Question 1:

(a)

$$\left(\begin{array}{ccc|c} 0 & 1 & -b & 1 \\ 1 & a & 0 & 1 \\ a & 1 & b & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & -b & 1 \\ a & 1 & b & 1 \end{array}\right) \xrightarrow{R_3 - aR_1} \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & -b & 1 \\ 0 & 1-a^2 & b & 1-a \end{array}\right)$$

$$\xrightarrow{R_3 - (1-a^2)R_2} \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & -b & 1 \\ 0 & 0 & b+(1-a^2)b & (1-a)-(1-a^2) \end{array}\right) = \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & 1 & -b & 1 \\ 0 & 0 & b(2-a^2) & a^2-a \end{array}\right)$$

Thus the linear system has infinitely many solutions if and only if

$$b(2-a^2) = 0 \quad \text{and} \quad a^2 - a = 0 \Leftrightarrow a = 0, b = 0 \quad \text{or} \quad a = 1, b = 0.$$

(b) (i) We find a linear system such that a general solution to the system is

$$\begin{cases} x &= 2+t \\ y &= 1-t \\ z &= t, \quad t \in \mathbb{R} \end{cases}$$

A row-echelon form of the system's augmented matrix can be

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

So the two planes can be $P_1 : x - z = 2$ and $P_2 : y + z = 1$.

(ii) We want $ax + by + cz = d$ such that the reduced row-echelon form of

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ a & b & c & d \end{array}\right) \quad \text{is} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ a & b & c & d \end{array}\right) \xrightarrow{R_3 - aR_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & b & c+a & d-2a \end{array}\right) \xrightarrow{R_3 - bR_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c+a-b & d-2a-b \end{array}\right) = \boldsymbol{R}.$$

Since we want $x = 1, y = 2, z = -1$ to be the unique solution, we can choose a, b, c, d such that $c+a-b = 1$ and $d-2a-b = -1$. For example $a = b = c = 1$ and $d = 2$. Verifying

$$\boldsymbol{R} = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R_2 - R_3]{R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

We thus have $P_3 : x + y + z = 2$.

Question 2

(i)

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{\boldsymbol{E_1}}{\overset{R_2-2R_1}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{\boldsymbol{E_2}}{\overset{R_3+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \underset{\boldsymbol{E_3}}{\overset{-R_2}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \underset{R_2-R_3\ (\boldsymbol{E_5})}{\overset{R_1-R_3\ (\boldsymbol{E_4})}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \underset{\boldsymbol{E_6}}{\overset{R_1+2R_2}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -5 & -3 \\ 0 & 1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

So $\boldsymbol{A}^{-1} = \begin{pmatrix} 11 & -5 & -3 \\ 4 & -2 & -1 \\ -2 & 1 & 1 \end{pmatrix}$.

(ii) $\boldsymbol{E_6}\boldsymbol{E_5}\cdots\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} = \boldsymbol{I}$. So,

$$\det(\boldsymbol{E_6})\cdots\det(\boldsymbol{E_1})\det(\boldsymbol{A}) = 1 \Rightarrow -\det(\boldsymbol{A}) = 1 \Rightarrow \det(\boldsymbol{A}) = -1.$$

(iii)

$$\operatorname{adj}(\boldsymbol{A}) = -\boldsymbol{A}^{-1} \Rightarrow \operatorname{adj}(\boldsymbol{A})^{-1} = -\boldsymbol{A}.$$

So

$$\operatorname{adj}(\boldsymbol{A})\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Leftrightarrow \boldsymbol{x} = \operatorname{adj}(\boldsymbol{A})^{-1}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

(iv) Note that $\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{U}$, an upper triangular matrix.

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \boldsymbol{L}\boldsymbol{U} \end{aligned}$$

where $\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ is a lower triangular matrix.

Question 3

(i) Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\boldsymbol{A}$ is anti-symmetric, then

$$\boldsymbol{A} = -\boldsymbol{A}^T \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \Leftrightarrow a = -a, d = -d, b = -c \Leftrightarrow a = d = 0, b = -c.$$

Thus $\boldsymbol{A} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$. Since $\boldsymbol{A}$ is non zero, $b \neq 0$. The determinant of $\boldsymbol{A}$ is $b^2 \neq 0$, so $\boldsymbol{A}$ is invertible.

(ii)
$$\boldsymbol{A} = -\boldsymbol{A}^T \Rightarrow \det(\boldsymbol{A}) = (-1)^n \det(\boldsymbol{A}^T) = -\det(\boldsymbol{A}^T) \quad \text{(since } n \text{ is odd)}$$

But since $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$, we have $\det(\boldsymbol{A}) = -\det(\boldsymbol{A})$ which implies that $\det(\boldsymbol{A}) = 0$. So $\boldsymbol{A}$ is singular.

Question 4

(a) (i)

$$\begin{aligned} S_1 &= \{(t,\ s-t,\ s) \mid s,t \in \mathbb{R}\} \\ &= \{t(1,-1,0) + s(0,1,1) \mid s,t \in \mathbb{R}\} \\ &= \operatorname{span}\{(1,-1,0),(0,1,1)\} \end{aligned}$$

Thus S_1 is a subspace of $\mathbb{R}^3$.

(ii) Let $\boldsymbol{u} = (0,1,2)$, $\boldsymbol{v} = (1,0,0)$. Then both $\boldsymbol{u}, \boldsymbol{v}$ belongs to S_2. But $\boldsymbol{u} + \boldsymbol{v} = (1,1,2)$ which does not belong to S_2. So S_2 is not a subspace of $\mathbb{R}^3$.

(b) (i) Note that $(1,0) \in U$ (since $(1,0) \in V$) and $(0,2) \in U$ (since $(0,2) \in W$). But $(1,0) + (0,2) = (1,2)$ which does not belong to U since $(1,2)$ does not belong to V nor W. Thus U is not a subspace.

(ii) If W is the zero space, the $U = V$ which is a subspace of $\mathbb{R}^2$. If $W = \mathbb{R}^2$, then we also have $U = \mathbb{R}^2$, which is a subspace of $\mathbb{R}^2$.