ST2334 (2018/2019 Semester 2) Solutions to Questions in Tutorial 7

Question 1

$f_{X,Y}(x,y)$		x = 2	x = 4	$f_Y(y)$
y	1	0.10	0.15	**0.25**
	3	0.20	0.30	**0.50**
	5	0.10	0.15	**0.25**
$f_X(x)$		**0.40**	**0.60**	**1**

(a) $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y. Hence, X and Y are independent

(b)

y	1	3	5
$f_{Y\|X}(y\|2)$	0.10/0.40 = 1/4	0.20/0.40 = 2/4	0.10/0.40 = 1/4

$E(Y|X=2) = 1(1/4) + 3(2/4) + 5(1/4) = 3$

(c)

X	2	4
$f_{X\|Y}(x\|3)$	0.20/0.50 = 2/5	0.30/0.50 = 3/5

$E(X|Y=3) = 2(2/5) + 4(3/5) = 16/5 = 3.2$

(d) $E(X) = 2(0.40) + 4(0.60) = 3.2$
$E(Y) = 1(0.25) + 3(0.50) + 5(0.25) = 3$
$E(2X - 3Y) = 2E(X) - 3E(Y) = 2(3.2) - 3(3) = -2.6$

(e) $E(XY) = (2)(1)(0.10) + (2)(3)(0.10) + (2)(5)(0.10) + (4)(1)(0.10) +$
$+(4)(3)(0.10) + (4)(5)(0.10) = 9.6$

(f) $V(X) = E(X^2) - [E(X)]^2 = 11.2 - (3.2)^2 = 0.96$
$V(Y) = E(Y^2) - [E(Y)]^2 = 11 - (3)^2 = 2$

(g) $\sigma_{X,Y} = 0$ as X and Y are independent.
Alternatively, $Cov(X,Y) = E(XY) - E(X)E(Y) = 9.6 - 3.2(3) = 0$
$\rho_{X,Y} = 0$ as $\sigma_{X,Y} = 0$. Alternatively, $\rho_{X,Y} = 0$ as X and Y are independent.

Question 2

$f_{X,Y}(x,y)$		x = 0	x = 1	x = 2	$f_Y(y)$
y	0	0.01	0.01	0.03	**0.05**
	1	0.03	0.08	0.07	**0.18**
	2	0.03	0.06	0.06	**0.15**
	3	0.07	0.07	0.13	**0.27**
	4	0.12	0.04	0.03	**0.19**
	5	0.08	0.06	0.02	**0.16**
$f_X(x)$		**0.34**	**0.32**	**0.34**	**1**

$E(X) = \sum x f_X(x) = 1$. $E(X^2) = \sum x^2 f_X(x) = 1.68$. $V(X) = E(X^2) - [E(X)]^2 = 0.68$
$E(Y) = \sum y f_Y(y) = 2.85$. $E(Y^2) = \sum y^2 f_Y(y) = 10.25$. $V(Y) = E(Y^2) - [E(Y)]^2 =$
2.1275.
Profit = $8X + 3Y - 10$. $E(\text{profit}) = 8E(X) + 3E(Y) - 10 = 6.55$
$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (2.47) - (1)(2.85) = -0.38$
$V(\text{profit}) = V[8X + 3Y - 10] = 8^2V(X) + 3^2E(Y) + 2(8)(3)Cov(X,Y) = 44.43$.

Question 3

(a) $f_X(x) = \int_0^1 \frac{2}{3}(x+2y)\, dy = \frac{2}{3}\left[xy + \frac{2y^2}{2}\right]_0^1 = \frac{2}{3}(x+1)$, for $0 \le x \le 1$

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)\, dx = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1 = \frac{2}{3}\left(\frac{1}{2} + 2y\right)$, for $0 \le y \le 1$

$f_{X,Y}(x,y) \ne f_X(x) f_Y(y)$ ➔ X and Y are dependent

(b) $E(X) = \frac{2}{3}\int_0^1 x(x+1)\, dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9}$

$E(X^2) = \frac{2}{3}\int_0^1 x^2(x+1)\, dx = \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{3}\right) = \frac{7}{18}$

$V(X) = E(X^2) - [E(X)]^2 = 13/162$

(c) $E(Y) = \frac{2}{3}\int_0^1 y\left(\frac{1}{2} + 2y\right)\, dy = \frac{2}{3}\left[\frac{y^2}{4} + \frac{2y^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{4} + \frac{2}{3}\right) = \frac{11}{18}$

$E(Y^2) = \frac{2}{3}\int_0^1 y^2\left(\frac{1}{2} + 2y\right)\, dx = \frac{2}{3}\left[\frac{y^3}{6} + \frac{y^4}{2}\right]_0^1 = \frac{2}{3}\left(\frac{1}{6} + \frac{1}{2}\right) = \frac{4}{9}$

$V(Y) = E(Y^2) - [E(Y)]^2 = 23/324$

(d) $E(XY) = \frac{2}{3}\int_0^1\int_0^1 xy(x+2y)\, dxdy = \frac{2}{3}\int_0^1\int_0^1 x^2y + 2xy^2\, dxdy$

$= \frac{2}{3}\int_0^1 \left[\frac{x^3y}{3} + \frac{2x^2y^2}{2}\right]_0^1 dy = \frac{2}{3}\int_0^1 \left(\frac{y}{3} + y^2\right) dy = \frac{2}{3}\left[\frac{y^2}{6} + \frac{y^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{1}{3}$

$\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (5/9)(11/18) = -1/162$

Question 4

(a) $f_X(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}(x^2 + \frac{1}{3})$, for $0 \le x \le 1$

$f_Y(y) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dx = \frac{3}{2}\left[\frac{x^3}{3} + xy^2\right]_0^1 = \frac{3}{2}(\frac{1}{3} + y^2)$, for $0 \le y \le 1$

$f_{X,Y}(x,y) \ne f_X(x) f_Y(y)$ ➔ X and Y are dependent

(b) $E(X) = \frac{3}{2}\int_0^1 x\left(x^2 + \frac{1}{3}\right)\, dx = \frac{3}{2}\left[\frac{x^4}{4} + \frac{x^2}{6}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5}{8}$

$E(X^2) = \frac{3}{2}\int_0^1 x^2\left(x^2 + \frac{1}{3}\right)\, dx = \frac{3}{2}\left[\frac{x^5}{5} + \frac{x^3}{9}\right]_0^1 = \frac{3}{2}\left(\frac{1}{5} + \frac{1}{9}\right) = \frac{7}{15}$

$V(X) = E(X^2) - [E(X)]^2 = 73/960$

(c) $E(Y) = \frac{3}{2}\int_0^1 y\left(y^2 + \frac{1}{3}\right)\, dy = \frac{3}{2}\left[\frac{y^4}{4} + \frac{y^2}{6}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{5}{8}$

$E(Y^2) = \frac{3}{2}\int_0^1 y^2\left(y^2 + \frac{1}{3}\right)\, dy = \frac{3}{2}\left[\frac{y^5}{5} + \frac{y^3}{9}\right]_0^1 = \frac{3}{2}\left(\frac{1}{5} + \frac{1}{9}\right) = \frac{7}{15}$

$V(Y) = E(Y^2) - [E(Y)]^2 = 73/960$

(d) $E(XY) = \frac{3}{2}\int_0^1\int_0^1 xy(x^2+y^2)\, dxdy = \frac{3}{2}\int_0^1\int_0^1 x^3y + xy^3\, dxdy$

$= \frac{3}{2}\int_0^1 \left[\frac{x^4y}{4} + \frac{x^2y^3}{2}\right]_0^1 dy = \frac{3}{2}\int_0^1 \left(\frac{y}{4} + \frac{y^3}{2}\right) dy = \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1 = \frac{3}{2}\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{3}{8}$

$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (3/8) - (5/8)(5/8 = -1/64$

(e) $E(X+Y) = E(X) + E(Y) = 5/8 + 5/8 = 5/4$

(f) $V[X+Y] = V(X) + V(Y) + 2(\sigma_{X,Y}) = \frac{73}{960} + \frac{73}{960} + 2\left(\frac{-1}{64}\right) = \frac{29}{240}$

Question 5

(a) $f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$, for $0 \le x \le 1$

$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{12}$

$f_Y(y) = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + xy\right]_0^1 = y + \frac{1}{2}$, for $0 \le y \le 1$

$E(Y) = \int_0^1 y\left(y + \frac{1}{2}\right)\,dy = \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1 = \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{12}$

$E(XY) = \int_0^1\int_0^1 xy(x+y,)\,dxdy = \int_0^1\int_0^1 x^2y + xy^2\,dxdy = \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$

$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{3}$

$\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (7/12)(7/12) = -1/144$

(b) $f_{Y|X}(y|x = 0.2) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x + {}^1/_2} = \frac{0.2+y}{0.7} = \frac{2+10y}{7}$, for $0 \le y \le 1$

$E(Y|X = 0.2) = \int_0^1 y\left(\frac{2+10y}{7}\right)\,dy = \frac{1}{7}\left[y^2 + \frac{10y^3}{3}\right]_0^1 = \frac{1}{7}\left(1 + \frac{10}{3}\right) = \frac{13}{21}$

(c) $f_{X|Y}(x|y = 0.5) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{y + {}^1/_2} = \frac{0.5+x}{1} = x + \frac{1}{2}$, for $0 \le x \le 1$

$E(X|Y = 0.5) = \int_0^1 x\left(x + \frac{1}{2}\right)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{12}$

Question 6

(a) $V(Z) = V[-2X + 4Y - 3] = (-2)^2V(X) + (4)^2V(Y) = 4(5) + 16(3) = 68$

(b) $V(Z) = V[-2X + 4Y - 3] = (-2)^2V(X) + (4)^2V(Y) + 2(-2)(4)Cov(X,Y)$

$= 4(5) + 16(3) + 2(-2)(4)(1) = 52$

(c) $\rho_{X,Y} = \frac{Cov\ (X,Y)}{\sigma_X\sigma_Y} = \frac{1}{\sqrt{5}\sqrt{3}} = 0.2582$

Question 7

$X \sim discrete\ uniform$

(a) $f_X(x) = \frac{1}{10}, \quad x = 1, 2, \cdots, 10$

(b) $\Pr(X < 4) = \sum_{x=1}^{3} f(x) = \frac{3}{10}$

(c) $\mu = \sum_{x=1}^{10} x\left(\frac{1}{10}\right) = 5.5$. $\sigma^2 = \sum_{x=1}^{10}(x - 5.5)^2\frac{1}{10} = 8.25$

Alternatively, $E(X^2) = \sum_{x=1}^{10} x^2\left(\frac{1}{10}\right) = 38.5$. $V(X) = 38.5 - 5.5^2 = 8.25$