JOURNAL OF APPROXIMATION THEORY 7, 41–47 (1973)

Error Bounds for Bicubic Spline Interpolation

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Communicated by Garrett Birkhoff

Received June 9, 1970

1. Introduction

In [3] Birkhoff gives a "broad-brush" survey of recent developments in piecewise bicubic spline interpolation and approximation. He alludes (p. 192) to a result of the authors on fourth order convergence of bicubic spline interpolants [1, p. 236] in rectangles independent of the mesh ratios. The main purpose of this paper is to document that result by giving explicit error bounds for bicubic spline interpolation.

In addition, we give similar bounds for the cubic spline-blended interpolation scheme of Gordon [7].

The present results improve those of Schultz [10, p. 179], which require quasi-uniform partitions and which involve an unknown constant K. They also imply corollaries on the orders of convergence of quasi-interpolants and approximants "by moments" proved by de Boor and Fix [6] and Birkhoff [2]. In the present paper it is assumed that the function being approximated is in $C^4[R]$ whereas proofs involving tensor product considerations require stronger smoothness constraints.

2. Cubic Spline Interpolation

The following slight generalization (to include the cases $m = 1, 2$) of [9, Theorem 1] is germane to the development of our main result.

THEOREM 1. *Let s_f be the unique cubic spline interpolant of $f \in C^m[x_0, x_n]$*

such that $s_f(x_i) = f(x_i)$, $0 \leqslant i \leqslant n$ *and* $s_f'(x_i) = f'(x_i)$, $i = 0, n$. *Let* π *be an arbitrary partition of* $[x_0, x_n]$. *Then for* $m = 1, 2, 3$, or 4

$$\|(s_f - f)^{(r)}\|_\infty \leqslant \epsilon_{m,r} \|f^{(m)}\|_\infty h_x^{m-r}, \qquad 0 \leqslant r \leqslant \min\{m, 3\}, \tag{1}$$

where $h_x = \max_{0 \leqslant i \leqslant n-1} (x_{i+1} - x_i)$ *and the* $\epsilon_{m,r}$ *are given in Table* 1.

TABLE 1[a]

ϵ_{mr}	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$m = 1$	15/4	14	—	—
$m = 2$	9/8	4	10^b	—
$m = 3$	71/216	31/27	5^b	$(63 + 8\beta x^2)/9$
$m = 4$	5/384	$(9 + \sqrt{3})/216$	5^b	$(2 + \beta x^2)/4$

[a] $\beta_x = h_x / \min_{0 \leqslant j \leqslant n-1} (x_{j+1} - x_j)$ is the mesh ratio.
[b] See [9, p. 210].

Proof. The proof of (1) for $m = 1$, 2, or 3 parallels the proof in [9, Theorem 1] for the case $m = 4$; hence, it will only be sketched here. As in [9] we write

$$s_f - f = (s_f - u_f) + (u_f - f) \tag{2}$$

where u_f is the unique cubic Hermite interpolant of f which satisfies $u_f^{(r)}(x_i) = f^{(r)}(x_i)$, $0 \leqslant i \leqslant n$, $r = 0, 1$. Bounds for the two terms on the right in (2) are derived from the following sequence of three lemmas. Since these lemmas are simple extensions of Lemmas 1–3 in [9, p. 211–213], we do not include their proofs.

LEMMA 1. *If* $f \in C^m[a, b]$, *then, for* $m = 1, 2, 3$, or 4,

$$\|(u_f - f)^{(r)}\|_\infty^1 \leqslant \alpha_{mr} \|f^{(m)}\|_\infty h_x^{m-r}, \qquad 0 \leqslant r \leqslant \min\{m, 3\}, \tag{3}$$

where the α_{mr} *are given in Table* 2.

TABLE 2

α_{mr}	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$m = 1$	5/4	4	—	—
$m = 2$	3/4	5/2	12	—
$m = 3$	7/24	1	5	7
$m = 4$	1/384	$\sqrt{3}/216$	1/12	1/2

[1] Although $u_{f_1}^{(2)}$ is only piecewise continuous, this symbol is used to denote the fact that both the right and left hand derivatives satisfy (3) at the points of discontinuity.

Lemma 1 establishes bounds for $\|(u_f - f)^{(r)}\|_\infty$ in (2). The next two lemmas establish bounds for $\|(s_f - u_f)^{(r)}\|_\infty$ in (2).

LEMMA 2. *If $f \in C^m[a, b]$, for $m = 1, 2, 3,$ or 4 then, for each mesh point $x = x_i$,*

$$| s_f(x_i) - f(x_i)| \leqslant \gamma_m \|f^{(m)}\|_\infty h_x^{m-1} \tag{4}$$

where $\gamma_1 = 10$, $\gamma_2 = 3/2$, $\gamma_3 = 4/27$, and $\gamma_4 = 1/24$.

LEMMA 3. *Let $f \in C^m[a, b]$, then for $m = 1, 2, 3,$ or 4,*

$$\|(s_f - u_f)^{(r)}\|_\infty \leqslant \nu_{mr} \|f^{(m)}\|_\infty h_x^{m-r}, \qquad 0 \leqslant r \leqslant m - 1,$$

where the ν_{mr} are given in Table 3.

TABLE 3

ν_{mr}	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$m = 1$	5/2	10	—	—
$m = 2$	3/8	3/2	$9\beta_x$	—
$m = 3$	1/27	4/27	$8\beta_x/9$	$8\beta_x^2/9$
$m = 4$	1/96	1/24	$\beta_x/4$	$\beta_x^2/4$

The ν_{mr} listed in Table 3 are bounds for the first term on the right side in (2). The proof is then completed by considering the triangle inequality.

Theorem 1 establishes the convergence of the sequence of univariate cubic spline interpolants s_f to $f \in C^m[a, b]$, $m = 1, 2, 3,$ or 4 as $h_x \to 0$ *independent* of the mesh ratios, i.e., independent of the way in which π is successively refined.

3. Cubic Spline Blended Interpolant

One of the most intriguing and versatile developments in interpolation and approximation theory has been the utilization of *blending function techniques* which have been developed by Gordon [7, 8]. Our discussion here will be limited to *cubic spline blended interpolants* over rectangular domains with rectangular partitions.

Let f be a sufficiently smooth function with domain

$$\Omega = [x_0, x_n] \times [y_0, y_{n'}].$$

The cubic spline blended interpolant, S_f, of the function f is given by

$$S_f = \Phi_f + \psi_f - s_f, \tag{5}$$

where

$$\Phi_f(x, y) = \sum_{i=-1}^{n+1} f(x_i, y)\, \phi_i(x), \tag{6}$$

$$\psi_f(x, y) = \sum_{j=-1}^{n'+1} f(x, y_j)\, \psi_j(y), \qquad \text{and} \tag{7}$$

$$s_f(x, y) = \sum_{j=-1}^{n'+1} \sum_{i=-1}^{n+1} f(x_i, y_j)\, \phi_i(x)\, \psi_j(y). \tag{8}$$

For ease of notation we use the convention that $f(x_{-1}, y) = \partial f(x_0, y)/\partial x$, $f(x, y_{-1}) = \partial f(x, y_0)/\partial y$ with the analogues for $f(x_{n+1}, y)$, $f(x, y_{n'+1})$, $f(x_{-1}, y_{-1})$ etc. The functions $\phi_i(x)$ and $\psi_j(y)$ are the usual (univariate) *cardinal* splines [see e.g. 1, 7].

We observe that

$$\begin{aligned} \Phi_f(x_i, y) &= f(x_i, y), \qquad -1 \leqslant i \leqslant n+1, \\ \psi_f(x, y_j) &= f(x, y_j), \qquad -1 \leqslant j \leqslant n'+1, \end{aligned} \tag{9}$$

and that $s_f(x, y)$ is the usual bicubic spline interpolant of f, [4, p. 214]. It follows from (5)–(9) that $S_f(x_i, y) = f(x_i, y)$, $-1 \leqslant i \leqslant n+1$, and $S_f(x, y_j) = f(x, y_j)$, $-1 \leqslant j \leqslant n'+1$.

Gordon has shown [8] that for $f \in C^{(4,4)}[\Omega]$,

$$\|(S_f - f)^{(k,l)}\|_\infty = O(h_x^{4-k} h_y^{4-l}),$$

where $g^{(k,l)} \equiv \partial g^{(k+l)}/\partial x^k\, \partial y^l$. We now give an alternate proof of this result in such a way as to establish explicit error bounds.

THEOREM 2. *Let $f \in C^{(4,4)}[\Omega]$, then*

$$\|(f - S_f)^{(k,l)}\|_\infty \leqslant \epsilon_{4k}\epsilon_{4l}\, \|f^{(4,4)}\|_\infty\, h_x^{4-k} h_y^{4-l}, \qquad 0 \leqslant k, l \leqslant 3$$

where values for ϵ_{4-} are given in Table 1.

Proof. It follows from (5) that

$$f - S_f = (f - \Phi_f) - (\psi_f - s_f) \equiv R_1 - R_2. \tag{10}$$

For each fixed $y = y^*$, $\Phi_f^{(0,l)}(x, y^*)$ is the unique cubic spline interpolant of $f^{(0,l)}(x, y^*)$, $0 \leqslant l \leqslant 4$. Since $f^{(0,l)}(x, y^*) \in C^{(4)}[x_0, x_n]$, it follows from Theorem 1 that

$$|R_1^{(k,l)}(x, y^*)| \leqslant \epsilon_{4k} \max_{x_0 \leqslant x \leqslant x_n} |f^{(4,l)}(x, y^*)|\, h_x^{4-k}, \qquad 0 \leqslant k \leqslant 3.$$

Next, consider $R_2(x, y)$ defined in (10). Substituting (7) and (8) into this expression we obtain

$$\begin{aligned} R_2(x, y) &= \sum_{j=-1}^{n'+1} \left\{ (f(x, y_j) - \sum_{i=-1}^{n+1} f(x_i, y_j)\, \phi_i(x) \right\} \psi_j(y) \\ &= \sum_{j=-1}^{n'+1} R_1(x, y_j)\, \psi_j(y) = \psi_{R_1}(x, y). \end{aligned} \tag{12}$$

Substituting (12) into (10) reveals

$$f - S_f = R_1 - \psi_{R_1}, \tag{13}$$

where $R_1 \in C^{(2,4)}[\Omega]$. For each fixed $x = x^*$, the right side of (13) is the difference of $R_1(x^*, y)$ and its cubic spline interpolant. Applying Theorem 1 and (11) to (13) we obtain

$$\begin{aligned} |(f - S_f)^{(k,l)}| &\leqslant |(R_1 - \psi_{R_1})^{(k,l)}| \leqslant \epsilon_{4l} \max_{y_0 \leqslant y \leqslant y_{n'}} |R_1^{(k,4)}|\, h_y^{4-l} \\ &\leqslant \epsilon_{4k}\epsilon_{4l} \| f^{(4,4)} \|_\infty h_x^{4-k} h_y^{4-l} \end{aligned} \tag{14}$$

and the proof of Theorem 2 is complete.

4. Bicubic Spline Interpolation

Our main result establishes the fourth order convergence of bicubic spline interpolation *independent* of the mesh ratio. This result, which includes *explicit* error bounds is given by:

THEOREM 3. *Let $f \in C^4[\Omega]$ where $\Omega = [x_0, x_n] \times [y_0, y_{n'}]$. Then*

$$\begin{aligned} \|(f - s_f)^{(k,l)}\|_\infty \leqslant{} & \epsilon_{4-l,k} \| f^{(4-l,l)} \|_\infty h_x^{4-k} + \epsilon_{2k}\epsilon_{2l} \| f^{(2,2)} \|_\infty h_x^{2-k} h_y^{2-l} \\ & + \epsilon_{4-k,l} \| f^{(k,4-k)} \|_\infty h_y^{4-l}, \qquad 0 \leqslant k, l \leqslant 2 \end{aligned} \tag{15}$$

where the ϵ_{ij} are given in Table 1.

Proof. Let ψ_f be the cubic spline blended interpolant defined by (7) and consider

$$f - s_f = (f - \psi_f) + (\psi_f - s_f). \tag{16}$$

For each fixed $x = x^*$, $\psi_f^{(k,0)}(x^*, y)$ is the cubic spline interpolant of $f^{(k,0)}(x^*, y)$. Thus, from Theorem 1

$$|(f - \psi_f)^{(k,l)}(x^*, y)| \leqslant \epsilon_{4-k,l} \max_{y_0 \leqslant y \leqslant y_{n'}} |f^{(k,4-k)}(x^*, y)|\, h_y^{(4-k)-l}. \tag{17}$$

Next consider

$$\psi_f - s_f = R_2 = \sum_{j=-1}^{n'+1} R_1(x, y_j)\, \psi_j(y) = \psi_{R_1} = (\psi_{R_1} - R_1) + R_1 . \tag{18}$$

For each fixed $y = y^*$, $R_1^{(0,l)}(x, y^*)$ has $(4 - l)$ continuous derivatives with respect to x. From Theorem 1, for $0 \leqslant k \leqslant 2$

$$| R_1^{(k,l)}(x, y^*)| \leqslant \epsilon_{4-l,k} \max_{x_0 \leqslant x \leqslant x_n} | f^{(4-l,k)}(x, y^*)|\, h_x^{(4-l)-k}. \tag{19}$$

Since $R_1 \in C^{(2,4)}[\Omega]$, we have for each fixed $x = x^*$ and each fixed l, $0 \leqslant l \leqslant 2$,

$$\begin{aligned}|(\psi_{R_1} - R_1)^{(k,l)}(x^*, y)| &\leqslant \epsilon_{2l} \max_{y_0 \leqslant y \leqslant y_{n'}} | R_1^{(k,2)}(x^*, y)|\, h_y^{2-l} \\ &\leqslant \epsilon_{2k}\epsilon_{2l} \| f^{(2,2)} \|_\infty h_x^{2-k} h_y^{2-l}, \qquad 0 \leqslant k \leqslant 2.\end{aligned}$$

The error bounds given in Theorem 3 then follow from (16) and the triangle inequality, and the proof is complete.

The following is an immediate consequence of the above Theorem.

COROLLARY. *Let s_f be the bicubic spline interpolant of $f \in C^4[\Omega]$, then as* $h = \max(h_x, h_y) \to 0$,

$$\|(s_f - f)^{(k,l)}\|_\infty = O(h^{4-(k+l)}), \qquad 0 \leqslant k, l \leqslant 2$$

independent of the mesh ratio.

The proof of Theorem 3 can easily be modified to establish:

THEOREM 4. *Let $f \in C^{(4,4)}[\Omega]$ where $\Omega = [x_0, x_n] \times [y_0, y_{n'}]$. Then*

$$\begin{aligned}\|(f - s_f)^{(k,l)}\|_\infty &\leqslant \delta_{4k} \| f^{(4,0)} \|_\infty h_x^{4-k} + \delta_{4l} \| f^{(0,4)} \|_\infty h_y^{4-l} \\ &\quad + \delta_{4k}\delta_{4l} \| f^{(4,4)} \|_\infty h_x^{4-k} h_y^{4-l}, \qquad 0 \leqslant k, l \leqslant 2.\end{aligned}$$

Hence as h_x and $h_y \to 0$

$$\|(f - s_f)^{(k,l)}\|_\infty = O(h_x^{4-k} + h_y^{4-l}), \qquad 0 \leqslant k, l \leqslant 2,$$

independent of the mesh ratios β_x and β_y.

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