Answers/Solutions of Exercise 7 (Version: November 14, 2018)

1. (a) T_1 is a linear transformation with standard matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

   (b) T_2 is not a linear transformation.

   (c) T_3 is a linear transformation with standard matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (d) T_4 is not a linear transformation.

   (e) T_5 is a linear transformation with standard matrix $\begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$.

   (f) T_6 is not a linear transformation.

2. (a) There is enough information.

   $$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y \\ 3x+2y+4z \\ -y+z \\ x+4y+6z \end{pmatrix} \quad \text{for} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

   The standard matrix is $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 4 & 6 \end{pmatrix}$.

   (b) The information is not enough because the two vectors do not form a basis for $\mathbb{R}^2$.

   (c) There is enough information.

   $$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-y \\ x+y \end{pmatrix} \quad \text{for} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

   The standard matrix is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

   (d) There is enough information.

   $$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} x+17y-8z \\ x+22y-8z \end{pmatrix} \quad \text{for} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

   The standard matrix is $\begin{pmatrix} \frac{1}{5} & \frac{17}{5} & \frac{-8}{5} \\ \frac{1}{5} & \frac{22}{5} & \frac{-8}{5} \end{pmatrix}$.

(e) There is enough information.

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y+z \\ z \\ x+z \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

The standard matrix is $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(f) The information is not enough because the three vectors do not form a basis for $\mathbb{R}^3$.

3. (a) $(S \circ T)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 2y \\ x+y \end{pmatrix}$.

$T \circ S$ is not defined.

(b) $(S \circ T)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -2x-y+3z \\ -x-y+3z \\ -3x-2y+6z \end{pmatrix}$.

$(T \circ S)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x+2y \\ 2x+y \end{pmatrix}$.

4. ($\Rightarrow$) It is a particular case of Theorem 7.1.4.2.

($\Leftarrow$) Suppose

$$T(c\boldsymbol{u} + d\boldsymbol{v}) = c\,T(\boldsymbol{u}) + d\,T(\boldsymbol{v}) \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n \text{ and } c, d \in \mathbb{R}. \qquad (*)$$

Let $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$ and let $\boldsymbol{A}$ be the $m \times n$ matrix $\begin{pmatrix} T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdots & T(\boldsymbol{e_n}) \end{pmatrix}$.

For any $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{u} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + \cdots + u_n\boldsymbol{e_n}$. By applying $(*)$ repeatedly, we have

$$\begin{aligned} T(\boldsymbol{u}) &= u_1T(\boldsymbol{e_1}) + u_2T(\boldsymbol{e_2}) + \cdots + u_nT(\boldsymbol{e_n}) \\ &= \begin{pmatrix} T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdots & T(\boldsymbol{e_n}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\ &= \boldsymbol{A}\boldsymbol{u}. \end{aligned}$$

Thus T is a linear transformation.

5. (a) For any $\boldsymbol{u} \in \mathbb{R}^n$, $(T_1 + T_2)(\boldsymbol{u}) = T_1(\boldsymbol{u}) + T_2(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = (\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u}$. So $T_1 + T_2$ is a linear transformation and the standard matrix for $T_1 + T_2$ is $\boldsymbol{A} + \boldsymbol{B}$.

   (b) For any $\boldsymbol{u} \in \mathbb{R}^n$, $(\lambda T)(\boldsymbol{u}) = \lambda T(\boldsymbol{u}) = \lambda \boldsymbol{A}\boldsymbol{u} = (\lambda \boldsymbol{A})\boldsymbol{u}$. So λT is a linear transformation and the standard matrix for λT is $\lambda \boldsymbol{A}$.

6. (a) (i) T is invertible and the inverse of T is T itself.

   (ii) T is not invertible. Assume there exists an inverse $S : \mathbb{R}^2 \to \mathbb{R}^2$. Then $(1, 0)^{\mathrm{T}} = S \circ T((1, 0)^{\mathrm{T}}) = S((1, 0)^{\mathrm{T}}) = S \circ T((0, 1)^{\mathrm{T}}) = (0, 1)^{\mathrm{T}}$, a contradiction.

   (b) $\boldsymbol{A}^{-1}$.

7. (a) Note that $(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}$ where LHS is the scalar $\boldsymbol{n} \cdot \boldsymbol{x}$ multiplied to the vector $\boldsymbol{n}$ while all operations on RHS are matrix multiplications. (To verify the equation, let $\boldsymbol{n} = (a_1, \ldots, a_n)^{\mathrm{T}}$ and $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ and then check that both sides give us the same vector.)

   For any $\boldsymbol{x} \in \mathbb{R}^n$, $P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = \boldsymbol{I}\boldsymbol{x} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{x}$. So P is a linear transformation and the standard matrix for P is $\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$.

   (b) Since for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$
\begin{aligned}
(P \circ P)(\boldsymbol{x}) = P(P(\boldsymbol{x})) &= P(\boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n}) \\
&= \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \{\boldsymbol{n} \cdot [\boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n}]\}\boldsymbol{n} \\
&= \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \{(\boldsymbol{n} \cdot \boldsymbol{x}) - (\boldsymbol{n} \cdot \boldsymbol{x})(\boldsymbol{n} \cdot \boldsymbol{n})\}\boldsymbol{n} \\
&= \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = P(\boldsymbol{x}),
\end{aligned}
$$

   $P \circ P = P$.

   Alternatively, since $\boldsymbol{n}$ is a unit vector, $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{n} = 1$. Thus

$$
(\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^2 = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} + \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}.
$$

   By Theorem 7.1.11, $P \circ P = P$.

8. (a) Suppose T is not the zero transformation. So there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $T(\boldsymbol{x}) \neq 0$. Define $\boldsymbol{u} = T(\boldsymbol{x})$. Then $\boldsymbol{u}$ is a nonzero vector and

$$
T(\boldsymbol{u}) = T(T(\boldsymbol{x})) = (T \circ T)(\boldsymbol{x}) = T(\boldsymbol{x}) = \boldsymbol{u}.
$$

   (b) Suppose T is not the identity transformation. So there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $T(\boldsymbol{y}) \neq \boldsymbol{y}$. Define $\boldsymbol{v} = T(\boldsymbol{y}) - \boldsymbol{y}$. Then $\boldsymbol{v}$ is a nonzero vector and

$$
T(\boldsymbol{v}) = T(T(\boldsymbol{y}) - \boldsymbol{y}) = (T \circ T)(\boldsymbol{y}) - T(\boldsymbol{y}) = T(\boldsymbol{y}) - T(\boldsymbol{y}) = \boldsymbol{0}.
$$

(c) Let $\boldsymbol{A}$ be the standard matrix for T. If T is not the zero transformation and the identity transformation, then by (a) and (b), 1 and 0 are the eigenvalues of $\boldsymbol{A}$. So by the result of Question 6.4,

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix} \text{ where } st = r(1-r).$$

9. (a) Similar to Question 7.7, for any $\boldsymbol{x} \in \mathbb{R}^n$, $F(\boldsymbol{x}) = \boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} = \boldsymbol{I}\boldsymbol{x} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x} = (\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{x}$. So F is a linear transformation and the standard matrix for F is $\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$.

(b) Since for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}(F \circ F)(\boldsymbol{x}) = F(F(\boldsymbol{x})) &= F(\boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n}) \\ &= \boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - 2\{\boldsymbol{n} \cdot [\boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n}]\}\boldsymbol{n} \\ &= \boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - 2\{(\boldsymbol{n} \cdot \boldsymbol{x}) - 2(\boldsymbol{n} \cdot \boldsymbol{x})(\boldsymbol{n} \cdot \boldsymbol{n})\}\boldsymbol{n} \\ &= \boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - 2\{-(\boldsymbol{n} \cdot \boldsymbol{x})\} = \boldsymbol{x},\end{aligned}$$

$F \circ F$ is the identity transformation.

Alternatively,

$$(\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^2 = (\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \boldsymbol{I} - 4\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} + 4\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \boldsymbol{I}.$$

By Theorem 7.1.11, $F \circ F$ is the identity transformation.

(c) Note that $(\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{I} - 2(\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$. Thus

$$(\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^{\mathrm{T}} = (\boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})^2 = \boldsymbol{I}$$

by (b). The standard matrix is an orthogonal matrix.

10. (a) By Theorem 7.1.4.2,

$$\begin{aligned}T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v}) &= (T(\boldsymbol{u}) + T(\boldsymbol{v})) \cdot (T(\boldsymbol{u}) + T(\boldsymbol{v})) \\ &= T(\boldsymbol{u}) \cdot T(\boldsymbol{u}) + 2(T(\boldsymbol{u}) \cdot T(\boldsymbol{v})) + T(\boldsymbol{v}) \cdot T(\boldsymbol{v}) \\ &= ||T(\boldsymbol{u})||^2 + ||T(\boldsymbol{v})||^2 + 2(T(\boldsymbol{u}) \cdot T(\boldsymbol{v})) \\ &= ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2(T(\boldsymbol{u}) \cdot T(\boldsymbol{v})). \qquad (1)\end{aligned}$$

On the other hand,

$$\begin{aligned}T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v}) &= ||T(\boldsymbol{u}+\boldsymbol{v})||^2 \\ &= ||\boldsymbol{u}+\boldsymbol{v}||^2 \\ &= (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v}) \\ &= \boldsymbol{u} \cdot \boldsymbol{u} + 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{v} \\ &= ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v}). \qquad (2)\end{aligned}$$

Thus (1) and (2) imply $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$.

(b) ($\Leftarrow$) Suppose $\boldsymbol{A}$ is an orthogonal matrix of order n. Then by Question 5.32, for all $\boldsymbol{u} \in \mathbb{R}^n$,

$$||T(\boldsymbol{u})|| = ||\boldsymbol{A}\boldsymbol{u}|| = ||\boldsymbol{u}||.$$

So T is an isometry.

($\Rightarrow$) Suppose T is an isometry on $\mathbb{R}^n$. Let $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$. Then

$$\begin{aligned}(\boldsymbol{A}\boldsymbol{e_i}) \cdot (\boldsymbol{A}\boldsymbol{e_j}) = (\boldsymbol{A}\boldsymbol{e_i})^{\mathrm{T}}\boldsymbol{A}\boldsymbol{e_j} &= \boldsymbol{e_i}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{e_j} \\ &= \text{the } (i,j)\text{-entry of } \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}. \end{aligned} \tag{3}$$

On the other hand, by (a),

$$(\boldsymbol{A}\boldsymbol{e_i}) \cdot (\boldsymbol{A}\boldsymbol{e_j}) = T(\boldsymbol{e_i}) \cdot T(\boldsymbol{e_j}) = \boldsymbol{e_i} \cdot \boldsymbol{e_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

By (3) and (4), $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}$. By Remark 5.4.4, $\boldsymbol{A}$ is an orthogonal matrix.

(c) All isometries on $\mathbb{R}^2$ are of the form

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x\cos(\theta) + \delta y \sin(\theta) \\ x\sin(\theta) - \delta y \cos(\theta) \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

where $\delta = \pm 1$ and $0 \leq \theta < 2\pi$.

11. The standard matrix of T is $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

(a) $\{(2,1)^{\mathrm{T}}, (1,-1)^{\mathrm{T}}\}$ is a basis for $\mathrm{R}(T)$. (For this example, any two linearly independent vectors in $\mathbb{R}^2$ is a basis for $\mathrm{R}(T)$. Why?)

(b) $\{(-\frac{1}{3}, \frac{2}{3}, 1)^{\mathrm{T}}\}$ is a basis for $\mathrm{Ker}(T)$.

(c) $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathrm{R}(T)) + \dim(\mathrm{Ker}(T)) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$.

(d) For example, $\{(-\frac{1}{3}, \frac{2}{3}, 1)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}\}$ is a basis for $\mathbb{R}^3$.

12. $$\begin{pmatrix} 3 & -1 & 2 & 7 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) $\{(3,1,0)^{\mathrm{T}},\ (-1,2,1)^{\mathrm{T}}\}$ is a basis for $\mathrm{R}(T)$.

(b) $\{(-1,-1,1,0)^{\mathrm{T}},\ (-2,1,0,1)^{\mathrm{T}}\}$ is a basis for $\mathrm{Ker}(T)$.

(c) $\mathrm{rank}(T)+\mathrm{nullity}(T) = \dim(\mathrm{R}(T))+\dim(\mathrm{Ker}(T)) = 2+2 = 4 = \dim(\mathbb{R}^4)$.

13. (a) 2. (b) 2. (c) 2.

14. (a) $\{\mathbf{0}\}$. (b) $\mathbb{R}^n$.

15. (a) Let $\{\boldsymbol{v_1},\ \boldsymbol{v_2},\ \dots,\ \boldsymbol{v_k}\}$ be an orthonormal basis for V. By Theorem 5.2.15,

$$\begin{aligned} P(\boldsymbol{u}) &= (\boldsymbol{u}\cdot\boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{u}\cdot\boldsymbol{v_2})\boldsymbol{v_2} + \cdots + (\boldsymbol{u}\cdot\boldsymbol{v_k})\boldsymbol{v_k} \\ &= \boldsymbol{v_1}\boldsymbol{v_1}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{v_2}\boldsymbol{v_2}^{\mathrm{T}}\boldsymbol{u} + \cdots + \boldsymbol{v_k}\boldsymbol{v_k}^{\mathrm{T}}\boldsymbol{u} \\ &= (\boldsymbol{v_1}\boldsymbol{v_1}^{\mathrm{T}} + \boldsymbol{v_2}\boldsymbol{v_2}^{\mathrm{T}} + \cdots + \boldsymbol{v_k}\boldsymbol{v_k}^{\mathrm{T}})\boldsymbol{u} \end{aligned}$$

Note that $\boldsymbol{v_1}\boldsymbol{v_1}^{\mathrm{T}} + \boldsymbol{v_2}\boldsymbol{v_2}^{\mathrm{T}} + \cdots + \boldsymbol{v_k}\boldsymbol{v_k}^{\mathrm{T}}$ is an $n\times n$ matrix. So P is a linear transformation.

(b) $\mathrm{Ker}(P) = \mathrm{span}\{(a,b,c)\}$ and $\mathrm{R}(P) = V$.

16. ($\Rightarrow$) Let $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n$ such that $T(\boldsymbol{u}) = T(\boldsymbol{v})$. Then $T(\boldsymbol{u}-\boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v}) = \mathbf{0}$ and hence $\boldsymbol{u}-\boldsymbol{v}\in\mathrm{Ker}(T)$. Since $\mathrm{Ker}(T) = \{\mathbf{0}\}$, $\boldsymbol{u}-\boldsymbol{v} = \mathbf{0}$, i.e. $\boldsymbol{u} = \boldsymbol{v}$. Thus T is one-to-one.

($\Leftarrow$) By Theorem 7.1.4.1, $T(\mathbf{0}) = \mathbf{0}$. Since T is one-to-one, for all $\boldsymbol{v}\in\mathbb{R}^n$, if $\boldsymbol{v}\neq\mathbf{0}$, $T(\boldsymbol{v})\neq T(\mathbf{0}) = \mathbf{0}$. Thus $\mathrm{Ker}(T) = \{\mathbf{0}\}$.

17. (a) Let $\boldsymbol{u}\in\mathrm{Ker}(S)$, i.e. $S(\boldsymbol{u}) = \mathbf{0}$. Then $T\circ S(\boldsymbol{u}) = T(S(\boldsymbol{u})) = T(\mathbf{0}) = \mathbf{0}$ and hence $\boldsymbol{u}\in\mathrm{Ker}(T\circ S)$.
Thus $\mathrm{Ker}(S)\subseteq\mathrm{Ker}(T\circ S)$.

(b) Let $\boldsymbol{v}\in\mathrm{R}(T\circ S)$, i.e. there exists $\boldsymbol{u}\in\mathbb{R}^n$ such that $\boldsymbol{v} = T\circ S(\boldsymbol{u})$. Put $\boldsymbol{w} = S(\boldsymbol{u})\in\mathbb{R}^m$. Then $\boldsymbol{v} = T(S(\boldsymbol{u})) = T(\boldsymbol{w})$. This means that $\boldsymbol{v}\in\mathrm{R}(T)$. Thus $\mathrm{R}(T\circ S)\subseteq\mathrm{R}(T)$.

18. For this question, it is helpful if you first compute $T((1,0)^{\mathrm{T}})$ and $T((0,1)^{\mathrm{T}})$ and then sketch the vectors on the xy-plane.

(a) T is the dilation by a factor of 3.

(b) T is the contraction by a factor of 0.5.

(c) $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(150^\circ) & \sin(150^\circ) \\ \sin(150^\circ) & -\cos(150^\circ) \end{pmatrix}$

T is the reflection about the line $y = x\tan(75^\circ)$.

(d) $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{pmatrix}$

T is the anti-clockwise rotation about the origin through an angle 30°.

(e) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{pmatrix}$

T is the clockwise rotation about the origin through an angle 30°.

(f) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(30^\circ) & \sin(30^\circ) \\ \sin(30^\circ) & -\cos(30^\circ) \end{pmatrix}$

T is the reflection about the line $y = x\tan(15^\circ)$.

(g) T is the scaling along axes in the directions of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by factors of 3 and 2 respectively.

19. The standard matrices for F_1 and F_2 are

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$$

respectively. Thus the standard matrices for $F_2 \circ F_1$ is

$$\begin{aligned} & \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \\ = & \begin{pmatrix} \cos(2\phi)\cos(2\theta) + \sin(2\phi)\sin(2\theta) & \cos(2\phi)\sin(2\theta) - \sin(2\phi)\cos(2\theta) \\ \sin(2\phi)\cos(2\theta) - \cos(2\phi)\sin(2\theta) & \sin(2\phi)\sin(2\theta) + \cos(2\phi)\cos(2\theta) \end{pmatrix} \\ = & \begin{pmatrix} \cos(2(\phi-\theta)) & -\sin(2(\phi-\theta)) \\ \sin(2(\phi-\theta)) & \cos(2(\phi-\theta)) \end{pmatrix} \end{aligned}$$

which is the standard matrix for the anti-clockwise rotation about the origin through an angle $2(\phi - \theta)$.

20. (a) The standard matrix for $T_1 \circ T_2$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and the standard matrix for $T_2 \circ T_1$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Hence $T_1 \circ T_2 = T_2 \circ T_1$.

(b) The standard matrix for $T_1 \circ T_2$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and the standard matrix for $T_2 \circ T_1$ is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$.

Hence $T_1 \circ T_2 \neq T_2 \circ T_1$.

(c) The standard matrix for $T_1 \circ T_2$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the standard matrix for $T_2 \circ T_1$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Hence $T_1 \circ T_2 \neq T_2 \circ T_1$.

21. (a) True. Let the standard matrices for R_1 and R_2 be $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and $\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$ respectively. Then the standard matrix for $R_2 \circ R_1$ is

$$\begin{aligned} &\begin{pmatrix} \cos(\theta)\cos(\phi) - \sin(\phi)\sin(\theta) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\theta)\sin(\theta) + \cos(\theta)\cos(\phi) \end{pmatrix} \\ = &\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}. \end{aligned}$$

Thus $R_2 \circ R_1$ is a rotation in $\mathbb{R}^2$.

(b) True. The standard matrix for $R_1 \circ R_2$ is $\begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}$ which is also the standard matrix for $R_2 \circ R_1$.

(c) True. Let the standard matrices for F and R be $\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$ and $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ respectively. Then the standard matrix for $F \circ R$ is

$$\begin{pmatrix} \cos(2\phi-\theta) & \sin(2\phi-\theta) \\ \sin(2\phi-\theta) & -\cos(2\phi-\theta) \end{pmatrix}.$$

Thus $F \circ R$ is a reflection in $\mathbb{R}^2$.

(d) False. For example, see Question 7.20(c).

(e) False. For example, let F_1 and F_2 be reflections about the x-axis and the line $y = x$ respectively. Then $F_2 \circ F_1$ is the anti-clockwise rotation about the origin through an angle $90°$.

(f) False. Using the example in (e), $F_1 \circ F_2$ is the clockwise rotation about the origin through an angle $90°$.

22. For this question, it is helpful if you first compute $T((1,0,0)^{\mathrm{T}})$, $T(0,1,0)^{\mathrm{T}})$ and $T((0,0,1)^{\mathrm{T}})$ and then figure out the positions of the vectors in $\mathbb{R}^3$.

(a) T is the dilation by a factor of 2.

(b) T is the contraction by a factor of $\frac{1}{3}$.

(c) T is the anti-clockwise rotation about the x-axis through an angle $\theta = \cos^{-1}(\frac{4}{5})$.

(d) T is the reflection about the plane spanned by $(1,0,0)^{\mathrm{T}}$ and $(0, \cos(\phi), \sin(\phi))^{\mathrm{T}}$ where $\phi = \frac{1}{2}\cos^{-1}(\frac{3}{5})$.

(e) T is the scaling along axes in the directions of $(1,0,0)^{\mathrm{T}}$, $(0, \frac{4}{5}, \frac{3}{5})^{\mathrm{T}}$ and $(0, -\frac{3}{5}, \frac{4}{5})^{\mathrm{T}}$ by factors of 2, 1 and 0.5 respectively.

23. (a) The standard matrix for $T_1 \circ T_2$ is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

and the standard matrix for $T_2 \circ T_1$ is

$$\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

Hence $T_1 \circ T_2 = T_2 \circ T_1$.

(b) The standard matrix for $T_1 \circ T_2$ is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the standard matrix for $T_2 \circ T_1$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence $T_1 \circ T_2 = T_2 \circ T_1$.

(c) The standard matrix for $T_1 \circ T_2$ is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the standard matrix for $T_2 \circ T_1$ is

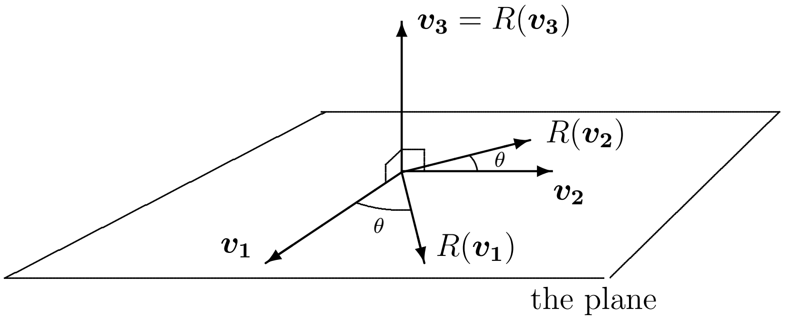
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $T_1 \circ T_2 \neq T_2 \circ T_1$.

24. By sketching the effect of the rotation on the standard basis vectors, we find that $R(\boldsymbol{e_1}) = -\boldsymbol{e_1}$, $R(\boldsymbol{e_2}) = \boldsymbol{e_3}$ and $R(\boldsymbol{e_3}) = \boldsymbol{e_2}$. So the standard matrix for R is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } R\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ z \\ y \end{pmatrix} \text{ for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

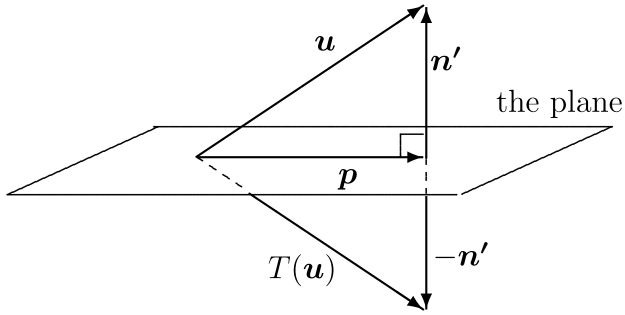
25. Let $\boldsymbol{v_1} = \frac{1}{\sqrt{2}}(-1, 1, 0)^{\mathrm{T}}$, $\boldsymbol{v_2} = \frac{1}{\sqrt{6}}(1, 1, 2)^{\mathrm{T}}$ and $\boldsymbol{v_3} = \frac{1}{\sqrt{3}}(1, 1, -1)^{\mathrm{T}}$ where $\boldsymbol{v_1}, \boldsymbol{v_2}$ are vectors on the plane $x + y - z = 0$ while $\boldsymbol{v_3}$ lies on the axis of rotation.



Thus $R(\boldsymbol{v_1}) = [\cos(\theta)]\boldsymbol{v_1} + [\sin(\theta)]\boldsymbol{v_2}$, $R(\boldsymbol{v_2}) = [-\sin(\theta)]\boldsymbol{v_1} + [\cos(\theta)]\boldsymbol{v_2}$ and $R(\boldsymbol{v_3}) = \boldsymbol{v_3}$.

(The answer for this question depends on the choices of $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$. A different choice may have a slightly different answer. For example, if $\boldsymbol{v_1'} = \frac{1}{\sqrt{2}}(1, -1, 0)^{\mathrm{T}}$, $\boldsymbol{v_2'} = \frac{1}{\sqrt{6}}(1, 1, 2)^{\mathrm{T}}$ and $\boldsymbol{v_3'} = \frac{1}{\sqrt{3}}(1, 1, -1)^{\mathrm{T}}$, then $R(\boldsymbol{v_1'}) = [\cos(\theta)]\boldsymbol{v_1'} - [\sin(\theta)]\boldsymbol{v_2'}$, $R(\boldsymbol{v_2'}) = [\sin(\theta)]\boldsymbol{v_1} + [\cos(\theta)]\boldsymbol{v_2'}$ and $R(\boldsymbol{v_3'}) = \boldsymbol{v_3'}$. Note that $\boldsymbol{v_1'} = -\boldsymbol{v_1}$, $\boldsymbol{v_2'} = \boldsymbol{v_2}$ and $\boldsymbol{v_3'} = \boldsymbol{v_3}$ in the diagram above.)

26. For any vector $\boldsymbol{u}$ in $\mathbb{R}^3$, we can write $\boldsymbol{u} = \boldsymbol{n'} + \boldsymbol{p}$ where $\boldsymbol{n'} = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right)\boldsymbol{n}$ and $\boldsymbol{p} = \boldsymbol{u} - \left(\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right)\boldsymbol{n}$. Since $\boldsymbol{n}$ is orthogonal to the plane, $\boldsymbol{n'}$ is also orthogonal to the plane. Also, since $\boldsymbol{p}\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n} - \left(\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right)\boldsymbol{n}\cdot\boldsymbol{n} = 0$, $\boldsymbol{p}$ is a vector in plane.



So $T(\boldsymbol{u}) = \boldsymbol{p} - \boldsymbol{n'}$ is the reflection about the plane. The standard matrix for T is

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}.$$

27. (a) The figure represented by $\boldsymbol{B}$ is "N".

(b) (i) The figure represented by $\boldsymbol{PB}$ is "N". The transformation is a contraction by a factor of 0.5.

(ii) The figure represented by $\boldsymbol{PB}$ is the inverted "Z". The transformation is a reflection about the line $y = x$.

(iii) The figure represented by $\boldsymbol{PB}$ is "Z". The transformation is a clockwise rotation about the origin through an angle 90°.

(iv) The figure represented by $\boldsymbol{PB}$ is "N". The transformation is a translation by distance of 1 in both the x and y direction.

28. (a) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

The transformation is a translation that moves $(x, y)^{\mathrm{T}}$ to $(x+1,\, y-2)^{\mathrm{T}}$.

(c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) Note that the point $(-1,2)^{\mathrm{T}}$ should be invariant under this transformation. So we first translate $(-1,2)^{\mathrm{T}}$ to $(0,0)^{\mathrm{T}}$ then do a rotation about the origin before performing the inverse translation.

The standard matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} - 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} + 2 \\ 0 & 0 & 1 \end{pmatrix}.$$