Review 3.2 - 3.4

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$k < n$

If $k < n$ the corresponding row-echelon form (see next page) has at least one zero row, then $span(S) \neq \mathbb{R}^n$.

Theorem

Let $S = \{u_1, \cdots, u_k\}$ be a set of vectors in $\mathbb{R}^n$. If $k < n$, then S cannot span $\mathbb{R}^n$.

Remark: This theorem says that a set of k vectors can at most span $\mathbb{R}^k$.

A useful method

Let $S = \{u_1, \cdots, u_k\} \subset \mathbb{R}^n$, where $u_i = (a_{i1}, \cdots, a_{in}), 1 \leq i \leq k$. To show that $span(S) = \mathbb{R}^n$ we need to verify that for any $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, v is contained in $span(S)$. Now $v \in span(S)$ if and only if the vector equation

$$c_1 u_1 + \cdots + c_k u_k = v$$

has a solution for $c_1, \cdots, c_k$, which means that the linear system

$$\begin{cases} a_{11}c_1 + a_{21}c_2 + \cdots + a_{k1}c_k & = c_1 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{k2}c_k & = c_2 \\ \quad \vdots & \quad \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{kn}c_k & = c_n \end{cases}$$

is consistent.

A useful method – continued

Write the linear system on the last page as matrix form $Ac = v$:

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

1. If a row-echelon form of A does not have any zero row, then the linear system always consistent regardless of the values of $v_1, \cdots, v_n$ and hence $span(S) = \mathbb{R}^n$.
2. If a row-echelon form of A has at least one zero row, then the linear system is not always consistent and hence $span(S) \neq \mathbb{R}^n$.
3. See Q1.

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Let $S_1 = \{u_1, u_2, \cdots, u_k\}, S_2 = \{v_1, v_2, \cdots, v_m\}$ be subsets of $\mathbb{R}^n$.

Theorem

$span(S_1) \subset span(S_2)$ if and only if each u_i is a linear combination of $v_1, v_2, \cdots, v_m$.

Whether $span(S_1) = span(S_2)$

- For two sets A and B, if we want to prove $A = B$, we need to show that $A \subset B$ and $B \subset A$.
- $S_1 = \{u_1, u_2, \cdots, u_k\}, S_2 = \{v_1, v_2, \cdots, v_m\}$ be subsets of $\mathbb{R}^n$.
- By the last theorem, to show that $span(S_1) = span(S_2)$, we need to show that $span(S_1) \subset span(S_2)$ and $span(S_2) \subset span(S_1)$.
- It suffice to show that:

  1. each u_i is a linear combination of $v_1, v_2, \cdots, v_m$;

  and

  2. each v_i is a linear combination of $u_1, u_2, \cdots, u_k$.

For instance, See Q3 and

Theorem

Let $u_1, u_2, \cdots, u_k$ be vectors in $\mathbb{R}^n$, if u_k is a linear combination of $u_1, u_2, \cdots, u_{k-1}$, then

$$span\{u_1, u_2, \cdots, u_{k-1}\} = span\{u_1, u_2, \cdots, u_k\}.$$

Find a set of vectors that span the solution space of homogeneous system.

1. Use Gaussian elimination to solve the homogeneous system.
2. Write down the general solution as

$$x = t_1 r_1 + \cdots t_k r_k,$$

   where $t_1, \cdots, t_k$ are arbitrary real numbers.
3. Then $\{r_1 \cdots, r_k\}$ span the solution space.
4. For details, see the proof of Theorem 3.3.6 in your textbook.

See Q2.

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Subspace

Definition

Let V be a subset of $\mathbb{R}^n$, then V is called a subspace of $\mathbb{R}^n$ if $V = span(S)$ where $S = \{u_1 \cdots, u_k\}$ for some vector $u_1 \cdots, u_k \in \mathbb{R}^n$. We called that V is the subspace spanned by S or S spans the subspace V.

1. If V is a subspace, then $0 \in V$.
2. (`Zero space`) $\{0\} = span\{0\}$.
3. $\mathbb{R}^n$ is itself a subspace, and

$$\mathbb{R}^n = span\{e_1, \cdots, e_n\}$$

where $e_1 = (1, 0, \cdots, 0), e_1 = (0, 1, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1)$.

4. How to verify that a set V is a subspace? See Q4.

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Linear Independent

Definition

Let $S = \{u_1 \cdots, u_k\}$ be a set of vectors in $\mathbb{R}^n$. Consider the equation

$$c_1 u_1 + \cdots c_k u_k = 0,$$

1. S is called a `linear independent set` and $u_1 \cdots, u_k$ are said to be `linear indenpendent` if the equation has only trivial solution $c_1 = \cdots = c_k = 0$.
2. Othterwise, S is called a `linear dependent set` and $u_1 \cdots, u_k$ are said to be `linear denpendent`

How to remove the redundant vector?

- As we may have that

$$span\{u_1 \cdots, u_k\} = span\{u_1 \cdots, u_{k-1}\}.$$

Which means that in this case u_k is redundant, i.e., if we let $S = \{u_1 \cdots, u_k\}$, then

$$span(S) = span(S - u_k).$$

- To remove the "redundant" vector in a set of vector S, we mean that first S must be linear dependent, or there is no vector that is redundant. Second, we need to find a vector u such that

$$span(S) = span(S - \{u\}).$$

- To do this, we need $u \in span(S - \{u\})$. (why?)