National University of Singapore
Department of Mathematics

Semester 1, 2018/2019 **MA1101R Linear Algebra I** **Homework 3**

Instruction

(a) This homework set consists of 3 pages and 8 questions.

(b) Do all the problems and submit on Oct. 15 (Monday) for SL1 group or on Oct. 16 (Tuesday) for SL2 group during lecture.

(c) Use A4 size writing paper. Write your full name, student number and tutorial group clearly on the first page of your answer scripts.

(d) Indicate the question numbers clearly (you do not need to copy the questions in your answer sheets).

(e) Show your steps of your working how the answers are derived, unless the questions state otherwise.

(f) Late Submission will not be accepted.

(g) **Warning**: If you are found to have copied answers from your friend(s), both you and your friend(s) will be penalized.

Problem Set (covering Lectures 9–14).

1. Let P represent a plane in $\mathbb{R}^3$ with equation $2x + y - 3z = 1$ and A, B, C represent three different lines given by the following set notation:

$$A = \{(at, bt, ct) \colon t \in \mathbb{R}\},\ B = \{(t+1, 2t-6, -t) \colon t \in \mathbb{R}\},\ C = \{(t, t, t) \colon t \in \mathbb{R}\}.$$

(a) Express the plane P in explicit set notation.

$P = \{((1 - s + 3t)/2, s, t) \colon s, t \in \mathbb{R}\}$.

(b) Write down the conditions on a, b, c so that the line A containing the origin with the direction (a, b, c) is parallel to the plane P, that is, the line has no intersection with P.

Show that (a, b, c) lies in the plane containing the origin and parallel to P.

An explicit form of the line A is given by $\{(at, bt, ct) \colon t \in \mathbb{R}\}$. If A has no intersection with P, then $2at + bt - 3ct = (2a + b - 3c)t \neq 1$ for all $t \in \mathbb{R}$. Thus the line A has no intersection with P if and only if $2a + b - 3c = 0$. Therefore, (a, b, c) lies in the plane $2x + y - 3z = 0$, which contains the origin and is parallel to P.

(c) Find all the points of intersection of the line B with the plane P.

We plug the explicit form of B into the plane P:

$$2(t + 1) + (2t - 6) - 3(-t) = 1.$$

And solve $t = \frac{5}{7}$ and then the intersection point is $(\frac{12}{7}, -\frac{32}{7}, -\frac{5}{7})$.

(d) Write down an explicit form of a plane P' containing the intersection point in Part (c) and the line C.

Since the line C contains the origin, P' contains the origin, i.e., $P' = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ for some $\boldsymbol{u}$ and $\boldsymbol{v}$. Because P' contains C, we may take $\boldsymbol{u} = (1, 1, 1)$. Since $(\frac{12}{7}, -\frac{32}{7}, -\frac{5}{7}) \in P'$ and P' contains the origin, P' also contain the vector $(\frac{12}{7}, -\frac{32}{7}, -\frac{5}{7})$. Hence $P' = \text{span}\{(1, 1, 1), (\frac{12}{7}, -\frac{32}{7}, -\frac{5}{7})\} = \{(s + \frac{12}{7}t, s - \frac{32}{7}t, s - \frac{5}{7}t) \colon s, t \in \mathbb{R}\}$.

2. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a set of linearly independent vectors in V. Assume that

$$\boldsymbol{v}_1 = a_{11}\boldsymbol{u}_1 + a_{12}\boldsymbol{u}_2 + a_{13}\boldsymbol{u}_3 \tag{1}$$
$$\boldsymbol{v}_2 = a_{21}\boldsymbol{u}_1 + a_{22}\boldsymbol{u}_2 + a_{23}\boldsymbol{u}_3 \tag{2}$$
$$\boldsymbol{v}_3 = a_{31}\boldsymbol{u}_1 + a_{32}\boldsymbol{u}_2 + a_{33}\boldsymbol{u}_3. \tag{3}$$

Denote

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly independent if and only if A is invertible.

Consider the vector equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$. Plugging in (1), (2) and (3), we have

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + a_{31}c_3 = 0 \\ a_{12}c_1 + a_{22}c_2 + a_{32}c_3 = 0 \\ a_{13}c_1 + a_{23}c_2 + a_{33}c_3 = 0 \end{cases}$$

The corresponding matrix equation is

$$A^T \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has only a trivial solution if and only if A^T is invertible, equivalently, A is invertible. Therefore, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly independent if and only if A is invertible.

3. Let

$$A = \begin{pmatrix} 1 & -3 & 2 & 0 \\ -4 & 2 & 1 & -1 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -5 & 1 & 3 & -3 \\ 5 & 7 & 1 & 19 \\ 2 & 2 & 0 & 6 \end{pmatrix}$$

(a) Is the column space of A equal to $\mathbb{R}^4$? Justify.

No, since $\det(A) = 0$, the columns of A is not a basis of $\mathbb{R}^4$.
Alternatively, we have the reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has a zero row. Then the column space of A is not the same with $\mathbb{R}^4$.

(b) Show that the column space of B is a subspace of the column space of A.

Write $A = (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4)$ and $B = (\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4)$, where $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are i-th columns of A and B respectively. We consider the augmented matrices

$$\begin{aligned}
&(\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4 \mid \boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3 \mid \boldsymbol{v}_4) \\
= &\left(\begin{array}{cccc|c|c|c|c} 1 & -3 & 2 & 0 & 1 & -1 & -1 & -1 \\ -4 & 2 & 1 & -1 & -5 & 1 & 3 & -3 \\ 2 & 1 & 0 & 3 & 5 & 7 & 1 & 19 \\ 1 & 1 & -1 & 1 & 2 & 2 & 0 & 6 \end{array}\right) \\
\to &\left(\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 1 & 2 & 2 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 & 3 & 1 & 7 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\boldsymbol{v}_1 &= 2\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 \\
\boldsymbol{v}_2 &= 2\boldsymbol{u}_1 + 3\boldsymbol{u}_2 + 3\boldsymbol{u}_3 \\
\boldsymbol{v}_3 &= \boldsymbol{u}_2 + \boldsymbol{u}_3 \\
\boldsymbol{v}_4 &= 6\boldsymbol{u}_1 + 7\boldsymbol{u}_2 + 7\boldsymbol{u}_3.
\end{aligned}$$

Hence, $\mathrm{span}\{\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4\} \subseteq \mathrm{span}\{\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4\}$.

(c) **(1 point)** Find bases S and T for the column spaces of A and B respectively. Are the two columns spaces the same?

Following the reduced row echelon form of A in Part(a), we have $\boldsymbol{u}_4 = -\boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ are linearly independent. Thus $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for the column space of A.

Consider the reduced row echelon form of B,

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -5 & 1 & 3 & -3 \\ 5 & 7 & 1 & 19 \\ 2 & 2 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\boldsymbol{v}_3 = \frac{1}{2}\boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2$ and $\boldsymbol{v}_4 = -\boldsymbol{v}_1 - 2\boldsymbol{v}_2$, so $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for the column space of B.

No, they have different dimensions. Thus,

$$\mathrm{span}\{\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4\} \subsetneq \mathrm{span}\{\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4\}.$$

(d) **(2 points)** Let $\boldsymbol{v} = (0, -4, 12, 4)$ and $\boldsymbol{u} = (2, 0, 3, 0)$. Find the coordinate $(\boldsymbol{v})_T$ and $(\boldsymbol{u})_S$. Based on your result in Part (b) and $(\boldsymbol{v})_T$, find $(\boldsymbol{v})_S$. Is it possible to find the coordinates $(\boldsymbol{u})_T$? Why?

To compute $(\boldsymbol{v})_T$, we need to consider the following augmented matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -5 & 1 & -4 \\ 5 & 7 & 12 \\ 2 & 2 & 4 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ and $(\boldsymbol{v})_T = (1, 1)$.
By Part (b),

$$\begin{aligned} \boldsymbol{v} =& \boldsymbol{v}_1 + \boldsymbol{v}_2 \\ =& (2\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3) + (2\boldsymbol{u}_1 + 3\boldsymbol{u}_2 + 3\boldsymbol{u}_3) \\ =& 4\boldsymbol{u}_1 + 4\boldsymbol{u}_2 + 4\boldsymbol{u}_3. \end{aligned}$$

So $(\boldsymbol{v})_S = (4, 4, 4)$.
To compute $(\boldsymbol{u})_S$, we need to consider the following augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ -4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + 2\boldsymbol{u}_3$ and $(\boldsymbol{u})_S = (1, 1, 2)$.
No. Let us consider the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ -5 & 1 & 0 \\ 5 & 7 & 3 \\ 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

Then the corrsponding linear system $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = \boldsymbol{u}$ is inconsistent. Hence $\boldsymbol{u}$ is not a linear combination of $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2\}$, equivalently, $\boldsymbol{u} \notin \text{span}\{\boldsymbol{v}_1,\ \boldsymbol{v}_2\}$. Therefore, $\boldsymbol{u}$ has no coordinate relative to T.

4. Discuss all the possibilities of the dimensions of the solution space V of the following homogeneous linear system.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & - & 5x_4 & = & 0 \\ -x_1 & & & + & 3x_3 & + & 5x_4 & = & 0 \\ x_1 & + & x_2 & + & ax_3 & + & bx_4 & = & 0 \end{array}\right.$$

(a) **(1 point)** Determine the values of a and b such that $\dim(V) = 2$ and find a basis of V;

Applying elementary row operations, we have

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 0 \\ -1 & 0 & 3 & 5 & 0 \\ 1 & 1 & a & b & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & -5 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & a+2 & b+5 & 0 \end{array}\right)$$

$\dim(V) = 2$ if and only if the solution space has two arbitrary parameters, which means two pivot columns. hence we have $a = -2$ and $b = -5$. In such case, $V = \{(3s+5t, -s, s, t) \colon s, t \in \mathbb{R}\}$ has a basis $\{(3, -1, 1, 0), (5, 0, 0, 1)\}$.

(b) Determine the values of a and b such that $\dim(V) = 1$ and find a basis of V;

Following the results in Part (a), a row echelon form of the augmented matrix has three pivot columns when $a \neq -2$ or $b \neq -5$. In such case, $\dim(V) = 1$. If $a \neq -2$, we have $V = \{((5 - \frac{3(5+b)}{2+a})t, \frac{5+b}{2+a}t, -\frac{5+b}{2+a}t, t) \colon t \in \mathbb{R}\}$ has a basis $\{(5 - \frac{3(5+b)}{2+a}, \frac{5+b}{2+a}, -\frac{5+b}{2+a}, 1)\}$. If $a = -2$ and $b \neq -5$, we have $V = \{(3t, -t, t, 0) \colon t \in \mathbb{R}\}$ has a basis $\{(3, -1, 1, 0)\}$.

(c) Is it possible that $\dim(V) = 0$ or 3? Justify.

No, $\dim(V) = 1$ or 2.

Regardless to the values of a and b, since the row echelon form of the augmented matrix has at least pivot columns, we need at most two arbitrary parameters, that is, $\dim(V)$ is at most 2. Hence $\dim(V) \neq 3$.

Recall that $\dim(V) = 0$ if and only if V is zero space, that is, the homogeneous linear system has only the trivial solution. However, the homogeneous linear system has more unknowns than equations, which has infinitely many solutions.

5. Find the condition on x, y and z such that

$$\operatorname{span}\{(2, 1, 1), (1, -1, 1), (x, y, z)\} = \mathbb{R}^3.$$

Write

$$A = \begin{pmatrix} 2 & 1 & x \\ 1 & -1 & y \\ 1 & 1 & z \end{pmatrix}.$$

By Theorem 3.6.7, $\{(2, 1, 1), (1, -1, 1), (x, y, z)\}$ spans $\mathbb{R}^3$ if and only if $\{(2, 1, 1), (1, -1, 1), (x, y, z)\}$ is a basis, that is, the columns of A form a basis for $\mathbb{R}^3$. By Theorem 3.6.11, the columns of A form a basis for $\mathbb{R}^3$ if and only if $\det(A) \neq 0$. Since $\det(A) = 2x - y - 3z$, $\{(2, 1, 1), (1, -1, 1), (x, y, z)\}$ spans $\mathbb{R}^3$ if and only if $2x - y - 3z \neq 0$.

6. (a) Let

$$S = \{(-3, 2, 4, 1), \qquad (0, 1, 5, -4), \qquad (2, -1, -1, 5)\}.$$

Extend S to be a basis for $\mathbb{R}^4$.

Form a matrix using the vectors in S as rows:

$$\begin{pmatrix} -3 & 2 & 4 & 1 \\ 0 & 1 & 5 & -4 \\ 2 & -1 & -1 & 5 \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So the third column is a non-pivot column. We may choose $(0, 0, 1, 0)$ and then $S \cup \{(0, 0, 1, 0)\}$ is a basis of $\mathbb{R}^4$.

(b) Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. Show that S is part of some basis of $\mathbb{R}^n$; that is, S may be extended to a basis of $\mathbb{R}^n$. (Hint: Let $T = \{e_1, e_2, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Show that there exists a subset S' of $S \cup T$ such that $S \subset S' \subset S \cup T$ is a basis of $\mathbb{R}^n$.)

We form a matrix A using the vectors in S as rows and then A is a $k \times n$ matrix. Consider a row-echelon form of A. Since S are linearly independent, we have k pivot columns and $n-k$ non-pivot columns. Suppose that the i_1-th, i_2-th,..., and i_{n-k}-th columns are all non-pivot columns of a row-echelon form of A. Define $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k, e_{i_1}, e_{i_2}, \ldots, e_{i_{n-k}}\}$. Form a matrix B using the vectors in S' as rows. Then every column is a pivot column and the reduced row-echelon form is I_n. By Theorem 3.6.11, the rows of B form a basis of $\mathbb{R}^n$, which is a desired basis.

(c) Let V be a vector space and T be a basis of V where $|T| = n$. Suppose that $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$ is a set of linearly independent vectors in V. Show that S is part of some basis of V. (Hint: Consider the coordinates of S relative to a basis of V.)

Let T be a basis of V and $(\boldsymbol{u}_i)_T$ be the coordinates of $\boldsymbol{u}_i$ relative to T. Since S are linearly independent, $\{(\boldsymbol{u}_1)_T, (\boldsymbol{u}_2)_T, \ldots, (\boldsymbol{u}_k)_T\}$ are linearly independent. By Part (a), $\{(\boldsymbol{u}_1)_T, (\boldsymbol{u}_2)_T, \ldots, (\boldsymbol{u}_k)_T\}$ may be extended to a basis of $\mathbb{R}^n$, say $\{(\boldsymbol{u}_1)_T, \ldots, (\boldsymbol{u}_k)_T, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}\}$. Then $(\boldsymbol{u}_1)_T, \ldots, (\boldsymbol{u}_k)_T, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}$ are linearly independent and $\text{span}\{(\boldsymbol{u}_1)_T, \ldots, (\boldsymbol{u}_k)_T, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}\} = \mathbb{R}^n$. Let $\boldsymbol{v}_i$ be a vector in V such that $(\boldsymbol{v}_i)_T = \boldsymbol{w}_i$. By Theorem 3.5.11, $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-k}$ are linearly independent and

$$\text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-k}\} = V.$$

Therefore $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-k}\}$ is a basis of V containing S.

7. Let

$$S = \{(3,0,7,5),\ (6,5,5,6),\ (5,2,5,-4)\}$$
$$T = \{(2,1,3,3),\ (1,1,0,-1),\ (0,1,-1,2)\}.$$

(a) **(1 point)** Find the transition matrix from S to T.

We consider the following augmented matrix, whose columns correspond the vectors of T and S:

$$\left(\begin{array}{ccc|c|c|c} 2 & 1 & 0 & 3 & 6 & 5 \\ 1 & 1 & 1 & 0 & 5 & 2 \\ 3 & 0 & -1 & 7 & 5 & 5 \\ 3 & -1 & 2 & 5 & 6 & -4 \end{array}\right) \xrightarrow[\textit{elimination}]{\textit{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So

$$P = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 2 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

is the transition matrix from S to T.

(b) **(1 point)** Let $\boldsymbol{w}$ be a vector in $\mathbb{R}^4$ such that $(\boldsymbol{w})_T = (3, 1, 4)$. Find $(\boldsymbol{w})_S$.

Let P be the transition matrix from S to T. Recall that $[\boldsymbol{w}]_T = P[\boldsymbol{w}]_S$, so $[\boldsymbol{w}]_S = P^{-1}[\boldsymbol{w}]_T$. By

$$P^{-1} = \begin{pmatrix} \frac{7}{23} & -\frac{5}{23} & -\frac{4}{23} \\ \frac{5}{23} & \frac{3}{23} & \frac{7}{23} \\ -\frac{1}{23} & \frac{4}{23} & -\frac{6}{23} \end{pmatrix},$$

we have

$$P^{-1}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

So $(\boldsymbol{w})_S = (0, 2, -1)$.

8. Let V and W be subspaces of $\mathbb{R}^n$. Recall

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \colon \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\}.$$

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m\}$ be bases of V and W respectively. Denote

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r, \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m\}.$$

(a) Assume $V \cap W = \{\boldsymbol{0}\}$:

(i) **(2 point)** Show that S is linearly independent.

(ii) **(1 point)** Show that $\dim(V + W) = \dim(V) + \dim(W)$.

(b) **(1 point)** Show that if $\dim(V \cap W) \geq 1$ then S is linearly dependent. (We may assume that $V \cap W$ is a subspace without proving it.)

Part (a)(i). We show that S is linearly independent. Consider the vector equation

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r + b_1\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_m\boldsymbol{w}_m = \boldsymbol{0}. \tag{4}$$

Then

$$-(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r) = b_1\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_m\boldsymbol{w}_m.$$

Since the left hand side of this equation is in V and the right hand side is in W, we have

$$-(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r) = b_1\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_m\boldsymbol{w}_m \in V \cap W.$$

By $V \cap W = \{\boldsymbol{0}\}$,

$$-(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r) = b_1\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_m\boldsymbol{w}_m = \boldsymbol{0}.$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m\}$ are linearly independent, $a_1 = a_2 = \cdots = a_r = b_1 = b_2 = \cdots = b_m = 0$. Then Equation (4) has only trivial solution and S is linearly independent.

Part (a)(ii). Following Tutorial 5, we have $V + W = \text{span}(S)$. Then S is a basis of $V + W$. Therefore, $\dim(V + W) = |S| = r + m = \dim(V) + \dim(W)$.

Part (b). By $\dim(V \cap W) \geq 1$, $V \cap W$ contains a nonzero vector $\boldsymbol{u}$ as $\dim(V \cap W) = 0$ iff $V \cap W = \{\mathbf{0}\}$. By $\boldsymbol{u} \in V$, $\boldsymbol{u} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r$ where not all of a_is are zero; By $\boldsymbol{u} \in W$, $\boldsymbol{u} = b_1\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_m\boldsymbol{w}_m$ where not all of b_js are zero. Thus

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_r\boldsymbol{v}_r - b_1\boldsymbol{w}_1 - b_2\boldsymbol{w}_2 - \cdots - b_m\boldsymbol{w}_m = \boldsymbol{u} - \boldsymbol{u} = \mathbf{0}.$$

Since not all of a_is and b_js are zero, S are linear dependent.