zig case:



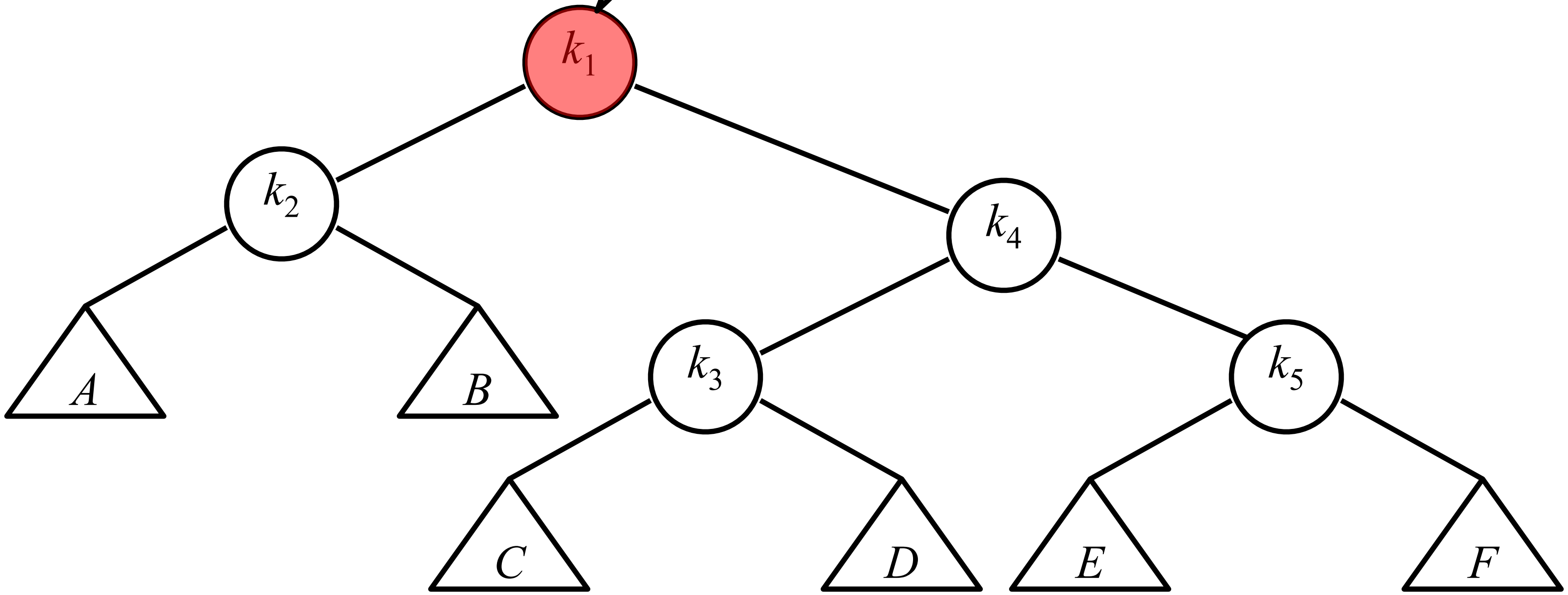
For zig-zig case, the right child of the node on splaying always goes deep.
The key is to make it go slower.


k_5
k_4
k_3
k_2
k_1
A
B
C
D
E
F


k_5
k_4
k_1
k_2
k_3
A
B
C
D
E
F


k_1
k_2
k_4
k_3
k_5
A
B
C
D
E
F

Splaying not only moves the accessed node to the root, but also roughly halves the depth of most nodes on the path.



Insert: 1, 2, 3, 4, 5, 6, 7

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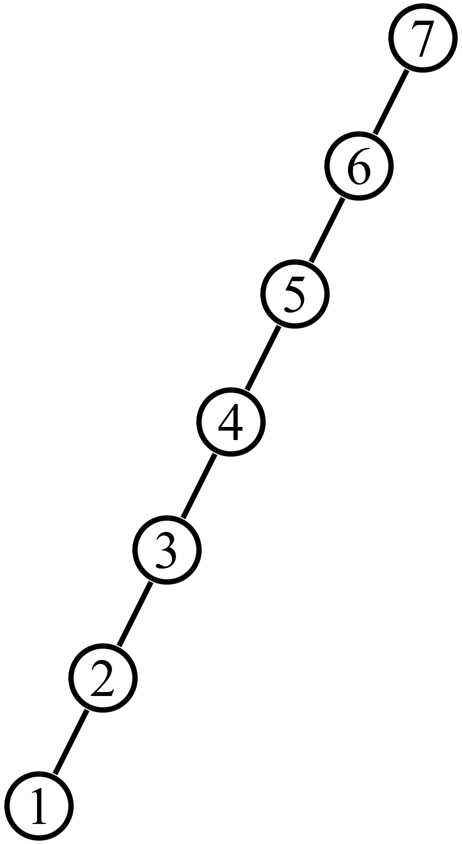
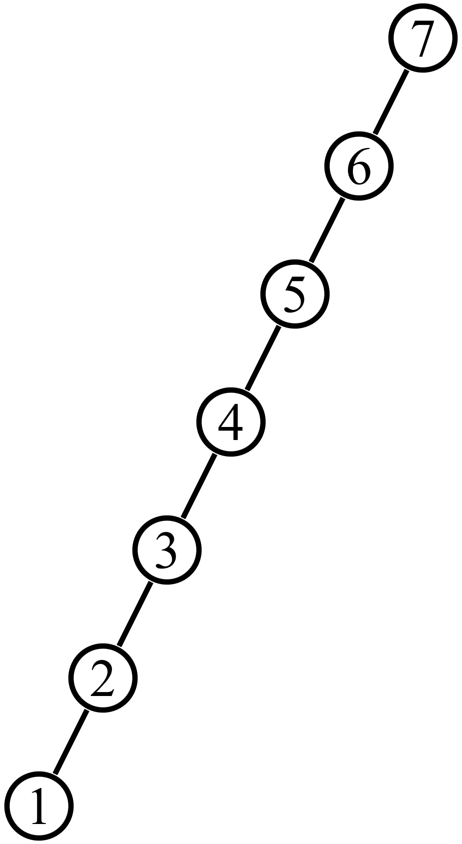


Insert: 1, 2, 3, 4, 5, 6, 7 Find: 1



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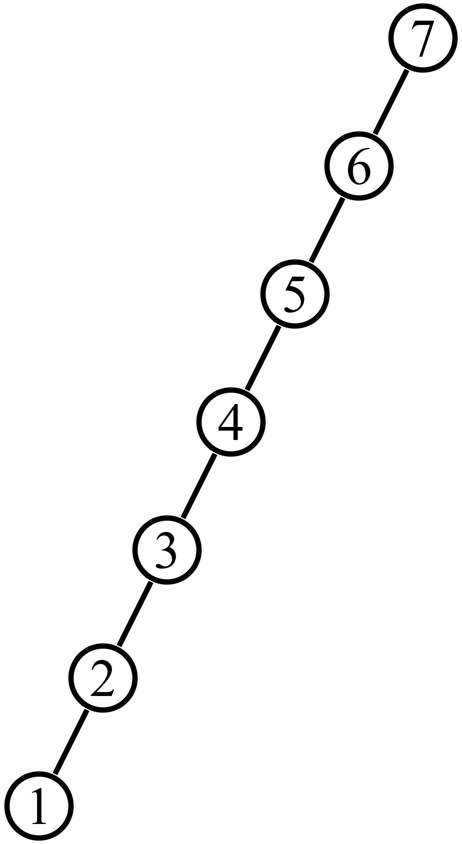

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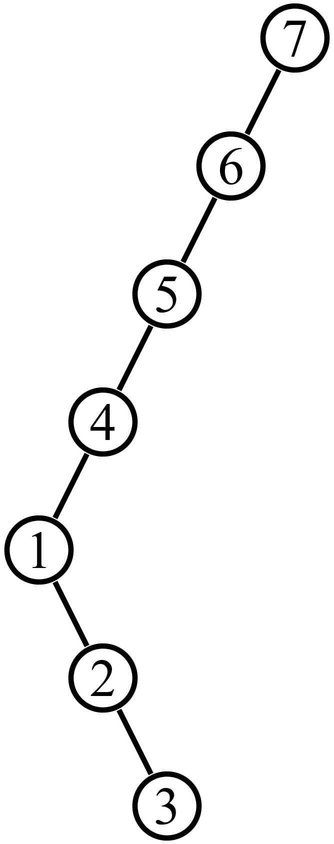
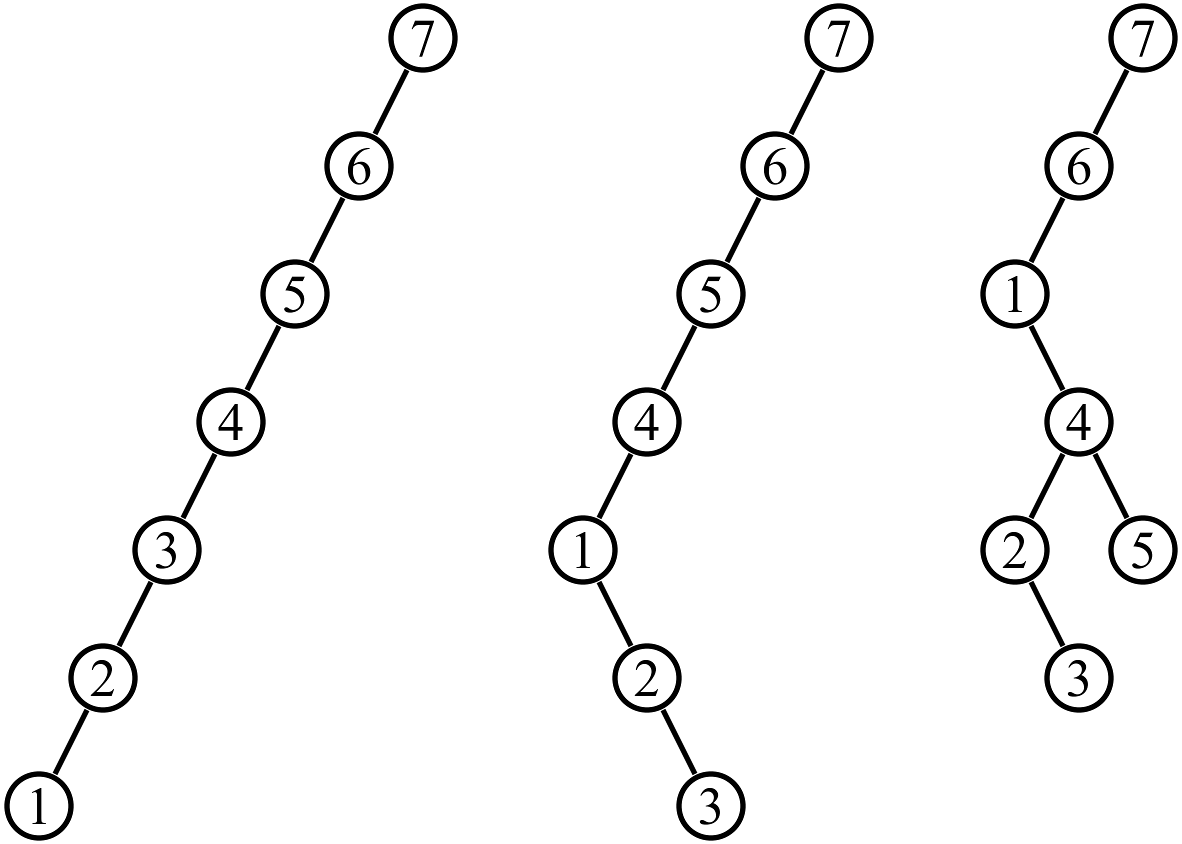
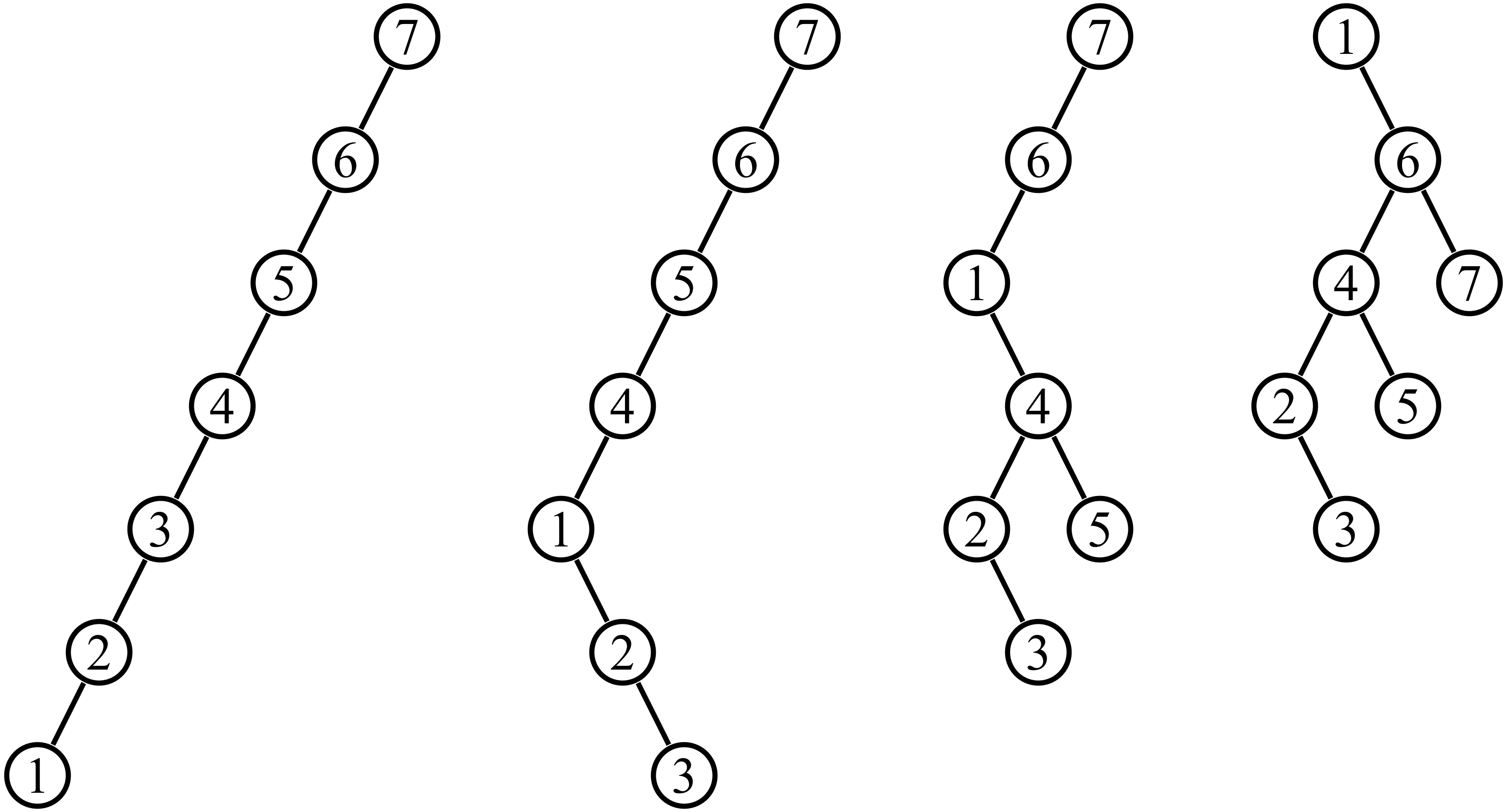


Insert: 1, 2, 3, 4, 5, 6, 7

Find: 1



Insert: 1, 2, 3, 4, 5, 6, 7   Find: 1



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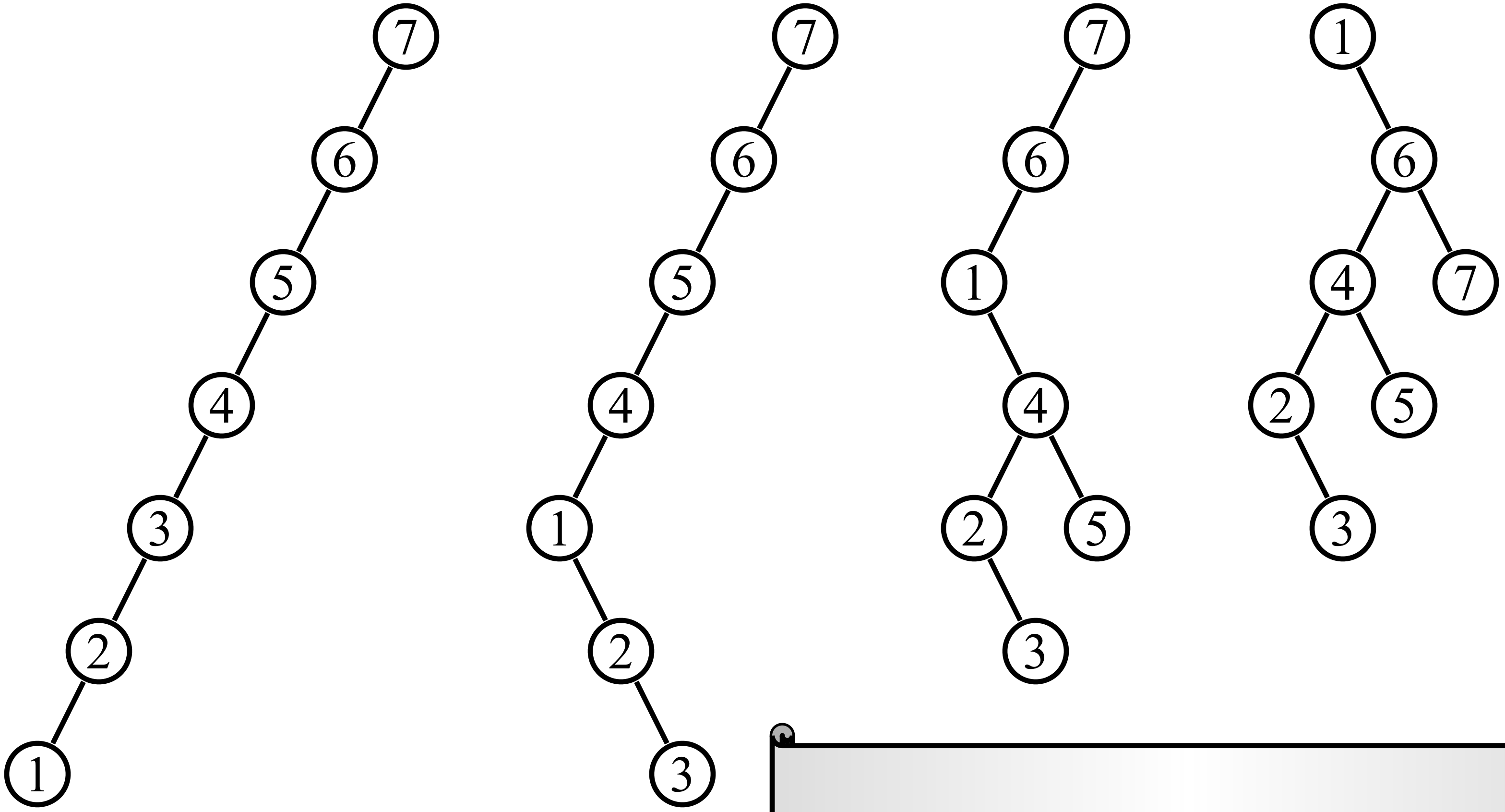

Read the 32-node example given in [Weiss] Figures 4.52 – 4.60

Operations on Splay Trees

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☞ Step 1: Find *X* ;

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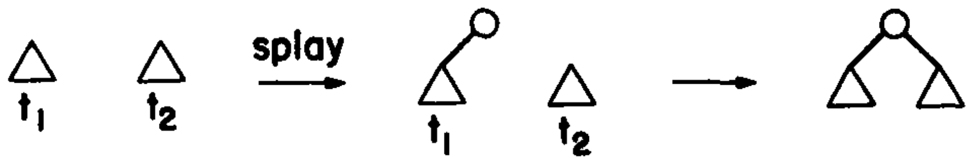
The largest element will be the root of T_L , and *has no right child.*

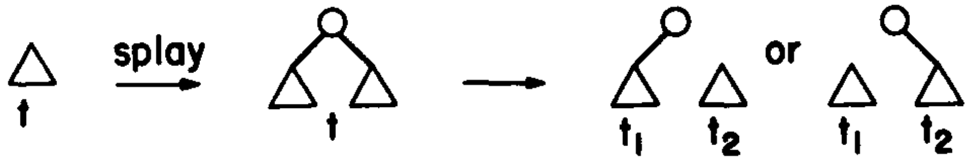
☞ Step 4: Make T_R the right child of the root of T_L .

Operations on Splay Trees

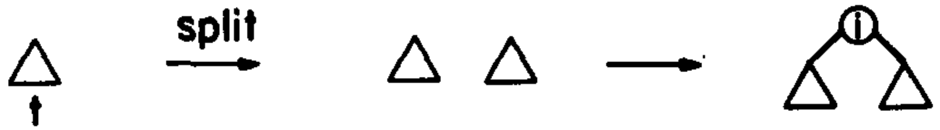
Join(t1, t2):



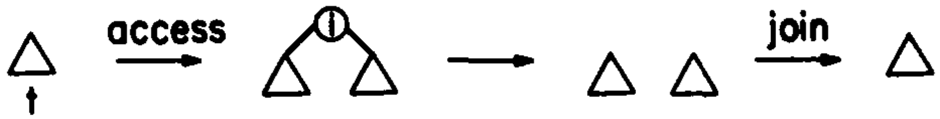
Split(i, t):



Insert(i, t):



Delete(i, t):



All operations involve a series of splay steps.
Check the details in the "Self-adjusting binary search trees" paper.
Next, we study the complexity of splay tree operations.

Outline:
Balanced Binary Search Trees (1)

- Binary search trees
- AVL trees
- Splay trees
- Amortized analysis
- Take-home messages

Amortized Analysis

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Target : Any M consecutive operations take at most $O(M \log N)$ time.

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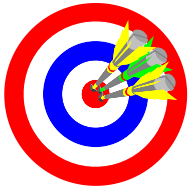
Amortized Analysis



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worst-case bound   amortized bound   average-case bound

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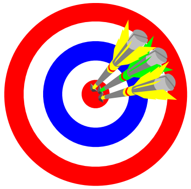

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worst-case bound ≥ amortized bound average-case bound

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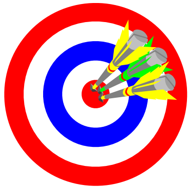

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Probability
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☞ Aggregate analysis

☞ Accounting method

☞ Potential method

Aggregate Method

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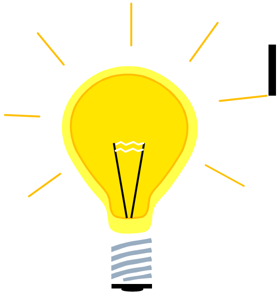
Idea : Show that for all *n*, a sequence of *n* operations takes *worst-case* time *T*(*n*) in total. In the worst case, the average cost, or *amortized cost*, per operation is therefore *T*(*n*)/*n*.

Aggregate Method



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〖Example〗 Stack with MultiPop(int k, Stack S)

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Algorithm  {
    while ( !IsEmpty(S) && k>0 ) {
        Pop(S);
        k - -;
    } /* end while-loop */
}
```

Aggregate Method

Idea : Show that for all *n*, a sequence of *n* operations takes *worst-case* time $T(n)$ in total. In the worst case, the average cost, or *amortized cost*, per operation is therefore $T(n)/n$.

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Algorithm  {
    while ( !IsEmpty(S) && k>0 ) {
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    } /* end while-loop */
}
        T = min ( sizeof(S), k )
```

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$\text{sizeof}(S) \leq n$

Aggregate Method

Idea : Show that for all *n*, a sequence of *n* operations takes *worst-case* time $T(n)$ in total. In the worst case, the average cost, or *amortized cost*, per operation is therefore $T(n)/n$.

Total = $O(n^2)$?

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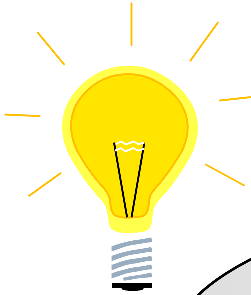

Idea : Show that for all n, a sequence of n operations takes *worst-case* time $T(n)$ in total. In the worst case, the average cost, operation is therefore $T(n)/n$.

We can pop each object from the stack *at most once* for each time we have pushed it onto the stack

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〚Example〛 Stack with Multi op(i Stack S)

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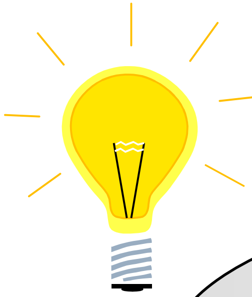

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$T_{amortized} = O(n)/n = O(1)$

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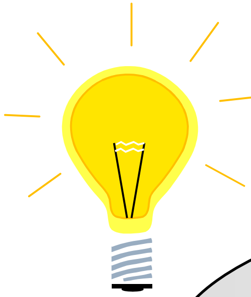

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〚Example〛 Stack with Multi op(Stack S)

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Consider a sequence of *n* Push, Pop, and MultiPop operations on an initially empty stack.

$\text{sizeof}(S) \le n$

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The total time of pop should be less than the total time of push.
The total time of push takes at most O(n).

Accounting Method

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Idea : When an operation's *amortized cost* $\hat{c}_i$ exceeds its *actual cost* c_i , we assign the difference to specific objects in the data structure as *credit*. Credit can help *pay* for later operations whose amortized cost is less than their actual cost.

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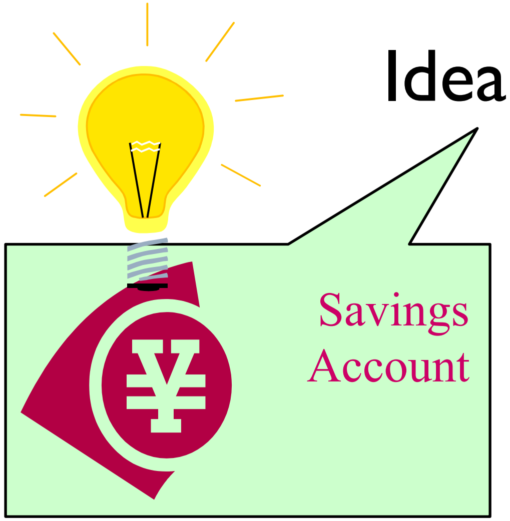

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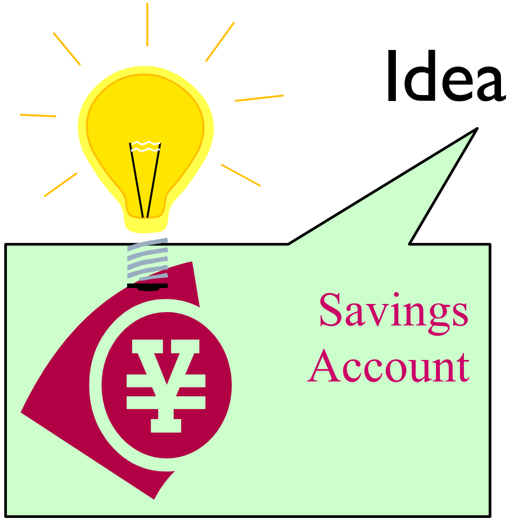

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$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$$

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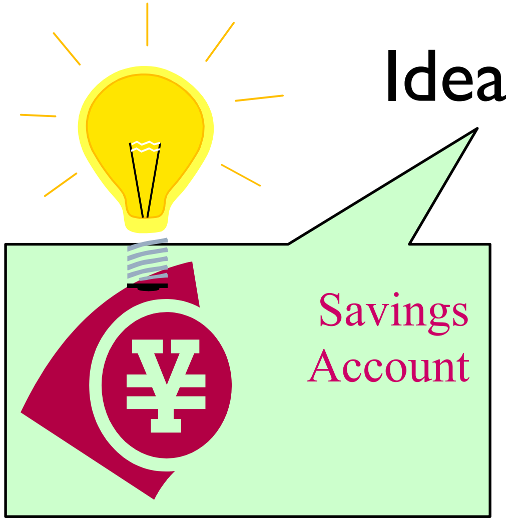

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$$T_{amortized} = \frac{1}{n} \sum_{i}^{n} \hat{c}_i$$

〖Example〗 Stack with MultiPop(int k, Stack S)

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c_i for Push: ; Pop: ; and MultiPop:

〖Example〗 Stack with MultiPop(int k, Stack S)

c_i for Push: 1 ; Pop: ; and MultiPop:

〖Example〗 Stack with MultiPop(int k, Stack S)

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〖Example〗 Stack with MultiPop(int k, Stack S)

c_i for Push: 1 ; Pop: 1 ; and MultiPop: $\min(\text{sizeof}(S), k)$

〖Example〗 Stack with MultiPop(int k, Stack S)

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$\hat{c}_i$ for Push: ; Pop: ; and MultiPop:

〖Example〗 Stack with MultiPop(int k, Stack S)

c_i for Push: 1 ; Pop: 1 ; and MultiPop: $\min(\text{sizeof}(S), k)$

$\hat{c}_i$ for Push: 2 ; Pop: ; and MultiPop:

〖Example〗 Stack with MultiPop(int k, Stack S)

c_i for Push: 1 ; Pop: 1 ; and MultiPop: $\min(\text{sizeof}(S), k)$

$\hat{c}_i$ for Push: 2 ; Pop: 0 ; and MultiPop:

〖Example〗 Stack with MultiPop(int k, Stack S)

c_i for Push: 1 ; Pop: 1 ; and MultiPop: $\min(\text{sizeof}(S), k)$

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The amortized costs of the operations may *differ* from each other

➡ $O(n) = \sum_{i=1} c_i \quad \sum_{i=1} c_i$

➡ $T_{amortized} = O(n)/n = O(1)$

Potential Method

- Why some problems have smaller amortized time cost?
  - The structure of the problem provides the constraints:

- Represent the states of the structure as potential functions.
  - The potential function is bounded by the structural constraints.
  - Bound the total cost by the increase of potential.

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All operations can not exceed the structural constraints.

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Potential Method

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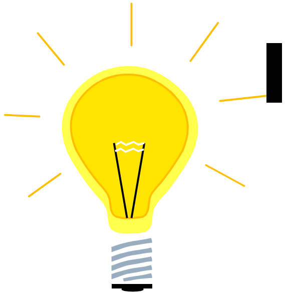

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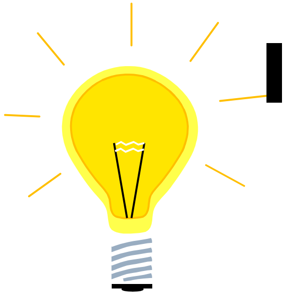

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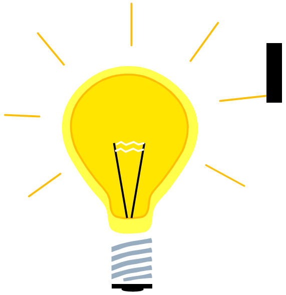

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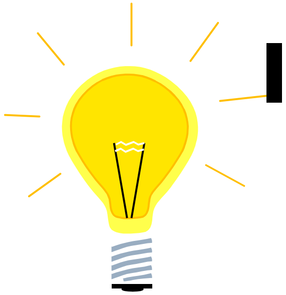

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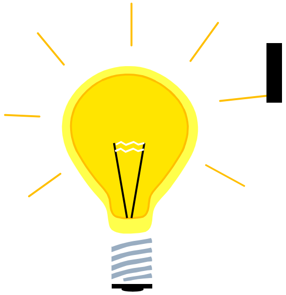

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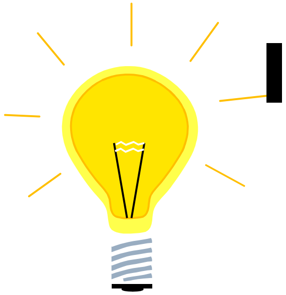

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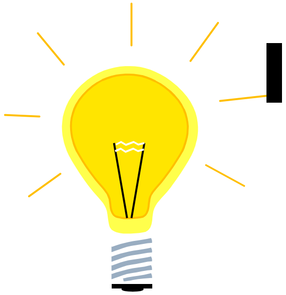

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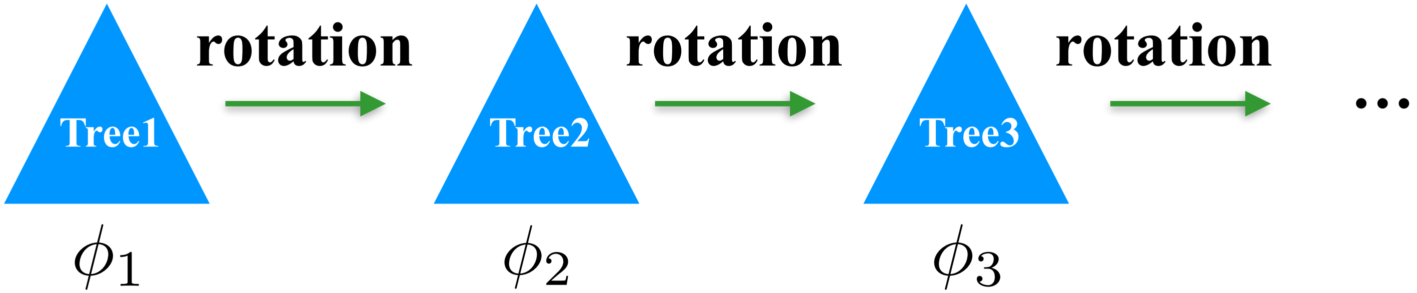
Analysis of Splay Trees

- What we want to bound?
  - The amortized cost of a sequence of operations, e.g. search, delete, insert, split…
  - Each operation involves slaying: a subsequence of rotations.
- The potential function is built on a state of tree. Let's consider the amortized cost of sequence of rotations first.



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Why not simply use the heights of the trees?

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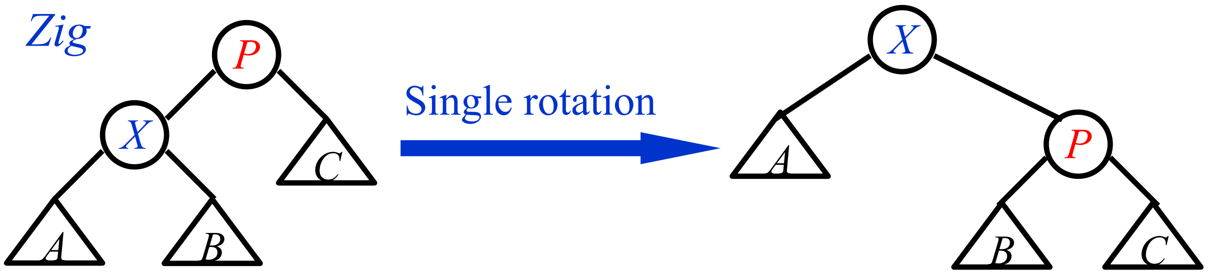
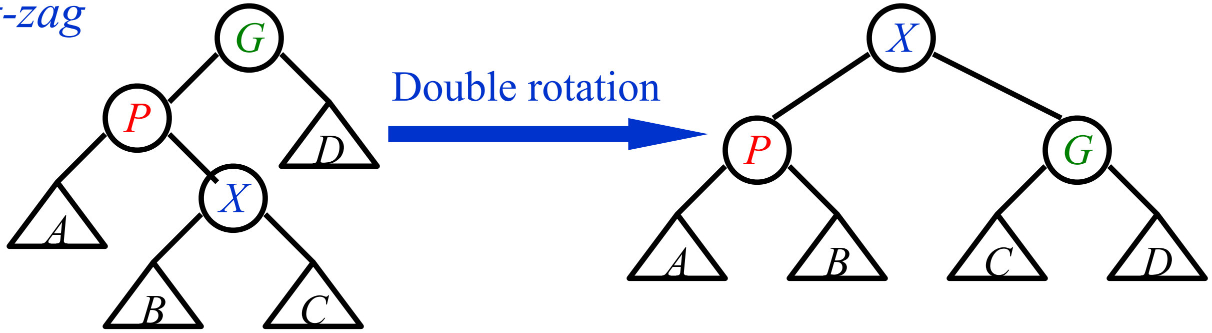
Zig


Single rotation

Zig-zag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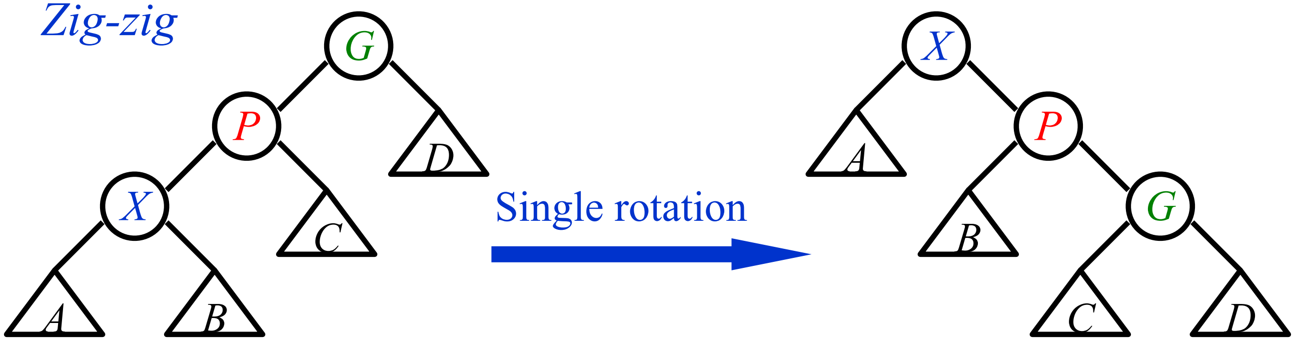
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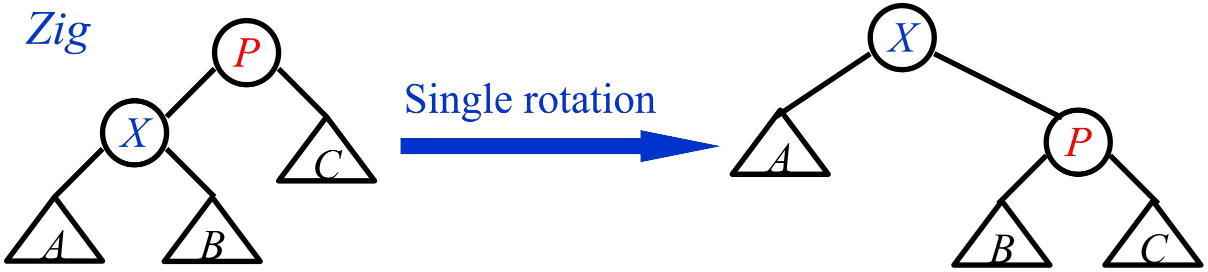

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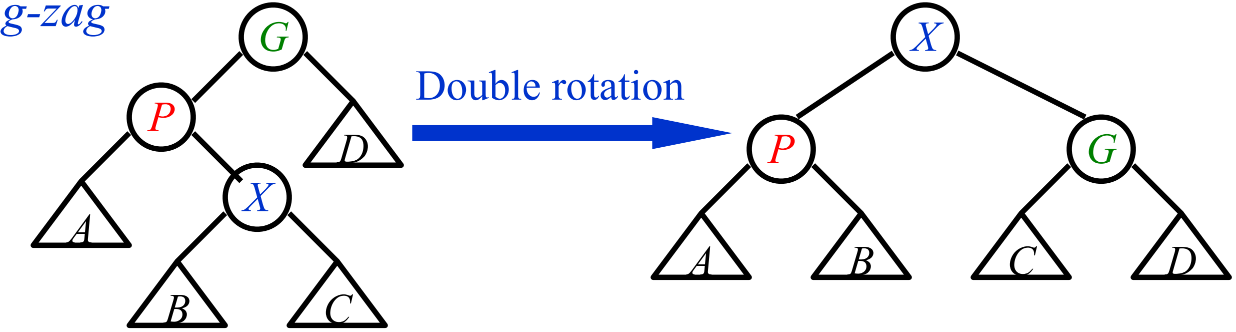

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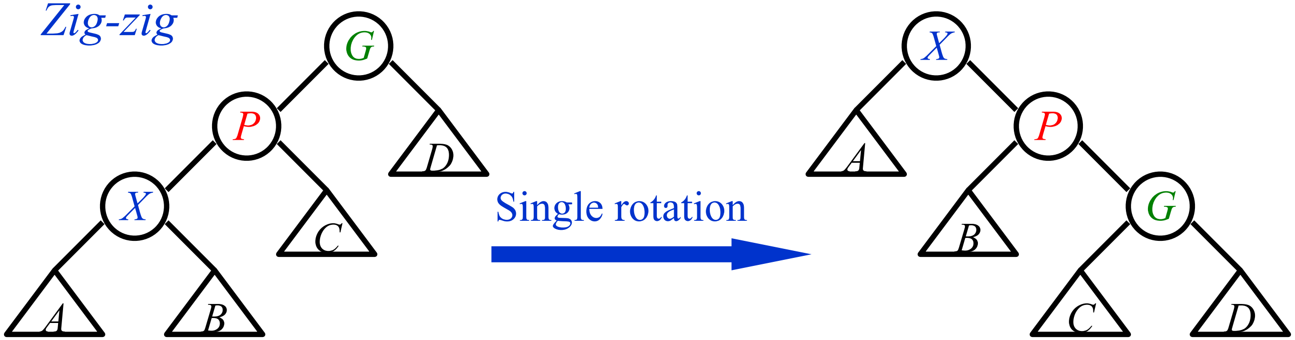

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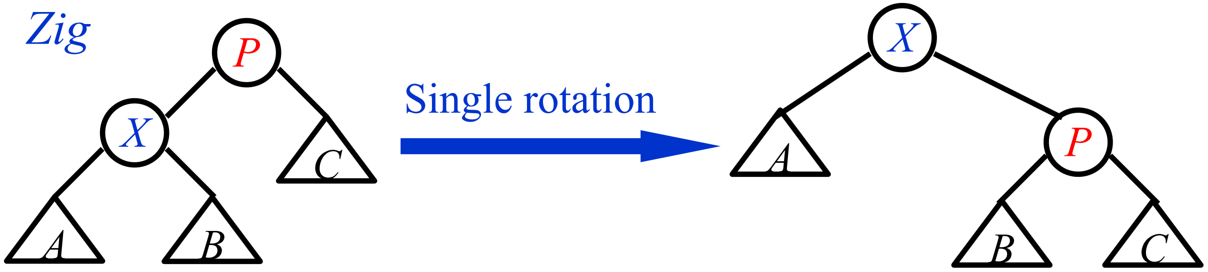

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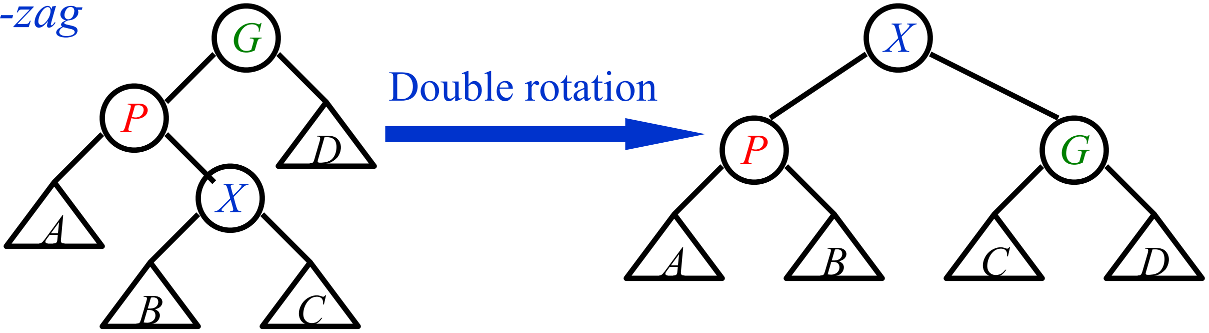

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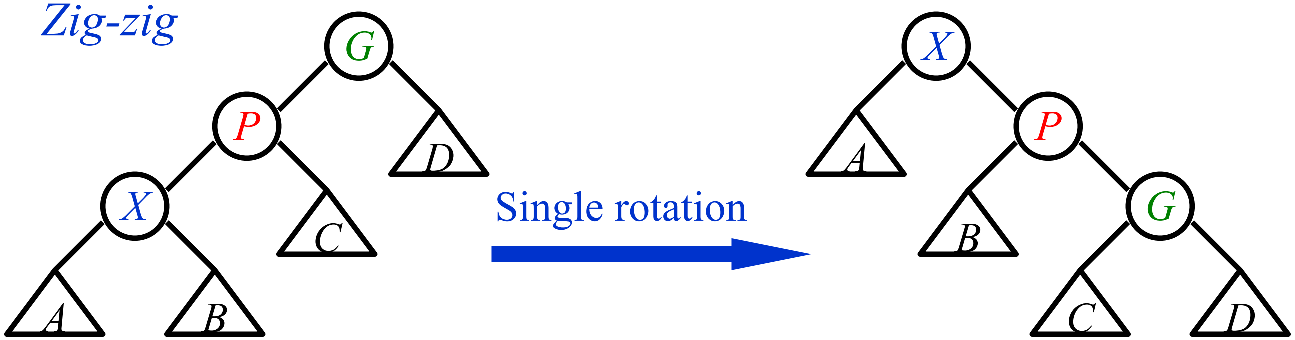

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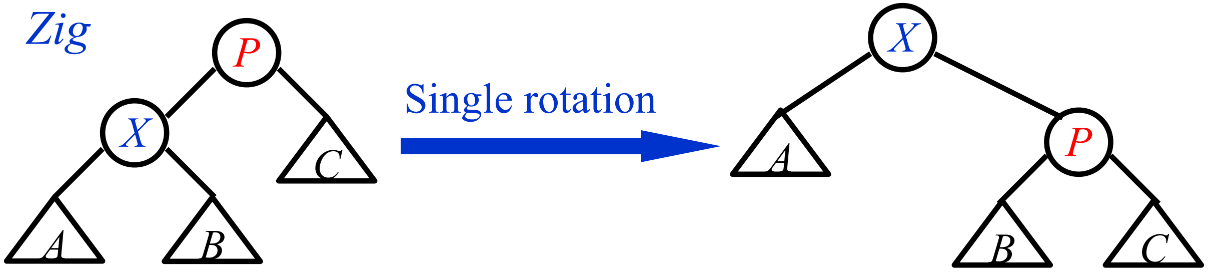
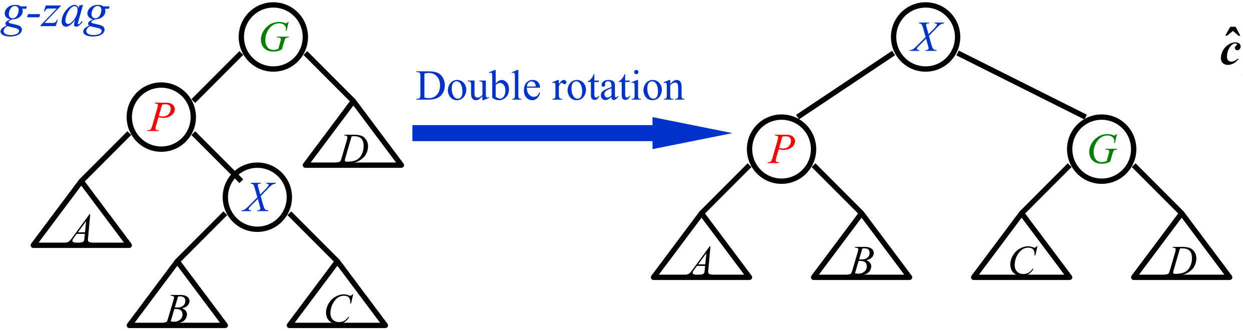


Single rotation

$$\begin{aligned}\hat{c}_i &= 1 + R_2(X) - R_1(X) \\ &\quad + \underline{R_2(P) - R_1(P)} \\ &\le 1 + R_2(X) - R_1(X)\end{aligned}$$

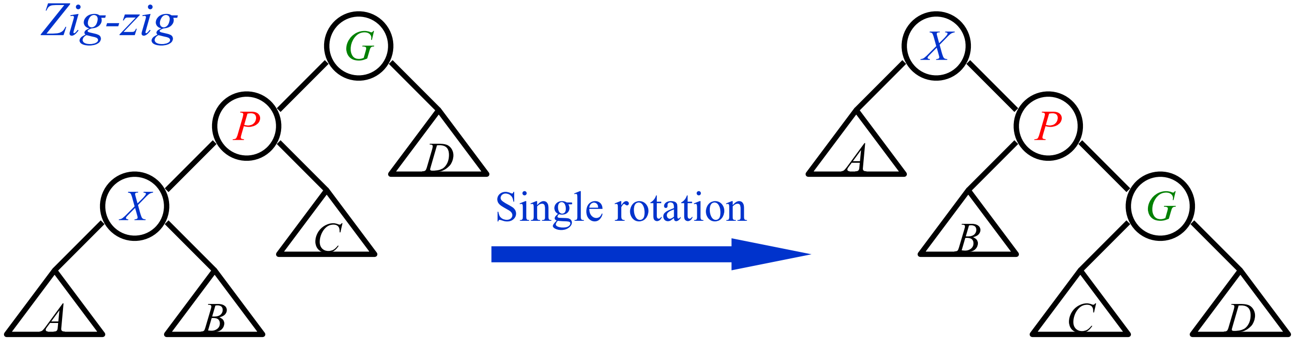
Zig-zag



Double rotation

$$\begin{aligned}\hat{c}_i &= 2 + R_2(X) - R_1(X) \\ &\quad + R_2(P) - R_1(P) \\ &\quad + R_2(G) - R_1(G) \\ &\le 2\big(R_2(X) - R_1(X)\big)\end{aligned}$$

Zig-zig



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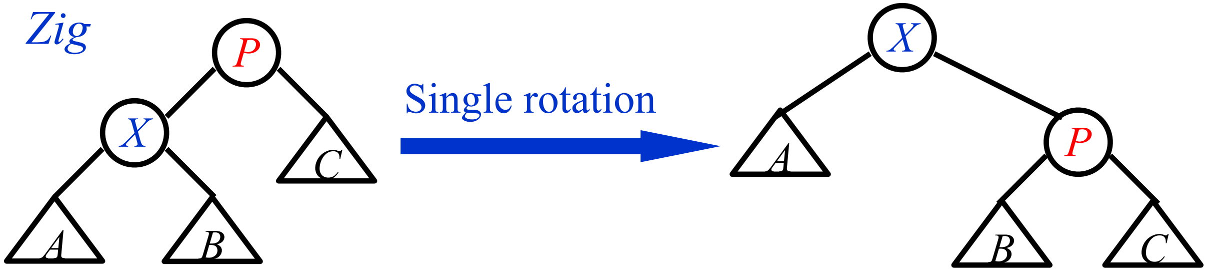

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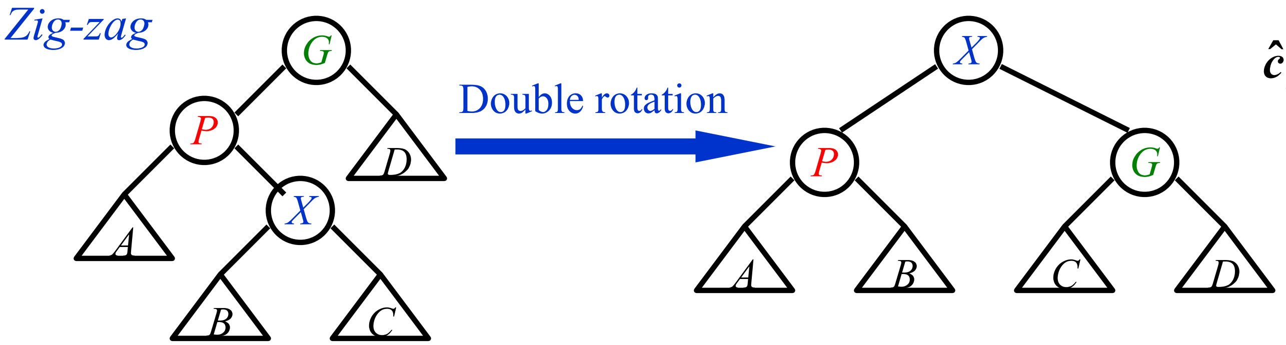
$$+ \underline{R_2(P) - R_1(P)}$$
$$\leq 1 + R_2(X) - R_1(X)$$



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$$+ R_2(G) - \cancel{R_1(G)}$$
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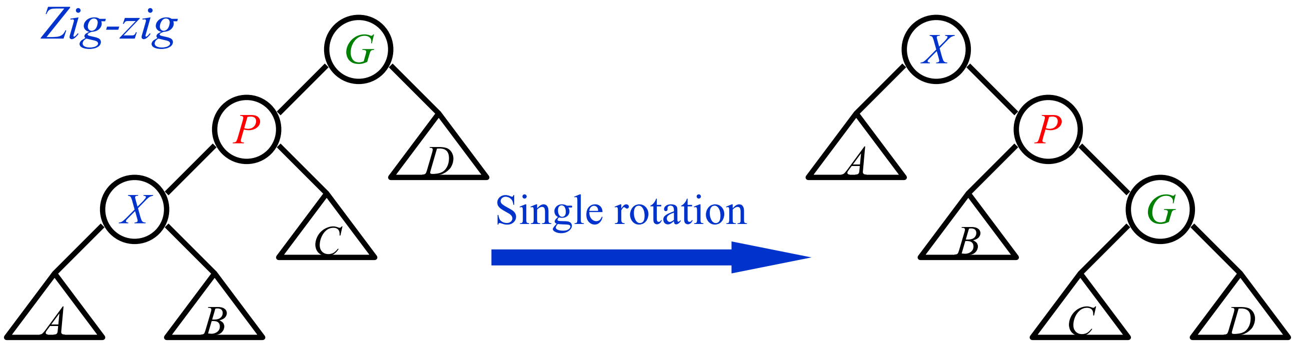
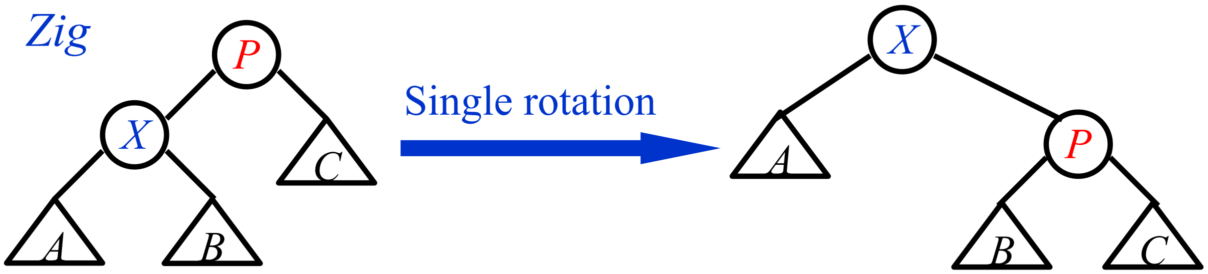


$$\Phi(T) = \sum_{i \in T} Rank(i)$$



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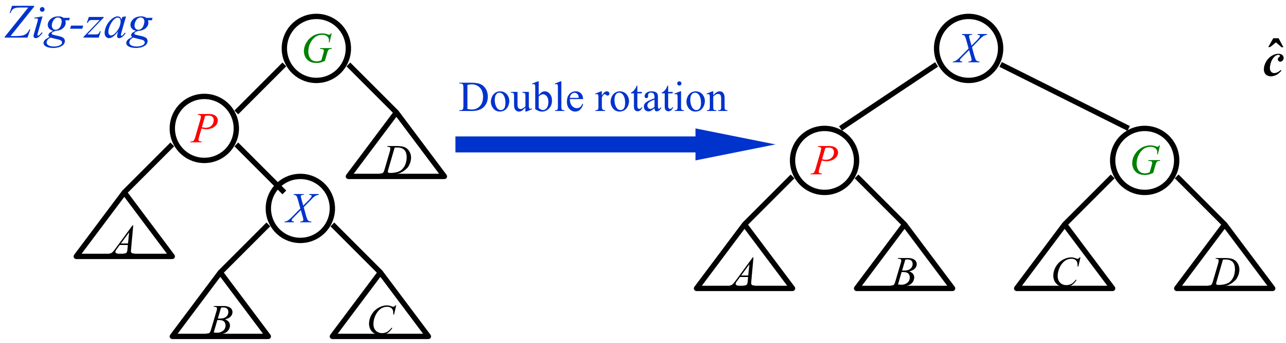
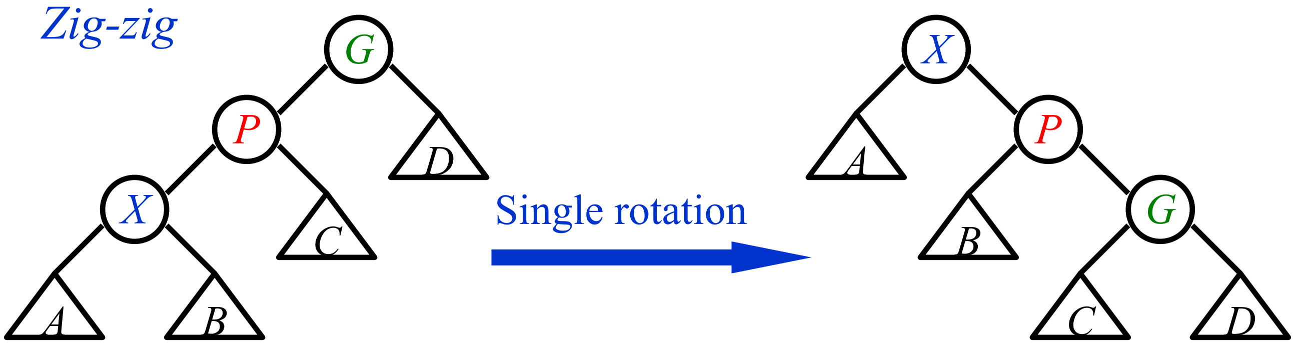
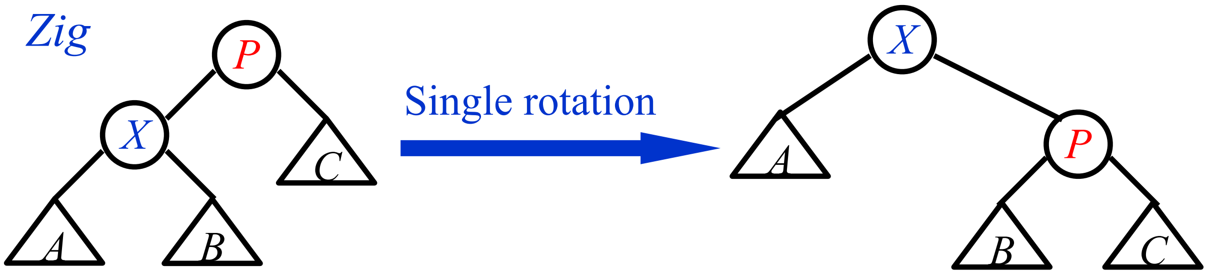


$$\begin{aligned} \hat{c}_i &= 2 + \cancel{R_2(X)} - \underline{R_1(X)} \\ &\quad + R_2(P) - \underline{R_1(P)} \\ &\quad + R_2(G) - \cancel{R_1(G)} \\ &\le 2\left(R_2(X) - \underline{R_1(X)}\right) \end{aligned}$$



$$\Phi(T) = \sum_{i \in T} Rank(i)$$


Zig

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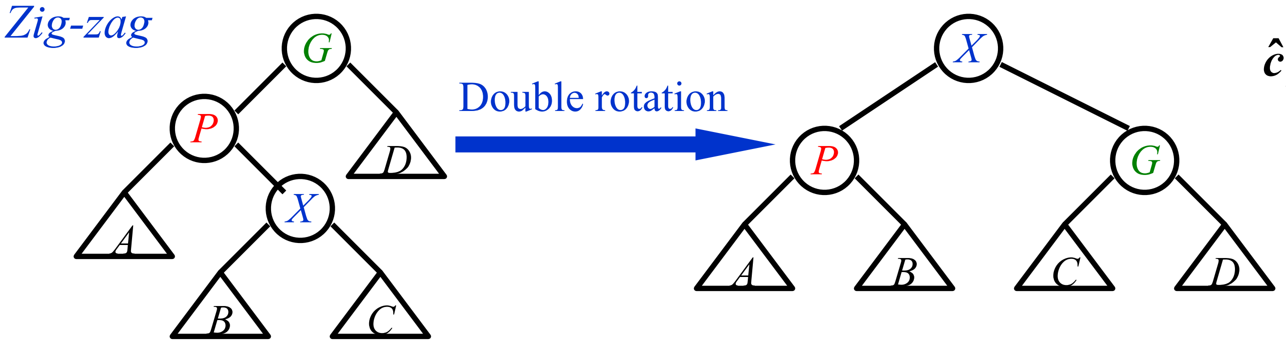
Zig-zag

Double rotation

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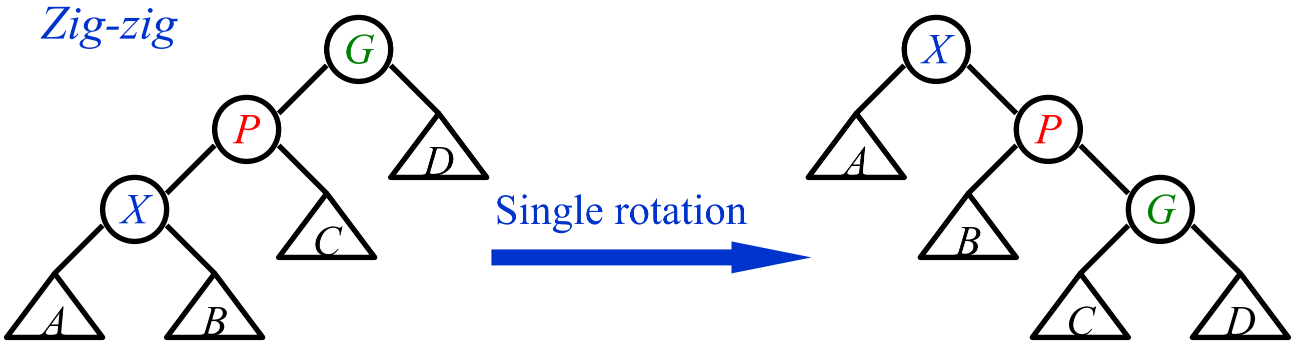
Lemma 11.4 on [Weiss] p.448


Zig-zig

Single rotation

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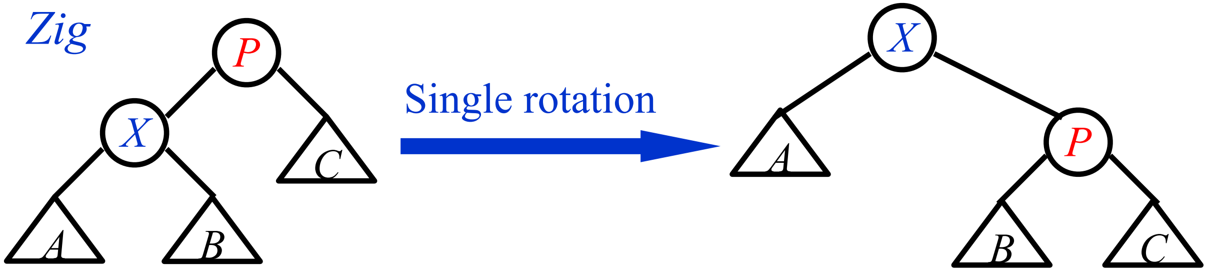
Zig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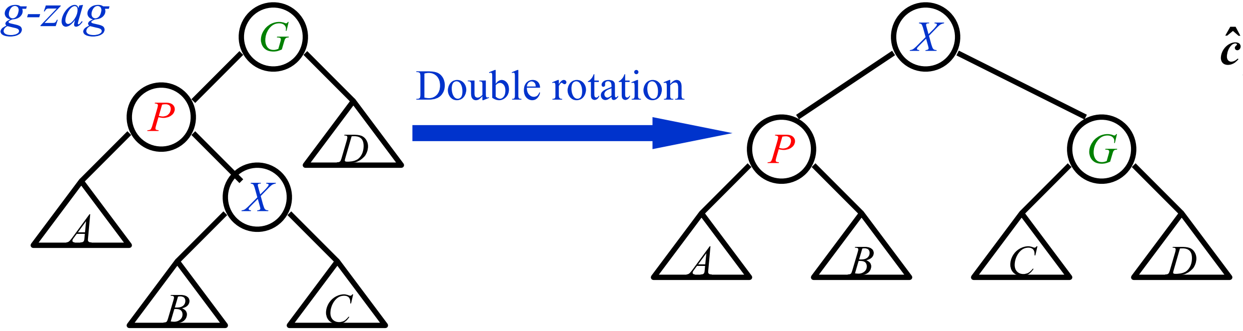
Zig-zag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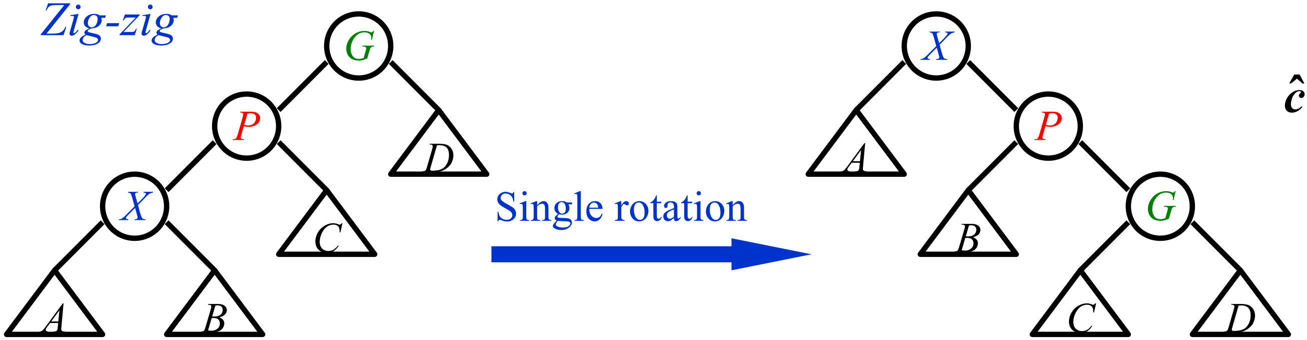
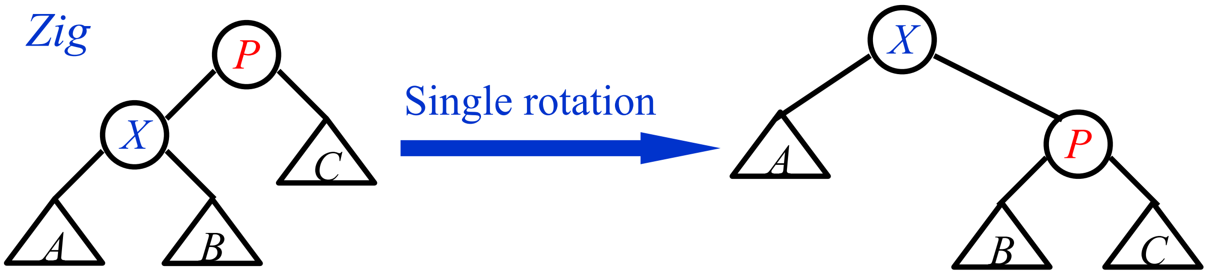
Zig-zig


Single rotation

$$\begin{aligned}\hat{c}_i &= 2 + R_2(X) - R_1(X) \\ &\quad + R_2(P) - R_1(P) \\ &\quad + R_2(G) - R_1(G) \\ &\le 3\left(R_2(X) - R_1(X)\right)\end{aligned}$$

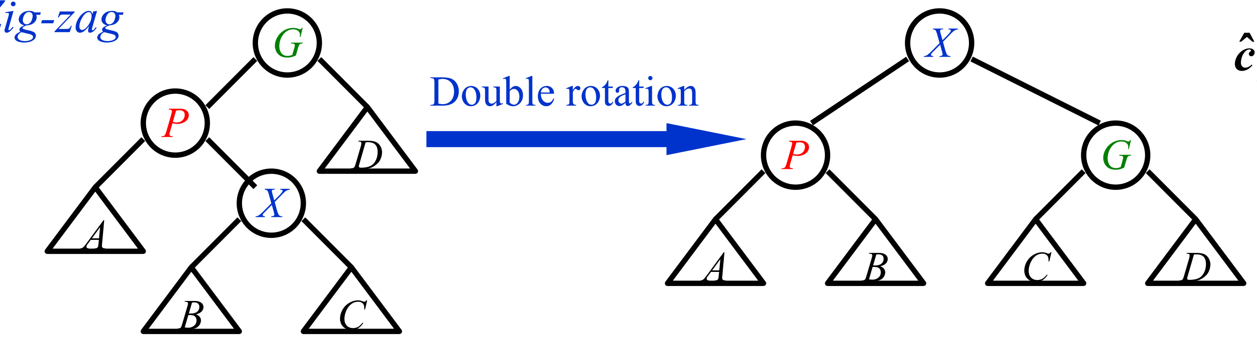
$$\Phi(T) = \sum_{i\in T} Rank(i)$$

Zig



$$\hat{c}_i = 1 + R_2(X) - R_1(X) + \underline{R_2(P) - R_1(P)} \le 1 + R_2(X) - R_1(X)$$

Zig-zag



$$\hat{c}_i = 2 + \cancel{R_2(X)} - \underline{R_1(X)} + R_2(P) - \underline{R_1(P)} + R_2(G) - \cancel{R_1(G)} \le 2\left(R_2(X) - \underline{R_1(X)}\right)$$

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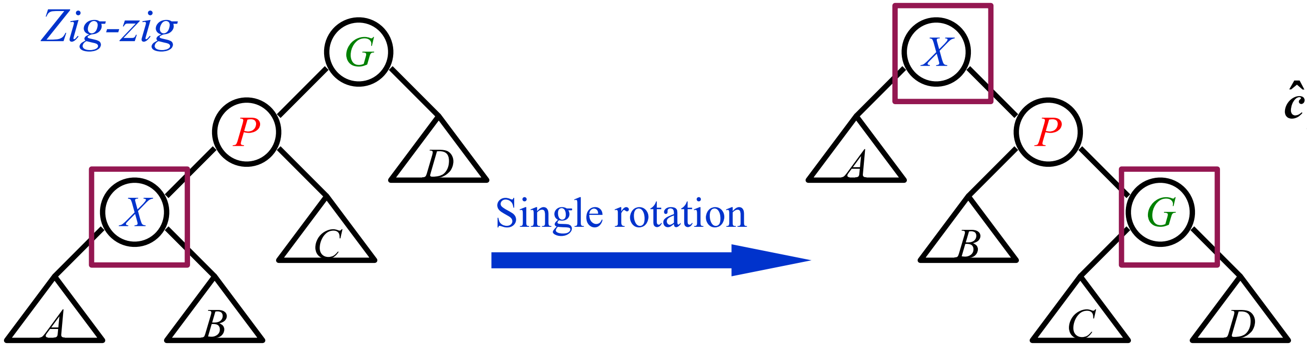
Zig-zig



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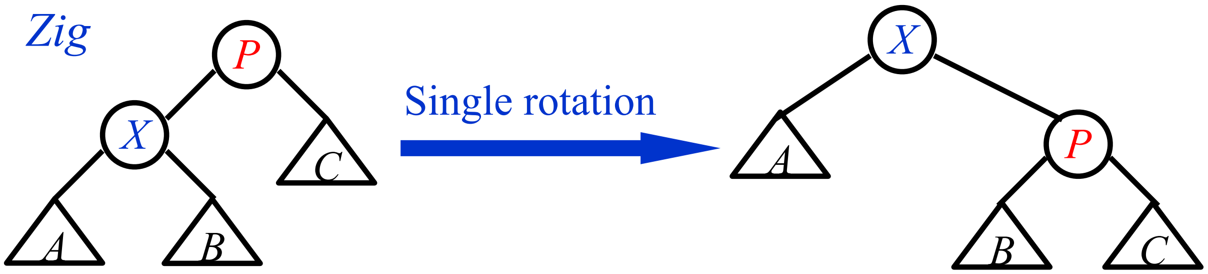

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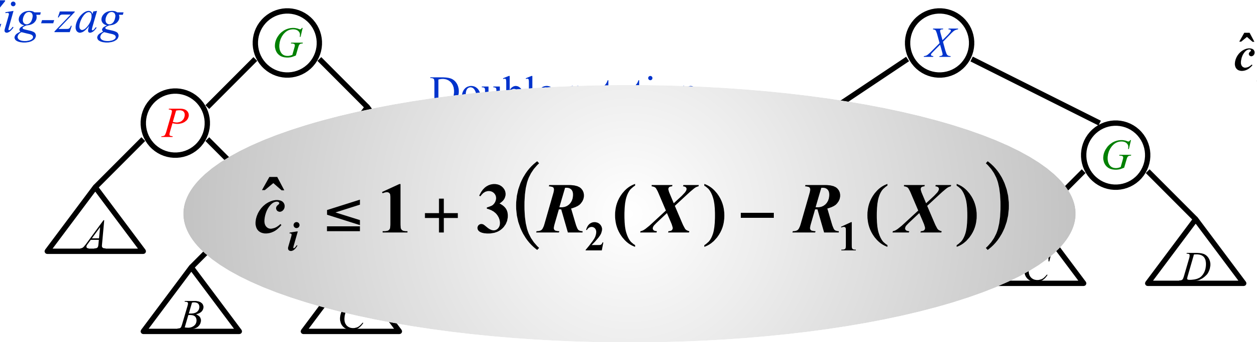
Zig-zag



$$\hat{c}_i \le 1 + 3\big(R_2(X) - R_1(X)\big)$$

$$\begin{aligned}\hat{c}_i &= 2 + \cancel{R_2(X)} - \underline{R_1(X)} \\ &\quad + R_2(P) - \underline{R_1(P)} \\ &\quad + R_2(G) - \cancel{R_1(G)} \\ &\le 2\big(R_2(X) - \underline{R_1(X)}\big)\end{aligned}$$

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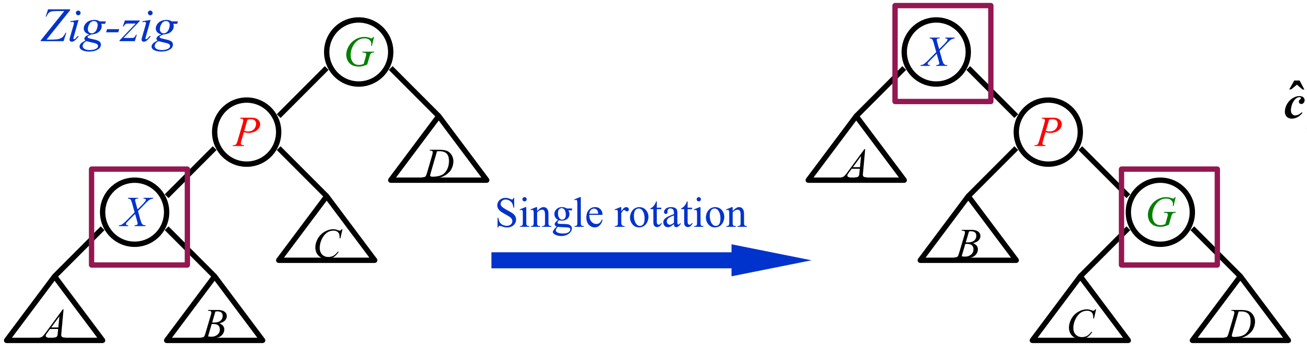
Zig-zig



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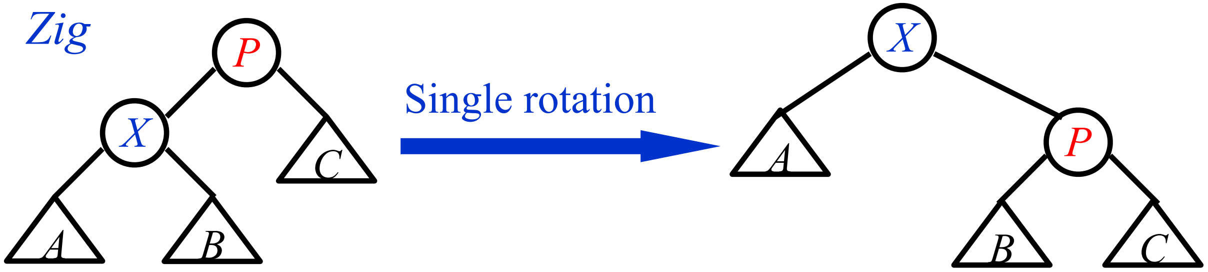

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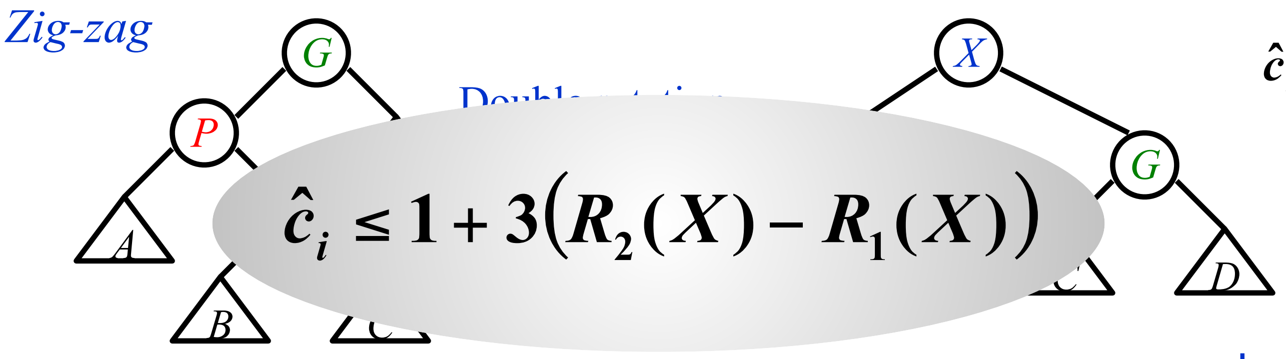

$$\hat{c}_i \le 1 + 3\left(R_2(X) - R_1(X)\right)$$

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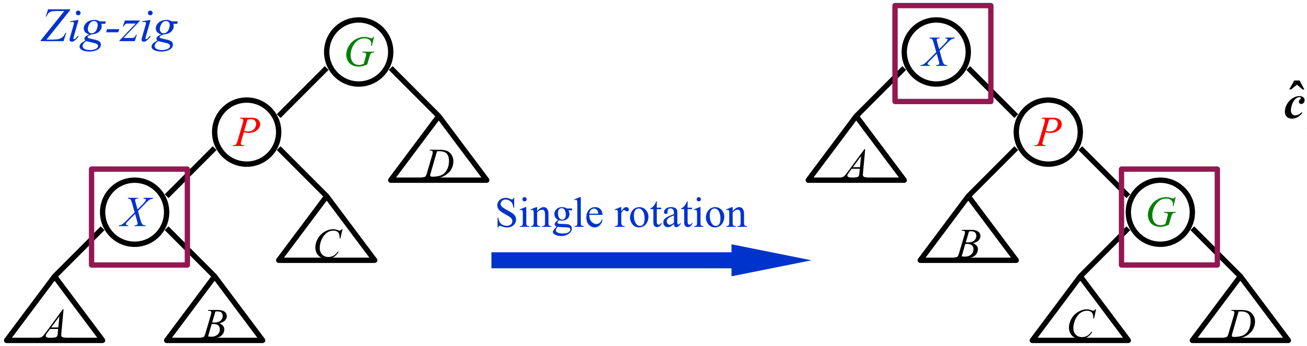

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【Lemma】 The total cost of $\sum \hat{c}_i$ to splay a tree by a series of rotations with root *T* at node *X* is at most 3(*R*(*T*) – *R* (*X*)) + 1.

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$$= \left(\sum_{i=1}^{n} c_i \right) + \underbrace{\Phi(D_n) - \Phi(D_0)}_{\geq 0}$$

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We should also consider the influences of other steps other than rotations on the potential functions.
Fortunately, their influences are minor.

Amortized Cost of Splay Trees

【Lemma】 The total cost of $\sum \hat{c}_i$ to splay a tree by a series of rotations with root T at node X is at most $3(\underbrace{R(T)}_{\text{bounded by log(N)}} - R(X)) + 1$.

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$$= \left(\sum_{i=1}^{n} c_i\right) + \underbrace{\Phi(D_n) - \Phi(D_0)}_{\geq 0}$$

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Fortunately, their influences are minor.

Theorem:
The amortized cost of a series of operations started from an empty splay tree is of order O(log N), where N is the number of all nodes involved in the operations.
Read the original splay tree paper for details.

Balanced Binary Search Trees (1)

- Binary search trees
- AVL trees
- Splay trees
- Amortized analysis
- Take-home messages

Take-Home Messages

- Balanced binary search trees:
    - Reduce depth to reduce cost of operations.
- AVL trees:
    - Satisfying height-balanced condition. Conduct rotations to achieve self-balancing once the condition is violated.
- Splay trees:
    - Achieving self-balancing by conducting splaying steps for any operations.
- Amortized analysis:
    - Averaging the total cost which is limited by the structure.

Thanks for your attention!
Discussions?

Reference

Data Structure and Algorithm Analysis in C (2nd Edition)： Chap. 4.4-4.5, 11.5.

Introduction to Algorithms (4th Edition): Chap.16.

Daniel Dominic Sleator, Robert Endre Tarjan:
Self-Adjusting Binary Search Trees. Journal of ACM 32(3): 652-686 (1985)