1 Differentiation, Integration

1.1 Limits

If $L = \lim f(x) \wedge M = \lim g(x)$ then

- $\lim kf(x) = kL$; $\lim(f(x) \pm g(x)) = L \pm M$
- $\lim f(x)g(x) = LM$
- $\lim f(x)/g(x) = L/M$ if $M \neq 0$
- $\lim_{x\to 0}(\sin x)/x = 1$
- $\sum_{n=0}^{\infty} 1/n! = e$; $\lim_{n\to\infty}(1+1/n)^n = e$

L'Hôpital's rule. For functions f, g differentiable on an open interval I except possibly at a point $c \in I$, if $\lim f(x) = \lim g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ for all $x \in I$, $x \neq c$, and $\lim f'(x)/g'(x)$ exists, then $\lim f(x)/g(x) = \lim f'(x)/g'(x)$.

1.2 Derivatives

- $(kf(x))' = kf'(x)$; $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $(f(x)/g(x))' = (f'(x)g(x) - f(x)g'(x))/((g(x))^2)$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} / \frac{\mathrm{d}x}{\mathrm{d}t}$
- $(e^x)' = e^x$; $(\ln x)' = 1/x$
- $(\sin x)' = \cos x$; $(\sin^{-1} x)' = 1/\sqrt{1-x^2}$
- $(\cos x)' = -\sin x$; $(\cos^{-1} x)' = -1/\sqrt{1-x^2}$
- $(\tan x)' = \sec^2 x$; $(\tan^{-1} x)' = 1/(1+x^2)$
- $(\csc x)' = -\csc x \cot x$; $(\csc^{-1} x)' = -1/|x|\sqrt{x^2-1}$
- $(\sec x)' = \sec x \tan x$; $(\sec^{-1} x)' = 1/|x|\sqrt{x^2-1}$
- $(\cot x)' = -\csc^2 x$; $(\cot^{-1} x)' = -1/(1+x^2)$
- $(x \ln x - x)' = \ln x$; $(\ln \sec x)' = \tan x$
- $(x \sin^{-1} x + \sqrt{1-x^2})' = \sin^{-1} x$
- $(x \cos^{-1} x - \sqrt{1-x^2})' = \cos^{-1} x$
- $(x \tan^{-1} x - \frac{1}{2}\ln(1+x^2))' = \tan^{-1} x$
- $(\ln(\sec x + \tan x))' = \sec x$
- $(-\ln(\csc x + \cot x))' = \csc x$
- $(\ln \sin x)' = \cot x$
- $f_x = \frac{\partial}{\partial x} f(x, \ldots)$; $f_{xx} = \frac{\partial^2}{\partial x^2} f(x, \ldots)$; and so on
- For $z(t) = f(x(t), y(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} z(t) = f_x \frac{\mathrm{d}}{\mathrm{d}t} x(t) + f_y \frac{\mathrm{d}}{\mathrm{d}t} y(t)$
- For $z(s,t) = f(x(s,t), y(s,t))$, $z_s = f_x x_s + f_y y_s$ and $z_t = f_x x_t + f_y y_t$
- $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$; $D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$ ($|\mathbf{u}| = 1$)
- In MA1521, $f_{xy} = f_{yx}$

1.3 Integrals

- $\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$; $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ where F is an antiderivative of f on $[a,b]$
- $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$, provided $g' \geq 0$ or $g' \leq 0$ in $[a,b]$.
- $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$; choose u as the harder-to-integrate function: log, $\sin^{-1}$ et al., algebraic, sin et al., and exponential, in decreasing order.
- Volume about x-axis $= \pi \int_a^b [f(x)]^2\,\mathrm{d}x$
- Volume about y-axis $= 2\pi \int_a^b x|f(x)|\,\mathrm{d}x$

Extreme values occur at interior points where $f'(x) = 0$ or does not exist, and domain endpoints. **Critical points** are interior points where $f'(x) = 0$ or does not exist, and where $f_x(a,b) = f_y(a,b) = 0$, or either $f_x(a,b)$ or $f_y(a,b)$ does not exist.

A graph is **concave down** on an interval if its shape looks like the graph of $y = -x^2$ i.e. $y'' < 0$, and **concave up** if it looks like $y = x^2$ i.e. $y'' > 0$. **Points of inflection** are points where f is continuous and its concavity changes.

First derivative test. If $f'(x) > 0$ for $x \in (a,c)$ and $f'(x) < 0$ for $x \in (c,b)$ then $f(c)$ is a local maximum. If $f'(x) < 0$ for $x \in (a,c)$ and $f'(x) > 0$ for $x \in (c,b)$ then $f(c)$ is a local minimum.

Second derivative test. If $f'(c) = 0 \wedge f''(c) < 0$, then f has a local maximum at $x = c$. If $f'(c) = 0 \wedge f''(c) > 0$, then f has a local minimum at $x = c$.

For functions of two variables, let $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$. If $D > 0 \wedge f_{xx}(a,b) > 0$, the point is a local minimum. If $D > 0 \wedge f_{xx}(a,b) < 0$, it is a local maximum. If $D < 0$, it is a saddle point. If $D = 0$, there is no conclusion.

f is **increasing** on an interval I if for any two points x_1, x_2 in I, $x_2 > x_1 \implies f(x_2) > f(x_1)$. If $x_2 > x_1 \implies f(x_2) < f(x_1)$, then f is **decreasing** on I.

Tests. f is increasing on I when $f'(x) > 0$ for all $x \in I$. f is decreasing on I when $f'(x) < 0$ for all $x \in I$.

2 Differential equations

Separable equations are those of the form $M(x)\mathrm{d}x = N(y)\mathrm{d}y$. They can be integrated directly.

Equations of the form $y' = g(y/x)$ can be made separable by $u = y/x$ and substituting $y = ux$, $y' = u + xu'$ to get $(g(u) - u)^{-1}\mathrm{d}u = x^{-1}\mathrm{d}x$.

Equations of the form $y' = f(ax + by + c)$ where f is continuous and $b \neq 0$ can be solved by substituting $u = ax + by + c$.

Equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ have general solution $y = R^{-1} \int RQ\,\mathrm{d}x$ where $R = \exp \int P\,\mathrm{d}x$.

Equations of the form $y' + P(x)y = Q(x)y^n$ can be reduced to the previous form by substituting $z = y^{1-n}$ to get $z' + (1-n)P(x)z = (1-n)Q(x)$.

2.1 Second order DEs

An equation of the form $y'' + ay' + by = 0$ has a characteristic equation $\lambda^2 + a\lambda + b = 0$ with roots λ_1, λ_2.

If λ_1, λ_2 are distinct and real, then the solution is
$y = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x)$.

If the roots are repeated, then the solution is $y = (c_1 + c_2 x)\exp(-\frac{1}{2}ax)$.

If the roots are complex, then if $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$, the solution is $y = c_1 \exp(\alpha x)\cos(\beta x) + c_2 \exp(\alpha x)\sin(\beta x)$.

2.2 Modelling

The Malthus model states that $N' = (B - D)N$ i.e. $N = N_0 \exp((B - D)t)$ where N is the population, B is the birth rate and D is the death rate.

The logistic model states that $D = sN$, so we have $N' = BN - sN^2$ i.e.

$$\frac{1}{N} = \frac{s}{B} + (\frac{1}{N_0} - \frac{s}{B})e^{-Bt}$$

and the long-term equilibrium population is B/s.

The harvesting model states that $N' = (B - sN)N - E$. To analyse this, consider $N'' = (B - 2sN)(BN - sN^2 - E) = -s(B - 2sN)(N - \beta_1)(N - \beta_2)$ where $\beta_1 \leq \beta_2$. If $E > B^2/4s$, N will go towards zero. If $0 < E < B^2/4s$, if $N_0 < \beta_1$, N will go towards zero in $T = \int_0^{N_0}(sN^2 - BN + E)^{-1}\,\mathrm{d}N$, or if $N_0 > \beta_1$, N will go towards β_2. If $E = B^2/4s$, if $N_0 > B/2s$, N will tend towards the latter, else it will go towards 0.

3 Series

An expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$ is an **infinite series**. a_n is the nth term. The sequence $s_1 = a_1, s_2 = a_1 + a_2, s_n = \sum_{k=1}^n a_k$ is the sequence of **partial sums** of the series, and s_n is the nth partial sum. If the sequence of partial sums converges to a limit L, the series is convergent and its sum is L.

A **geometric series** is one of the form $a + ar + ar^2 + \cdots = \sum_{n=1}^\infty ar^{n-1}$ and its nth partial sum $s_n = a(1 - r^n)/(1 - r)$. If $|r| < 1$ then the series converges and its sum is $a/(1-r)$.

A **power series** is one of the form $c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^\infty c_n x^n$. Such a series can be said to converge for all $x \in (a-h, a+h)$ and diverges elsewhere except possibly at $x = a - h$ or $x = a + h$; h could be zero, in which case the series converges only at $x = a$, or h could be ∞, in which case the series converges everywhere. h is known as the radius of convergence.

A power series is a function with the domain being the values of x for which the series converes. This function can be differentiated or integrated term-by-term.

Ratio test. Let $\rho = \lim_{n\to\infty}|a_{n+1}/a_n|$. If $\rho < 1$, the series converges; $\rho > 1$, the series diverges; $\rho = 1$, there is no conclusion.

3.1 Taylor series

At $x = a$, for all x u.o.s.

- $f(x) = f(a) + f'(a)(x-a) + \cdots = \sum_{k=0}^\infty \frac{f^{(k)}(a)}{k!}(x-a)^k$
- $1/x = \frac{1}{a} - \frac{x-a}{a^2} + \frac{(x-a)^2}{a^3} - \cdots$
  $= \sum_{n=0}^\infty((-1)^n/a^{n+1})(x-a)^n$ for $|1 - x/a| < 1$
- $\frac{1}{1-cx} = \frac{1}{1-ac} + \frac{c(x-a)}{(1-ac)^2} + \frac{c^2(x-a)^2}{(1-ac)^3} + \cdots$
  $= \sum_{n=0}^\infty(c^n/(1-ac)^{n+1})(x-a)^n$
  for $|c(a-x)/(ac-1)| < 1$
- $\frac{1}{1+cx} = \frac{1}{1+ac} - \frac{c(x-a)}{(1+ac)^2} + \frac{c^2(x-a)^2}{(1+ac)^3} - \cdots$
  $= \sum_{n=0}^\infty((-c)^n/(1+ac)^{n+1})(x-a)^n$
  for $|c(a-x)/(ac+1)| < 1$
- $e^{cx} = e^{ac}(1 + c(x-a) + c^2(x-a)^2/2! + \cdots)$
  $= \sum_{n=0}^\infty(e^{ac}c^n/n!)(x-a)^n$
- $\sin cx = \sin ac + c(\cos ac)(x-a) - c^2(\sin ac)(x-a)^2/2! - c^3(\cos ac)(x-a)^3/3! - \cdots = \sum_{n=0}^\infty(c^n \sin(ac + n\pi/2)/n!)(x-a)^n$
- $\cos cx = \cos ac - c(\sin ac)(x-a) - c^2(\cos ac)(x-a)^2/2! + c^3(\sin ac)(x-a)^3/3! + \cdots = \sum_{n=0}^\infty(c^n \cos(ac + n\pi/2)/n!)(x-a)^n$
- $\ln x = \ln a + (x-a)/a - (x-a)^2/2a^2 + (x-a)^3/3a^3 - \cdots = \ln a + \sum_{n=1}^\infty((-1)^{n+1}/(na^n))(x-a)^n$
  for $|1 - x/a| < 1$
- $\tan^{-1} x = x - x^3/3 + x^5/5 - \cdots$
  $= \sum_{n=0}^\infty(-1)^n x^{2n+1}/(2n+1)$ for $|x| < 1$ at $x = 0$

4 Identities

- $\sin^2 x + \cos^2 x = \sec^2 x - \tan^2 x = \csc^2 x - \cot^2 x = 1$
- $\frac{\sin}{\tan}(-x) = -\frac{\sin}{\tan}x$; $\cos(-x) = \cos x$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha\tan\beta)$
- $\cot(\alpha \pm \beta) = (\cot\alpha\cot\beta \mp 1)/(\cot\beta \pm \cot\alpha)$
- $\sin(2x) = 2\sin x\cos x = (2\tan x)/(1 + \tan^2 x)$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = (1 - \tan^2 x)/(1 + \tan^2 x)$
- $\tan(2x) = (2\tan x)/(1 - \tan^2 x)$
- $\cot(2x) = (\cot^2 x - 1)/(2\cot x)$
- $\sec(2x) = (\sec^2 x)/(2 - \sec^2 x)$
- $\csc(2x) = (\sec x \csc x)/2$
- $\sin(3x) = -4\sin^3 x + 3\sin x$
- $\cos(3x) = 4\cos^3 x - 3\cos x$
- $\tan(3x) = (3\tan x - \tan^3 x)/(1 - 3\tan^2 x)$
- $\cot(3x) = (3\cot x - \cot^3 x)/(1 - 3\cot^2 x)$
- $\sin^2 x = (1 - \cos(2x))/2$; $\cos^2 x = (1 + \cos(2x))/2$
- $\sin^2 x\cos^2 x = (1 - \cos(4x))/8$
- $\sin x \pm \sin y = 2\sin((x \pm y)/2)\cos((x \mp y)/2)$
- $\cos x \pm \cos y = \pm 2\frac{\cos}{\sin}((x+y)/2)\frac{\cos}{\sin}((x-y)/2)$
- $\frac{\cos}{\sin}x\frac{\cos}{\sin}y = (\cos(x-y) \pm \cos(x+y))/2$
- $\frac{\sin}{\cos}x\frac{\cos}{\sin}y = (\sin(x+y) \pm \sin(x-y))/2$
- $\tan x\tan y = (\cos(x-y) - \cos(x+y))/(\cos(x-y) + \cos(x+y))$