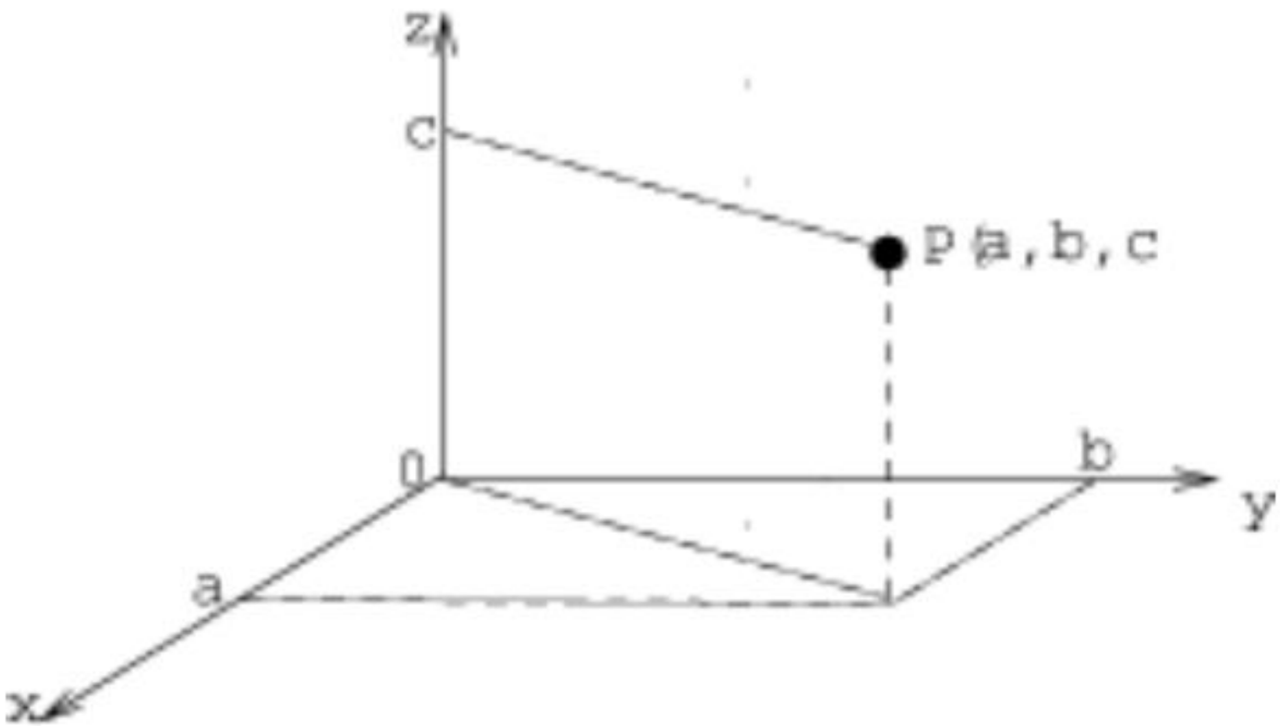
Chapter 5. Three Dimensional Space

5.1 The Coordinate System of the 3D Space

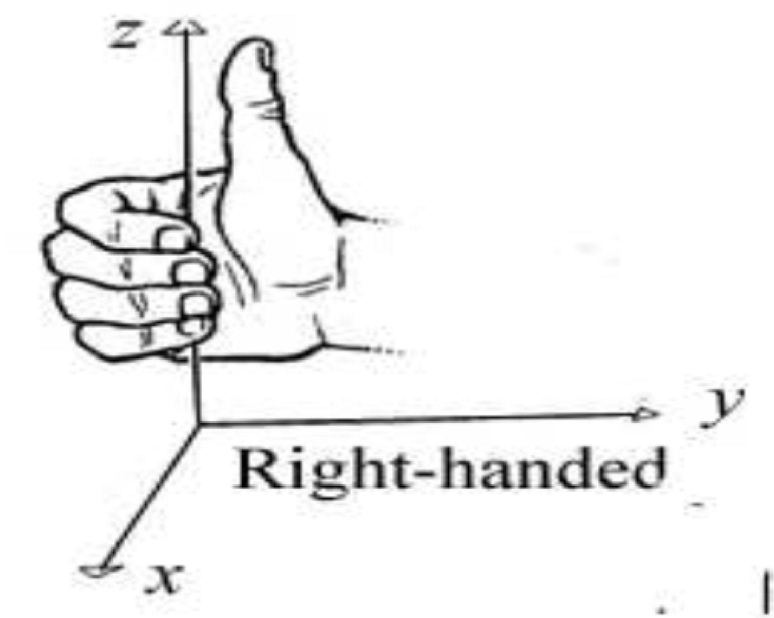
For three dimensional space, we first fix a coordinate system by choosing a point called the **origin**, and three lines, called the coordinate axes, so that each line is perpendicular to the other two. These lines are called the x-. y- and z-axes.


z
c
P(a, b, c
0
b
y
a
x

Associated with a point P in three dimensional space is an ordered triple (a, b, c) where a, b and c are the projections of P on the x-, y- and z-axes respectively.

This is the **Cartesian coordinate system** for three dimensional space. We also call this space the xyz-space.

By convention, we use the **right-handed coordinate system**. A right-handed coordinate system fix the orientation of the axes as follow:



5.2 Vectors in xyz-Space

A vector is measurable quantity with a *magnitude* and a *direction*. It is geometrically represented by an arrow in the xyz-space with an initial point and a terminal point. The direction of the arrow gives the direction of the vector; and the length of the arrow gives the magnitude of the vector.

5.2.1 Terminologies and notations

(1) Let P and Q be points in the xyz-space with co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then the vector $\overrightarrow{PQ}$ is algebraically given by

$$\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

The vector $\overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is called the position vector of P.

(2) The zero vector in the xyz-space is $\mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(3) The sum of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

[Note that $\mathbf{v}_1 + \mathbf{O} = \mathbf{O} + \mathbf{v}_1 = \mathbf{v}_1$.]

(4) The negative of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is $-\mathbf{v}_1 = \begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix}$.

[Note that $\mathbf{v}_1 - \mathbf{v}_1 = -\mathbf{v}_1 + \mathbf{v}_1 = \mathbf{O}$.]

(5) The difference $\mathbf{v}_1 - \mathbf{v}_2$ is

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -y_2 \\ -z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}.$$

(6) If c is a real number, the scalar $c\mathbf{v}_1$ of $\mathbf{v}_1$ by c is

$$c\mathbf{v}_1 = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

If $c > 0$, then $c\mathbf{v}_1$ is in the same direction as $\mathbf{v}_1$.

If $d < 0$, then $d\mathbf{v}_1$ is in the opposite direction as

$\mathbf{v}_1$.

(7) The magnitude of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is

$$||\mathbf{v}_1|| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

[Note that $||c\mathbf{v}_1|| = |c|\ ||\mathbf{v}_1||$ for a real number c.]

5.2.2 Example

Let P_1, P_2, Q_1 and Q_2 be the points $(3,2,-1)$, $(0,0,0)$, $(5,5,4)$ and $(2,3,5)$ respectively.

$$\overrightarrow{P_1Q_1} = \begin{bmatrix} 5-3 \\ 5-2 \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\overrightarrow{P_2Q_2} = \begin{bmatrix} 2-0 \\ 3-0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

Hence

$$\overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}.$$

The magnitude of $\overrightarrow{P_1Q_1}$ is

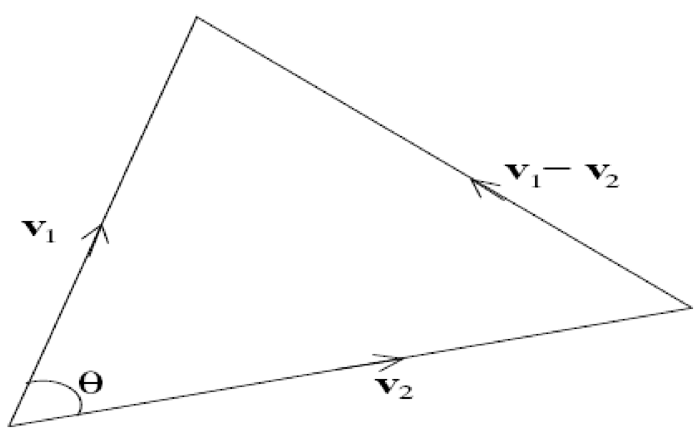
$$||\overrightarrow{P_1Q_1}|| = \sqrt{(2)^2 + (3)^2 + (5)^2} = \sqrt{38}.$$

So the magnitude of $5\overrightarrow{P_1Q_1}$ is

$$5||\overrightarrow{P_1Q_1}|| = 5\sqrt{38}.$$

5.2.3 Angle between two vectors

The angle between the nonzero vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is the angle θ, $(0 \leq \theta \leq 180^0)$ as shown below.



Applying the law of cosines to this triangle, we obtain

$$||\mathbf{v}_1 - \mathbf{v}_2||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \cos\theta. \quad (1)$$

Now LHS of (1) $||\mathbf{v}_1 - \mathbf{v}_2||^2$ is given by

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$$

$$= ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2(x_1x_2 + y_1y_2 + z_1z_2).$$

If we substitute this expression in (1) and solve for $\cos\theta$, we obtain

$$\cos\theta = \frac{x_1\ x_2 + y_1\ y_2 + z_1\ z_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||} \qquad (2)$$

5.2.4 Scalar or dot product

The **scalar product** or **dot product** of the vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is defined by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Thus we can rewrite (2), where $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors, as

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||}, \quad (0 \le \theta \le 180^0)$$

and notice that

$$\mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are perpendicular} \iff \mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

5.2.5 Example

If $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (4)(2) + (5)(3) = 21.$$

Also

$$||\mathbf{v}_1|| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45},$$

$$||\mathbf{v}_2|| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}.$$

Hence

$$\cos\theta = \frac{21}{\sqrt{45}\sqrt{14}} = \frac{\sqrt{7}}{\sqrt{10}}.$$

Thus θ is approximately $33°13'$.

The vectors $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ are perpendicular since their dot product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(4) + (-5)(2) + (1)(2) = 0.$$

5.2.6 Properties of scalar product

If $\mathbf{v_1}$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors in xyz-space and c is a real number, then

(a) $\mathbf{v}_1 \cdot \mathbf{v}_1 = ||\mathbf{v}_1||^2 \geq 0$.

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$.

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3$.

(d) $(c\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (c\mathbf{v}_2) = c(\mathbf{v}_1 \cdot \mathbf{v}_2)$.

5.2.7 Unit vector

A **unit vector** in xyz-space is a vector of magnitude or length 1. The vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors along the positive x-, y- and z-axes respectively.

Notice that every vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

For example,

$$\mathbf{w} = \begin{bmatrix} 4 \\ -5 \\ 22 \end{bmatrix} = 4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}.$$

The unit vector with the same direction as $\mathbf{w}$ is

$$\begin{aligned}
\frac{1}{||\mathbf{w}||}\mathbf{w} &= \frac{1}{\sqrt{4^2+5^2+22^2}}(4\mathbf{i}-5\mathbf{j}+22\mathbf{k}) \\
&= \frac{4}{\sqrt{525}}\mathbf{i}-\frac{5}{\sqrt{525}}\mathbf{j}+\frac{22}{\sqrt{525}}\mathbf{k}.
\end{aligned}$$