Review of 4.2 - 5.2

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Basis for column space and basis for row space

- Recalled that we have two methods to find a basis for $V = span(S), S = \{u_1, \cdots, u_k\}$ (consider row space and column space respectively).
- A be a matrix and R is its row-echelon form, then row space of A is equal to that of R (Remark 4.1.9), dimension of row space of A is equal to dimension of row space of R, which is the number of non-zero rows in R.
- The number of non-zero rows in R equals to the number of pivot columns in R. By Remark 4.1.13, it is the same as the dimension of column space of A.

Theorem (4.2.1)

The row space and column space of a matrix have the same dimension.

Ranks

Definition (Rank)

The *rank* of a matrix A is the dimension of its row space (or column space). Denoted by **rank(A)**.

For concise, for any $m \times n$ matrix A we have

1. $rank(A) = \dim(\text{column space of A})$.
2. $rank(A) = \dim(\text{row space of A})$.
3. $rank(A) \leq \min\{m, n\}$, if equality holds then A is said to have *full rank*.
4. A square matrix A has full rank if and only if $\det(A) \neq 0$.
5. $rank(A) = rank(A^T)$.
6. A linear system $Ax = b \Leftrightarrow b$ is in the column space of $A \Leftrightarrow rank(A) = rank(A|b)$,

Rank inequalities

Theorem (4.2.8)

Let A and B be $m \times n$ and $n \times p$ matrices respectively. Then

$$rank(AB) \leq min\{rank(A), rank(B)\}.$$

Theorem (Q1)

Let A and B be two matrices of the same size, then

$$rank(A + B) \leq rank(A) + rank(B).$$

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Nullspace and Nullities

Definition (4.3.1)

Let A be an $m \times n$ matrix. The solution space of $Ax = 0$ is called the *nullspace of A*. And the dimension of the nullspace is called the *nullity* of A.

Question: How to find the nullspace and nullity of A?

1. Write down the augmented matrix of $Ax = 0$, and use the Gauss-Jordan Algorithm to solve this linear system.
2. Write the general solution as
$$x = t_1 u_1 + \cdots + t_k u_k.$$
3. Then the solution space $V = span(S)$, where $S = \{u_1, \cdots, u_k\}$. And $u_1, \cdots, u_k$ are linear independent. Hence $dim(V) = k$.
4. By the definition of nullity, we have the nullity of A is exactly k.

Theorem (4.3.4 (Dimension Theorem for Matrices))

Let A be a matrix with n columns. Then

$$rank(A) + nullity(A) = n.$$

Proof.

nullity(A) = number of arbitary variables = number of columns minus number of non-zero rows. Which is $n - rank(A)$. □

Properties for Nullspace — 2

Suppose now we have a paticular solution u for $Ax = b$, then if v is any solution for the homogeneous linear system, we have

$$A(u + v) = Au + Av = b + 0 = b.$$

So $u + v, \forall v$ in the solution space of $Ax = 0$ is a solution of $Ax = b$.

Theorem (4.3.6)

Suppose that the system of linear equations $Ax = b$ has a solution v. Then the solution set of the system is given by

$$M = \{u + v | u \textit{is an element of the null space of } A\}.$$

That is, $Ax = b$ has a general solution

$$x = (\textit{a general solution for } Ax = 0) + (\textit{one particular solution of } Ax = b).$$

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The dot product

Let $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$ and $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$.

1. *(Dot Product or Inner Product)*
$$u \cdot v = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + \cdots + u_n v_n.$$
2. *(Norm)*
$$\|u\| = \sqrt{u \cdot u} = \sqrt{\sum_{i=1}^{n} u_i^2} = \sqrt{u_1^2 + \cdots + u_n^2}.$$
3. *(Distance)*
$$d(u, v) = \|u - v\| = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2} = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$
4. *(Angle between u and v)*
$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|}\right).$$

Norm

Theorem

Let $u, v \in \mathbb{R}^n$ and c is a scalar, then we have

1. $\|u\| = 0 \Leftrightarrow u = \mathbf{0}$.
2. $\|cu\| = |c|\|u\|$.
3. *(Triangle Inequality)*

$$\|u + v\| \leq \|u\| + \|v\|.$$

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Orthogonal Sets

Definition (5.2.1)

1. Two vector u and v in $\mathbb{R}^n$ are called *orthogonal* if $u \cdot v = 0$.
2. A set of vectors S in $\mathbb{R}^n$ is called *orthogonal* if every pair of distinct vectors in S are orthogonal.
3. A set of vectors S in $\mathbb{R}^n$ is called *orthonormal* if S is orthogonal and every vector in S is a unit vector.

If u and v are orthogonal, then

$$\|u+v\|^2 = (u+v)\cdot(u+v) = (u\cdot u)^2 + u\cdot v + v\cdot u + (v\cdot v)^2 = \|u\|^2 + \|v\|^2.$$