THE CAUCHY-KOWALEVSKAYA-KASHIWARA THEOREM

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This is not a self-contained note. It should be used as a supplement for the book [Kas03]. We fill the detail and prove some propositions whose proof are omitted in the book. We try to understand some complicated material by considering in the case of the Wely algebra.

Notations

X: complex manifold (or smooth variety)
$\mathcal{O}_X$: sheaf of holomorphic functions on X (or regular functions on X)
$\mathcal{D}_X$: sheaf of ring of differential operators on X
Θ_X: sheaf of vector fields on X

1. D-modules

Let X be an n-dimensional complex manifold, and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. A differential operator on X is a sheaf morphism P from $\mathcal{O}_X$ to $\mathcal{O}_X$. Let $(x_1, \dots, x_n)$ be the local coordinate then P can be locally written as

$$Pu(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha(x) \partial_x^\alpha u(x) \quad (u \in \mathcal{O}_X)$$

where $a_\alpha(x)$ are holomorphic functions. Here for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we set

$$\partial_x^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

and assume $a_\alpha(x) = 0$ except for finitely many α. We call P a differential operator of order at most m if $a_\alpha(x) = 0$ for all α with $|\alpha| := \alpha_1 + \cdots + \alpha_n > m$. We denote by $\mathcal{D}_X$ the sheaf of differential operators on X. The sheaf $\mathcal{D}_X$ has a ring structure with the composition of operators as multiplication, and $\mathcal{O}_X$ is a subring of $\mathcal{D}_X$. Let Θ_X denote the sheaf of vector fields on X. Then $\mathcal{D}_X$ contains Θ_X. Note that Θ_X is a left $\mathcal{O}_X$-submodule of $\mathcal{D}_X$, but not a right $\mathcal{O}_X$-submodule. Let $v \in \Theta_X$ and $a \in \mathcal{O}_X$. Then

$$[v, a] := va - av = v(a),$$

where we denote by $v(a)$ the holomorphic function obtained by operating by v on a. This is an immediate consequence of the Leibniz rule $v(ab) = v(a)b + av(b)$, and shows that $\mathcal{D}_X$ is a non-commutative ring.

The simplest $\mathcal{D}_X$-module is $\mathcal{O}_X$. It is generated by 1 as a $\mathcal{D}_X$-module, and the kernel of the map $\mathcal{D}_X \to \mathcal{O}_X, P \mapsto P \cdot 1$ is $\mathcal{D}_X\Theta_X \subset \mathcal{D}_X$. Hence we have an exact sequence of $\mathcal{D}_X$-modules

$$\mathcal{D}_X \otimes \mathcal{O}_X \Theta_X \xrightarrow{\delta_1} \mathcal{D}_X \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where

$$\delta_1(P \otimes v) = Pv, \quad P \in \mathcal{D}_X, v \in \Theta_X.$$

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If $\mathcal{F}$ is a $\mathcal{D}_X$-module, then by the left-exactness of $\mathcal{H}om$ we obtain the exact sequence

$$0 \to \mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \to \mathcal{H}om(\mathcal{D}_X, \mathcal{F}) \to \mathcal{H}om(\mathcal{D}_X \otimes \Theta_X, \mathcal{F}).$$

Given the coordinate system $x = (x_1, \cdots, x_n)$, we then have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F}) = \left\{ u \in \mathcal{F} : \frac{\partial}{\partial x_i} u = 0 (i = 1, \cdots, n) \right\}.$$

We will see more about this when talking about linear PDEs.

2. Linear PDEs

In the chapter 6 of [Cou95], we learned how to build the associated D-modules for a system of PDEs that has polynomial solutions. We extend this idea to more general PDEs:

$$\sum_{j=1}^{p} P_{ij}(x, \partial) u_j = 0, \ i = 1, \dots q, \tag{2.1}$$

where $P_{ij}(x, \partial) = \sum a_\alpha(x) \partial^\alpha$ are the linear partial differential operators.

Let D be the ring of linear differential operators and $P : D^{\oplus q} \to D^{\oplus p}$ given by

$$(Q_1, \dots, Q_q) \mapsto (\sum_{i=1}^{q} Q_i P_{i1}, \dots, \sum_{i=1}^{q} Q_i P_{ip}).$$

The cokernel $M = D^{\oplus p}/\mathrm{Im}\ P$ is the D-module corresponding to the system of PDEs. In particular, if $D = D(K[x_1, \dots, x_n]) = K[x, \partial] = A_n$ and $P : A_n \to A_n$ by $Q \mapsto \sum QP_i$, then

$$M = A_n / \sum A_n P_i.$$

Let F be the space where we want to find solutions. We regard F as a left D-module. Then

$$\mathcal{H}om_D(D^{\oplus p}, F) = \bigoplus \mathcal{H}om_D(D, F) = F^{\oplus p} = \{(u_1, \dots u_p) : u_1, \dots, u_p \in F\}.$$

Consider the finite presentation of M,

$$D^{\oplus q} \to D^{\oplus p} \to M \to 0.$$

By the left exactness of $\mathcal{H}om_D$,

$$0 \to \mathcal{H}om_D(M, F) \to \mathcal{H}om_D(D^{\oplus p}, F) \xrightarrow{\bar{P}} \mathcal{H}om_D(D^{\oplus q}, F).$$

Thus,

$$\mathcal{H}om_D(M, F) = \mathrm{Ker}\ \bar{P} = \{(u_1, \dots, u_p) \in F^{\oplus p} : \sum_j P_{ij} u_i = 0\}.$$

Remark 2.1. The map $\bar{P}$ can be viewed as P by the following correspondences

$$\begin{tikzcd}
\mathcal{H}om_D(D^{\oplus p}, F) \arrow[r] \arrow[d, leftrightarrow] & \mathcal{H}om_D(D^{\oplus q}, F) \arrow[d, leftrightarrow] \\
F^{\oplus p} \arrow[r] & F^{\oplus q}
\end{tikzcd}$$

Hence, the solution space of the system of linear PDEs is $\mathcal{H}om_D(M, F)$.

Conversely, given a D-module M, for each isomorphism

$$M \simeq \text{Coker}\left(D^{\oplus q} \xrightarrow{P} D^{\oplus p}\right),$$

we obtain equation (2.1) Then (2.1) is considered an explicit presentation of M corresponding to the isomorphism. There are many such isomorphisms for a given D-module. Each isomorphism gives a different explicit presentation (0.1).

3. Rings of Differential Operators

Let R be a commutative K-algebra. The ring of differential operators of R is defined, inductively, as a subring of $End_K(R)$. We identify an element $a \in R$ with the operator of $End_K(R)$ defined by the rule $r \longmapsto ar$, for every $r \in R$.

We now define, inductively, the order of an operator. An operator $P \in End_K(R)$ has order zero if $[a, P] = 0$, for every $a \in R$. Suppose we have defined operators of order $< n$. Let $D^n(R)$ denote the set of all operators of $End_K(R)$ of order $\leq n$. We define D^n as

$$D^0(R) = R$$
$$D^n(R) = \left\{P \in \text{End}_K(R) : [a, P] \in D^{n-1}(R) \text{ for all } a \in R\right\}.$$

It is easy to check, from the definitions, that $D^n(R)$ is a K-vector space.

A derivation of the K-algebra R is a linear operator D of R which satisfies Leibniz's rule: $D(ab) = aD(b) + bD(a)$ for every $a, b \in R$. Let $\text{Der}_K(R)$ denote the K-vector space of all derivations of R. Of course $\text{Der}_K(R) \subseteq End_K(R)$. If $D \in \text{Der}_K(R)$ and $a \in R$, we define a new derivation aD by $(aD)(b) = aD(b)$ for every $b \in R$. The vector space $\text{Der}_K(R)$ is a left R-module for this action.

Moreover, composing a differential operator of order at most n with one of order at most m gives a differential operator of order at most $n + m$. i.e.

$$D^n(R)D^m(R) \subset D^{n+m}(R).$$

The ring of differential operators $D(R)$ of K-algebra R is defined by

$$D(R) = \bigcup_{n=0}^{\infty} D^n(R).$$

4. Weyl Algebras

In this section we consider special case of rings of differential operators $\mathcal{D}_X$ over affines spaces. In this case, they are called the Weyl algebra. But we first introduce them as a ring of operators on a vector space of infinite dimension. Let K denote a field of characteristic zero and $K[X]$ the ring of polynomials $K[x_1, \ldots, x_n]$ in n commuting indeterminates over K.

The ring $K[X]$ is a vector space of infinite dimension over K. Consider its endomorphisms $End_K(K[X])$. Recall that the algebra operations in the endomorphism ring are the addition and composition of operators. The Weyl algebra will be defined as a subalgebra of $End_K(K[X])$.

Let $\hat{x}_1, \ldots, \hat{x}_n$ be the operators of $K[X]$ which are defined on a polynomial $f \in K[X]$ by the formulae $\hat{x}_i(f) = x_i \cdot f$. Similarly, $\partial_1, \ldots, \partial_n$ are the operators defined by $\partial_i(f) = \partial f / \partial x_i$. These are linear operators of $K[X]$. The n-th Weyl algebra A_n is the K-subalgebra of

$End_K(K[X])$ generated by the operators $\hat{x}_1, \dots, \hat{x}_n$ and $\partial_1, \dots, \partial_n$. For the sake of consistency, we write $A_0 = K$. Similar to the general cases, these generators satisfy:

$$[\partial_i, \hat{x}_j] = \delta_{ij} \cdot 1,$$
$$[\partial_i, \partial_j] = [\hat{x}_i, \hat{x}_j] = 0.$$

For simplicity, we write these generators as x and ∂. A simple observation is that $\{x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a basis for A_n as a K-vector space.

Equivalently, Weyl algebras are the rings of differential operators on the affine spaces. Precisely,

Theorem 4.1. *The ring of differential operators of $K[X]$ is $A_n(K)$.*

For a proof, see [Cou95].

5. Generators of D

Let Θ_X denote the sheaf of vector fields on a manifold X. Then Θ_X is a subsheaf of $\mathcal{D}_X$, and $\Theta_X \to \mathcal{D}_X$ is (left) $\mathcal{O}_X$-linear. The ring $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X, and their fundamental relations are the following:

$\mathcal{O}_X \to \mathcal{D}_X$ is a ring homomorphism,

$\Theta_X \to \mathcal{D}_X$ is left $\mathcal{O}_X$-linear,

$\Theta_X \to \mathcal{D}_X$ is a Lie algebra homomorphism,

$[v, a] = v(a)$ for all $v \in \Theta_X, a \in \mathcal{O}_X$,

where we denote by $v(a)$ the element of $\mathcal{O}_X$ obtained by differentiating a with respect to v.

The following lemma is a very useful tool for defining the actions on $\mathcal{D}$-modules.

Lemma 5.1. *Let R be a sheaf of rings on X, and $\iota : \mathcal{O}_X \to R$ and $\varphi : \Theta_X \to R$ sheaf morphisms such that*

(1) $\iota : \mathcal{O}_X \to R$ is a ring homomorphism,

(2) $\varphi : \Theta_X \to R$ is left $\mathcal{O}_X$-linear, where a left $\mathcal{O}_X$-module structure of R is given through ι,

(3) $\varphi : \Theta_X \to R$ is a Lie algebra homomorphism, where Lie brackets in R are commutators, and

(4) $[\varphi(v), \iota(a)] = \iota(v(a))$ for all $v \in \Theta_X, a \in \mathcal{O}_X$.

Then there exists a unique ring homomorphism $\Phi : \mathcal{D}_X \to R$ such that $\mathcal{O}_X \to \mathcal{D}_X \xrightarrow{\Phi} R$ coincides with ι, and $\Theta_X \longrightarrow \mathcal{D}_X \xrightarrow{\Phi} R$ with φ.

This lemma means if the morphisms $\mathcal{O}_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ and $\Theta_X \xrightarrow{\varphi} \mathcal{E}nd_{\mathbb{C}}(M)$ satisfy the conditions, then they can be uniquely extended to a ring homomorphism $\mathcal{D}_X \to \mathcal{E}nd_{\mathbb{C}}(M)$.

If we apply this lemma to Weyl algebras, $\mathcal{O}_X = K[x_1, \dots, x_n], \Theta_X = K[\partial_1, \dots, \partial_n]$ and (1)(2)(3) are satisfied superfluously. And clearly, $[\partial_j, x_j] = \partial_j(x_j)$.

There is another point of view on Weyl algebras discussed in Appendix 1 of [Cou95].

6. The Category of D-modules

This section correspond to chapter 1 of [Kas03]. The goal is to prove the equivalence between the category $\mathrm{Mod}(\mathcal{D}_X)$ of left $\mathcal{D}_X$-modules and the category $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$ of right $\mathcal{D}_X$-modules.

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules.

Definition 6.1. A $\mathbb{C}$-linear sheaf homomorphism $f : \mathcal{F} \to \mathcal{G}$ is called a *differential homomorphism* if for each $s \in \mathcal{F}$ there exists finitely many $P_j \in \mathcal{D}_X$ and $v_j \in \mathcal{G}$ such that

$$f(as) = \sum_j P_j(a)v_j$$

for all $a \in \mathcal{O}_X$.

Let $\mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G})$ be the sheaf of differential homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. Note that $\mathcal{D}_X = \mathcal{D}\mathit{iff}(\mathcal{O}_X, \mathcal{O}_X)$.

Proposition 6.2. *There is an isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G}).$$

To prove Corollary 1.4, we first state a well-known theorem.

Lemma 6.3. *(Adjoint Associativity) Let R and S be rings and $A_R, {}_RB_S, C_S$,-bimodules. Then there is an isomorphism of abelian groups*

$$\alpha : \mathcal{H}om_S(A \otimes_R B, C) \xrightarrow{\sim} \mathcal{H}om_R(A, \mathcal{H}om_S(B, C)),$$

defined for each $f : A \otimes_R B \to C$ by

$$((\alpha f)(a))(b) = f(a \otimes b).$$

Proof. See [Hun89]. □

Corollary 6.4 (Corollary 1.4)**.**

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G}).$$

Proof. It suffices to show that

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

Since the right $\mathcal{D}_X$-modules induces left D_X^{op}-modules,

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

By the adjoint associativity,

$$\begin{aligned} \mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)) \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)) \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \end{aligned}$$

□

We talk about the invertible modules. The references are [Eis95] §11.3 and [Ati94] Chapter 9.

Definition 6.5. Let R be a ring and I be an R-module. Then I is invertible if I is finitely generated and locally free of rank 1; that is, for all prime ideals $\mathfrak{p}$ of R, $I_\mathfrak{p} \cong R_\mathfrak{p}$.

Definition 6.6. Let S be the set of non zero-divisors, and $K(R) = S^{-1}R$ be the total quotient ring of R. An R-submodule I of $K(R)$ is a fractional ideal if there exists $r \in R$ such that $rI \subset R$.

Theorem 6.7. *(*[Eis95] *Theorem 11.6) Let R be a noetherian ring.*

(a) *If I is an R-module, then I is invertible iff the natural map $\mu : I^* \otimes I \to R$ is an isomorphism.*

(b) *Every invertible module is isomorphic to a fractional ideal of R. Every invertible fractional ideal contains a nonzerodivisor of R.*

(c) *If $I, J \subset K(R)$ are invertible modules, then the natural maps $I \otimes J \to IJ$, taking $s \otimes t$ to st, and $I^{-1}J \to \operatorname{Hom}_R(I, J)$, taking $t \in I^{-1}J$ to $\varphi_t : I \to J$ defined by $\varphi_t(a) = ta$, are isomorphisms. In particular, $I^{-1} \cong I^*$.*

There are some remarks about Theorem 11.6 in [Eis95]. First, every invertible module is a fractional ideal. Let $I^{-1} = \{s \in K(R) : sI \subset R\}$. Then invertibility is a "dual" property; that is, if I is invertible then so is I^{-1}. To see that $I^{-1} \cong I^* = \mathcal{H}om_R(I, R)$, notice that

$$I^{-1} \cong I^{-1} \otimes R \cong I^{-1}R \cong \mathcal{H}om_R(I, R).$$

As an application in [Kas03] page 9, let $R = \mathcal{O}, I = \mathcal{L}$. Then $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^*$ is invetible. By the theorem above, $\mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

To get $\mathcal{D}iff(\Omega_X, \Omega_X) = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ in [Kas03] Proposition 1.10, we use Corollary 6.4 and hence

$$\begin{aligned}
\mathcal{D}iff(\Omega_X, \Omega_X) &= \mathcal{H}om_{\mathcal{D}_X^{op}}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \\
&= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)) \\
&= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \\
&= \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}
\end{aligned}$$

7. Coherent Modules

Definition 7.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is of finite type if for each $x \in X$ there exists an open neighborhood U such that $\mathcal{M}|_U$ is generated by finitely many sections. i.e. there is a surjective morphism $\mathcal{O}_X^n \to \mathcal{M}|_U$ for some n.

Definition 7.2. We say that $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module if the following conditions hold:

(1) $\mathcal{M}$ is of finite type

(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{M}(U),\ i = 1, \dots, n$, the kernel of the map $\mathcal{O}_X^n \to \mathcal{M}|_U$ is of finite type.

Let $\mathcal{F}$ be a sheaf on X. The support of $s \in \mathcal{F}(U)$ is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P is the germ of s in the stalk $\mathcal{F}_P$. We can show that $\operatorname{Supp}(s)$ is a closed subset of U. For an element m in a module $\mathcal{M}$, we have $\operatorname{Supp}(m) = V(\operatorname{Ann} m)$. (See [Har77])

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and Z is a closed analytic subset of a manifold X. Since $\operatorname{Supp}(m) = V(\operatorname{Ann} m)$, we can see that $\Gamma_Z(\mathcal{F}) = \{u \in \mathcal{F} : \operatorname{Supp}(u) \subset Z\}$ is a finitely generated submodule and hence is coherent.

8. Cotangent Bundles

Let

$$F_m(\mathcal{D}_X) = \{P \in \mathcal{D}_X : P = \sum_{\substack{|\alpha| \leq m \\ \alpha \in \mathbb{Z}_{\geq 0}^n}} a_\alpha(x) \partial_x^\alpha\}$$

where $|\alpha| = \alpha_1 + \dots \alpha_n$. Then $\{F_m\}$ is the *order filtration* of $\mathcal{D}_X$. It gives rise to the graded ring

$$\operatorname{Gr}^F(\mathcal{D}_X) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Gr}_n^F(\mathcal{D}_X)$$

where

$$\mathrm{Gr}_n^F(\mathcal{D}_X) = F_n(\mathcal{D}_X)/F_{n-1}(\mathcal{D}_X).$$

Note that $\mathrm{Gr}^F(\mathcal{D}_X) = \bigoplus_\alpha \mathcal{O}_X \partial_x^\alpha$ is a sheaf of commutative algebra over $\mathcal{O}_X$.

Take a coordinate system $(x_1, \dots, x_n)$. We introduce the symbols

$$\xi_i = \frac{\partial}{\partial x_i} \mod F_0(\mathcal{D}_X) = \mathcal{O}_X.$$

i.e. $\xi_i - \frac{\partial}{\partial x_i}$ is a holomorphic function. Denote by Θ_X the sheaf of vector fields on X. With local coordinates, $\Theta_X = \bigoplus_{i=1}^n \mathcal{O}_X \frac{\partial}{\partial x_i}$. Then, there is an isomorphism

$$\mathrm{Sym}_{\mathcal{O}_X}(\Theta_X) \xrightarrow{\sim} \mathcal{O}_X \otimes \mathbb{C}[\xi_1, \dots, \xi_n] = \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

by $\frac{\partial}{\partial x_i} \to \xi_i$. Similarly, we can identify $\mathrm{Gr}^F(\mathcal{D}_X)$ with $\mathcal{O}_X[\xi_1, \dots, \xi_n]$. We have thus obtained the following theorem.

Theorem 8.1. *There is an isomorphism of graded rings*

$$\mathrm{Sym}_{\mathcal{O}_X}(\Theta_X) \xrightarrow{\sim} \mathrm{Gr}^F(\mathcal{D}_X).$$

Consider the the ring homomorphisms

$$\sigma_m : F_m(\mathcal{D}_X) \to \mathrm{Gr}_m^F(\mathcal{D}_X) \subset \mathrm{Gr}^F(\mathcal{D}_X) \simeq \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

given by

$$\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \in \mathrm{Sym}^m_{\mathcal{O}_X}(\Theta_X).$$

The image $\sigma_m(P)$ is call the *principal symbol* of P.

Let $\pi : T^*X \to X$ be the cotangent bundle. It induces a map $\mathcal{O}_{T^*X} \to \pi_*\mathcal{O}_{T^*X}$. We may regard $\xi_1, \dots, \xi_n$ as a coordinate system of T_x^*X.(see [THT07]) Indeed, if $\theta_x = \sum_j \xi_j dx^j \big|_x \in T_x^*X$, then

$$\xi_i(\theta_x) = \theta_x \left(\frac{\partial}{\partial x_i} \right) = \sum_j \xi_j dx^j \bigg|_x \left(\frac{\partial}{\partial x_i} \right) = \xi_i.$$

These $\xi_1, \dots, \xi_n$ gives the *natural chart* $(\pi^{-1}(U), \{\xi_i\})$ of T^*X. Hence, we have a canonical identification

$$\pi_*\mathcal{O}_{T^*X}(V) = \mathcal{O}_{T^*X}(\pi^{-1}(V)) \simeq \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

as $\mathcal{O}_X$-algebras. Thus, we have the proved the following

Corollary 8.2. *There exists isomorphisms of $\mathcal{O}_X$-algebras*

$$\mathrm{Gr}^F(\mathcal{D}_X) \simeq \pi_*\mathcal{O}_{T^*X} \simeq \mathrm{Sym}_{\mathcal{O}_X}(\Theta_X).$$

Thus, for all $P \in F_m(\mathcal{D}_X)$ we can associate to it a regular function $\sigma_m(P)$ defined on the cotangent bundle T^*X.

Let $P \in F_{m_1}(\mathcal{D}_X), Q \in F_{m_2}(\mathcal{D}_X)$. Then, $[P, Q] \in F_{m_1+m_2-1}(\mathcal{D}_X)$. This induces a map

$$\mathrm{Gr}^F_{m_1}(\mathcal{D}_X) \times \mathrm{Gr}^F_{m_2}(\mathcal{D}_X) \to \mathrm{Gr}^F_{m_1+m_2-1}(\mathcal{D}_X).$$

Given functions $f(x, \xi)$ and $g(x, \xi)$, set

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial x_i} \right).$$

Then,

$$\sigma_{m_1+m_2-1}([P,Q]) = \{\sigma_{m_1}(P), \sigma_{m_2}(Q)\}.$$

Let $p \in T^*X$ be a point. Let ω_X be a 1-form (i.e. a section of $T^*(T^*X)$) on T^*X defined by $\omega_X(p) = \pi_X^*\omega_p$, where ω_p is a 1-form (i.e. a section of T^*X) at the point $\pi(p) \in X$. In local coordinates,

$$\omega_X = \sum_i \xi_i dx_i.$$

At every point f, the 2-form $\theta_X = d\omega_X$ gives an anti-symmetric bilinear form on $T_p(T^*X)$. This is nondegenerate. Let $H : T_p(T^*X) \xrightarrow{\sim} T_p(T^*X)$ be given by the paring

$$\langle \theta_X, v \wedge H(\eta) \rangle = \langle \eta, v \rangle$$

for $v \in T_p(T^*X), \eta \in T_p^*(T^*X)$. In local coordinates,

$$d\xi_i \xrightarrow{H} \frac{\partial}{\partial x_i}$$
$$dx_i \xrightarrow{H} -\frac{\partial}{\partial \xi_i}$$

In particular, $H_f = H(df)$ is a vector field on T^*X, which is called the *Hamiltonian* of f.

Definition 8.3. For functions f, g on T^*X, $\{f, g\} = H_f(g)$ is called the *Poisson bracket* of f and g.

Definition 8.4. Let X be a manifold, and θ be a closed 2-form on X that gives a nondegenerate anti-symmetric bilinear form on T_pX for every p. A tuple (X, θ) is called a *symplectic manifold.*

For example, (T^*X, θ_X) is a symplectic manifold.

If X is the affine space $\mathbb{C}^n$, then $T^*X = \mathbb{C}^{2n}$. Let

$$\Omega = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Given $u, v \in \mathbb{C}^{2n}$, define

$$\omega(u, v) = u\Omega v^t.$$

This is called the standard symplectic structure.

9. Characteristic Varieties

Definition 9.1. Let M be an R-module. The support of M is

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : M_{\mathfrak{p}} \neq 0\}$$

Proposition 9.2. *If M is a finitely generated R-module, then*

$$\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{Ann}(M) \subset \mathfrak{p}\}.$$

Proof. Since M is finitely generated, we write $M = (x_1, \ldots, x_n)$. Then, $\mathfrak{p} \in \operatorname{Supp}(M) \Leftrightarrow M_{\mathfrak{p}} \neq 0 \Leftrightarrow \exists i$ such that $x_i \neq 0$ in $M_{\mathfrak{p}} \Leftrightarrow \exists i$ such that $\operatorname{ann}(x_i) \subset \mathfrak{p} \Leftrightarrow \operatorname{ann}(M) = \bigcap_{i=1}^n \operatorname{ann}(x_i) \subset \mathfrak{p}$ □

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.

Definition 9.3. A sequence $\{F_m(\mathcal{M})\}_{m\in\mathbb{Z}}$ of subsheaves of $\mathcal{M}$ is called a filtration as a D-module if it satisfies $\mathcal{M} = \bigcup_m F_m(\mathcal{M})$ and $F_m(\mathcal{D}_X) F_l(\mathcal{M}) \subset F_{m+l}(\mathcal{M})$, and a coherent filtration if in addition $\bigoplus_m F_m(\mathcal{M})$ is locally finitely generated as a $\bigoplus_m F_m(\mathcal{D}_X)$-module.

As an example, leet $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ is locally generated by a finite number of sections $u_1, \ldots, u_N$. Consider

$$F_m(\mathcal{M}) = \sum_{\nu=1}^{N} F_m\left(\mathcal{D}_X\right) u_\nu.$$

This is a coherent filtration.

Let $\{F_m(\mathcal{M})\}$ be a coherent filtration of a coherent $\mathcal{D}_X$-module $\mathcal{M}$.

Set

$$\mathrm{Gr}^F(\mathcal{M}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Gr}_n^F(\mathcal{M})$$

where

$$\mathrm{Gr}_n^F(\mathcal{D}_X) = F_n(\mathcal{M})/F_{n-1}(\mathcal{M}).$$

Then, $\mathrm{Gr}^F(\mathcal{M})$ is $\mathrm{Gr}^F(\mathcal{D}_X)$-module.

Denote

$$\widetilde{\mathrm{Gr}^F\mathcal{M}} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_X} \pi^{-1}\mathrm{Gr}^F\mathcal{M}.$$

This is an $\mathcal{O}_{T^*X}$-module. i.e. $^\sim$ is an exact functor from $\mathrm{Mod}(\mathrm{Gr}^F(\mathcal{D}_X))$ to $\mathrm{Mod}(\mathcal{O}_{T^*X})$. Hence, we can define a variety in T^*X.

Definition 9.4. The characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is

$$\mathrm{Ch}(\mathcal{M}) = \mathrm{Supp}(\widetilde{\mathrm{Gr}^F\mathcal{M}}) = \mathrm{Supp}(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_X} \pi^{-1}\mathrm{Gr}^F\mathcal{M}).$$

Following is Theorem 2.6 on [Kas03], which implies this definition is valid.

Theorem 9.5. $\mathrm{Supp}\left(\left(\mathrm{Gr}^F\,\mathcal{M}\right)^{\sim}\right) \subset T^*X$ *is independent of the choice of a coherent filtration* $F(\mathcal{M})$ *of* $\mathcal{M}$.

Denote

$$J_{\mathcal{M}} = \sqrt{\mathrm{Ann}_{\mathcal{O}_X}(\widetilde{\mathrm{Gr}^F\mathcal{M}})}.$$

By the lemma above,

$$\mathrm{Supp}(\widetilde{\mathrm{Gr}^F\mathcal{M}}) = V(\mathrm{Ann}(\widetilde{\mathrm{Gr}^F\mathcal{M}})) = V(J_{\mathcal{M}}) \subset T^*X.$$

If U is an open set of X, then

$$\begin{aligned}\mathrm{Ch}(\mathcal{M}) \cap T^*U &= \mathrm{Supp}(\mathcal{O}_{T^*U} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_U} \pi^{-1}\mathrm{Gr}^F\mathcal{M}|_U)\\ &= V(J_{\mathcal{M}(U)})\\ &= \{p \in T^*U : f(p) = 0 \text{ for all } f \in J_{\mathcal{M}(U)}\}.\end{aligned}$$

Clearly, $\mathrm{Ch}(\mathcal{M})$ is a closed subset.

If X is affine space $\mathbb{A}^n$, then

$$J_{\mathcal{M}} = \sqrt{\mathrm{Ann}(\mathcal{O}_{T^*X} \otimes_{\mathcal{O}_{T^*X}} \pi^{-1}\mathrm{Gr}^F(\mathcal{M}))} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}^F\mathcal{D}_X}(\mathrm{Gr}^F(\mathcal{M}))} \subset T^*X = \mathbb{A}^{2n}.$$

Note that $\mathrm{Gr}^F\mathcal{D}_X = \mathrm{Gr}^F A_n \simeq k[y_1, \ldots, y_{2n}]$. (See [Cou95])

We can further write

$$\mathrm{Ch}(\mathcal{M}) \cap T^*U = \bigcup_{\mathfrak{p}\in\mathrm{Supp}(\mathcal{M}(U))} \{p \in T^*U : f(p) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

Then,

$$\pi(\mathrm{Ch}(\mathcal{M})) = \bigcup_{\mathfrak{p}\in \mathrm{Supp}(\mathcal{M}(U))} \{\pi(p) \in T^*U : f(p) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

Thus, we obtain the following properties of $\mathrm{Ch}(\mathcal{M})$.

Proposition 9.6.

$$\mathrm{Supp}(\mathcal{M}) = \pi(\mathrm{Ch}(\mathcal{M})).$$

Lemma 9.7. *If J is a homogeneous ideal of a graded ring R, then $\sqrt{J}$ is also homogeneous.*

Proof. Let $a \in \sqrt{J}$, then $a^n \in J$ for some n. Write $a = a_0 + \cdots a_s$. Then, $a^n = (a_0 + \cdots a_s)^n \in J$. Clearly, $a_s^n \in J$ i.e. $a_s \in \sqrt{J}$. Thus, $a' = a - a_s \in \sqrt{J}$. Apply the same process to a'. By induction, $a_i \in \sqrt{J}$ for all i. □

By the lemma, $J_{\mathcal{M}}$ is an homogeneous ideal. Thus, $\mathrm{Ch}(\mathcal{M})$ is a homogeneous closed set.

10. Involutivity

Suppose that a $\mathcal{D}_X$-module $\mathcal{M}$ is generated by a single element u. Let $\mathcal{I} = \mathrm{Ann}(u) = \{P \in \mathcal{D}_X : Pu = 0\}$. Then $\mathcal{M} \simeq \mathcal{D}_X/\mathcal{I}$.

Introduce a filtration by

$$F_m(\mathcal{M}) = F_m(\mathcal{D}_X)u$$
$$F_m(\mathcal{I}) = \mathcal{I} \cap F_m(\mathcal{D}_X).$$

Then $\mathrm{Gr}^F\mathcal{M} = \mathrm{Gr}^F\mathcal{D}_X/\mathrm{Gr}^F\mathcal{I}$. Since $\sigma_m(P) \in \mathrm{Ann}(\mathrm{Gr}^F\mathcal{M})$ for $P \in F_m(\mathcal{I})$, we have

$$\mathrm{Ch}(\mathcal{M}) = \{p \in T^*X : \sigma_m(P)(p) = 0 \text{ for all } m \text{ and for all } P \in F_m(\mathcal{I})\}.$$

If $\mathcal{I} = \sum \mathcal{D}_X P_j$ with $P_j \in F_{m_j}(\mathcal{D}_X)$, then

$$\mathrm{Ch}(\mathcal{M}) \subset \bigcap_j \sigma_{m_j}(P_j)^{-1}(0).$$

The equality does not hold in general.

Definition 10.1. The generators $\{P_j\}$ is an involutive system of generators of $\mathcal{I}$ if

$$\mathrm{Gr}^F\mathcal{I} = \sum(\mathrm{Gr}^F\mathcal{D}_X)\sigma_{m_j}(P_j).$$

Lemma 2.1 of [Kas03] gives a characterization of this definition.

In general, we have the following definititon.

Definition 10.2. A closed subset V of T^*X is involutive if its defining ideal

$$I_V = \{a \in \mathcal{O}_{T^*X} : a|_V = 0\}$$

satisfies $\{I_V, I_V\} \subset I_V$.

If $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$, then $\mathrm{Ch}(\mathcal{M}) = \{p \in T^*X : a(p) = 0 \text{ for all } a \in \mathrm{Gr}^F\mathcal{I}\}$. Conversely, suppose that $\mathrm{Gr}^F\mathcal{I} = \{a \in \mathrm{Gr}^F\mathcal{D}_X : a|_{\mathrm{Ch}(\mathcal{M})} = 0\}$. Let $a, b \in \mathrm{Gr}^F\mathcal{I}$ and $A, B \in \mathcal{I}$ be such that $\sigma(A) = a, \sigma(B) = b$. Here σ apply to $\mathcal{I}$ by apply σ_m to the graded pieces. Since $[A, B] \in \mathcal{I}$, we have

$$\{a, b\} = \sigma([A, B])$$

vanishing on $\mathrm{Ch}(\mathcal{M})$. Hence, $\mathrm{Ch}(\mathcal{M})$ is an involutive variety.

Generally, we have the following theorem.

Theorem 10.3. *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then* $\mathrm{Ch}(\mathcal{M})$ *is involutive.*

For the special case of affine spaces, a similar result holds. If U is a subspace of $\mathbb{C}^{2n}$, then its skew-orthogonal complement is

$$U^{\perp} = \left\{ v \in \mathbb{C}^{2n} : \omega(u, v) = 0 \text{ for all } u \in U \right\}.$$

In the book [Cou95], the involutive subspaces are the ones that contain their skew-complement. Then the following proposition shows that this definition is equivalent to the definition 10.2.

Proposition 10.4. *An affine variety V in $\mathbb{C}^{2n}$ is involutive if and only if its ideal I_V is closed for the Poisson bracket.*

11. Tensor Products

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be left $\mathcal{D}_X$-modules. Then there exists a natural left $\mathcal{D}_X$-module structure in $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$. For $v \in \Theta_X, s_1 \in \mathcal{M}_1$, and $s_2 \in \mathcal{M}_2$, similarly to Leibniz's rule, set

$$v\,(s_1 \otimes s_2) = v s_1 \otimes s_2 + s_1 \otimes v s_2.$$

Then this is well-defined, satisfies the conditions in Lemma 5.1, and hence gives a $\mathcal{D}_X$-module structure to $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2$.

Similarly to $\bullet \overset{D}{\otimes} \bullet, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ has a left $\mathcal{D}_X$-module structure for left $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$. On $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}), v \in \Theta_X$ acts by

$$(v\varphi)(u) = v(\varphi(u)) - \varphi(v \cdot u) \quad (u \in \mathcal{M})$$

This can be extended to an action of $\mathcal{D}_X$ by Lemma 5.1.

12. Inverse Images

Let $F : X \to Y$ be a polynomial map and M a left A_m-module. Since $Y = K^m$, it follows that M is in particular a $K[Y]$-module. Thus we may compute the inverse image of M by F defined by:

$$F^{\star}(M) = K[X] \otimes_{K[\eta]} M.$$

It is a module over $K[X]$. We want to make this module into an A_n-module.

We know how a polynomial of $K[X]$ acts on $F^{\star}(M)$. We now give the recipe for the action of ∂_{x_i}. Let

$$q \otimes u \in F^{\star}(M),$$

where $q \in K[X]$ and $u \in M$. Let $F_1, \ldots, F_m$ be the coordinate functions of F. The action of ∂_{x_i} is defined by

$$\partial_{x_i}(q \otimes u) = \partial_{x_i}(q) \otimes u + \sum_{k=1}^{m} q \partial_{x_i}(F_k) \otimes \partial_{y_k} u$$

for $i = 1, \ldots, n$. The x's and ∂_x's generate A_n, and we know how they act on $F^{\star}(M)$. According to Appendix 1, these formulae will make an A_n-module of $F^{\star}(M)$.

13. Cauchy-Kovlevskaya Theorem

Let
$$P=\sum_{|\alpha|\leq r}a_\alpha(x)\partial_x^\alpha.$$
Its principal symbol
$$\sigma_r(P)=\sum_{|\alpha|=r}a_\alpha(x)\xi^\alpha$$
is a homogeneous polynomial of degree r in the variables $\xi_1,\dots,\xi_n$.

Definition 13.1. We say a hyperplane $X\subset\mathbb{A}^n$ (e.g. $X=\{x_n=0\}$) is non-characteristic with respect to P if $a_{(0,\dots,0,r)}(x_1,x_2,\dots,x_{n-1},0)\neq 0$ for every $x_1,\dots,x_{n-1}$.

The following is the classical Cauchy-Kowalevskaya theorem.

Theorem 13.2 (Cauchy-Kowalevskaya)**.** *Let $Y=\mathbb{C}^n$ and let X be the hypersurface of Y defined by $X=\{x_n=0\}$. Let $P\in D_X$ be a differential operator of order $r\geq 0$ on X whose coefficients are holomorphic near the origin. If X is noncharacteristic with respect to P, then for every holomorphic functions $g_0,\dots,g_{r-1}$, there exists a unique holomorphic solution $u\in\mathcal{O}_X$ to the Cauchy problem*
$$\begin{cases} Pu=0,\\ \partial_1^j u\Big|_X=g_j \quad (j=0,1,\dots,r-1).\end{cases}$$

Next, we want to state the slightly general Cauchy-Kowalevskaya-Kashiwara theorem over the affine spaces.

We begin by considering the embedding $\iota: X\to X\times Y$ defined by $\iota(X)=(X,0)$, where 0 denotes the origin of Y. We will call ι the standard embedding. The comorphism $\iota^\sharp: K[X,Y]\to K[X]$ is defined by $\iota^\sharp(g(X,Y))=g(X,0)$. It may be used to make $K[X]$ into a $K[X,Y]$-module; as such, $K[X]$ is generated by 1 , since
$$g(X,Y)\cdot 1=\iota^\sharp(g(X,Y))\cdot 1=g(X,0).$$

The annihilator of 1 in $K[X,Y]$ is the ideal generated by $y_1,\dots,y_m$. Denoting this ideal by (Y) we have that $K[X]\cong K[X,Y]/(Y)$ as $K[X,Y]$-modules.

Let A_n and A_m be the Weyl algebras over X and Y respectively. Consider a left A_{m+n}-module M. Since $A_n\subseteq A_{m+n}$, the A_{m+n}-module M is also an A_n-module. The elements of A_n commute with the variables $y_1,\dots,y_m$, thus $(Y)M$ is an A_n–submodule of M. Hence $M/(Y)M$ is an A_n–module. Note however that it is not an A_{m+n}-submodule.

Lemma 13.3. *We have the following isomorphism of A_n-modules*
$$\iota^\star M=K[X]\otimes_{K[X,Y]}M\cong M/(Y)M.$$

Proof. If $u\in M$, let $\bar{u}$ denote its image in $M/(Y)M$. Define the map
$$\phi: K[X]\otimes_{K[X,Y]}M\to M/(Y)M$$
by $\phi(q\otimes u)=q\bar{u}$. It is a homomorphism of $K[X]$-modules. We want to show that the action of the derivations $\partial_{x_1},\dots,\partial_{x_n}$ is compatible with ϕ. We have that
$$\partial_{x_i}(q\otimes u)=\frac{\partial q}{\partial x_i}\otimes u+\sum_{k=1}^n q\frac{\partial x_k}{\partial x_i}\otimes\partial_{x_k}u$$

and so

$$\partial_{x_i}(q \otimes u) = \frac{\partial q}{\partial x_i} \otimes u + q \otimes \partial_{x_i} u.$$

The right hand side is mapped by ϕ onto

$$\frac{\partial q}{\partial x_i}\bar{u} + q\overline{\partial_{x_i} u}$$

which is equal to

$$\partial_{x_i}(q\bar{u}) = \partial_{x_i}(\phi(q \otimes u)).$$

Thus ϕ is an isomorphism of A_n-modules. □

Set the base field to be $\mathbb{C}$. Let $X = \mathbb{A}^{n-1}$ and $Y = \mathbb{A}^n$. Set $M = A_n/A_nP$. By the Lemma 13.3, we have

$$\begin{aligned} K[x_1, \ldots, x_{n-1}] \otimes_{K[x_1,\ldots,x_n]} M &\cong M/x_nM \\ &= \frac{A_n/A_nP}{x_n\,(A_n/A_nP)} \\ &\cong A_n/x_nA_n + A_nP. \end{aligned}$$

We claim that the morphism of left A_{n-1}-modules

$$\begin{aligned} \varphi : A_{n-1}^{\oplus r} &\to A_n/\left(x_nA_n + A_nP\right) \\ (Q_0, Q_1, \ldots, Q_{r-1}) &\mapsto Q_0 + Q_1\partial_n + \cdots + Q_{r-1}\partial_n^{r-1}. \end{aligned}$$

is an isomorphism. The proof of the claim is taken from [Sch19].

To prove that φ is injective, note that this is equivalent to saying that if

$$Q_0 + Q_1\partial_n + \cdots + Q_{r-1}\partial_n^{r-1} = x_nS + TP$$

for some $Q_0, \ldots, Q_{r-1} \in A_{n-1}$ and $S, T \in A_n$, then actually $Q_0 = \cdots = Q_{r-1} = 0$. We can write $T = x_nT_0 + T_1$, in such a way that x_n does not appear in T_1; since $x_nS + TP = x_n\,(S + T_0) + T_1P$, we can therefore assume without loss of generality that T does not involve x_n. Since P is non-characteristic, P contains ∂_n^r. Since T does not contain x_n, ∂_n^r cannot be cancelled unless $T = 0$. But then $Q_0 = \cdots = Q_{r-1} = 0$.

To prove the surjectivity, we first show that ∂_n^r is in the image. We can write our differential operator $P \in A_n$ uniquely in the form

$$P = f\partial_n^r - P_{r-1}\partial_n^{r-1} - \cdots - P_1\partial_n - P_0,$$

where $f \in k\,[x_1, \ldots, x_n]$ and where $P_0, \ldots, P_{r-1} \in A_n$ do not involve ∂_n. The fact that P is non-characteristic means that f is nowhere vanishing on $\mathbb{A}^{n-1}$; after rescaling, we can write $f = 1 - x_ng$. Let Q_j be such that $P_j = Q_n + x_nR_j$, then we get

$$\partial_n^r = \sum_{j=0}^{r-1} Q_j\partial_n^j + x_n\left(g\partial_n^r + \sum_{j=0}^{r-1} R_j\partial_n^j\right) + P$$

Hence ∂_n^r belongs to the image of φ. By the similar process, we see that this is true for all powers of ∂_n, and so φ is surjective.

With this claim, we obtain

$$\mathcal{O}_X^{\oplus r} \simeq \mathcal{H}om_{A_{n-1}}(\iota^*M, \mathcal{O}_X).$$

We have seen that $\mathcal{H}om_{A_n}(M, \mathcal{O}_Y) = \{u \in \mathcal{O}_Y : Pu = 0\}$. Then,

$$\iota^{-1}\mathcal{H}om_{A_n}(M, \mathcal{O}_Y) = \{u \in \mathcal{O}_Y|_X : Pu = 0\}.$$

With those fancy language, Theorem 13.2 actually implies

Theorem 13.4 (Cauchy-Kowalevskaya-Kashiwara Theorem on Affine Spaces)**.** *Let $Y = \mathbb{C}^n$, X be the hypersurface of Y defined by $X = \{x_n = 0\}$ and $\iota : X \to Y$ be the canonical embedding. Let $P \in D_X$ be a differential operator of order $r \geq 0$ on X whose coefficients are holomorphic near the origin. If X is noncharacteristic with respect to P, then the morphism*

$$\begin{aligned} \iota^{-1}\mathcal{H}om_{A_n}(\mathcal{M}, \mathcal{O}_Y) &\to \mathcal{H}om_{A_{n-1}}(\iota^*\mathcal{M}, \mathcal{O}_X) \\ u &\mapsto \left(u|_X, \partial_n u|_X, \ldots, \partial_n u^{r-1}|_X\right) \end{aligned}$$

is an isomorphism.

Note that the above isomorphism means that for every set of boundary conditions $\partial_1^j u|_X = g_j, (j = 0, 1, \ldots, r-1)$, there exists a solution u for the equation $Pu = 0$.

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