"How To" for MA1101R

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This cute file is provided for the students in MA1101R. I hope it is helpful to "kill" the Final Examination. Any corrections and suggestions are always welcome. Please feel free to email me.

User guide: For each how-to question, I will list the most powerful methods, related examples (chosen from Examples in the textbook[1] and Exercises in the tutorial questions) and remarks if necessary.

1 Chapter 1

Question 1. *How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?*

Answer. [Page 8] A matrix is said to be in row-echelon form (REF) if it has properties (1) and (2):

(1) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(2) In any two successive rows that do not consist entirely of zeros, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

A matrix is said to be in reduced row-echelon form (RREF) if it is has properties (1), (2), (3), and (4):

(3) The leading entry of every nonzero row is 1.

(4) In each pivot column, except the pivot point, all other entries are zero.

Properties (3) and (4) are the differences between a REF and a RREF.

➢ Example: Exercises 1.28 and 1.29.

♣ Remark: In the textbook, the REF and RREF are defined for the augmented matrix, but they can be generalized for any arbitrary matrix. □

Question 2. *Given a matrix, how to obtain a row-echelon form or a reduced row-echelon form for it?*

Answer. [Page 11] If we want to get a REF, we should apply Gaussian Elimination. If we want to get a RREF, then we should apply Gauss-Jordan Elimination.

➢ Example: Example 1.4.4.

♣ Remark: In the textbook, Gaussian Elimination and Gauss-Jordan Elimination are defined for the augmented matrix, but they can be generalized for any arbitrary matrix, too. □

Question 3. *How to tell the number of solutions of a linear system from* REF*?*

Answer. [Page 15–16] There are the following three cases:

- A linear system has no solution if the last column of a REF of the augmented matrix is a pivot column, i.e. there is a row with nonzero last entry but zero elsewhere.

[1] Ma Siu Lun, Ng Kah Loon, Victor Tan: Linear Algebra – Concepts and Techniques on Euclidean Spaces (Revised Edition), McGraw-Hill, 2014

- A linear system has exactly one solution if except the last column, every column of a REF of the augmented matrix is a pivot column. That is, a linear system has exactly one solution if it is consistent and (♯ variables = ♯ nonzero rows).
- A linear system has infinitely many solutions if apart from the last column, a REF of the augmented matrix has at least one more non-pivot column. That is, a linear system has infinitely many solutions if it is consistent and (♯ variables > ♯ nonzero rows).

  In this case, its general solution has (♯ variables − ♯ nonzero rows) arbitrary parameter(s).

□

➢ Example: Example 1.4.10 and Exercise 1.22.

Question 4. *If a linear system is consistent, how to find a general solution of it?*

Answer. We need follow this process:

1. Transfer the linear system to the corresponding augmented matrix;
2. Apply Gaussian Elimination (resp. Gauss-Jordan Elimination) to obtain a REF (resp. a RREF) of the augmented matrix;
3. Identify the pivot columns and non-pivot columns;
4. For any i, if the i-th column is a non-pivot column, then take the i-th variable to be a parameter.
5. Express each of the remaining variable(s) as an expression of the parameter(s).

□

➢ Example: Example 1.4.7.

2 Chapter 2

Question 5. *How to determine whether an $m \times n$ matrix A is invertible?*

Answer. If A is not a square matrix, it cannot be invertible. If A is a square matrix, we have the following four methods:

- First method: [Page 65] A is invertible if and only if $\det(A) \neq 0$. So after computing the determinant of A, it is easy to identify the invertibility of A.
- Second method: [Page 181] A is invertible if and only if $\text{rank}(A) = n$. So we may apply Gaussian Elimination to obtain a REF for A, then find the $\text{rank}(A)$ and determine whether $\text{rank}(A) = n$.
- Third method: [Page 45–46] If we can find a matrix B such that $AB = I$ or $BA = I$, then it follows from the definition that A is invertible.

  ➢ Example: Examples 2.3.3 and 2.3.8.
- Fourth method: [Page 53] A is invertible if and only if the RREF of A is an identity matrix.

  ➢ Example: Example 2.4.11 and Exercise 2.43.

□

♣ Remark: The second method and the fourth method are essentially the same.

Question 6. *How to find the inverse of an invertible matrix A?*

Answer. There are the following three methods:

- First method: [Page 54]
  1. Consider the $n \times 2n$ matrix $(A \mid I)$.
  2. By Gauss-Jordan Elimination, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I$, that is,
  $$A^{-1} = E_k \cdots E_2 E_1.$$

  ➢ Example: Example 2.4.9.

- Second method: [Page 67] When A is sparse (nonzero entries are few), we may apply $\mathrm{adj}(A)$ to compute A^{-1}:

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A).$$

  ➢ Example: Example 2.5.26.

- Third method: [Page 45–46] If we can find a matrix B such that $AB = I$ or $BA = I$, then by definition A is invertible and $A^{-1} = B$.

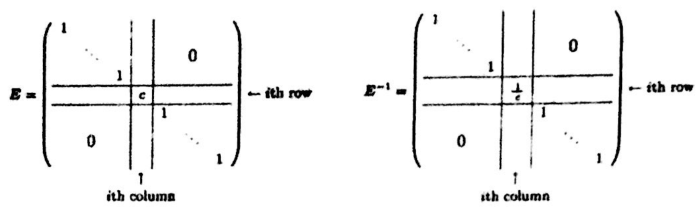
  ➢ Example: Exercises 2.26, 2.27, and 2.29.

□

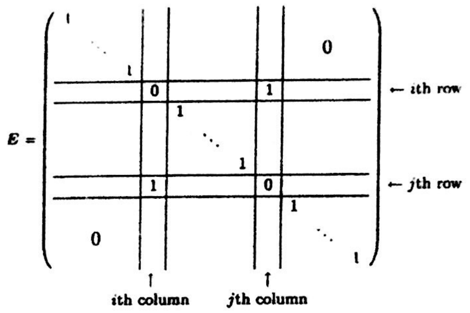
Question 7. *How to convert elementary row operations to elementary matrices, and vice versa?*

Answer. [Page 48–51] There are the following three cases:

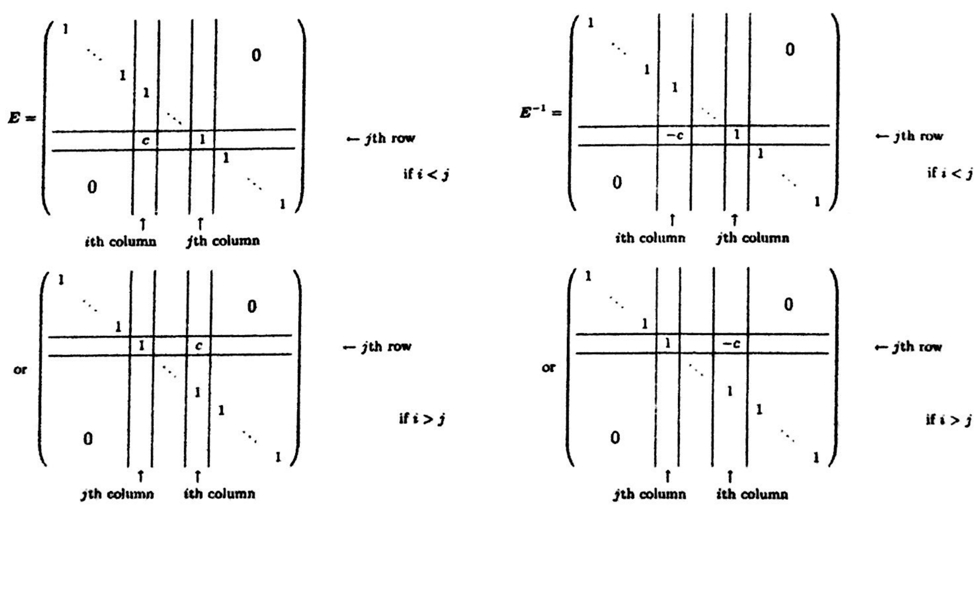
- Multiply a row by a constant:



- Interchange two rows:



- Add a multiple of a row to another row:



□

Question 8. *Given two row equivalent matrices A and B, how to find an invertible matrix D such that $DA = B$?*

Answer. [Page 52] B is obtained from A by elementary row operations. For each of these elementary row operations, we write down the corresponding elementary matrices $E_1, E_2, \dots, E_k$:

$$A \xrightarrow{E_1} \xrightarrow{E_2} \cdots \xrightarrow{E_k} B.$$

It follows that $E_k \cdots E_2 E_1 A = B$. So we can take $D = E_k \cdots E_2 E_1$.

➢ Example: Example 2.4.5 and Exercise 2.33. □

Question 9. *How to compute the determinant of a matrix A using various methods (and not just cofactor expansion)?*

Answer. We have the following three methods:

- First method: [Page 58–60] Apply the definition or cofactor expansion.

  ➢ Example: Examples 2.5.4 and 2.5.7.

- Second method: [Page 59] For the 2×2 and 3×3 matrices, we have the following formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

- Third method: [Page 62–65] Apply Gauss Elimination or Gauss-Jordan Elimination to get a REF or a RREF R for A, that is, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$E_k \cdots E_2 E_1 A = R.$$

  Then we have

$$\det(A) = \det(E_1)^{-1} \det(E_2)^{-1} \cdots \det(E_k)^{-1} \det(R),$$

  where R is an upper triangular matrix whose determinant is the product of the diagonal entries.

  ➢ Example: Example 2.5.17 and Exercise 2.52.

□

Question 10. *How does a row (column) operation change the determinant?*

Answer. [Page 62–64] There are the following three cases:

- If B is obtained from A by multiplying one row of A by a constant k, then $\det(B) = k\det(A)$;
- If B is obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$;
- If B is obtained from A by adding a multiple of one row of A to another row, then $\det(B) = \det(A)$.

□

Question 11. *How are invertibility of a matrix and the homogeneous system related?*

Answer. [Page 53] A matrix A is invertible if and only if the homogeneous linear system $Ax = \mathbf{0}$ has only the trivial solution.

➢ Example: Exercise 2.41.

□

Question 12. *How are invertibility and determinant of a matrix related?*

Answer. [Page 65] A matrix A is invertible if and only if $\det(A) \neq 0$.

➢ Example: Example 2.5.20.

□

3 Chapter 3

Question 13. *How to write down implicit and explicit set notations for lines and planes in $\mathbb{R}^3$?*

Answer. [Page 87] An implicit form for a line in $\mathbb{R}^3$ is

$$\{ (x,y,z) \mid a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2 \},$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are real constants, a_1, b_1, c_1 are not all zero, and a_2, b_2, c_2 are not all zero.

An explicit form for a line in $\mathbb{R}^3$ is

$$\{ (a_0, b_0, c_0) + t(a,b,c) \mid t \in \mathbb{R} \},$$

where a_0, b_0, c_0, a, b, c are real constants and a, b, c are not all zero.

An implicit form for a plane in $\mathbb{R}^3$ is

$$\{ (x,y,z) \mid ax + by + cz = d \},$$

where a, b, c, d are real constants and a, b, c are not all zero.

An explicit form for a plane in $\mathbb{R}^3$ is

$$\begin{cases} \{ (\frac{d-bs-ct}{a}, s, t) \mid s,t \in \mathbb{R} \}, & \text{if } a \neq 0; \\ \{ (s, \frac{d-as-ct}{b}, t) \mid s,t \in \mathbb{R} \}, & \text{if } b \neq 0; \\ \{ (s, t, \frac{d-as-bt}{c}) \mid s,t \in \mathbb{R} \}, & \text{if } c \neq 0. \end{cases}$$

□

Question 14. *How to find a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ when you are given the direction of the line and a point on the line?*

Answer. [Page 87] In $\mathbb{R}^2$, given a point (a_0, b_0) on the line and the direction (a, b), then the line is

$$\{ (a_0, b_0) + t(a,b) \mid t \in \mathbb{R} \}.$$

In $\mathbb{R}^3$, given a point (a_0, b_0, c_0) on the line and the direction (a, b, c), then the line is

$$\{ (a_0, b_0, c_0) + t(a,b,c) \mid t \in \mathbb{R} \}.$$

If we want to get an implicit form for the line $\{ (a_0, b_0, c_0) + t(a,b,c) \mid t \in \mathbb{R} \}$, we should cancel the parameter t, and construct two equations in terms of x, y, z. Indeed, let $x = a_0 + ta$, $y = b_0 + tb$, and $z = c_0 + tc$. Now we need to remove t. From $x = a_0 + ta$ and $y = b_0 + tb$, we will obtain

$$bx - ay = a_0b - b_0a. \tag{1}$$

Similarly, from $y = b_0 + tb$ and $z = c_0 + tc$, we will have

$$cy - bz = b_0c - c_0b. \tag{2}$$

Then the set

$$\{ (x,y,z) \mid bx - ay = a_0b - b_0a,\ cy - bz = b_0c - c_0b \}$$

gives us an implicit form for the line. □

Question 15. *How to find the equation of a plane in $\mathbb{R}^3$ when you are given three points on the plane?*

Answer. [Page 87] Assume the plane is represented by the equation $ax + by + cz = d$, where a, b, c are not all zero. Since there are three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) on the plane, they satisfy the equation $ax + by + cz = d$. Then consider the following linear system in terms of a, b, c, d:

$$\begin{cases} x_1a + y_1b + z_1c - d = 0 \\ x_2a + y_2b + z_2c - d = 0 \\ x_3a + y_3b + z_3c - d = 0 \end{cases}$$

We may have a general solution for a, b, c, d (should be determined by one parameter). By taking a particular one, we will get an equation for the plane.

♣ Remark: if the three points are on a line, the plane cannot be determined uniquely. □

Question 16. *How to determine whether a vector v is a linear combination of a given set of vectors $\{u_1, u_2, \ldots, u_k\}$?*

Answer. [Page 88–89] Assume $v = c_1u_1 + c_2u_2 + \cdots + c_ku_k$. Then we will have a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = v,$$

where $u_1, u_2, \ldots, u_k, v$ are column vectors.

Since we have that v can be expressed as a linear combination of $u_1, u_2, \ldots, u_k$ if and only if the above linear system is consistent, it suffices to see whether the linear system is consistent.

➢ Example: Example 3.2.2. □

Question 17. *How to express a vector v as a linear combination of a given set of vectors $\{u_1, u_2, \ldots, u_k\}$?*

Answer. Following the process in the last question, if the linear system is consistent, and we can find a solution $(c_1, c_2, \ldots, c_k)$, then we have

$$v = c_1u_1 + c_2u_2 + \cdots + c_ku_k.$$

➢ Example: Example 3.2.2. □

Question 18. *How to show a linear span* $\mathrm{span}(S_1)$ *is contained in another one* $\mathrm{span}(S_2)$*, where $S_1 = \{u_1, u_2, \ldots, u_k\}$, $S_2 = \{v_1, v_2, \ldots, v_m\}$?*

Answer. [Page 92–93] Since $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$ if and only if each u_i is a linear combination of $v_1, v_2, \ldots, v_m$, it suffices to show that each of the last k columns in the following REF is not a pivot column:

$$\left(v_1 \quad v_2 \quad \cdots \quad v_m \mid u_1 \mid u_2 \mid \cdots \mid u_k\right),$$

where $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m$ are column vectors.

➢ Example: Example 3.2.11 and Exercise 3.12. □

Question 19. *How to show that a set V is a subspace of $\mathbb{R}^n$?*

Answer. Firstly, we should show that V is a subset of $\mathbb{R}^n$. Then there are the following three methods:

- First method: [Page 95] If we can find a set S which spans V, then V is a subspace of $\mathbb{R}^n$ by definition directly.

  ➢ Example: Example 3.3.3, and Exercises 3.20, 5.7.

- Second method: [Page 99] V is a subspace of $\mathbb{R}^n$ if and only if $V \neq \emptyset$ and for all $u, v \in V$ and $c, d \in \mathbb{R}$, $cu + dv \in V$.

  ➢ Example: Exercises 3.20, 3.21, 3.22, 3.24, and 5.7.

- Third method: [Page 97] If V is the solution set of a homogeneous linear system, then V is a subspace of $\mathbb{R}^n$.

  ➢ Example: Exercises 3.21, 3.22, and 5.7.

□

Question 20. *How to show that a set $\{u_1, u_2, \ldots, u_k\}$ is linearly (in)dependent?*

Answer. [Page 99–100] Apply the working definition. The equation

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$$

gives us a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0},$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \mathbf{0}$ are column vectors.

By Gaussian Elimination, we can identify whether this linear system has only the trivial solution, and we have the fact that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ is linearly independent if and only if this linear system has only the trivial solution. □

➢ Example: Example 3.4.3.

Question 21. *How to show that a set S is a basis for a vector space V?*

Answer. [Page 103–111] We have the following four conditions:

- Condition (1): $S \subseteq V$;
- Condition (2-1): S is linearly independent;
- Condition (2-2): S spans V;
- Condition (2-3): $|S| = \dim(V)$.

If condition (1) and any <u>two</u> of conditions (2-1), (2-2) and (2-3) hold, then S is a basis for V.

♣ Remark: if we know $\dim(V)$, then we can choose conditions (1), (2-1) and (2-3) to check; if we do not know $\dim(V)$, we need to check conditions (1), (2-1), and (2-2). □

➢ Example: Examples 3.5.5, 3.6.8, and Exercises 3.32, 3.38, 3.45.

Question 22. *How to find a basis for a vector space V?*

Answer. [Page 104–110]

1. Find a set $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ which spans V, where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are column vectors;
2. Apply Gaussian Elimination to the matrix $\begin{pmatrix} \boldsymbol{u_1}^T \\ \boldsymbol{u_2}^T \\ \vdots \\ \boldsymbol{u_k}^T \end{pmatrix}$, we obtain a REF $\boldsymbol{R}$;
3. Let S' be the set of nonzero rows in $\boldsymbol{R}$, then S' is a basis for V.

♣ Remark: S' is not necessarily unique. □

➢ Example: Examples 3.4.6, 3.6.4, and Exercise 4.7.

Question 23. *Given a basis $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ for a vector space V and a vector $\boldsymbol{v} \in V$, how to find coordinate vectors $[\boldsymbol{v}]_S$ and $(\boldsymbol{v})_S$?*

Answer. [Page 106–107] Let $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$. By solving the following linear system

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v},$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{v}$ are column vectors, we will find a solution $(c_1, c_2, \ldots, c_k)$. So the coordinate vectors are

$$[\boldsymbol{v}]_S^T = (\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k).$$

□

➢ Example: Example 3.5.9.

Question 24. *How to compute the dimension for a vector space?*

Answer. [Page 109–111] Following the process in **Question 22**, S' is a basis for V . Then we have

$$\dim(V) = |S'|.$$

➢ Example: Example 3.6.4.

♣ Remark: Except $\mathbb{R}^n$ and $\{\mathbf{0}\}$, we can identify $\dim(V)$ only when we have found a basis for it. □

Question 25. *How to compute the transition matrix from $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ to $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$, where S and T are two bases for a vector space V?*

Answer. There are the following two methods:

- First method: [Page 114–115] By solving linear systems, we can find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T, \ldots, [\boldsymbol{u}_k]_T$. Then the transition matrix from S to T is

  $$\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{u}_1]_T & [\boldsymbol{u}_2]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix}.$$

  ➢ Example: Example 3.7.4.

- Second method: [Page 116] If we know the transition matrix from T to S is $\boldsymbol{Q}$, then the transition matrix from S to T is $\boldsymbol{P} = \boldsymbol{Q}^{-1}$.

  ➢ Example: Example 3.7.6.

□

4 Chapter 4

Question 26. *How to find bases for the row space and the column space of $\boldsymbol{A}$?*

Answer. [Page 129–134] We may apply the similar method in **Question 22**:

Apply Gaussian Elimination to $\boldsymbol{A}$ to obtain a REF $\boldsymbol{R}$ of $\boldsymbol{A}$. Then the set of nonzero rows in $\boldsymbol{R}$ forms a basis for the row space of $\boldsymbol{A}$.

Besides, by choosing the pivot columns, we will find a set of columns of $\boldsymbol{R}$ which forms a basis for the column space of $\boldsymbol{R}$. Then the set of corresponding columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$.

➢ Example: Examples 4.1.12, 4.1.14, and Exercises 4.3, 4.4. □

Question 27. *How to extend a linearly independent set S to a basis for $\mathbb{R}^n$?*

Answer. [Page 134] We need follow this procedure:

1. Form a matrix $\boldsymbol{A}$ using the vectors in S as rows;
2. Reduce $\boldsymbol{A}$ to a REF $\boldsymbol{R}$;
3. Identify the non-pivot columns in $\boldsymbol{R}$;
4. For each non-pivot column identified in Step 3, get a vector such that the leading entry of the vector is at that column;
5. Now, $S \cup$ (the set of vectors obtained in Step 4) is a basis for $\mathbb{R}^n$.

➢ Example: Example 4.1.14. □

Question 28. *How to find a basis for the nullspace of $\boldsymbol{A}$?*

Answer. [Page 138–139] Solve the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. Then find a basis for the solution space.

➢ Example: Example 4.3.3. □

Question 29. *How to find the rank and the nullity of a matrix $\boldsymbol{A}$?*

Answer. [Page 136–140] Apply Gaussian Elimination to reduce $\boldsymbol{A}$ to a REF $\boldsymbol{R}$. Then

$$\text{rank}(\boldsymbol{A}) = (\sharp \text{ pivot columns in } \boldsymbol{R}),$$

and

$$\begin{aligned}\text{nullity}(\boldsymbol{A}) &= (\text{the dimension of the solution space of } \boldsymbol{Ax} = \boldsymbol{0}) \\ &= (\sharp \text{ columns of } \boldsymbol{A}) - \text{rank}(\boldsymbol{A}).\end{aligned}$$

➢ Example: Examples 4.2.2, 4.3.3, and Exercise 4.15. □

5 Chapter 6

Question 30. *How to find eigenvalues and eigenvectors of a matrix $\boldsymbol{A}$?*

Answer. There are the following three methods:

- First method: [Page 180] Solve the characteristic equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$.

  ➢ Example: Examples 6.1.7, 6.1.10, and Exercise 6.3.

- Second method: [Page 178] By definition, if we find a scalar λ and a nonzero column vector $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $\boldsymbol{Au} = \lambda \boldsymbol{u}$, then λ is an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

  ➢ Example: Exercise 6.16.

- Third method: If we have such an equation

  $$\boldsymbol{A}^k + a_{k-1}\boldsymbol{A}^{k-1} + \cdots + a_1\boldsymbol{A} + a_0\boldsymbol{I} = \boldsymbol{0},$$

  then any solution for the equation $\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0 = 0$ is an eigenvalue of $\boldsymbol{A}$.

  ➢ Example: Exercise 6.4.

♣ Remark: If $\boldsymbol{A}$ is an "abstract" matrix (e.g. a stochastic matrix), we can only apply the second method. □

Question 31. *How to find a basis for the eigenspace of a matrix $\boldsymbol{A}$?*

Answer. [Page 182–184] Suppose we are given an eigenvalue λ of $\boldsymbol{A}$. By solving the linear system

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0},$$

we will have a general solution, say $\boldsymbol{x} = s_1\boldsymbol{u_1} + s_2\boldsymbol{u_2} + \cdots + s_k\boldsymbol{u_k}$, where $s_1, s_2, \ldots, s_k$ are arbitrary parameters. Then the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue λ is $E_\lambda = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$.

➢ Example: Example 6.1.12. □

Question 32. *How are eigenvalues and the invertibility of a matrix related?*

Answer. A matrix $\boldsymbol{A}$ is invertible if and only if 0 is <u>not</u> an eigenvalue of $\boldsymbol{A}$. □

Question 33. *How to determine whether a matrix $\boldsymbol{A}$ is diagonalizable?*

Answer. There are the following two methods:

- First method: [Page 188] If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable.

  ➢ Example: Examples 6.2.8 and 6.2.11.

- Second method: [Page 186–188] Apply Algorithm 6.2.4.

  ➢ Example: Example 6.2.6 and Exercise 6.13.

Question 34. *How to diagonalize a matrix?*

Answer. [Page 186–188] Apply Algorithm 6.2.4.

➢ Example: Example 6.2.6 and Exercise 6.11. □

Question 35. *How to compute powers of a matrix A using diagonalization?*

Answer. Assume there exists an invertible matrix P such that

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Then for any positive integer m, we have

$$A^m = \underbrace{A \cdots A}_{\text{m times}} = \underbrace{(PDP^{-1}) \cdots (PDP^{-1})}_{\text{m times}} = PD^mP^{-1} = P \begin{pmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{pmatrix} P^{-1}.$$

➢ Example: Example 6.2.11. □

Question 36. *How to solve linear recurrence relations using diagonalization?*

Answer. [Page 190] We need follow this process:

1. Transfer the linear recurrence relation to a matrix equation. For example, given a linear recurrence relation $a_{n+1} = a_n + a_{n-1} + a_{n-2}$, then the related matrix equation is

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = A \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix},$$

   where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

2. Applying Algorithm 6.2.4, we will find an invertible matrix P such that $P^{-1}AP = D$, where $D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$.

3. Thus

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = A \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = P \begin{pmatrix} \lambda_1^{n-1} & & \\ & \lambda_2^{n-1} & \\ & & \lambda_3^{n-1} \end{pmatrix} P^{-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

4. By computing the last expression, we will get an explicit form for a_n.

➢ Example: Example 6.2.11 and Exercise 6.21. □

6 Chapter 5

Question 37. *Given an orthogonal basis S or an orthonormal basis T for a vector space V and a vector $w \in V$, how to find $[w]_S$, $(w)_S$, $[w]_T$, and $(w)_T$?*

Answer. [Page 154] We discuss by cases:

- If $S = \{u_1, u_2, \dots, u_k\}$ is an orthogonal basis for V, then

$$w = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k,$$

  and hence

$$[w]_S^T = (w)_S = \left(\frac{w \cdot u_1}{u_1 \cdot u_1}, \frac{w \cdot u_2}{u_2 \cdot u_2}, \dots, \frac{w \cdot u_k}{u_k \cdot u_k} \right).$$

- If $T = \{v_1, v_2, \dots, v_k\}$ is an orthonormal basis for V, then

$$w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k,$$

  and hence

$$[w]_T^T = (w)_T = (w \cdot v_1, w \cdot v_2, \dots, w \cdot v_k).$$

➢ Example: Example 5.2.9. □

Question 38. *How to find an orthogonal basis and an orthonormal basis for a vector space V?*

Answer. [Page 158] Following the process in **Question 22**, we have a basis $\{u_1, u_2, \dots, u_k\}$ for V. Then applying Gram-Schmidt Process, we will get an orthogonal basis $\{v_1, v_2, \dots, v_k\}$ for V. Then by normalizing every vector v_i, we will obtain an orthonormal basis $\{w_1, w_2, \dots, w_k\}$ for V.

➢ Example: Example 5.2.20. □

Question 39. *How to find the projection of a vector w onto a subspace V of $\mathbb{R}^n$?*

Answer. [Page 156] Suppose we have an orthogonal basis $\{u_1, u_2, \dots, u_k\}$ and an orthonormal basis $\{v_1, v_2, \dots, v_k\}$ for V. Then

$$\frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k,$$

and

$$(w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$$

are the projection of w onto V.

➢ Example: Examples 5.2.14 and 5.2.16. □

Question 40. *How to find the least squares solution to a system $Ax = b$?*

Answer. [Page 162–163] Solve the linear system $A^T A x = A^T b$.

➢ Example: Example 5.3.11. □

Question 41. *How to identify a matrix to be an orthogonal matrix?*

Answer. [Page 166] Given a square matrix A, then the following statements are equivalent:

- (Definition 5.4.3) A is orthogonal, i.e. $A^{-1} = A^T$;
- (Remark 5.4.4) $AA^T = I$;
- (Remark 5.4.4) $A^T A = I$;
- (Theorem 5.4.6) The rows of A form an orthonormal basis for $\mathbb{R}^n$;
- (Theorem 5.4.6) The columns of A form an orthonormal basis for $\mathbb{R}^n$;
- (Exercise 5.32) $\|Au\| = \|u\|$ for any vector $u \in \mathbb{R}^n$;
- (Exercise 5.32) $Au \cdot Av = u \cdot v$ for any vectors $u, v \in \mathbb{R}^n$.

➢ Example: Example 5.4.5. □

Question 42. *How is an orthogonal matrix related to an orthonormal basis?*

Answer. [Page 166] $\boldsymbol{A}$ is an orthogonal matrix if and only if the columns (rows) of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$. □

➢ Example: Exercise 5.34 (a), (b).

Question 43. *How is a transition matrix related to an orthogonal matrix?*

Answer. [Page 167] Let S and T be two orthonormal bases for a vector space, and let $\boldsymbol{P}$ be the transition matrix from S to T. Then $\boldsymbol{P}$ is orthogonal. □

Question 44. *How to orthogonally diagonalize a symmetric matrix?*

Answer. [Page 192] Apply Algorithm 6.3.5.

➢ Example: Example 6.3.7, and Exercises 6.25, 6.29. □

7 Chapter 7

Question 45. *How are linear transformations related to matrices?*

Answer. [Page 209] Apply Definition 7.1.1.

➢ Example: Exercises 7.7 and 7.9. □

Question 46. *How to show that a mapping $T: V \to W$ is a linear transformation?*

Answer. There are the following two methods:

- First method: [Page 209] If we can find the standard matrix for T, then T is a linear transformation.
- Second method: [Page 210] If $T(c\boldsymbol{u}+d\boldsymbol{v}) = cT(\boldsymbol{u})+dT(\boldsymbol{v})$ for all $\boldsymbol{u},\boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, then T is a linear transformation.

➢ Example: Example 7.1.2. □

Question 47. *How to find a basis for $R(T)$ (resp.* $\mathrm{Ker}(T)$*)?*

Answer. [Page 215–218] Let $\boldsymbol{A}$ be the standard matrix for T. Note that

$$\begin{aligned} R(T) &= \text{the column space of } \boldsymbol{A}, \\ \mathrm{Ker}(T) &= \text{the nullspace of } \boldsymbol{A}. \end{aligned}$$

Then follow the process in **Question 26** (resp. **Question 28**).

➢ Example: Examples 7.2.6 and 7.2.11. □

Question 48. *How to find* $\mathrm{rank}(T)$ *and* $\mathrm{nullity}(T)$*?*

Answer. [Page 218] Let $\boldsymbol{A}$ be the standard matrix for T. Note that

$$\begin{aligned} \mathrm{rank}(T) &= \mathrm{rank}(\boldsymbol{A}), \\ \mathrm{nullity}(T) &= \mathrm{nullity}(\boldsymbol{A}). \end{aligned}$$

Then follow the process in **Question 29**.

➢ Example: Exercise 7.12. □