Lecture 08 recap

1) Geometric vectors and Euclidean $n-$space.

2) Identifying a vector with a matrix.

3) Subsets of $\mathbb{R}^n$. Implicit and explicit representations.

4) Linear combination

5) Linear span (set of all linear combinations).

Lecture 09

Linear combinations and linear spans (cont'd)

Subspaces

Learning points for Lecture 09

Section 3.2 Linear combinations and linear spans

1) (Discussion 3.2.5) How to determine whether span(S) is the entire Euclidean $n-$space? How this links back to determining whether a linear system is consistent or not?

2) (Theorem 3.2.7) What is the minimum number of vectors required to span $\mathbb{R}^n$?

3) (Theorem 3.2.9) Two characteristics of linear spans.

4) A necessary and sufficient condition for one linear span to be entirely contained inside another linear span.

Learning points for Lecture 09

Section 3.2 Linear combinations and linear spans

5) (Theorem 3.2.12) If $\boldsymbol{u}_k$ is a linear combination of the vectors in $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\}$ then span(S) and span($S \cup \{\boldsymbol{u}_k\}$) are equal (notion of redundancy).

6) How to express all lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$ (those through the origin or not) using linear spans.

Discussion 3.2.5

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if $\operatorname{span}(S)=\mathbb{R}^n$.

$$\boldsymbol{u}_1=(a_{11},a_{12},\dots,a_{1n})\quad \boldsymbol{u}_2=(a_{21},a_{22},\dots,a_{2n})\quad \dots\quad \boldsymbol{u}_k=(a_{k1},a_{k2},\dots,a_{kn})$$

For any $\boldsymbol{v}=(v_1,v_2,\dots,v_n)\in\mathbb{R}^n$, consider the equation:

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\dots+c_k\boldsymbol{u}_k=\boldsymbol{v}$$

$$c_1(a_{11},a_{12},\dots,a_{1n})+c_2(a_{21},a_{22},\dots,a_{2n})+\dots+c_k(a_{k1},a_{k2},\dots,a_{kn})$$
$$=(v_1,v_2,\dots,v_n)$$

Discussion 3.2.5

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\mathrm{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots + c_k\boldsymbol{u}_k = \boldsymbol{v}$$

$$c_1(a_{11}, a_{12}, \dots, a_{1n}) + c_2(a_{21}, a_{22}, \dots, a_{2n}) + \dots + c_k(a_{k1}, a_{k2}, \dots, a_{kn})$$
$$= (v_1, v_2, \dots, v_n)$$

$$\begin{cases} a_{11}c_1 & + & a_{21}c_2 & + & \dots & + & a_{k1}c_k & = & v_1 \\ a_{12}c_1 & + & a_{22}c_2 & + & \dots & + & a_{k2}c_k & = & v_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{1n}c_1 & + & a_{2n}c_2 & + & \dots & + & a_{kn}c_k & = & v_n \end{cases}$$

Discussion 3.2.5

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\text{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \boxed{(*)}$$

$$c_1(a_{11}, a_{12}, \ldots, a_{1n}) + c_2(a_{21}, a_{22}, \ldots, a_{2n}) + \ldots + c_k(a_{k1}, a_{k2}, \ldots, a_{kn})$$
$$= (v_1, v_2, \ldots, v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ \begin{array}{c} n \\ \text{rows} \end{array} & \left(\begin{array}{c|c} & v_1 \\ & v_2 \\ \boldsymbol{A} & \vdots \\ & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ does not have a zero row, $\boxed{(*)}$ is always consistent regardless of $\boldsymbol{v}$

$$\boxed{\Rightarrow \text{span}(S) = \mathbb{R}^n}$$

Discussion 3.2.5

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\operatorname{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \boxed{(*)}$$

$$c_1(a_{11}, a_{12}, \ldots, a_{1n}) + c_2(a_{21}, a_{22}, \ldots, a_{2n}) + \ldots + c_k(a_{k1}, a_{k2}, \ldots, a_{kn})$$
$$= (v_1, v_2, \ldots, v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ \begin{array}{c} n \\ \text{rows} \end{array} & \left(\begin{array}{cccc|c} & & & & v_1 \\ & \boldsymbol{A} & & & v_2 \\ & & & & \vdots \\ 0 & 0 & \ldots & 0 \; 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, $\boxed{(*)}$ is not always consistent

$$\boxed{\Rightarrow \operatorname{span}(S) \neq \mathbb{R}^n}$$

Example 3.2.6

From earlier example:

Show that $\text{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

No zero row

Show that $\text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Has zero row

Theorem 3.2.7

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

If $k < n$, then S cannot span $\mathbb{R}^n$.

If $k < n$, then a row-echelon form of $\boldsymbol{A}$ has at least one zero row $\Rightarrow S$ cannot span $\mathbb{R}^n$.

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\text{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \text{(*)}$$

k columns

$$\begin{array}{cc} & k \text{ columns} \\ n \text{ rows} & \left(\begin{array}{ccccc|c} & & & & & v_1 \\ & & \boldsymbol{A} & & & v_2 \\ & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, (*) is not always consistent

Example 3.2.8

1) One vector cannot span $\mathbb{R}^2$.

2) One or two vectors cannot span $\mathbb{R}^3$.

Theorem 3.2.9

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$.

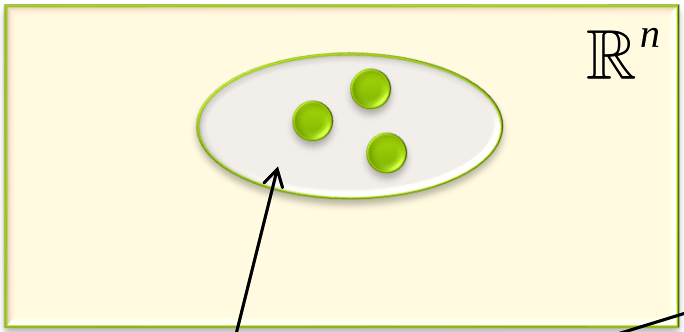
1) $\mathbf{0} \in \text{span}(S)$

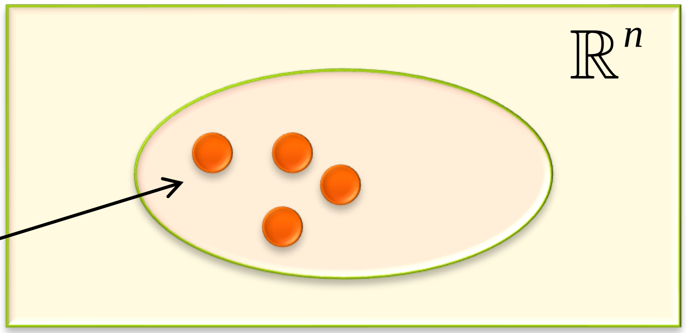
2) For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_r\boldsymbol{v}_r \in \text{span}(S).$$

Theorem 3.2.10

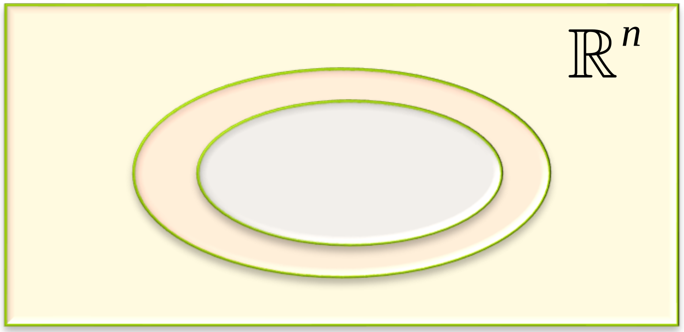
Let $S_1=\{u_1,u_2,\ldots,u_k\}$ and $S_2=\{v_1,v_2,\ldots,v_m\}$ be subsets of $\mathbb{R}^n$.





Then span$(S_1)\subseteq$ span$(S_2)\Leftrightarrow$ each u_i is a linear combination of

$v_1,v_2,\ldots,v_m$.



Example 3.2.11.1

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

$$= a(1,2,3) + b(2,-1,1)$$

Example 3.2.11.1

$$\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c|c} 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{-4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$(1,0,1) = \quad (1,2,3) + \quad (2,-1,1)$$

$$(1,1,2) = \quad (1,2,3) + \quad (2,-1,1)$$

$$(-1,2,1) = \quad (1,2,3) + \quad (2,-1,1)$$

Since each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$

is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$,

span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Example 3.2.11.1

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Can we show

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \supseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$?

Shown:

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$= a(1,0,1) + b(1,1,2) + c(-1,2,1)$$

Example 3.2.11.1

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boldsymbol{v}_1 = a(1,0,1) + b(1,1,2) + c(-1,2,1)$$

Example 3.2.11.1

Let $\boldsymbol{u}_1=(1,0,1),\boldsymbol{u}_2=(1,1,2),\boldsymbol{u}_3=(-1,2,1),\boldsymbol{v}_1(1,2,3),\boldsymbol{v}_2=(2,-1,1)$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since each of $\boldsymbol{v}_1,\boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$,

$$\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}\supseteq\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}.$$

Together with $\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}\subseteq\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$, we have shown

$$\boxed{\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}=\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}.}$$

Example 3.2.11.2*

Let $\boldsymbol{u}_1 = (1,1,0,2), \boldsymbol{u}_2 = (1,0,0,1), \boldsymbol{u}_3 = (0,1,0,1),$

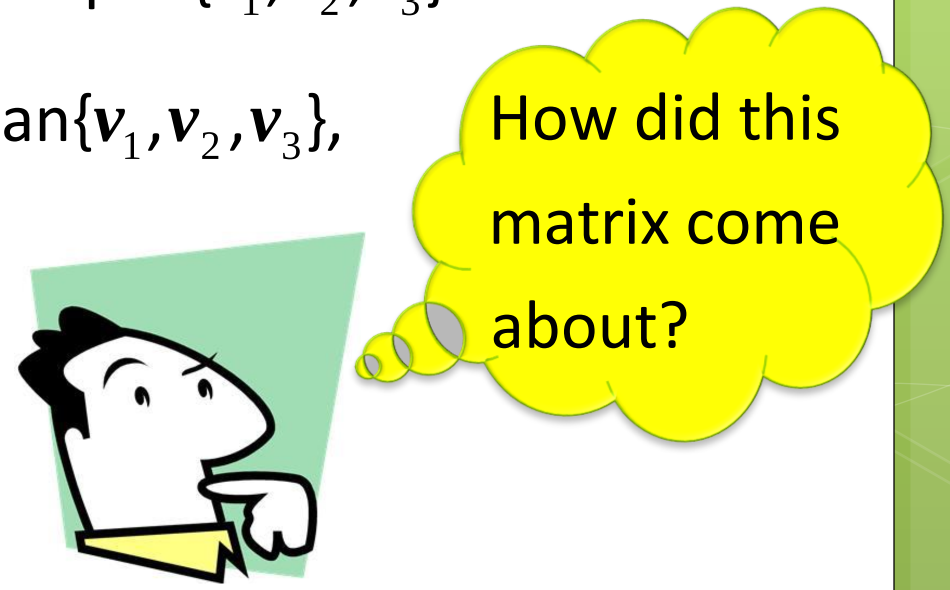
$\boldsymbol{v}_1 = (1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1).$

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ but

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\},$

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 1 & 1 \end{array}\right)$$



Example 3.2.11.2*

Let $\boldsymbol{u}_1 = (1,1,0,2), \boldsymbol{u}_2 = (1,0,0,1), \boldsymbol{u}_3 = (0,1,0,1)$,

$\boldsymbol{v}_1 = (1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ but

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Example 3.2.11.2*

Let $\boldsymbol{u}_1 = (1,1,0,2), \boldsymbol{u}_2 = (1,0,0,1), \boldsymbol{u}_3 = (0,1,0,1),$

$\boldsymbol{v}_1 = (1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1).$

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ but

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

$$\boldsymbol{v}_1 = (1,1,1,1) = a(1,1,0,2) + b(1,0,0,1) + c(0,1,0,1)$$

has no solution.

So $\boldsymbol{v}_1$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

Example

Let $\boldsymbol{u}_1 = (1,0,0,1), \boldsymbol{u}_2 = (0,1,-1,2), \boldsymbol{u}_3 = (2,1,-1,4)$,
$\boldsymbol{v}_1 = (1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1)$.

Show that $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \not\subset \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Which $\boldsymbol{v}_i$ is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

Theorem 3.2.12 ('useless vector')

Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

set of ALL linear combinations of

=

set of ALL linear combinations of

Theorem 3.2.12 ('useless vector')

Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

set of ALL linear combinations of

=

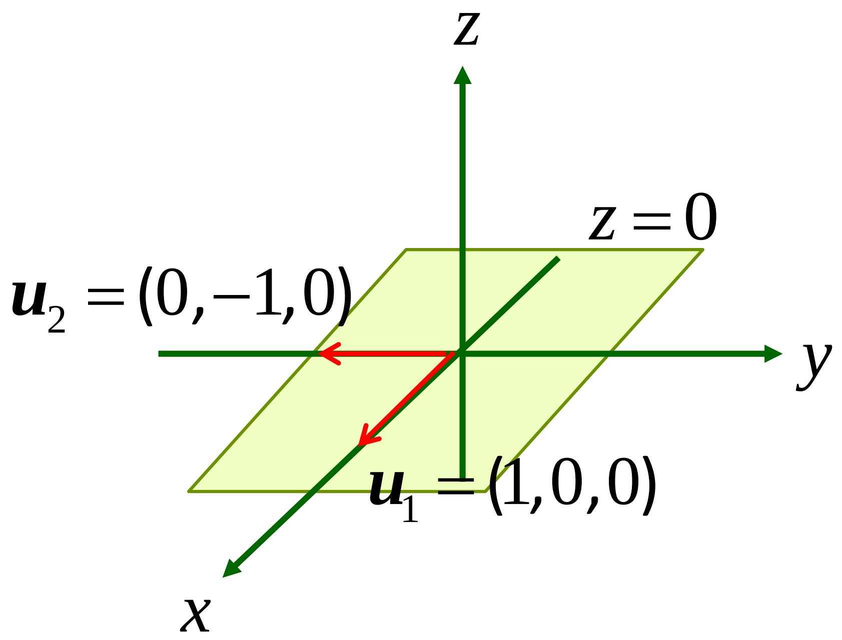
set of ALL linear combinations of

Example 3.2.13*

Let $\boldsymbol{u}_1 = (1,0,0), \boldsymbol{u}_2 = (0,-1,0), \boldsymbol{u}_3 = (2,3,0)$.

Clearly, $\boldsymbol{u}_3 = 2\boldsymbol{u}_1 - 3\boldsymbol{u}_2$. So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

Can you describe $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ geometrically?

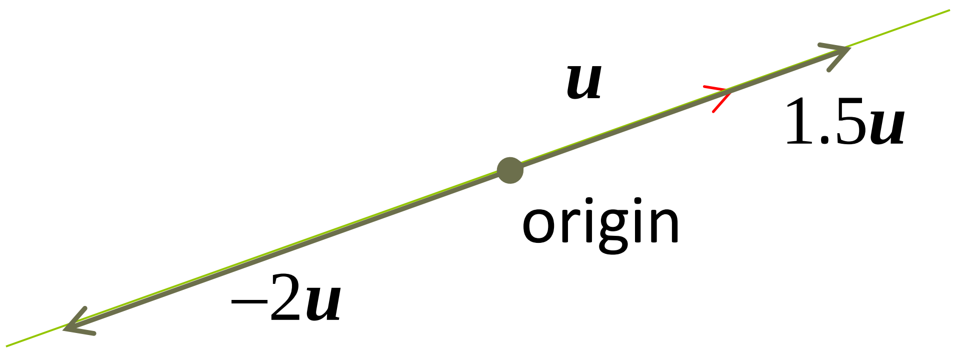


Discussion 3.2.14

Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span{$\boldsymbol{u}$} is the set of all linear combinations (or scalar multiples) of $\boldsymbol{u}$.

Geometrically, span{$\boldsymbol{u}$} is a straight line passing through the origin.



Discussion 3.2.14

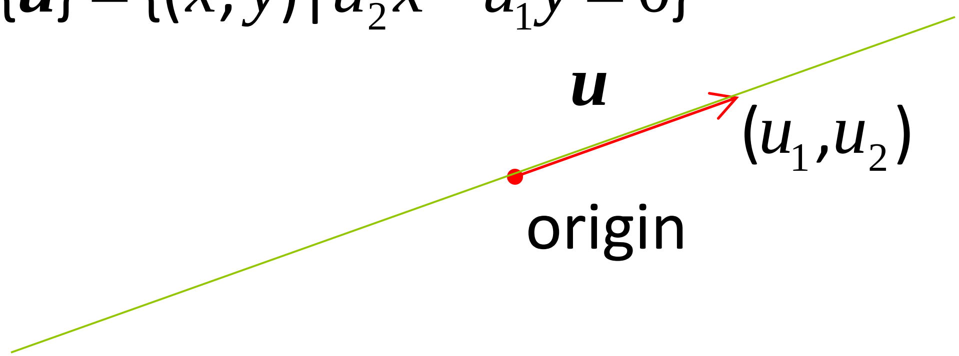
Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(In $\mathbb{R}^2$) $\boldsymbol{u} = (u_1, u_2)$, span$\{\boldsymbol{u}\} = \{(cu_1, cu_2) \mid c \in \mathbb{R}\}$

(explicit representation)

(implicit representation i.e. equation of line?)

span$\{\boldsymbol{u}\} = \{(x, y) \mid u_2 x - u_1 y = 0\}$



$$ax + by = 0$$

$$(u_1, u_2)$$

$$au_1 + bu_2 = 0$$

One solution for a, b is $a = u_2$ and $b = -u_1$

Discussion 3.2.14

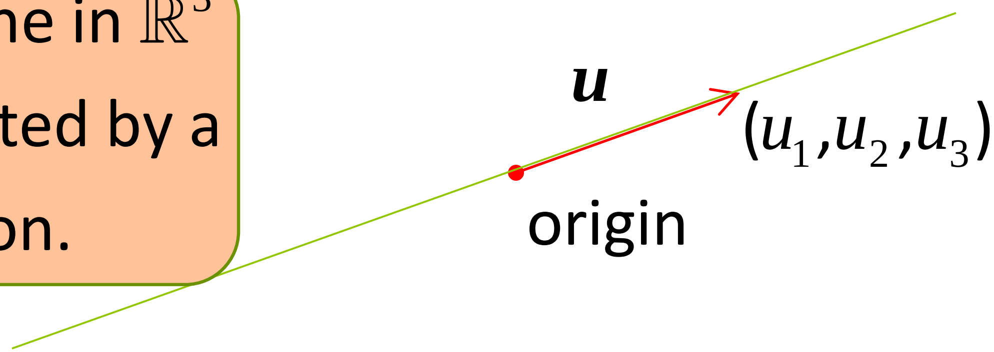
Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(In $\mathbb{R}^3$) $\boldsymbol{u}=(u_1,u_2,u_3)$, span$\{\boldsymbol{u}\}=\{(cu_1,cu_2,cu_3)\,|\,c\in\mathbb{R}\}$

(explicit representation)

$$\{(0,0,0)+c(u_1,u_2,u_3)\,|\,c\in\mathbb{R}\}=\{(cu_1,cu_2,cu_3)\,|\,c\in\mathbb{R}\}$$

Remember that a line in $\mathbb{R}^3$ cannot be represented by a single linear equation.



Discussion 3.2.14

Let $\boldsymbol{u},\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span$\{\boldsymbol{u},\boldsymbol{v}\}$ is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$=\{s\boldsymbol{u}+t\boldsymbol{v} \mid s,t\in\mathbb{R}\}$$

What if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel?

span$\{\boldsymbol{u},\boldsymbol{v}\}=$ span$\{\boldsymbol{u}\}$

$=$ straight line passing through the origin.

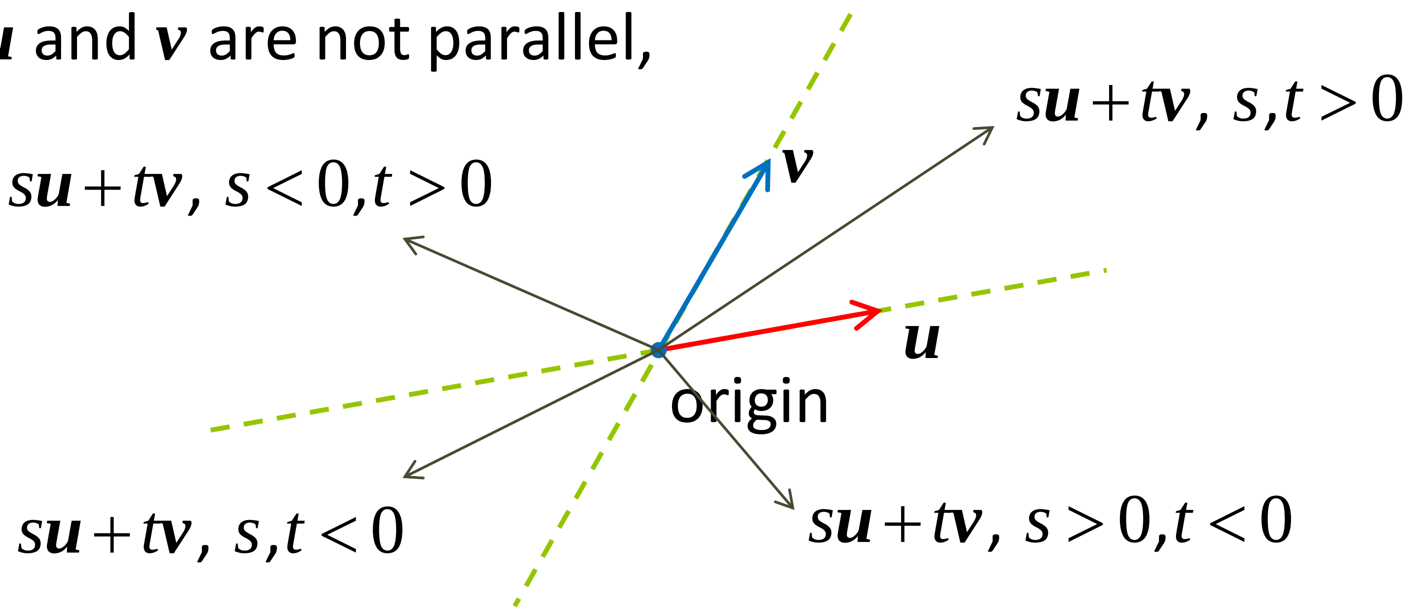
Discussion 3.2.14

Let $\boldsymbol{u},\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span{$\boldsymbol{u},\boldsymbol{v}$} is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$=\{s\mathbf{u}+t\mathbf{v} \mid s,t\in\mathbb{R}\}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel,



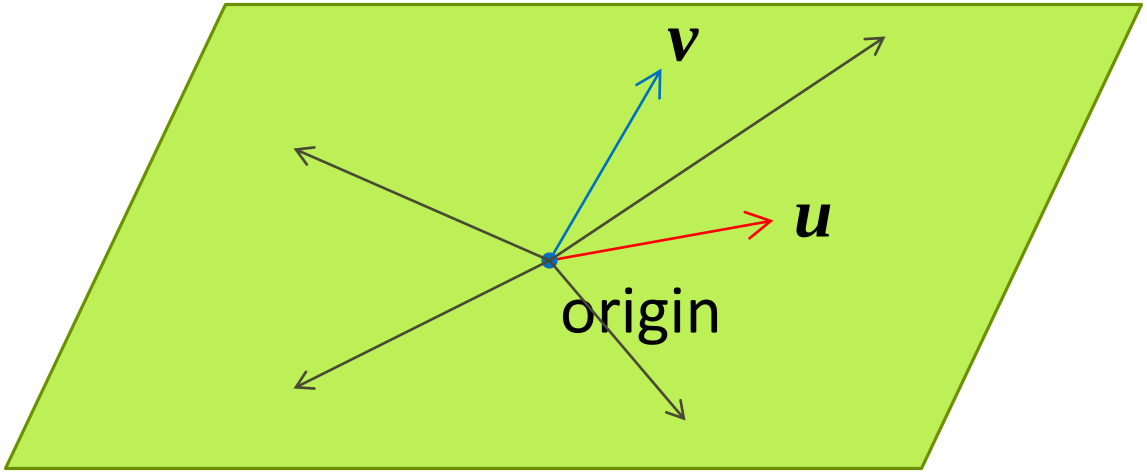
Discussion 3.2.14

Let $\boldsymbol{u}, \boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$= \{s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbb{R}\}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a plane containing the origin.



Discussion 3.2.14

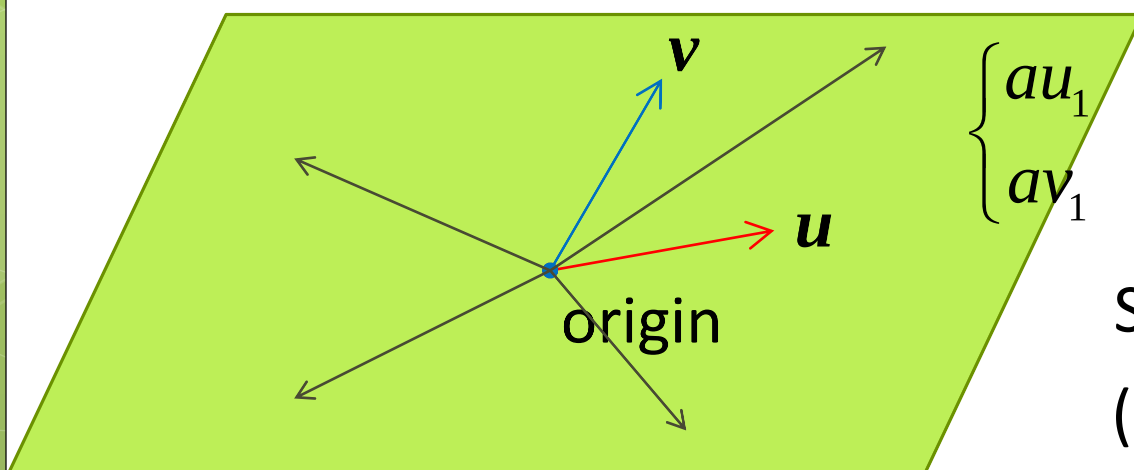
If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel,

(In $\mathbb{R}^2$) $\text{span}\{\boldsymbol{u},\boldsymbol{v}\} = \mathbb{R}^2$.

(In $\mathbb{R}^3$) $\text{span}\{\boldsymbol{u},\boldsymbol{v}\} = \{s\boldsymbol{u} + t\boldsymbol{v} \mid s,t \in \mathbb{R}\}$ (explicit representation)

(implicit representation, i.e. equation of the plane?)

$\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ $\qquad ax + by + cz = 0$



$$\begin{cases} au_1 + bu_2 + cu_3 = 0 \\ av_1 + bv_2 + cv_3 = 0 \end{cases}$$

Solve for a, b, c

(non-trivial solutions)

Discussion 3.2.15

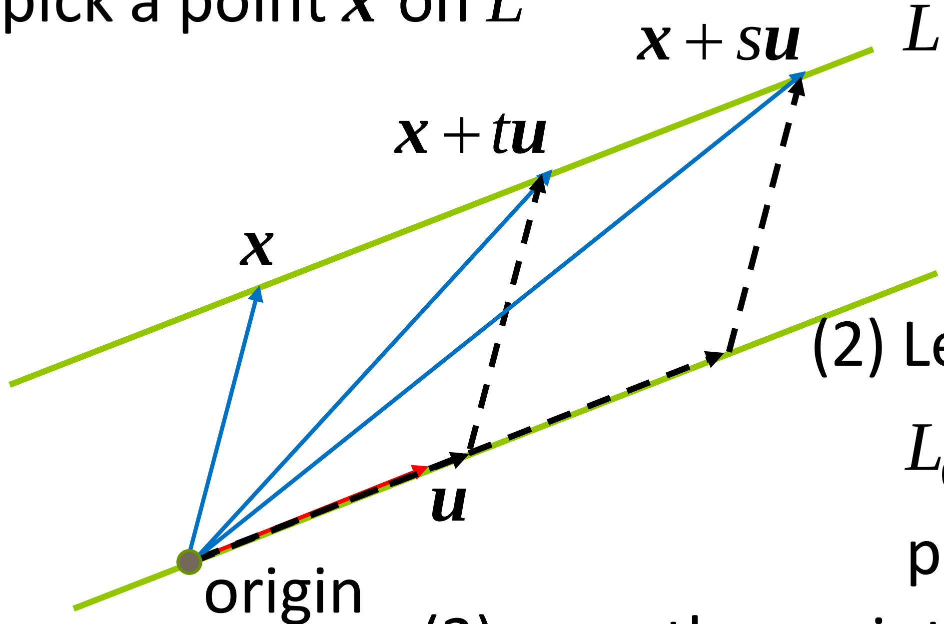
The previous discussion shows that linear spans (in $\mathbb{R}^2$ or $\mathbb{R}^3$) are, geometrically, lines or planes (in $\mathbb{R}^2$ or $\mathbb{R}^3$) that passes through (or contains) the origin.

What about the converse? Can all lines and planes in $\mathbb{R}^2$ or $\mathbb{R}^3$ be expressed in set notation using linear spans?

Discussion 3.2.15

(1) pick a point $\boldsymbol{x}$ on L

$$L = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}\}$$
$$= \{\boldsymbol{x} + t\boldsymbol{u} \mid t \in \mathbb{R}\}$$



$L_0 = \text{span}\{\boldsymbol{u}\}$

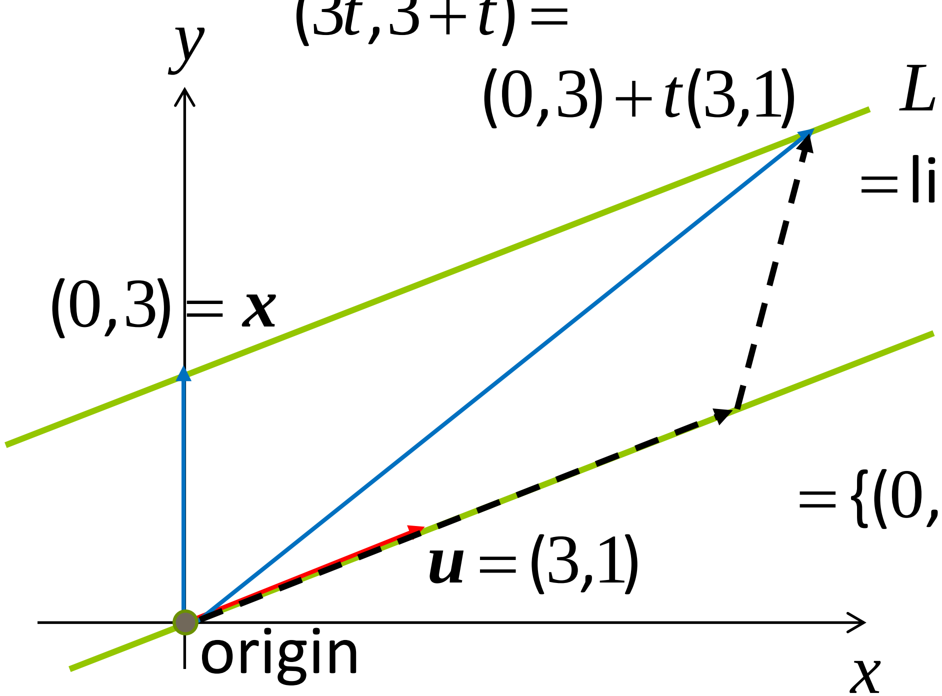
(2) Let $\boldsymbol{u} \in L_0$ $(\boldsymbol{u} \neq \boldsymbol{0})$, where L_0 is parallel to L and passes through origin

(3) any other point on L can be expressed as $\boldsymbol{x} + \boldsymbol{w}$ for some $\boldsymbol{w} \in L_0$

$= \boldsymbol{x} +$ (a scalar multiple of $\boldsymbol{u}$)

$= \boldsymbol{x} + \boldsymbol{w}$ for some $\boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}$

Discussion 3.2.15



L $=$ line in $\mathbb{R}^2$ with equation

$$x - 3y = -9$$

$$= \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in L_0\}$$

$$= \{(0,3) + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{(3,1)\}\}$$

$$= \{(0,3) + t(3,1) \mid t \in \mathbb{R}\}$$

$$= \{(3t, 3+t) \mid t \in \mathbb{R}\}$$

$L_0 =$ line in $\mathbb{R}^2$ with equation

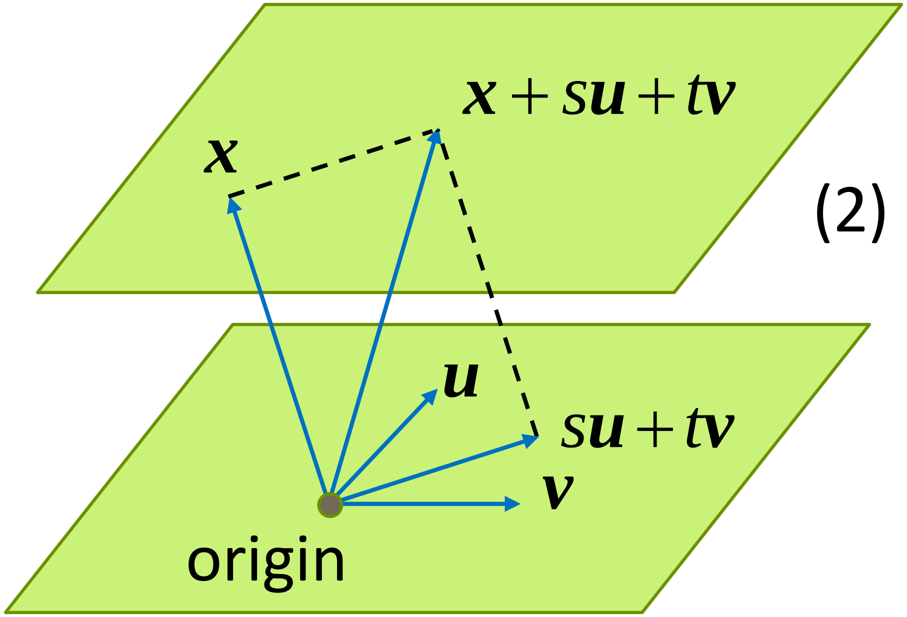
$$x - 3y = 0$$

$$= \text{span}\{(3,1)\}$$

Discussion 3.2.15

(1) pick a point $\boldsymbol{x}$ on P

$$P = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}\}$$
$$= \{\boldsymbol{x} + s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbb{R}\}$$



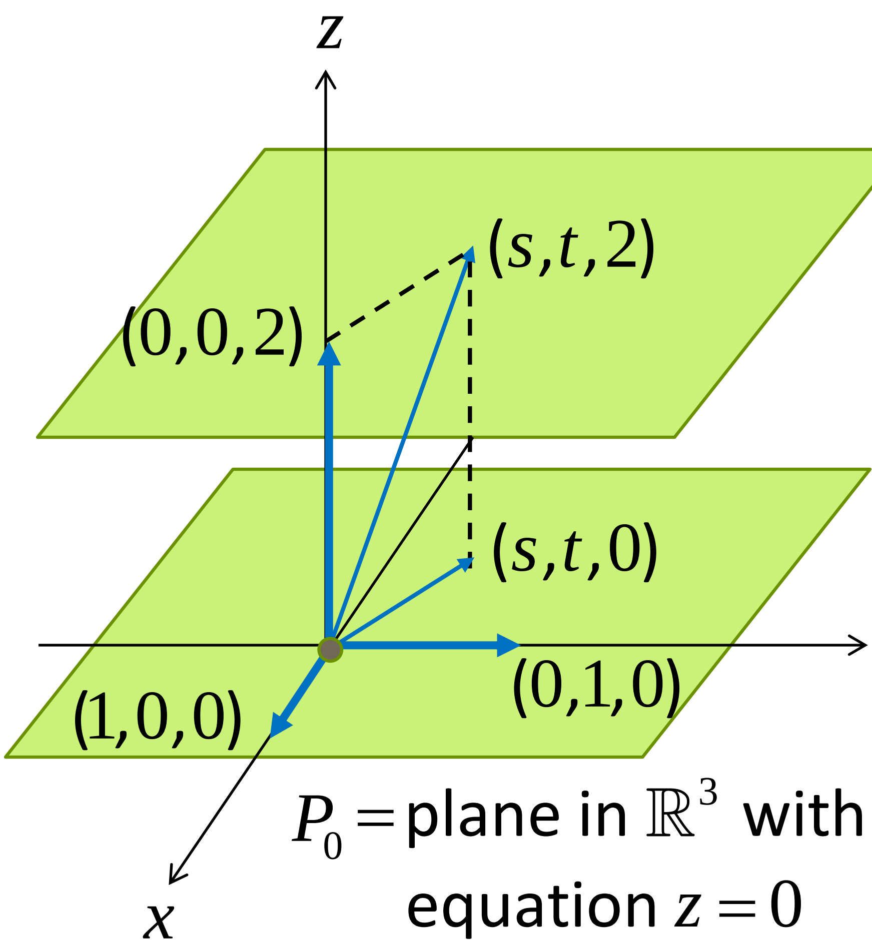
(2) Let $\boldsymbol{u}, \boldsymbol{v} \in P_0$ $(\boldsymbol{u}, \boldsymbol{v} \neq \mathbf{0})$, where $P_0 = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is parallel to P and contains the origin

$P_0 = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$

(3) any other point on P can be expressed as $\boldsymbol{x} + \boldsymbol{w}$ for some $\boldsymbol{w} \in P_0$

$= \boldsymbol{x} +$ (a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$)

Discussion 3.2.15



$P =$ plane in $\mathbb{R}^3$ with equation $z = 2$

$$= \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in P_0\}$$

$$= \{(0,0,2) + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{(1,0,0),(0,1,0)\}\}$$

$$= \{(0,0,2) + s(1,0,0) + t(0,1,0) \mid s,t \in \mathbb{R}\}$$

$$= \{(s,t,2) \mid s,t \in \mathbb{R}\}$$

$P_0 =$ plane in $\mathbb{R}^3$ with equation $z = 0$

$$= \text{span}\{(0,1,0),(1,0,0)\}$$

Discussion 3.2.15

Can we have lines and planes in $\mathbb{R}^n$ when $n \geq 4$?

1) Take $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$, where $\boldsymbol{u} \neq \mathbf{0}$. The set

$$L = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}\}$$ is a line in $\mathbb{R}^n$.

2) Take $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, where $\boldsymbol{u}, \boldsymbol{v} \neq \mathbf{0}$ and $\boldsymbol{u}$ is not a scalar multiple of $\boldsymbol{v}$. The set

$$P = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}\}$$

is a plane (or more generally, a $2-$plane) in $\mathbb{R}^n$.

Discussion 3.2.15

2) Take $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, where $\boldsymbol{u}, \boldsymbol{v} \neq \boldsymbol{0}$ and $\boldsymbol{u}$ is not a scalar multiple of $\boldsymbol{v}$. The set

$$P = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}\}$$

is a plane (or more generally, a $2-$plane) in $\mathbb{R}^n$.

3) Take $\boldsymbol{x}, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r \in \mathbb{R}^n$. The set

$$Q = \{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}\}$$

is called a $k-$plane in $\mathbb{R}^n$ where k is the "dimension" of span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$.

(Section 3.6)

End of Lecture 09

Lecture 10:

Subspaces (cont'd)

Linear independence (till Example 3.4.3)