NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 5**

1. Determine which of the following sets of vectors span $\mathbb{R}^4$.

   (a) $S_1 = \{(2,3,2,0),(0,2,1,1)\}$.
   (b) $S_2 = \{(2,1,1,0),(1,2,-1,0),(0,3,0,3),(0,1,-1,3)\}$
   (c) $S_3 = \{(3,2,-1,2),(4,0,0,2),(5,6,-3,2),(0,4,-2,-1)\}$
   (d) $S_4 = \{(1,2,-2,1),(4,0,4,0),(1,-1,-1,-1),(1,1,1,1),(0,1,0,1)\}$.

   (a) S_1 does not span $\mathbb{R}^4$ as you need at least 4 vectors to span $\mathbb{R}^4$.

   (b) Following Discussion 3.2.5 in lectures, we check if the vector equation

   $$c_1(2,1,1,0) + c_2(1,2,-1,0) + c_3(0,3,0,3) + c_4(0,1,-1,3) = (w,x,y,z) \quad (*)$$

   is consistent for all $(w,x,y,z) \in \mathbb{R}^4$. To do this, consider the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 3 & 3 \end{pmatrix}$ and find a row-echelon form of $\boldsymbol{A}$. In this case

   $$\boldsymbol{A} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \boldsymbol{R}$$

   Since $\boldsymbol{R}$ has no zero rows, $(*)$ is always consistent and hence S_2 spans $\mathbb{R}^4$.

   (c) Following the same method as (b),

   $$\boldsymbol{A} = \begin{pmatrix} 3 & 4 & 5 & 0 \\ 2 & 0 & 6 & 4 \\ -1 & 0 & -3 & -2 \\ 2 & 2 & 2 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} -1 & 0 & -3 & -2 \\ 0 & 2 & -4 & -5 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}$$

   Since $\boldsymbol{R}$ has a zero row, we conclude that S_3 does not span $\mathbb{R}^4$.

   (d) S_4 spans $\mathbb{R}^4$.

2. Find a set of vectors that spans the solution space of the following homogeneous linear system:

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ -2x_1 - 2x_2 + x_3 - 5x_4 = 0 \\ x_1 + x_2 - x_3 + 3x_4 = 0 \\ 4x_1 + 4x_2 - x_3 + 9x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ -2 & -2 & 1 & -5 & 0 \\ 1 & 1 & -1 & 3 & 0 \\ 4 & 4 & -1 & 9 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A general solution is

$$\begin{cases} x_1 &= -s-2t \\ x_2 &= s \\ x_3 &= t \\ x_4 &= t, \quad s,t \in \mathbb{R} \end{cases}$$

So $\{(-1,1,0,0),(-2,0,1,1)\}$ spans the solution space of the homogeneous linear system.

3. For each of the following sets S_1 and S_2, determine whether

   (i) $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$;

   (ii) $\mathrm{span}(S_2) \subseteq \mathrm{span}(S_1)$;

   (iii) $\mathrm{span}(S_1) = \mathrm{span}(S_2)$.

   (a) $S_1 = \{(2,-2,0),(-1,1,-1),(0,0,9)\}$ and $S_2 = \{(1,1,-1),(-2,-2,1),(1,5,-2)\}$.

   (b) $S_1 = \{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$, $S_2 = \{\boldsymbol{u},\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$ where $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are vectors in $\mathbb{R}^4$.

   (a) Following Theorem 3.2.10 (see Example 3.2.11), to check $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$, we evaluate the reduced row-echelon form of the following matrix

$$\left(\begin{array}{ccc|c|c|c} 1 & -2 & 1 & 2 & -1 & 0 \\ 1 & -2 & 5 & -2 & 1 & 0 \\ -1 & 1 & -2 & 0 & -1 & 9 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & \frac{3}{2} & -18 \\ 0 & 1 & 0 & -1 & \frac{3}{2} & -9 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 0 \end{array}\right)$$

   All 3 linear systems are consistent, so $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$. Similarly, consider

$$\left(\begin{array}{ccc|c|c|c} 2 & -1 & 0 & 1 & -2 & 1 \\ -2 & 1 & 0 & 1 & -2 & 5 \\ 0 & -1 & 9 & -1 & 1 & -2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & -\frac{9}{2} & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -9 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right)$$

   All 3 linear systems are inconsistent, so $\mathrm{span}(S_2) \not\subseteq \mathrm{span}(S_1)$. Hence $\mathrm{span}(S_1) \neq \mathrm{span}(S_2)$.

   (b) Clearly, each vector in S_2 is a linear combination of the vectors in S_1, so $\mathrm{span}(S_2) \subseteq \mathrm{span}(S_1)$ is immediate. Conversely, we have

$$\boldsymbol{u} = 1\boldsymbol{u} + 0(\boldsymbol{u}+\boldsymbol{v}) + 0(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$$

$$\boldsymbol{v} = -\boldsymbol{u} + 1(\boldsymbol{u}+\boldsymbol{v}) + 0(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$$

$$\boldsymbol{w} = 0\boldsymbol{u} - 1(\boldsymbol{u}+\boldsymbol{v}) + 1(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$$

   So each vector in S_1 is a linear combination of vectors in S_2 and thus $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$. Together with the above, we have $\mathrm{span}(S_1) = \mathrm{span}(S_2)$.

4. Let V and W be subspaces of $\mathbb{R}^n$. Define

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\}.$$

(a) Show that $V + W$ is a subspace of $\mathbb{R}^n$.
(**Hint:** Since V and W are subspaces, $V = \text{span}(S)$ and $W = \text{span}(T)$ for sets S and T in $\mathbb{R}^n$. Use S and T to find a set R such that $V + W = \text{span}(R)$.)

(b) Write down the subspace $V + W$ explicitly (that is, find a finite set S such that $V + W = \text{span}(S)$) if

(i) $V = \{(t, 0) \mid t \in \mathbb{R}\}$ and $W = \{(0, t) \mid t \in \mathbb{R}\}$.

(ii) $V = \{(t, 2t, 3t) \mid t \in \mathbb{R}\}$ and $W = \{(t, 0, -t) \mid t \in \mathbb{R}\}$.

(iii) V is the line spanned by $(1, 1, 1)$ in $\mathbb{R}^3$ and W is the plane with equation $x + y - z = 0$ in $\mathbb{R}^3$.

(a) Following the hint, let S and T be sets of vectors in $\mathbb{R}^n$ such that $V = \text{span}(S)$ and $W = \text{span}(T)$. We may let $S = \{\boldsymbol{s_1}, \cdots, \boldsymbol{s_k}\}$ and $T = \{\boldsymbol{t_1}, \cdots, \boldsymbol{t_r}\}$. Now for any vector $\boldsymbol{x} \in V + W$, we have

$$\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w} \quad \text{where } \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W.$$

Since S spans V and $\boldsymbol{v} \in V$, we can write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{s_1}, \cdots, \boldsymbol{s_k}$, say

$$\boldsymbol{v} = a_1\boldsymbol{s_1} + a_2\boldsymbol{s_2} + \cdots + a_k\boldsymbol{s_k}.$$

Similarly, we can write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{t_1}, \cdots, \boldsymbol{t_r}$, say

$$\boldsymbol{w} = b_1\boldsymbol{t_1} + b_2\boldsymbol{t_2} + \cdots + b_r\boldsymbol{t_r}.$$

This implies that $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$ can be written as a linear combination of $\boldsymbol{s_1}, \cdots, \boldsymbol{s_k}, \boldsymbol{t_1}, \cdots, \boldsymbol{t_r}$. Thus $S \cup T$ spans $V + W$ and $V + W$ is a subspace.

(b) (i) $V + W = \{(x, y) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2$.

(ii) $V = \text{span}\{(1, 2, 3)\}$ and $W = \text{span}\{(1, 0, -1)\}$. So $V + W = \text{span}\{(1, 2, 3), (1, 0, -1)\}$.

(iii) $V = \text{span}\{(1, 1, 1)\}$. Solving $x + y - z = 0$, we know that $W = \text{span}\{(1, 0, 1), (-1, 1, 0)\}$. So $V + W = \text{span}\{(1, 1, 1), (1, 0, 1), (-1, 1, 0)\} = \mathbb{R}^3$.

5. For each of the sets $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ in Question 1,

(i) determine if S is a linearly independent set.

(ii) If S is a linearly dependent set, find a non-trivial solution to the equation

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0}.$$

Hence or otherwise, find a vector $\boldsymbol{x}$ in S such that

$$\text{span}(S) = \text{span}(S - \{\boldsymbol{x}\}).$$

(i) S_1 and S_2 are linearly independent. S_3 and S_4 are linearly dependent.

(ii) Consider S_3:

$$a(3,2,-1,2)+b(4,0,0,2)+c(5,6,-3,2)+d(0,4,-2,-1)=(0,0,0,0)$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrcrcl} 3a & + & 4b & + & 5c & & & = & 0 \\ 2a & & & + & 6c & + & 4d & = & 0 \\ -a & & & - & 3c & - & 2d & = & 0 \\ 2a & + & 2b & + & 2c & - & d & = & 0 \end{array}\right.$$

Solving the linear system, we have a general solution

$$\left\{\begin{array}{rcl} a & = & s \\ b & = & \frac{s}{2} \\ c & = & -s \\ d & = & s, \quad s \in \mathbb{R} \end{array}\right.$$

A possible non trivial solution is $(a,b,c,d)=(2,1,-2,2)$. So

$$(4,0,0,2)=-2(3,2,-1,2)+2(5,6,-3,2)-2(0,4,-2,-1).$$

We can choose $\boldsymbol{x}=(4,0,0,2)$, then $\mathrm{span}(S_3)=\mathrm{span}(S_3-\{\boldsymbol{x}\})$.
For S_4, we follow the same procedure as above and obtain

$$(4,0,4,0)=4(1,1,1,1)-4(0,1,0,1).$$

So we can choose $\boldsymbol{x}=(4,0,4,0)$, then $\mathrm{span}(S_4)=\mathrm{span}(S_4-\{\boldsymbol{x}\})$.