NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I
2018-2019 (Semester 1) **Tutorial 8**

1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of the same size. Show that
$$\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}).$$

2. Determine the possible rank, nullity and nullspace of the following matrix:
$$\boldsymbol{A} = \begin{pmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{pmatrix}$$

3. Let
$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$
Show that $\boldsymbol{A}$ has rank 2 if and only if one or more of the following determinants is nonzero.
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{13} & a_{13} \\ a_{23} & a_{23} \end{vmatrix}. \tag{1}$$

4. Let W be a subspace of $\mathbb{R}^n$ and
$$\boldsymbol{u}_1 = (1,0,1,-1),\ \boldsymbol{u}_2 = (0,1,0,-1),\ \boldsymbol{u}_3 = (-2,3,-3,1).$$
Define $W^\perp = \{\boldsymbol{u} \in \mathbb{R}^n \colon \boldsymbol{u} \text{ is orthogonal to } W\}$.
   (a) Compute $\|\boldsymbol{u}_1\|$, $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2$ and $d(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.
   (b) Let $W = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. Find $W^\perp$.
   (c) Find the equation of the 3-plane W in $\mathbb{R}^4$.
   (d) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$ and $\dim(W) + \dim(W^\perp) = n$.

5. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be an orthogonal set of vectors in a vector space. Show that
$$\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n\|^2 = \|\boldsymbol{u}_1\|^2 + \|\boldsymbol{u}_2\|^2 + \cdots + \|\boldsymbol{u}_n\|^2.$$