Reminder

1. Lab quiz next week.
2. Lab quiz duration: 50 minutes
3. Please bring along your student (matriculation) card.
4. You are allowed to bring your own rough paper for working.
5. Open book quiz.
6. Please arrive at least 10 minutes before the hour and wait **outside** the lab.
7. Remember to check your seat number and log in using the designated PC beforehand.

Lecture 21 Recap

1) Definition of a linear transformation.

2) The standard matrix and formula for a linear transformation.

3) Abstract definition of a linear transformation.

4) Two properties that linear transformations have.

5) What do we need to know to completely determine a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$.

6) Composition of linear transformations.

Lecture 22

Ranges and Kernels

Learning points for Lecture 22

Section 7.2 Ranges and Kernels

1) What is the range, $R(T)$, of a linear transformation T?

2) If $\boldsymbol{A}$ is the standard matrix for T, how is the range of T related to $\boldsymbol{A}$?

3) What is the dimension of $R(T)$? What is the rank of a linear transformation T?

Learning points for Lecture 22

Section 7.2 Ranges and Kernels

4) What is the kernel, Ker(T), of a linear transformation T?

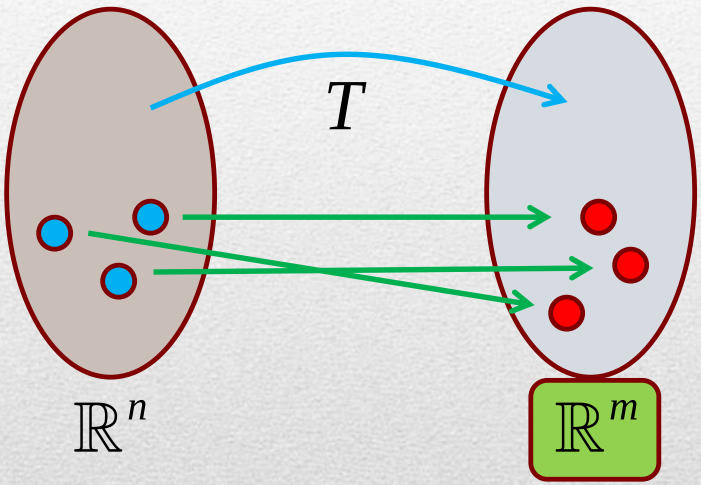
5) If $\boldsymbol{A}$ is the standard matrix for T, how is the kernel of T related to $\boldsymbol{A}$?

6) What is the dimension of Ker(T)? What is the nullity of a linear transformation T?

7) The dimension theorem for linear transformations.

Definition 7.2.1

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.



$$R(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^n\} \boxed{\subseteq \mathbb{R}^m}$$

$$R(T) = \{ \bullet \ \bullet \qquad \bullet \}$$ 'set of all images'

The range, denoted by $R(T)$ of T is the set of images of T.

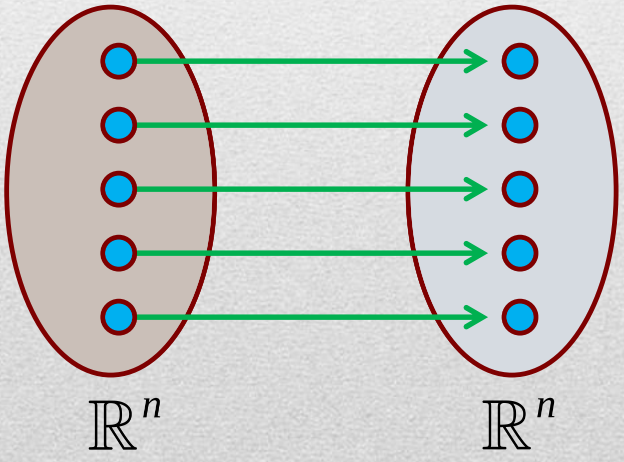
Example 7.2.2*

Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'what you put into I,

you get back the same thing'



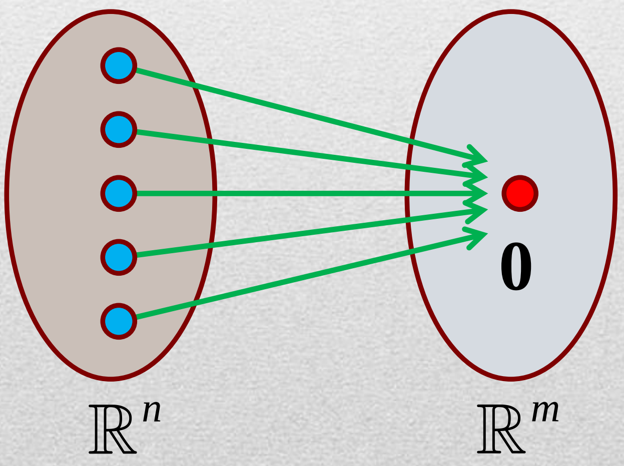
$$R(I) = \mathbb{R}^n$$

Example 7.2.2*

Let $O : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'whatever you put into O, you get back the zero vector' (in $\mathbb{R}^m$)



$$R(O) = \{\mathbf{0}\} \subseteq \mathbb{R}^m$$

Example 7.2.2

Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

$$R(T) = \left\{ \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

'all the images are of the form $\begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$ for some real numbers x, y'

Example 7.2.2

$$R(T)=\left\{\begin{pmatrix} x+y \\ y \\ x \end{pmatrix}\middle|\, x,y\in\mathbb{R}\right\} = \left\{x\begin{pmatrix}1\\0\\1\end{pmatrix}+y\begin{pmatrix}1\\1\\0\end{pmatrix}\middle|\, x,y\in\mathbb{R}\right\}$$

Equation of the plane?

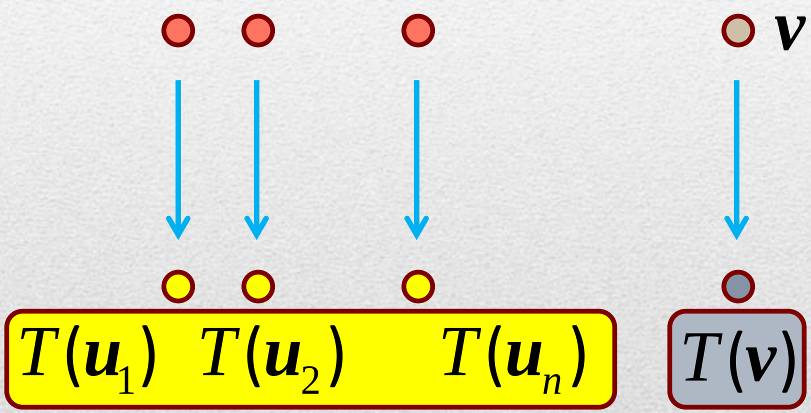
$$=\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \text{ (a plane in } \mathbb{R}^2\text{)}$$

$$ax+by+cz=0$$

$$\boxed{x-y-z=0}$$

Discussion 7.2.3

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



For any $\boldsymbol{v} \in \mathbb{R}^n$, we have already observed that $\boxed{T(\boldsymbol{v})}$ is some linear combination of $\boxed{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n).}$

So $R(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\}$

$\subseteq \text{span}\{T(\boldsymbol{u}_1), \ldots, T(\boldsymbol{u}_n)\}$

Each $T(\boldsymbol{v})$ is a linear combination of $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

Discussion

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and

$\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$. Conversely,

$$\boldsymbol{w} \in \text{span}\{T(\boldsymbol{u}_1),T(\boldsymbol{u}_2)\ldots,T(\boldsymbol{u}_n)\}$$

since $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_n$ is a basis for $\mathbb{R}^n$

$$\Rightarrow \boldsymbol{w} = a_1T(\boldsymbol{u}_1),+a_2T(\boldsymbol{u}_2)+\ldots+a_nT(\boldsymbol{u}_n)$$

$$= T(a_1\boldsymbol{u}_1+a_2\boldsymbol{u}_2+\ldots+a_n\boldsymbol{u}_n)$$

$= T(\boldsymbol{v})$ for some $\boldsymbol{v}\in\mathbb{R}^n$

$$\Rightarrow \boldsymbol{w}\in R(T)$$

$$\Rightarrow \text{span}\{T(\boldsymbol{u}_1),T(\boldsymbol{u}_2),\ldots,T(\boldsymbol{u}_n)\}\subseteq R(T)$$

So $R(T)=\{T(\boldsymbol{v})\,|\,\boldsymbol{v}\in\mathbb{R}^n\}$

$$\subseteq \text{span}\{T(\boldsymbol{u}_1),\ldots,T(\boldsymbol{u}_n)\}$$

$R(T)$

$= \text{span}\{T(\boldsymbol{u}_1),T(\boldsymbol{u}_2),\ldots,T(\boldsymbol{u}_n)\}$

What does this mean?

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.

$$\boxed{\begin{aligned}&R(T)\\ &= \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)\}\end{aligned}}$$

1) $R(T)$ is always a subspace.

2) $R(T) = \text{span}\{T(\boldsymbol{e}_1), T(\boldsymbol{e}_2) \ldots, T(\boldsymbol{e}_n)\}$

$= \text{column space of } \boldsymbol{A}$

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$

$=$ standard basis for $\mathbb{R}^n$

Remember

standard matrix for $T = \boldsymbol{A} = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}$

Theorem 7.2.4

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$R(T) = \text{column space of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^m$.

Definition 7.2.5

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$R(T) = \text{column space of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^m$.

$\dim(R(T))$ is called the $\boxed{\text{rank}}$ of T and is denoted by $\boxed{\text{rank}(T)}$.

By theorem above, $\boxed{\text{rank}(A) = \text{rank}(T)}$ where $\boldsymbol{A}$ is the standard matrix for T.

Example 7.2.6

Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

Find a basis for $R(T)$ and determine rank(T).

If $\boldsymbol{A}$ is the standard matrix for T...

Find a basis for the column space of $\boldsymbol{A}$ and determine rank($\boldsymbol{A}$).

Example 7.2.6

Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

We will find a basis for the column space of and rank of

$$= \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{matrix}} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

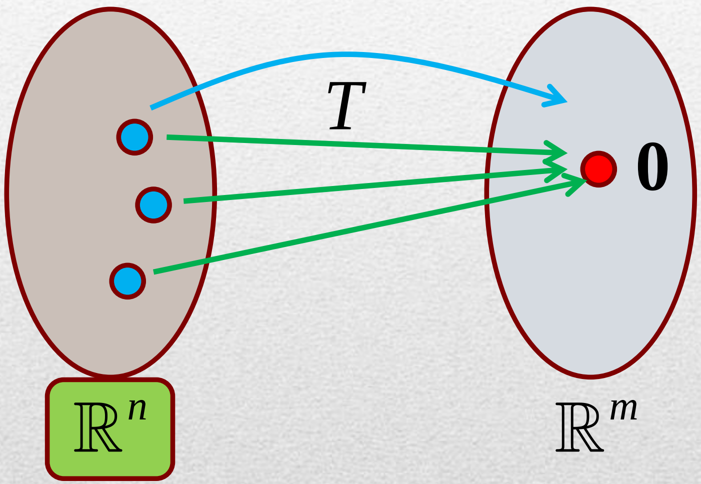
Example 7.2.6

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for $R(T)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \right\}$ and $\operatorname{rank}(T) = 2$.

Definition 7.2.7

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.



$$\text{Ker}(T) = \{\boldsymbol{u} \in \mathbb{R}^n \mid T(\boldsymbol{u}) = \mathbf{0}\}$$

$$\boxed{\subseteq \mathbb{R}^n}$$

The kernel, denoted by $\text{Ker}(T)$ of T is the set of vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$.

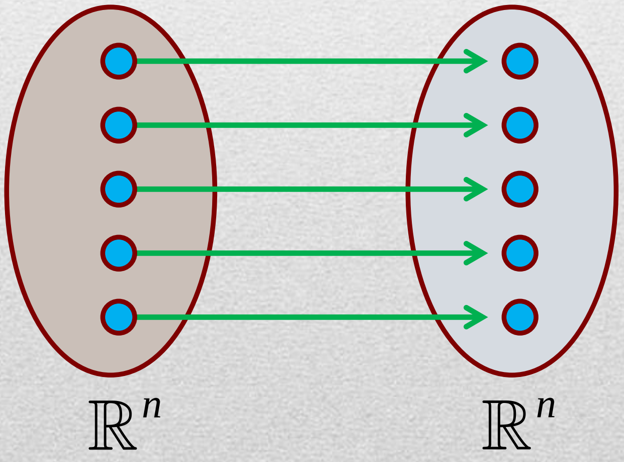
Example 7.2.8*

Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'what you put into I,

you get back the same thing'



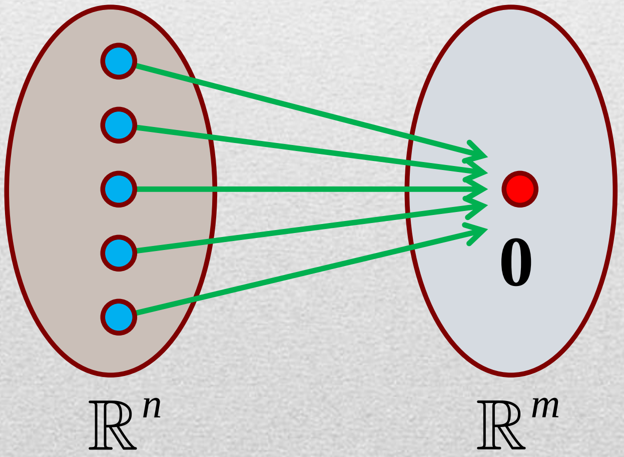
$\text{Ker}(I) = ?$

Example 7.2.8*

Let $O:\mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'whatever you put into O, you get back the zero vector' (in $\mathbb{R}^m$)



$\text{Ker}(O) = ?$

Example 7.2.8*

Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

$\text{Ker}(T) = ?$

Example 7.2.8.1

Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ be the transformation defined by

$$T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x-z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

Solve!

To find Ker(T_1), we need to

find all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Example 7.2.8.1

$$\begin{cases} 2x & - & y & & & = & 0 \\ x & - & y & + & 3z & = & 0 \\ -5x & + & y & & & = & 0 \\ x & & & - & z & = & 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x = 0, y = 0, z = 0$

So $\mathrm{Ker}(T_1)$ contains only the zero vector,

that is, $\mathrm{Ker}(T_1) = \{\mathbf{0}\}$.

Example 7.2.8.2

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

To find Ker(T_2), we need to

find all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Example 7.2.8.2

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} z-y=0 \\ x=0 \end{cases} \quad \Rightarrow \begin{cases} y=z \\ x=0 \end{cases}$$

$$\text{Ker}(T_2) = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \middle|\ y \in \mathbb{R} \right\}$$

$$\text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \left\{ y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \middle|\ y \in \mathbb{R} \right\}$$

Theorem 7.2.9

Since Ker(T) is simply the solution space of $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$ (where $\boldsymbol{A}$ is the standard matrix for T),

the following theorem is obvious.

nullspace of $\boldsymbol{A}$

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$\text{Ker}(T) = \text{nullspace of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^n$.

Definition 7.2.10

Let T be a linear transformation. The dimension of Ker(T) is called the nullity of T and is denoted by nullity(T).

Since Ker(T) = nullspace of $\boldsymbol{A}$,

By theorem 7.2.9, nullity($\boldsymbol{A}$) = nullity(T) where $\boldsymbol{A}$ is the standard matrix for T.

Example 7.2.11*

Let $I:\mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$I(\boldsymbol{u})=\boldsymbol{u}, \quad \forall \boldsymbol{u}\in\mathbb{R}^n$ Ker$(I)=\{\mathbf{0}\}$ $\Rightarrow$ nullity$(I)=0$

Let $O:\mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$O(\boldsymbol{u})=\mathbf{0}, \quad \forall \boldsymbol{u}\in\mathbb{R}^n$ Ker$(O)=\mathbb{R}^n$ $\Rightarrow$ nullity$(O)=n$

Example 7.2.11.1

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

$$\text{Ker}(T_2) = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \middle| \; y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\text{Ker}(T_2)$ and

$\text{nullity}(T_2) = 1$

Example 7.2.11.2

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

Find a basis for Ker(T) and determine its dimension.

Example 7.2.11.2

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

The standard matrix for T is $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

Let's find a basis for (and determine the dimension of) the nullspace of $\boldsymbol{A}$.

Example 7.2.11.2

Solving $\boldsymbol{Ax} = \mathbf{0}$:

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} w & = & s \\ x & = & -3t \\ y & = & t \\ z & = & t, \quad s,t \in \mathbb{R}. \end{cases}$$

So $\mathrm{Ker}(T) = \left\{ \begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$

Example 7.2.11.2

So $\operatorname{Ker}(T) = \left\{ \begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \middle|\ s,t \in \mathbb{R} \right\} = \left\{ s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \middle|\ s,t \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\operatorname{Ker}(T)$

and $\dim(\operatorname{Ker}(T)) = \operatorname{nullity}(T) = 2.$

Theorem 7.2.12

If $T:\mathbb{R}^{n}\to\mathbb{R}^{m}$ is a linear transformation then

$$\text{rank}(T)+\text{nullity}(T)=n.$$

Proof:

Let $\boldsymbol{A}$ be the standard matrix for T. So $\boldsymbol{A}$ is a $m\times n$ matrix.

By dimension theorem for matrices:

$$\text{rank}(\boldsymbol{A}) \quad + \quad \text{nullity}(\boldsymbol{A})=n$$

by earlier observation

$$\Rightarrow \text{rank}(T) \quad + \quad \text{nullity}(T)=n$$

End of Lecture 22