Chapter 3. Integration

3.1 Indefinite Integral

Integration can be considered as the antithesis of differentiation, and they are subtly linked by the **Fundamental Theorem of Calculus**. We first introduce indefinite integration as an "inverse" of differentiation.

3.1.1 Antiderivatives

A (differentiable) function $F(x)$ is an *antiderivative* of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f.

The set of all antiderivatives of f is the *indefinite integral* of f with respect to x, denoted by

$$\int f(x)\,dx.$$

Terminology:

f : *integrand* of the integral x : *variable* of integration

3.1.2 Constant of Integration

Any constant function has zero derivative. Hence the antiderivatives of the zero function are all the constant functions.

If $F'(x) = f(x) = G'(x)$, then $G(x) = F(x) + C$, where C is some constant. So

$$\int f(x)dx = F(x) + C.$$

C here is called the *constant of integration* or an *arbitrary constant.* Thus,

$$\int f(x)\,dx = F(x) + C$$

means the same as

$$\frac{d}{dx}F(x) = f(x).$$

In words,

indefinite integral and antiderivative (of a function) *differ by an arbitrary constant.*

3.1.3 Integral formulas

1. $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1, \; n \text{ rational}$$

$$\int 1 \, dx = \int dx = x + C \qquad \text{(Special case, } n = 0\text{)}$$

2. $$\int \sin kx \, dx = -\frac{\cos kx}{k} + C$$

3. $$\int \cos kx \, dx = \frac{\sin kx}{k} + C$$

4. $\displaystyle\int \sec^2 x\, dx = \tan x + C$

5. $\displaystyle\int \csc^2 x\, dx = -\cot x + C$

6. $\displaystyle\int \sec x \tan x\, dx = \sec x + C$

7. $\displaystyle\int \csc x \cot x\, dx = -\csc x + C$

3.1.4 Rules for indefinite integration

1. $$\int k f(x)\, dx = k \int f(x)\, dx,$$

$k =$ constant (independent of x)

2. $$\int -f(x)\, dx = -\int f(x)\, dx$$

(Rule 1 with $k = -1$)

3. $\displaystyle\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

3.1.5 Example

Find the curve in the xy-plane which passes through the point $(9, 4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

Solution. The curve is given by $y = y(x)$, satisfying

$$\text{(i)}\ \ \frac{dy}{dx} = 3\sqrt{x} \qquad \text{and} \qquad \text{(ii)}\ \ y(9) = 4.$$

Solving (i), we get

$$y = \int 3\sqrt{x}\,dx = 3\,\frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

By (ii), $\qquad 4 = (2)9^{3/2} + C = (2)27 + C,$

$C = 4 - 54 = -50.$

Hence $\qquad y = 2x^{3/2} - 50.$

3.2 Riemann Integrals

3.2.1 Area under a curve

Let $f = f(x)$ be a non-negative continuous function $f = f(x)$ on an interval $[a, b]$.

Partition $[a, b]$ into n consecutive sub-intervals $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ each of length $\Delta x = \dfrac{b-a}{n}$, where we set $a = x_0$, $b = x_n$, and $x_1, x_2, \cdots, x_{n-1}$ to be successive points between a and b with $x_k - x_{k-1} = \Delta x$.

Let c_k be any intermediate point in the sub-interval

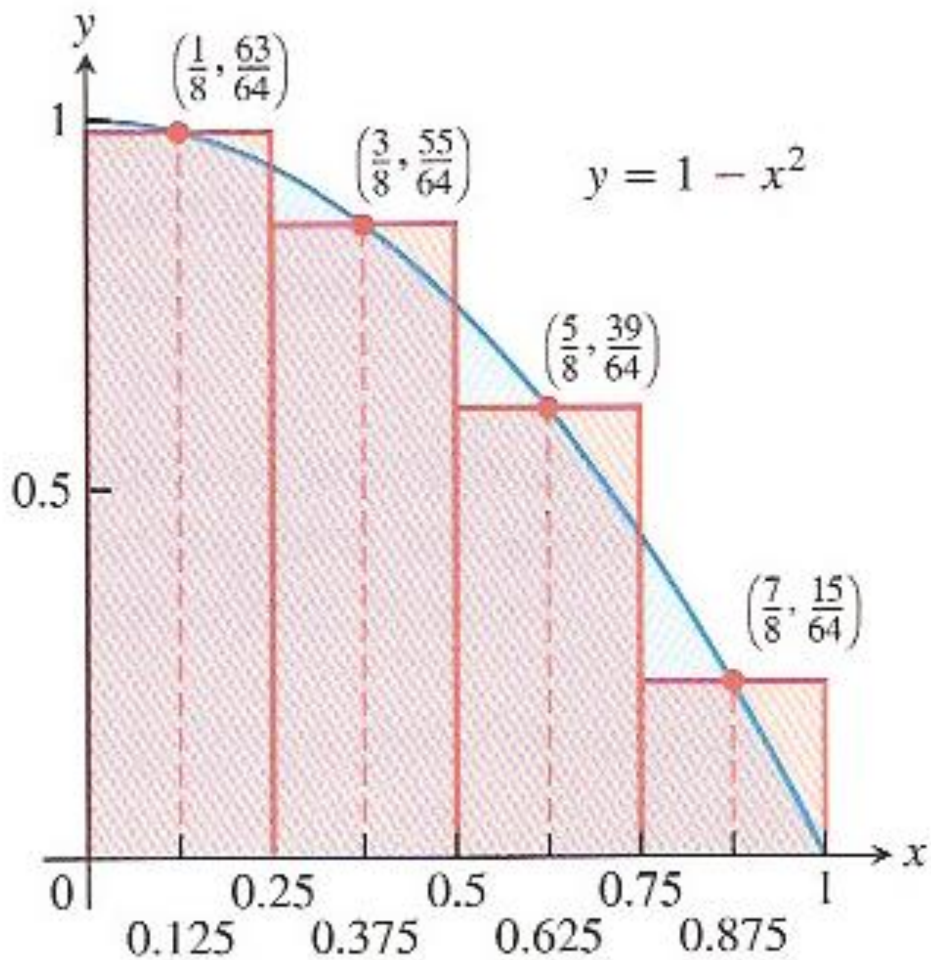
$[x_{k-1}, x_k]$.

Then the sum

$$S = \sum_{k=1}^{n} f(c_k)\Delta x$$

gives an approximate area under the curve of $y =$

$f(x)$ from $x = a$ to $x = b$.



y
x
$\left(\frac{1}{8}, \frac{63}{64}\right)$
$\left(\frac{3}{8}, \frac{55}{64}\right)$
$\left(\frac{5}{8}, \frac{39}{64}\right)$
$\left(\frac{7}{8}, \frac{15}{64}\right)$
$y = 1 - x^2$
1
0.5
0
0.25
0.5
0.75
1
0.125
0.375
0.625
0.875

The *exact* area A under the curve of $y = f(x)$ is achieved by letting the partition of the interval $[a, b]$ tends to infinity:

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x.$$

3.2.2 Riemann Integral

Let us continue with the notation as in the previous section and denote the limit by I.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = I.$$

We call I the **Riemann integral** (or **definite

integral) of f over $[a, b]$ and we write

$$I = \int_a^b f(x)\, dx.$$

3.2.3 Terminology

$$\int_a^b f(x)dx$$

$[a, b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

x : variable of integration

$f(x)$: the integrand

x is a *dummy* variable, i.e.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \quad \text{etc.}$$

3.2.4 Rules of algebra for definite integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{(any constant } k\text{)}$

   $$\left(\text{In particular, } \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx\right)$$

4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

In particular, if $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

6. If f is continuous on the interval joining a, b and c, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

3.3 The Fundamental Theorem of Calculus

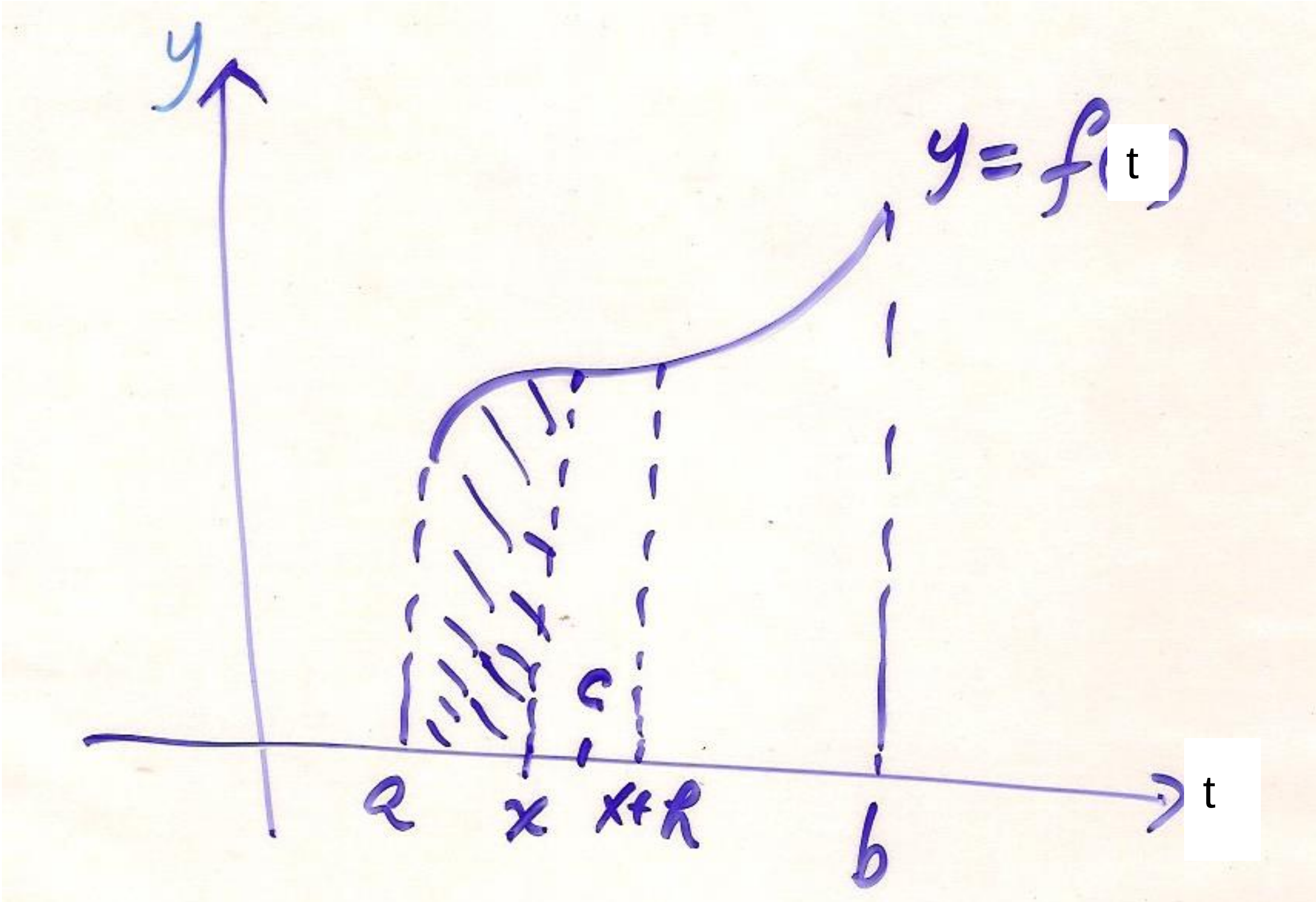
3.3.1 Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

has a derivative at every point of $[a, b]$, and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$



$$F(x) = \int_a^x f(t)\,dt$$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{f(c)\, h}{h}$$

c between x and $x+h$

$$= \lim_{h \to 0} f(c)$$

$$= f(x)$$

3.3.2 Examples

$$\frac{d}{dx}\int_{-\pi}^{x}\cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_{0}^{x}\frac{dt}{1+t^2} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\int_{1}^{x^2}\cos t\,dt = \left[\frac{d}{d(x^2)}\int_{1}^{x^2}\cos t\,dt\right]\frac{d(x^2)}{dx} = (\cos x^2)\,2x$$

$$= \underline{\underline{2x\cos(x^2)}}$$

3.3.3 Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$,

then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Set $G(x) = \int_a^x f(t)\,dt.$

By the Fundamental Theorem of Calculus, Part 1,

above,

$$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We also know that $F'(x) = f(x)$. Thus $G'(x) = F'(x)$ for $x \in [a, b]$.

Hence we have $F(x) = G(x) + c$ throughout $[a, b]$

for some constant c. Thus

$$\begin{aligned}
F(b) - F(a) &= G(b) + c - \big(G(a) + c\big) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt.
\end{aligned}$$

3.3.4 Examples

$$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = \sin \pi - \sin 0 = 0$$

$$\int_0^{2} t^2 \, dt = \frac{1}{3} t^3 \Big|_0^{2} = \frac{8}{3}$$

$$\int_{-2}^{2} (4 - u^2)\, du = \left[4u - \tfrac{1}{3}u^3\right]\Big|_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= \frac{32}{3}$$

3.4 Integration by substitution

To evaluate $\displaystyle\int f\big(g(x)\big)\,g'(x)\,dx$ where f and g' are continuous:

1. Set $u = g(x)$. Then $g'(x) = \dfrac{du}{dx}$, the given integral becomes $\displaystyle\int f(u)\,du$.

2. Integrate with respect to u.

3. Replace u by $g(x)$ in the result of step 2.

3.4.1 Examples

$$I = \int (x^2+2x-3)^2 (x+1)\, dx$$

Let $u = x^2 + 2x - 3$

$$du = (2x+2)\, dx$$

$$= 2(x+1)\, dx$$

$$I = \int u^2 \frac{1}{2} du = \frac{1}{2} \int u^2 du$$

$$= \frac{1}{6} u^3 + C$$

$$= \frac{1}{6} (x^2 + 2x - 3)^3 + C$$

$$I = \int \sin^4 x \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$I = \int u^4 \, du$$

$$= \frac{1}{5} u^5 + C$$

$$= \underline{\underline{\frac{1}{5} \sin^5 x + C}}$$

3.4.2 Substitution in definite integrals

The limits change accordingly:

$$\int_a^b f\left(g(x)\right) g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Note that in general we require $g' \geq 0$ or $g' \leq 0$ in

$[a, b]$.

3.4.3 Example

$$I = \int_0^{\pi/4} \tan x \sec^2 x \, dx$$

Let $u = \tan x$

$$x = 0 \implies u = 0$$

$$x = \frac{\pi}{4} \implies u = 1$$

$$du = \sec^2 x \, dx$$

$$I = \int_0^1 u\,du$$

$$= \frac{1}{2}u^2 \Big|_0^1$$

$$= \frac{1}{2}$$

3.5 Integration by parts

Integration by parts is a technique for evaluating integrals of the form

$$\int f(x)g(x)\ dx$$

in which f can be differentiated repeatedly and g can be integrated without difficulty.

Recall the product rule

$$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u\,dv + v\,du$$

or, equivalently,

$$u\,dv = d(uv) - v\,du.$$

Thus we have the **Integration-by-parts Formula**:

$$\int u\,dv = uv - \int v\,du$$

or,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Example

Evaluate $I = \int x \cos x \, dx$.

Solution.

$$
\begin{aligned}
I &= \int x \cos x \, dx = \int x \, d(\sin x) \\
&= x \sin x - \int \sin x \, dx \\
&= x \sin x + \cos x + C
\end{aligned}
$$

$$\int x^2 e^x \, dx$$

$$\int x^2 e^x dx$$
$$= \int x^2 d(e^x)$$
$$= x^2 e^x - 2\int x e^x dx$$
$$= x^2 e^x - 2\int x \, d(e^x)$$
$$= x^2 e^x - 2x e^x + 2\int e^x dx$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$

Example

Find $\int_1^e (\ln x)^2 \, dx$

$$\int \underbrace{(\ln x)^2}_{u} \underbrace{dx}_{dv}$$

$$= x(\ln x)^2 - \int x\,[2\ln x]\,\frac{1}{x}\,dx$$

$$= x(\ln x)^2 - 2\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv}$$

$$= x(\ln x)^2 - 2\left\{x\ln x - \int x\left(\frac{1}{x}\right)dx\right\}$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

$$\int_1^e (\ln x)^2 dx = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e$$

$$= e - 2e + 2e - 2$$

$$= \underline{\underline{e - 2}}$$

3.6 Area between two curves

If f_1 and f_2 are continuous functions with $f_1(x) \leq f_2(x)$ in the interval $a \leq x \leq b$, then the area of the region between the curves $y = f_1(x)$ and $y = f_2(x)$ from a to b is the integral of $f_2 - f_1$ from a to b,

i.e.

$$\text{Area} = \int_a^b \left[f_2(x) - f_1(x) \right] dx. \tag{1}$$

This is the basic formula.

If the curves only cross at one or both end points of $[a, b]$, we apply (1) once to find the area. If the curves cross within the interval $[a, b]$, we need to apply (1) more than once. Thus, to find the area of the region between two curves

(i) Sketch the curves and determine the crossing points.

(ii) Evaluate the area(s) using (1). **Or**, integrate $|f_2 - f_1|$ over $[a, b]$.

3.6.1 Example

Find area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

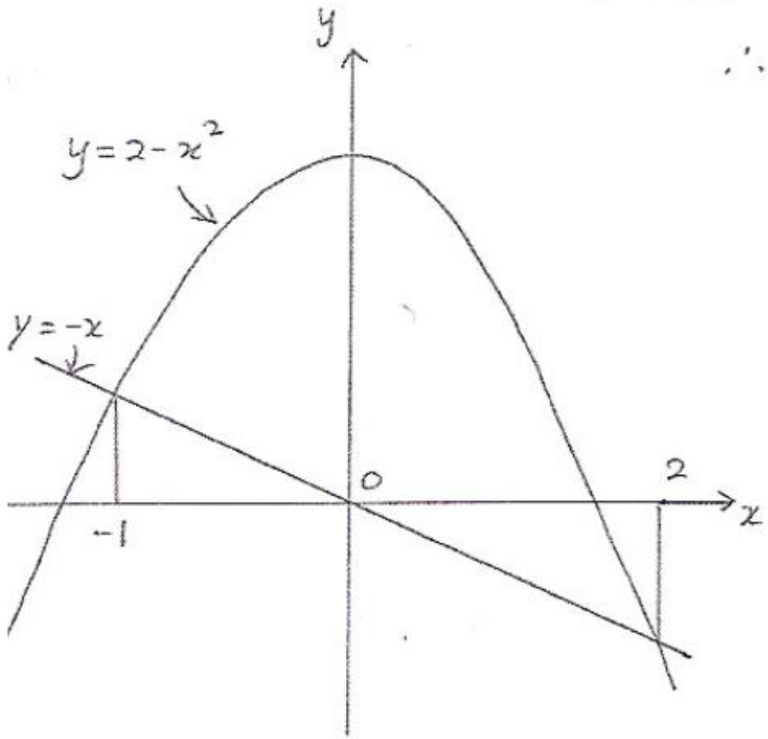
$y = 2 - x^2, \quad y = -x$

Points of intersection: Set $2 - x^2 = -x$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$\therefore x = -1, \quad x = 2.$$



$$\text{Area} = \int_{-1}^{2} \{(2-x^2) - (-x)\}\, dx$$

$$= \int_{-1}^{2} (2 - x^2 + x)\, dx$$

$$= \left[2x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2} \;//$$

Remark.

Sometimes we may like to view the curve as $x = g(y)$

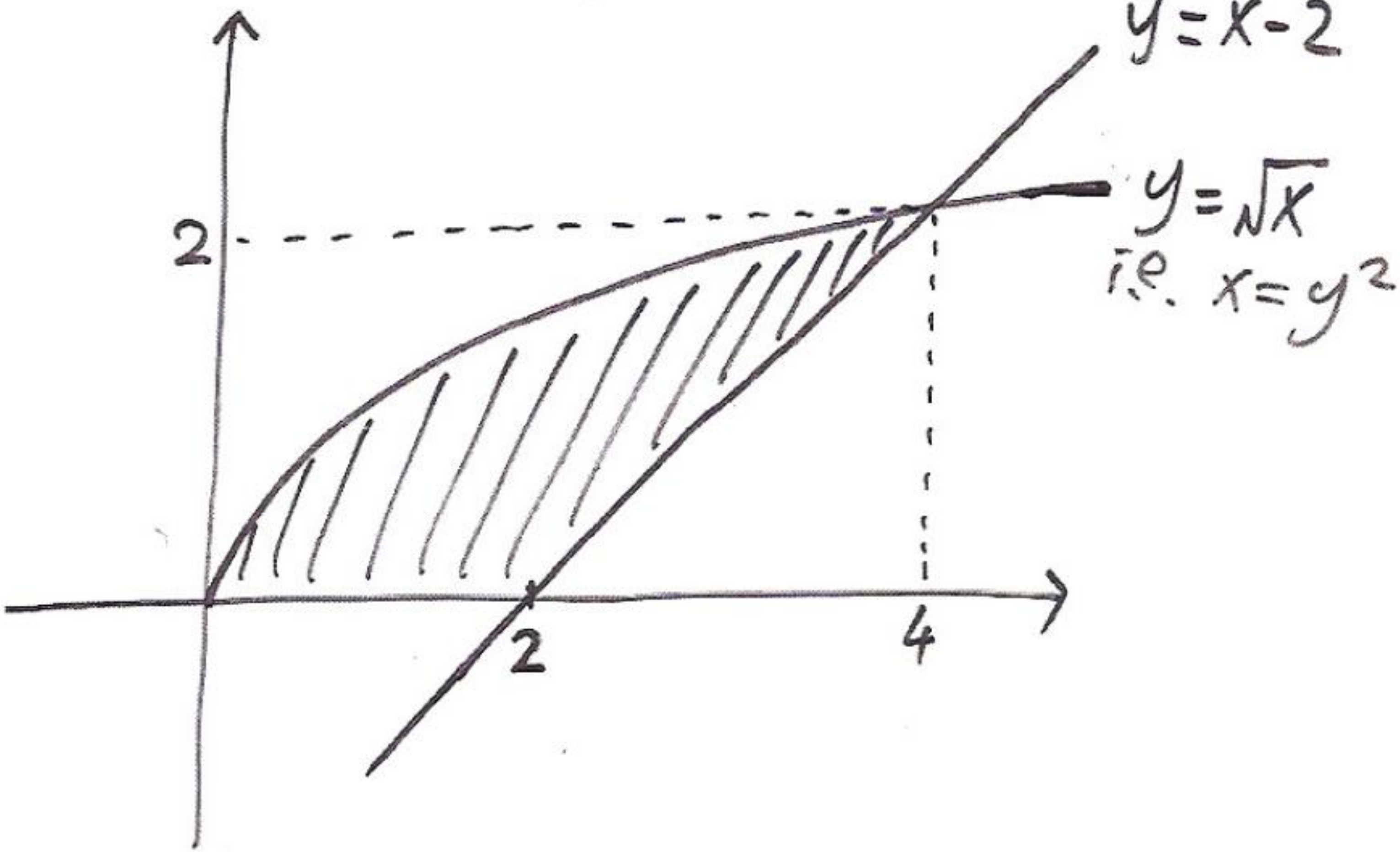
(instead of $y = f(x)$) when evaluating area.

The area will be $A = \displaystyle\int_c^d \left[g_2(y) - g_1(y)\right] dy.$

3.6.2 Example

Find area of the region in the first quadrant bounded by $y = \sqrt{x}$ and $y = x - 2$.

View the curve as $x = g(y)$



$$\text{Area} = \int_0^2 \left\{ (y+2) - (y^2) \right\} dy$$

$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right]_0^2$$

$$= 2 + 4 - \frac{8}{3}$$

$$= \underline{\underline{\frac{10}{3}}}$$

3.7 Volume of solids of revolution

In general, solids of revolutions are solids which are generated by revolving plane regions about x- or y-axis.

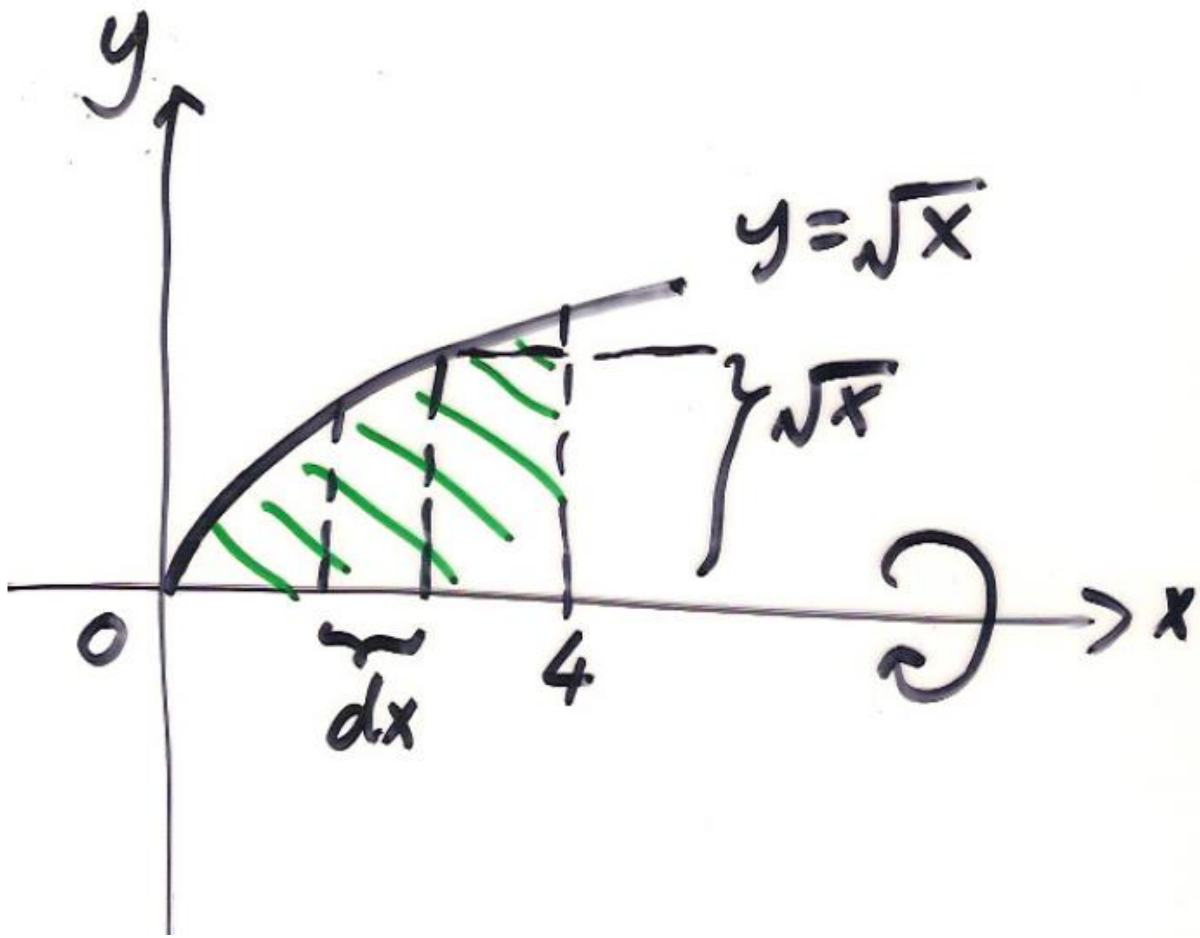
3.7.1 Revolution about x-axis

The volume of a solid generated by revolving *about the* x-axis the region between the graph of a continuous function $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is

$$\text{Volume} = \int_a^b \pi \left[f(x) \right]^2 dx.$$

3.7.2 Example

The region between $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis. Find the volume of the solid generated.


y
$y = \sqrt{x}$
$\sqrt{x}$
0
dx
4
x

$$\text{Vol} = \int_0^4 \pi(\sqrt{x})^2\,dx$$

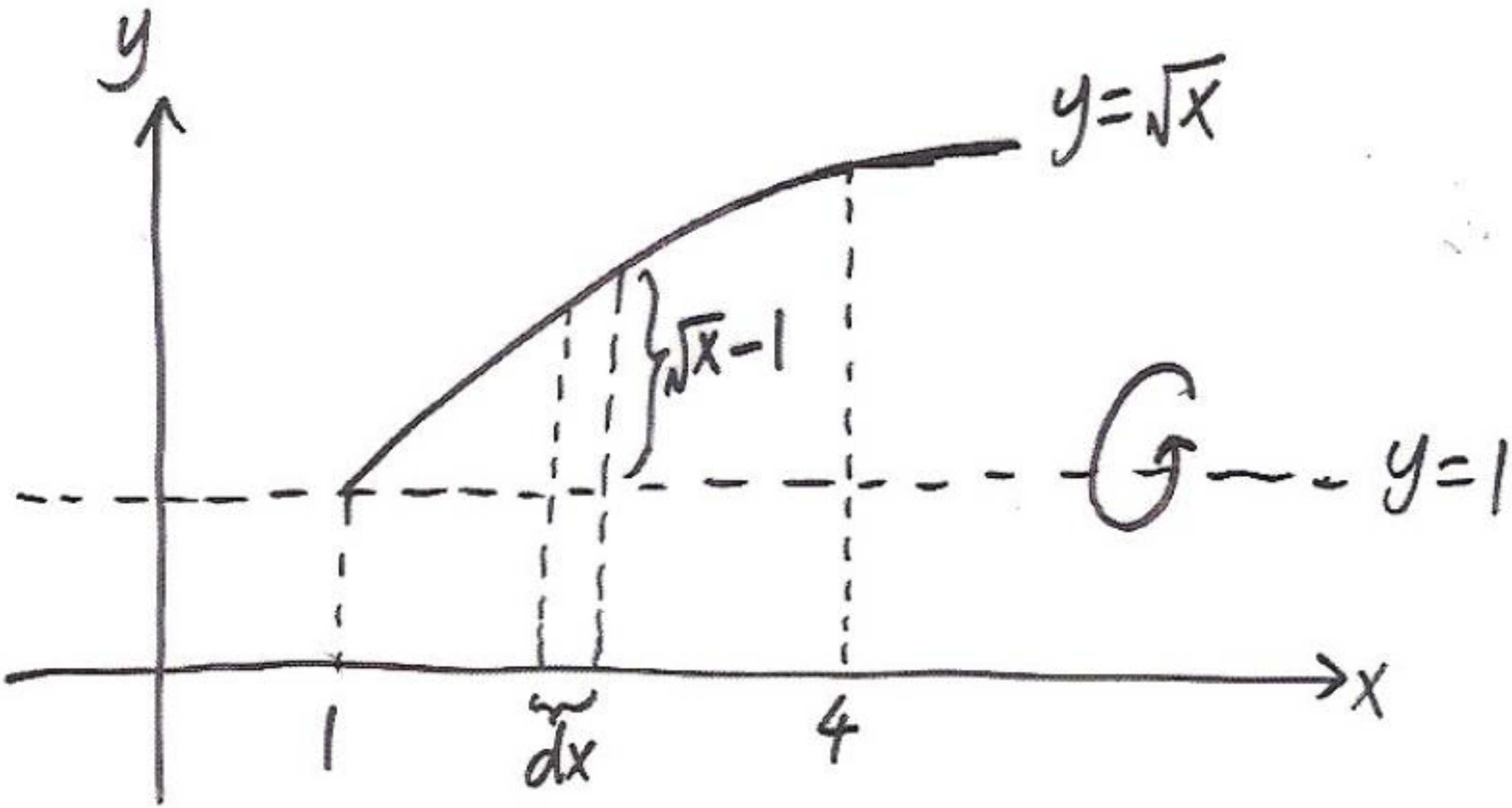
$$= \pi\int_0^4 x\,dx$$

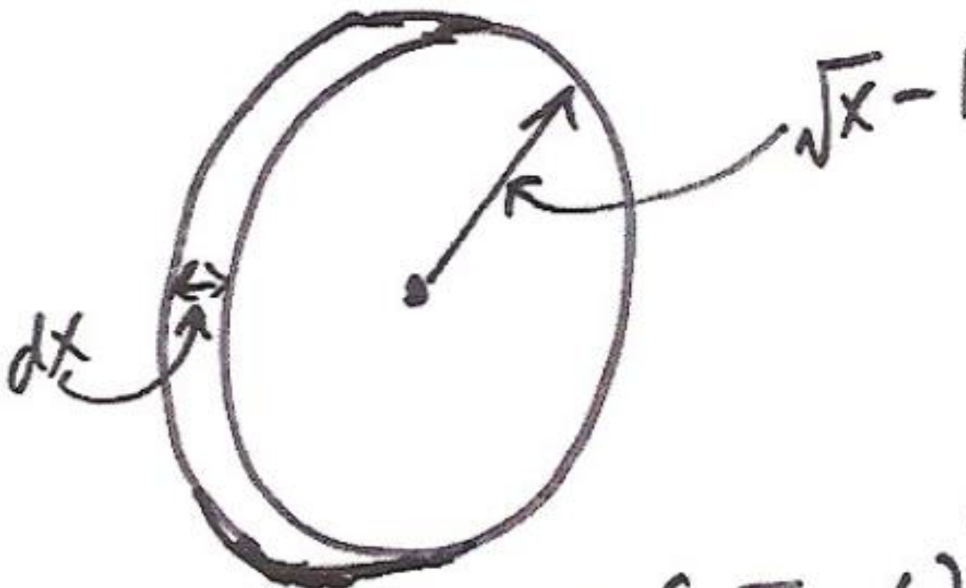
$$= \pi\,\tfrac{1}{2}x^2\Big|_0^4$$

$$= \underline{\underline{8\pi}}$$

3.7.3 Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.


y
y=√x
√x-1
y=1
x
1
dx
4



$$dV = \pi(\sqrt{x}-1)^2\,dx$$

$$\text{Vol.} = \int_1^4 \pi . (\sqrt{x} - 1)^2 \, dx$$

$$= \int_1^4 \pi \left(x - 2\sqrt{x} + 1 \right) dx$$

$$= \pi \left[\frac{1}{2} x^2 - \frac{4}{3} x^{\frac{3}{2}} + x \right]_1^4$$

$$= \frac{7}{6} \pi$$

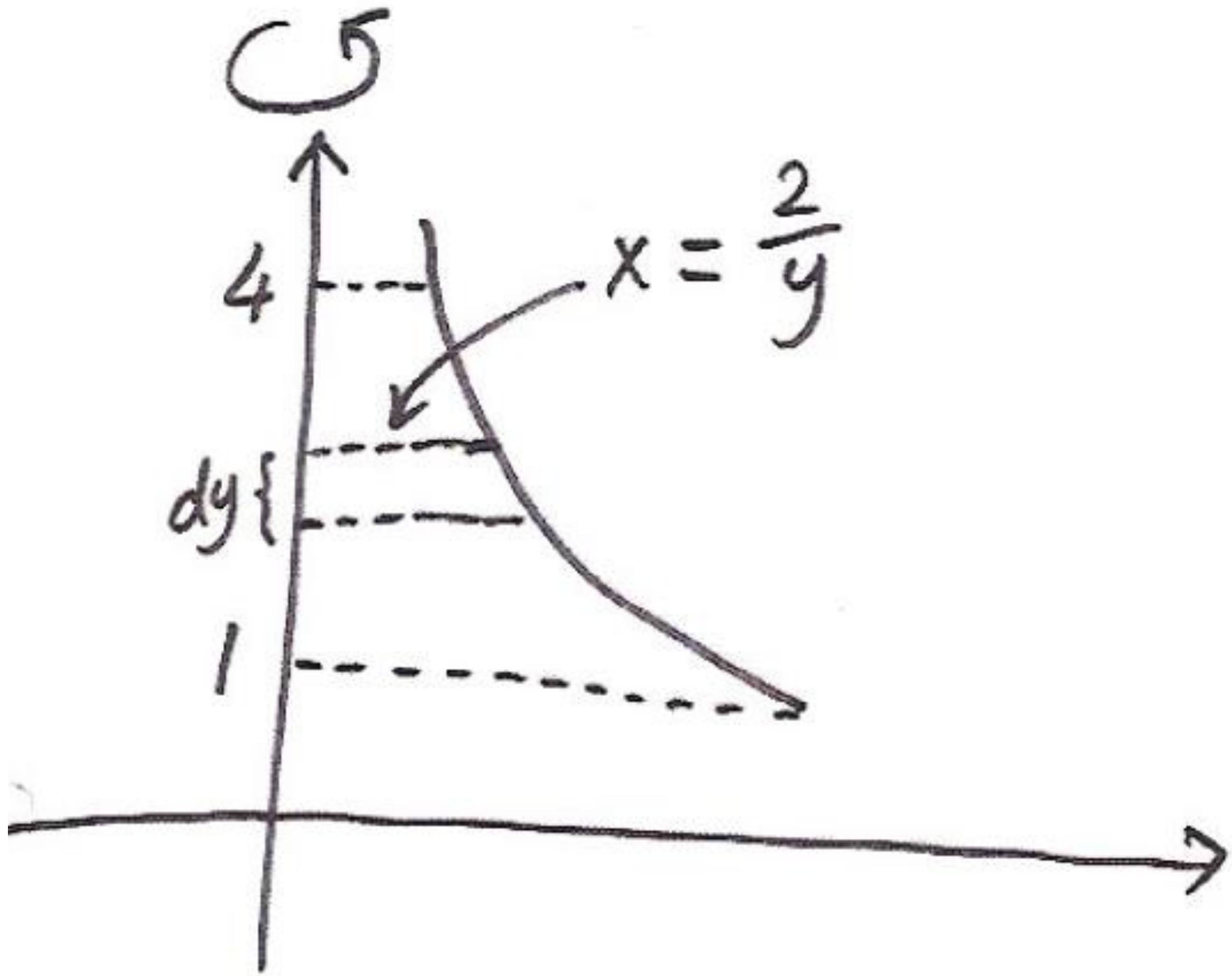
3.7.4 Revolution about y-axis

The volume of a solid generated by revolving about the y-axis the region between the graph of $x = g(y)$ and the y-axis from $y = c$ to $y = d$ is

$$\text{Volume} = \int_c^d \pi \left[g(y)\right]^2 dy.$$

3.7.5 Example

The region between the curve $x = \frac{2}{y}$, $1 \leq y \leq 4$ and the y-axis is revolved about the y-axis to generate a solid. Find its volume.


$x = \frac{2}{y}$
4
dy
1

$$
\begin{aligned}
\text{Vol.} &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy \\
&= 4\pi \int_1^4 \frac{1}{y^2}\, dy \\
&= 4\pi \left(\frac{3}{4}\right) \\
&= \underline{\underline{3\pi}}
\end{aligned}
$$

More Examples

Example

$$\int_0^1 \frac{1}{1+x^{\frac{1}{3}}}\,dx$$

Solution: Using direct substitution with $u = 1 + x^{\frac{1}{3}}$, and $du = \frac{1}{3}x^{\frac{-2}{3}}\,dx$, so $dx = 3x^{\frac{2}{3}}\,du = 3(u-1)^2\,du$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 2$. We have that:

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^{\frac{1}{3}}}\,dx &= \int_1^2 \frac{3(u-1)^2}{u}\,du = \int_1^2 (3u - 6 + \frac{3}{u})\,du \\
&= (\frac{3}{2}u^2 - 6u + 3\ln|u|)\,\Big|_1^2 \\
&= (6 - 12 + 3\ln 2) - (\frac{3}{2} - 6 + 3\ln 1) = -\frac{3}{2} + 3\ln 2
\end{aligned}$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^{\frac{1}{3}}}\,dx = -\frac{3}{2} + 3\ln 2.$$

Example

Evaluate

$$\int_{\frac{1}{e}}^{e} |\ln x| \, dx$$

$$\int_{1/e}^{e} |\ln x|\,dx = \int_{1/e}^{1} -\ln x\,dx + \int_{1}^{e} \ln x\,dx$$

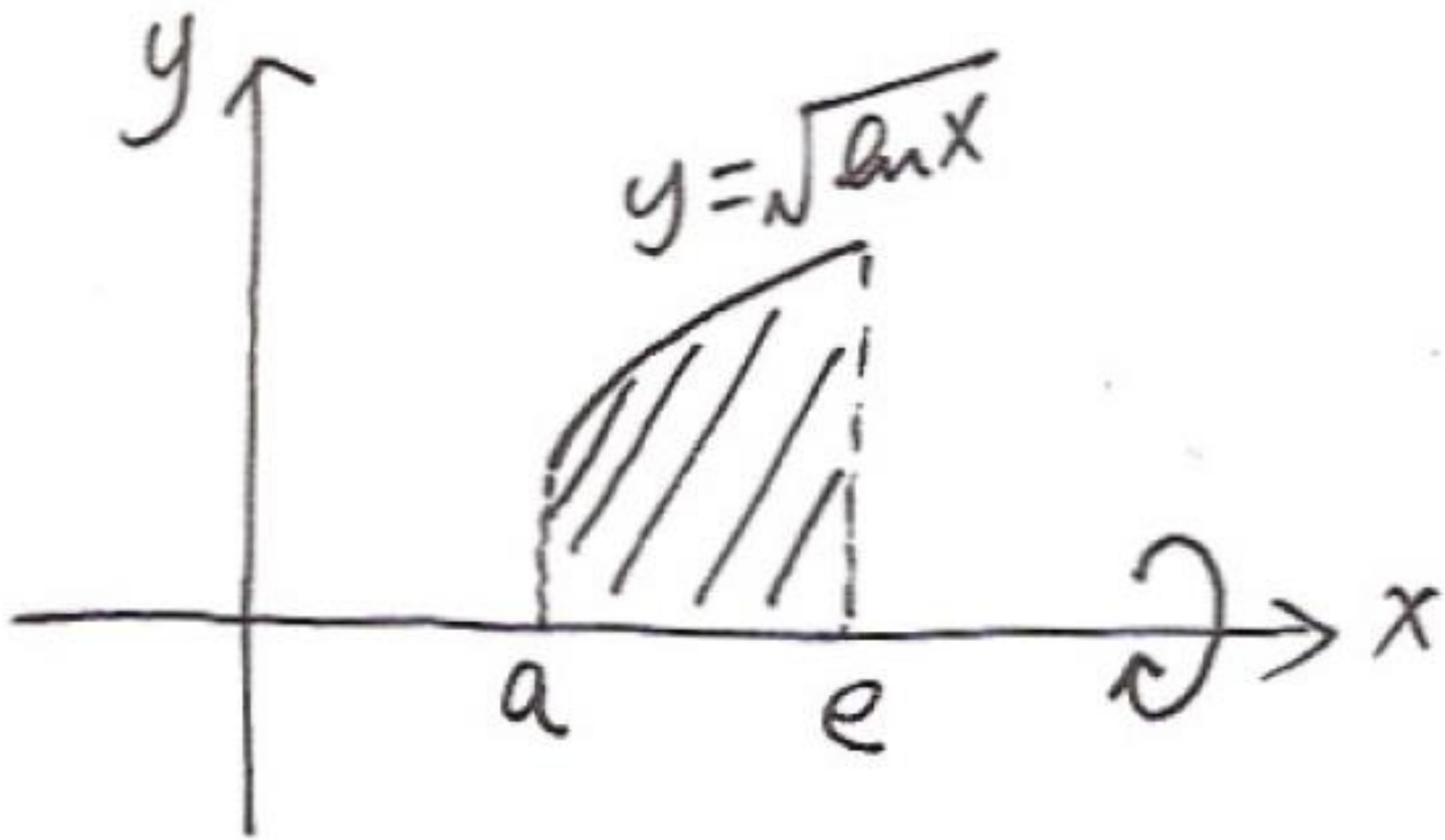
$$= -x\ln x\Big|_{1/e}^{1} + \int_{1/e}^{1} dx + x\ln x\Big|_{1}^{e} - \int_{1}^{e} dx$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = \underline{\underline{2(1-e^{-1})}}$$

Example

Let a be a positive constant and $1 < a < e$. Let R denote the finite region in the first quadrant bounded by the curve $y = \sqrt{\ln x}$, the x-axis, the line $x = a$ and the line $x = e$. Find the **exact value** of the volume of the solid formed by revolving R one complete round about the x-axis. Leave your answer in terms of a.



y
y = \sqrt{\ln x}
x
a
e

$$
\begin{aligned}
\text{Vol} &= \int_a^e \pi y^2 \, dx \\
&= \pi \int_a^e \ln x \, dx \\
&= \pi \left[x \ln x - x \right]_a^e \\
&= \underline{\underline{\pi (a - a \ln a)}}
\end{aligned}
$$