CHAPTER 6

SECTION 6.1

DEFINITION:

A **GRAPH** $G = (V, E)$ consists of:

- V, a nonempty finite set of **VERTICES** AND
- E, a set of **EDGES**.

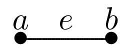
Each edge is associated with either one or two vertices called its **ENDPOINTS**. An edge with one endpoint is called a **LOOP**.

Sometimes, there may be more than one edge associated with a pair of vertices. Such edges are called **MULTIPLE EDGES**.

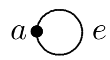
GRAPHS AS DRAWINGS

In general we visualize graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where an edge associated with the vertices a and b is represented by a line segment joining a to b.

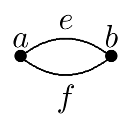
EDGE: has 1 or 2 ENDPOINTS, joins a to b. Call it e or ab.



LOOP: has 1 endpoint; joins a vertex to itself. Call it edge e or loop at a.



MULTIPLE EDGES: Call them edges e, f or edge ab with multiplicity 2.



DEFINITION:

A graph with no loops or multiple edges is **SIMPLE**.

A graph with no loops but admits multiple edges is called a **MULTIGRAPH**.

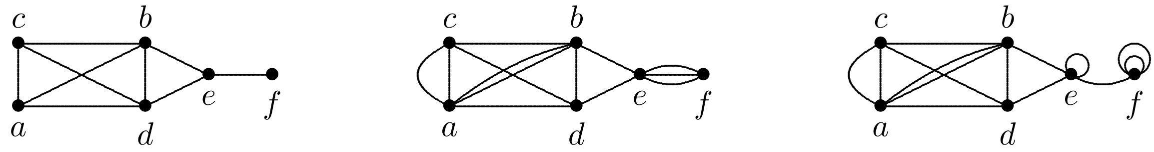
Chapter 9 in text

REMARK

In general a graph may have both loops and multiple edges.

In a simple graph, when there is an edge joining a to b, we say ab is an edge of the graph. Note that ab and ba refer to the same edge. We sometimes give it a name, say e, and call it edge e.

In a multigraph, if there are m edges joining a to b, we say ab is an edge of multiplicity m. We sometimes give names, say $e, f, \ldots$, to these edges and call them edges $e, f, \ldots$.



G: Simple Graph $\quad$ H: Multigraph $\quad$ J: Graph

Sometimes, we need to give directions to the edges, i.e., to think of edges as ordered pairs. (This happens when we use graphs to models one-way streets.) This leads to the notion of a digraph.

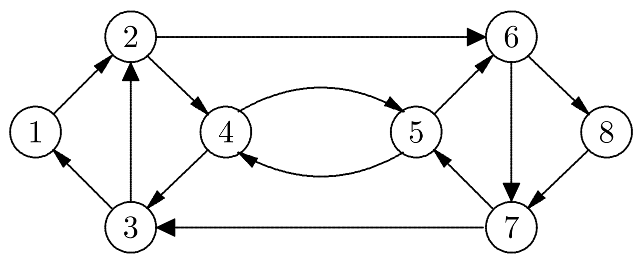
DEFINITION:

A **DIGRAPH** $G = (V, E)$ consists of:

- V, a nonempty finite set of **VERTICES** AND
- E, a set of **DIRECTED EDGES**.

Each directed edge is associated with an ordered pairs of vertices. The directed edge associated with the ordered pair (u, v) is said to **START** from u and **END** at v and is denoted as $\overline{uv}$.

Directed edges are depicted in a diagram using arrows.



A Digraph

- Vertices: 1, 2, ..., 8. Edges: $\overline{12}$, $\overline{31}$, $\overline{32}$, $\overline{45}$, $\overline{54}$, etc.

GRAPH MODELS

• **ACQUAINTANCESHIP GRAPHS**: Vertices represent people. ab is an edge if a and b know each other.

• **INFLUENCE GRAPHS**: In studies of group behaviour, it is observed that certain people can influence the thinking of others. A digraph can be used to model this: $\overline{uv}$ is a directed edge if u can influence v.

• **CALL GRAPHS**: Directed multigraphs can be used to model telephone calls in a network. Here telephones are represented as vertices and each call from a to b is represented by $\overline{ab}$.

• **WEB GRAPHS**: The World Wide Web can be modelled as a digraph where each Web page is represented by a vertex. There is a directed edge $\overline{ab}$ is there is a link on a pointing to b.

• **PRECEDENCE GRAPHS**: Computer programs can be executed more rapidly by executing certain statements concurrently. However, certain statements depends on the results from other statements. Thus we to create a precedence graphs. Here each vertex represents a statement. A directed edge $\overline{ab}$ means that a must be executed before b.

SECTION 6.2 TERMINOLOGY

DEFINITION:

Let G be an undirected graph.

• Two distinct vertices a, b in G are **ADJACENT** or **NEIGHBOURS** if ab is an edge.

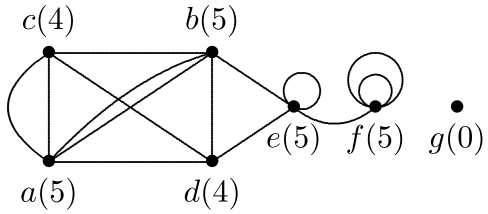
• An edge e and a vertex a in G are **INCIDENT** if $e = ax$ for some vertex x.

• The **DEGREE** of a vertex u in G, $\deg(u)$, is the number of edges incident with u, with each loop counted as 2.

• A vertex is **ISOLATED** if its degree is 0.

EXAMPLE

The number in (.) is the degree.



J: Graph

(HANDSHAKING THEOREM):

Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

PROOF: Every edge contributes 2 to the sum (since it has two endpoints). Thus the degree sum is twice the number of edges.

COROLLARY:

In any graph there is an even number of vertices with odd degrees.

Let $\sum_{\text{odd}}$ and $\sum_{\text{even}}$ denote the summation over vertices with odd and even degrees, respectively. Then

$$\begin{aligned} \sum \deg(v) &= \sum_{\text{odd}} \deg(v) + \sum_{\text{even}} \deg(v) \\ \Rightarrow \quad \sum_{\text{odd}} \deg(v) &= 2|E(G)| - \sum_{\text{even}} \deg(v). \end{aligned}$$

Since $\sum_{\text{even}} \deg(v)$, being the sum of even numbers, is even. The rhs is even.

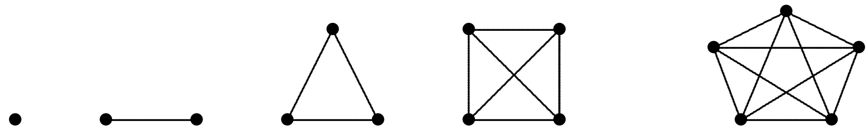
The lhs is the sum of odd numbers, but the sum is even. Therefore the number of terms is even, i.e., the number of vertices with odd degrees is even.

SOME SPECIAL GRAPHS

The **COMPLETE GRAPH** on n vertices, denoted by K_n is the simple graph such that every two distinct vertices are adjacent. Note that:

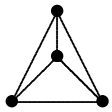
$$|E(K_n)| = \binom{n}{2}.$$

The figure below gives the complete graphs on 1-5 vertices.



K_1 K_2 K_3 K_4 K_5

Another drawing of K_4:

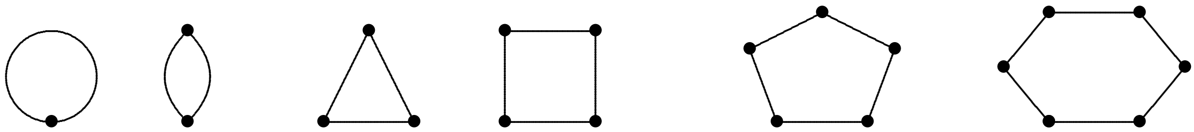


K_4

The **CYCLE** C_n, $n \geq 1$, consists of

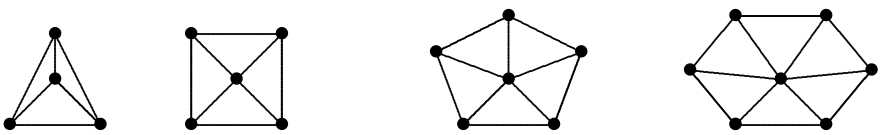
n vertices: $v_1, v_2, \ldots, v_n$ and

n edges: $v_1v_2, v_2v_3, \ldots, v_nv_1$.



C_1 C_2 C_3 C_4 C_5 C_6

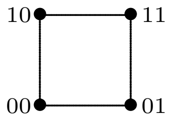
The **WHEEL** W_n, $n \geq 3$, is the simple graph obtained from C_n by adding a new vertex and connect it to $v_1, \ldots, v_n$ by new edges.



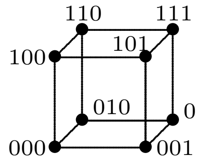
W_3 W_4 W_5 W_6

The n-**DIMENSIONAL HYPERCUBE** or n-**CUBE**, denoted by Q_n, is the simple graph whose vertices represent the 2^n bit strings of length n. Two vertices are adjacent iff the bit strings that they represent differ in exactly one bit position.

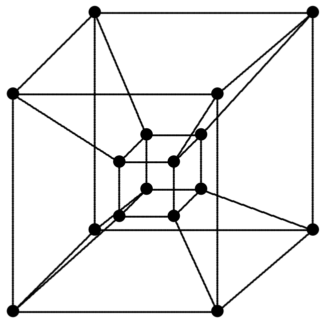
Q_1



Q_2



Q_3



Q_4

Label the vertices yourself.

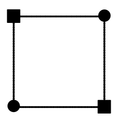
BIPARTITE GRAPHS

DEFINITION:

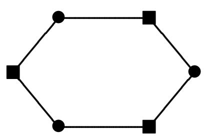
A simple graph $G = (V, E)$ is called **BIPARTITE** if V can be divided into two disjoint sets V_1, V_2 such that every edge connects a vertex in V_1 to a vertex in V_2.

When this condition holds, we call (V_1, V_2) a **BIPARTITION** of V.

The figure below shows that C_4 and C_6 are bipartite. (The two types of vertices indicate the bipartitions). However, C_3 is not bipartite since in any division of the vertex into two disjoint subsets, one of them must contain two vertices. But any two vertices in C_3 are adjacent. In a similar way, we can show that C_5 is not bipartite. (Later, we'll see a better way to show this.)

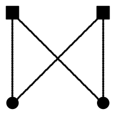


C_4

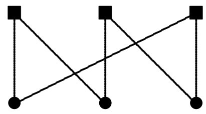


C_6

Usually, bipartite graphs are drawn so that the bipartitions are on different sides:



C_4



C_6

THEOREM:

A graph if bipartite iff it contains no odd cycles.

PROOF: Suppose G is bipartite. Since the vertices in a cycle must alternate between V_1 and V_2. It must have an even number of vertices.

We omit the proof of the converse.

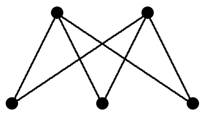
Thus we see that W_n, K_n, $n \geq 3$ are not bipartite since they all contain C_3.

DEFINITION:

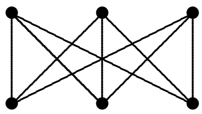
Let $m, n \in \mathbb{Z}^+$. A **COMPLETE BIPARTITE GRAPH** on (m, n) vertices, denoted $K_{m,n}$, is a simple graph

- with vertices $v_1, v_2, \ldots, v_m$, $w_1, w_2, \ldots, w_n$;
- edges $v_i w_j$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Note that: $|E(K_{m,n})| = mn$.



$K_{2,3}$



$K_{3,3}$

REMARK

These special graphs have applications such as job assignments, local area network, interconnection network for parallel computing. Read about these in the text.

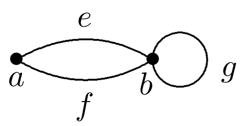
DEFINITION:

A graph $H = (W, F)$ is a **SUBGRAPH** of a graph $G = (V, E)$ if

$$W \subseteq V \quad \text{and} \quad F \subseteq E.$$

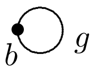
EXAMPLE

All the subgraphs of

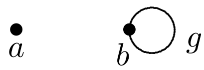


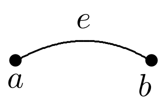
(1) 

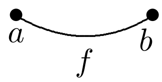
(2) 

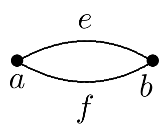
(3) 

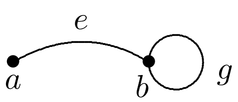
(4) 

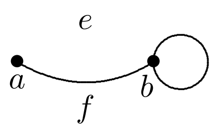
(5) 

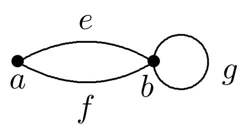
(6) 

(7) 

(8) 

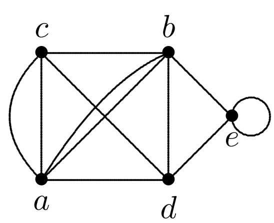
(9) 

(10) 

(11) 

SECTION 6.3 REPRESENTING GRAPHS

One way to represent a graph is to **LIST** all its edges, including multiplicity. Another way is by means of an **ADJACENCY LIST** as shown below.

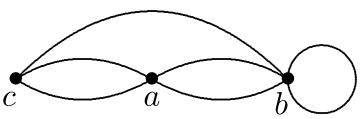


Vert.	*Adj. vert.*
a	*bbccd*
b	*aacde*
c	*aabd*
d	*abce*
e	*bde*

DEFINITION:

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. The **ADJACENCY MATRIX** of G is the matrix $A = (a_{i,j})$ where the rows and columns both correspond to the vertices

$$a_{i,j} = \text{the number of edges joining } v_i \text{ and } v_j.$$



$$\begin{array}{c|c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

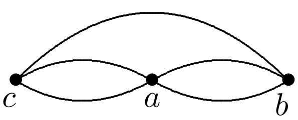
• Note that the adjacency matrix of a graph is symmetric.

• Conversely, given any $n \times n$ symmetric matrix with nonnegative integer entries, one can obtain a unique graph with n vertices. Thus the adjacency matrix uniquely determines the corresponding graph.

DEFINITION:

Let $G = (V, E)$ be a multigraph with $V(G) = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The **INCIDENCE MATRIX** of G is the $n \times m$ matrix $M = (m_{i,j})$ where the rows correspond to the vertices, the columns correspond to the edges and

$$m_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{array}{c|c} & \begin{array}{ccccc} ac & ac & ab & ab & bc \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

SECTION 6.4 CONNECTEDNESS IN GRAPHS

DEFINITION:

Let G be a graph and $n \in \mathbb{Z}^*$. A **PATH OF LENGTH** n from vertex u to vertex v is an alternating sequence of vertices and edges of G:

$$v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$$

where $u = v_0$, $v = v_n$, and $e_i = v_{i-1}v_i$ for $i = 1, \dots, n$.

A **CIRCUIT** is a path with at least one edge and with $u = v$.

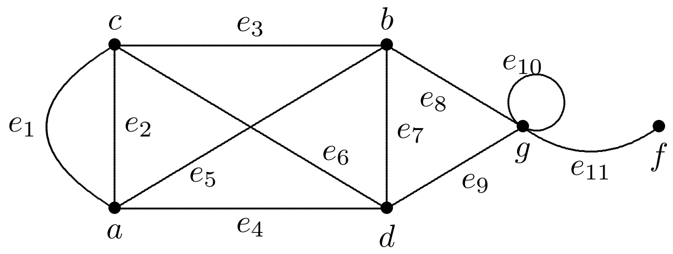
A path or circuit is **SIMPLE** if the edges it traverses are pairwise distinct.

Note: A cycle is a simple circuit where $v_0, v_1, \dots, v_{n-1}$ are distinct.

When the graph is simple or when it is not necessary to distinguish between multiple edges, we will denote a path by the vertices it passes through:

$$v_0 v_1 \dots v_n.$$

EXAMPLE



- a — path of length 0.
- $ae_1ce_2ae_4d$ — simple path
- $be_8ge_{11}fe_{11}ge_9de_7b$ — circuit
- ce_3be_3c — circuit
- $ge_{10}g$ — simple circuit. (It is C_1).
- $be_8ge_{10}ge_9de_6ce_3b$ — simple circuit (not a cycle)
- $ce_6de_7be_3c$ — simple circuit. (It is C_3).
- ce_1ae_2c — simple circuit. (It is C_2).

THEOREM:

Let

$$v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$$

where $v_n = v_0$ be a circuit. Then, for any i,

$$v_i e_{i+1} v_{i+1} \ldots e_n v_n e_1 v_1 e_2 \ldots v_{i-1} e_i v_i$$

is also a circuit.

CONNECTEDNESS IN GRAPHS

DEFINITION:

A graph G is **CONNECTED** if there is a path between every pair of distinct vertices.

THEOREM:

Let G be a connected graph. If e is an edge that belongs to a cycle, then deleting e from G yields a graph G' which remains connected.

PROOF: Let

$$v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$$

where $v_n = v_0$ be a cycle containing the edge e. Without loss of generality, we can assume $e_1 = e$. Let u, v be a pair of distinct vertices in G'. Then there is a simple path connecting u to v in G. If this path does not contain e, then it is a path in G'. If it contains e, i.e., the path contains the sequence $v_1 e v_0$. Note that e appears exactly once in the path. Now replace the sequence $v_1 e v_0$ by

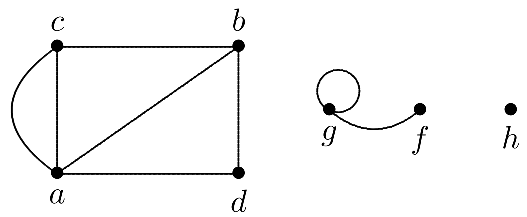
$$v_1 e_2 \ldots v_{n-1} e_n v_n.$$

This gives a path in G' connecting u to v. Thus G' is connected.

DEFINITION:

A maximal connected subgraph H of a graph G is called a **CONNECTED COMPONENT** of G.

This means any vertex not in H is not connected to H.



Has 3 connected components

THEOREM:

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $A = (a_{ij})$ its adjacency matrix. Then for each nonnegative integer k, the (i, j)-entry of $A^k = (a_{ij}^k)$ is the number of paths of length k from vertex v_i to vertex v_j.

PROOF: Use mathematical induction.

For $k = 1$, $a_{i,j}$ is the number edges joining v_i to v_j and is thus the number of paths of length 1 from v_i to v_j.

Now for any $k \geq 1$, assume that $a_{i,j}^k$, the (i, j)-entry of A^k gives the number of paths of length k from v_i to v_j.

First note that in order to go from v_i to v_j along a path of length k, one must first get to a neighbour of v_j along a path of length $k - 1$.

Thus The number of paths of length $k + 1$ from v_i to v_j is the sum of

No. of paths of length k from v_i to $v_1 \times$ No. of edges from v_1 to v_j
No. of paths of length k from v_i to $v_2 \times$ No. of edges from v_2 to v_j
...
No. of paths of length k from v_i to $v_n \times$ No. of edges from v_n to v_j

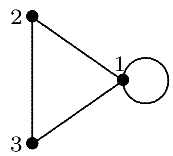
This sum is equal to

$$\sum_{m=1}^{n} a_{i,m}^k a_{m,j} = a_{i,j}^{k+1}.$$

This completes the proof by induction.

EXAMPLE

For the graph below, we have



$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad A^3 = \begin{pmatrix} 7 & 5 & 5 \\ 5 & 3 & 4 \\ 5 & 4 & 3 \end{pmatrix}$$

Since $a_{11}^2 = 3$, there are 3 paths of length 2 from v_1 to v_1: $v_1v_2v_1$, $v_1v_3v_1$, $v_1v_1v_1$.

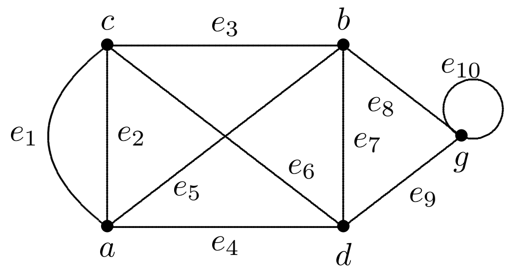
Since $a_{23}^3 = 4$, there are 4 paths of length 3 from v_2 to v_3: $v_2v_1v_1v_3$, $v_2v_3v_1v_3$, $v_2v_1v_2v_3$, $v_2v_3v_2v_3$.

Problem: Since $a_{1,1}^3 = 7$, there are 7 paths of length 3 from v_1 to v_1. Find them.

SECTION 6.5 EULER CIRCUITS

DEFINITION:

An **EULER CIRCUIT** a graph G is a simple circuit that contains every vertex and every edge of G.



An Euler circuit is $e_3e_1e_2e_6e_7e_5e_4e_9e_{10}e_8$.

THEOREM:

A connected graph has an Euler circuit iff every vertex is of even degree.

PROOF: Suppose that G has an Euler circuit. When one traverses an Euler circuit, each time one enters a vertex via one edge, one must exit via another edge. Since the edges at every vertex must be paired, the number must be even. Hence the degree of every vertex is even.

We outline a proof of the converse.

We suppose that G is a connected graph such that the degree of every vertex is even.

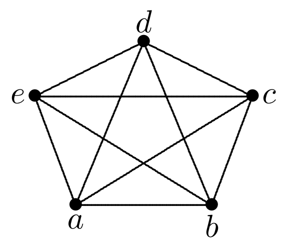
Start from any vertex a. Each time your traverse an edge to go to another vertex $b(\neq a)$, you remove the edge. Since the degree of b was even, so upon the deletion of the edge, the degree of b becomes odd. That means there is still at least one edge incident to b. So you can leave b via that edge. Since the graph is finite, you must eventually return to a. Thus we get a circuit.

If there are still edges remaining, one of them must be incident to a vertex, say c, of the circuit that we have just deleted.

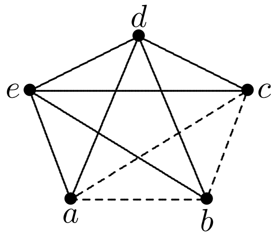
Now repeat the procedure with c to obtain another circuit. This circuit can be merge with the earlier circuit to form a larger circuit.

Repeat this until all edges have been exhausted, and what we get is an Euler circuit.

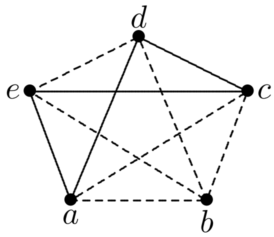
EXAMPLE



First we get *abca*.



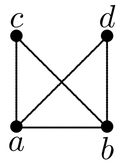
Then we get *bdeb*, which is combined with the previous circuit to get $a(bdeb)ca$ or *abdebca*.



Finally we get *daecd*, which is merged to get $ab(daecd)ebca$ which is an Euler circuit.

DEFINITION:

An **EULER PATH** of a graph G is a simple path which is not a circuit and which contains all the edges and all the vertices of G.



Euler Path: *acbdab*

COROLLARY

A graph has an Euler path iff it is connected and all vertices, but two, have even degrees.

PROOF: Let G be the graph and the two vertices of odd degree be u and w. Add a new edge joining u and w to obtain a new graph G'. Then G' is connected and every vertex if of even degree. Thus G' contains an Euler circuit. An Euler path in G is now obtained by removing the added edge.