Chapter 1

Basic Concepts of Probability

Overview

- Basic probability concepts and definitions
  - Sample space and events
- Operations of events
  - Complement events, mutually exclusive events
  - Union of events, Intersection of events
- Counting methods
  - Multiplication principle
  - Addition principle

Overview (Continued)

- Permutation
- Combination
- Relative frequency
- Axioms of Probability
- Basic properties of probability
- Conditional probability
- Multiplicative rule of probability

Overview (Continued)

- The Law of Total Probability
- Bayes’ Theorem
- Independent events

1.1 Sample Space and Sample Points

1.1.1 Sample Space

- **Observation:** We refer to any recording of information, whether it is *numerical or categorical*, as an observation.

- **Statistical Experiment:** Any procedure that generates a set of data (observations).

Sample Space and Sample Points (Continued)

1.1.1 Sample Space (Continued)

- **Sample Space:** The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Examples

1. Consider an experiment of tossing a die.
   - If we are interested in the number that shows on the top face, then the sample space would be

$$S = \{1, 2, 3, 4, 5, 6\}.$$

   - If we are interested only in whether the number is even or odd, then the sample space is simply

$$\text{S} = \{\text{even}, \text{odd}\}.$$

Examples (Continued)

2. Consider an experiment of tossing two dice.
   - If we are interested in the numbers that show on the top faces, then the sample space would be

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \cdots, (6,5), (6,6)\}.$$

Examples (Continued)

3. An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once.

   The sample space is

   $$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

4. Recording the Straits Times Index.

   $$S = \{\cdots, 3328.95, 3286.32, 3265.73 \cdots\}.$$

Examples (Continued)

5. An experiment consists of drawing two balls from a box containing a blue, a white and a red ball.

   If we are interested in the colours of the two balls drawn, then the sample space is

$$S = \{(B, W), (B, R), (W, B), (W, R), (R, B), (R, W)\}.$$

1.1.2 Sample Points

Sample Points

- Every outcome in a **sample space** is called an element of the sample space or simply a **sample point**.

Examples (Continued)

1. $S = \{1, 2, 3, 4, 5, 6\}$.

   Sample point: 1 or 2 or 3 or 4 or 5 or 6.

   $S = \{\text{even}, \text{odd}\}$.

   Sample point: even or odd

2. $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), \cdots, (6,5), (6,6)\}$.

   Sample point: (1,1) or (1,2) or $\cdots$ or (6,5) or (6,6).

Examples (Continued)

3. $S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

Sample point: (H,H), or (H,T) or $(T,1)$ or $(T,2)$ or $(T,3)$ or $(T,4)$ or $(T,5)$ or $(T,6)$.

Examples (Continued)

4. $S = \{\cdots, 3328.95, 3286.32, 3265.73, \ \cdots\}$.

   Sample point: $\cdots$ or 3328.95 or 3286.32 or 3265.73 or $\cdots$

5. $S = \{(B,W), (B,R), (W,B), (W,R), (R,B), (R,W)\}$.

   Sample point: (B,W) or (B,R) or (W,B) or (W,R) or (R,B) or (R,W)

1.1.3 Events

An **event** is a subset of a sample space.

Examples

1. (a) $S = \{1, 2, 3, 4, 5, 6\}$.

   An event that an odd number occurs $= \{1, 3, 5\}$

   An event that a number greater than 4 occurs $= \{5, 6\}$

   (b) $S = \{\text{even}, \text{odd}\}$.

   An event that an odd number occurs $= \{\text{odd}\}$

Events (Continued)

Examples (Continued)

2. In rolling a pair of dice, if event $A = \{\text{the sum of the dice equals 7}\}$, then

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

3. $S = \{(H,H), (H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

   An event that no die is thrown

$$\{(H,H), (H,T)\}$$

Events (Continued)

Examples (Continued)

4. $S = \{(B,W),(B,R),(W,B),(W,R),(R,B),(R,W)\}$.

   If event $A =$ { a white ball is chosen }, then

   $$A = \{(W,B),(W,R),(B,W),(R,W)\}.$$

5. In tossing two coins, $S = \{(H,H),(H,T),(T,H),(T,T)\}$.

   Getting no head or one head or two heads =

   $$\{(T,T),(H,T),(T,H),(H,H)\}.$$

   Getting no head or one head = $\{(T,T),(H,T),(T,H)\}$.

   Getting two heads or two tails = $\{(T,T),(H,H)\}$.

1.1.4 Simple and Compound Events

- **Simple Event:** An event is said to be **simple** if it consists of exactly **one** outcome (i.e. one sample point)

- **Compound Event:** An event is said to be **compound** if it consists of more than one outcomes (or sample points).

Examples (Continued)

1. (a) $S = \{1, 2, 3, 4, 5, 6\}$.

   Compound events:

   (a) An odd number occurs $= \{1, 3, 5\}$.

   (b) $\{\text{Obtain a number} > 4\} = \{5, 6\}$.

   Simple event:

   Obtain a "six" $= \{6\}$.

   (b) $S = \{\text{even}, \text{odd}\}$.

   Simple event: An odd number occurs $= \{\text{odd}\}$

Examples (Continued)

2. In rolling a pair of dice,

   Compound event:

   $$A = \{\text{the sum of the dice equals } 7\}$$
   $$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

   Simple event:

   $$B = \{\text{the sum of the dice equals } 2\} = \{(1,1)\}.$$

Examples (Continued)

3. $S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

   Compound event:

   No die is thrown $= \{(H,H),(H,T)\}$

   Simple event:

   Obtain a "one" $= \{(T,1)\}$

More Example

- Let $S = \{t: t \geq 0\}$, where t is the life in years of a certain electronic component.
- Find the event A that the component fails before the end of the fifth year.
- $A = \{t: 0 \leq t < 5\}$.

Remarks

1. The sample space is itself an event, and is usually called a ***sure event***.

2. A subset of S that contains no elements at all is the empty set, denoted by $\emptyset$, and is usually called a ***null event***.

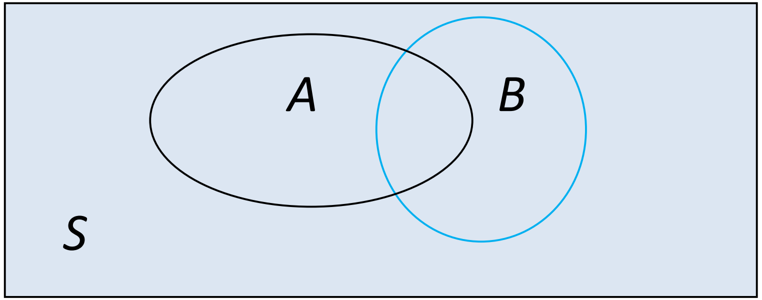
1.2 Operations with Events

1.2.1. Union and Intersection Events

Let S denote a sample space, A and B are any two events of S.

- **Union:** The **Union** of two events A and B, denoted by $A \cup B$, is the event containing all the elements that belong to A or B or to both. That is,

$$A \cup B = \{x: x \in A \textbf{ or } x \in B\}$$



Operations with Events (Continued)

Union and Intersection Events (Continued)

Let S denote a sample space, A and B are any two events of S.

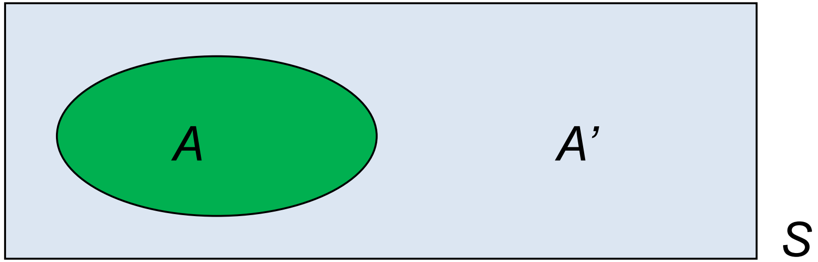
- **Intersection:** The **intersection** of two events A and B, denoted by $A \cap B$ or simply AB, is the event containing all elements that are common to A and B. That is

$$A \cap B = \{x: x \in A \textbf{ and } x \in B\}$$

1.2.2 Complement Event

Complement: The **complement** of event A with respect to S, denoted by A' or A^C, is the set of all elements of S that are not in A. That is

$$A' = \{x : x \in S \text{ and } \boldsymbol{x} \notin \boldsymbol{A}\}$$



Examples

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

Then

- $A \cup B = \{1, 2, 3, 5\}$
- $A \cup C = \{1, 2, 3, 4, 6\}$
- $B \cup C = \{1, 2, 3, 4, 5, 6\} = S$
- $A \cup B \cup C = A \cup (B \cup C) = A \cup S = S$
- $A \cup B \cup C = (A \cup B) \cup C = \{1, 2, 3, 5\} \cup \{2, 4, 6\} = S$
- $A \cap B = \{1, 3\}$

Examples (Continued)

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

Then

- $A \cap C = \{2\}$
- $B \cap C = \emptyset$
- $A \cap B \cap C = (A \cap B) \cap C = \{1, 3\} \cap \{2, 4, 6\} = \emptyset$
- $(A \cap B) \cup C = \{1, 3\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 6\}$
- $A \cap (B \cup C) = A \cap S = A = \{1, 2, 3\}$

Notice that $(A \cap B) \cup C \neq A \cap (B \cup C)$

Examples (Continued)

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

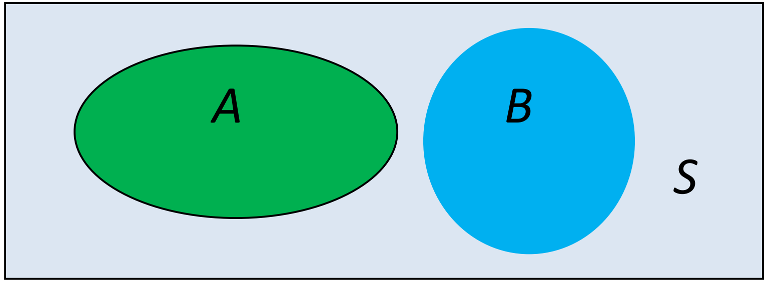
Then

- $A' = \{4, 5, 6\}$
- $B' = \{2, 4, 6\} = C$
- $A' \cap B' = \{4, 5, 6\} \cap \{2, 4, 6\} = \{4, 6\}$
- $(A \cup B)' = \{1, 2, 3, 5\}' = \{4, 6\}$

Notice that both $A' \cap B'$ and $(A \cup B)'$ equal $\{4, 6\}$ in this example. Is it true that $A' \cap B' = (A \cup B)'$ in general?

1.2.3 Mutually Exclusive Events

Two events A and B are said to be **mutually exclusive** or **mutually disjoint** if $A \cap B = \emptyset$, that is, if A and B have no elements in common.



Examples

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

- Events B and C are mutually exclusive events since $B \cap C = \emptyset$.
- Events A and B are not mutually exclusive events since $A \cap B = \{1,3\}$.
- Having a "one" and having a "six" are mutually exclusive events since $\{1\} \cap \{6\} = \emptyset$.

Note:

- Events A and A' are mutually exclusive.

1.2.4 Union of n events

<u>Union</u>:

The **Union** of n events $A_1, A_2, \cdots, A_n$, denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n,$$

is the event containing all the elements that belong to one or more of the events A_1, or A_2, or $\cdots$, or A_n. That is,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x: x \in A_1 \text{ or} \cdots \text{or } x \in A_n\}$$

1.2.5 Intersection of n events

Intersection:

The **intersection** of n events $A_1, A_2, \cdots, A_n$, denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n,$$

is the event containing all the elements that are common to all the events A_1, and A_2, and $\cdots$, and A_n,. That is,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x: x \in A_1 \text{ and} \cdots \text{and } x \in A_n\}$$

1.2.6 Some Basic Properties of Operations of Events

1. $A \cap A' = \emptyset$.
2. $A \cap \emptyset = \emptyset$.
3. $A \cup A' = S$.
4. $(A')' = A$
5. $(A \cap B)' = A' \cup B'$

Some Basic Properties of Operations of Events (Continued)

6. $(A \cup B)' = A' \cap B'$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup B = A \cup (B \cap A')$
10. $A = (A \cap B) \cup (A \cap B')$

1.2.7 De Morgan's Law

For any n events $A_1, A_2, \cdots, A_n$,

1.
$$(A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n'$$

2.
$$(A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n'$$

Example

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

$A' = \{4,5,6\}$, $B' = \{2,4,6\} = C$ and $C' = \{1,3,5\} = B$.

$$\begin{aligned}(A \cup B \cup C)' &= (\{1,2,3\} \cup \{1,3,5\} \cup \{2,4,6\})' \\ &= \{1,2,3,4,5,6\}' = \emptyset.\end{aligned}$$

On the other hand,

$$A' \cap B' \cap C' = \{4,5,6\} \cap \{2,4,6\} \cap \{1,3,5\} = \emptyset.$$

Example (Continued)

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

$A' = \{4,5,6\}$, $B' = \{2,4,6\} = C$ and $C' = \{1,3,5\} = B$.

$$
\begin{aligned}
(A \cap B \cap C)' &= (\{1,2,3\} \cap \{1,3,5\} \cap \{2,4,6\})' \\
&= \emptyset' = \{1,2,3,4,5,6\}.
\end{aligned}
$$

On the other hand,

$$A' \cup B' \cup C' = \{4,5,6\} \cup \{2,4,6\} \cup \{1,3,5\} = \{1,2,3,4,5,6\}.$$

1.2.8 Contained ($\subset$)

- If **all** of the elements in event A are also in event B, then event A is contained in event B, denoted by

$$A \subset B.$$

  (or equivalently, $B \supset A$).

- If $A \subset B$ and $B \subset A$, then $A = B$.

  (i.e. Event A is equivalent with event B).

1.2.9 More Examples

Tossing a die and then flipping a coin if the number on the die is even and twice if the number on the die is odd.

$$S = \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\}$$

Let $A = \{\text{Number shown on the die} < 3\}$

$$= \{1HH, 1HT, 1TH, 1TT, 2H, 2T\},$$

$$B = \{\text{2 tails occur}\} = \{1TT, 3TT, 5TT\}$$

$$C = \{\text{Even number shown on the die}\}$$

$$= \{2H, 2T, 4H, 4T, 6H, 6T\}$$

More Examples (Continued)

- $A' = \{1HH, 1HT, 1TH, 1TT, 2H, 2T\}'$
  $= \{3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\}$
- $B' = \{1TT, 3TT, 5TT\}'$
  $= \{1HH, 1HT, 1TH, 2H, 2T, 3HH, 3HT, 3TH, 4H, 4T, 5HH, 5HT, 5TH, 6H, 6T\}$
- $C' = \{2H, 2T, 4H, 4T, 6H, 6T\}'$
  $= \{1HH, 1HT, 1TH, 1TT, 3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}$

More Examples (Continued)

- $A \cap B = \{1TT\}$
- $A \cap C = \{2H, 2T\}$
- $B \cap C = \emptyset$
- $A' \cap B' = \{3HH, 3HT, 3TH, 4H, 4T, 5HH, 5HT, 5TH, 6H, 6T\}$
- $A' \cap C' = \{3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}$
- $B' \cap C' = \{1HH, 1HT, 1TH, 3HH, 3HT, 3TH, 5HH, 5HT, 5TH\}$

More Examples (Continued)

$$
\begin{aligned}
A \cup B &= \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 3TT, 5TT\} \\
&= (A' \cap B')' \\
A \cup C &= \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 4H, 4T, 6H, 6T\} \\
&= (A' \cap C')' \\
B \cup C &= \{1TT, 2H, 2T, 3TT, 4H, 4T, 5TT, 6H, 6T\} \\
&= (B' \cap C')' \\
A' \cup B' &= \{1HH, 1HT, 1TH, 2H, 2T, 3HH, 3HT, 3TH, 3TT, \\
&\quad 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\} \\
&= (A \cap B)'
\end{aligned}
$$

1.3 Counting Methods

1.3.1 Multiplication Principle

- If an operation can be performed in $\boldsymbol{n_1}$ ways, and
- if for **each of these ways** a second operation can be performed in $\boldsymbol{n_2}$ ways, then
- the two operations can be performed together in

$$\boldsymbol{n_1\, n_2}$$

ways.

Counting Methods (Continued)

<u>Multiplication Principle</u> (Continued)

<u>An alternative statement</u>:

- Suppose that two experiments are to be performed.
- If the experiment 1 can result in any one of the $\boldsymbol{n_1}$ possible outcomes and
- if for **each outcome of experiment 1**, there are $\boldsymbol{n_2}$ possible outcomes of experiment 2, then
- together there are $\boldsymbol{n_1 n_2}$ possible outcomes of the two experiments.

Examples

1. How many sample points are there in the sample space when a die and a coin are thrown together?

   <u>Solution</u>

- There are 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$ for throwing a die.
- For each outcome of the die, there are two possible outcomes for throwing a coin $\{H, T\}$.
- Hence the sample space is given by

$$S = \{(x, y)\colon\ x = 1, 2, 3, 4, 5, \text{or } 6;\, y = H \text{ or } T\}.$$

Examples (Continued)

- There are 6 possible outcomes in the first operation (throwing a die).
- For ***each outcome in the first operation***, there are 2 outcomes in the second operation (throwing a coin).
- Hence there are altogether $6 \times 2 = 12$ elements in the sample space.

Examples (Continued)

2. A small community consists of 10 men, each of whom has 3 sons. If one man and one of his sons are to be chosen as father and son of the year, how many different choices are possible?

   Solution

   There are 10 possible men to be chosen.

   For each of the men, there are 3 sons to be chosen.

   Therefore the number of choices is $10 \times 3 = 30$.

Remark

Note

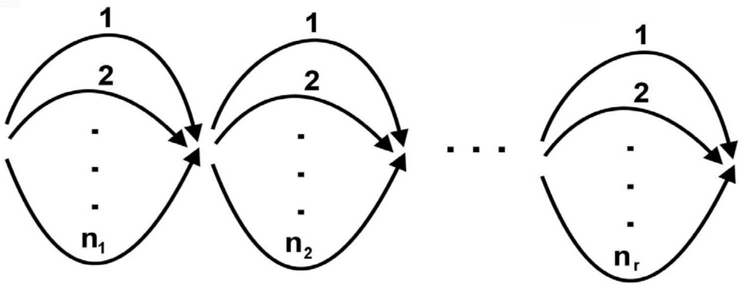
The above principle can be generalized to a sequence of **more than two** operations.

Generalized Multiplication Rule

- If an operation can be performed in n_1 ways, and
- if for each of these ways, a second operation can be performed in n_2 ways, and
- for each of the first two ways, a third operation can be performed in n_3 ways, and so forth, then
- the sequence of k operations can be performed in

$$n_1\, n_2 \cdots n_k$$

ways.



Generalized Multiplication Rule (continued)

An alternative statement:

- If k experiments that are to be performed are such that the first one may result in n_1 possible outcomes, and
- if for **each** of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and
- if for **each** of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and so on, then there are a total of

$$n_1 \, n_2 \cdots n_k$$

possible outcomes of the k experiments.

Example 1

How many **even three-digit numbers** can be formed from the digits 1, 2, 5, 6, and 9 if each digit can be used **only once**?

<u>Solution</u>

For the ones, we can use digits 2 and 6 only. (why?)

For tens and hundreds, we can freely use whatever digits left.

	Hundreds	Tens	Ones
Number of ways	3	4	2
Possible digits			{ 2or 6}

Hence the number of even three-digit numbers is $3 \times 4 \times 2 = 24$.

Example 2

A test consists of 10 multiple choice questions with each has 4 possible answers.

(a) How many possible ways are there in which a student can choose one answer to each question?

(b) Among all these cases, how many are such that all answers are wrong?

Solution to Example 2

(a) There are 4 possible answers for the first question.

For each of these answers, there are 4 possible answers for the second question.

For each of the ways in answering the first and second questions, there are 4 possible answers for the third questions and so on.

Therefore there are $4 \times 4 \times \cdots \times 4 = 4^{10} = 1048576$ possible ways in answering the test.

Solution to Example 2 (Continued)

(b) There are 3 wrong answers for the first question.

For each of these answers, there are 3 wrong answers for the second question.

For each of the ways in getting wrong answers for the first and second questions, there are 3 wrong answers for the third questions and so on.

Therefore there are $3 \times 3 \times \cdots \times 3 = 3^{10} = 59049$ possible ways in getting all wrong answers in the test.

Example 3

A college freshman must take a science course, a humanity course, and a mathematics course.

If she may select any of 6 science courses, any of 4 humanity courses, and any of 4 mathematics courses, in how many ways can she arrange her program?

<u>Solution</u>

The number of ways that the student can arrange her program is $6 \times 4 \times 4 = 96$.

Example 4

A person can travel from Singapore to San Francisco by 4 different airlines and each airlines can go via 3 different routes.

Solution

The number of different routes that the person can travel from Singapore to San Francisco is

$$4 \times 3 = 12.$$

Example 5

Find the number of even 3-digit numbers to be formed from digits 1, 2, 5, 6, 7 and 8.

(a) if each digit can be used once; and

(b) if no restriction on how many times a digit is used.

Example 5 (Continued)

<u>Solution</u>

(a) There are 3 choices for the ones place, namely $\{2, 6, 8\}$.

For each choice of ones place, we have $6 - 1 = 5$ choices for the hundreds places

For each choice of ones and hundreds places, we have $6 - 2 = 4$ choices for the tens place.

Hence, there are $5 \times 4 \times 3 = 60$ different even numbers.

Example 5 (Continued)

Solution

(b) There are 3 choices for the ones place, namely $\{2, 6, 8\}$.

For each choice of ones place, we have 6 choices, $\{1, 2, 5, 6, 7, 8\}$ for the hundreds place as any of the 6 digits can be used again.

For each choice of ones and hundreds places, we have 6 choices for the tens place.

Hence, there are $6 \times 6 \times 3 = 108$ different even numbers.

Example 6

How many set-lunches consisting of a bowl of soup, a sandwich, a dessert and a drink are possible if we can select from 4 choices of soup, 3 choices of sandwich, 5 choices of dessert and 4 choices of drink?

Solution

There are $4 \times 3 \times 5 \times 4 = 240$ different set-lunches.

Example 7

In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

Example 7 (Continued)

Solution

Arrangement of boys and girls: G B G B G B G B G

The number of ways:

$$
\begin{aligned}
&5(4)(4)(3)(3)(2)(2)(1)(1) \\
&= 5(4)(3)(2)(1)(4)(3)(2)(1) = 5!\,4! = 2880
\end{aligned}
$$

where $n! = n(n-1)(n-2)\cdots(2)(1)$.

Question : What happens if there are 5 boys and 5 girls?

1.3.2 Addition Principle

- Suppose that a procedure, designated by 1 can be performed in n_1 ways.
- Assume that a procedure, designated by 2 can be performed in n_2 ways.
- Suppose furthermore that it is **NOT possible** that both procedures 1 and 2 are **performed together**.
- Then the number of ways in which we can perform **1 or 2** is

$$n_1 + n_2$$

ways.

Addition Principle (Continued)

It may be generalized as follows.

- If there are k procedures and the i^{th} procedure may be performed in n_i ways, $i = 1, 2, \cdots, k$,
- then the number of ways in which we may perform procedure 1 **or** procedure 2 **or** $\cdots$ **or** procedure k is given by

$$n_1 + n_2 + \cdots + n_k,$$

assuming that **no** two procedures may be performed together.

Example 1

- We may take MRT **or** bus from home to Orchard Road.
- If there are three bus routes and two MRT routes, then there are $3 + 2 = 5$ different routes available for the trip.

Bus Route	MRT
1 2 3	1 2

Example 2

How many **even** three-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

Solution

We have to consider two cases.

Case A: 0 is used for the ones.

- There are $5 \times 4 = 20$ ways to arrange the hundreds and tens places.
- Hence the number of even numbers formed with 0 at the ones place is 20.

Example 2 (Continued)

Case B: 0 is not used for the ones.

- We have to use either 2 or 6 in the ones.
- Next, we *deal with the hundreds*. As we cannot put 0 in the hundreds, there are only 4 eligible digits to be put in the hundreds.
- Then we can put whatever digits left in the tens. Therefore the number of even 3-digit numbers is $4 \times 4 \times 2 = 32$.

Example 2 (Continued)

Since both Cases A and B lead to an even 3-digit number and they cannot occur together, by applying the addition rule, we have $20 + 32 = 52$ even 3-digit numbers.

Example 3

Consider the digits 0, 1, 2, 3, 4, 5 and 6.

(a) If each digit can be used only once,

(i) how many 3-digit numbers are **even**?

(ii) how many of these are **greater than 420**?

Example 3 (Continued)

Consider the digits 0, 1, 2, 3, 4, 5 and 6.

(b) If each digit can be used more than once,

(i) how many 3-digit numbers can be formed?

(ii) how many 3-digit numbers formed are **odd**? even?

(iii) How many 3-digit numbers formed are **greater than 420**?

Solution to Example 3

(a) (i)

Case 1: Number of 3-digit even numbers with last digit being 0:

$$6(5)(1) = 30.$$

Case 2: Number of 3-digit even numbers with last digit other than 0 :

$$5(5)(3) = 75.$$

Solution to Example 3 (Continued)

(a) (i)

The required 3-digit even number either comes from Case 1 **or** Case 2

Hence, the number of 3-digit even numbers $= 30 + 75 = 105$.

Solution to Example 3 (Continued)

(a) (ii)

Case 1: Hundreds is 4. Tens is 2.

Number of 3-digit numbers = $1(1)(4) = 4$.

Case 2: Hundreds is 4. Tens is 3, 5 or 6.

Number of 3-digit numbers = $1(3)(5) = 15$.

Case 3: Hundreds is 5 or 6.

Number of 3-digit numbers =$2(6)(5) = 60$.

Solution to Example 3 (Continued)

(a) (ii)

The required 3 digit number comes from Case 1, Case 2 **or** Case 3

Hence, the number of 3-digit numbers formed greater than $420 = 4 + 15 + 60 = 79$.

Solution to Example 3 (Continued)

(b) (i)

6 choices for hundreds place (1 to 6), 7 choices for tens place (0 to 6) and 7 choices for ones place (0 to 6)

Hence, the number of 3-digit numbers is given by

$$6(7)(7) = 294.$$

Solution to Example 3 (Continued)

(b) (ii)

3 digit **odd** number

3 choices for the ones place (1, 3, & 5), 6 choices for the hundreds place (1 to 6) and 7 choices for tens place (0 to 6)

Number of odd 3-digit numbers is given by

$$6(7)(3) = 126.$$

Solution to Example 3 (Continued)

(b) (ii)

3 digit **even** number

4 choices for the units (0, 2, 4, & 6), 6 choices for the hundreds (1 to 6) and 7 choices for tens (0 to 6)

Number of 3-digit even numbers is given by

$$6(7)(4) = 168.$$

Solution to Example 3 (Continued)

(b) (iii)

Case 1: Hundreds is 4. Tens is 2.

Number of 3-digit numbers $= 1(1)(6) = 6.$

Case 2 : Hundreds is 4. Tens is 3, 4, 5 or 6.

Number of 3-digit numbers $= 1(4)(7) = 28.$

Case 3 : Hundreds is 5 or 6.

Number of 3-digit numbers $= 2(7)(7) = 98.$

Solution to Example 3 (Continued)

(b) (iii)

The required 3-digit number comes from Case 1, Case 2 **or** Case 3

Hence, the number of 3-digit numbers formed greater than 420 =

$$6 + 28 + 98 = 132.$$

1.3.3 Permutation

A **permutation** is an arrangement of r objects from a set of n objects, where $r \leq n$.

(Note that the **order** is taken into consideration in permutation.)

Permutation (Continued)

Example

Given a set $\{a, b, c\}$, the possible arrangements or permutations are:

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

Permutation (Continued)

- There are 3 choices for the first place.
- For each of these choices, there are two choices for the second place.
- For each of these choices for the first and second places, there is only 1 choice left for the last place.
- Hence the number of possible arrangements is

$$3 \times 2 \times 1 = 6.$$

Permutation (Continued)

- In general n distinct objects can be arranged in

$$n(n-1)(n-2)\cdots\times 2\times 1 = n!$$

ways which is read as ***n factorial*** ways.

Note: $1! = 1$ and $0! = 1$.

1.3.3.1 Permutations of n distinct objects taken r at a time

Number of **permutations** of n distinct objects taken r at a time is denoted by

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-(r-1)) = n!/(n-r)! $$

We may consider by putting n distinct objects in r compartments:

r compartments

n ways	$n-1$ ways	$n-2$ ways	....	$n-(r-1)$ ways

By the multiplication principle, there are

$$ n(n-1)(n-2)\cdots(n-(r-1)) \text{ ways.} $$

Permutations of *n* distinct objects

A special case is when $r = n$.

- The number of permutations of n distinct objects taken all together is

$$ {}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! $$

Example 1

Find the number of all possible four-letter code words in which all letters are different.

Solution

There are 26 distinct letters. $n = 26$

We want a 4 letter code word. $r = 4$

The number of all possible four-letter code words is

$$ {}_{26}P_4 = \frac{26!}{(26-4)!} = \frac{26!}{22!} = 26(25)(24)(23) = 358800 $$

Example 2

Twenty horses take part in a horse race.

How many possible ways to arrange the first, second and third places?

Example 2 (Continued)

Solution

There are 20 distinct horses. $n = 20$

We want first 3 places. $r = 3$

The number of ways to arrange the first, second and third places is given by

$$_{20}P_3 = \frac{20!}{17!} = 20(19)(18) = 6840$$

Example 3

(a) How many ways can 6 persons line up to get on a bus?

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Example 3 (continued)

(a) How many ways can 6 persons line up to get on a bus?

Solution

(a) There are 6 persons (objects). Hence $n = 6$.

There are 6 places in the queue to permute. Hence $r = 6$.

Therefore there are

$$_6P_6 = 6! = 720 \text{ ways}.$$

Example 3 (continued)

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

Solution

(b) Let a, b, c, d, e, f denote the 6 persons.

- Without loss of generality, we assume a, b, c insisting on following each other.
- Group a, b and c into one group. Denote this group by A. i.e. $A = \{a, b, c\}$
- Then we permute 4 objects, A, d, e, f. Hence the number of permutations is ${}_4P_4 = 4! = 24$.

Example 3 (continued)

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

Solution (Continued)

(b) However, for each permutation such as (A, d, e, f), we can permute a, b, c within the group A.

The number of permutations within the group A is

$$_3P_3 = 3! = 6.$$

Therefore, applying the multiplicative rule, the number of ways to form the line up is

$$24 \times 6 = 144.$$

Example 3 (Continued)

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Solution

(c) Firstly, find the number of ways of line up with these 2 persons follow each other.

Similar to part (b), the number of line up with these 2 persons follow each other is

$$_5P_5 \times {}_2P_2 = 5! \times 2! = 240.$$

Example 3 (Continued)

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Solution (Continued)

(c) From part (a), the total number of line up is 720.

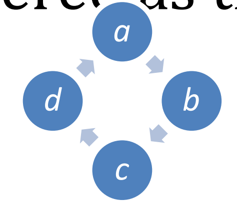
Therefore there are $720 - 240 = 480$ possible ways of line up with the 2 persons not following each other.

1.3.3.2 Permutations of n distinct objects arranged in a circle

- The number of permutations of n distinct objects arranged in **a circle** is $(n-1)!$.
- By considering 1 person in a fixed position and arranging the other $n-1$ persons, therefore there are $(n-1)!$ ways.
- Consider the following 4 different ways of arrangement in a line

  $abcd \qquad bcda \qquad cdab \qquad dabc$

  And these arrangements are considered as the same arrangement in a circle



Permutations of n distinct objects arranged in a circle (Continued)

Example

How many different arrangements are possible for sitting 4 persons around a table?

Solution

There are $(4-1)! = 6$ ways for sitting 4 persons around a table.

1.3.3.3 Permutations when not all n objects are distinct

- Suppose we have n objects such that there are n_1 of one kind, n_2 of second kind, $\cdots$, n_k of a k^{th} kind, where

$$n_1 + n_2 + \cdots + n_k = n\,.$$

- Then the number of distinct permutations of these n objects taken all together is given by

$${}_nP_{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Example 1

In how many ways can 3 red, 4 yellow and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets?

Solution

There are $3 + 4 + 2 = 9$ objects. i.e. $n = 9$

There are 3 red, 4 yellow and 2 blue bulbs.

i.e. $n_1 = 3, n_2 = 4$ and $n_3 = 2$

The total number of distinct arrangements is

$$\frac{9!}{3!\,4!\,2!} = 1260$$

Example 2

In how many ways can 4 Mathematics books, 3 Physics books, and 5 Chemistry books are arranged in a bookshelf ?

Solution

There are 12 books. i.e. $n = 12$

There are 4 Maths, 3 Physics and 5 Chemistry books.

i.e. $n_1 = 4, n_2 = 3$ and $n_3 = 5$

The total number of distinct arrangements is

$$\frac{12!}{4!\,3!\,5!} = 27720$$

Example 3

How many ways can 7 people be assigned to 1 triple and 2 double rooms? Assume that the double rooms are different

Solution

There are 7 people. i.e. $n = 7$

There are 1 group of 3 persons, 2 groups of 2 persons.

i.e. $n_1 = 3, n_2 = 2$ and $n_3 = 2$

Number of ways of assignments is given by

$$\frac{7!}{3!\,2!\,2!} = 210$$

1.3.4 Combination

- In many problems we are interested in the number of ways of selecting r objects from n objects **without regard to the order**.
- These selections are called **combinations.**
- A combination creates a partition with 2 groups, one group containing the r objects selected and the other group containing the $n - r$ objects that are left.
- The number of such combinations is denoted by

$$\binom{n}{r} \quad \text{or} \quad {}_nC_r \quad \text{or} \quad C_r^n$$

Combination (Continued)

- The number of **combinations** of n **distinct** objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Combination (Continued)

- To develop the formula, let us first examine the following ${}_4P_3$ permutations of the 4 letters a, b, c, d taken 3 at a time:

abc	acb	bac	bca	cab	cba
abd	adb	bad	bda	dab	dba
acd	adc	cad	cda	dac	dca
bcd	bdc	cbd	cdb	dbc	dcb

Combination (Continued)

- If we are not concerned with the order in which 3 letters are chosen from the 4 letters a, b, c, d, there are only 4 ways in which the selection can be made.
- They are those shown in the first column.
- Each row of the table merely contains the $3! = 6$ different permutations of the letters shown in the first column. Hence

$$4 \times 3! = {}_4P_3.$$

Combination (Continued)

- In general, there are $r!$ permutations of any r objects we select from a set of n distinct objects.
- Let $\binom{n}{r}$ or ${}_nC_r$ denote the number of ways of choosing r out of n distinct objects, disregarding order.
- Then we have

$$ {}_nC_r \, r! = {}_nP_r $$

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!}\frac{1}{r!} = \frac{n!}{r!\,(n-r)!} $$

Binomial Coefficient

- The quantity $\binom{n}{r}$ is called a **binomial coefficient** because it is the coefficient of the term $a^r b^{(n-r)}$ in the binomial expansion of $(a+b)^n$.
- It can be verified that the following hold:

1. $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, \cdots, n$
2. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for $1 \leq r \leq n$
3. $\binom{n}{r} = 0$ for $r < 0$ or $r > n$

Example 1

A box contains 25 good apples and 5 bad apples. If a sample of 5 apples are selected from the box (**without** replacement).

(a) How many samples can be selected?

(b) How many of the samples in (a) involve exactly one bad apple?

(c) How many of the samples in (a) involve at least 2 bad apples?

Solution to Example 1

(a) There are $25 + 5$ distinct apples. i.e. $n = 30$

We select a group of 5 apples. i.e. $r = 5$

The number of samples can be selected is given by

$$_{30}C_5 = \frac{30!}{5!\,25!} = 142506.$$

Solution to Example 1 (Continued)

(b) The number of ways a bad apple being selected from the 5 bad ones is ${}_5C_1 = 5$.

The number of ways 4 good apples being selected from 25 good ones is ${}_{25}C_4 = 25!/(4!\,21!) = 12650$.

There are 2 groups in the sample: One group of bad apple and another group of good apples.

Solution to Example 1 (Continued)

(b) (Continued)

For each choice of the first group of bad apple, there are ${}_{25}C_4$ ways of selecting 4 good apples.

By the **multiplication principle**, the number of samples involving exactly one bad apple is

$$ {}_5C_1 \times {}_{25}C_4 = 5(12650) = 63250 $$

Solution to Example 1 (Continued)

(c) Number of samples with exactly two bad apples is

$$_5C_2 \times {}_{25}C_3 = 23000.$$

Number of samples with exactly three bad apples is

$$_5C_3 \times {}_{25}C_2 = 3000.$$

Number of samples with exactly four bad apples is

$$_5C_4 \times {}_{25}C_1 = 125.$$

Number of samples with exactly five bad apples is

$$_5C_5 \times {}_{25}C_0 = 1.$$

Solution to Example 1 (Continued)

(c) Since **at least** two bad apples in the sample implies there are two or three or four or five bad apples in the sample, therefore by **the addition principle**, the number of samples with at least two bad apples is given by

$$
\begin{aligned}
{}_5C_2 \times {}_{25}C_3 + {}_5C_3 \times {}_{25}C_2 + {}_5C_4 \times {}_{25}C_1 + {}_5C_5 \times {}_{25}C_0 \\
= 23000 + 3000 + 125 + 1 \\
= 26126.
\end{aligned}
$$

Example 2

From 4 women and 3 men, find the number of committees of 3 that can be formed with 2 women and 1 man.

Solution to Example 2

Number of ways of selecting 2 women from 4 is given by

$$ {}_4C_2 = 6 $$

Number of ways of selecting 1 man from 3 is given by

$$ {}_3C_1 = 3 $$

By the **multiplication principle**, the number of possible committees formed with 2 women and 1 man is given by

$$ {}_4C_2 \times {}_3C_1 = 6(3) = 18. $$

Example 3

From a group of 4 men and 5 women, how many committees of size 3 are possible

(a) with **no restriction**?

(b) with 2 men and 1 woman if a ***certain man*** *must be on the committee*?

(c) with 2 men and 1 woman if ***2 of the men*** are feuding and ***refuse*** *to serve* on the committee **together**?

Solution to Example 3

(a) Number of committees = ${}_9C_3 = 9!/(3!\,6!) = 84$.

(b) Since a particular man must be on the committee, therefore we can only choose one man from the remaining 3 men to the committee.

Hence the number of ways to choose 2 men to sit on the committee with a certain man in the committee is

$$ {}_1C_1 \times {}_3C_1 = 3. $$

Therefore the number of committees is

$$ {}_1C_1 \times {}_3C_1 \times {}_5C_1 = 15. $$

Solution to Example 3 (continued)

(c) There are ${}_2C_2 \times {}_5C_1 = 5$ ways to form a committee on which these 2 “particular” men serve together.

As such a case is undesirable, thus the number of desirable ways of forming a committee is given by

$$ {}_4C_2 \times {}_5C_1 - {}_2C_2 \times {}_5C_1 = 30 - 5 = 25. $$

Example 4

Shortly after being put into service, some buses manufactured by a certain company have developed cracks on the underside of the main frame.

Suppose a particular bus company has 20 of these buses, and the cracks have actually appeared in 8 of them.

(a) How many ways are there to select a sample of 5 buses from the 20 for a thorough inspection?

(b) In how many ways can a sample of 5 buses contain **exactly** 4 buses with visible cracks?

Solution to Example 4

(a) Number of ways selecting 5 buses out of 20 buses

$$ {}_{20}C_5 = 20!/(5!\,15!) = 15504. $$

Solution to Example 4 (Continued)

(b) Number of ways selecting 4 buses from the 8 buses with visible cracks is

$$_8C_4 = 8!/(4!\,4!) = 70.$$

Number of ways selecting 1 bus from the remaining 12 buses with no cracks is

$$_{12}C_1 = 12!/(1!\,11!) = 12.$$

Number of samples of 5 buses that contains exactly 4 buses with visible cracks

$$_8C_4 \times {}_{12}C_1 = 840.$$

Example 5

How many bridge hands are possible containing 4 spades, 6 diamonds, 1 club and 2 hearts?

Solution

Number of possible hands:

$$
{}_{13}C_4 \times {}_{13}C_6 \times {}_{13}C_1 \times {}_{13}C_2
$$

$$
= 715(1716)(13)(78)
$$

$$
= 1{,}244{,}117{,}160.
$$

1.4 Relative frequency and definition of probability

1.4.1 Introduction

- In an experiment, we do not know which particular outcome will occur when the experiment is performed.
- If A is an event associated with the experiment, then we cannot state with certainty that A will or will not occur.
- Hence it becomes very important to try to associate a number with the event A which will **measure how likely** the event A occurs.

Relative frequency and definition of probability

(Continued)

Introduction (Continued)

- This task leads us to the theory of probability.
- You may refer to p.154-5 on computing probability based on the sample space.

1.4.2 Relative Frequency

- Alternatively, probability can be derived based on the relative frequency
- Suppose we repeat the experiment E for n times and let A be an event associated with E.
- We let n_A be the number of times that the event A has occurred among the n repetitions respectively.
- Then $f_A = \frac{n_A}{n}$ is called the ***relative frequency*** of the event A in the n repetitions of E.

Properties of Relative Frequency

The relative frequency f_A has the following important properties:

(1) $0 \leq f_A \leq 1$.

(2) $f_A = 1$ if and only if A occurs every time among the n repetitions.

(3) $f_A = 0$ if and only if A never occurs among the n repetitions.

Properties of Relative Frequency (Continued)

(4) If A and B are two **mutually exclusive** events and if $f_{A\cup B}$ is the relative frequency associated with the event $A \cup B$, then $f_{A\cup B} = f_A + f_B$.

(5) f_A "**stabilizes**" near some definite numerical value as the experiment is repeated more and more times.

For example, when tossing a die, the relative frequency of having a "1" stabilizes to $1/6$ when the number of experiments is very large.

1.4.3 Axioms of Probability

- Consider an experiment whose sample space is S.
- The objective of probability is to assign to each event A, a number $\Pr(A)$, called the **probability** of the event A, which will give a precise measure of the chance that A will occur.
- Consider the collection of all events and denote it by $\mathcal{P}$.
- For each event A of the sample space S we assume that a number $\Pr(A)$, which is called the **probability** of the event A, is defined and satisfies the following three axioms:

Axioms of Probability (Continued)

Axiom 1: $0 \leq \Pr(A) \leq 1$.

Axiom 2: $\Pr(S) = 1$.

Axiom 3: If $A_1, A_2, \cdots$ are **mutually exclusive** (disjoint) events (that is, $A_i \cap A_j = \emptyset$ when $i \neq j$), then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

In particular, if A and B are two mutually exclusive events then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Example 1

Consider an experiment of tossing a coin. Let H denote the event of having a head.

Find $\Pr(H)$ if

(a) the coin is fair;

(b) if the coin is biased and a head is twice as likely to appear as a tail.

Solution to Example 1

Solution

(Refer to p.154-5 on computing probability based on the sample space)

Sample space $= \{H, T\}$

(a) $\Pr(H) + \Pr(T) = 1$

Since the coin is fair, therefore $\Pr(H) = \Pr(T)$

Solving the 2 equations, we have $\Pr(H) = 1/2$.

Hence $\Pr(H) = 1/2$ for a fair coin.

Solution to Example 1 (Continued)

Solution (Continued)

(b) If a head is twice as likely to appear as a tail, then $\Pr(H) = 2\Pr(T)$ and $\Pr(H) + \Pr(T) = 1$.

Solving the 2 equations for $\Pr(H)$, we have

$$\Pr(H) = \frac{2}{3}.$$

Example 2

- A fair die is tossed.
- Let A be the event that an even number turns up and let B be the event that either an "1" or a "3" occurs.
- Find $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cup B)$.

Solution to Example 2

Sample space = {1, 2, 3, 4, 5, 6}

- A = {2, 4, 6}. Hence $\Pr(A) = 3/6 = 1/2$.
- B = {1, 3}. Hence $\Pr(B) = 2/6 = 1/3$.
- Since $A \cap B = \emptyset$, hence A and B are mutually exclusive.
- Thus,

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) \\ &= 1/2 + 1/3 = 5/6.\end{aligned}$$

Example 3

Refer to example 2.

- Let C be the event that a number divisible by 3 occurs.
- Find $\Pr(C)$ and $\Pr(A \cup C)$.

Solution to Example 3

Sample space = {1, 2, 3, 4, 5, 6} and C = {3, 6}

Hence $\Pr(C) = 2/6$.

$A \cup C = \{2, 3, 4, 6\}$. Hence $\Pr(A \cup C) = 2/3$.

Notice that

$$\Pr(A) + \Pr(C) = 1/2 + 1/3 = 5/6 \neq \Pr(A \cup C) = 2/3$$

This is because A and C are **not mutually exclusive** events as $A \cap C = \{6\} \neq \emptyset$.

1.5 Basic Properties of Probability

1.5.1 Some Basic properties of probability

1. $\Pr(\emptyset) = 0$.

   Take $A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset, \cdots\cdots$

   then $A_1 \cup A_2 \cup A_3 \cup \cdots\cdots = \emptyset$.

   Hence, by Axiom 3

$$\Pr(\emptyset) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{\infty} \Pr(\emptyset)$$

   which can hold only if $\Pr(\emptyset) = 0$.

Basic Properties of Probability (Continued)

2. If $A_1, A_2, \cdots, A_n$ are **mutually exclusive events**, then

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$$

Let $A_{n+1} = \emptyset, A_{n+2} = \emptyset, \cdots\cdots$, then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i \qquad \text{and}$$

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{n} \Pr(A_i)$$

Basic Properties of Probability (Continued)

3. For any event A,

$$\Pr(A') = 1 - \Pr(A).$$

Since $S = A \cup A'$ and $A \cap A' = \emptyset$, therefore

$$\Pr(S) = \Pr(A \cup A') = \Pr(A) + \Pr(A').$$

But $\Pr(S) = 1$ by Axiom 2, the result follows.

Basic Properties of Probability (Continued)

4. For any two events A and B,

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B').$$

Since

$$A = (A \cap B) \cup (A \cap B')$$

and

$$(A \cap B) \cap (A \cap B') = \emptyset,$$

therefore

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B').$$

Basic Properties of Probability (Continued)

5. For any two events A and B,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Since $A \cup B = B \cup (A \cap B')$ and $B \cap (A \cap B') = \emptyset$, therefore

$$\Pr(A \cup B) = \Pr(B) + \Pr(A \cap B')$$

But from Property (4), we have

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$$

Hence $\Pr(A \cup B) = \Pr(B) + \Pr(A) - \Pr(A \cap B)$.

Basic Properties of Probability (Continued)

6. For any three events A, B, C,

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

Write $A \cup B \cup C = A \cup (B \cup C)$ and then by applying Property (5), we have

$$\begin{aligned}
&\Pr\big(A \cup (B \cup C)\big) \\
&= \Pr(A) + \Pr(B \cup C) - \Pr(A \cap (B \cup C)) \\
&= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(B \cap C) - \Pr(A \cap (B \cup C)).
\end{aligned}$$

Basic Properties of Probability (Continued)

6. (Continued)

But

$$\Pr\big(A \cap (B \cup C)\big) = \Pr((A \cap B) \cup (A \cap C))$$
$$= \Pr(A \cap B) + \Pr(A \cap C) - \Pr((A \cap B) \cap (A \cap C))$$
$$= \Pr(A \cap B) + \Pr(A \cap C) - \Pr(A \cap B \cap C).$$

Hence

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B)$$
$$- \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

Basic Properties of Probability (Continued)

The above property can be extended to n events

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \Pr(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\left(A_i \cap A_j\right)$$

$$+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Pr\left(A_i \cap A_j \cap A_k\right) - \cdots\cdots$$

$$+(-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

It can be proved by mathematical induction.

The above identity is also known as **"The Inclusion-Exclusion Principle"**.

Basic Properties of Probability (Continued)

7. If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.

Since $B = (B \cap A) \cup (B \cap A')$ and $B \cap A = A$,

so $B = A \cup (B \cap A')$ and $A \cap (B \cap A') = \emptyset$.

Therefore

$$\Pr(B) = \Pr(A) + \Pr(B \cap A') \geq \Pr(A).$$

The result follows by noting that $\Pr(B \cap A') \geq 0$.

Example 1

- A retail establishment accepts either the American Express or the VISA credit card.
- A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both.
- What percentage of its customers carries a credit card that the establishment will accept?

Solution to Example 1

- Let A and V represent the events that a customer carries an American Express and Visa card respectively.
- $\Pr(A) = 0.24, \Pr(V) = 0.61$ and $\Pr(A \cap V) = 0.11$.
- The desired probability that customers carry a credit card that the establishment will accept is given by

$$\begin{aligned}\Pr(A \cup V) &= \Pr(A) + \Pr(V) - \Pr(A \cap V) \\ &= 0.24 + 0.61 - 0.11 \\ &= 0.74.\end{aligned}$$

Example 2

- A poll of statisticians in USA was conducted to ascertain their professional responsibilities.
- An analysis of their responses gave the following distribution of professional responsibilities :

A = {Research}	40%
B = {Professional consultation}	64%
C = {Data collection and analysis}	36%

Example 2 (Continued)

- Suppose that 10% are involved in all three activities; 15% are involved in both A and C; and 17% are involved in both A and B.
- Use this information to find the percentage of all statisticians in USA that are involved in both B and C.
- Assume that $\Pr(A \cup B \cup C) = 1$.

Solution to Example 2

- $\Pr(A \cap B \cap C) = 0.1$, $\Pr(A \cap C) = 0.15$ and $\Pr(A \cap B) = 0.17$.
- Since $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$, therefore
$$1 = 0.4 + 0.64 + 0.36 - 0.17 - 0.15 - \Pr(B \cap C) + 0.1$$
- Hence,
$$\begin{aligned} \Pr(B \cap C) &= 0.4 + 0.64 + 0.36 - 0.17 - 0.15 + 0.1 - 1 \\ &= 0.18. \end{aligned}$$

Example 3

- The probabilities that a gas station pumps gas into 0, 1, 2, 3, 4, or 5 or more cars during a certain 30-minute period are 0.03, 0.18, 0.24, 0.28, 0.10 and 0.17 respectively.

- Find the probability that in a 30-minute period

  (a) more than 2 cars receive gas;

  (b) at most 4 cars receive gas.

Solution to Example 3

- Let A_i be the event of pumping i cars, $i = 0, 1, 2, 3, 4$ and A_5 be the event of pumping 5 or more cars.
- Note that all A_i's are mutually exclusive events.

(a) Pr(more than 2 cars receive gas)

$$= \Pr(A_3 \cup A_4 \cup A_5)$$
$$= \Pr(A_3) + \Pr(A_4) + \Pr(A_5)$$
$$= 0.28 + 0.1 + 0.17$$
$$= 0.55.$$

Solution to Example 3 (Continued)

(b) $\Pr(\text{at most 4 cars receive gas})$

$$= \Pr(A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)$$
$$= \Pr(A_0) + \Pr(A_1) + \Pr(A_2) + \Pr(A_3) + \Pr(A_4)$$
$$= 0.03 + 0.18 + 0.24 + 0.28 + 0.1$$
$$= 0.83.$$

Alternatively,

$$\Pr(\text{at most 4 cars receive gas})$$
$$= 1 - \Pr(\text{5 or more cars receive gas})$$
$$= 1 - \Pr(A_5) = 1 - 0.17 = 0.83.$$

Example 4

(Hall Pageant)

- Audrey is taking part in her hall's pageant.
- The probability that she will win the crown is 0.14.
- The probability that she will win Miss Photogenic is 0.3.
- The probability that she will win both is 0.11.

(a) What is the probability that she wins at least one of the two?

(b) What is the probability that she wins only one of two?

Solution to Example 4

Let A be the event that she wins the crown, and B that she wins Miss Photogenic.

(a) The probability that she wins at least one of the two titles

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 0.14 + 0.3 - 0.11 = 0.33.\end{aligned}$$

(b) The probability that she wins only one of the two titles

$$\begin{aligned}\Pr\big((A \cup B) \backslash (A \cap B)\big) &= \Pr(A \cup B) - \Pr(A \cap B) \\ &= 0.33 - 0.11 = 0.22.\end{aligned}$$

Example 5

(Birthday Problem)

- Here's a useful party trick: walk into a room or bar with at least 50 people.
- Boldly claim that you sense two people sharing the same birthday. Act awesome afterwards. How often are you right?
- We can cast this as a probability question:
- There are *n* people in a room, what is the probability that there are at least two people with the same birthday?

Solution to Example 5

We assume each day is **equally likely** to be a birthday of everyone, and there is no leap year.

A person can have his birthday on any of the 365 days. So there are a total of $(365)^n$ outcomes, i.e. $\#(S) = (365)^n$

Let A denote the event that there are at least two people among the n people sharing the same birthday.

We will work out the event that no two people sharing the same birthday which is denoted A'.

Solution to Example 5 (Continued)

To count A', note that

$$\#(A') = 365(364)\cdots[365-(n-1)].$$

Therefore,

$$\Pr(A') = \frac{365(364)\cdots(365-n+1)}{365^n}.$$

Hence

$$\Pr(A) = 1 - \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right).$$

Remarks on Example 5

Let $q_n = \Pr(A')$ when there are n people, and

$p_n = \Pr(A) = 1 - q_n$.

The values of p_n and q_n for selected values of n are tabulated on the right.

n	q_n	p_n
1	1	0
2	0.99726	0.00274
3	0.99180	0.00820
10	0.88305	0.11695
15	0.74710	0.25290
20	0.58856	0.41144
21	0.55631	0.44369
22	0.52430	0.47570
23	0.49270	0.50730
30	0.29368	0.70632
40	0.10877	0.89123
50	0.029626	0.979374
100	$3.0725(10)^{-7}$	1
253	$6.9854(10)^{-53}$	1

Remarks on Example 5 (Continued)

- We see that for 50 people, 98% of the time you will be able to find two people with the same birthday.
- Take note that the probability of having two people sharing the same birthday exceeds 1/2 once you have 23 people.

Example 6

(Inverse Birthday Problem)

- How large does a group of (randomly selected) people have to be such that the probability that someone is sharing his or her birthday **with you** is larger that 0.5?

Solution to Example 6

- The probability that n persons all have different birthdays **from you** is $\left(\frac{364}{365}\right)^n$.
- So we need n such that $1 - \left(\frac{364}{365}\right)^n \geq 0.5$.
- Solving, we obtain

$$n \geq \frac{\log(0.5)}{\log(364/365)} = 252.7.$$

- We need at least 253 people (excluding yourself).

Remarks on Birthday Problems

Why there is a big difference in the answers between the two birthday problems?

- The inverse birthday problem requires the sharing of **a particular day** as the common birthday
- The birthday problem allows that **any day** is the shared birthday.

1.5.2 Sample Spaces Having Finite Outcomes

Consider the sample space S which contains a finite number of k outcomes. That is,

$$S = \{a_1, a_2, \cdots, a_k\}$$

Let $\Pr(a_i) = p_i$ be the probability of $\{a_i\}$ and

(1) $0 \leq p_i \leq 1$, for $i = 1, 2, \cdots, k$.

(2) $p_1 + p_2 + \cdots + p_k = 1$.

Sample Spaces Having Finite Outcomes (Continued)

Let an event A consists of r outcomes, $1 \leq r \leq k$, say

$$A = \{a_{j_1}, a_{j_2}, \cdots, a_{j_r}\}$$

where $j_1, j_2, \cdots, j_r$ represent any r indices from $1, 2, \cdots, k$.

Then

$$\Pr(A) = p_{j_1} + p_{j_2} + \cdots + p_{j_r},$$

where $\Pr\left(a_{j_l}\right) = p_{j_l},\ l = 1, \cdots, r$.

That is, the probability of an event A equals the sum of the probabilities of the various individual outcomes making up the event A.

Example 1

- Let S be the sample space of the sum of the numbers when a pair of dice is tossed. The outcomes of tossing 2 dice are

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

- Hence $S = \{2, 3, 4, \cdots, 11, 12\}$, and

Example 1 (Continued)

- $\Pr(\{2\}) = p_2 = 1/36, \Pr(\{3\}) = p_3 = 2/36,$
  $\Pr(\{4\}) = p_4 = 3/36, \Pr(\{5\}) = p_5 = 4/36,$
  $\Pr(\{6\}) = p_6 = 5/36, \Pr(\{7\}) = p_7 = 6/36,$
  $\Pr(\{8\}) = p_8 = 5/36, \Pr(\{9\}) = p_9 = 4/36,$
  $\Pr(\{10\}) = p_{10} = 3/36, \Pr(\{11\}) = p_{11} = 2/36,$
  $\Pr(\{12\}) = p_{12} = 1/36.$
- Let $A = \{5, 7, 8, 11\}$. Then

$$\Pr(A) = p_5 + p_7 + p_8 + p_{11}$$
$$= 4/36 + 6/36 + 5/36 + 2/36 = 17/36.$$

Example 2

A die is loaded in such a way that an even number is twice likely to occur as an odd number.

(a) If E is the event that a number is less than 4 on a single toss, find $\Pr(E)$.

(b) Let A = {even numbers} and

B = {numbers divisible by 3}.

What is $\Pr(A \cap B)$ and $\Pr(A \cup B)$?

Solution to Example 2

(a) Since $\Pr(S) = 1$, so $\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) = 1$. We also have and $\Pr(\text{even number}) = 2 \times \Pr(\text{odd number})$. Hence

$$p + 2p + p + 2p + p + 2p = 1.$$

Therefore,

$$p = 1/9.$$

$E = \{1, 2, 3\}$. So

$$\Pr(E) = \Pr(1) + \Pr(2) + \Pr(3)$$
$$= p + 2p + p = 1/9 + 2/9 + 1/9 = 4/9.$$

Solution to Example 2 (Continued)

(b) $A = \{2, 4, 6\}$ and $B = \{3, 6\}$.

Therefore $A \cap B = \{6\}$ and $A \cup B = \{2, 3, 4, 6\}$.

$\Pr(A \cap B) = \Pr(6) = 2p = 2/9$.

$$\begin{aligned}\Pr(A \cup B) &= \Pr(\{2, 3, 4, 6\}) \\ &= \Pr(2) + \Pr(3) + \Pr(4) + \Pr(6) \\ &= 2p + p + 2p + 2p = 7/9.\end{aligned}$$

Alternatively,

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 6/9 + 3/9 - 2/9 = 7/9.\end{aligned}$$

1.5.3 Sample Spaces Having Equally Likely Outcomes

- Consider an experiment whose sample space S is a finite set, say, $S = \{a_1, a_2, \cdots, a_k\}$.
- Assume that all outcomes in the sample space are **equally likely** to occur.
- That is

$$\Pr(a_1) = \Pr(a_2) = \cdots = \Pr(a_k).$$

- Since $\Pr(a_1) + \Pr(a_2) + \cdots + \Pr(a_k) = 1$ and $\Pr(a_i)$, $i = 1, 2, \cdots, k$ are the same, therefore,

$$\Pr(a_i) = 1/k, \qquad i = 1, 2, \cdots, k.$$

Sample Spaces Having Equally Likely Outcomes

(Continued)

- Obviously, if an experiment can result in any one of the k different **equally likely** outcomes, then for any event A

$$\Pr(A) = \frac{\text{Number of sample points in } A}{\text{Number of sample points in } S}$$

Examples

1. For outcomes which are equally likely

   (a) Tossing a fair coin, then

   $$\Pr(\text{Head}) = \Pr(\text{Tail}) = 1/2.$$

   (b) Tossing a fair die,

   let event A = {numbers greater than 4} = {5, 6},

   then $\Pr(A) = 2/6$.

2. For outcomes which are not equally likely

   Tossing a **bias coin** such that a Head is twice as likely as a Tail to occur, then

   $$\Pr(\text{Head}) = 2/3 \text{ and } \Pr(\text{Tail}) = 1/3.$$

Example 3

- A box contains 50 bolts and 150 nuts.
- Half of the bolts and half of the nuts are rusted.
- If one item is chosen at random, what is the probability that it is rusted or is a bolt?

Solution

- Let A = {the item is rusted} and B = {the item is a bolt}.
- Then

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 100/200 + 50/200 - 25/200 = 5/8.\end{aligned}$$

Example 4

- If two fair dice are rolled, what is the probability that the sum of the upturned faces will equal 8?

Solution

- Let A be the event that the sum of the upturned faces is 8.
- Then $A = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
- Since each outcome $(\bullet,\bullet)$ are equally likely with probability $1/36$, therefore

$$\Pr(A) = 5/36.$$

Example 5

- If 2 balls are "randomly drawn" from an urn containing 6 white and 5 black balls,
- what is the probability that one of the drawn balls is white and the other black?

Solution to Example 5

- Number of elements in the sample space (i.e. all possible outcomes) is given by ${}_{11}C_2 = 11(10)/2 = 55$.
- Number of sample points that have one white and one black ball is given by ${}_6C_1 \times {}_5C_1 = 6(5) = 30$.
- So, the probability that one of the drawn balls is white and the other black is given by

$$\frac{{}_6C_1 \times {}_5C_1}{{}_{11}C_2} = \frac{30}{55}.$$

Example 6

- In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution

- Number of all possible outcomes of drawing 5 cards is given by ${}_{52}C_5 = 52!/(5!\,47!) = 2598960$.

Example 6 (continued)

Solution (Continued)

- Number of outcomes that have 2 aces and 3 jacks is given by

$$ {}_4C_2 \times {}_4C_3 \times {}_{44}C_0 = \big(4!/(2!\,2!)\big) \times \big(4!/(3!\,1!)\big) \times 1 = 24. $$

- So, the probability that there are 2 aces and 3 jacks in a poker hand is given by

$$ \frac{{}_4C_2 \times {}_4C_3 \times {}_{44}C_0}{{}_{52}C_5} = \frac{24}{2598960}. $$

Example 7

If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems and a dictionary, what is the probability that

(a) a dictionary is selected?

(b) 2 novels and 1 book of poems are selected?

Solution to Example 7

(a) Number of all possible selections of 3 books is given by

$$_9C_3 = 9!/(3!6!) = 84.$$

Number of ways selecting a dictionary and 2 other books is

$$_1C_1 \times {}_8C_2 = 1 \times \left(8!/(2!\,6!)\right) = 28.$$

The probability that a dictionary is selected is given by

$$\frac{\text{\# of ways selecting a dictionary and 2 other books}}{\text{\# of all possible selections of 3 books}}$$

$$= \frac{28}{84} = \frac{1}{3}.$$

Solution to Example 7 (Continued)

(b) Number of selecting 2 novels and 1 book of poem is

$$ {}_5C_2 \times {}_3C_1 \times {}_1C_0 = \frac{5!}{(2!\,3!)} \times \frac{3!}{2!\,1!} \times \frac{1!}{1!\,0!} = 10(3) = 30. $$

- The required probability is given by

$$ \frac{30}{84} = \frac{5}{14}. $$

Example 8

- A professor hands out a list of 10 topics, 5 of which will appear in a test.
- One student has enough time to prepare only 7 of them.
- If the professor chooses the 5 topics at random, then what is the probability that the student will be prepared for

  (a) all five topics that appear in the test?

  (b) less than 3 topics?

  (c) exactly 4 topics?

Solution to Example 8

(a) Let S be the sample space and

A = {Prepared for all five topics in the test}.

Number of elements in $S = \#(S) = {}_{10}C_5 = 10!/(5!\,5!)$.

$\#(A) = {}_7C_5 \times {}_3C_0 = [7!/(5!\,2!)] \times [3!/(3!\,0!)]$.

$\Pr(A) = \#(A)/\#(S) = [7!/(5!\,2!)]/[10!/(5!\,5!)] = 1/12$.

(b) B = {Prepared for less than 3 topics in the test}.

$\#(B) = {}_7C_2 \times {}_3C_3 = [7!/(2!\,5!)] \times [3!/(3!\,0!)]$.

$\Pr(B) = \#(B)/\#(S) = [7!/(2!\,5!)]/[10!/(5!\,5!)] = 1/12$.

Solution to Example 8 (Continued)

(c) C = {Prepared for exactly four topics in the test}.

$$\#(C) = {}_7C_4 \times {}_3C_1 = \frac{7!}{4!\,3!} \times \frac{3!}{2!\,1!}.$$

$$\Pr(C) = \frac{\#(C)}{\#(S)} = \frac{\dfrac{7!}{4!\,3!} \times \dfrac{3!}{2!\,1!}}{\dfrac{10!}{5!\,5!}} = \frac{5}{12}.$$

Example 9

- In a class of 200 students, 108 study economics, 138 study chemistry and 70 study both chemistry and economics.
- If a student is selected at random, what is the probability that the student

  (a) takes economics **or** chemistry;

  (b) doesn't take **neither** of these subjects;

  (c) takes chemistry **but not** economics.

Solution to Example 9

- Let E = {The student takes economics} and
  C = {The student takes chemistry}.
- From the info given, we have $\Pr(E) = 108/200$, $\Pr(C) = 138/200$ and $\Pr(E \cap C) = 70/200$.

(a) $\Pr(E \cup C) = \Pr(E) + \Pr(C) - \Pr(E \cap C)$

$$= 108/200 + 138/200 - 70/200 = 0.88.$$

(b) $\Pr\left((E \cup C)'\right) = 1 - \Pr(E \cup C) = 1 - 0.88 = 0.12.$

Solution to Example 9 (Continued)

(c) Since $C = (E' \cap C) \cup (E \cap C)$ and $(E' \cap C) \cap (E \cap C) = \emptyset$, therefore

$$\Pr(C) = \Pr(E' \cap C) + \Pr(E \cap C).$$

Hence $\Pr(E' \cap C) = \Pr(C) - \Pr(E \cap C)$

$$= 138/200 - 70/200$$

$$= 0.34.$$

1.6 Conditional probability

1.6.1 Introduction

- We sometimes encounter in calculating probabilities of events when some partial information concerning the result of the experiment is available; in such a situation the desired probabilities are **conditional ones**.
- Let A and B be two events associated with an experiment E. We denote

$$\Pr(A|B)$$

the **conditional probability** of the event A, given that event B has occurred.

An Illustrative Example

- Two fair dice are tossed, the outcome being recorded as (x_1, x_2), where x_i is the outcome of the i^{th} die, $i = 1, 2$.
- Hence the sample space is

$$S = \{(1,1), (1,2), (1,3), \cdots\cdots, (6,5), (6,6)\}.$$

- Let $A = \{(x_1, x_2) \mid x_1 + x_2 = 10\} = \{(5,5), (4,6), (6,4)\}$
- $B = \{(x_1, x_2) \mid x_1 > x_2\} = \{(2,1), (3,1), (3,2), \cdots, (6,4), (6,5)\}$
- $\#(S) = 36, \#(A) = 3, \#(B) = 15.$
- $\Pr(A) = 3/36 = 1/12, \Pr(B) = 15/36 = 5/12.$

An Illustrative Example (Continued)

- Suppose that we know the outcome satisfies $x_1 > x_2$.
- What is the probability that $x_1 + x_2 = 10$?

  That is, knowing that event B has occurred, what is the chance that event A occurs? (i.e. $\Pr(A|B) = ?$)
- Since the event B has occurred, instead of considering the set of all possible outcomes (i.e. S), we consider the set of sample points for event B.
- Among the sample points in B, figure out those that favor A.
- That is, find out the elements in $A \cap B$.

An Illustrative Example (Continued)

- Hence the conditional probability of A given B can be obtained by

$$\Pr(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

- If we divide both the numerator and denominator by $\#(S)$, then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

- Similarly,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}.$$

An Illustrative Example (Continued)

- Since $A \cap B = \{(6,4)\}$, hence $\#(A \cap B) = 1$.
- Therefore

$$\Pr(A|B) = \frac{\#(A \cap B)}{\#(B)} = \frac{1}{15}.$$

- Similarly,

$$\Pr(B|A) = \frac{\#(A \cap B)}{\#(A)} = \frac{1}{3}.$$

1.6.2 Conditional Probability

Definition

- The **conditional probability of B given A**, is defined as

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}, \qquad \text{if } \Pr(A) \neq 0.$$

- Intuitively $\Pr(B|A)$ means that probability of the occurrence of the event B, under the assumption that event A has already occurred.

Remarks

Note: For fixed A, $\Pr(B|A)$ satisfies the various postulates of probability.

- That is, we have
  1. $0 \leq \Pr(B|A) \leq 1$.
  2. $\Pr(S|A) = 1$.
  3. If $B_1, B_2, \cdots$ are **mutually exclusive** (disjoint) events (that is, $B_i \cap B_j = \emptyset$ when $i \neq j$), then
  $$\Pr(\cup_{i=1}^{\infty} B_i \,|A) = \sum_{i=1}^{\infty} \Pr(B_i|A).$$
  In particular, if B_1 and B_2 are disjoint events, then
  $$\Pr(B_1 \cup B_2|A) = \Pr(B_1|A) + \Pr(B_2|A) .$$

Example 1

- Roll 2 balanced dice.
- Suppose that the first die is a 3.
- What is the probability that the sum of the 2 dice equals 8?

Solution to Example 1

- Let $A = \{(3, y): 1 \le y \le 6\}$, which is the event that the first die is a 3.
- Let $B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$, the event that the sum of the 2 dice equals 8.
- Hence, $A \cap B = \{(3, 5)\}$.
- We would like to find the ***conditional*** probability $\Pr(B|A)$, which is given by

$$\frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/36}{6/36} = \frac{1}{6}.$$

Example 2

- Roll an unbalanced die.
  (The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.)
- Suppose that the respective probabilities are $1/12$, $1/12$, $1/6$, $1/6$, $1/6$, and $1/3$.

  Note: The sum of the probabilities equals 1

  (a) If the number obtained is even, what is the probability that it is a 6?

  (b) What is the probability that the number obtained is a perfect square number given that a number greater than 3 has obtained?

Solution to Example 2

(a) Let $A = \{2, 4, 6\}$, the event that the number obtained is even.

$$\Pr(A) = 1/12 + 1/6 + 1/3 = 7/12$$

$B = \{6\}$ and $A \cap B = \{6\}$.

$$\Pr(A \cap B) = \Pr(\{6\}) = 1/3.$$

Hence,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/3}{7/12} = \frac{4}{7}.$$

Solution to Example 2 (Continued)

(b) Let C = {Number > 3} = {4, 5, 6} and

D = {A perfect square number} = {1, 4}.

Hence, $C \cap D = \{4\}$ and

$$\Pr(C) = \Pr(\{4\}) + \Pr(\{5\}) + \Pr(\{6\})$$
$$= 1/6 + 1/6 + 1/3 = 2/3$$

$$\Pr(C \cap D) = \Pr(\{4\}) = 1/6.$$

Hence

$$\Pr(D|C) = \frac{\Pr(C \cap D)}{\Pr(C)} = \frac{1/6}{2/3} = \frac{1}{4}.$$

Example 3

- A couple has 2 children.
- What is the probability that both are boys if it is known that they have at least one son?

Solution to Example 3

- Let us, say, assume it is **equally likely** to be a boy or a girl for every new born baby.
- Put $E = \{\text{Both are boys}\}$ and

  $F = \{\text{At least one son}\}$.
- Then $E = \{(B, B)\}$, $F = \{(B, G), (G, B), (B, B)\}$ and $E \cap F = E$.
- Hence

$$\Pr(E|F) = \frac{\#(E \cap F)}{\#(F)} = \frac{1}{3}.$$

Example 4

- The probability that a regularly scheduled flight departs on time is 0.83 $(= \Pr(D))$;
- the probability that it arrives on time is 0.82 $(= \Pr(A))$; and
- the probability that it departs and arrives on time is 0.78 $(= \Pr(D \cap A))$.
- Find the probability that a plane arrives on time given that it departed on time.

<u>Solution</u>

$$\Pr(A|D) = \frac{\Pr(D \cap A)}{\Pr(D)} = \frac{0.78}{0.83} = 0.9398.$$

Example 5

- The following data were obtained in a study

	Non-smokers	Moderate smokers	Heavy smokers	Row Total
Hypertension	21	36	30	87
No hyper-tension	48	26	19	93
Column total	69	62	49	180

Example 5 (Continued)

- If one of these individuals is selected at random, find the probability that the person is

(a) experiencing hypertension;

(b) experiencing hypertension given that the person is a heavy smoker;

(c) a non-smoker, given that the person is experiencing no hypertension.

Solution to Example 5

(a) Let H and H' denote the events that the selected person is experiencing hypertension and no hypertension respectively.

Out of 180 persons, there are 87 of them with hypertension. Therefore

$$\Pr(H) = 87/180$$

Out of 180 persons, there are 93 of them with no hypertension. Hence

$$\Pr(H') = 93/180.$$

Alternatively, $\Pr(H') = 1 - \Pr(H) = 1 - \frac{87}{180} = \frac{93}{180}$.

Solution to Example 5 (Continued)

(b) Let A denote the event that the selected person is a heavy smoker.

There are 49 heavy smokers among 180 persons. That is,

$$\Pr(A) = 49/180.$$

Also there are 30 heavy smokers with hypertension among the 180 persons. Hence,

$$\Pr(H \cap A) = 30/180.$$

Hence the required conditional probability is given by

$$\Pr(H|A) = \frac{\Pr(H \cap A)}{\Pr(A)} = \frac{(30/180)}{49/180} = \frac{30}{49}.$$

Solution to Example 5 (Continued)

(c) Let B denote the event that the selected person is a non-smoker.

There are 48 non-smokers with no hypertension among the 180 persons

$$\Pr(B \cap H') = 48/180$$

From part (a), we have $\Pr(H') = 93/180$

Therefore

$$\Pr(B|H') = \frac{\Pr(B \cap H')}{\Pr(H')} = \frac{48/180}{93/180} = \frac{48}{93} = \frac{16}{31}.$$

Example 6

- For married couples living in a certain district, the probability that the husband will vote in a referendum is 0.21,
- the probability that his wife will vote is 0.28, and
- the probability that both will vote is 0.15.
- What is the probability that

  (a) at least one member of a married couple will vote?

  (b) a wife will vote, given that her husband votes?

  (c) a husband will vote, given that his wife does not vote?

Solution to Example 6

- Let H represent the event that the husband votes and W represent the event that the wife votes.
- It is given that $\Pr(H) = 0.21$, $\Pr(W) = 0.28$, and $\Pr(H \cap W) = 0.15$.

(a) $\Pr(H \cup W) = \Pr(H) + \Pr(W) - \Pr(H \cap W)$

$= 0.21 + 0.28 - 0.15 = 0.34.$

(b) $\Pr(W|H) = \Pr(H \cap W)\,/\Pr(H)$

$= 0.15/0.21 = 5/7.$

Solution to Example 6 (continued)

(c) $\Pr(H|W') = \Pr(H \cap W')\,/\Pr(W')$

$$= [\Pr(H) - \Pr(H \cap W)]/(1 - \Pr(W))$$

$$= (0.21 - 0.15)/(1 - 0.28) = 1/12.$$

- Notice that $H = (H \cap W) \cup (H \cap W')$ and $(H \cap W) \cap (H \cap W') = \emptyset$.
- Hence $\Pr(H) = \Pr(H \cap W) + \Pr(H \cap W')$ or $\Pr(H \cap W') = \Pr(H) - \Pr(H \cap W)$.

Example 7

- A statistics professor is teaching both a morning and an afternoon sections of introductory statistics.
- Let A = {the professor gives a bad morning lecture}

  and B = {the professor gives a bad afternoon lecture}.
- Suppose that

$$\Pr(A) = 0.3, \Pr(B) = 0.2 \text{ and } \Pr(A \cap B) = 0.1,$$

Example 7 (Continued)

Calculate the following probabilities.

(a) $\Pr(B|A)$.

(b) $\Pr(B'|A)$.

(c) $\Pr(B|A')$.

(d) $\Pr(B'|A')$.

(e) If at the conclusion of the afternoon class, the professor is heard to mutter “what a rotten lecture”, what is the probability that the morning lecture was also bad?

Solution to Example 7

(a) $\Pr(B|A) = \Pr(A \cap B) \,/\, \Pr(A) = 0.1/0.3 = 1/3.$

(b) $\Pr(B'|A) = 1 - \Pr(B \mid A) = 1 - 1/3 = 2/3.$

(c) $\Pr(B|A') = \Pr(A' \cap B) \,/\, \Pr(A')$

$$= [\Pr(B) - \Pr(B \cap A)]/(1 - \Pr(A))$$

$$= (0.2 - 0.1)/(1 - 0.3) = 1/7.$$

(d) $\Pr(B'|A') = 1 - \Pr(B|A') = 1 - 1/7 = 6/7.$

(e) $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = 0.1/0.2 = 1/2.$

1.6.3 Multiplication Rule of Probability

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A) \quad \text{or}$$

$$\Pr(A \cap B) = \Pr(B)\Pr(A|B),$$

providing $\Pr(A) > 0, \Pr(B) > 0$.

- This rule enables us to calculate the probability that two events will both occur.
- The probability that both events occur is the product of the probability of one event occurs and the conditional probability that the other event occurs given that the first event has occurred.

Multiplication Rule of Probability (Continued)

- It can be extended to more than 2 events:

$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B|A)\Pr(C|A \cap B),$$

providing that $\Pr(A \cap B) > 0$.

- In general

$$\Pr(A_1 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2 \mid A_1)$$
$$\Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}),$$

providing that $\Pr(A_1 \cap \cdots \cap A_{n-1}) > 0$.

Example 1

- Suppose that among 12 shirts, 3 are white.
- Two shirts are chosen randomly one by one **without replacement**.

  (a) What is the probability that **both** of the shirts that being picked are white?

  (b) What is the probability that there is **only one** white shirt being picked?

  (c) If 3 shirts are chosen at random, what is the probability that they are **all** white?

Solution to Example 1

Let A_1 = {the first shirt is white} and

A_2 = {the second shirt is white}.

$$\Pr(A_1) = 3/12$$

If the first shirt is white, then there are 2 white shirts among the remaining 11 shirts.

$$\Pr(A_2|A_1) = 2/11$$

(a) $\Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2|A_1)$

$$= \left(\frac{3}{12}\right)\left(\frac{2}{11}\right) = \frac{1}{22}.$$

Solution to Example 1 (Continued)

(b) We have $\Pr(A_1) = 3/12$ and $\Pr(A_1') = 1 - 3/12 = 9/12.$

$\Pr(A_2|A_1) = 2/11$ and $\Pr(A_2'|A_1) = 1 - 2/11 = 9/11.$

If the first shirt is not white, then there are 3 white shirts among the remaining 11 shirts. Hence

$$\Pr(A_2|A_1') = 3/11.$$

$$\begin{aligned}\Pr((A_1 \cap A_2') \cup (A_1' \cap A_2)) &= \Pr(A_1 \cap A_2') + \Pr(A_1' \cap A_2) \\ &= \Pr(A_1)\Pr(A_2'|A_1) + \Pr(A_1')\Pr(A_2|A_1') \\ &= \left(\frac{3}{12}\right)\left(\frac{9}{11}\right) + \left(\frac{9}{12}\right)\left(\frac{3}{11}\right) = \frac{9}{22}.\end{aligned}$$

Solution to Example 1 (Continued)

(c) Let $A_1 = \{(W,\ *,\ *)\}$,

$A_2 = \{(*,\ W,\ *)\}$ and

$A_3 = \{(*,\ *,\ W)\}$

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)$$
$$= \left(\frac{3}{12}\right)\left(\frac{2}{11}\right)\left(\frac{1}{10}\right) = \frac{1}{220}.$$

Example 2

- If 2 balls are "randomly drawn" **without replacement** from an urn containing 6 white and 5 black balls,
- what is the probability that the first drawn ball is black and the second one is white?

Solution to Example 2

- Let B_1 represent the event that a black ball is drawn in the first draw and
- W_2 represent the event that a white ball is drawn in the second draw.

$$\Pr(B_1 \cap W_2) = \Pr(B_1)\Pr(W_2|B_1)$$
$$= \left(\frac{5}{11}\right)\left(\frac{6}{10}\right) = \frac{3}{11}.$$

Example 3

- Two cards are drawn in succession from an ordinary deck of 52 playing cards **without replacement**.
- What is the probability that both cards are greater than 3 but less than 8?

Solution to Example 3

- There are 16 cards that are greater than 3 but less than 8. (i.e. There are 16 cards with numbers 4, 5, 6 and 7.)
- Let A_i represent the event the i^{th} card drawn is greater than 3 but less than 8.

$$\Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2|A_1)$$
$$= \left(\frac{16}{52}\right)\left(\frac{15}{51}\right) = \frac{20}{221}.$$

Example 4

- The probability that a doctor correctly diagnoses a particular illness is 0.7.
- Given that the doctor makes an incorrect diagnosis, the patient will enter a law suit is 0.9.
- What is the probability that the doctor makes an incorrect diagnosis and the patient sues?

Solution to Example 4

- Let D = {Correct diagnosis} and U = {Patient sues}.
- It is given that $\Pr(D) = 0.7$ and $\Pr(U|D') = 0.9$

$$\begin{aligned}\Pr(D' \cap U) &= \Pr(D')\Pr(U|D') \\ &= (1 - 0.7)(0.9) \\ &= 0.27.\end{aligned}$$

Example 5

- Four individuals have responded to a request by a blood bank for blood donations.
- All of them forget their blood types.
- Suppose only type A+ is desired and only one of the four actually has this type.
- If the potential donors are selected at random order for typing, what is the probability that

  (a) at least three individuals must be typed to obtain the desired type?

  (b) the third donor's blood type is A+?

Solution to Example 5

(a) Let A = {First donor not A+} and

B = {Second donor not A+}.

$\Pr(A) = 3/4$ and $\Pr(B|A) = 2/3$. Hence

{At least 3 individuals are typed } $\equiv$ {the first 2 donors are not A+ type}

$\Pr(\text{at least 3 individuals are typed}) = \Pr(A \cap B)$

$$= \Pr(A)\Pr(B|A) = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{2}.$$

Solution to Example 5 (Continued)

(b) Pr(Third donor is A+) = Pr(First donor isn't A+) $\times$

Pr(Second donor isn't A+ | First donor isn't A+) $\times$

Pr(Third donor is A+ | First 2 donors are not A+)

$$= \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

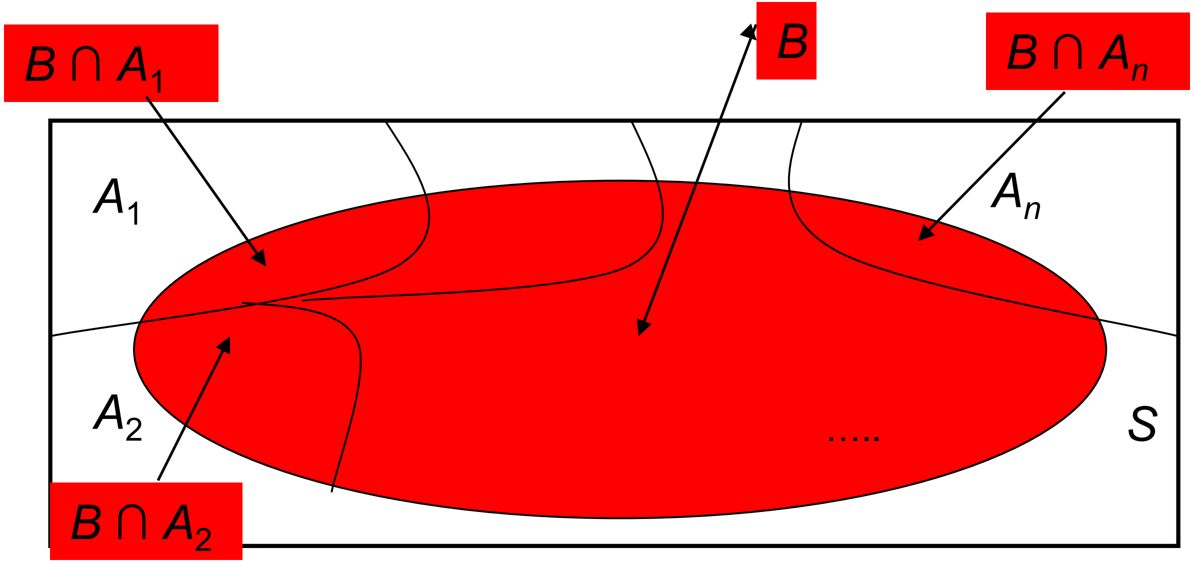
1.6.4 The Law of Total Probability

- Let $A_1, A_2, \cdots, A_n$ be a **partition** of the sample space S.
- That is $A_1, A_2, \cdots, A_n$ are **mutually exclusive and exhaustive events** such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = S$.

- Then for any event B

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap A_i) = \sum_{i=1}^{n} \Pr(A_i) \Pr(B|A_i)$$

The Law of Total Probability (Continued)



Example 1

- In a certain factory, Machines A, B, and C produce 2%, 1% and 3% defective screws respectively.
- Of the total production, Machines A, B, and C produce 35%, 25% and 40% respectively.
- If a screw is selected randomly,

  (a) what is the probability that it is a defective?

  (b) what is the probability that it is produced by the Machine C given that it is defective?

Solution to Example 1

Let D be the event that a defective screw is selected.

Let A, B and C be the event that the selected screw come from Machine A, B and C respectively.

A, B and C are mutually exclusive and exhaustive events

(a) $\Pr(D) = \Pr(A)\Pr(D|A) + \Pr(B)\Pr(D|B) + \Pr(C)\Pr(D|C)$

$$= (0.35)(0.02) + (0.25)(0.01) + (0.4)(0.03) = 0.0215.$$

(b) $\Pr(C|D) = \Pr(C \cap D)/\Pr(D)$

$$= \Pr(C)\Pr(D|C)/\Pr(D)$$

$$= (0.4)(0.03)/0.0215 = 0.5581.$$

Example 2

- One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls.
- One ball is drawn from the first bag and placed unseen in the second bag.
- What is the probability that a ball now drawn from the second bag is black?

Solution to Example 2

- Let A_1 = {the 1st ball drawn is white} and
  A_2 = {the 1st ball drawn is black} $= A_1'$.
- Let B = {the ball drawn from the 2nd bag is black}.
- $\Pr(A_1) = 4/7$ and
- $\Pr(A_2) = \Pr(A_1') = 1 - 4/7 = 3/7$

Solution to Example 2 (Continued)

- There are 4 white balls among 9 balls in the 2nd bag after a white ball was drawn and put in the 2nd bag. Hence
$$\Pr(B|A_1) = 5/9.$$
- Similarly, $\Pr(B|A_2) = 6/9.$
- Use the Law of Total Probability.
$$\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)$$
$$= \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) + \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) = \frac{38}{63}.$$

Example 3

- Traffic police plan to enforce speed limits by using speed cameras at 4 different locations of the expressway.
- The speed cameras at each of the locations L_1, L_2, L_3 and L_4 are operated 40%, 30%, 20% and 30% of the time respectively.
- If a driver who is speeding on his way to work has probabilities 0.2, 0.1, 0.5 and 0.2 respectively, of passing through these locations, what is the probability that
  (a) he will receive a speeding ticket?
  (b) he passed through the radar trap at location L_2 if he received a speeding ticket?

Solution to Example 3

Let B = {The driver receives a speeding ticket}

A_i = {He past through the location L_i} for $i = 1, \cdots, 4$

(a) $\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2) + \Pr(A_3)\Pr(B|A_3) + \Pr(A_4)\Pr(B|A_4)$

$= 0.2(0.4) + 0.1(0.3) + 0.5(0.2) + 0.2(0.3) = 0.27.$

(b) $\Pr(A_2|B) = \Pr(A_2 \cap B)/\Pr(B)$

$= \Pr(A_2)\Pr(B|A_2)/\Pr(B)$

$= [0.1(0.3)]/0.27 = 1/9.$

Example 4

- A bag contains one 50 cent, one 20 cent and one 10 cent coins.
- The 10 cent coin is a fake coin with two heads.
- A coin is chosen at random from the bag and tossed four times in succession.
- If the result is 4 heads, what is the probability that the fake 10 cent coin is used?

Solution to Example 4

- Let A = {The 10 cent coin is chosen},

  B = {The 20 cent coin is chosen},

  C = {The 50 cent coin is chosen},

  $4H$ = {Having 4 heads in 4 tosses}.
- If the 10 cent coin is chosen, $\Pr(H) = 1$ and

  $\Pr(4H) = 1^4 = 1$.
- That is, $\Pr(4H|A) = 1^4 = 1$.

Solution to Example 4 (Continued)

- If the 20 cent soin is chosen, then $\Pr(H) = 1/2$ and
  $\Pr(4H) = (1/2)^4 = 1/16$.
  That is $\Pr(4H|B) = 1/16$.
- Similarly, $\Pr(4H|C) = 1/16$.

- A coin is chosen at random implies that

$$\Pr(A) = \Pr(B) = \Pr(C) = 1/3$$

Solution to Example 4 (Continued)

- Applying the law of total probability, we have

$$\begin{aligned}\Pr(4H) &= \Pr(A)\Pr(4H|A) + \Pr(B)\Pr(4H|B) \\ &\quad + \Pr(C)\Pr(4H|C) \\ &= (1/3)(1 + 1/16 + 1/16) = 3/8.\end{aligned}$$

$$\begin{aligned}\Pr(A|4H) &= \Pr(A \cap 4H)/\Pr(4H) \\ &= \Pr(A)\Pr(4H|A)/\Pr(4H) \\ &= [(1/3)(1)]/[3/8] = 8/9.\end{aligned}$$

1.6.5 Bayes' Theorem

<u>Bayes' Theorem</u>

- Let $A_1, A_2, \cdots, A_n$ be **a partition** of the sample space S. Then

$$\Pr(A_k | B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\sum_{i=1}^{n} \Pr(A_i)\Pr(B \mid A_i)}$$

for $k = 1, \cdots, n$.

Note : The denominator is just $\Pr(B)$. That is,

$$\Pr(A_k | B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\Pr(B)}$$

Example 1

- Suppose that there is a chance for a newly constructed house to collapse whether the design is faulty or not.
- The chance that the design is faulty is 1%.
- The chance that the house collapses if the design is faulty is 75% and otherwise it is 0.01%.
- It is seen that the house collapsed.
- What is the probability that it is due to faulty design?

Solution to Example 1

- Let B = {The design is faulty},

  A = {The house collapses}.

- Hence

$$\Pr(B) = 0.01,$$
$$\Pr(A|B) = 0.75,$$

  and

$$\Pr(A|B') = 0.0001$$

Solution to Example 1 (Continued)

- Applying the Law of Total Probability, we have

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B)$$
$$= 0.01(0.75) + 0.99\,(0.0001) = 0.007599$$

Therefore,

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$$
$$= \frac{0.01(0.75)}{0.007599} = 0.9870.$$

Example 2

- An insurance company believes that people can be divided into two classes — those who are accident prone and those who are not.
- Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.04, whereas this probability decreases to 0.02 for a non-accident-prone person.
- Assume that 30% of the population is accident prone.

Example 2 (Continued)

(a) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

(b) If a new policyholder has an accident within a year of purchasing a policy, what is the probability that he or she is accident prone?

Solution to Example 2

- Let B to be the event that a new policy holder has an accident within a year of purchasing a policy.
- Let A_1 and A_2 be the events that a (randomly selected) new policy holder is accident prone and non-accident prone respectively.
- Then $A_2 = A_1'$.
- It is given that $\Pr(A_1) = 0.3$, and hence $\Pr(A_2) = 0.7$.
- Also, $\Pr(B|A_1) = 0.04$, and $\Pr(B|A_2) = 0.02$.

Solution to Example 2 (Continued)

- Using the Law of Total Probability, we obtain

$$\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)$$
$$= 0.3(0.04) + 0.7(0.02) = 0.26.$$

- Using the Bayes' Rule, we have

$$\Pr(A_1|B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)}$$
$$= \frac{0.3(0.04)}{0.3(0.04) + 0.7(0.02)} = \frac{6}{13}.$$

Example 3

- Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black.
- The 3 cards are mixed up in a box, and 1 card is randomly selected and put down on the table.
- If the upper side of the chosen card is colored red,
- what is the probability that the other side is colored black?

Solution to Example 3

- Let A_i denote the event that the card i is selected for $i = 1, 2, 3$ and
- let B denote the event that upper side of the chosen card is colored red.

- From the given info, we have,

$$\Pr(A_1) = \Pr(A_2) = \Pr(A_3) = \frac{1}{3},$$

$$\Pr(B|A_1) = 1, \Pr(B|A_2) = 0, \text{and } \Pr(B|A_3) = \frac{1}{2}.$$

Solution to Example 3 (Continued)

- The desired probability is the conditional probability of the card 3 being selected, given that B occurs.
- That is, we are to evaluate $\Pr(A_3|B)$.
- Applying the Bayes' rule, we have

$$\Pr(A_3|\,B) = \frac{\Pr(A_3)\Pr(B\mid A_3)}{\sum_{i=1}^{3}\Pr(A_i)\Pr(B\mid A_i)}$$

$$= \frac{(1/3)(1/2)}{1/3\,(1+0+1/2)} = \frac{1}{3}$$

Example 4

- Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed.
- The test is such that, when an individual actually has the disease, a positive result will occur 99% of the time,
- while an individual without the disease will show a positive test only 2% of the time.
- If a randomly selected individual is tested and the result is **positive**, what is the probability that the individual has the disease?

Solution to Example 4

- Let A_1 = {An individual has the disease},

  A_2 = {An individual doesn't have the disease} = A_1',

  B = {positive test result}
- Using the Bayes' Theorem, we have

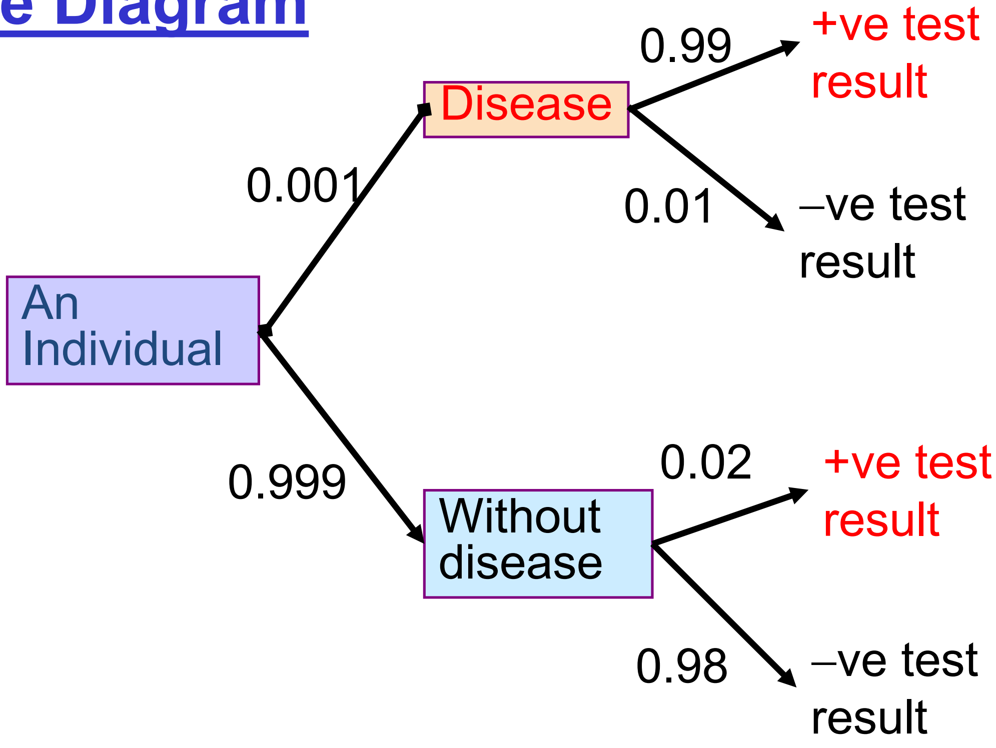
$$\Pr(A_1|B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)}$$

$$= \frac{0.001(0.99)}{0.001(0.99) + 0.999(0.02)}$$

$$= \frac{0.00099}{0.00099 + 0.01998} = 0.0472.$$

Solution to Example 4 (Continued)

Tree Diagram



Pr (+ve test result and with disease) = 0.001(0.99) = 0.00099

Pr (+) = 0.00099 + 0.01998 = 0.02097

Pr (+ve test result and without disease) = 0.999(0.02) = 0.01998

Pr (D|+) = 0.00099/(0.00099 + 0.01998) = 0.047

Example 5

- A student answers a multiple-choice examination question that offers four possible answers.
- Suppose that the probability that the student knows the answer to the question is 0.8 and
- the probability that the student will guess is 0.2.
- Assume that if the student guesses, the probability of selecting the correct answer is 0.25.
- If the student correctly answers a question, what is the probability that the student really knew the correct answer?

Solution to Example 5

- Let A_1 = {A student knows the answer},
  A_2 = {A student doesn't know the answer} = A_1',
  B = {A student answers correctly}

- Using the Bayes' Theorem, we have

$$\Pr(A_1 | B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)}$$

$$= \frac{0.8(1)}{0.8(1) + 0.2(0.25)} = 0.9412.$$

Example 6

- Components of a certain type are shipped to a supplier in batches of ten.
- Suppose that 50% of all such batches contain no defective components,

  30% contain one defective component, and

  20% contain two defective components.
- Two components from a batch are randomly selected and tested.
- What are the probabilities associated with 0, 1 and 2 defective components being in the batch if neither tested component is defective?

Solution to Example 6

- Let A_i = $\{i$ defectives in batch$\}$, for $i = 0, 1, 2$ and

  D_i = $\{i$ defectives in sample$\}$ for $i = 0, 1, 2$.

- From the given info, we have

  $\Pr(A_0) = 0.5, \Pr(A_1) = 0.3, \Pr(A_2) = 0.2$.

  The number of ways to select 2 items out of a batch of 10 items $= {}_{10}C_2 = 45$.

Solution to Example 6 (Continued)

- The number of ways to select 2 nondefective items out of a batch with 10 nondefective items $= {}_{10}C_2 = 45$.
- Therefore $\Pr(D_0|A_0) = {}_{10}C_2/{}_{10}C_2 = 1$.
- Similarly, $\Pr(D_0|A_1) = ({}_9C_2 \times {}_1C_0)/{}_{10}C_2 = 0.8$ and $\Pr(D_0|A_2) = ({}_8C_2 \times {}_2C_0)/{}_{10}C_2 = 0.6222$.

  Applying the Law of Total Probability, we have

  $\Pr(D_0)$

  $= \Pr(A_0)\Pr(D_0|A_0) + \Pr(A_1)\Pr(D_0|A_1) + \Pr(A_2)\Pr(D_0|A_2)$

  $= 0.5(1) + 0.3(0.8) + 0.2(0.6222) = 0.86444.$

Solution to Example 6 (continued)

- Applying the Bayes' Theorem, we have

$$\Pr(A_0|D_0) = \frac{\Pr(A_0)\Pr(D_0|A_0)}{\Pr(D_0)} = \frac{0.5(1)}{0.86444} = 0.5784.$$

Similarly,

$$\Pr(A_1|D_0) = \frac{0.3(0.8)}{0.86444} = 0.2776.$$

$$\Pr(A_2|D_0) = \frac{0.2(0.622)}{0.86444} = 0.1440.$$

Example 7

(The Monty Hall Problem)

- Suppose you’re on a game show, and you’re given the choice of three doors: behind one door is a car; behind the others, goats.
- You pick a door, say No. 1, and the host Monty, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, “Do you want to pick door No. 2?"
- Is it to your advantage to switch your choice?

Solution to Example 7

Let's look at the probabilities of winning for the two strategies.

(a) **Stick strategy**

I choose a door with the car initially with probability 1/3. Monty will always open a door with a goat, so if I don't switch, I'll win.

I choose a door with a goat initially with probability 2/3. Monty will always open a door with a goat, so if I don't switch, I'll lose.

That is, the probability of winning is $\left(\frac{1}{3}\right)1 + \frac{2}{3}(0) = \frac{1}{3}$.

Solution to Example 7 (Continued)

(b) Switch strategy

I choose a door with the car initially with probability 1/3. Monty will always open a door with a goat, so if I switch, I'll lose.

I choose a door with the goat initially with probability 2/3. Monty will always open a door with a goat, so if I switch, I'll win.

That is, the probability of winning is $\frac{1}{3}(0) + \frac{2}{3}(1) = \frac{2}{3}$.

Therefore, the "switch" strategy will bear a better chance of winning.

Remark on Monty Hall Problem

Still confused? Watch the following videos:

http://www.youtube.com/watch?v=mhlc7peGlGg

http://www.youtube.com/watch?v=P9WFKmLK0dc

1.7 Independent Events

1.7.1 Introduction

- Suppose that a fair die is tossed twice.
- Define the events A and B as follows:

  A = {the first toss shows an even number}

  B = {the second toss shows a 5 or a 6}.
- The number of outcomes in tossing a die twice is given by

$$ {}_6C_1 \times {}_6C_1 = 36. $$

1.7.1 Introduction (Continued)

- The number of outcomes in the event $A = {}_3C_1 \times {}_6C_1$

$$\Pr(A) = \frac{{}_3C_1 \times {}_6C_1}{{}_6C_1 \times {}_6C_1} = \frac{3}{6} = \frac{1}{2}.$$

Similarly,

$$\Pr(B) = \frac{{}_6C_1 \times {}_2C_1}{{}_6C_1 \times {}_6C_1} = \frac{2}{6} = \frac{1}{3}.$$

$$\Pr(A \cap B) = \frac{{}_3C_1 \times {}_2C_1}{{}_6C_1 \times {}_6C_1} = \frac{6}{36} = \frac{1}{6}.$$

Introduction (Continued)

- Therefore

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{1/3} = \frac{1}{2}.$$

On the other hand, $\Pr(A) = 1/2$. Hence $\Pr(A|B) = \Pr(A)$.

- Similarly,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

On the other hand, $\Pr(B) = 1/3$. Hence $\Pr(B|A) = \Pr(B)$.

Introduction (Continued)

- Generally speaking, $\Pr(B|A)$ is not equal to $\Pr(B)$.

- In other words, knowing that event A has occurred generally gives a different view on the chance of event B's occurrence.

- However, the above example shows that there are some special cases where $\Pr(B|A)$ does in fact equal $\Pr(B)$.

1.7.2 Independent Events

Definition:

- Two events A and B are said to be ***independent*** if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

- Two events A and B that are not independent are said to be **dependent**.

Independent Events (Continued)

- Hence the events A and B are independent if the occurrence of one event does not in any way influence the occurrence of the other event.
- Therefore

$$
\begin{aligned}
& A \text{ is independent of } B \\
\Leftrightarrow\ & B \text{ is independent of } A \\
\Leftrightarrow\ & \Pr(A \cap B) = \Pr(A)\Pr(B).
\end{aligned}
$$

1.7.3 Properties of Independent Events

1. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are independent, then

   $$\Pr(B|A) = \Pr(B) \text{ and } \Pr(A|B) = \Pr(A).$$

   The above equalities are sometimes used as the definition of two independent events.

   That is, the conditional probability of event B given event A has occurred is the same as the unconditional probability of event B.

Properties of Independent Events (continued)

2. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are **independent** events, then events A and B **cannot be** **mutually exclusive**.

   Proof: Since A and B are independent events, $\Pr(A) > 0$ and $\Pr(B) > 0$, therefore

   $$\Pr(A \cap B) = \Pr(A)\Pr(B) > 0.$$

   Since $\Pr(A \cap B) \neq 0$, therefore $A \cap B \neq \emptyset$.

   Hence A and B are not mutually exclusive events.

Properties of Independent Events (Continued)

3. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are **mutually exclusive**, then A and B **cannot be independent**.

   Proof: A and B are **mutually exclusive** events implies

   $$\Pr(A \cap B) = 0.$$

   On the other hand, $\Pr(A) > 0$ and $\Pr(B) > 0$ implies

   $$\Pr(A)\Pr(B) > 0$$

   Therefore $\Pr(A \cap B) \neq \Pr(A)\Pr(B)$ and hence events A and B are not independent.

Properties of Independent Events (Continued)

4. The sample space S as well as the empty set $\emptyset$ are independent of any event.

<u>Proof</u>: For any event A, $A \cap S = A$, and $A \cap \emptyset = \emptyset$.

$$\Pr(A \cap S) = \Pr(A) = \Pr(A)\Pr(S)$$

since $\Pr(S) = 1$.

$$\Pr(A \cap \emptyset) = \Pr(\emptyset) = \Pr(\emptyset)\Pr(A)$$

since $\Pr(\emptyset) = 0$.

Properties of Independent Events (Continued)

5. If $A \subset B$, then A and B are dependent unless $B = S$.

   Proof: $A \subset B$ implies that $A \cap B = A$.

   $\Pr(A \cap B) = \Pr(A) \neq \Pr(A)\Pr(B)$ unless $\Pr(B) = 1$ which implies that $B = S$.

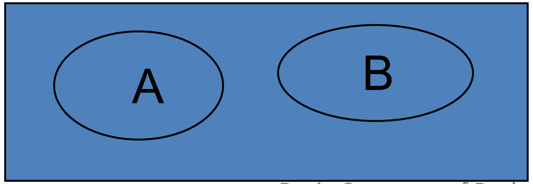
Some remarks about independent events

- The properties of independence, unlike the mutually exclusive property, **cannot** be shown on a Venn diagram.

  This means you can't trust your intuition.

  In general, <u>the only way to check for independence is by performing the calculations of the probabilities in the definition</u>.

- Mutually exclusive events are dependent events since $\Pr(A)\Pr(B) \neq \Pr(A \cap B) = 0$.



Some remarks about independent events (Continued)

Mutually exclusive and not independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

D = event of even numbers = {2, 4, 6}

A and D are **mutually exclusive** events

- $\Pr(A) = 1/2, \Pr(D) = 1/2, \Pr(A \cap D) = 0.$

Since $\Pr(A)\Pr(D) \neq \Pr(A \cap D)$, therefore A and D **are not independent** events.

Some remarks about independent events (Continued)

Not mutually exclusive and not independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

B = event of numbers ≤ 3 = {1, 2, 3}

- $A \cap B = \{1, 3\}$, hence A and B are **not mutually exclusive** events
- $\Pr(A) = 1/2, \Pr(B) = 1/2, \Pr(A \cap B) = 1/3.$

Since $\Pr(A)\Pr(B) \neq \Pr(A \cap B)$, hence A and B **are not independent** events.

Some remarks about independent events (Continued)

Not mutually exclusive and independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

C = event of numbers ≤ 4 = {1, 2, 3, 4}

- $A \cap C = \{1, 3\}$, hence A and C are not mutually exclusive events
- $\Pr(A) = 1/2, \Pr(C) = 2/3, \Pr(A \cap C) = 1/3.$

Since $\Pr(A)\Pr(C) = 1/3 = \Pr(A \cap C)$, hence A and C **are independent** events.

Example 1

- If the probability is 0.1 that a person will make a mistake on his state income tax return,
- find the probability that two totally unrelated persons each make a mistake.

<u>Solution</u>

- Let M_i = {Person i makes a mistake in his tax return} for $i = 1, 2$.
- Since M_1 and M_2 are independent events, therefore
$$\Pr(M_1 \cap M_2) = \Pr(M_1)\Pr(M_2) = (0.1)(0.1) = 0.01.$$

Theorem

- If A and B are **independent**, then so are A and B', A' and B, A' and B'.

Example 2:

- The probability that Tom will be alive in 20 years is 0.7, and
- the probability that Jack will be alive in 20 years is 0.9.
- What is the probability that neither will be alive in 20 years?

Solution to Example 2

- Let A and B respectively to be events that Tom and Jack would be alive in 20 years.
- Note that these two events are independent.
- Hence A' and B' are independent, too.
- The desired probability is given by

$$\begin{aligned}\Pr(A' \cap B') &= \Pr(A') \Pr(B') \\ &= (1 - 0.7)(1 - 0.9) = 0.03.\end{aligned}$$

1.7.4 n Independent Events

Pairwise Independent Events

Definition :

- A set of events $A_1, A_2, \cdots, A_n$ are said to be **pairwise independent** if and only if
$$\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$$
for $i \neq j$ and $i, j = 1, \cdots, n$.

n Mutually Independent Events

- The events $A_1, A_2, \cdots, A_n$ are called **mutually independent** (or simply **independent**)

  if and only if for **any subset** $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ of $A_1, A_2, \cdots, A_n$,

$$\Pr\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = \Pr\left(A_{i_1}\right)\Pr\left(A_{i_2}\right)\cdots\Pr\left(A_{i_k}\right)$$

Remarks

1. When one says that events $A_1, A_2, \cdots, A_n$ are mutually independent, it means that
   - firstly, for any pair of events A_j, A_k where $j \neq k$, the multiplication rule holds, and
   - secondly, for any three events A_i, A_j and A_k for distinct i, j, k, the multiplication rule holds,
   - and so on.
   - Of course, the following multiplication rule also holds:
     $\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = \Pr(A_1) \Pr(A_2) \cdots \Pr(A_n)$
   - There are in total $2^n - n - 1$ different cases.

Remarks (Continued)

2. Mutually independence implies pairwise independence, but pairwise independence does **not** imply mutually independence.

3. Suppose $A_1, A_2, \cdots, A_n$ are mutually independent events. Let

$$B_i = A_i \text{ or } A_i', \ \ i = 1, 2, \cdots, n.$$

Then $B_1, B_2, \cdots, B_n$ are also mutually independent events.

Example 1

- A fair die is tossed three times.
- What is the probability that the first toss gives an odd number, the second toss gives an even number and the third gives a value greater than 4?

<u>Solution</u>

- Let

  A = {odd number obtained in 1st toss}

  B = {even number obtained in 2nd toss}

  C = {5 or 6 obtained in 3rd toss}

Example 1 (Continued)

- $\Pr(A) = ({}_3C_1 \times {}_6C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(B) = ({}_6C_1 \times {}_3C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(C) = ({}_6C_1 \times {}_6C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/3.$
- $\Pr(A \cap B) = ({}_3C_1 \times {}_3C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/4$
  $= (1/2)(1/2) = \Pr(A)\Pr(B).$
- $\Pr(A \cap C) = ({}_3C_1 \times {}_6C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/6$
  $= (1/2)(1/3) = \Pr(A)\Pr(C).$
- $\Pr(B \cap C) = ({}_6C_1 \times {}_3C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/6$
  $= (1/2)(1/3) = \Pr(B)\Pr(C).$

Example 1 (Continued)

- $\Pr(A \cap B \cap C) = ({}_3C_1 \times {}_3C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1)$
  $= 1/12 = (1/2)(1/2)(1/3) = \Pr(A)\Pr(B)\Pr(C).$
- Since $\Pr(A \cap B) = \Pr(A)\Pr(B)$,
  $\Pr(A \cap C) = \Pr(A)\Pr(C)$,
  $\Pr(B \cap C) = \Pr(B)\Pr(C)$ and
  $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$,
  therefore events A, B and C are mutually independent.

Example 2

- Suppose that we toss 2 dice. Define

  A = {1st die shows an even number}

  B = {2nd die shows an odd number}

  C = {the 2 dice show both odd or both even numbers}
- $\Pr(A) = ({}_3C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(B) = ({}_6C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(C) = \Pr(\text{both dice are odd}) + \Pr(\text{both dice are even})$

  $= ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) + ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1)$

  $= 1/2.$

Example 2 (Continued)

- $\Pr(A \cap B) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(A)\Pr(B)$.
- $\Pr(A \cap C) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(A)\Pr(C)$.
- $\Pr(B \cap C) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(B)\Pr(C)$.

- Hence A, B and C are pairwise independent.
- However, A, B and C are not mutually independent since
  $\Pr(A \cap B \cap C) = 0$ and
  $\Pr(A)\Pr(B)\Pr(C) = (1/2)^3 = 1/8 \neq \Pr(A \cap B \cap C) = 0$.

Example 3

- Three fire engines operate independently.
- The probability that a specific engine is available when needed is 0.9.
- What is the probability that no engine is available when needed?

Solution to Example 3

- Let E_i = {i-th engine available when needed}.
- {No engine is available} $= E_1' \cap E_2' \cap E_3'$.
- Since E_1, E_2 and E_3 are independent, therefore E_1', E_2' and E_3' are also **independent.**

$$\Pr(E_i') = 1 - \Pr(E_i) = 1 - 0.9 = 0.1 \text{ for } i = 1, 2, 3$$

Hence

$$\Pr(E_1' \cap E_2' \cap E_3') = \Pr(E_1')\Pr(E_2')\Pr(E_3')$$
$$= (0.1)^3 = 0.001.$$

Example 4

- The probability that a grader will make a marking error on any particular question of a multiple-choice question exam is 0.05.
- If there are ten questions and questions are marked **independently,**
  (a) what is the probability that no errors are made?
  (b) That at least one error is made?
  (c) If there are n questions and the probability of marking an error is p rather than 0.05, give expressions for these two probabilities.

Solution to Example 4

Let E_i denote the event of making an error in marking the i-th question.

(a) Since $E_1, E_2, \cdots$, and E_{10} are independent, therefore $E_1', E_2', \cdots$, and E_{10}' are also **independent.**

$$
\begin{aligned}
&\Pr(\text{no error in any of the 10 questions}) \\
&= \Pr(E_1' \cap E_2' \cap \cdots \cap E_{10}') \\
&= \Pr(E_1')\Pr(E_2')\cdots\Pr(E_{10}') \\
&= (1-0.05)^{10} = 0.95^{10} = 0.5987.
\end{aligned}
$$

Solution to Example 4 (Continued)

(b) $\text{Pr(at least one error)}$
$$= 1 - \text{Pr(no error in 10 questions)}$$
$$= 1 - (0.95)^{10}$$
$$= 0.4013.$$

(c) For p replacing 0.05 and n replacing 10, the two probabilities are
$$(1-p)^n \text{ and } 1-(1-p)^n.$$