NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 10**

1. Let $\boldsymbol{A}$ be a symmetric matrix. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two eigenvectors of $\boldsymbol{A}$ associated with eigenvalues λ and μ, respectively, where $\lambda \neq \mu$, show that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ by following the following steps.

   (a) Show that $\boldsymbol{v}^T \boldsymbol{A} = \mu \boldsymbol{v}^T$.

   (b) Show that $\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{u} = \mu \boldsymbol{v} \cdot \boldsymbol{u}$.

   (c) Show that $\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{u} = \lambda \boldsymbol{v} \cdot \boldsymbol{u}$.

   (d) Show that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

2. Let
$$\boldsymbol{A} = \begin{pmatrix} b & a & a \\ a & b & a \\ a & a & b \end{pmatrix}.$$

   Find a matrix $\boldsymbol{P}$ that orthogonally diagonalize $\boldsymbol{A}$ and determine $\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P}$.

   (a) Show that $\boldsymbol{A}$ has eigenvalues $b - a$ and $2a + b$.

   (b) Find an orthogonal basis of the eigenspace E_{b-a}.

   (c) Find an orthogonal basis of the eigenspace E_{2a+b}.

   (d) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalize $\boldsymbol{A}$. and determine $\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P}$.

3. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

   (a) Compute $\boldsymbol{A}^4$.

   (b) Find a matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^2 = \boldsymbol{A}$.

4. A square matrix $(a_{ij})_{n \times n}$ is called a stochastic matrix if all the entries are non-negative and the sum of entries of each column is 1, i.e. $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

   Let $\boldsymbol{A}$ be a stochastic matrix.

   (a) Show that $(1, 1, \ldots, 1)^T$ is an eigenvector of $\boldsymbol{A}^T$.

   (b) Show that 1 is an eigenvalue of $\boldsymbol{A}$.

   (c) Show that $\boldsymbol{A}^k$ for $k \geq 0$ is a stochastic matrix.

(d) Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$ be an eigenvector of $\boldsymbol{A}^T$ associated an eigenvalue λ and denote

$$u_{\max} = \max\{u_1, u_2, \ldots, u_n\} \text{ and } u_{\min} = \max\{u_1, u_2, \ldots, u_n\},$$

that is, $u_{\max}$ and $u_{\min}$ are the maximum and minimum elements in the set $\{u_1, u_2, \ldots, u_n\}$, respectively.

Show that $\lambda u_j \leq u_{\max}$ and $u_{\min} \leq \lambda u_j$ for all $1 \leq j \leq n$.

(e) If λ is an eigenvalue of $\boldsymbol{A}$, then $\lambda \leq 1$.

(f) If λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq 1$. (Hint: Apply Question (4e) to $\boldsymbol{A}^2$.)

5. (This is an induction step for proving Remark 6.2.5.3.) Let $\boldsymbol{A}$ be a square matrix of order n. By Theorem 6.2.3, to diagonalize $\boldsymbol{A}$, we need to find n linearly independent eigenvectors.

Suppose we already have $m(< n)$ linearly independent eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, say, $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ for $i = 1, 2, \ldots, m$ where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are not necessarily distinct. For a new eigenvalue μ ($\mu \neq \lambda_i$ for $i = 1, 2, \ldots, m$) of $\boldsymbol{A}$, let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ be a basis for the eigenspace E_μ. Prove that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ is linearly independent.

(Hint: Consider the vector equation

$$a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \cdots + a_m\boldsymbol{u}_m + b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + \cdots + b_p\boldsymbol{v}_p = \mathbf{0}.$$

By using the property of eigenvectors, show that

$$a_1(\lambda_1 - \mu)\boldsymbol{u}_1 + a_2(\lambda_2 - \mu)\boldsymbol{u}_2 + \cdots + a_m(\lambda_m - \mu)\boldsymbol{u}_m = \mathbf{0}.$$

Then make use of the linearly independent assumption on $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, as well as $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$, to finish the proof.)