CHAPTER 5: COUNTING

SECTION 5.1 BASICS

PRODUCT RULE

Suppose an operation can be broken down into a sequence of 2 steps. If the first step can be done in r ways and the second step can be done in s ways (regardless of how the first step was done), then the entire operation can be performed in rs ways.

EXAMPLE

• The chairs of an auditorium are to be labelled with a letter and a positive integer ≤ 100. What is the largest number of seats that can be labelled differently?

SOLN: The letters can be chosen in 26 ways and the numbers can be chosen in 100 ways. Thus the answer is $26 \cdot 100 = 2600$.

• How many bit strings of length 7 are there?

SOLN: Each bit (represented by X) can be chosen in 2 ways.

$$XXXXXXX$$

Thus the answer is $2^7 = 128$.

• How many functions from A to B are there if $|A| = m$ and $|B| = n$?

SOLN: Each member of A has a choice of n images. Thus the answer is n^m.

• How many 1-1 functions from A to B are there if $|A| = m$ and $|B| = n$?

SOLN: The answer is 0 if $n < m$. If $n \geq m$, then the first element of A has a choice of n images, the second has a choice of $n-1$ images, ..., the n^{th} element of A has a choice of $n-m+1$ images. Thus the answer is $n(n-1)\cdots(n-m+1)$. (Thus if $n = m$, the answer is $n!$.)

• (**NUMBER OF SUBSETS**) Prove that the number of subsets of a finite set S is $2^{|S|}$.

SOLN: Recall that subsets of S are in 1-1 correspondence with bit strings of length $|S|$. Thus the answer is $2^{|S|}$.

• How many elements are there in $\{1, 2\} \times \{1, 2, 3\}$?

SOLN: This is solved by asking how many ways there are to form the ordered pair (x, y), $x \in \{1, 2\}$, $y \in \{1, 2, 3\}$.

There are 2 ways to write x and 3 ways to write y. Thus there are 6 ways to write the ordered pair (x, y).

• (**NUMBER OF ORDERED PAIRS**) For any finite sets A, B, $|A \times B| = |A||B|$.

More generally if each A_i is a finite set, then

$$|A_1 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

THE SUM RULE

If the number of objects with property 1 is m and the number of objects with property 2 is n and there are no objects with both property 1 and 2, the total number of objects is $m + n$.

Formulation in terms of sets:

Let A and B be finite set. Then

$$|A \cup B| = |A| + |B| \qquad \text{if } A \cap B = \emptyset.$$

THE DIFFERENCE RULE

Suppose there are m objects with properties 1 or 2 and there are n objects with property 2. Then the number of objects with property 1 but not property 2 is $m - n$.

Alternatively, it can be formulated in terms of sets as follows:

Let A be a finite set. Then

$$|A - B| = |A| - |B| \qquad \text{if } B \subseteq A.$$

EXAMPLE

- How many n-letter passwords are there where $1 \leq n \leq 3$?

SOLN: When $n = 1$, there are 26 passwords. When $n = 2$, there are $26^2 = 676$ passwords. When $n = 3$, there are $26^3 = 17576$ passwords. By the sum rule, the total number of passwords is

$$26 + 676 + 17576 = 18278.$$

• Each character of a PIN is either an upper case letter or a digit.

(1) Find the number of 4-character PIN with repeated letter or digit.

(2) Find the number of 6-, 7- or 8-character PIN with at least 1 digit.

SOLN: (1) There are $(26+10)^4 = 1,679,616$ PINs.

Among them, $36 \times 35 \times 34 \times 33 = 1,413,720$ are PINs with distinct characters.

Thus the number PINs with one or more repeated characters is

$$1,679,616 - 1,413,720 = 265,896.$$

(2) Let P_i be the number of the required PIN of length i. The number of PIN with no restrictions is 36^i while the number of PIN without any digit is 26^i. Thus $P_i = 36^i - 26^i$. Hence the answer is

$$P_6 + P_7 + P_8 = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8) = 2,612,282,842,880.$$

• How many integers from 100 to 999 inclusive are divisible by 5?

SOLN: The number of multiples of 5 in $[1,999]$ is $\lfloor 999/5 \rfloor = 199$.

The number of multiples of 5 in $[1,99]$ is $\lfloor 99/5 \rfloor = 19$.

So the answer is $199 - 19 = 180$.

ALT SOLN: Observe that an integer is divisible by 5 if its unit digit is either 0 or 5.

There are $9 \times 10 = 90$ numbers of the form $xy0$.

There are $9 \times 10 = 90$ numbers of the form $xy5$.

Therefore there are 180 numbers that are divisible by 5.

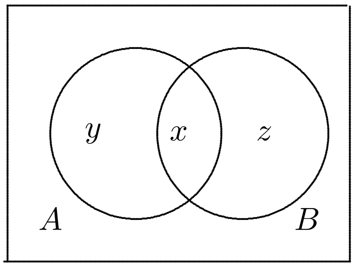
The sum rule is not applicable if there are objects with both property 1 and 2. We need to subtract off the over count.

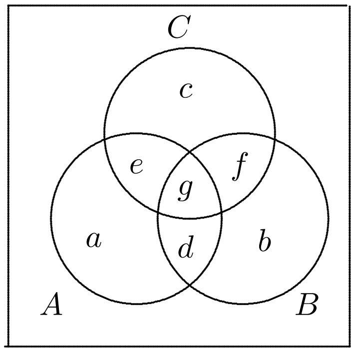
THE INCLUSION/EXCLUSION RULE: 2, 3 SETS

Let A, B, C be finite sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

PROOF: (2 sets): Referring to the Venn diagram below, we have $|A \cup B| = x + y + z$, $|A| = x + y$, $|B| = x + z$ and $|A \cap B| = x$. The result thus follows.





(3 sets): The result follows from

$$|A \cup B \cup C| = a + b + c + d + e + f + g$$
$$|A| = a + d + e + g, \quad |B| = b + d + f + g, \quad |C| = c + e + f + g$$
$$|A \cap B| = d + g, \quad |B \cap C| = f + g, \quad |A \cap C| = e + g$$
$$|A \cap B \cap C| = g.$$

The result then follows.

EXAMPLE

- How many integers from 1 to 1,000 inclusive are multiples of 3 or 5?

SOLN: Let A_i be the set of multiples of i. The required integers are members of $A_3 \cup A_5$. Now

$$|A_3| = \lfloor 1000/3 \rfloor = 333$$
$$|A_5| = \lfloor 1000/5 \rfloor = 200$$
$$|A_3 \cap A_5| = |A_{15}| = \lfloor 1000/15 \rfloor = 66$$
$$|A_3 \cup A_5| = 333 + 200 - 66 = 467$$

- How many bit strings of length 8 either start with a 1 or end with two 0?

SOLN: Let A be the set of bit strings of length 8 that start with a 1, B be the set of bit strings of length 8 that end with two 0. Then $A \cap B$ is the set of bit strings of length 8 that start with a 1 and end with two 0. We want the number of bit strings in $A \cup B$. We have $|A| = 2^7$, $|B| = 2^6$ and $|A \cap B| = 2^5$. Thus $|A \cup B| = 2^7 + 2^6 - 2^5 = 160$.

SECTION 5.3 PERMUTATIONS & COMBINATIONS

DEFINITION:

A **PERMUTATION** of a set of distinct objects is an ordering of the objects.

More generally, an r-**PERMUTATION** of a set of n distinct objects is an ordering of r elements from the set.

The number of r-permutations of a set of n elements is denoted $P(n, r)$.

Thus a permutation is just an n-permutation.

EXAMPLE

- There are 6 permutations of $\{1, 2, 3\}$:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

- There are 6 2-permutations of $\{1, 2, 3\}$:

$$12, 13, 21, 23, 31, 32$$

THEOREM:

The number of permutations of n distinct objects is $n!$.

PROOF: There are n ways to select the first object.

There are $n - 1$ ways to select the second object, regardless of how the first object was chosen. In general there are $n - k$ ways to select the $(k + 1)$th object.

By the product rule, there are all together $n(n - 1) \cdots 1 = n!$ permutations.

Similarly, we have

THEOREM: $P(n, r) = n(n - 1) \ldots (n - r + 1) = \frac{n!}{(n-r)!}$.

EXAMPLE

- There are $5 \times 4 \times 3 = 60$ ways of arranging 3 of the 5 letters of BYTES.
- There are $4 \times 3 = 12$ ways of arranging 3 of the 5 letters of BYTES and the first letter must be B.
- How many permutations of the letters $ABCDEFGH$ contain the string ABC?

SOLN: Treat ABC as a single block. Then such a permutation is a permutation of the block ABC and the 5 letters $DEFGH$. Thus the answer is $6! = 720$.

COMBINATIONS

Here we deal with questions such as "How many committees of 3 can be formed from a group of 4 students?" The answer can be found by listing all committees. If the students are A, B, C, D, then the committees are:

$$ABC, ABD, ACD, BCD$$

and the answer is therefore 4.

DEFINITION:

Let n, r be integers with $0 \leq r \leq n$. An r-**COMBINATION** of a set of n (distinct) objects is a subset of r objects.

The notation $\binom{n}{r}$, read as n choose r, denotes the number of r-combinations of a set of n objects.

THEOREM:

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

PROOF: The r-permutations of a set of n elements can be obtained from a 2-step process.

1) Find an r-combination.

2) Permute all the r members of a r-combination to obtain its r-permutations.

Thus

$$P(n,r) = \binom{n}{r} \times r!$$

or

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

THEOREM: $\binom{n}{r} = \binom{n}{n-r}.$

PROOF: The result follows from the fact that they are both $\frac{n!}{r!(n-r)!}$.

EXAMPLE

- How many 8-bit strings have exactly 3 1s?

SOLN: Such a string can be formed by choosing the 3 positions for the 1's. So the answer is $\binom{8}{3} = 56$.

EXAMPLE

You are given a group of 12 persons and want to form 5-person teams.

- The number of 5-person teams is

$$\binom{12}{5} = \frac{12!}{7!5!} = 792.$$

- What is the answer if A and B must be together?

SOLN: Either they are both out of the team or they are both in the team. Thus the answer is

$$\binom{10}{5} + \binom{10}{3} = 252 + 120 = 372.$$

- What is the answer if A and B cannot be together in the team?

SOLN: The complement is when they are both in the team. So the answer is

$$792 - 120 = 672.$$

EXAMPLE

There are 5 men and 7 women.

- The number of 5-person teams consisting of 3 men and 2 women is

$$\binom{5}{3} \times \binom{7}{2} = 210$$

- The number of 5-person teams including at least one man is, by the difference rule:

$$\binom{12}{5} - \binom{7}{5} = 771$$

since the number of all women teams is $\binom{7}{5}$.

- The number of 5-person teams including at most one man is, by the addition rule:

$$\binom{5}{1}\binom{7}{4} + \binom{5}{0}\binom{7}{5} = 196.$$

SECTION 5.4 BINOMIAL COEFFICIENTS

Consider the expansion:

$$
\begin{aligned}
(a+b)^4 &= (a+b)(a+b)(a+b)(a+b) = (a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4) \\
&= (a_1a_2a_3a_4) + (a_1a_2a_3b_4 + a_1a_2b_3a_4 + a_1b_2a_3a_4 + b_1a_2a_3a_4) \\
&\quad + (a_1a_2b_3b_4 + a_1b_2a_3b_4 + a_1b_2b_3a_4 + b_1a_2a_3b_4 + b_1a_2b_3a_4 + b_1b_2a_3a_4) \\
&\quad + (a_1b_2b_3b_4 + b_1a_2b_3b_4 + b_1b_2a_3b_4 + b_1b_2b_3a_4) + (b_1b_2b_3b_4) \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}
$$

(The subscripts are included to track where the terms come from.) We see that each term comprises one variable from each of 4 factors in $(a+b)^4$. Thus the coefficient of a^3b is 4 because there are $\binom{4}{3} = 4$ ways to choose 3 a's from the 4 factors and the coefficient of a^2b^2 is 6 because there are $\binom{4}{2} = 6$ ways to choose 2 a's from the 4 factors.

THE BINOMIAL THEOREM

For any $n \in \mathbb{Z}^+$,

$$
\begin{aligned}
(x+y)^n &= x^n + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1y^{n-1} + y^n \\
&= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i = \sum_{i=0}^{n}\binom{n}{n-i}x^{n-i}y^i.
\end{aligned}
$$

PROOF: In the expansion of $(x+y)^n$, the term $x^{n-k}y^k$ occurs $\binom{n}{k}$ times because the k y's can come from any k of the n factors in $(x+y)^n$. Thus its coefficient is $\binom{n}{k}$.

EXAMPLE

- The expansion of $(x+y)^4$ is

$$
\begin{aligned}
(x+y)^4 &= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{}x^1y^3 + \binom{4}{4}x^0y^4 \\
&= x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + y^4
\end{aligned}
$$

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

SOLN: Note that $(2x-3y)^{25} = \big((2x)+(-3y)\big)^{25}$. Thus the term in $x^{12}y^{13}$ is $\binom{25}{12}(2x)^{12}(-3y)^{13}$. So its coefficient is

$$
\binom{25}{12}(2)^{12}(-3)^{13} = -\frac{25!2^{12}3^{13}}{13!12!}.
$$

THEOREM: Let $n \in \mathbb{Z}^*$. Then

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

PROOF: Put $x = y = 1$ in the binomial theorem, we get the result.

Remark The number of subsets of a set A with n elements can be enumerated as follows. It has $\binom{n}{0}$ subsets with no element, $\binom{n}{1}$ subsets with one element, ..., $\binom{n}{n}$ subsets with n elements. Thus

$$|P(A)| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

Thus the above theorem gives yet another proof that $|P(A)| = 2^{|A|}$.

THEOREM: Let $n \in \mathbb{Z}^*$. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$

PROOF: Put $x = 1$, $y = -1$ in the binomial theorem, the result follows.

The result above implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Combinatorial, this says that the number of subsets with an odd number of elements is equal to the number of subsets with an even number of elements.

THEOREM: Let $n \in \mathbb{Z}^*$. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

PROOF: Put $x = 1$, $y = 2$ in the binomial theorem, the result follows.

PASCAL'S TRIANGLE

THEOREM:

PASCAL'S IDENTITY: Let $n, r \in \mathbb{Z}^+$ with $n \geq r$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

PROOF: Consider $A = \{0, 1, \ldots, n\}$. The number of r-element subsets of A is $\binom{n+1}{r}$. On the other hand, the number of r-element subsets that contain the number 0 is $\binom{n}{r-1}$ and the number of r-element subsets that do not contain the number 0 is $\binom{n}{r}$. Thus

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Note: The above can also be proved by using the formula for $\binom{n}{r}$.

Using this identity, we can construct the **PASCAL'S TRIANGLE**, where each entry is the sum of the two entries above. Note that in the figure on the left, in each small triangle, the bottom is the sum of the two at the top by Pascal's identity. So the Pascal's triangle can be constructed starting with the figure on the right. Then by using the Pascal's identity, each subsequent row can be constructed. The final result is shown in the third figure.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}
\end{array}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1
\end{array}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \\
1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1
\end{array}$$

SECTION 5.5 GENERALIZED PERMUTATIONS & COMBINATIONS

PERMUTATIONS WITH REPETITIONS

EXAMPLE

How many words of length r can be formed using the 26 letters of the alphabets?

SOLN: Each of the word cab be chosen in 26 ways. Thus by the product rule, the answer is 26^r.

THEOREM:

The number of r-permutations of a set of n elements with repetition allowed is n^r.

PERMUTATIONS WITH INDISTINGUISHABLE OBJECTS

EXAMPLE

How many ways are there to arrange (permute) all the letters of MISSISSIPPI?

SOLN: Note that the answer is **not** 11! because the 11 letters are not distinct.

Form the permutation as follows:

1) Choose 4 positions for the S's.

2) Choose 4 positions for the I's.

3) Choose 2 positions for the P's.

4) Choose 1 positions for the M.

Ans: $\binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = 34650$.

Alternative soln: Another approach is to treat all letters as different. Then there are 11! permutations. The 4 S can permute among themselves in 4! ways. This contributes to an over count by a factor of 4!. Similarly, the 4 I and 2 P contribute to a duplication by a factor of 4!2!. Thus the answer is $\frac{11!}{4!4!2!} = 34650$.

THEOREM:

Suppose a collection consists of n objects of which n_i are of type-i, $i = 1, 2, \ldots, k$ where $n = n_1 + \cdots + n_k$. The number of ways of permuting the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\ldots n_k!}.$$

Remark: The first formula is obtained by following the first approach while the second is obtained by the second approach.

COMBINATIONS WITH REPETITION

EXAMPLE

How many non-negative integral solutions are there for

$$x_1 + x_2 + x_3 = 5?$$

Discussion: $(x_1, x_2, x_3) = (1, 1, 3)$ is a solution. It can be represented as

$$* + * + * * *$$

On the other hand, $* * + + * * *$ gives rise to the solution $(x_1, x_2, x_3) = (2, 0, 3)$.

SOLN: Each solution can be represented as a permutation of 5 copies of $*$ and 2 copies of $+$.

The answer is $\frac{7!}{5!2!} = 21$.

Number of integer solutions

The number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r$$

is

$$\frac{(r + n - 1)!}{r!(n - 1)!}$$

EXAMPLE

- How many positive integral solutions are there for $x_1 + x_2 + x_3 = 6$?

SOLN: The solutions corresponds to permutations of 6 copies of $*$ and 2 copies of $+$ with the condition that there are no consecutive copies of $+$. For example $* * * + + * * **$ does not correspond to a solution as $x_2 = 0$. The required permutations can be constructed by placing the 6 copies of $*$ in a row

$$* \quad * \quad * \quad * \quad * \quad *$$

and then inserting the 2 copies of $+$ into the 5 spaces between consecutive copies off $*$. The answer is $\binom{5}{2} = \frac{5!}{3!2!}$.

Alternatively, we can let $x_i = y_i + 1$. Then $x_i \geq 1 \Leftrightarrow y_i \geq 0$. Thus the equation becomes $y_1 + y_2 + y_3 = 3$ and we seek the nonnegative integral solutions. The answer is now $\frac{5!}{3!2!}$.

• How many ways are there to select 4 pieces of fruits from a basket containing apples, oranges and pears if the order in which the fruits are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least 4 pieces of each type?

SOLN: The answer is given by the number of nonnegative integral solutions of $x_1 + x_2 + x_3 = 4$ which is $\frac{6!}{4!2!}$.

THEOREM:

The number of r-combinations with repetition allowed that can be selected from a set of n objects is

$$\frac{(r+n-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

PROOF: The number of such r-combinations is the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r$$

which is

$$\frac{(r+n-1)!}{r!(n-1)!}$$

EXAMPLE

• The number of ways to select 15 cans of Soft Drinks of Five Different Types is

$$\frac{(15+5-1)!}{15!(5-1)!} = 3876.$$

• The number of triples (i, j, k), with $1 \leq i \leq j \leq k \leq n$, is equal to the number of selecting 3 integers from the set $\{1, 2, \ldots, n\}$ with repetition allowed. Thus the number is

$$\frac{(3+n-1)!}{3!(n-1)!} = \frac{(n+2)(n+1)(n)}{3!}$$

SECTION 5.5 Discrete Probability

To say that a process is random means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

DEFINITION: A **SAMPLE SPACE** is the set of all possible outcomes of a random process or experiment.

An **EVENT** is a subset of a sample space.

EXAMPLE

• Tossing two coins. Sample Space is {(Head, Head), (Head, Tail), (Tail, Head), (Tail, Tail)}. "Tail occurs exactly one time" is an event = {(Head, Tail), (Tail, Head)}

DEFINITION:

If S is a finite sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the **PROBABILITY** of E is

$$p(E) = \frac{|E|}{|S|}$$

EXAMPLE

• Tossing two coins. Let E be "Tail occurs exactly one time". The probability of E is

$$\frac{2}{4} = \frac{1}{2}.$$

• If letters of the word COMPUTER are randomly arranged in a row, what is the probability that the letters CO remain next to each other (in order) as a unit?

SOLN:

$$\frac{7!}{8!}$$