MA 1521

Tutorial 6 Solutions

1. $V = I \times R \implies I = \frac{V}{R}$.

(i) $\dfrac{\partial I}{\partial V} = \dfrac{1}{R}$.

If $R = 15$, then $\dfrac{\partial I}{\partial V} = \dfrac{1}{15} \approx 0.0667$ A/V.

(ii) $\dfrac{\partial I}{\partial R} = -\dfrac{V}{R^2}$. If $V = 120$ and $R = 20$, then $\dfrac{\partial I}{\partial R} = -\dfrac{120}{20^2} = -0.3$ A/Ω.

(iii) By Chain rule,

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt}.$$

Since $R = 400$, $I = 0.08$, $V = 32$, $\dfrac{dV}{dt} = -0.01$, $\dfrac{dR}{dt} = 0.03$, so

$$\frac{dI}{dt} = \frac{1}{400}(-0.01) - \frac{(0.08)(400)}{400^2}(0.03) = -3.1 \times 10^{-5}.$$

2. $f_x = e^{2y-x} + xe^{2y-x}(-1) = e^{2y-x}(1-x)$ and $f_y = 2xe^{2y-x}$.

So $f_x(-2,-1) = 3$ and $f_y(-2,-1) = -4$.

(i) $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$ is a unit vector. Thus,

$$D_{\mathbf{u}}f(-2,-1) = 3 \times \frac{1}{\sqrt{2}} - 4 \times \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

(ii) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{j}$ is $\mathbf{u} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$. Thus,

$$D_{\mathbf{u}}f(-2,-1) = 3 \times \frac{3}{5} - 4 \times \frac{4}{5} = -\frac{7}{5}.$$

Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ be a unit vector. Then

$$\begin{aligned} D_{\mathbf{u}}f(-2,-1) &= f_x(-2,-1) \times a + f_y(-2,-1) \times b \\ &= (f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}) \bullet (a\mathbf{i} + b\mathbf{j}) \\ &= \|f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}\| \, \|\mathbf{u}\| \cos\theta \quad (*) \end{aligned}$$

where θ is the angle between $f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}$ and $\mathbf{u}$.

Since the largest value of $\cos\theta$ is 1 which occurs when $\theta = 0$, this means that the largest possible value of $D_{\mathbf{u}}f(-2,-1)$ occurs when $\mathbf{u}$ is in the same direction as $f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$.

3. $f_x = yz\cos(xyz)$, $\quad f_y = xz\cos(xyz)\quad$ and $\quad f_z = xy\cos(xyz)$.

So $f_x(\frac{1}{2},\frac{1}{3},\pi) = \frac{\sqrt{3}}{6}\pi$, $\quad f_y(\frac{1}{2},\frac{1}{3},\pi) = \frac{\sqrt{3}}{4}\pi\quad$ and $\quad f_z(\frac{1}{2},\frac{1}{3},\pi) = \frac{\sqrt{3}}{12}$.

(i) Let $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

Thus, the rate of change of f at P in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f\left(\frac{1}{2},\frac{1}{3},\pi\right) = \frac{\sqrt{3}}{6}\pi\times\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}\pi\times\left(-\frac{1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{12}\times\frac{1}{\sqrt{3}} = \frac{1}{12}(1-\pi).$$

(ii) The change in the value of f:

$$df \approx D_{\mathbf{u}}f\left(\frac{1}{2},\frac{1}{3},\pi\right)\times dt = \frac{1}{12}(1-\pi)\times 0.1 \approx -0.01785.$$

So the value of f will *decrease* by about 0.01785 unit.

4. (i) Let $f(x,y) = \ln(x^2y) - xy - 2x = 2\ln x + \ln y - xy - 2x$, where $x > 0,\ y > 0$.

We have $f_x = \frac{2}{x} - y - 2$ and $f_y = \frac{1}{y} - x$. Also, $f_{xx} = -\frac{2}{x^2}$, $f_{xy} = -1$ and $f_{yy} = -\frac{1}{y^2}$. Then $f_y = 0$ implies that $x = \frac{1}{y}$, and substitution into $f_x = 0$ gives $2y - y - 2 = 0$, i.e. $y = 2$. So $x = 1/2$. Thus, the only critical point is $(1/2, 2)$.

Now $D(1/2,2) = (-8)(-1/4) - 1^2 = 1 > 0$, $\;f_{xx}(1/2,2) = -8 < 0$ implies that $f(1/2,2) = -\ln 2 - 2$ is a local maximum.

(ii) Let $g(x,y) = xy(1-x-y)$.

We have $g_x = y - 2xy - y^2$, $g_y = x - x^2 - 2xy$, $g_{xx} = -2y$, $g_{yy} = -2x$ and $g_{xy} = 1 - 2x - 2y$.

Then $g_x = 0$ implies $y = 0$ or $y = 1-2x$. Substituting $y = 0$ into $g_y = 0$ gives $x = 0$ or $x = 1$. Substituting $y = 1-2x$ into $g_y = 0$ gives $x = 0$ or $x = 1/3$.

Thus the critical points are $(0,0), (1,0), (0,1)$ and $(1/3,1/3)$.

Now $D(0,0) = D(1,0) = D(0,1) = -1$ while $D(1/3,1/3) = 1/3$ and $g_{xx}(1/3,1/3) = -2/3 < 0$.

Thus $(0,0), (1,0)$ and $(0,1)$ are saddle points, and $g(1/3,1/3) = 1/27$ is a local maximum.

(iii) Let $h(x,y) = x^2 + y^2 + x^{-2}y^{-2}$.

We have $h_x = 2x - 2x^{-3}y^{-2}$, $h_y = 2y - 2x^{-2}y^{-3}$, $h_{xx} = 2 + 6x^{-4}y^{-2}$, $h_{yy} = 2 + 6x^{-2}y^{-4}$ and $h_{xy} = 4x^{-3}y^{-3}$.

Then $h_x = 0$ implies that $2x^4y^2 - 2 = 0$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $h_y = 0$ implies that $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$, that is $x = \pm 1$. If $x = 1$, then $y = \pm 1$. If $x = -1$, then $y = \pm 1$.

So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

Now $D(\pm 1,\pm 1) = D(\pm 1,\mp 1) = 64 - 16 > 0$ and h_{xx} is always greater than zero. Thus $h(\pm 1,\pm 1) = h(\pm 1,\mp 1) = 3$ are local minima.