Properties

All useful properties

$$AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1B \\ a_2B \\ \vdots \\ a_nB \end{pmatrix}$$

$$\sum_{x=1}^{m}\sum_{y=1}^{n} a_{ix}b_{yj} = \sum_{y=1}^{n}\sum_{x=1}^{m} a_{ix}b_{yj}$$

$$(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$$

$$(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$$

$$(\boldsymbol{A}^{\mathrm{T}})^{-1} = (\boldsymbol{A}^{-1})^{\mathrm{T}}$$

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

$$\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$$

$$\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$$

$$\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$$

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A})$$

$$\det(\mathbf{adj}(\boldsymbol{A})) = \det(\boldsymbol{A})^{n-1}$$

$$\mathbf{adj}(\mathbf{adj}(\boldsymbol{A})) = \det(\boldsymbol{A})^{n-2}\boldsymbol{A}$$

$\dim(V) = |S|$, S is a basis for V.

$$\mathcal{CS}(\boldsymbol{A}_{m\times n}) = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n\}$$

$$\mathcal{NS}(\boldsymbol{A}_{m\times n}) = \{\boldsymbol{u} \mid \boldsymbol{Au} = \boldsymbol{0}\}$$

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}$$

$$(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w}$$

$$\boldsymbol{u}\cdot\boldsymbol{u} \geq 0, \text{and } \boldsymbol{u}\cdot\boldsymbol{u} = 0$$

$$\Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$$

$$\theta = \arccos\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

Theorems

Jesus' Theorem

1. $\boldsymbol{A}$ is invertible.
2. $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$.
4. $\boldsymbol{A}$ is a product of elementary matrices.
5. $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution $\boldsymbol{b}, \forall \boldsymbol{b} \in \mathbb{R}^n$.
6. $\det(\boldsymbol{A}) \neq 0$
7. Columns and rows of $\boldsymbol{A}$ are linearly independent.
8. Columns and rows of $\boldsymbol{A}$ span $\mathbb{R}^n$.
9. Columns and rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
10. $\mathrm{rank}(\boldsymbol{A}) = n$
11. $\mathrm{nullity}(\boldsymbol{A}) = 0$
12. $\mathcal{NS}(\boldsymbol{A})^{\perp} = \mathbb{R}^n$
13. $\mathcal{RS}(\boldsymbol{A})^{\perp} = \{\boldsymbol{0}\}$
14. $\ker(T_{\boldsymbol{A}}) = \{\boldsymbol{0}\}$
15. $\mathrm{range}(T_{\boldsymbol{A}}) = \mathbb{R}^n$
16. $T_{\boldsymbol{A}}$ is one-one.
17. $\lambda = 0$ is not an eigenvalue.

Bases

Let S be a basis for a vector space V where $|S| = k$. Then, if $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k \in V$,

1. $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ are linearly dependent in V $\Leftrightarrow (\boldsymbol{v}_1)_S, \cdots, (\boldsymbol{v}_k)_S$ are linearly dependent in $\mathbb{R}^k$.
2. $\mathrm{span}\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\} = V \Leftrightarrow \mathrm{span}\{(\boldsymbol{v}_1)_S, \cdots, (\boldsymbol{v}_k)_S\} = \mathbb{R}^k$

Let V be a vector space of dimension k and S a subset of V. Then (equiv.):

1. S is a basis for V.
2. S is lin. indep and $|S| = k$.
3. S spans V and $|S| = k$.

Dimensions

If U is a subspace of V, then $\dim(U) \leq \dim(V)$. If $\dim(U) = \dim(V)$, then $U = V$.

Row/Col Spaces

EROs preserve the row space but not the column space (in general). EROs preserve the linear relations between cols.

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a lin. sys. Then if $\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n$ are the cols. of A, (equiv.)

1. $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent
2. $\boldsymbol{b} \in \mathrm{span}\{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n\}$
3. $\mathrm{span}\{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n\} = \mathrm{span}\{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n, \boldsymbol{b}\}$
4. $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}|\boldsymbol{b})$

Ranks

$\boldsymbol{A}_{m\times n}$; $\mathrm{rank}(\boldsymbol{A}) = \min\{m, n\}$.

$\mathrm{rank}(\boldsymbol{AB}) \leq \min\{\mathrm{rank}(\boldsymbol{A}), \mathrm{rank}(\boldsymbol{B})\}$

$\mathcal{CS}(\boldsymbol{AB}) \subseteq \mathcal{CS}(A)$

$\mathcal{RS}(\boldsymbol{AB}) \subseteq \mathcal{RS}(B)$

If $\mathcal{CS}(\boldsymbol{AB}) \subsetneq \mathcal{CS}(\boldsymbol{A})$, then $\boldsymbol{B}$ is not invertible.

Orthogonality

If $S = \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then $\forall \boldsymbol{w} \in \mathbb{R}^n$,

$$\boldsymbol{p} = \frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u}_1 + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\boldsymbol{u}_k\cdot\boldsymbol{u}_k}\boldsymbol{u}_k$$

where $\boldsymbol{p}$ is the projection of $\boldsymbol{w}$ onto V and $\boldsymbol{w} = \boldsymbol{n} + \boldsymbol{p}$, where $\boldsymbol{n}$ is orthogonal to V. $\boldsymbol{w}$ is said to be orthogonal to V if $\boldsymbol{w}\cdot\boldsymbol{u}_i = 0, \forall i$. $\boldsymbol{u}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ iff $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Au} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

A square matrix $\boldsymbol{A}$ is orthogonal iff $\boldsymbol{AA}^{\mathrm{T}} = \boldsymbol{I}$. Then, (equiv.); the rows/cols of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$. The transition matrix between two orthonormal bases is orthogonal.

Diagonalisation

If $\boldsymbol{A}_{n\times n}$ is a square matrix, then a **nonzero** $\boldsymbol{u} \in \mathbb{R}^n$ is an eigenvector of $\boldsymbol{A}$ if $\boldsymbol{Au} = \lambda\boldsymbol{u}$, for some $\lambda \in \mathbb{R}$. Char. poly. of $\boldsymbol{A}$ is $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$.

If $\boldsymbol{A}$ is triangular, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries. $\boldsymbol{A}$ is diagonalisable $\Leftrightarrow \boldsymbol{A}$ has n lin. indep. eigenvectors.

In the char. poly φ, $\varphi(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2}\cdots$, the # of eigenvectors associated with λ_i is bounded above by r_i. If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalisable.

$\boldsymbol{A}$ is ortho-diag if can $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP} = \boldsymbol{D}$.

$\boldsymbol{A}$ is symmetric $\Leftrightarrow \boldsymbol{A}$ is ortho-diag.

Lin. Transforms

If T is a lin. transform, then $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$ and $T(\boldsymbol{0}) = \boldsymbol{0}$.

For T defined by standard matrix $\boldsymbol{A}$, $\mathrm{range}(T) = \mathcal{CS}(\boldsymbol{A})$ and $\ker(T) = \mathcal{NS}(\boldsymbol{A})$.

If $T\colon \boldsymbol{B}$ and $S\colon \boldsymbol{A}$, then $(T \circ S)(\boldsymbol{x}) = \boldsymbol{BAx}$.

Algorithms

Gauss-Jordan Elimination

1. Find first nonzero row (interchange if needed) and delete downwards.
2. Move to next nonzero row and delete, repeat until last nonzero row. If zero row reached, move to bottom - stop for Gaussian Elimination.
3. Make all leading entries 1.
4. Start with last nonzero row and delete rightmost entry upwards, repeat until top row.

Values for Consistency

Always ensure no DIV/0.

1. Convert augmented matrix to REF.

2. If any possible DIV/0 in REF conversion, isolate values causing DIV/0 and sub into original augmented matrix $\Rightarrow$ New Case
3. Continue where left off, considering all cases except isolated values.

Inverse

1. Through adjoint:
   (a) For each entry, find the cofactor and replace the entry.
   (b) Transpose
   (c) Divide by determinant of original
2. Through GJE
   (a) Form $(\boldsymbol{A}|\boldsymbol{I})$ and convert to RREF to obtain $(\boldsymbol{I}|\boldsymbol{A}^{-1})$.

Determinant

1. Choose any row or column and calculate the cofactor of all entries in the row.
2. Take the dot product of the row/column and the set of cofactors.

$$A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$$

Determinant of a triangular matrix is the product of the diagonal entries.

Cramer's Rule

For a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x}_i$ is obtained by replacing the i-th column of $\boldsymbol{A}$ with b, then find $\frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}$.

Show Span $\subseteq$ Span

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. To show $\text{span}(S) \subseteq \text{span}(T)$, form $\left(\begin{array}{ccc|c|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array}\right)$ and show consistency. In essence, for $S \subseteq T$, show $(T|S)$ consistent.

Transition Matrix

Transition matrix from S to T is the matrix whose columns are the vectors of S expressed in the basis of T. $\boldsymbol{P}_{S\to T}$ is given by converting $(T|S)$ into $(\boldsymbol{I}|\boldsymbol{P}_{S\to T})$.

Finding Basis for $\boldsymbol{V}$

- With new vectors:
  1. Place the vectors as the rows of a matrix and convert to REF
  2. Take the nonzero vectors as the basis vectors
- With existing vectors:
  1. Place the vectors as the columns of a matrix and convert to REF
  2. Take the pivot columns of the REF and use the corresponding columns of the original

To extend the basis to some n-space, insert the standard basis vectors corresponding to nonpivot cols.

Gram-Schmidt

Choose $\boldsymbol{v}_1 = \boldsymbol{u}_1$. Then,

$$\boldsymbol{v}_k = \boldsymbol{u}_k - \sum_{i=1}^{k-1} \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_i}{\boldsymbol{v}_i \cdot \boldsymbol{v}_i} \boldsymbol{v}_i$$

To normalise, take $\boldsymbol{v}_i' = \frac{1}{\boldsymbol{v}_i \cdot \boldsymbol{v}_i} \boldsymbol{v}_i$.

Least Squares Solution

Compute $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$ then solve $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

Diagonalisation

1. Find all eigenvalues λ_i by solving the characteristic equation.
2. For each eigenvalue, find a basis B_{λ_i} for the eigenspace E_{λ_i}.
3. Let $S = \bigcup_{i=1}^{k} S_{\lambda_i}$.
4. If $|S| < n$, then not diagonalisable.
5. If $|S| = n$, then diagonalisable. Take $\boldsymbol{P}$ as the matrix whose columns are vectors of S, and $\boldsymbol{D}$ as the eigenvalues associated with S vectors *in the same order*.

Recurrence Relations

If the sequence is $a_0 = c_0, a_1 = c_1, \cdots$ with some $a_n = p(a_{n-1}) + q(a_{n-2})$, form the linear system of $\begin{pmatrix} a_n & a_{n+1} & a_{n+2} \end{pmatrix}^{\mathrm{T}} = \boldsymbol{A}\begin{pmatrix} a_{n-1} & a_n & a_{n+1} \end{pmatrix}$, where $\boldsymbol{A}$ is the coefficient matrix formed through info given. Find the eigenvalues of $\boldsymbol{A}$ and diagonalise $\boldsymbol{A}$. Then, express $\boldsymbol{x}_n$ as $\boldsymbol{A}^n\boldsymbol{x}_0 = \boldsymbol{PD}^n\boldsymbol{P}^{-1}\boldsymbol{x}_0$.

Ortho-Diagonalisation

Same as normal diagonalisation except send every eigenbasis to the Gram-Schmidt. If the matrix isn't symmetric initially, it can't be OD.

Show Mapping is LT

Show that $T(c\boldsymbol{u}+d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$. To show not LT, try $T(\boldsymbol{0}) \neq 0$ first.

Finding Formula of LT

Find the solution set of column space of T for some $(x_1, x_2, \cdots, x_k)^{\mathrm{T}}$. This is really the general formula for the subspace that forms range(T).

Proofs

Idempotent/Projection
Nilpotent
Stochastic (and doubly)
Permutation
Persymmetric
Antisymmetric
Involutory
Orthogonal
Unipotent