internal

$n = 40i + 1$

$n \bmod 40 = 1$

(11) Let T be a full 40-ary tree. How many among the numbers 121, 202, 313, 434, can be the number of vertices of T? (Your answer ranges from 0 to 4.)

$\boxed{1}$

(12) How many edges are there in a forest of t trees containing a total of n vertices?

tree with n vertices has n−1 edges.

$\boxed{n-t.}$

(13) (i) Find the **minimum** values of m if an m-ary tree has at least 600 leaves and height 4.

$m^4 \geq 600$ $\quad 5^4 \geq 600 \geq 4^4$

$\boxed{5}$

(ii) Find the **value(s)** of n if a full and balanced n-ary tree has 81 leaves and height 4.

$h = \lceil \log_m l \rceil$

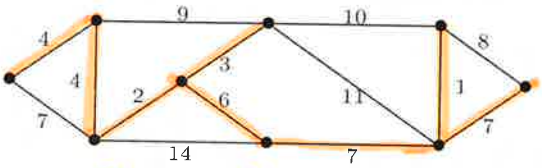
$4 \quad n \quad 81$

$n = 3$

$\boxed{3}$

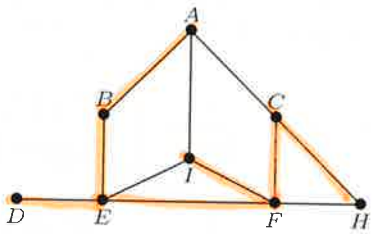
(14) Find the **weight** of a minimum spanning tree in the following graph.

$\boxed{34}$

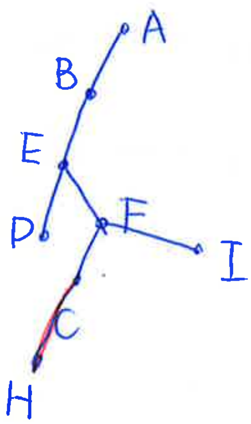


$4 + 4 + 2 + 3 + 6 + 7 + 1 + 7$

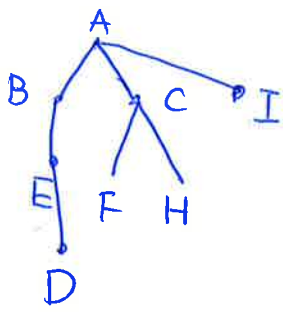
(15) Let G be the following graph. Using the alphabetical ordering, find a spanning tree by **depth first** search. Draw the tree below.



Graph G



DFS



BFS

Question B [5 marks]. Prove by using mathematical induction that for any integer $n \geq 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Let $P(n)$ be $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Basis Step. $P(1)$ is " $1 \leq 2 - \frac{1}{1}$ " which is true.

Inductive Step Suppose $P(1) \cdots P(k)$ all true., where $P(k)$ is

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k} \quad ①$$

Now we check $P(k+1)$.

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \overset{①}{\leq} 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{(k+1)^2 - k}{k(k+1)^2}$$

$$\leq 2 - \frac{k^2+k+1}{k(k+1)^2}$$

$$\leq 2 - \frac{1}{k+1}\left[\frac{k^2+k+1}{k(k+1)}\right]$$

$$\leq 2 - \frac{1}{k+1}\left[\frac{k^2+k+1}{k^2+k}\right]$$

$$\leq 2 - \frac{1}{k+1}\left[1 + \frac{1}{k^2+k}\right]$$

$$\leq 2 - \frac{1}{k+1}$$

$\therefore$ $P(k+1)$ is true.

Question C [5 marks]. Prove that for any positive integer n,

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

Consider the set A with n elements $A = \{a_1\ a_2\ \cdots\ a_n\}$

the set B with another n elements

$B = \{b_1\ b_2\ \cdots\ b_n\}$.

$A \cup B = \{a_1\ a_2\ \cdots\ a_n,\ b_1\ b_2\ \cdots\ b_n\}$ has $2n$ elements.

Consider the number of subsets of $A \cup B$ of size n.

Method 1 Because $A \cup B$ has $2n$ elements, choosing subsets of size n has $\binom{2n}{n}$ ways.

Method 2 From A choose r elements and from B choose $n-r$ elements. we have $\binom{n}{r}\binom{n}{n-r} = \binom{n}{r}^2$ ways

Case 0 $r=0$. we have $\binom{n}{0}^2$ ways

Case 1 $r=1$ we have $\binom{n}{1}^2$ ways

$\vdots$

Case n $r=n$ we have $\binom{n}{n}^2$ ways.

$\therefore$ From Method 1, the ~~numeber~~ number of subsets of $A \cup B$ of size n is $\binom{2n}{n}$

Two Ans should be the same

From Method 2, the number of subsets of $A \cup B$ of size n is $\sum_{r=0}^{n} \binom{n}{r}^2$.

Question D [5 marks]. Suppose that T_1 and T_2 are spanning trees of a simple graph G with at least 3 vertices. Moreover, suppose that e_1 is an edge in T_1 that is not in T_2. Show that there is an edge e_2 in T_2 that is not in T_1 such that T_1 remains a spanning tree if e_1 is removed from it and e_2 is added to it, and T_2 remains a spanning tree if e_2 is removed from it and e_1 is added to it.

Suppose $e_1 = \{u, v\}$ is as speciified.

Then $T_2 \cup \{e_1\}$ contains a simple circuit C containing e_1.

The graph $T - \{e_1\}$ has two connected components:

the endpoints of e_1 are in different components.

Travel C from u in the direction opposite to e_1 until you come to the first vertex in the same component as v.

The edge just crossed is e_2.

Clearly, $T_2 \cup \{e_1\} - \{e_2\}$ is a tree, because e_2 is on C.

Also, $T_1 - \{e_1\} \cup \{e_2\}$ is a tree because e_2 re united the two components.