NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 7**

1. Consider the set $V = \{(x, y, z) \mid ax + by + cz = 0\} \subseteq \mathbb{R}^3$.

   (a) Describe the set V geometrically. Is V a subspace of $\mathbb{R}^3$?

   (b) If V contains $\boldsymbol{v_1} = (1, 4, -6)$ and $\boldsymbol{v_2} = (0, 2, -4)$, use Gaussian Elimination to find a, b, c.

   (c) Is $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ a basis for V? Justify your answer.

   (d) Show that $T = \{\boldsymbol{u_1}, \boldsymbol{u_2}\}$, where $\boldsymbol{u_1} = (1, 1, 0)$, $\boldsymbol{u_2} = (1, 5, -8)$, is also a basis for V.

   (e) Find the transition from T to S.

   (f) Is it possible to compute $(\boldsymbol{v})_S$ for the vector $\boldsymbol{v} = (1, 1, 2)$? Justify your answer.

   (a) V represents a plane in $\mathbb{R}^3$ that contains the origin. Yes, V is a subspace of $\mathbb{R}^3$.

   (b) Substitute $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ into the equation $ax + by + cz = 0$:

$$\left(\begin{array}{ccc|c} 1 & 4 & -6 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

   A general solution is $a = -2t, b = 2t, c = t, t \in \mathbb{R}$. So we may choose $a = -2, b = 2, c = 1$. The equation of the plane is thus $-2x + 2y + z = 0$.

   (c) Since V is a subspace of $\mathbb{R}^3$ with dimension 2 and $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ are two vectors in V that are not multiples of each other, we can conclude that $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is a basis for V.

   (d) Again, note that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are not multiples of each other, meaning that they are linearly independent. For T to be a basis for V, it suffices to show that $\boldsymbol{u_1}, \boldsymbol{u_2} \in V$. Indeed both $(1, 1, 0)$ and $(1, 5, -8)$ satisfies the equation $-2x + 2y + z = 0$, thus T is a basis for V.

   (e)

$$\left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 4 & 2 & 1 & 5 \\ -6 & -4 & 0 & -8 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   So $\boldsymbol{u_1} = \boldsymbol{v_1} - \frac{3}{2}\boldsymbol{v_2}$ and $\boldsymbol{u_2} = \boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_2}$ and the transition matrix from T to S is $\begin{pmatrix} 1 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

   (f) No, since $\boldsymbol{v} = (1, 1, 2)$ does not belong to V (that is, $(1, 1, 2)$ does not satisfy the equation of the plane).

2. (a) Suppose $\boldsymbol{P}$ is the transition matrix from T to S, where $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, $T = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ are bases for $\mathbb{R}^2$. If

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{v_2} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \boldsymbol{P} = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix},$$

find $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$.

(b) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ where

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \boldsymbol{v_1} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(i) Verify that S and T are both bases for $\mathbb{R}^3$.

(ii) Find the transition matrix from T to S.

(c) Suppose $\boldsymbol{Q}$ is the transition matrix from S to T, where $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, $T = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ are bases for $\mathbb{R}^2$. If

$$\boldsymbol{v_1} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \boldsymbol{w_1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix},$$

find $\boldsymbol{v_2}$ and $\boldsymbol{w_2}$.

(a) Since $\boldsymbol{P}$ is the transition matrix from T to S, we have $(\boldsymbol{w_1})_S = (3, 1)$ and $(\boldsymbol{w_2})_S = (5, -2)$. So $\boldsymbol{w_1} = 3(1, 2) + 1(2, 3) = (5, 9)$ and $\boldsymbol{w_2} = 5(1, 2) - 2(2, 3) = (1, 4)$.

(b) (i)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 4 & 6 & 7 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 3 \neq 0.$$

Thus both S and T are bases for $\mathbb{R}^3$.

(ii) To find the transition matrix from T to S, we need to write each $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ as a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 0 & 1 \\ 1 & 2 & 3 & 6 & 1 & 0 \\ 1 & 2 & 4 & 7 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right).$$

So $(\boldsymbol{v_1})_S = (1, 1, 1)$, $(\boldsymbol{v_2})_S = (-1, 1, 0)$, $(\boldsymbol{v_3})_S = (1, -2, 1)$ and the required transition matrix is $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$.

(c) $(\boldsymbol{v_1})_T = (4, 2) \Rightarrow \boldsymbol{v_1} = 4\boldsymbol{w_1} + 2\boldsymbol{w_2}$. Thus

$$\begin{pmatrix} 2 \\ 6 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2\boldsymbol{w_2} \Rightarrow 2\boldsymbol{w_2} = \begin{pmatrix} -2 \\ -10 \end{pmatrix} \Rightarrow \boldsymbol{w_2} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}.$$

Similarly, $(\boldsymbol{v_2})_T = (1, 1) \Rightarrow \boldsymbol{v_2} = \boldsymbol{w_1} + \boldsymbol{w_2}$. Thus

$$\boldsymbol{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

3. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$ where

$$\boldsymbol{u_1} = (0,1,0,0), \;\; \boldsymbol{u_2} = (-1,0,2,-3), \;\; \boldsymbol{u_3} = (0,1,0,0)$$

$$\boldsymbol{u_4} = (1,1,-2,3), \;\; \boldsymbol{u_5} = (1,6,2,0), \;\; \boldsymbol{u_6} = (0,7,0,2).$$

(a) By finding a row-echelon form of $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 3 \\ 1 & 6 & 2 & 0 \\ 0 & 7 & 0 & 2 \end{pmatrix}$, find a basis for $V =$ span(S).

(b) Find another basis T for $V = \text{span}(S)$ such that T is a subset of S.

(a)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 3 \\ 1 & 6 & 2 & 0 \\ 0 & 7 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So a basis for $V = \text{span}(S)$ is $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}\}$.

(b) Consider the transpose of $\boldsymbol{A}$ and its reduced row-echelon form:

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 6 & 7 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the pivot columns are at the first, second, fifth and sixth columns, $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$ is a basis for $V = \text{span}(S)$. (**Remark:** Students should not take $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ which corresponds to rows 1 to 4 of the reduced row-echelon form of $\boldsymbol{A}$ in part (a), as this is not a linearly independent set.)

4. Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{pmatrix}.$$

(a) Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$. Which are the non pivot columns of $\boldsymbol{R}$? Write each of the non pivot columns of $\boldsymbol{R}$ as a linear combination of the pivot columns of $\boldsymbol{R}$.

(b) Which columns of $\boldsymbol{A}$ corresponds to the pivot columns of $\boldsymbol{R}$? Recall that these columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$. Write each of the remaining columns of $\boldsymbol{A}$ as a linear combination of these basis vectors.

(c) What do you observe when comparing the answers in (a) and (b)?

(a)

$$\boldsymbol{A} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{R}.$$

The non pivot columns of $\boldsymbol{R}$ are the second, fourth and fifth columns. Let $\boldsymbol{r_1}, \cdots, \boldsymbol{r_6}$ be the columns of $\boldsymbol{R}$, where $\boldsymbol{r_i}$ is the i-th column of $\boldsymbol{R}$. Thus the non pivot columns are $\boldsymbol{r_2}, \boldsymbol{r_4}, \boldsymbol{r_5}$ and the pivot columns are $\boldsymbol{r_1}, \boldsymbol{r_3}, \boldsymbol{r_6}$.

$$\begin{aligned} \boldsymbol{r_2} &= 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2\boldsymbol{r_1} + 0\boldsymbol{r_3} + 0\boldsymbol{r_6} \\ \boldsymbol{r_4} &= 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 5\boldsymbol{r_1} - \boldsymbol{r_3} + 0\boldsymbol{r_6} \\ \boldsymbol{r_5} &= -3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\boldsymbol{r_1} + 2\boldsymbol{r_3} + 0\boldsymbol{r_6} \end{aligned}$$

(b) Let $\boldsymbol{a_1}, \cdots, \boldsymbol{a_6}$ be the columns of $\boldsymbol{A}$. So the colmuns of $\boldsymbol{A}$ that corresponds to the pivot columns of $\boldsymbol{R}$ are $\boldsymbol{a_1}, \boldsymbol{a_3}, \boldsymbol{a_6}$.

$$\begin{aligned} \boldsymbol{a_2} &= \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2\boldsymbol{a_1} + 0\boldsymbol{a_3} + 0\boldsymbol{a_6} \\ \boldsymbol{a_4} &= \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = 5\boldsymbol{a_1} - \boldsymbol{a_3} + 0\boldsymbol{a_6} \\ \boldsymbol{a_5} &= \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = -3\boldsymbol{a_1} + 2\boldsymbol{a_3} + 0\boldsymbol{a_6} \end{aligned}$$

(c) We see that the linear relationship between the vectors $\boldsymbol{r_1}, \cdots, \boldsymbol{r_6}$ is the same as that between the vectors $\boldsymbol{a_1}, \cdots, \boldsymbol{a_6}$.

5. Prove Theorem 4.1.11 (from the textbook):

   Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Prove the following statements.

   (a) A given set of columns of $\boldsymbol{A}$ is linearly independent if and only if the set of corresponding columns of $\boldsymbol{B}$ is linearly independent.

   (b) A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the set of corresponding colmuns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{pmatrix}$ and $\boldsymbol{B}$ be $m \times n$ matrices where $\boldsymbol{a_i}$ is the ith column of $\boldsymbol{A}$. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent, i.e. there exists elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \ldots, \boldsymbol{E_k}$ such that

$$\boldsymbol{B} = \boldsymbol{E_k} \cdots \boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A}.$$

Define $\boldsymbol{P} = \boldsymbol{E_k} \cdots \boldsymbol{E_2}\boldsymbol{E_1}$. Then $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{P}\boldsymbol{a_1} & \boldsymbol{P}\boldsymbol{a_2} & \cdots & \boldsymbol{P}\boldsymbol{a_n} \end{pmatrix}$ where $\boldsymbol{P}\boldsymbol{a_i}$ is the ith column of $\boldsymbol{B}$. By Theorem 2.4.7, $\boldsymbol{P}$ is invertible.

Let $S_1 = \{\boldsymbol{a_{i_1}}, \boldsymbol{a_{i_2}}, \ldots, \boldsymbol{a_{i_r}}\}$ be a set of columns of $\boldsymbol{A}$. Note that $S_2 = \{\boldsymbol{P}\boldsymbol{a_{i_1}}, \boldsymbol{P}\boldsymbol{a_{i_2}}, \ldots, \boldsymbol{P}\boldsymbol{a_{i_r}}\}$ is the set of corresponding columns of $\boldsymbol{B}$.

(a) Since $\boldsymbol{P}$ is invertible, by Question 3.30 (Chapter 3 Problem 30), S_1 is linearly independent if and only if S_2 is linearly independent.

(b) Suppose S_1 is a basis for the column space of $\boldsymbol{A}$. We want to show that S_2 is a basis for the column space of $\boldsymbol{B}$:

  (i) By (a), S_2 is linearly independent.

  (ii) It is obvious that $\text{span}(S_2) \subseteq$ the column space of $\boldsymbol{B}$.

  Take any $\boldsymbol{u} \in$ the column space of $\boldsymbol{B}$, i.e. for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$,

  $$\boldsymbol{u} = c_1\boldsymbol{P}\boldsymbol{a_1} + c_2\boldsymbol{P}\boldsymbol{a_2} + \cdots + c_n\boldsymbol{P}\boldsymbol{a_n}.$$

  Since $\text{span}(S_1) =$ the column space of $\boldsymbol{A}$,

  $$\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n} \in \text{span}(S_1) = \text{span}\{\boldsymbol{a_{i_1}}, \boldsymbol{a_{i_2}}, \ldots, \boldsymbol{a_{i_r}}\}$$

  and hence

  $$\boldsymbol{P}\boldsymbol{a_1}, \boldsymbol{P}\boldsymbol{a_2}, \ldots, \boldsymbol{P}\boldsymbol{a_n} \in \text{span}\{\boldsymbol{P}\boldsymbol{a_{i_1}}, \boldsymbol{P}\boldsymbol{a_{i_2}}, \ldots, \boldsymbol{P}\boldsymbol{a_{i_r}}\} = \text{span}(S_2).$$

  By Theorem 3.2.9.2, $\boldsymbol{u} \in \text{span}(S_2)$. So the column space of $\boldsymbol{B} \subseteq \text{span}(S_2)$.

  We have shown that $\text{span}(S_2) =$ the column space of $\boldsymbol{B}$.

By (i) and (ii), S_2 is a basis for the column space of $\boldsymbol{B}$.

Similarly, follow the arguments above by replacing $\boldsymbol{a_i}$ by $\boldsymbol{P}\boldsymbol{a_i}$ and $\boldsymbol{P}$ by $\boldsymbol{P}^{-1}$. We conclude that if S_2 is a basis for the column space of $\boldsymbol{B}$, then S_1 is a basis for the column space of $\boldsymbol{A}$.