Review of 5.2 - 5.3

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$V = span\{u_1, \cdots, u_k\}$

Now we would like to consider the case that $S = \{u_1, \cdots, u_k\}$, where S is a orthogonal set.

Theorem

Let S be an orthogonal set of nonzero vectors in a vector space. Then S is linearly independent.

This theorem tells us that S is a basis for $V = span(S)$. Therefore, we can compute the coordinate vector $(v)_S$ for any vector $v \in V$.

Theorem (5.2.8)

For any $v \in V$,

$$(v)_S = (\frac{v \cdot u_1}{u_1 \cdot u_1}, \cdots, \frac{v \cdot u_k}{u_k \cdot u_k}).$$

Which means for any $v \in V$, we have

$$v = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{v \cdot u_k}{u_k \cdot u_k} u_k.$$

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Orthogonal Complement

Definition (Perpendicular)

Let V be a subspace of $\mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is said to be *orthogonal (or perpendicular)* to V if u is orthogonal to all vectors in V. In general, if $V = span\{u_1, \cdots, u_k\}$ is a subspace of $\mathbb{R}^n$, then a vector $u \in \mathbb{R}^n$ is orthogonal to V if and only if

$$u \cdot u_i = 0, \quad \forall i = 1, 2, \cdots, k.$$

Let $V^\perp$ denote all the vector u that is orthogonal to V, then $V^\perp$ is the solution space of the following homogeneous linear system

$$\begin{cases} u \cdot u_1 = 0, \\ \vdots \\ u \cdot u_k = 0. \end{cases}$$

Then $V^\perp$ is a subspace (why?) and it is called the orthogonal complement of V.

Dimension of $V^\perp$

We now know that $V^\perp$ is the solution space of a homogeneous linear system, if further we denoted

$$u_i = (a_{i1}, \cdots, a_{in}), \quad i = 1, \cdots, k.$$

And let

$$A = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

be a $k \times n$ matrix, then $V^\perp$ is the same as the nullspace of A. Hence

$$dim(V^\perp) = dim(\text{nullspace of A}) = nullity(A) = n - rank(A) = n - dim(V)$$

Orthogonal Projection

Let V be a subspace of $\mathbb{R}^n$. Every vector $u \in \mathbb{R}^n$ can be written uniquely as

$$u = n + p$$

such that n is a vector orthogonal to V and p is a vector in V. The vector p is called the orthogonal projection of u onto V.

Theorem (5.2.15)

Let V be a subspace of $\mathbb{R}^n$ and w be a vector in $\mathbb{R}^n$. If $\{u_1, \cdots, u_k\}$ is an orthogonal basis for V, then

$$\frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k$$

is the projection of w onto V.

(Check $n = w - p$ is orthogonal to V directly, then by the uniqueness of such expression, p is the orthogonal projection of w onto V.)

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How to obtained an orthogonal basis from any basis?

Knowing the orthogonal basis for a vector space is very useful, since we can write down the coordinate vector relative to this basis directly (see Theorem 5.2.8) and we can obtain the orthogonal projection easily (see Theorem 5.2.15). So given an arbitrary basis $S = \{u_1, \cdots, u_k\}$ (not necessarily a orthogonal basis) for a vector space V. We would like to find an orthogonal basis which can be computed from S.
See next page...

How to obtained a orthogonal basis from any basis? — continued.

Theorem (5.2.19)

Let $S = \{u_1, \cdots, u_k\}$ be a basis for a vector space V. Let

$$
\begin{aligned}
v_1 &= u_1, \\
v_2 &= u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\
&\vdots \\
v_k &= u_k - \frac{u_k \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_k \cdot v_2}{v_2 \cdot v_2} v_2 + \cdots + \frac{u_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}} v_{k-1}
\end{aligned}
$$

Then $\{v_1, \cdots .v_k\}$ is an orthogonal basis for V. And we can divide the norm of each vector to obtain a orthonormal basis for V.

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Best Approximation

Theorem (5.3.2)

Let V be a subspace in $\mathbb{R}^n$. If u is a vector in $\mathbb{R}^n$ and p is the projection of u onto V, then

$$d(u, p) \leq d(u, v) \quad \forall v \in V.$$

Which means that p is the best approximation of u in V.

So we first need to find a orthogonal basis for V and then compute the projection of u onto V, then we can have the best approximation.

Least Squares Method

Definition (Least squares solution)

Let $Ax = b$ be a linear system where A is an $m \times n$ matrix. A vector $u \in \mathbb{R}^n$ is called a least squares solution to the linear system if

$$\|b - Au\| \leq \|b - Av\| \quad \forall v \in \mathbb{R}^n.$$

Note that this definition does not give any hint for us to compute the least squares solution, so we need the following theorem to show us how to compute it.

Theorem (5.3.10)

Let $Ax = b$ be a linear system. Then u is a least squares solution to $Ax = b$ if and only if u is a solution to $A^T Ax = A^T b$.

So all we need to do is to solve the linear system $A^T Ax = A^T b$.

In this slide we are interested in using least squares method to compute the orthogonal projection of a vector u onto V.

1. Let $V = span\{u_1, \cdots, u_k\}$, which means that we need a set of vector $S = \{u_1, \cdots, u_k\}$ that can span the vector space.
2. Let $A = (u_1^T, \cdots, u_k^T)$ (u_i here is row vector, and we write A as column form). And let $b = u^T$ be a column vector.
3. Solve $A^T Ax = A^T b$, then choose any particular solution x.
4. The vector $p = Ax$ (column vector, need to be transposed to a row vector if needed) is the projection of u onto V.

See Example 5.3.11 (3).