

1. (a) The characteristic equation is $(\lambda + 1)(\lambda - 3) = 0$; eigenvalues are -1 and 3 ; $\{(0, 1)^T\}$ is a basis for E_{-1} and $\{(1, 2)^T\}$ is a basis for E_3 .
- (b) The characteristic equation is $(\lambda - 2)^2 = 0$; the eigenvalue is 2 ; $\{(1, 1)^T\}$ is a basis for E_2 .
- (c) The characteristic equation is $\lambda^2 - 4 = 0$; eigenvalues are -2 and 2 ; $\{(-2, 1)^T\}$ is a basis for E_{-2} and $\{(2, 1)^T\}$ is a basis for E_2 .
- (d) The characteristic equation is $\lambda^2 = 0$; the eigenvalue is 0 ; $\{(1, 0), (0, 1)^T\}$ is a basis for E_0 .
- (e) The characteristic equation is $\lambda(\lambda - 2)^2 = 0$; eigenvalues are 0 and 2 ; $\{(-1, 1, 0)^T\}$ is a basis for E_0 and $\{(1, 1, 0)^T\}$ is a basis for E_2 .
- (f) The characteristic equation is $(\lambda - 2)(\lambda^2 - 9) = 0$; eigenvalues are $2, -3$ and 3 ; $\{(0, 0, 1)^T\}$ is a basis for E_2 , $\{(-1, 3, 0)^T\}$ is a basis for E_{-3} and $\{(1, 3, 0)^T\}$ is a basis for E_3 .
- (g) The characteristic equation is $(\lambda - 1)^3 = 0$; the eigenvalue is 1 ; $\{(0, 0, 1)^T\}$ is a basis for E_1 .
- (h) The characteristic equation is $(\lambda + 1)(\lambda - 1)^2 = 0$; eigenvalues are -1 and 1 ; $\{(-1, -1, 1)^T\}$ is a basis for E_{-1} and $\{(1, 2, 0)^T, (1, 0, 2)^T\}$ is a basis for E_1 .
- (i) The characteristic equation is $(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0$; eigenvalues are $1, 2, 3$ and 4 ; $\{(0, 0, 0, 1)^T\}$ is a basis for E_1 , $\{(0, 0, 1, 1)^T\}$ is a basis for E_2 , $\{(0, 2, 4, 3)^T\}$ is a basis for E_3 and $\{(3, 9, 12, 8)^T\}$ is a basis for E_4 .
- (j) The characteristic equation is $\lambda^4 - 2\lambda^2 + 1 = 0$; eigenvalues are -1 and 1 ; $\{(-1, 0, 1, 0)^T, (0, -1, 0, 1)^T\}$ is a basis for E_{-1} and $\{(1, 0, 1, 0)^T, (0, 1, 0, 1)^T\}$ is a basis for E_1 .

$$2. (a) \det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 + (-a - d)\lambda + (ad - bc)$$

Hence $m = -a - d = -\text{tr}(\mathbf{A})$ and $n = \det(\mathbf{A})$.

(b) Direct verification shows that $\mathbf{A}^2 + m\mathbf{A} + n\mathbf{I} = \mathbf{0}$.

3. (a) Let \mathbf{x} be an eigenvector of \mathbf{A} associated with λ , i.e. $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. We prove that $\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$ by induction on n .

It is given that $\mathbf{A}^1\mathbf{x} = \lambda^1\mathbf{x}$. Assume that $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$. Then

$$\mathbf{A}^{k+1}\mathbf{x} = \mathbf{A}(\mathbf{A}^k\mathbf{x}) = \mathbf{A}(\lambda^k\mathbf{x}) = \lambda^k\mathbf{A}\mathbf{x} = \lambda^k\lambda\mathbf{x} = \lambda^{k+1}\mathbf{x}.$$

By mathematical induction, $\mathbf{A}^n \mathbf{x} = \lambda^n \mathbf{x}$ and hence λ^n is an eigenvalue of \mathbf{A} for all positive integer n .

(b) Let \mathbf{x} be an eigenvector of \mathbf{A} associated with λ . Then

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{A}^{-1}\mathbf{x} \Rightarrow \frac{1}{\lambda}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}.$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of \mathbf{A}^{-1} .

(c) λ is an eigenvalue of $\mathbf{A} \Rightarrow \det(\lambda\mathbf{I} - \mathbf{A}) = 0$
 $\Rightarrow \det((\lambda\mathbf{I} - \mathbf{A})^T) = 0$
 $\Rightarrow \det(\lambda\mathbf{I} - \mathbf{A}^T) = 0$
 $\Rightarrow \lambda$ is an eigenvalue of \mathbf{A}^T .

4. (a) Let \mathbf{x} be an eigenvector of \mathbf{A} associated with λ , i.e. $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and \mathbf{x} is a nonzero vector. Then

$$\mathbf{A}^2 = \mathbf{A} \Rightarrow \mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{x} \Rightarrow \lambda^2\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda(\lambda - 1)\mathbf{x} = \mathbf{0}$$

Since \mathbf{x} is nonzero, $\lambda = 0$ or 1 .

(b) Since \mathbf{A} has 2 distinct eigenvalues, it is diagonalizable. Let $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

be an invertible matrix such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} ad & -ab \\ cd & -cb \end{pmatrix} \text{ where } ad - bc \neq 0.$$

We can simplify the expression to $\mathbf{A} = \begin{pmatrix} r & s \\ t & 1 - r \end{pmatrix}$ where $st = r(1 - r)$.

5. (a) Let \mathbf{x} be a nonzero eigenvector of \mathbf{A} associated with λ , i.e. $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

$$\mathbf{A}^2 = \mathbf{0} \Rightarrow \mathbf{A}^2\mathbf{x} = \mathbf{0}\mathbf{x} \Rightarrow \mathbf{A}(\lambda\mathbf{x}) = \mathbf{0} \Rightarrow \lambda^2\mathbf{x} = \mathbf{0}$$

Since \mathbf{x} is nonzero, $\lambda = 0$.

(b) No. Suppose \mathbf{A} is diagonalizable. Then there exists invertible \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{0}$. Then $\mathbf{A} = \mathbf{P}\mathbf{0}\mathbf{P}^{-1} = \mathbf{0}$, a contradiction.

(c) Consider the vector equation

$$a\mathbf{u} + b\mathbf{A}\mathbf{u} = \mathbf{0}. \quad (*)$$

Pre-multiplying \mathbf{A} to both side of $(*)$, we have

$$\mathbf{A}(a\mathbf{u} + \mathbf{A}\mathbf{u}) = \mathbf{A}\mathbf{0} \Rightarrow a\mathbf{A}\mathbf{u} = \mathbf{0}. \quad (\because \mathbf{A}^2 = \mathbf{0}.)$$

As $\mathbf{A}\mathbf{u} \neq \mathbf{0}$, $a = 0$. Substituting $a = 0$ into (*), we have $b\mathbf{A}\mathbf{u} = \mathbf{0}$ and hence $b = 0$. Since (*) has only the trivial solution, \mathbf{u} and $\mathbf{A}\mathbf{u}$ are linearly independent.

(d) Let $\mathbf{P} = (\mathbf{u} \ \mathbf{A}\mathbf{u})$. By (c), \mathbf{P} is invertible. Since

$$\mathbf{A}\mathbf{P} = (\mathbf{A}\mathbf{u} \ \mathbf{A}^2\mathbf{u}) = (\mathbf{A}\mathbf{u} \ \mathbf{0})$$

and

$$\mathbf{P} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0\mathbf{u} + \mathbf{A}\mathbf{u} \ 0\mathbf{u} + 0\mathbf{A}\mathbf{u}) = (\mathbf{A}\mathbf{u} \ \mathbf{0}),$$

$$\mathbf{A}\mathbf{P} = \mathbf{P} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ which implies } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

6. (a) Since $\det(-\mathbf{I} - \mathbf{A}) = 0$, -1 is an eigenvalue of \mathbf{A} .

(b) $\{(1, 1, 0)^T, (0, 0, 1)^T\}$ is a basis for E_{-1} and hence $\dim(E_{-1}) = 2$.

(c) For example, $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

7. (a) Since $\det(2\mathbf{I} - \mathbf{A}) = 0$, 2 is an eigenvalue of \mathbf{A} .

(b) $\{(1, 2, 0)^T, (-3, 0, 1)^T\}$ is a basis for the eigenspace associated with 2 .

(c) Let E_2 be the eigenspace of \mathbf{A} associated with 2 and let E'_λ be the eigenspace of \mathbf{B} associated with λ .

Since E_2 and E'_λ are subspaces of \mathbb{R}^3 and have dimension 2 , they are two planes in \mathbb{R}^3 that contain the origin. So $E_2 \cap E'_\lambda$ is either a line through the origin or a plane containing the origin. In both cases, we can find a nonzero vector $\mathbf{u} \in E_2 \cap E'_\lambda$, i.e. $\mathbf{A}\mathbf{u} = 2\mathbf{u}$ and $\mathbf{B}\mathbf{u} = \lambda\mathbf{u}$, such that

$$(\mathbf{A} + \mathbf{B})\mathbf{u} = \mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u} = 2\mathbf{u} + \lambda\mathbf{u} = (2 + \lambda)\mathbf{u}.$$

So $2 + \lambda$ is an eigenvalue of $\mathbf{A} + \mathbf{B}$.

8. Note that for $i = 1, 2, \dots, n$, $\mathbf{A}^n\mathbf{u}_i = \mathbf{A}^{n-1}\mathbf{u}_{i+1} = \dots = \mathbf{A}^i\mathbf{u}_n = \mathbf{0}$.

Let $\mathbf{v} \in \mathbb{R}^n$ be an eigenvector of \mathbf{A} associated with eigenvalue λ , i.e. $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for \mathbb{R}^n ,

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_n\mathbf{u}_n$$

for some $c_1, c_2, \dots, c_n \in \mathbb{R}$. Then

$$\mathbf{A}^n\mathbf{v} = c_1\mathbf{A}^n\mathbf{u}_1 + c_2\mathbf{A}^n\mathbf{u}_2 + \dots + c_n\mathbf{A}^n\mathbf{u}_n = \mathbf{0}.$$

From the proof of ??(a), $\mathbf{A}^n \mathbf{v} = \lambda^n \mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, $\lambda = 0$. Hence we have shown that \mathbf{A} has only one eigenvalue 0.

As $\lambda = 0$, we get $\mathbf{A}\mathbf{v} = \mathbf{0}$. Then

$$\mathbf{0} = \mathbf{A}\mathbf{v} = c_1 \mathbf{A}\mathbf{u}_1 + c_2 \mathbf{A}\mathbf{u}_2 + \cdots + c_n \mathbf{A}\mathbf{u}_n = c_1 \mathbf{u}_2 + c_2 \mathbf{u}_3 + \cdots + c_{n-1} \mathbf{u}_n.$$

Since $\mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$ are linearly independent, $c_1 = 0, c_2 = 0, \dots, c_{n-1} = 0$, i.e. $\mathbf{v} = c_n \mathbf{u}_n$. Hence all eigenvectors of \mathbf{A} are scalar multiples of \mathbf{u}_n .

9. (a) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.
- (b) Not diagonalizable.
- (c) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.
- (d) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- (e) Not diagonalizable.
- (f) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
- (g) Not diagonalizable.
- (h) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- (i) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 9 \\ 0 & 1 & 4 & 12 \\ 1 & 1 & 3 & 8 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.
- (j) Diagonalizable. Let $\mathbf{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
10. (a) Eigenvalues are $-i$ and i .
- Let $\mathbf{P} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.
- (b) Eigenvalues are $2 - i$ and $2 + i$.
- Let $\mathbf{P} = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}$.

- (c) Eigenvalues are 0, $2 - i$ and $2 + i$.

$$\text{Let } \mathbf{P} = \begin{pmatrix} 1 & 1+3i & 1-3i \\ 0 & 5i & -5i \\ 0 & 5 & 5 \end{pmatrix}. \text{ Then } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2+i \end{pmatrix}.$$

11. (a) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(b) $\mathbf{A}^{10} = \begin{pmatrix} 1 & 0 & 4^{10} - 1 \\ 0 & 4^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}$

(c) For example, let $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then

$$\mathbf{B}^2 = \mathbf{A}.$$

12. Let $\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then the matrix $\mathbf{P}\mathbf{D}\mathbf{P}^{-1} =$

$$\begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ has the required eigenvalues and eigenvectors.}$$

13. The matrix is diagonalizable if and only if $a \neq b$.

14. (a) The eigenvalues are 2, 0, 1 and -1 .

- (b) \mathbf{u}_1 is an eigenvector associated with 2.

\mathbf{u}_2 is an eigenvector associated with 0.

$\mathbf{u}_3 + \mathbf{u}_4$ is an eigenvector associated with 1.

$\mathbf{u}_3 - \mathbf{u}_4$ is an eigenvector associated with -1 .

- (c) Note that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_4, \mathbf{u}_3 - \mathbf{u}_4$ are linearly independent eigenvectors. By ??, \mathbf{B} is diagonalizable.

Alternatively Solution: Since \mathbf{B} has 4 distinct eigenvalues, by ??, \mathbf{B} is diagonalizable.

15. (a) (i) $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \Rightarrow \mathbf{B}^n = \underbrace{(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \cdots (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})}_{n \text{ times}} = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}$

So \mathbf{A}^n is similar to \mathbf{B}^n .

- (ii) $B = P^{-1}AP \Rightarrow B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$
 So A^{-1} is similar to B^{-1} .
- (iii) Suppose there exists an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. Let $R = P^{-1}Q$. Then R is invertible and $R^{-1}BR = Q^{-1}PBP^{-1}Q = Q^{-1}AQ$ is a diagonal matrix.

- (b) Since A is a triangular matrix, its eigenvalues are 0, 1 and -1 . Also it is easy to find from the characteristic equation of B that the eigenvalues of B are 0, 1 and -1 . By ??, both A and B are diagonalizable. So there exist invertible matrices R and Q such that

$$R^{-1}AR = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = Q^{-1}BQ.$$

Let $P = RQ^{-1}$. Then P is invertible matrix and $P^{-1}AP = QR^{-1}ARQ^{-1} = B$.

16. (a) Let $A = (a_{ij})_{n \times n}$. Then $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \dots, n$.

$$(i) \quad A^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus 1 is an eigenvalue of A^T . By ??, 1 is an eigenvalue of A .

- (ii) By ??, λ is an eigenvalue of A^T .

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of A^T associated with the eigenvalue λ , i.e. $A^T \mathbf{x} = \lambda \mathbf{x}$. Choose $k \in \{1, 2, \dots, n\}$ such that $|x_k| = \max\{|x_i| \mid i = 1, 2, \dots, n\}$, i.e. $|x_k| \geq |x_i|$ for $i = 1, 2, \dots, n$. Since \mathbf{x} is a nonzero vector, $|x_k| > 0$.

By comparing the k th coordinate of both sides of $A^T \mathbf{x} = \lambda \mathbf{x}$, we have

$$\begin{aligned} a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n &= \lambda x_k \\ \Rightarrow |\lambda| |x_k| &= |a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n| \\ &\leq |a_{1k}x_1| + |a_{2k}x_2| + \cdots + |a_{nk}x_n| \\ &\leq a_{1k}|x_1| + a_{2k}|x_2| + \cdots + a_{nk}|x_n| \quad (\because a_{ij} \geq 0 \text{ for all } i, j) \\ &\leq (a_{1k} + a_{2k} + \cdots + a_{nk})|x_k| \\ &= |x_k| \\ \Rightarrow |\lambda| &\leq 1. \end{aligned}$$

(b) (i) Yes.

(ii) Let $\mathbf{P} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{BP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}$.

17. Let a_n (respectively, b_n) be the number of customers who pay late (respectively, early) in month n . Then for $n = 1, 2, \dots$,

$$\begin{cases} a_n = \frac{1}{2}a_{n-1} + \frac{2}{10}b_{n-1} \\ b_n = \frac{1}{2}a_{n-1} + \frac{8}{10}b_{n-1}. \end{cases}$$

Let $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$. Then $\mathbf{x}_n = \mathbf{Ax}_{n-1} = \dots = \mathbf{A}^{n-1}\mathbf{x}_1$ where

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}.$$

By ??, we find a matrix $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}$. Then

$$\mathbf{x}_n = \mathbf{P} \begin{pmatrix} 1 & 0 \\ 0 & 0.3^{n-1} \end{pmatrix} \mathbf{P}^{-1}\mathbf{x}_1 = \frac{10000}{7} \begin{pmatrix} 2 - 2(0.3)^{n-1} \\ 5 + 2(0.3)^{n-1} \end{pmatrix}.$$

So the number of customers that will pay on time in April is $b_4 = \frac{10000}{7}[5 + 2(0.3)^3] = 7220$.

The number of customers that will pay on time will stabilize in the long run and $\lim_{n \rightarrow \infty} b_n = \frac{50000}{7} \approx 7143$.

18. Let a_n , b_n and c_n be the percentage of customers choosing brand A, B and C, respectively, after n months. Then for $n = 1, 2, \dots$,

$$\begin{cases} a_n = 0.97a_{n-1} + 0.01b_{n-1} + 0.02c_{n-1} \\ b_n = 0.01a_{n-1} + 0.97b_{n-1} + 0.02c_{n-1} \\ c_n = 0.02a_{n-1} + 0.02b_{n-1} + 0.96c_{n-1}. \end{cases}$$

Let $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$.

Then $\mathbf{x}_n = \mathbf{Ax}_{n-1} = \dots = \mathbf{A}^n\mathbf{x}_0$ where $\mathbf{x}_0 = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

By Algorithm 6.2.4, we find $\mathbf{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \end{pmatrix}$.

Then

$$\mathbf{x}_n = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96^n & 0 \\ 0 & 0 & 0.94^n \end{pmatrix} \mathbf{P}^{-1} \mathbf{x}_0 = \frac{50}{3} \begin{pmatrix} 2 + 3 \cdot 0.96^n + 0.94^n \\ 2 - 3 \cdot 0.96^n + 0.94^n \\ 2 - 2 \cdot 0.94^n \end{pmatrix}.$$

The present market shares are $\frac{50}{3}[2 + 3 \cdot 0.96^4 + 0.94^4]\% \approx 88.8\%$, $\frac{50}{3}[2 - 3 \cdot 0.96^4 + 0.94^4]\% \approx 3.9\%$ and $\frac{50}{3}[2 - 2 \cdot 0.94^4]\% \approx 7.3\%$ for brand A, B and C, respectively.

The market shares will stabilize after a long run and $\lim_{n \rightarrow \infty} \mathbf{x}_n = \begin{pmatrix} \frac{100}{3} \\ \frac{100}{3} \\ \frac{100}{3} \end{pmatrix}$.

19. Note that $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for $x \in \mathbb{R}$.

(a) Since $\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ for $n = 1, 2, \dots$,

$$e^{\mathbf{A}} = \begin{pmatrix} 1 + \frac{1}{1!} + \frac{1}{2!} + \dots & 0 & 0 \\ 0 & 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \dots & 0 \\ 0 & 0 & 1 + \frac{1}{1!}3 + \frac{1}{2!}3^2 + \dots \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$

(b) Let $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Since $\mathbf{A}^n = \mathbf{P} \begin{pmatrix} 2^n & 0 \\ 0 & 4^n \end{pmatrix} \mathbf{P}^{-1}$ for $n = 1, 2, \dots$,

$$e^{\mathbf{A}} = \mathbf{P} \begin{pmatrix} 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \dots & 0 \\ 0 & 1 + \frac{1}{1!}4 + \frac{1}{2!}4^2 + \dots \end{pmatrix} \mathbf{P}^{-1} = \frac{1}{2} \begin{pmatrix} e^4 + e^2 & e^4 - e^2 \\ e^4 - e^2 & e^4 + e^2 \end{pmatrix}.$$

(c) Let $\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\mathbf{A}^n =$

$$\mathbf{P} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{P}^{-1} \text{ for } n = 1, 2, \dots,$$

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{P} \begin{pmatrix} 1 - \frac{1}{1!} + \frac{1}{2!} - \dots & 0 & 0 \\ 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \dots & 0 \\ 0 & 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \dots \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} e^{-1} & \frac{1}{2}(e - e^{-1}) & \frac{1}{2}(e - e^{-1}) \\ -e + e^{-1} & \frac{1}{2}(3e - e^{-1}) & \frac{1}{2}(e - e^{-1}) \\ e - e^{-1} & \frac{1}{2}(-e + e^{-1}) & \frac{1}{2}(e + e^{-1}) \end{pmatrix}. \end{aligned}$$

20. In the following, we use the procedure discussed in ??.

(a) Let $\mathbf{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. Then $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} = \dots = \mathbf{A}^n \mathbf{x}_0$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Thus

$$\begin{aligned} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} &= \mathbf{x}_n = \mathbf{P} \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P}^{-1} \mathbf{x}_0 \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^n - 1 \\ 2^{n+1} - 1 \end{pmatrix}. \end{aligned}$$

Thus $a_n = 2^n - 1$.

(b) Let $\mathbf{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. Then $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} = \dots = \mathbf{A}^n \mathbf{x}_0$.

Let $\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. Thus

$$\begin{aligned} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} &= \mathbf{x}_n = \mathbf{P} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \mathbf{P}^{-1} \mathbf{x}_0 \\ &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}[2^n + 2(-1)^n] \\ \frac{1}{3}[2^{n+1} - 2(-1)^n] \end{pmatrix}. \end{aligned}$$

Thus $a_n = \frac{1}{3}[2^n + 2(-1)^n]$.

21. Use cofactor expansion along the first row:

$$\begin{aligned}
 d_n &= \begin{vmatrix} 3 & 1 & & & 0 \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 3 & 1 \\ 0 & & & & 1 & 3 \end{vmatrix}_{n \times n} \\
 &= 3 \begin{vmatrix} 3 & 1 & & & 0 \\ 1 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1) \times (n-1)} - \begin{vmatrix} 1 & 1 & & & 0 \\ 0 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1) \times (n-1)}.
 \end{aligned}$$

The first determinant above is d_{n-1} . By using cofactor expansion along the first column, we find that the second determinant is d_{n-2} . So

$$d_n = 3d_{n-1} - d_{n-2}.$$

Note that $d_1 = 3$ and $d_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$.

By the procedure discussed in ??, we obtain

$$d_n = \left(\frac{5 + 3\sqrt{5}}{10} \right) \left(\frac{3 + \sqrt{5}}{2} \right)^n + \left(\frac{5 - 3\sqrt{5}}{10} \right) \left(\frac{3 - \sqrt{5}}{2} \right)^n.$$

22. Consider the vector equation

$$a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_m \mathbf{u}_m + b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p = \mathbf{0}. \quad (1)$$

Pre-multiplying \mathbf{A} to both side of (1), we have

$$a_1 \lambda_1 \mathbf{u}_1 + a_2 \lambda_2 \mathbf{u}_2 + \cdots + a_m \lambda_m \mathbf{u}_m + b_1 \mu \mathbf{v}_1 + b_2 \mu \mathbf{v}_2 + \cdots + b_p \mu \mathbf{v}_p = \mathbf{0}. \quad (2)$$

Subtracting (2) by μ times of (1), we obtain

$$a_1(\lambda_1 - \mu) \mathbf{u}_1 + a_2(\lambda_2 - \mu) \mathbf{u}_2 + \cdots + a_m(\lambda_m - \mu) \mathbf{u}_m = \mathbf{0}.$$

Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ are linearly independent, $a_1(\lambda_1 - \mu) = 0$, $a_2(\lambda_2 - \mu) = 0$, \dots , $a_m(\lambda_m - \mu) = 0$. As $\lambda_i \neq \mu$ for $i = 1, 2, \dots, m$, we have $a_1 = 0$, $a_2 = 0$, \dots , $a_m = 0$.

Substituting $a_1 = 0, a_2 = 0, \dots, a_m = 0$ into (2), we have

$$b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_p \mathbf{v}_p = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, $b_1 = 0, b_2 = 0, \dots, b_p = 0$.

We have shown that the vector equation (1) has only the trivial solution. Thus $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is linearly independent.

23. (a) True. Let \mathbf{P} be an invertible matrix that diagonalizes \mathbf{A} , i.e. $\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \mathbf{D}$ where \mathbf{D} is a diagonalizable matrix. Then

$$\mathbf{D} = \mathbf{D}^T = (\mathbf{P}^{-1} \mathbf{A} \mathbf{P})^T = \mathbf{P}^T \mathbf{A}^T (\mathbf{P}^{-1})^T = \mathbf{P}^T \mathbf{A}^T (\mathbf{P}^T)^{-1}.$$

Thus the matrix $(\mathbf{P}^T)^{-1}$ diagonalizes \mathbf{A}^T .

- (b) False. For example, $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable

but $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

- (c) False. For example, $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable

but $\mathbf{A} \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

24. (a) Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

- (b) Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$.

- (c) Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 + \sqrt{2} & 0 \\ 0 & 0 & 2 - \sqrt{2} \end{pmatrix}$.

- (d) Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

- (e) Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

$$(f) \text{ Let } \mathbf{P} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \text{ Then } \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

$$(g) \text{ Let } \mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \text{ Then } \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$(h) \text{ Let } \mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{pmatrix}. \text{ Then } \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

25. (a) Since $(\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}\mathbf{u}^T$, $\mathbf{u}\mathbf{u}^T$ is symmetric. Hence $\mathbf{I} - \mathbf{u}\mathbf{u}^T$ is also symmetric and thus is orthogonally diagonalizable.

$$(b) \text{ When } \mathbf{u} = (1, -1, 1)^T, \mathbf{I} - \mathbf{u}\mathbf{u}^T = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

$$\text{Let } \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \text{ Then } \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

26. By the given conditions, we have $\mathbf{A}^T = \mathbf{A}$, $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$ and $\mathbf{A}\mathbf{v} = \mu\mathbf{v}$. We compute $\mathbf{v}^T \mathbf{A}\mathbf{u}$ in two ways:

$$\mathbf{v}^T \mathbf{A}\mathbf{u} = \mathbf{v}^T (\lambda\mathbf{u}) = \lambda \mathbf{v}^T \mathbf{u} = \lambda(\mathbf{v} \cdot \mathbf{u}),$$

$$\mathbf{v}^T \mathbf{A}\mathbf{u} = \mathbf{v}^T \mathbf{A}^T \mathbf{u} = (\mathbf{A}\mathbf{v})^T \mathbf{u} = (\mu\mathbf{v})^T \mathbf{u} = \mu \mathbf{v}^T \mathbf{u} = \mu(\mathbf{v} \cdot \mathbf{u}).$$

Thus $\lambda(\mathbf{v} \cdot \mathbf{u}) = \mu(\mathbf{v} \cdot \mathbf{u})$ which implies $(\lambda - \mu)(\mathbf{v} \cdot \mathbf{u}) = 0$. Since $\lambda \neq \mu$, we have $\mathbf{v} \cdot \mathbf{u} = 0$.

27. Since

$$E_1 = \{(x, y, z)^T \mid x + y - z = 0\} = \text{span}\{(-1, 1, 0)^T, (1, 0, 1)^T\},$$

$\{(-1, 1, 0)^T, (1, 0, 1)^T\}$ is a basis for E_1 .

Let \mathbf{u} be an eigenvector associated with -1 . Since \mathbf{A} is symmetric, by ??, \mathbf{u} is orthogonal to E_1 , i.e. \mathbf{u} is perpendicular to $x + y - z = 0$. Hence \mathbf{u} is a scalar multiple of $(1, 1, -1)^T$. This means

$$E_{-1} = \text{span}\{(1, 1, -1)^T\}$$

and $\{(1, 1, -1)^T\}$ is a basis for E_{-1} .

Let $\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Hence

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

28. Suppose the eigenvalues associated with the eigenspaces $\text{span}\{(1, 0, 1, 0)^T, (1, 1, 1, 1)^T\}$ and $\text{span}\{(1, 1, -1, -1)^T, (1, -1, -1, 1)^T\}$ are λ and μ respectively.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{P}\mathbf{A} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$. So

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2}(\lambda + \mu) & 0 & \frac{1}{2}(\lambda - \mu) & 0 \\ 0 & \frac{1}{2}(\lambda + \mu) & 0 & \frac{1}{2}(\lambda - \mu) \\ \frac{1}{2}(\lambda - \mu) & 0 & \frac{1}{2}(\lambda + \mu) & 0 \\ 0 & \frac{1}{2}(\lambda - \mu) & 0 & \frac{1}{2}(\lambda + \mu) \end{pmatrix}$$

which is a symmetric matrix.

Alternative Solution: Since

$$\begin{aligned} (1, 0, 1, 0) \cdot (1, 1, -1, -1) &= 0, \\ (1, 0, 1, 0) \cdot (1, -1, -1, 1) &= 0, \\ (1, 1, 1, 1) \cdot (1, 1, -1, -1) &= 0, \\ (1, 1, 1, 1) \cdot (1, -1, -1, 1) &= 0, \end{aligned}$$

any vector from $\text{span}\{(1, 0, 1, 0)^T, (1, 1, 1, 1)^T\}$ is orthogonal to any vector from $\text{span}\{(1, 1, -1, -1)^T, (1, -1, -1, 1)^T\}$.

Take any orthonormal bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $\text{span}\{(1, 0, 1, 0)^T, (1, 1, 1, 1)^T\}$ and $\text{span}\{(1, 1, -1, -1)^T, (1, -1, -1, 1)^T\}$ respectively. By the observation above, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is orthonormal. Let $\mathbf{P} = (\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{v}_1 \ \mathbf{v}_2)$. Then \mathbf{P} is an orthogonal matrix that diagonalizes \mathbf{A} . By ??, \mathbf{A} is symmetric.

29. (a) Since $\mathbf{A}\mathbf{u} = 4\mathbf{u}$, \mathbf{u} is an eigenvector of \mathbf{A} associated with the eigenvalue 4.
 (b) $\mathbf{v} \cdot \mathbf{u} = 0 \Rightarrow a + b + c + d = 0$.

Thus $\mathbf{A}\mathbf{v} = \mathbf{0} = 0\mathbf{v}$, \mathbf{v} is an eigenvector of \mathbf{A} associated with the eigenvalue 0.

- (c) Since \mathbf{P} is an orthogonal matrix, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot (a_i, b_i, c_i, d_i) = 0$ for $i = 1, 2, 3$. By (a), the first column of \mathbf{P} is the eigenvector of \mathbf{A} associated with the eigenvalue 4. By (b), the other four columns of \mathbf{P} are eigenvectors of \mathbf{A} associated with the eigenvalue 0. So

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

30. (a) True. Since \mathbf{A} and \mathbf{B} are orthogonally diagonalizable, they are both symmetric. Then $\mathbf{A} + \mathbf{B}$ is also symmetric and hence orthogonally diagonalizable.

- (b) False. For example, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are both orthogonally diagonalizable but $\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not orthogonally diagonalizable.

31. (a) (i) $Q_1(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$

- (ii) Let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$Q_1(x, y) = 3x'^2 + 7y'^2 = \frac{3}{2}(x + y)^2 + \frac{7}{2}(x - y)^2.$$

- (b) (i) $Q_2(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

- (ii) Let $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\begin{aligned} Q_2(x, y, z) &= 3x'^2 + 6y'^2 + 9z'^2 \\ &= \frac{1}{3}(-x - 2y + 2z)^2 + \frac{2}{3}(2x + y + 2z)^2 + (-2x + 2y + z)^2. \end{aligned}$$

32. (a) (i) With $(x_1, x_2, x_3) = (1, 0, 0)$, we have $x_1^2 + x_2^2 + x_3^2 = 1$ and $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = \lambda_1$. So $\min\{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \leq \lambda_1$.

On the other hand, for any x_1, x_2, x_3 satisfying $x_1^2 + x_2^2 + x_3^2 = 1$,

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \geq \lambda_1 x_1^2 + \lambda_1 x_2^2 + \lambda_1 x_3^2 = \lambda_1(x_1^2 + x_2^2 + x_3^2) = \lambda_1.$$

So $\min\{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \mid x_1^2 + x_2^2 + x_3^2 = 1\} = \lambda_1$.

(ii) The proof is similar to Part (i) above.

(b) (i) Let $\mathbf{u} = (x_1, x_2, x_3)^T$. Then $\mathbf{u}^T \mathbf{Q} \mathbf{u} = x_1^2 + 2x_2^2 + 3x_3^2$ and $\mathbf{u}^T \mathbf{u} = x_1^2 + x_2^2 + x_3^2$. Thus by (a), the minimum value is 1 and the maximum value is 3.

(ii) The eigenvalues of \mathbf{Q} are $2 - \sqrt{2}$, 2 and $2 + \sqrt{2}$. There exists an orthogonal matrix \mathbf{P} such that $\mathbf{P}^T \mathbf{Q} \mathbf{P} = \begin{pmatrix} 2 - \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 + \sqrt{2} \end{pmatrix}$.

Let $\mathbf{P}^T \mathbf{u} = (x_1, x_2, x_3)^T$. Then

$$\begin{aligned} \mathbf{u}^T \mathbf{Q} \mathbf{u} &= \mathbf{u}^T (\mathbf{P} \mathbf{P}^T) \mathbf{Q} (\mathbf{P} \mathbf{P}^T) \mathbf{u} \\ &= (\mathbf{P}^T \mathbf{u})^T (\mathbf{P}^T \mathbf{Q} \mathbf{P}) (\mathbf{P}^T \mathbf{u}) \\ &= (2 - \sqrt{2})x_1^2 + 2x_2^2 + (2 + \sqrt{2})x_3^2 \end{aligned}$$

and

$$\mathbf{u}^T \mathbf{u} = \mathbf{u}^T (\mathbf{P} \mathbf{P}^T) \mathbf{u} = (\mathbf{P}^T \mathbf{u})^T (\mathbf{P}^T \mathbf{u}) = x_1^2 + x_2^2 + x_3^2.$$

Thus by (a), the minimum value is $2 - \sqrt{2}$ and the maximum value is $2 + \sqrt{2}$.

33. (a) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$x^2 + 2y^2 - 2x + 8y + 8 = 0 \Leftrightarrow (x - 1)^2 + \frac{(y + 2)^2}{1/2} = 1.$$

The conic is an ellipse.

(b) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$x^2 - 4x + 4y + 4 = 0 \Leftrightarrow (x - 2)^2 = -4y.$$

The conic is a parabola.

(c) The quadratic form is $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$2x^2 - 4xy - y^2 + 8 = 0 \Leftrightarrow -\frac{x'^2}{8/3} + \frac{y'^2}{4} = 1.$$

The conic is a hyperbola.

(d) The quadratic form is $(x \ y) \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$.

Then

$$x^2 + xy + y^2 = 6 \Leftrightarrow \frac{x'^2}{12} + \frac{y'^2}{4} = 1.$$

The conic is an ellipse.

(e) The quadratic form is $(x \ y) \begin{pmatrix} 11 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$.

Then

$$11x^2 + 24xy + 4y^2 - 15 = 0 \Leftrightarrow \frac{x'^2}{3/4} - \frac{y'^2}{3} = 1.$$

The conic is a hyperbola.

(f) The quadratic form is $(x \ y) \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0 \Leftrightarrow \frac{(x' - \sqrt{5})^2}{6} + \frac{y'^2}{3} = 1.$$

The conic is an ellipse.

(g) The quadratic form is $(x \ y) \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$. Then $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$9x^2 + 6xy + y^2 - 10\sqrt{10}x + 10\sqrt{10}y + 90 = 0 \Leftrightarrow (y' - 1)^2 = -4(x' + 2).$$

The conic is a parabola.

34. Since \mathbf{A} is a symmetric matrix with eigenvalues 1 and 4, there exists an orthogonal matrix \mathbf{P} such that $\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Define $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} x \\ y \end{pmatrix}$, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P} \begin{pmatrix} x' \\ y' \end{pmatrix}$. Then

$$\begin{pmatrix} x & y \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = 8 \Leftrightarrow \begin{pmatrix} x' & y' \end{pmatrix} \mathbf{P}^T \mathbf{A} \mathbf{P} \begin{pmatrix} x' \\ y' \end{pmatrix} = 8 \Leftrightarrow \frac{x'^2}{8} + \frac{y'^2}{2} = 1.$$

The conic is an ellipse.