NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 2**

1. Determine which of the following statements are true. Justify your answer.

   (1) If an augmented matrix of a linear system has only one row, then it is always in row-echelon form.

   Answer. True.

   (2) A (nonzero) linear equation is always consistent.

   Answer. True. By definition, the coefficients of variables are not all zero. So we do not have the cases such as $\sum_{i=1}^{n} 0x_i = b$.

   (3) For a homogeneous system, the number of arbitrary parameters in a general solution equals the difference of the number of variables and the number of the pivot columns.

   Answer. True.

   (4) For an arbitrary augmented matrix in row-echelon form, the number of pivot columns equals the number of nonzero rows.

   Answer. True.

   (5) For an arbitrary augmented matrix in row-echelon form, the number of pivot columns equals the number of nonzero columns.

   Answer. False. For example, $(1\ 2\ 3 \mid 0)$ is in row-echelon form, which has 3 nonzero columns. But it has only one pivot column.

2. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{cases} -2x^2 & + & y^2 & - & 3z^2 & = & -13 \\ 4x^2 & - & y^2 & + & 5z^2 & = & 25 \\ -x^2 & - & 3y^2 & + & 3z^2 & = & -1 \end{cases}$$

   (1) Use the substitutions $x_1 = x^2$, $x_2 = y^2$ and $x_3 = z^2$, and solve the linear systems in 3 variables x_1, x_2 and x_3.

   Answer. First, using the substitutions $x_1 = x^2$, $x_2 = y^2$ and $x_3 = z^2$, we have a linear system

$$\begin{cases} -2x_1 & + & x_2 & - & 3x_3 & = & -13 \\ 4x_1 & - & x_2 & + & 5x_3 & = & 25 \\ -x_1 & - & 3x_2 & + & 3x_3 & = & -1 \end{cases}$$

Secondly, we apply Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} -2 & 1 & -3 & -13 \\ 4 & -1 & 5 & 25 \\ -1 & -3 & 3 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & -3 & 3 & -1 \\ 4 & -1 & 5 & 25 \\ -2 & 1 & -3 & -13 \end{array}\right)$$

$$\xrightarrow[R_3-2R_1]{R_2+4R_1} \left(\begin{array}{ccc|c} -1 & -3 & 3 & -1 \\ 0 & -13 & 17 & 21 \\ 0 & 7 & -9 & -11 \end{array}\right) \xrightarrow{R_3+\frac{7}{13}R_2} \left(\begin{array}{cccc} -1 & -3 & 3 & -1 \\ 0 & -13 & 17 & 21 \\ 0 & 0 & \frac{2}{13} & \frac{4}{13} \end{array}\right)$$

$$\xrightarrow{(-1)R_1,\ \frac{1}{13}R_2,\ \frac{13}{2}R_3} \left(\begin{array}{cccc} 1 & 3 & -3 & 1 \\ 0 & 1 & -\frac{17}{13} & -\frac{21}{13} \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow[R_2+\frac{17}{13}R_3]{R_1+3R_3} \left(\begin{array}{cccc} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1-3R_2} \left(\begin{array}{cccc} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Finally, we have

$$\begin{cases} x_1 &= 4 \\ x_2 &= 1 \\ x_3 &= 2 \end{cases}$$

(2) Following the calculation in Part 1, solve the system of equations

$$\begin{cases} x^2 &= x_1 \\ y^2 &= x_2 \\ z^2 &= x_3 \end{cases}$$

Answer.

$$\begin{cases} x^2 &= 4 \\ y^2 &= 1 \\ z^2 &= 2 \end{cases} \Rightarrow \begin{cases} x &= \pm 2 \\ y &= \pm 1 \\ z &= \pm\sqrt{2} \end{cases}$$

3. Determine the values of a for which the following homogeneous linear system has non-trivial solutions.

$$\begin{cases} 6x - y + z = 0 \\ ax + z = 0 \\ y + az = 0 \end{cases}$$

Answer. Let us apply Gauss elimination:

$$\left(\begin{array}{ccc|c} 6 & -1 & 1 & 0 \\ a & 0 & 1 & 0 \\ 0 & 1 & a & 0 \end{array}\right) \xrightarrow{R_2-\frac{a}{6}R_1} \left(\begin{array}{ccc|c} 6 & -1 & 1 & 0 \\ 0 & \frac{a}{6} & 1-\frac{a}{6} & 0 \\ 0 & 1 & a & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 6 & -1 & 1 & 0 \\ 0 & 1 & a & 0 \\ 0 & \frac{a}{6} & 1-\frac{a}{6} & 0 \end{array}\right)$$

$$\xrightarrow{R_3-\frac{a}{6}R_2} \left(\begin{array}{ccc|c} 6 & -1 & 1 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1-\frac{a}{6}-\frac{a^2}{6} & 0 \end{array}\right).$$

Then the homogeneous linear system has non-trivial solutions if and only if $1 - \frac{a}{6} - \frac{a^2}{6} = 0$, i.e., $a = -3$ or $a = 2$.

4. Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} 3 & 1 & 5 \\ 2 & -1 & -3 \end{pmatrix}.$$

(1) Evaluate the following, whenever possible.

$$(a)\ 2\boldsymbol{B}, \quad (b)\ \boldsymbol{A} + 2\boldsymbol{B}, \quad (c)\ \boldsymbol{AB}, \quad (d)\ \boldsymbol{A}^2, \quad (e)\ \boldsymbol{BA}.$$

Answer.

$$2\boldsymbol{B} = \begin{pmatrix} 6 & 2 & 10 \\ 4 & -2 & -6 \end{pmatrix}, \quad \boldsymbol{AB} = \begin{pmatrix} 12 & -1 & 1 \\ 1 & -3 & -11 \end{pmatrix}, \quad \boldsymbol{A}^2 = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix}.$$

$\boldsymbol{A} + 2\boldsymbol{B}$ and $\boldsymbol{BA}$ are not defined.

(2) Is $\boldsymbol{AB} = \boldsymbol{BA}$?

Answer. No, $\boldsymbol{AB} \neq \boldsymbol{BA}$ since $\boldsymbol{BA}$ is not defined.

5. Write each of the following systems of equations as a matrix equation.

$$(a) \begin{cases} 2x_1 + x_2 - x_3 = 6 \\ 3x_1 - 2x_2 + 3x_3 = 7 \end{cases} \qquad (b) \begin{cases} 2x_1 + x_2 + x_3 = 5 \\ x_1 + x_2 - 2x_3 = 1 \\ 3x_1 - 2x_2 + x_3 = 2 \end{cases}$$

Answer.

$$(a) \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

$$(b) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$