National University of Singapore
Department of Mathematics

Semester 1, 2018/2019 **MA1101R Linear Algebra I** **Homework 3**

Instruction

(a) This homework set consists of 3 pages and 8 questions.

(b) Do all the problems and submit on Oct. 15 (Monday) for SL1 group or on Oct. 16 (Tuesday) for SL2 group during lecture.

(c) Use A4 size writing paper. Write your full name, student number and tutorial group clearly on the first page of your answer scripts.

(d) Indicate the question numbers clearly (you do not need to copy the questions in your answer sheets).

(e) Show your steps of your working how the answers are derived, unless the questions state otherwise.

(f) Late Submission will not be accepted.

(g) **Warning**: If you are found to have copied answers from your friend(s), both you and your friend(s) will be penalized.

Problem Set (covering Lectures 9–14).

1. Let P represent a plane in $\mathbb{R}^3$ with equation $2x + y - 3z = 1$ and A, B, C represent three different lines given by the following set notation:

$$A = \{(at, bt, ct) : t \in \mathbb{R}\},\ B = \{(t+1, 2t-6, -t) : t \in \mathbb{R}\},\ C = \{(t, t, t) : t \in \mathbb{R}\}.$$

   (a) Express the plane P in explicit set notation.

   (b) Write down the conditions on a, b, c so that the line A containing the origin with the direction (a, b, c) is parallel to the plane P, that is, the line has no intersection with P.

   Show that (a, b, c) lies in the plane containing the origin and parallel to P.

   (c) Find all the points of intersection of the line B with the plane P.

   (d) Write down an explicit form of a plane P' containing the intersection point in Part (c) and the line C.

2. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a set of linearly independent vectors in V. Assume that

$$\boldsymbol{v}_1 = a_{11}\boldsymbol{u}_1 + a_{12}\boldsymbol{u}_2 + a_{13}\boldsymbol{u}_3 \tag{1}$$
$$\boldsymbol{v}_2 = a_{21}\boldsymbol{u}_1 + a_{22}\boldsymbol{u}_2 + a_{23}\boldsymbol{u}_3 \tag{2}$$
$$\boldsymbol{v}_3 = a_{31}\boldsymbol{u}_1 + a_{32}\boldsymbol{u}_2 + a_{33}\boldsymbol{u}_3. \tag{3}$$

   Denote

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

   Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly independent if and only if A is invertible.

3. Let

$$A = \begin{pmatrix} 1 & -3 & 2 & 0 \\ -4 & 2 & 1 & -1 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -5 & 1 & 3 & -3 \\ 5 & 7 & 1 & 19 \\ 2 & 2 & 0 & 6 \end{pmatrix}$$

(a) Is the column space of A equal to $\mathbb{R}^4$? Justify.

(b) Show that the column space of B is a subspace of the column space of A.

(c) Find bases S and T for the column spaces of A and B respectively. Are the two columns spaces the same?

(d) Let $\boldsymbol{v} = (0, -4, 12, 4)$ and $\boldsymbol{u} = (2, 0, 3, 0)$. Find the coordinate $(\boldsymbol{v})_T$ and $(\boldsymbol{u})_S$. Based on your result in Part (b) and $(\boldsymbol{v})_T$, find $(\boldsymbol{v})_S$. Is it possible to find the coordinates $(\boldsymbol{u})_T$? Why?

4. Discuss all the possibilities of the dimensions of the solution space V of the following homogeneous linear system.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & - & 5x_4 & = & 0 \\ -x_1 & & & + & 3x_3 & + & 5x_4 & = & 0 \\ x_1 & + & x_2 & + & ax_3 & + & bx_4 & = & 0 \end{array}\right.$$

(a) Determine the values of a and b such that $\dim(V) = 2$ and find a basis of V;

(b) Determine the values of a and b such that $\dim(V) = 1$ and find a basis of V;

(c) Is it possible that $\dim(V) = 0$ or 3? Justify.

5. Find the condition on x, y and z such that

$$\operatorname{span}\{(2, 1, 1), (1, -1, 1), (x, y, z)\} = \mathbb{R}^3.$$

6. (a) Let

$$S = \{(-3, 2, 4, 1), \qquad (0, 1, 5, -4), \qquad (2, -1, -1, 5)\}.$$

Extend S to be a basis for $\mathbb{R}^4$.

(b) Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. Show that S is part of some basis of $\mathbb{R}^n$; that is, S may be extended to a basis of $\mathbb{R}^n$. (Hint: Let $T = \{e_1, e_2, \cdots, e_n\}$ be the standard basis of V. Show that there exists a subset S' of $S \cup T$ such that $S \subset S' \subset T$.)

(c) Let V be a vector space and T be a basis of V where $|T| = n$. Suppose that $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ is a set of linearly independent vectors in V. Show that S is part of some basis of V. (Hint: Consider the coordinates of S relative to T.)

7. Let

$$\begin{aligned} S &= \{(3, 0, 7, 5),\ (6, 5, 5, 6),\ (5, 2, 5, -4)\} \\ T &= \{(2, 1, 3, 3),\ (1, 1, 0, -1),\ (0, 1, -1, 2)\}. \end{aligned}$$

(a) Find the transition matrix from S to T.

(b) Let $\boldsymbol{w}$ be a vector in $\mathbb{R}^4$ such that $(\boldsymbol{w})_T = (3, 1, 4)$. Find $(\boldsymbol{w})_S$.

8. Let V and W be subspaces of $\mathbb{R}^n$. Recall

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \colon \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\}.$$

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m\}$ be bases of V and W respectively. Denote

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m\}.$$

(a) Assume $V \cap W = \{\mathbf{0}\}$:

   (i) Show that S is linearly independent.

   (ii) Show that $\dim(V + W) = \dim(V) + \dim(W)$.

(b) Show that if $\dim(V \cap W) \geq 1$ then S is linearly dependent. (We may assume that $V \cap W$ is a subspace without proving it.)