CHAPTER 1

SECTION 1.1 PROPOSITIONAL LOGIC

PROPOSITIONS

DEFINITION:

A **PROPOSITION** is a declarative sentence that is true or false.

This quality of being true (T) or false (F) is called the **TRUTH VALUE** of the proposition.

REMARK

We use letters, such as $p, q, r \ldots$ to denote propositional variables. The area of logic that deals with propositions is called propositional calculus or propositional logic. It was first developed by the Greek philosopher Aristotle more than 2300 years ago.

True propositions.

- $1 + 1 = 2$.

False propositions.

- $4 + 5 = 6$.

Non-propositions

Non-propositions include questions, commands, exclamations, sentences with undefined or ambiguous words:

- How is the lecture?
- Stop talking!
- $x + y > 0$.

(Later we'll learn that this is a predicate, and it becomes a proposition when the values of x and y are known.)

We now turn our attention to methods of producing new propositions from those we already have.

DEFINITION:

The **NEGATION** of a proposition p is written $\neg p$. It denotes the statement "not p" or "it is not the case p". The propositions p and $\neg p$ have opposite truth values.

p	$\neg p$
T	F
F	T

DEFINITION:

The **CONJUNCTION** of two propositions p and q is written $p \wedge q$. It denotes the proposition "p and q". It is true when both p and q are true; it is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

DEFINITION:

The **DISJUNCTION** of two propositions p and q is written $p \vee q$. It denotes the proposition "p or q". It is false when both p and q are false; it is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE

Let p be "it is hot" and q be "it is sunny."

- $\neg p$ is the proposition "It is not hot."
- The conjunction $\neg p \wedge q$ is "It is not hot but it is sunny" or " It is sunny and is not hot."

This proposition is true only when both $\neg p$ and q are true, i.e., when it is both "not hot" and "sunny".

- Express "it is neither hot nor sunny" as a logical expression.

SOLN: $\neg p \wedge \neg q$.

EXAMPLE

Let x be some known real number. Let p be "$x > 0$", q be "$x < 3$", and r be "$x = 3$".

- $q \vee r$: $x < 3$ or $x = 3$, i.e., $x \leq 3$.

The proposition is true when $x = 3$. It is also true when $x = 2$.

- $p \wedge (q \vee r)$: $x > 0$ and $x \leq 3$, i.e., $0 < x \leq 3$

CONDITIONAL PROPOSITIONS

EXAMPLE

- The government promises "taxes will be lowered if the growth is $\geq 3\%$".

Let p be "growth is $\geq 3\%$" and q be "taxes will be lowered". Then promise is if p then q or $p \to q$. We have the following scenario.

Growth	Taxes	Promise
$\geq 3\%$	L	Kept
$\geq 3\%$	$\neg$L	Not kept
$< 3\%$	L	Not broken (very happy)
$< 3\%$	$\neg$L	Not broken

Thus we see that the promise is deemed to be broken only when the growth is $\geq 3\%$ but the taxes are not lowered.

This is leads to the following definition.

DEFINITION:

Let p and q be propositions. The **CONDITIONAL PROPOSITION**

$$p \to q$$

is false when p is true and q is false; and is true otherwise.

Here p is called the **HYPOTHESIS** or **PREMISE** and q is called the **CONCLUSION** or **CONSEQUENCE**.

The truth table for $p \to q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

REMARK

$p \to q$ is "if p then q".

DEFINITION: Given $p \to q$

- the **CONTRAPOSITIVE** form is $\neg q \to \neg p$;
- the **CONVERSE** form is $q \to p$;
- the **INVERSE** form is $\neg p \to \neg q$.

DEFINITION:

Two compound propositions p and q are **EQUIVALENT** and we write $p \equiv q$ if they always have the same truth value.

EXAMPLE

Show that $(p \vee \neg q) \to (p \wedge q) \equiv q$.

We denote the LHS by R.

p	q	$p \vee \neg q$	$p \wedge q$	R
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	T	F	F

From the table we see R and q have identical columns, i.e., they same the same truth values. Therefore they are equivalent.

THEOREM:

(i) $p \to q \equiv \neg q \to \neg p$, (equivalent to contrapositive.)

(ii) $q \to p \equiv \neg p \to \neg q$, (converse equivalent to inverse.)

(iii) $p \to q \not\equiv q \to p$ (not equivalent to converse.)

Proof

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

EXAMPLE

(1) If the units digit of an integer n is 0, then n is a multiple of 2.

(2) Contrapositive: If an integer n is **NOT** a multiple of 2, then its units digit is **NOT** 0.

(3) Converse: If an integer n is a multiple of 2, then its units digit is 0.

(4) Inverse: If the units digit of an integer n is **NOT** 0, then it is **NOT** a multiple of 2.

If $n = 10$, then all are true. If $n = 12$, then (1) and (2) are true but (3), (4) are false. If $n = 1$, then all are true.

THEOREM: $p \to q \equiv \neg p \vee q$.

PROOF: The proof follows from the following truth table as the columns corresponding to the two propositions are identical.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

DEFINITION:

Let p and q be propositions. The **BICONDITIONAL PROPOSITION**

$$p \leftrightarrow q$$

is true when p and q have the same truth values and is false otherwise.

The truth table for $p \leftrightarrow q$ is

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

"IF", "ONLY IF "

"IF"

When we say "if p then q" it means that whenever p is true, it would follow that q is true. Thus it is the same as p implies q. Hence logically, it is the same as $p \to q$.

We can also say that p is a sufficient condition for q.

Thus $p \to q$ is the same as

- if p then q
- p implies q
- p is sufficient for q.

"only if"

When we say

"You can have ice cream only if you finish dinner"

it means that

"if you don't finish you dinner, then you can't have ice cream."

Thus p only if q means

$$\neg q \to \neg p.$$

But $\neg q \to \neg p$ is equivalent to $p \to q$. Thus p only if q is

$$p \to q.$$

We also see that for p to be true, it is necessary that q be true. Thus we say that q is a necessary condition for p to be true (because if q is false, then p is false as well).

In summary, p only if q is

- $p \to q$
- $\neg q \to \neg p$
- q is a necessary condition for p.

if and only if

If and only if is often abbreviated to iff. Thus p iff q is $q \to p$ (the "if" part) and $p \to q$ (the "only if" part). Note that

$$(p \to q) \wedge (q \to p)$$

is true when both p and q have the same truth values (verify it yourself using a truth table). Thus p iff q is $p \leftrightarrow q$. Here we say that p is both necessary and sufficient for q. Logically $p \leftrightarrow q$ is true when $p \equiv q$.

LOGICAL OPERATOR PRECEDENCE

The order of precedence, listed in decreasing order from top to bottom, is

$$\neg$$
$$\wedge, \quad \vee$$
$$\rightarrow, \quad \leftrightarrow$$

- $\wedge$ and $\vee$ are of the same order of precedence, as are $\rightarrow$ and $\leftrightarrow$
- Parentheses should be used to resolve ambiguities.
- Note that $\neg p \vee q$ means $(\neg p) \vee q$ and not the negation of $p \vee q$.
- You can't write $p \vee q \wedge r$, instead, you should write either $(p \vee q) \wedge r$ or $p \vee (q \wedge r)$.

TRANSLATING ENGLISH SENTENCES

EXAMPLE

Translate the following into a logical expression.

"You can access the internet from campus only if you are a computer science major or you are not a freshman."

SOLN: Let a, c and f represent "you can access the Internet from campus", "you are a computer science major" and "you are a freshman", respectively. Then the answer is $a \rightarrow (c \vee \neg f)$.

• Translate the following into logical expressions. Find an instance in which all are simultaneously true.

-The diagnostic message is stored in the buffer or it is retransmitted.

-The diagnostic message is not stored in the buffer.

-If the diagnostic message is stored in the buffer then it is retransmitted.

SOLN: Let s and r denote the diagnostic message is stored in the buffer and retransmitted, respectively. Then the three specifications are

$$s \vee r, \qquad \neg s, \qquad s \to r.$$

We see that the above are all true if s is false and r is true. (We can also use the truth table.)

SECTION 1.2 TAUTOLOGY and CONTRADICTION

DEFINITION:

A **TAUTOLOGY** is a compound proposition that is always true.

EXAMPLE

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

DEFINITION:

A **CONTRADICTION** is a compound proposition that is always false.

EXAMPLE

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

DEFINITION:

A **CONTINGENCY** is a compound proposition that is neither a tautology nor a contradiction.

EXAMPLE

$p \wedge q$ is a contingency.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The theorem below are some basic logical equivalences which are useful

THEOREM

Given any propositional variables p, q, and r, a tautology $\mathbf{T}$ and a contradiction $\mathbf{C}$, the following logical equivalences hold:

- Commutative laws:
$$p \wedge q \equiv q \wedge p, \qquad p \vee q \equiv q \vee p.$$
- Associative laws:
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r), \qquad (p \vee q) \vee r \equiv p \vee (q \vee r).$$
- Distributive laws:
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
- Identity laws:
$$p \wedge \mathbf{T} \equiv p, \qquad p \vee \mathbf{C} \equiv p.$$
- Negation laws:
$$p \wedge \neg p = \mathbf{C} \qquad p \vee \neg p = \mathbf{T}.$$
- Double negation laws:
$$\neg(\neg p) \equiv p.$$
- Idempotent laws:
$$p \wedge p \equiv p, \qquad p \vee p \equiv p.$$
- Universal bound laws:
$$p \wedge \mathbf{C} \equiv \mathbf{C}, \qquad p \vee \mathbf{T} \equiv \mathbf{T}.$$
- De Morgan's laws:
$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \neg(p \vee q) \equiv \neg p \wedge \neg q.$$
- Absorption laws:
$$p \wedge (p \vee q) \equiv p, \qquad p \vee (p \wedge q) \equiv p.$$
- Negations of $\mathbf{T}$ and $\mathbf{C}$:
$$\neg\mathbf{C} \equiv \mathbf{T}, \qquad \neg\mathbf{T} \equiv \mathbf{C}.$$

EXAMPLE

- The distributive law: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

SOLN:

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	L	R
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

- $\neg(p \to q) \equiv p \wedge \neg q$.

SOLN: Using the theorem:

$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q \equiv p \wedge \neg q$.

Alternative solution:

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

- What is the negation of $-1 < x \leq 4$?

SOLN: Let p be "$-1 < x$", then $\neg p$ is "$-1 \geq x$". Let q be "$x \leq 4$", then $\neg q$ is "$x > 4$". The given statement is equivalent to $p \wedge q$. Thus

$$\neg(p \wedge q) = \neg p \vee \neg q \quad \text{i.e.,} \quad -1 \geq x \quad \text{or} \quad x > 4.$$

- Show that $\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p$.

SOLN:

$$\begin{aligned} \neg(\neg p \wedge q) \wedge (p \vee q) &\equiv (\neg(\neg p) \vee \neg q) \wedge (p \vee q) \\ &\equiv (p \vee \neg q) \wedge (p \vee q) \\ &\equiv p \vee (\neg q \wedge q) \equiv p \vee \mathbf{C} \equiv p \end{aligned}$$

Note: Can use truth table as well.

SECTION 1.3 PREDICATES & QUANTIFIERS

DEFINITION:

A **PREDICATE** is a sentence that contains a finite number of variables and becomes a proposition when the variables are specialized. The **DOMAIN** of a predicate variable is the set of (all) values that may be substituted in place of the variable.

EXAMPLE

• Let $P(x)$ be the predicate "x is student at NUS" with domain the residents of Clementi. When you replace x by John Tan who is a resident of Clementi, it becomes a proposition.

• Let $Q(x)$ be the predicate "$x^2 > x$" with domain $\mathbb{R}$. Then $Q(2)$ is T, $Q(.5)$ is F.

• Let $R(x, y, z)$ be the predicate $x + y > z$ with domain $\mathbb{R}$. Then $R(1, 2, 3)$ is F and $R(0, 1, -1)$ is T.

UNIVERSAL & EXISTENTIAL QUANTIFIERS

DEFINITION:

The **UNIVERSAL QUANTIFICATION** of $P(x)$ with domain D is:

$$\forall\, x \in D,\ P(x).$$

It is **TRUE** if $P(x)$ is true for all $x \in D$.

It is **FALSE** if $P(x)$ is false for at least one x in D.

An x for which $P(x)$ is false is called a **COUNTER EXAMPLE** to the universal statement.

The symbol $\forall$(for all) is called the **UNIVERSAL QUANTIFIER**.

We often write "$\forall x P(x)$" when the domain is understood.

EXAMPLE

• "$\forall\, x \in \{1, 2, 3, 4, 5\},\ x^2 \geq x$" is true since

$$1^2 \geq 1, 2^2 \geq 2, 3^2 \geq 3, 4^2 \geq 4, 5^2 \geq 5.$$

• "$\forall\, x \in \mathbb{R},\ x^2 \geq x$" is false since $x = .5$ is a counter example.

REMARK

Some other ways of expressing the universal quantifier are: "for every", "all of", "for each", "given any", "for arbitrary", "for each".

• "Every integer is also rational" is an universal quantification since it can be rewritten as:

$$\forall x \in \mathbb{Z}, x \in \mathbb{Q}.$$

DEFINITION:

The **EXISTENTIAL QUANTIFICATION** of $P(x)$ with domain D is:

$$\exists\, x \in D,\ P(x).$$

It is **TRUE** if there is at least one $x \in D$ such that $P(x)$ is true.

It is **FALSE** if $P(x)$ is false for all $x \in D$.

The symbol $\exists$(there exists) is called the **EXISTENTIAL QUANTIFIER**.

We often write "$\exists x P(x)$" when the domain is understood.

EXAMPLE

• "$\exists\, x \in \mathbb{Z}$ such that $x^2 = x$" is true because $0^2 = 0$.

• "$\exists\, x \in \{-1, 2, 3\}$ such that $x^2 = x$" is false because there is no such x in the domain:

• "The integer 24 can be written as a sum of two integers" is an existential quantification since it can be written as

$$\exists m, n \in \mathbb{Z}(24 = m + n).$$

TRANSLATING FROM ENGLISH

Translating English into logical expressions is crucial in mathematics and computer science. Here are some examples.

EXAMPLE

• "Every student in this class has studied calculus."

SOLN: What we need is a logical expression which is true when every student in the class has studied calculus and is false when there is one student in the class who has not studied calculus.

Let $P(x)$ be "x has studied calculus", $Q(x)$ be "x is in the class", C be the set of students in the class and U be the set of students in the university. If you take the domain as C, the answer is

(i) $\forall x \in C, P(x)$; or

If you take the domain as U, the answer is

(ii) $\forall x \in U, Q(x) \to P(x)$".

It is clear that (i) is a correct answer.

That (ii) is also correct is seen as follows. Consider the case where every student in the class has studied calculus:

If $x \notin C$, $Q(x)$ is false and thus $Q(x) \to P(x)$ is true. If $x \in C$, then $Q(x)$ and $P(x)$ are both true. Thus $Q(x) \to P(x)$ is also true. Hence the universal quantification is true.

Consider the case where there is one student, say John, who has not studied calculus:

John is a counter example. Thus the universal quantification is false.

REMARK

It's common mistake to give the answer as $\forall x \in U, Q(x) \wedge P(x)$. Note that $Q(x) \wedge P(x)$ is false if $x \notin C$. Thus every student who is not a student in C serves as a counter example. Hence, as long as there is a student who is not in$C$, the quantification is false, even if every student in C has studied calculus.

- "Some students in this class has not studied calculus."

SOLN: "$\exists x \in C, \neg P(x)$" or "$\exists x \in U, Q(x) \wedge \neg P(x)$".

- "All mathematicians wear glasses."

SOLN: Let $G(x)$ be "x wears glasses", $P(x)$ be "x is a mathematician", M be the set of mathematicians, H be the set of all human beings. Then the answers is either

(i) $\forall x \in M, G(x)$; or

(ii) $\forall x \in H, P(x) \to G(x)$.

- "No mathematician wears glasses."

SOLN: $\forall x \in M, \neg G(x)$ or $\forall x \in H, P(x) \to \neg G(x)$.

- "At least one mathematician does not wear glasses."

SOLN: $\exists x \in M, \neg G(x)$ or $\exists x \in H, P(x) \wedge \neg G(x)$.

- Every even integer is divisible by 2

SOLN: $\forall n \in \mathbb{Z}, E(n) \to D(n)$ where $E(n)$ is "n is even" and $D(n)$ is "n is divisible by 2".

- Every multiple of 5 ends in either 0 or 5.

SOLN: $\forall n \in \mathbb{Z}, M(n) \to Z(n) \vee F(n)$ where $M(n)$ is "n is a multiple of 5", $Z(n)$ is "n is ends in 0" and $F(n)$ is "n ends in 5".

NEGATING QUANTIFIED EXPRESSIONS

Consider:

(1): "All mathematicians wear glasses."

(2): "No mathematicians wear glasses."

(3): "At least one mathematician does not wear glasses."

	(1)	(2)	(3)
All wear	T	F	F
Some wear, some don't	F	F	T
None wear	F	T	T

There are three different cases and in all cases, (1) and (3) have opposite true values. Thus they are the negations of each other.

This example shows that, with the same domain for both quantification,

$$\boxed{\neg\forall\, xP(x) \equiv \exists\, x(\neg P(x))}$$

Since for every p, $\neg(\neg p) = p$, we have $\forall\, xP(x) \equiv \neg(\neg\forall\, xP(x)) \equiv \neg\exists\, x\neg P(x)$. Thus, replacing $P(x)$ by its negation,

$$\boxed{\neg\exists\, xP(x) \equiv \forall\, x\neg P(x)}$$

EXAMPLE

- Consider: "for all primes p, p is odd". Its logical expression is

$$\forall p, O(p)$$

where the domain is the set of all primes and $O(p)$ is "p is odd". Its negation is

$$\exists p, \neg O(p) \quad \text{or} \quad \exists p, E(p)$$

where $E(p)$ is "p is even".

So the negation is "there is a prime p, such that p is not odd" or "there is a prime p, such that p is even"

- The negation of "There exists a triangle such that the sum of its angles is 200°" is

  "For all triangles T, the sum of the angles of T is not 200°."

- Show that $\neg(\forall x P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$.

SOLN: Note that $P(x) \to Q(x) \equiv \neg P(x) \vee Q(x)$. Thus

$$\neg(P(x) \to Q(x)) \equiv \neg(\neg P(x) \vee Q(x)) \equiv P(x) \wedge \neg Q(x).$$

Therefore

$$\neg(\forall x P(x) \to Q(x)) \equiv \exists x(\neg(P(x) \to Q(x))) \equiv \exists x(P(x) \wedge \neg Q(x)).$$

- Find the negation of "Every person who is blond has blue eyes."

SOLN: Let $P(x)$ be "x is blond" and $Q(x)$ be "x has blue eyes". Then the given expression is "$\forall x(P(x) \to Q(x))$". Thus its negation is $\exists x(P(x) \wedge \neg Q(x))$. Hence the required negation is"There is a person who is blond but does not have blue eyes."

EXAMPLE

Suppose you have 5 blue balls, 5 red balls and a bowl. Consider the proposition "All balls in the bowl are blue".

- If you put 2 B and 1 R balls in the bowl, the proposition is false.
- If the bowl is empty, what happens?

SOLN: This is quite hard to decide by intuition. Thus we turn to logic. Let C be the set of all balls, B be the set of balls in bowl, $P(x)$ be " ball x is blue" and $Q(x)$ be "ball x is in the bowl". The given proposition translates into "$\forall x \in B, P(x)$". Its negation is $\exists x \in B, \neg P(x)$ which says there is at least one ball in the bowl and this is not true. Therefore the original proposition is true.

Alternatively, the proposition translates into $\forall x \in C, Q(x) \to P(x)$. Since $Q(x)$ is false for every x, $Q(x) \to P(x)$ is true for every x. Thus the proposition is true.

SECTION 1.4 NESTED QUANTIFIERS

In this section we'll study **NESTED QUANTIFIERS**. Two quantifiers are nested if one is within the scope of the other, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example, the example is the same as $\forall x Q(x)$ where $Q(x)$ is $\exists y (x + y = 0)$.

REMARK

When the domain for x and y are different, say D and E, respectively, we write

$$\forall x \in D \exists y \in E (x + y = 0).$$

EXAMPLE

- $\forall x \in \mathbb{R} \exists y \in \mathbb{R} (x + y = 0)$

This means for every real number x, there is a real number y such that $x + y = 0$. This is true since we can take y to be $-x$.

- $\forall x \in \mathbb{R} \forall y \in \mathbb{R} (x + y = y + x)$.

This means that very real number x and every real number y satisfy $x + y = y + x$ and is a true proposition.

- $\forall x \in \mathbb{R} \exists y \in \mathbb{R} (xy = 1)$.

This is false since when $x = 0$, y doesn't exist.

ORDER OF QUANTIFIERS

When an expression contains more than one quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur. Reversing the order of the quantifiers can change the truth value of a statement unless the quantifiers are all universal or all existential.

For example, "everyone loves someone" is translated into:

$$\forall x \exists y P(x, y) \quad \text{where } P(x, y) \text{ is "} x \text{ loves } y.\text{"}$$

However

$$\exists y \forall x P(x, y)$$

is "there is someone who is loved by everyone". These two are clearly different. For example if the domain consists of 3 persons a, b, c and a loves b, b loves c and c loves a. The the former is true but the latter is false.

We summarize the quantification of 2 variables as follows

• $\forall x \forall y P(x, y)$, $\forall y \forall x P(x, y)$.

True when $P(x, y)$ is true for every pair x, y.

False when there is a pair x, y for which $P(x, y)$ is false.

• $\forall x \exists y P(x, y)$.

True when for every x, there is a y for which $P(x, y)$ is true.

False when there is a x for which $P(x, y)$ is false for every y.

• $\exists x \forall y P(x, y)$.

True when there is a x for which $P(x, y)$ is true for every y.

False when for every x there is a y for which $P(x, y)$ is false.

• $\exists x \exists y P(x, y)$, $\exists y \exists x P(x, y)$

True when there is a pair x, y for which $P(x, y)$ is true.

False when $P(x, y)$ is false for every pair x, y.

EXAMPLE

• For any even number x, there is an integer y such that $2y = x$. ($\forall x \in D \exists y \in \mathbb{Z}(2y = x)$, where D is the set of even integers). This is true since we can take $y = x/2$ which is an integer.

• For any person x, there is a person y such that y is the father of x. ($\forall x \exists y(P(x, y))$ where $P(x, y)$ is y is the father of x.")

• $\forall x \in \mathbb{R} \exists y \in \mathbb{Z}(x < y)$. This is true.

• $\exists y \in \mathbb{R}^+ \forall x \in \mathbb{R}^+ (y \leq x)$.

Translating into English, this means there is a positive real number y which is less than every positive real number. This is false because y is not $\leq y/2$.

NEGATION OF MULTIPLE QUANTIFIED STATEMENTS

Negation of $\forall$ $\exists$

$$\begin{aligned}\neg(\forall x \exists y P(x,y)) &\equiv \exists x \neg(\exists y P(x,y)) \\ &\equiv \exists x \forall y \neg P(x,y)\end{aligned}$$

Negation of $\exists$ $\forall$

$$\begin{aligned}\neg(\exists x \forall y P(x,y)) &\equiv \forall x \neg(\forall y P(x,y)) \\ &\equiv \forall x \exists y \neg P(x,y)\end{aligned}$$

EXAMPLE

• Negate $\forall x \exists y (xy = 1)$.

SOLN: $\exists x \forall y (xy \neq 1)$.

• Use quantifiers to express "There does not exist a woman who has taken a flight on every airline in the world."

SOLN: The negation is "There is a woman who has taken a flight on every airline in the world." Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a." The logical expression is

$$\exists w \forall a \exists f (P(w,f) \wedge Q(f,a)).$$

Its negation, which is what we want, is

$$\begin{aligned}\neg(\exists w \forall a \exists f (P(w,f) \wedge Q(f,a))) &\equiv \forall w \exists a \neg(\exists f (P(w,f) \wedge Q(f,a))) \\ &\equiv \forall w \exists a \forall f \neg(P(w,f) \wedge Q(f,a))) \\ &\equiv \forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a)))\end{aligned}$$

Translating into English: "For every woman there is an airline such that for all flights, either this woman has not taken that flight or that flight is not on this airline."

SECTION 1.5 VALID AND INVALID ARGUMENTS

DEFINITION:

An **ARGUMENT FORM** is a sequence of propositions

$$p_1, p_2, \ldots, p_n, c.$$

The propositions

$$p_1, p_2, \ldots, p_n$$

are called **PREMISES**, or **ASSUMPTIONS**, or **HYPOTHESES**. The last proposition, c is called the **CONCLUSION**.

An argument form is **VALID** when all premises are true, the conclusion is also true.

A valid argument form is also known as a rule of inference. To highlight the conclusion we precede it with the $\therefore$ symbol.

VALID ARGUMENT FORMS

TESTING FOR A VALID ARGUMENT FORM

1. Identify the premises and conclusion.

2. Construct a truth table.

3. If the truth table contains a row in which all the premises are true and the conclusion is false, the argument form is invalid. Otherwise the form is valid.

MODUS PONENS (Method of Affirming)

$$\begin{array}{rl} & p \to q \\ & p \\ \therefore & q \end{array}$$

p	q	$p \to q$	p	q	
T	T	T	T	T	*
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	
		premises		conclusion	

- This is a valid argument form.

MODUS TOLLEN (Method of Denying)

$$\begin{array}{rl} & p \to q \\ & \neg q \\ \therefore & \neg p \end{array}$$

p	q	$p \to q$	$\neg q$	$\neg p$	
T	T	T	F	F	
T	F	F	T	F	
F	T	T	F	T	
F	F	T	T	1	*
		premises		conclusion	

- This is a valid argument form.

EXAMPLE

- If there are more pigeons than there are pigeonholes, then two pigeons roost in the same hole.

There are 5 pigeons and 3 pigeonholes.

Therefore there are 2 pigeons in the same hole.

- If an integer is divisible by 6 then it is divisible by 3.

870232 is not divisible by 3.

Therefore 870232 is not divisible by 6.

SPECIALIZATION, GENERALIZATION

$$\begin{array}{rl} \textbf{SPECIALZATION}: & p \wedge q, \quad \therefore p \\ & p \wedge q, \quad \therefore q \\ \textbf{GENERALIZATION}: & p, \quad \therefore p \vee q \\ & q, \quad \therefore p \vee q \end{array}$$

ELIMINATION

$$\begin{array}{l} p \vee q, \quad \neg p, \quad \therefore q \\ p \vee q, \quad \neg q, \quad \therefore p \end{array}$$

$\boxed{\textbf{TRANSITIVITY}}$

$$p \to q, \quad q \to r, \quad \therefore p \to r.$$

$\boxed{\textbf{PROOF BY CASES}}$

$$p \vee q, \quad p \to r, \quad q \to r, \quad \therefore r$$

If at least one of two (or several) possibilities is true and each possibility leads to the same conclusion, then the conclusion must also be true.

$\boxed{\textbf{PROOF BY CONTRADICTION}}$

$$\neg p \to \mathbf{C} \quad \text{which is a contradiction} \quad , \quad \therefore p$$

If an assumption leads to a contradiction, then that assumption must be false.

EXAMPLE

Knights always tell the truth, knaves always lie.

Can we tell what A and B are, from what they say below?

A: B is a knight.

B: A and I are of opposite type.

- To analyze, let

$a =$ "A is a knight".

$b =$ "B is a knight".

The first fact translates to $a \to b$ and $\neg a \to \neg b$ or equivalently $\neg a \vee b$ and $a \vee \neg b$.

The second fact translates to $b \to \neg a$ and $\neg b \to \neg a$ or equivalently $\neg b \vee \neg a$ and $b \vee \neg a$.

There are many ways to proceed.

Solution 1.

1.	$a \to b$	
2.	$b \to \neg a$	
3.	$\therefore a \to \neg a$	From 1, 2 (transitivity)
4.	$\therefore \neg a \vee \neg a = \neg a$	From 3
5.	$\neg a \to \neg b$	
6.	$\therefore \neg b$	From 4,5 (modus ponens)

The above argument tells us that if there is a solution, it must be that both A and B are knaves. This can be checked to be consistent with the given facts. Thus this is the solution.

Solution 2.

1.	$\neg b \vee a$	
2.	$\neg b \vee \neg a$	
3.	$\therefore (\neg b \vee a) \wedge (\neg b \vee \neg a)$	
4.	$\therefore \neg b \vee (a \wedge \neg a) = \neg b \vee \mathbf{C} = \neg b$	From 3 (distributive law)
5.	$b \vee \neg a$	
6.	$\therefore \neg a$	From 5 (Elimination)

EXAMPLE

Will (a), (b), (c) and (d) lead to the conclusion (e)?

(a) It is cold and not sunny this afternoon.

(b) We will go swimming only if it is sunny,

(c) If we do not go swimming, then we will take a canoe trip,

(d) If we take a canoe trip, then we will be home by sunset,

(e) We will be home by sunset.

SOLN: Let

sun be "It is sunny this afternoon",

cold be "It is cold this afternoon",

swim be "We will go swimming",

canoe be "We will take a canoe trip", and

home be "We will be home by sunset".

The answer is yes by the following argument.

1.	$\neg$sun $\wedge$ cold	Hypothesis (a)
2.	$\neg$sun	From 1(specialization)
3.	swim $\rightarrow$ sun	Hypothesis (b)
4.	$\neg$swim	From 2, 3 (Modus tollens)
5.	$\neg$swim $\rightarrow$ canoe	Hypothesis (c)
6.	canoe	From 4, 5 (Modus ponens)
7.	canoe $\rightarrow$ home	Hypothesis (d)
8.	home	From 6, 7 (Modus ponens)