Review 2.5 - 3.2

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Cofactor Expansion

Recalled that

$$det(A) = a_{11}A_{11} + \cdots + a_{1n}A_{1n}, \quad A_{ij} = (-1)^{i+j} det(M_{ij}).$$

A_{ij} is called the cofactor of a_{ij}, $det(M_{ij})$ is called the minor of a_{ij}.

Theorem

For an $n \times n$ matrix $A = (a_{ij})$,

$$det(A) = a_{i1}A_{i1} + \cdots + a_{in}A_{in}, \quad \textit{(Along the ith row).}$$

Or

$$det(A) = a_{1j}A_{1j} + \cdots + a_{nj}A_{nj}, \quad \textit{(Along the jth column).}$$

Mathematical Induction

- Goal: To prove that a property $P(n)$ holds for every natrual number n. (Sometime can be extended to all integers.)
- Steps.
  1. The Initial Step: Prove that the statment holds for the first natural number n. (eg. $n = 0$ or $n = 1$ or larger one.)
  2. The Induction Step: Assume that the statement holds for some natrual number k, and prove that then the statement hold for $k + 1$,
  3. The Conclusion Step: Hence, $P(n)$ is true for every $n \geq 0$ (or $n \geq 1$).

A example for illustrating mathematical induction

Theorem

If A is a square matrix, then $det(A) = det(A^T)$.

Proof.

First we restate the statement. For all $n \geq 1$, and A is a square matrix of order n, we have that $det(A) = det(A^T)$ holds.

1. **The initial step**. *The first number is 1. So if A is 1×1 matrix, then A is a number, so $A = A^T$. Therefore, $det(A) = A = A^T = det(A^T)$.*
2. **The induction step**. *Assume that the statement hold for $n = k$, i.e., $det(A) = det(A^T)$ holds for any $k \times k$ matrix. Now for $n = k + 1$, which means that A is $(k+1) \times (k+1)$ matrix. Then we expand along the first row of A to get $det(A)$ and expand the first column of A^T to get $det(A^T)$. And find that $det(A) = det(A^T)$ by assumption for $n = k$. So the statement holds for $n = k + 1$.*
3. **The conclusion step**. *The statement holds for all $n \geq 1$.*

$\square$

Determinants of Elementary Matrices

$$E_i(k) = \begin{bmatrix} \mathbb{I}_{i-1} & & \\ & {\color{red}k} & \\ & & \mathbb{I}_{n-i} \end{bmatrix}, E_{ij} = \begin{bmatrix} \mathbb{I}_{i-1} & & & & \\ & 0 & & {\color{red}1} & \\ & & \ddots & & \\ & {\color{red}1} & & 0 & \\ & & & & \mathbb{I}_{n-j} \end{bmatrix},$$

$$E_{ij}(k) = \begin{bmatrix} \mathbb{I}_{i-1} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & {\color{red}k} & & 1 & \\ & & & & \mathbb{I}_{n-j} \end{bmatrix}$$

1. (Multiply ith row by a constant k) ${\color{red}det(E_i(k)) = k}$.
2. (Interchange ith row and jth row) ${\color{red}det(E_{ij}) = -1}$.
3. (Add a multiple of ith row to jth row) ${\color{red}det(E_{ij}(k)) = 1}$.

The most important theorem of determinant

Theorem

Let A be a square matrix and E be a elementary matrix of the same size of A. Then

$$det(EA) = det(E)det(A).$$

1. If B obtained from A by multiplying ith row of A by a constant k, then $det(B) = det(E_i(k)A) = det(E_i(k))det(A) = kdet(A)$.
2. If B obtained from A by interchanges ith and jth rows of A, then $det(B) = det(E_{ij}A) = det(E_{ij})det(A) = -det(A)$.
3. If B obtained from A by adding k times of ith row of A to jth row, then $det(B) = det(E_{ij}(k)A) = det(E_{ij}(k))det(A) = det(A)$.
4. If the ith row and jth row of A are equal, then $det(A) = det(E_{ij}A) = -det(A)$. So $det(A) = 0$.
5. $det(AB) = det(A)det(B)$.

Determinants of Row Equivalent Matrices

If A and B are row equivalent, i.e., B be obtained from A by perform a sequence of elementary row operations. Which meams there exist a sequence of elementary matrices $E_1, \cdots, E_k$ such that

$$B = E_k \cdots E_1 A.$$

Hence,

$$det(B) = det(E_k) \cdots det(E_1) det(A).$$

Consequently:

1. $det(A) = 0$ if and only if $det(B) = 0$.
2. A is invertible (I and A are row equivalent) if and only if $det(A) \neq 0$.
3. See Q1, Q2 and Q4.

The Adjoint of Matrix

Let $A = (a_{ij})_{n\times n}$, we consider $b_{ij} = a_{i1}A_{j1} + \cdots + a_{in}A_{jn}$.

1. If $i = j$, then by the definition of determinant, $b_{ij} = det(A)$.
2. If $i \neq j$, suppose that $i < j$, consider

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

cofactor expanding along jth row of B, then we can find that $b_{ij} = det(B) = 0$ (since B has two rows identical).

The Adjoint of Matrix

Definition

Let A be a square matrix of order n. Then the *adjoint* of A is the $n \times n$ matrix

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^{T} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Theorem

Let A be a square matrix, if A is invertible. Then

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Cramer's Rule

Let A is of order n, then if A is invertible, the unique solution of linear system $Ax = b$ can be written as

$$x = A^{-1}b = \frac{1}{det(A)} adj(A)b.$$

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then we have

$$x_i = \frac{b_1 A_{i1} + \cdots + b_n A_{ni}}{det(A)} = \frac{det(A_i)}{det(A)}.$$

Where A_i is the matrix obtained from A by replacing the ith column of A by b. See Q3 (a), (b) and (c).

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Linear Combinations and Linear Spans

Definition

Let $u_1, u_2, \cdots, u_k$ be vectors in $\mathbb{R}^n$. For any real numbers $c_1, c_2, \cdots, c_k$, the vector

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

is called a *linear combination* of $u_1, u_2, \cdots, u_k$.

Definition

Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of vectors in $\mathbb{R}^n$. Then the set of all linear combinations of $u_1, u_2, \cdots, u_k$,

$$\{c_1 u_1 + c_2 u_2 + \cdots + c_k u_k | c_1, c_2, \cdots, c_k \in \mathbb{R}\},$$

is called the *linear span* of S, denoted by $span(S)$ ($span\{u_1, u_2, \cdots, u_k\}$).

Determine if $x \in span(S)$.

Let $S = \{u_1, u_2, \cdots, u_k\}$, and $u_i = (u_{i1}, u_{i2}, \cdots, u_{in})$.

1. Write all the vector $u_1, u_2, \cdots, u_k$ in $\mathbb{R}^n$ in the column form. Denote by $u_i^T, 1 \leq i \leq k$.
2. Then find $c_1, \cdots, c_k$ by solving the following linear system

$$c_1 u_1^T + c_2 u_2^T + \cdots + c_k u_k^T = x^T.$$

3. Write the linear system in (2) in the matrix form,

$$\begin{bmatrix} u_{11} & u_{21} & \cdots & u_{k1} \\ u_{12} & u_{22} & \cdots & u_{k2} \\ \vdots & \vdots & & \vdots \\ u_{1n} & u_{2n} & \cdots & u_{kn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

4. If the linear system is not consistent, then $x \notin span(S)$. Otherwise, $c_1, \cdots, c_k$ is a solution to the above linear system.