National University of Singapore
School of Computing
CS3244: Machine Learning
Solution to Tutorial 2

Decision Trees and Ensemble Methods

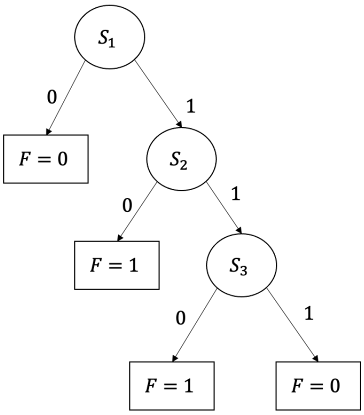
1. **Introduction to Decision Trees.**

The data in table 1 represents the three states (S_1, S_2, S_3) which contribute to the lighting of a bulb (the final state F). Each state takes value from the set $\{0, 1\}$.

S_1	S_2	S_3	F
0	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

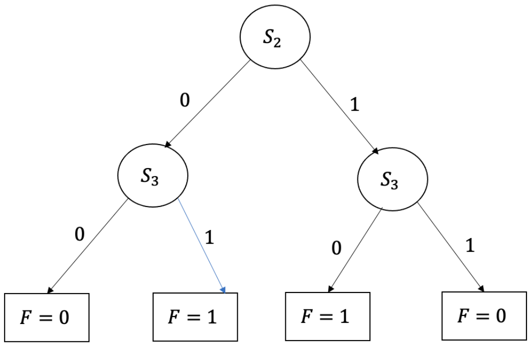
Table 1: States and the final outcome

(a) Construct a decision tree to classify the final outcome (F) from the three initial states S_1, S_2 and S_3. Follow a greedy way to construct a decision tree with feature order S_1, S_2 and S_3 which can fit the dataset with 0 training error.



(b) Comment on the tree from part (a). Is the tree optimal? If it is not optimal construct an optimal decision tree. (Here optimality is decided from the depth of a DT.)

F is just function which can be represented as $S_2 \bigoplus S_3$. Using this we can build an optimal tree with depth 2 as shown below.



(c) Alice tries to implement a XOR function which has d inputs using decision tree (DT). Why using DT is not a scalable solution? Explain your answer. If we implement AND or OR function with d inputs, do we get any advantage over the XOR function?

To implement the given XOR using DTs, we need to have 2^d leaf nodes and $2^d - 1$ internal nodes are needed. The required space grows exponentially with d. Hence, this is not a scalable solution. Moreover, to calculate the final outcome of the XOR function, every input/feature needs to be considered. Hence, pruning is also not possible.

Implementation of AND or OR function gets the advantage of pruning. For example, if a particular variable is false in AND function, then the outcome is always false. Similarly, if a particular variable is true in OR function, then the outcome is always false.

Think about how many DT internal nodes are needed to model the AND function.

2. **Bank on Decision Trees.**

The loans department of DBN (Development Bank of NUS) has the following past loan processing records each containing an applicant's income, credit history, debt, and the final approval decision. Details are shown in Table 2.

Income	Credit History	Debt	Decision
0 – 5K	Bad	Low	Reject
0 – 5K	Good	Low	Approve
0 – 5K	Unknown	High	Reject
0 – 5K	Unknown	Low	Approve
0 – 5K	Unknown	Low	Approve
0 – 5K	Unknown	Low	Reject
5 – 10K	Bad	High	Reject
5 – 10K	Good	High	Approve
5 – 10K	Unknown	High	Approve
5 – 10K	Unknown	Low	Approve
Over 10K	Bad	Low	Reject
Over 10K	Good	Low	Approve

Table 2: Loan processing records

(a) Construct a decision tree based on the above training examples. (Note: $\log_2 \frac{x}{y} = \log_2$ x - $\log_2$ y, $\log_2 1 = 0$, $\log_2 2 = 1$, $\log_2 3 = 1.585$, $\log_2 4 = 2$, $\log_2 5 = 2.322$, $\log_2 6 = 2.585$, $\log_2 7 = 2.807$, $\log_2 8 = 3$, $\log_2 9 = 3.170$, $\log_2 10 = 3.322$, $\log_2 11 = 3.459$, and $\log_2 12 = 3.585$)

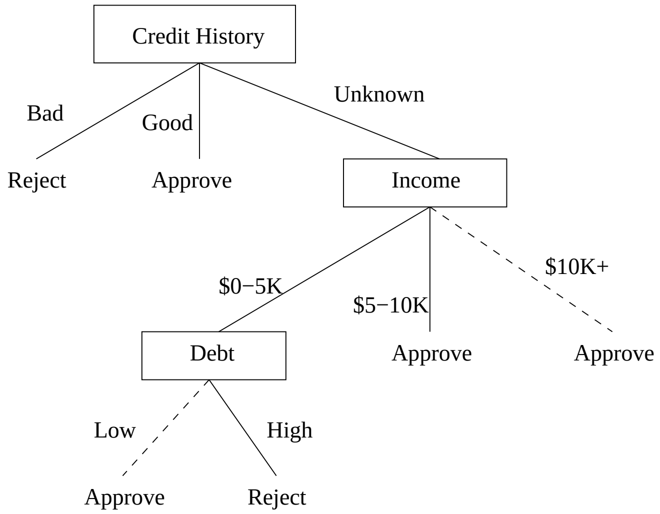
$$\begin{aligned}
IG(Income) &= H(\frac{5}{12}, \frac{7}{12}) - remainder(Income) \\
remainder(Income) &= \frac{6}{12}(-\frac{3}{6}\log_2\frac{3}{6} - \frac{3}{6}\log_2\frac{3}{6}) + \frac{4}{12}(-\frac{1}{4}\log_2\frac{1}{4} \\
&\quad -\frac{3}{4}\log_2\frac{3}{4}) + \frac{2}{12}(-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}) \\
&= 0.937 \\
IG(Income) &= 0.98 - 0.937 = 0.043 \\
remainder(CreditHistory) &= \frac{3}{12}(-\frac{3}{3}\log_2\frac{3}{3} - \frac{0}{3}\log_2\frac{0}{3}) + \frac{3}{12}(-\frac{3}{3}\log_2\frac{3}{3} \\
&\quad -\frac{0}{3}\log_2\frac{0}{3}) + \frac{6}{12}(-\frac{2}{6}\log_2\frac{2}{6} - \frac{4}{6}\log_2\frac{4}{6}) \\
&= 0.459 \\
IG(CreditHistory) &= 0.98 - 0.459 = 0.521 \\
remainder(Debt) &= \frac{8}{12}(-\frac{3}{8}\log_2\frac{3}{8} - \frac{5}{8}\log_2\frac{5}{8}) + \frac{4}{12}(-\frac{2}{4}\log_2\frac{2}{4} - \frac{2}{4}\log_2\frac{2}{4}) \\
&= 0.970 \\
IG(Debt) &= 0.98 - 0.970 = 0.01
\end{aligned}$$

Since Credit History has the highest gain, choose it as the root, which has three values, i.e., "Bad", "Good", and "Unknown". Since all examples for "Bad" have the same classification (i.e., "Reject") and all examples for "Good" have the same classification (i.e., "Approve"), both nodes have no further subtree. For "Unknown", a subtree for the following subset of examples is to be constructed:

Income	*Debt*	*Decision*
$0 - 5K$	*High*	*Reject*
$0 - 5K$	*Low*	*Approve*
$0 - 5K$	*Low*	*Approve*
$0 - 5K$	*Low*	*Reject*
$5 - 10K$	*High*	*Approve*
$5 - 10K$	*Low*	*Approve*

$$\begin{aligned}
H(\frac{2}{6}, \frac{4}{6}) &= -\frac{2}{6}\log_2\frac{2}{6} - \frac{4}{6}\log_2\frac{4}{6} = 0.918 \\
remainder(Income) &= \frac{4}{6}(-\frac{2}{4}\log_2\frac{2}{4} - \frac{2}{4}\log_2\frac{2}{4}) + \frac{2}{6}(-\frac{2}{2}\log_2\frac{2}{2} - \frac{0}{2}\log_2\frac{0}{2}) = 0.667 \\
gain(Income) &= 0.918 - 0.667 = 0.251 \\
remainder(Debt) &= \frac{2}{6}(-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}) + \frac{4}{6}(-\frac{1}{4}\log_2\frac{1}{4} - \frac{3}{4}\log_2\frac{3}{4}) = 0.874 \\
gain(Debt) &= 0.918 - 0.874 = 0.044
\end{aligned}$$

Since Income has a higher gain than Debt, Income is chosen as the root of the subtree under Credit History=Unknown.



The two decisions on dotted lines are those that are not quite so straight forward. In one case, there is insufficient data for us to decide what to do, so we do something reasonable. In the other case, the resulting sample is not decisive. There are two approves and one reject. We can use a simple majority to decide.

Question is: why do we have such a node where there seems to be ambiguity? There is no clear answer (or at least we don't have the information to determine). One possibility is that we are missing some attribute from the data that will allow us to differentiate at this last node. Another possibility is that there is some non-determinism in the underlying function that we are trying to approximate with this decision tree.

(b) Construct 3 different DTs, where each of the three DTs is fully grown from two of the three attributes, again based on the same set of examples: {Income, Credit History}, {Credit History, Debt} and {Debt, Income}.

Based on the earlier answers for the decision tree, we know that Credit History will be the root node as it has a better information gain. This leads to pure "Bad" and "Good" leaves, leading to only using the secondary attribute for the remaining six cases of "Unknown".

The {Income, Credit History} tree is identical to the one for the full decision tree, except that the bottom Debt node is left off and replaced by the mode of the examples that fulfill the tests (Credit History == "Unknown" $\cap$ Income == "0 – 5K"). This is a tie, so the parent node's test of just Credit History == "Unknown" is used to obtain "Approve".

For the {Credit History, Debt} tree, the Credit History == "Unknown" takes a Debt test, resulting "Approve" for "Low" values (by the mode of examples), and also "Approve" for "High" values (by the mode of the parent, since the mode of the examples is tied).

For the {Debt, Income} tree, we can recall that the information gain of the previous decision tree showed that Income was better than Debt. All three subcases of Income are not pure, and each needs a test for Debt. In the “0 – 5K” Income, Debt == “Low” yields “Approve” by the mode of the examples, and yields “Reject” for “High” Debt cases. In the “5 – 10K” Income, Debt == “High” yields “Approve” by the mode of the examples, and yields “Approve” by purity when Debt is “Low”. In the “Over 10K” Income, there are no examples of “High”, so it defaults to “Approve”. Similarly, since there is a tied outcome for Debt == “Low”, we default to the mode of all examples, hence “Approve”.

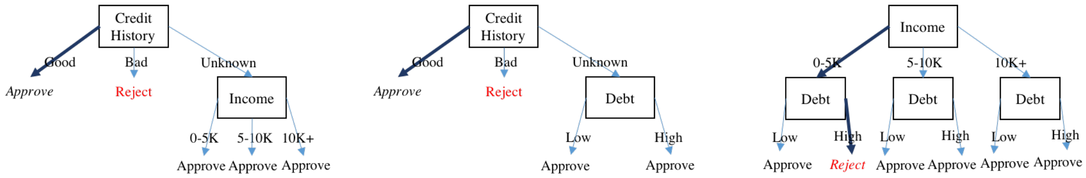


Figure 1: 3 DTs in Part (b). The dark edges and italicized labels correspond to the decision for each tree for the individual described in Part (c).

(c) What is the DT classifier’s (part (a)) decision for a person who has 4K yearly income, a good credit history and a high amount of debt? Is your result different if we use 3 DTs in part (b) to make a decision?

For the individual, the DT from part (a) approves the application from the root node of the tree.

As done in Part (b), the first two trees vote “Approve”, but the final {Income, Debt} tree votes “Reject” (as it satisfies the Income ==“0 – 5K” → Debt == “High” → “Reject” path. If we employ uniform voting from all 3 DTs, so the $\frac{2}{3}$rds vote wins to “Approve” the applicant too.

(Optional) How could the decisions (possibly different) given by the 3 DTs be collated together?

3. **Scaling the Decision Trees.**

   “The management of a company that I shall call Stygian Chemical Industries, Ltd., must decide **whether** to build a small plant or a large one to manufacture a new product with an expected market life of ten years. The decision hinges on what size the market for the product will be.

   If the company builds a big plant, it must live with it whatever the size of market demand. If it builds a small plant, management has the option of expanding the plant in two years in the event that demand is high during the introductory period.... These decisions are growing more important at the same time that they are increasing in complexity....

   In this article I shall present one recently developed concept called the “decision tree”, which has tremendous potential as a decision-making tool....”

Decision Trees for Decision Making
John F. Magee
Harvard Business Review
July 1964

The Decision Tree, developed in 1963, quickly became an exciting tool for businesses. Over the years, it has been improved to alleviate its few, yet noticeable, shortcomings.

Discuss the disadvantages (and their possible resolutions) of a Decision Tree based on past loan processing records of Development Bank of NUS (*Question 2*) in the following context:

(a) Income and Debt are dependent on each other.

The term Debt is relative to the Income of the person. For instance, Person A with an income of 5K has a debt of 4K and Person B with an income of 15K also has a debt of 4K. For the same amount of debt, Person A's debt is considered **High** *whereas Person B's debt is considered* **Low**. *In this categorical definition of Debt, the quantifiable information is missing making its explainability ambiguous.* (**Explainability in AI** *will be discussed in future lectures.*)

(b) Due to a storage fault, four of the twelve rows have one or more *missing* cells in its attributes.

Traditionally, Decision Tree does not behave well with missing values. Replacing with most common attribute value skews the dataset. Altogether dropping the four affected rows reduces the size of the training set. A possible way of addressing it is by assuming **missing** (or a dummy variable) *to be another value of that attribute, which might further have some grave repercussions.*

(c) Recent additions were made to the loan processing records where all of the loans were rejected by DBN due to the bad economy.

Decision Tree does not take the temporal factor into account. These added rejected loans rows will significantly tamper the decisions even when the economy improves. A possible way of addressing it is by removing these rows after a certain time. But this might result in information loss of those cases which would have been rejected even when the economy was better.

4. [[1]**] **Uniform Blending (UB).**

One of the simplest ensemble methods is UB. Given a set of hypothesis: $h_1, h_2, h_3, ..., h_T$, UB makes predictions simply by mixing the predictions given by $h_1, h_2, h_3, ..., h_T$ uniformly. Concretely, for binary classification, UB predicts by:

$$H(x) = \text{sign}(\sum_{t=1}^{T} h_t(x)), \tag{1}$$

and for regression, UB predicts by:

$$H(x) = \frac{1}{T}\sum_{t=1}^{T} h_t(x) \tag{2}$$

[1]** question is harder than other questions in this tutorial.

Taking regression as an example, show that the performance (measured by out-of-sample error) of UB is no worse than the average performance over $h_1, h_2, h_3, ..., h_T$; i.e.:

$$\frac{1}{T}\sum_{t=1}^{T} L_{\text{test}}(h_t(x)) \geq L_{\text{test}}(H(x)) \tag{3}$$

Assume we evaluate the testing error by mean square error.

Hint: Start by calculating the average error over $h_1, h_2, h_3, ..., h_T$ for one fixed data point x.

Proving Equation 3 can give us an intuition on why ensembling can help to reduce the out-of-sample error.

Let's first see the case for only one fixed data point x. Suppose the ground truth function is $f(x)$. The average error over $h_1, h_2, h_3, ..., h_T$ for point x is

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}[h_t(x)-f(x)]^2 &= \frac{1}{T}\sum_{t=1}^{T}\left[h_t(x)^2 - 2h_t(x)f(x) + f(x)^2\right] && (4)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - \frac{2f(x)}{T}\sum_{t=1}^{T}h_t(x) + \frac{1}{T}\sum_{t=1}^{T}f(x)^2 && (5)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - 2f(x)H(x) + \frac{1}{T}\sum_{t=1}^{T}f(x)^2 && H(x)=\tfrac{1}{T}\textstyle\sum_{t=1}^{T}h_t(x) \quad (6)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - 2f(x)H(x) + f(x)^2 && f(x) \text{ does not depend on } t \quad (7)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - H(x)^2 + H(x)^2 - 2f(x)H(x) + f(x)^2 && (8)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - H(x)^2 + [H(x)-f(x)]^2 && (9)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - 2H(x)^2 + H(x)^2 + [H(x)-f(x)]^2 && (10)\\
&= \frac{1}{T}\sum_{t=1}^{T}h_t(x)^2 - \frac{1}{T}\sum_{t=1}^{T}2h_t(x)H(x) + \frac{1}{T}\sum_{t=1}^{T}H(x)^2 + [H(x)-f(x)]^2 && (11)\\
&= \frac{1}{T}\sum_{t=1}^{T}\left[h_t(x)^2 - 2h_t(x)H(x) + H(x)^2\right] + [H(x)-f(x)]^2 && (12)\\
&= \frac{1}{T}\sum_{t=1}^{T}[h_t(x)-H(x)]^2 + [H(x)-f(x)]^2 && (13)
\end{aligned}$$

To compute the out-of-sample error, we only need to compute the expected error over all x, *hence, the average out-of-sample error over* $h_1, h_2, h_3, ..., h_T$ *is*

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} L_{test}(h_t(x)) &= \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_x\left\{[h_t(x) - f(x)]^2\right\} && (14)\\
&= \mathbb{E}_x\left\{\frac{1}{T}\sum_{t=1}^{T} [h_t(x) - f(x)]^2\right\} && (15)\\
&= \mathbb{E}_x\left\{\frac{1}{T}\sum_{t=1}^{T} [h_t(x) - H(x)]^2 + [H(x) - f(x)]^2\right\} && (16)\\
&= \mathbb{E}_x\left\{\frac{1}{T}\sum_{t=1}^{T} [h_t(x) - H(x)]^2\right\} + \mathbb{E}_x\left\{[H(x) - f(x)]^2\right\} && (17)\\
&= \mathbb{E}_x\left\{\frac{1}{T}\sum_{t=1}^{T} [h_t(x) - H(x)]^2\right\} + L_{test}(H(x)) && (18)\\
&\geq L_{test}(H(x)) && (19)
\end{aligned}$$

From the above deduction, we can see, the average out-of-sample error over $h_1, h_2, h_3, ..., h_T$ *is greater than or equal to the out-of-sample error of* $H(x)$*. Hence, the performance (measured by out-of-sample error) of UB is no worse than the average performance over* $h_1, h_2, h_3, ..., h_T$*.*