ST2334 (2018/2019 Semester 2) Solutions to Questions in Tutorial 9

Question 1
X = time between two successive arrivals. $X \sim Exp(1)$.
(a) $E(X) = 1/\lambda = 1$.
(b) $\sigma = 1/\lambda = 1$.
(c) $\Pr(X \le 4) = 1 - e^{-1(4)} = 0.9817$.
$\Pr(2 \le X \le 5) = 1 - e^{-5} - (1 - e^{-2}) = 0.1286$.

Question 2
X = time until failure for the fan. $X \sim Exp(1/25000)$
(a) $\Pr(X > 20000) = e^{-20000/25000} = 0.4493$.
$\Pr(20000 \le X \le 30000) = (1 - e^{-30000/25000}) - 0.4493 = 0.1481$.
(b) $\sigma = 1/\lambda = 25000$. Therefore $\Pr(X - \mu > 2\sigma) = \Pr(X - 25000 > 2(125000)) =$
$\Pr(X > 75000) = e^{-75000/25000} = 0.0498$.

Question 3
$X =$ length of time to fail, in years. $E(X) = 2$. $X \sim Exp(1/2)$
(a) $V(X) = [E(X)]^2 = [2]^2 = 4$
(b) $\Pr(X < 1) = 1 - e^{-(1/2)(1)} = 0.3935$
$Y =$ number of electrical switch that fail during the first year.
$Y \sim Binomial(n = 100, p = 0.3935)$. Hence $E(Y) = np = 39.35$, $V(Y) =$
$np(1-p) = 23.87$. $\Pr(Y \le 30) = \Pr(Y \le 30.5) = \Pr\left(Z \le \frac{30.5 - 39.35}{\sqrt{23.87}}\right) =$
$\Pr(Z < -1.81) = 0.0351$. [Exact probability: $\Pr(Y \le 30) = 0.03347$.]

Question 4
$\Pr(\mu - 3\sigma < X < \mu + 3\sigma) = \Pr(-3 < Z < 3) = 1 - 2\Pr(Z > 3) = 1 - 2(0.00135) =$
0.9973.
[Compare with $\Pr(\mu - 3\sigma < X < \mu + 3\sigma) \ge 8/9$ using Chebyshev's Inequality]

Question 5
$X =$ amount of the soft drink. $X \sim N(200, 15^2)$
(a) $\Pr(X > 224) = \Pr\left(\frac{X - 200}{15} > \frac{224 - 200}{15}\right) = \Pr(Z > 1.60) = 0.0548$
(b) $\Pr(191 < X < 209) = \Pr(-0.60 < Z < 0.60) = 1 - 2(0.2743) = 0.4514$
(c) $\Pr(X > 230) = \Pr(Z > 2.00) = 0.02275 = p$
$Y =$ number of cups that overflow. $Y \sim Binomial\ (1000, 0.02275)$
$E(Y) = np = 1000(0.02275) = 22.75 = 23$
(d) $\Pr(Z < z_{0.25}) = 0.25; z_{0.25} = -0.6745$. $Z = \frac{X - \mu}{\sigma}$ or $X = \mu + \sigma Z$
$x_{0.25} = \mu + z_{0.25}\sigma = 200 + (-0.6745)(15) = 189.88\ ml$

Question 6
$X =$ commute time from home to office. $X \sim N(24, 3.8^2)$
(a) $\Pr(X > 30) = \Pr\left(\frac{X - 24}{3.8} > \frac{30 - 24}{3.8}\right) = \Pr(Z > 1.58) = 0.0571$
(b) $\Pr(X > 15) = \Pr(Z > -2.37) = 1 - 0.00889 = 0.99111 = 99.11\%$
(c) $Y =$ number of trips that take at least half an hour. $Y \sim Binomial\ (3, 0.0571)$.
$\Pr(Y = 2) = \binom{3}{2}(0.0571)^2(1 - 0.0571)^1 = 0.00922$

Question 7

Y = number of head in 400 tosses of a coin. $Y \sim Binomial\ (400, 0.5)$

$E(Y) = np = 400(0.5) = 200.\ V(Y) = np(1-p) = 400(0.5)(0.5) = 100$

$Y \sim Normal\ (200, 100)$ approximately

(a) $\Pr(185 \leq Y \leq 210) = \Pr(184.5 < Y < 210.5) = \Pr(-1.55 < Z < 1.05)$
$= 1 - 0.0606 - 0.1469 = 0.7925$

(b) $\Pr(Y = 205) = \Pr(204.5 < Y < 205.5) = \Pr(0.45 < Z < 0.55)$
$= 0.3261 - 0.2912 = 0.0352$

(c) $\Pr(Y < 176$ or $Y > 227) = \Pr(Y < 175.5) + \Pr(Y > 227.5)$
$= \Pr(Z < -2.45) + \Pr(Z > 2.75) = 0.00714 + 0.00298 = 0.01012$

Question 8

Y = number of drunk driver. $Y \sim Binomial\ (400, 0.1)$

$E(Y) = np = 400(0.1) = 40.\ V(Y) = np(1-p) = 400(0.1)(0.9) = 36.$

$Y \sim N(40, 6^2)$ approximately

(a) $\Pr(Y < 32) = \Pr(Y < 31.5) = \Pr\left(\frac{Y-40}{\sqrt{36}} < \frac{31.5-40}{\sqrt{36}}\right) \approx \Pr(Z < -1.42) = 0.0778$

(b) $\Pr(Y > 49) = \Pr(Y > 49.5) \approx \Pr(Z > 1.58) = 0.0571$

(c) $\Pr(35 \leq Y < 47) = \Pr(34.5 < Y < 46.5) \approx \Pr(-0.92 < Z < 1.08)$
$= 1 - 0.1788 - 0.1401 = 0.6811$

Question 9

Y = number of defective parts. $Y \sim Binomial\ (100, 0.05)$

$E(Y) = np = 100(0.05) = 5.\ V(Y) = np(1-p) = 100(0.05)(0.95) = 4.75$

$Y \dot\sim N(5, 4.75)$

(a) $\Pr(Y > 2) = \Pr(Y > 2.5) = \Pr\left(\frac{Y-5}{\sqrt{4.75}} < \frac{2.5-5}{\sqrt{4.75}}\right) \approx \Pr(Z > -1.15) = 1 - 0.1251 =$
0.8749

(b) $\Pr(Y > 10) = \Pr(Y > 10.5) = \Pr(Z > 2.52) = 0.00587$

Question 10

(a) $\mu = \sum x f_X(x) = 4(0.2) + 5(0.4) + 6(0.3) + 7(0.1) = 5.3$
$\sigma^2 = \sum (x-\mu)^2 f(x) = (4-5.3)^2(0.2) + (5-5.3)^2(0.4) + (6-5.3)^2(0.3) + (7-$
$5.3)^2(0.1) = 0.81$

(b) With $n = 36$, $\mu_{\bar{X}} = \mu = 5.3$; ${\sigma_{\bar{X}}}^2 = \frac{\sigma^2}{n} = \frac{0.81}{36} = 0.0225$

(c) Applying the Central Limit Theorem, $\bar{X}\ approx \sim N(5.3, 0.0255)$
$\Pr(\bar{X} < 5.5) = \Pr\left(Z < \frac{5.5-5.3}{\sqrt{0.0225}}\right) = \Pr(Z < 1.33) = 1 - 0.0918 = 0.9082$

Question 11

X = amount of benzene. $E(X) = \mu$ and $V(X) = 100^2$

(a) $n = 25$. By the CLT, $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$. $\Pr(\bar{X} > 7950 |\ \mu = 7950) = \Pr(\bar{X} > \mu) = 0.5$

(b) $X \sim N(\mu, 100^2)$. Hence $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$ approximately.
$\Pr(\bar{X} \geq 7960 |\ \mu = 7950) = \Pr\left(Z > \frac{7960-7950}{100/\sqrt{25}}\right) = \Pr(Z > 0.5) = 0.3085$
No, there is no strong evidence that the population mean exceeds the government limit as it is likely to see a sample mean is equal to or larger than 7960 if the population mean equals to the government limit 7950.