ST2334 (2018/2019 Semester 2) Solutions to Questions in Tutorial 5

Question 1

(a) $E(X) = \sum x\, f_X(x) = 4.11$. $E(X^2) = \sum x^2\, f_X(x) = 17.63$

(b) $V(X) = \sum (x-\mu)^2 f_X(x) = 0.7379$.

Alternatively, $V(X) = E(X^2) - [E(X)]^2 = 0.7379$.

(c) $E(Z) = 3E(X) - 2 = 10.33$. $V(X) = 3^2 V(X) = 6.6411$

(d)

X	2	3	4	5	6
Z	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

$E(Z) = \sum z f_Z(z) = 10.33$. $V(Z) = \sum (z-\mu)^2\, f_Z(z) = 6.6411$

(e) $W = aZ + b$. $E(W) = aE(Z) + b = 10.33a + b$. $V(W) = a^2 V(Z) = 6.6411a^2$

Question 2

X	0	1	2	3	4	5
$f_X(x)$	1/15	2/15	2/15	3/15	4/15	3/15

$E(X) = \sum x\, f_X(x) = 3.0667$. Profit = revenue – cost $= 1.65X + \frac{3}{4}(1.20)(5-X) - 5(1.20) = 0.75X - 1.50$. Expected Profit, E(Profit) $= 0.75X - 1.50 = \$0.80$

Question 3

We find $\Pr(M \geq k)$, where $k = 1, 2, \cdots, 6$.

Let X_i be the number that turns up on die i, where $i = 1, 2, 3$. Then

$\Pr(M \geq k) = \Pr(X_1 \geq k, X_2 \geq k, X_3 \geq k) = \Pr(X_1 \geq k)\Pr(X_2 \geq k)\Pr(X_3 \geq k) = \left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right) = \left(\frac{7-k}{6}\right)^3$

Hence, by the tail sum formula

$E(M) = \sum_{k=1}^{\infty} \Pr(M \geq k) = \sum_{k=1}^{6} \Pr(M \geq k) = \sum_{i=1}^{6} \left(\frac{7-k}{6}\right)^3 = \frac{1^3+2^3+\cdots+6^3}{6^3} = 2.0417$

Question 4

(a) $E(X) = \int_0^1 x\, f_X(x)\, dx = \int_0^1 x(2-2x)\, dx = \int_0^1 (2x - 2x^2) dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}$

$V(X) = \int_0^1 \left(x - \frac{1}{3}\right)^2 (2-2x) dx = \int_0^1 \left(x^2 - \frac{2}{3}x + \frac{1}{9}\right)(2-2x) dx = \int_0^1 \left(-2x^3 + \frac{10}{3}x^2 - \frac{14}{9}x + \frac{2}{9}\right) dx = \left[-\frac{1}{2}x^4 + \frac{10}{9}x^3 - \frac{7}{9}x^2 + \frac{2}{9}x\right]_0^1 = \frac{1}{18}$.

(b) $Y = 3X - 2$. $E(Y) = 3E(X) - 2 = 3\left(\frac{1}{3}\right) - 2 = -1$. $V(Y) = 3^2 V(X) = 9\left(\frac{1}{18}\right) = \frac{1}{2}$.

Question 5

We make use of the properties (1) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$ and (2) $E(X) = 3/5$ to set up 2 equations with 2 unknowns a and b. $\int_0^1 (a + bx^2)\, dx = 1$ gives $a + \frac{b}{3} = 1$ and $\int_0^1 x(a + bx^2)\, dx = 3/5$ gives $a/2 + b/4 = 3/5$. Solving these 2 equations, we have $a = 3/5$ and $b = 6/5$.

Question 6

$E[(X-1)^2] = 10$ implies $E[X^2 - 2X + 1] = E(X^2) - 2E(X) + 1 = 10$ (1)

$E[(X-2)^2] = 6$ implies $E[X^2 - 4X + 4] = E(X^2) - 4E(X) + 4 = 6$ (2)

Eq(1) – Eq(2), we have $2E(X) - 3 = 4$ or $E(X) = 7/2$.

Substitute $E(X) = 7/2$ into Eq(1), we have $E(X^2) = 16$.

Hence $V(X) = E(X^2) - [E(X)]^2 = 15/4$.

Question 7

$E(X) = 10$ and $V(X) = 4$ or $\sigma = 2$.

(a) $\Pr(5 < X < 15) = \Pr[10 - (5/2)(2) < X < 10 + (5/2)(2)] = \Pr\big(|X - 10| < (5/2)(2)\big)$
Applying Chebyshev's Inequality with $k = 5/2$, we have $\Pr\big(|X - 10| < (5/2)(2)\big) \geq 1 - \frac{1}{(5/2)^2} = \frac{21}{25}$.

(b) $\Pr(6 < X < 14) = \Pr[10 - 2(2) < X < 10 + 2(2)] = \Pr\big(|X - 10| < 2(2)\big)$
Applying Chebyshev's Inequality with $k = 2$, we have $\Pr\big(|X - 10| < 2(2)\big) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$. Hence, $\Pr(5 < X < 14) \geq \Pr(6 < X < 14) \geq \frac{3}{4}$.

(c) $\Pr(|X - 10| < 3) = \Pr(|X - 10| < (3/2)2)$
Applying Chebyshev's Inequality with $k = 3/2$, we have $\Pr[10 - (3/2)(2) < X < 10 + (3/2)(2)] \geq 1 - \frac{1}{(3/2)^2} = \frac{5}{9}$.

(d) $\Pr(|X - 10| \geq 3) = \Pr\big(|X - 10| \geq (3/2)(2)\big)$
Applying Chebyshev's Inequality with $k = 3/2$, we have $\Pr\big(|X - 10| \geq (3/2)(2)\big) \leq \frac{1}{(3/2)^2} = \frac{4}{9}$.

(e) $\Pr(|X - 10| \geq c) \leq 0.04 = \frac{1}{5^2}$. Hence, $k = 5$. $c = k\sigma = 5(2) = 10$

Question 8

$\mu = 900$ and $\sigma = 50$. Hence $700 = \mu - 4\sigma$ or $k = 4$.
$\Pr(X < 700 \text{ or } X > 1100) \leq 1/4^2 = 0.0625$. Since X is symmetric, $\Pr(X < 700) = \Pr(X > 1100)$ and $\Pr(X < 700) = (1/2)(0.0625) = 0.03125$.

Question 9

$E(X) = \int_0^1 x\, f_X(x)\, dx = \int_0^1 6x^2(1 - x)dx = \left[2x^3 - \frac{3}{2}x^4\right]_0^1 = 0.5$

$E(X^2) = \int_0^1 x^2 f_X(x)\, dx = \int_0^1 6x^3(1 - x)\, dx = \left[\frac{3}{2}x^4 - \frac{6}{5}x^5\right]_0^1 = 0.3$

$V(X) = E(X^2) - [E(X)]^2 = 0.05$. $\sigma = \sqrt{0.05}$.

$\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \Pr(0.5 - 2\sqrt{0.05} < X < 0.5 + \sqrt{0.05})) = \Pr(0.0528 < X < 0.9472) = \int_{0.0528}^{0.9472} 6x(1 - x)dx = [3x^2 - 2x^3]_{0.0528}^{0.9472} = 0.9839$.

Chebyshev's theorem: $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \geq 1 - 1/2^2 = 0.75$

The exact probability is 0.9839 while Chebyshev's theorem states that it is at least 0.75.

Question 10

$f_{X,Y}(x, y) = c\lvert x - y\rvert$		y	
		-2	3
x	-2	$0c$	$5c$
	0	$2c$	$3c$
	2	$4c$	$1c$

$\sum_i \sum_j f_{X,Y}(x_i, y_j) = 1$ implies $15c = 1$. Hence, $c = 1/15$.