NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 8**

1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of the same size. Show that

$$\text{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B}).$$

Let $\boldsymbol{A} = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots\ \boldsymbol{a}_n)$ and $\boldsymbol{B} = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots\ \boldsymbol{b}_n)$ where $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are the i-th columns of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Then $\boldsymbol{A}+\boldsymbol{B} = (\boldsymbol{a}_1+\boldsymbol{b}_1\ \boldsymbol{a}_2+\boldsymbol{b}_2\ \cdots\ \boldsymbol{a}_n+\boldsymbol{b}_n)$. Since $\boldsymbol{a}_i+\boldsymbol{b}_i$ is a linear combination of $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n,\boldsymbol{b}_1,\dots,\boldsymbol{b}_n\}$ for all $1 \leq i \leq n$, $\boldsymbol{a}_i+\boldsymbol{b}_i$ is in $\text{span}\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n,\boldsymbol{b}_1,\dots,\boldsymbol{b}_n\}$. By Theorem 3.2.10,

$$\text{span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\} \subseteq \text{span}\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n,\boldsymbol{b}_1,\dots,\boldsymbol{b}_n\}.$$

Since the column space of $\boldsymbol{A}+\boldsymbol{B}$ equals $\text{span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\}$,

$$\text{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \dim(\text{span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\}).$$

As $\text{span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\}$ is a subspace of $\text{span}\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_n,\boldsymbol{b}_1,,\boldsymbol{b}_n\}$,

$$\begin{aligned}
&\text{rank}(\boldsymbol{A}+\boldsymbol{B})\\
=&\dim(\text{span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\})\\
\leq&\dim(\text{span}\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n,\boldsymbol{b}_1,\dots,\boldsymbol{b}_n\})\\
\leq&\dim(\text{span}\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n\}) + \dim(\text{span}\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_n\})\\
\leq&\text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B}).
\end{aligned}$$

2. Determine the possible rank, nullity and nullspace of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$$

Let us perform the elementary row operations on $\boldsymbol{A}$ and obtain its row echelon form:

$$\boldsymbol{A} = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gauss}} \begin{pmatrix} -1 & -3 & t \\ 0 & -3 & 3t-2 \\ 0 & 0 & -2t^2+5t-3 \end{pmatrix}.$$

We have two cases: $-2t^2+5t-3 \neq 0$ or $-2t^2+5t-3 = 0$.

If $-2t^2+5t-3 = 0$, then $t = 1$ or $t = \frac{3}{2}$, and the nullity of $\boldsymbol{A}$ is 1; If $-2t^2+5t-3 \neq 0$ (i.e., $t \neq 1$ and $t \neq \frac{3}{2}$), then the nullity of $\boldsymbol{A}$ is 0.

When $t = 1$, the null space is $\text{span}\{(0,1,3)^T\}$; When $t = \frac{3}{2}$, the null space is $\text{span}\{(-6,5,6)^T\}$; When $t \neq 1$ and $t \neq \frac{3}{2}$, the nullspace is $\{\boldsymbol{0}\}$.

3. Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Show that $\boldsymbol{A}$ has rank 2 if and only if one or more of the following determinants is nonzero.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \tag{1}$$

Equivalently, we show that $\boldsymbol{A}$ has rank < 2 if and only if all of the determinants of 2×2-matrices in (1) are zero. Denote

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix} = (\boldsymbol{c}_1 \ \boldsymbol{c}_2 \ \boldsymbol{c}_3),$$

where $\boldsymbol{r}_i$ is the i-th row of $\boldsymbol{A}$ and $\boldsymbol{c}_i$ is the i-th column of $\boldsymbol{A}$.

Suppose that $\boldsymbol{A}$ has rank < 2. $\text{rank}(\boldsymbol{A}) = 0$ if and only if $\boldsymbol{A}$ is the zero matrix. In this case, all matrices in (1) are zero matrices, whose determinants are 0. If $\text{rank}(\boldsymbol{A}) = 1$, then $\dim(\text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2\}) = 1$, i.e., $\boldsymbol{r}_1 = a_0\boldsymbol{r}_2$ or $\boldsymbol{r}_2 = a_0\boldsymbol{r}_1$ for some $a_0 \in \mathbb{R}$. If $\boldsymbol{r}_1 = a_0\boldsymbol{r}_2$, then

$$\begin{vmatrix} a_0a_{21} & a_0a_{22} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_0a_{21} & a_0a_{23} \\ a_{21} & a_{23} \end{vmatrix} = \begin{vmatrix} a_0a_{23} & a_0a_{23} \\ a_{23} & a_{23} \end{vmatrix} = 0.$$

Similarly, if $\boldsymbol{r}_2 = a_0\boldsymbol{r}_1$, we also have all determinants in (1) are zero.

Suppose all of the determinants of 2×2-matrices in (1) are zero, that is,

$$\det(\boldsymbol{c}_1 \ \boldsymbol{c}_2) = \det(\boldsymbol{c}_1 \ \boldsymbol{c}_3) = \det(\boldsymbol{c}_2 \ \boldsymbol{c}_3) = 0.$$

Thus, $\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$, $\{\boldsymbol{c}_1, \boldsymbol{c}_3\}$ and $\{\boldsymbol{c}_2, \boldsymbol{c}_3\}$ are linearly dependent. If all vectors $\boldsymbol{c}_i$ are zero vectors, then $\boldsymbol{A}$ is the zero matrix and $\text{rank}(\boldsymbol{A}) = 0$. Assume that one of vectors $\boldsymbol{c}_i$ is nonzero, say c_1. Then $\boldsymbol{c}_2 = x_1\boldsymbol{c}_1$ and $\boldsymbol{c}_3 = x_2\boldsymbol{c}_1$ for some x_1 and x_2 in $\mathbb{R}$. Therefore, the column space of $\boldsymbol{A}$ equals $\text{span}\{\boldsymbol{c}_1, x_1\boldsymbol{c}_1, x_2\boldsymbol{c}_1\} = \text{span}\{\boldsymbol{c}_1\}$, that is, $\text{rank}(\boldsymbol{A}) = 1$. In sum, we have $\text{rank}(\boldsymbol{A}) < 2$.

4. Let W be a subspace of $\mathbb{R}^n$ and

$$\boldsymbol{u}_1 = (1, 0, 1, -1), \ \boldsymbol{u}_2 = (0, 1, 0, -1), \ \boldsymbol{u}_3 = (-2, 3, -3, 1).$$

Define $W^\perp = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u} \text{ is orthogonal to } W\}$.

(a) Compute $\|\boldsymbol{u}_1\|$, $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2$ and $d(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

$$\begin{aligned}
&\|\boldsymbol{u}_1\| = \sqrt{1^2 + 0^2 + 1^2 + (-1)^2} = \sqrt{3} \\
&\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 1 \times 0 + 0 \times 1 + 1 \times 0 + (-1) \times (-1) = 1 \\
&d(\boldsymbol{u}_1, \boldsymbol{u}_2) = \sqrt{(1-0)^2 + (0-1)^2 + (1-0)^2 + ((-1)-(-1))^2} = \sqrt{3} \\
&\theta = \cos^{-1}\left(\frac{\sqrt{3}^2 + \sqrt{2}^2 - \sqrt{3}^2}{2\sqrt{3} \cdot \sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3} \cdot \sqrt{2}}\right) = \cos^{-1}\frac{\sqrt{6}}{6}.
\end{aligned}$$

(b) Let $W = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. Find $W^{\perp}$.

A vector $\boldsymbol{u}$ is orthogonal to W if and only if $\boldsymbol{u} \cdot \boldsymbol{u}_i = 0$ for all $\boldsymbol{u}_i$. Let $\boldsymbol{u} = (x, y, z, w)$. Consider

$$\begin{cases} x + z - w = 0 \\ y - w = 0 \\ -2x + 3y - 3z + w = 0 \end{cases}$$

We have a general solution $x = -t$, $y = t$, $z = 2t$ and $w = t$. Thus, $W^{\perp} = \text{span}\{(-1, 1, 2, 1)\}$.

(c) Find the equation of the 3-plane W in $\mathbb{R}^4$.

Note that $(W^{\perp})^{\perp} = W$. Thus, the equation of the plan W is $x - y - 2z - w = 0$.

(d) Show that $W^{\perp}$ is a subspace of $\mathbb{R}^n$ and $\dim(W) + \dim(W^{\perp}) = n$.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ be a basis of W. Then $W^{\perp}$ consists of all vectors $\boldsymbol{w}$ such that $\boldsymbol{w} \cdot \boldsymbol{u}_i = 0$ for all $1 \le i \le r$. Assume that $\boldsymbol{u}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $\boldsymbol{w} = (x_1, x_2, \ldots, x_n)$. Then we have a homogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = 0 \end{cases} \tag{2}$$

Thus $W^{\perp}$ is identical to the solution space of the homogeneous linear system (2). Therefore $W^{\perp}$ is a subspace and $\dim(W^{\perp})$ equals to the number of unknowns minus the rank of the coefficient matrix, which is $n - \dim(W)$. So $\dim(W^{\perp}) = n - \dim(W)$.

5. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be an orthogonal set of vectors in a vector space. Show that

$$\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n\|^2 = \|\boldsymbol{u}_1\|^2 + \|\boldsymbol{u}_2\|^2 + \cdots + \|\boldsymbol{u}_n\|^2.$$

First, we prove the statement holds for $n = 2$, that is, if $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$, then $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$. In fact,

$$\begin{aligned} &\|\boldsymbol{u} + \boldsymbol{v}\|^2 \\ =&(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) \\ =&\boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} \\ =&\|\boldsymbol{u}\|^2 + 2\boldsymbol{u} \cdot \boldsymbol{v} + \|\boldsymbol{v}\|^2 \\ =&\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 \qquad \text{as } \boldsymbol{u} \cdot \boldsymbol{v} = 0 \end{aligned}$$

For arbitrary n, let $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_{n-1}$ and $\boldsymbol{v} = \boldsymbol{u}_n$. Note that $(\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_{n-1}) \cdot \boldsymbol{u}_n = \boldsymbol{u}_1 \cdot \boldsymbol{u}_n + \boldsymbol{u}_2 \cdot \boldsymbol{u}_n + \cdots + \boldsymbol{u}_{n-1} \cdot \boldsymbol{u}_n$. By $\boldsymbol{u}_i \cdot \boldsymbol{u}_n = 0$ for all $1 \leq i \leq n-1$, $(\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_{n-1}) \cdot \boldsymbol{u}_n = 0$, that is, $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. Thus,

$$\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n\|^2 = \|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_{n-1}\|^2 + \|\boldsymbol{u}_n\|^2.$$

By induction, $\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_{n-1}\|^2 = \|\boldsymbol{u}_1\|^2 + \|\boldsymbol{u}_2\|^2 + \cdots + \|\boldsymbol{u}_{n-1}\|^2$. Therefore,

$$\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n\|^2 = \|\boldsymbol{u}_1\|^2 + \|\boldsymbol{u}_2\|^2 + \cdots + \|\boldsymbol{u}_n\|^2.$$

Alternatively, one can also show this directly.

$$\begin{aligned} ||\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}||^2 &= (\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}) \cdot (\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}) \\ &= (\boldsymbol{u_1} \cdot \boldsymbol{u_1}) + \cdots + (\boldsymbol{u_n} \cdot \boldsymbol{u_n}) \quad \text{since } \boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0 \text{ for } i \neq j \\ &= ||\boldsymbol{u_1}||^2 + \cdots + ||\boldsymbol{u_n}||^2 \end{aligned}$$