Review 3.4 - 3.6

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Recall...

Let $S = \{u_1, \cdots, u_k\}$, then S is called linear independent if the equation

$$c_1 u_1 + \cdots + c_k u_k = 0$$

which is a homogeneous linear system, has only trivial solution. Otherwise, S is called linear dependent.

Theorem (3.4.4)

S is linear independent if and only if no vector in S can be written as a linear combination of other vectors in S.

Theorem (3.4.10)

If S is linear independent, and another vector u_{k+1} is not a linear combination of vectors in S, then $S \cup \{u_{k+1}\}$ is linear independent.

How to remove a redundant vector in S?

1. Requirement: S must be linear dependent. Based on the Theorem 3.4.4, S is linear dependent if and only if at least **one** vector u_i can be written as a linear combination of other vectors in S.
2. A simple example is that: when $k > n$, then S must be linear dependent. (see Theorem 3.4.7)
3. Solve the following homogeneous linear system:

   $$c_1 u_1 + \cdots + c_k u_k = 0, \quad \text{(always have non-trivial solutions, why?)}$$

   then find a number c_i such that $c_i \neq 0$, we have

   $$u_i = -\frac{1}{c_i} \sum_{j \neq i} c_j u_j.$$

4. So u_i is redundant, in this case (see Q5 in Tutorial 5),

   $$span(S) = span(S - \{u_i\}).$$

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Vector Space and Subspace

1. V is called a *vector space*, if either, (1) $V = \mathbb{R}^n$ or, (2) V is a subspace of $\mathbb{R}^n$ for some positive integer n. In this sense, the "whole space" and "subspace" are all vector spaces.
2. Let W be a vector space, then V is called a subspace of W if V is a vector space contained in W.

Bases

Definition (Plural Bases)

Let $S = \{u_1, \cdots, u_k\}$ be a subset of a vector space V. Then S is called a basis for V, if

1. S is linear independent and
2. S spans V.

Remarks:

1. *A* basis for a vector space V contains the smallest possible number of vectors that can span V.
2. The empty set $\emptyset$ is the basis for the zeros space.
3. Except the zero space, any vector space has infinitely many different bases.

Coordinates relative to the basis S.

Let $S = \{u_1, \cdots, u_k\}$ be a basis for a vector space V, then

Theorem (3.5.7)

Every vector $v \in V$ can be expressed in the form

$$v = c_1 u_1 + \cdots + c_k u_k$$

in exactly one way, where $c_1, \cdots, c_k$ are real numbers.

Definition (Coordinates)

The coefficients $c_1, \cdots, c_k$ are called the coordinates of v relative to the basis S. The vector

$$(v)_S = (c_1, c_2, \cdots, c_k) \in \mathbb{R}^k$$

is called the coordinates vectors of v relative to the basis S.

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Dimension

By the remark 1 in Page 7 in this slide, we can see that all bases for a vector space have the same number of vectors. So we can define the following:

Definition (Dimensions)

The dimensions of a vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the dimension of the zero space to be zero.

A Theorem that we must know.

Let V be a vector space, we want to find a set of vector $S \subset V$ such that $V = span(S)$, and S is a basis for V.

Theorem (3.6.7)

The following are equivalent:

1. *S is a basis for V.*
2. *S is linearly independent and $|S| = \dim(V)$.*
3. *S spans V and $|S| = \dim(V)$.*

How to find a basis T for $V = span(S)$?

Suppose $S = \{u_1, \cdots, u_k\}, u_i = (a_{i1}, \cdots, a_{in}) \in \mathbb{R}^n, 1 \leq i \leq k$, and $V = span(S)$, then V is a vector space. Now, we want to find a basis T for V. We can do this by the following way:

1. First we check whether S is a basis for V. By Theorem 3.6.7 (3), if $k > n$, then S is impossible to be a basis for V (why?). (see Q2 (a))
2. Remove all the redundant vector in S, refer to Page 4 in this slide.
3. Then the set of vectors that remainedn (denoted by T) is a basis for V.

How to find a basis for the solution space of homogeneous system $Ax = 0$ (for non-trivial case)?

We can follow the way in Discussion 3.6.5.

1. Write down the augmented matrix for this homogeneous linear system.
2. Use Gaussian elimination or Gauss-Jordan elimination to solve this system.
3. Write down the general solution in the following form

$$x = t_1 u_1 + \cdots + t_k u_k, \quad t_i \in \mathbb{R}^n, u_i \text{ fixed vetor}, 1 \leq i \leq k.$$

4. Then the solution space is equal to $span\{u_1, \cdots, u_k\}$. Furthermore, $\{u_1, \cdots, u_k\}$ obtained in this way is always linear independent (Why?). So the dimension of the solution space is k.
5. Recalled that k is the same as the number of non-pivot columns in the row-echelon form of A.