CSC578C: Assignment 3

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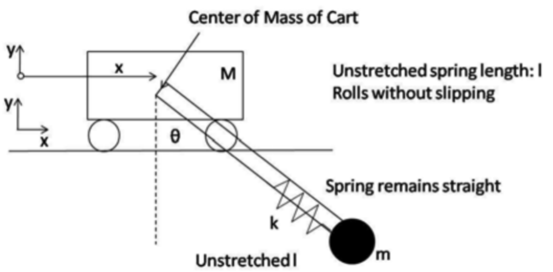
1 Cube with Spring-Mass-Pendulum

This system is a spring-mass-pendulum system. As shown in the picture, a cube of mass M is able to move along a frictionless, horizontal track. The movement distance of this cube is x. A spring, of force constant k, is attached between the cube and a sphere of mass m. The upstretched length of this spring is l and the angle of inclination is θ.



2 Problem Explanation

In this problem, there are three generalized coordinates, x, θ and s.

2.1 Kinetic Energy

First we need to find the velocities of the cube and sphere.

For CUBE:
$x_M = x$
$y_M = 0$
Then we can get the velocity of cube:
$v_M = \dot{x}$
$v_M^2 = \dot{x}^2$

For SPHERE:
$x_m = x + s\sin\theta$
$y_m = -s\cos\theta$
Then we can get the velocity of sphere:
$\dot{x_m} = \dot{x} + \dot{s}\sin\theta + s\dot{\theta}\cos\theta$
$\dot{y_m} = -\dot{s}\cos\theta + s\dot{\theta}\sin\theta$
$v_m^2 = \dot{x_m}^2 + \dot{y_m}^2 = \dot{x}^2 + \dot{s}^2 + s^2\dot{\theta}^2 + 2\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)$
Therefore we can get the kinetic energy:
$T = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$
$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[\dot{x}^2 + \dot{s}^2 + s^2\dot{\theta}^2 + 2\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)]$
$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(\dot{s}^2 + s^2\dot{\theta}^2) + m\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)$

2.2 Potential Energy

In this system there are two forces: gravity and elastic force.
$U = -mgs\cos\theta + \frac{1}{2}k\Delta s^2$
$= -mgs\cos\theta + \frac{1}{2}k(s-l)^2$

2.3 Lagrange Equation

From the following formulas:

$$L = T - U \tag{1}$$

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{2}$$

we can get $L = T - U = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(\dot{s}^2 + s^2\dot{\theta}^2) + m\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta) + mgs\cos\theta - \frac{1}{2}k(s-l)^2$

As we mentioned, there are three generalized coordinates, x, θ and s and we need to do partial derivative for each of them.

Find L for x:
$\frac{\partial L}{\partial x} = 0$
$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)$
$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = (M+m)\ddot{x} + m(\ddot{s}\sin\theta + \dot{s}\dot{\theta}\cos\theta + \dot{s}\dot{\theta}\cos\theta + s\ddot{\theta}\cos\theta - s\dot{\theta}^2\sin\theta)$
Then we can get: $(M+m)\ddot{x} + m(\ddot{s}\sin\theta + 2\dot{s}\dot{\theta}\cos\theta + s\ddot{\theta}\cos\theta - s\dot{\theta}^2\sin\theta) = 0$

$\ddot{x}$

$\ddot{x} = -m(\ddot{s}\sin\theta + 2\dot{s}\dot{\theta}\cos\theta + s\ddot{\theta}\cos\theta - s\dot{\theta}^2\sin\theta)/(M+m)$

Find L for θ:
$\frac{\partial L}{\partial \theta} = m\dot{x}(\dot{s}\cos\theta - s\dot{\theta}\sin\theta) - mgs\sin\theta$
$\frac{\partial L}{\partial \dot{\theta}} = ms^2\dot{\theta} + m\dot{x}s\cos\theta$
$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = 2ms\dot{s}\dot{\theta} + ms^2\ddot{\theta} + m(\ddot{x}s\cos\theta + \dot{x}\dot{s}\cos\theta - \dot{x}s\dot{\theta}\sin\theta)$
Then we can get: $m\dot{x}(\dot{s}\cos\theta - s\dot{\theta}\sin\theta) - mgs\sin\theta - 2ms\dot{s}\dot{\theta} - ms^2\ddot{\theta} - m(\ddot{x}s\cos\theta + \dot{x}\dot{s}\cos\theta - \dot{x}s\dot{\theta}\sin\theta) = 0$
$s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0$

$\ddot{\theta}$

$\ddot{\theta} = -(2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta)/s$

Find L for s:
$\frac{\partial L}{\partial s} = m\dot{\theta}^2 s + m\dot{x}\dot{\theta}\cos\theta + mg\cos\theta - k(s-l)$
$\frac{\partial L}{\partial \dot{s}} = m\dot{x}\sin\theta + m\dot{s}$
$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} = m\ddot{x}\sin\theta + m\dot{x}\dot{\theta}\cos\theta + m\ddot{s}$
Then we can get: $m\dot{\theta}^2 s + m\dot{x}\dot{\theta}\cos\theta + mg\cos\theta - k(s-l) - m\ddot{x}\sin\theta - m\dot{x}\dot{\theta}\cos\theta - m\ddot{s} = 0$
$\dot{\theta}^2 s + g\cos\theta - \frac{k}{m}(s-l) - \ddot{x}\sin\theta - \ddot{s} = 0$

$\ddot{s}$

$\ddot{s} = \dot{\theta}^2 s + g\cos\theta - \frac{k}{m}(s-l) - \ddot{x}\sin\theta$

3 Implementation

From Lagrange Equation we can get values of $\ddot{x}$,$\ddot{\theta}$ and $\ddot{s}$(i.e. accelerations).Then we can update velocities $\dot{x}$,$\dot{\theta}$,$\dot{s}$ correspondingly. Finally we will get new values of x, θ and s and update the positions. The screenshots of implementation are shown as follows.