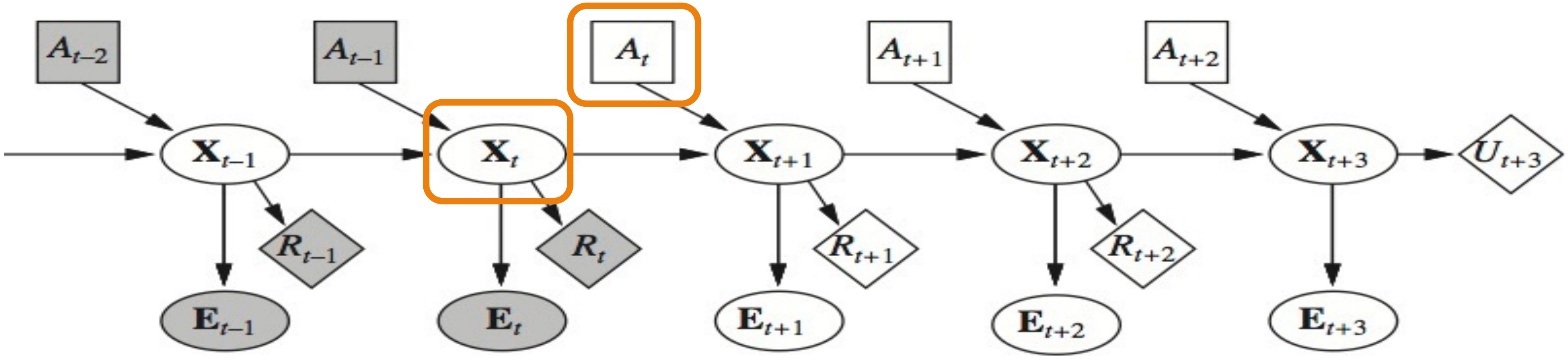
Filtering

The FORWARD function

A Dynamic Decision Network for POMDP

Note:
- Variables with known values are shaded
- Current time is t and agent must decide what to do

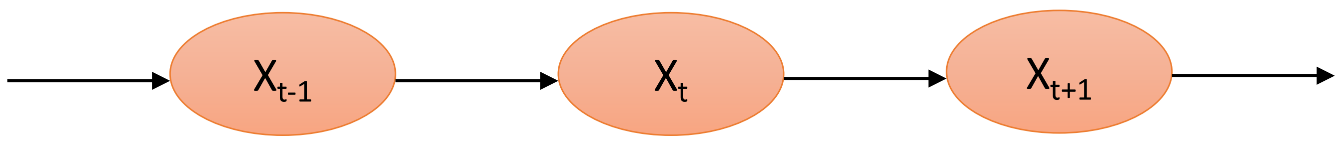


What is X_t? E_t? A_t? R_t? U_t?

Source: RN 3e Figure 17.10

Transition Model

- Transition model for first-order process
  - For all t: $$\mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1})$$



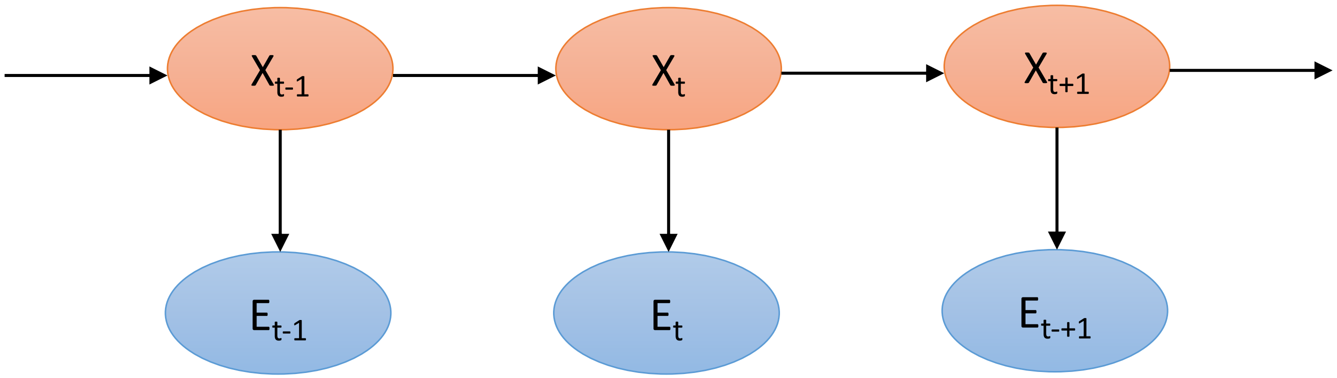
Assumption: Let’s ignore the actions for now

Sensor (Observation) Model

Assumption: Let's ignore the actions for now

- Sensor model:

$$\mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t)$$



  - Describes how the "sensors" – the evidence variables – are affected by the actual state of the world
- Question:
  - Why does the direction of the "edge" goes from state to sensor values?

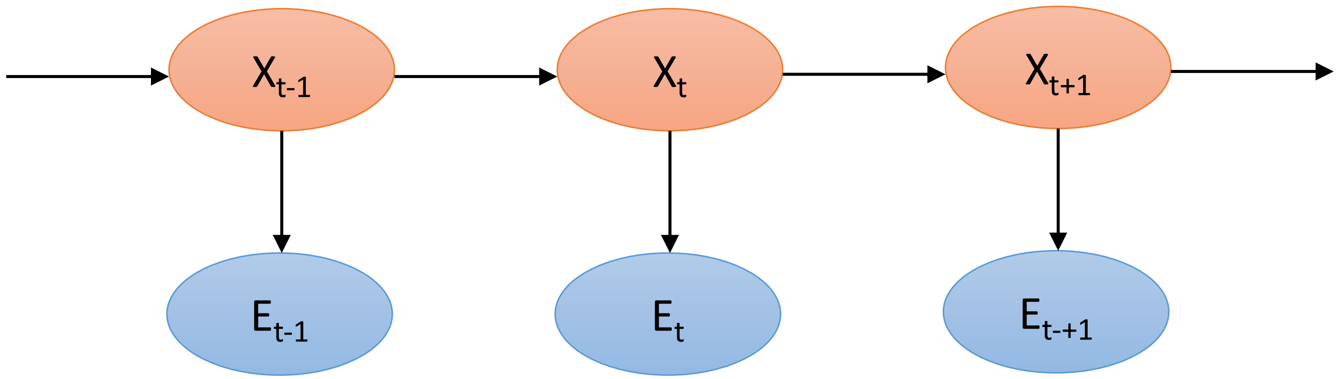
Example: Transition Model and Sensor Model

Prior Probability

$P(X_0) = 0.5$

X_{t-1}	$P(X_t \mid \bullet)$
T	0.7
F	0.2

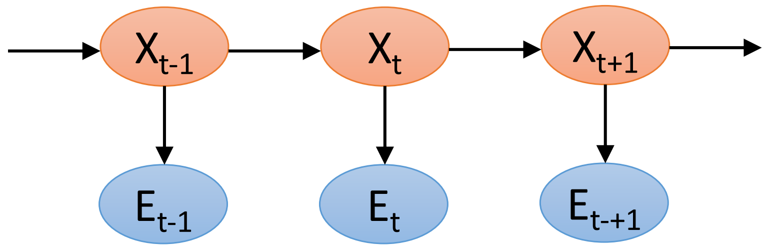
Transition Model



X_t	$P(E_t \mid \bullet)$
T	0.9
F	0.2

Sensor Model

Full Joint Distribution

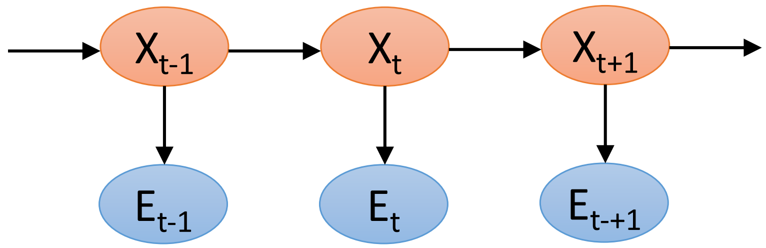


- Full joint distribution over all the variables are defined by:
  - Prior probability
  - Transition model
  - Sensor model

$$\mathbf{P}(\mathbf{X}_0,\mathbf{X}_1,\ldots,\mathbf{X}_t,\mathbf{E}_1,\ldots,\mathbf{E}_t) = \mathbf{P}(\mathbf{X}_0)\prod_{i=1}^{t}\mathbf{P}(\mathbf{X}_i \mid \mathbf{X}_{i-1})\mathbf{P}(\mathbf{E}_i \mid \mathbf{X}_i)$$

- Assumption:
  - Evidence arrives only starting at time t = 1

Full Joint Distribution



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Prior

Transition model

Sensor model

- Assumption:
  - Evidence arrives only starting at time $t = 1$

Filtering or Monitoring

- Computing the belief state
  - Posterior distribution over the current state, given all evidence to date
- Compute:

$$\mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{1:t})$$

- Assume
  - evidence arrives continuously from t =1

Filtering

- Recursive estimation:
  - Given the result of filtering up to time t, compute the result for t + 1 from new evidence $\mathbf{e}_{t+1}$:

$$P(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t \mid \mathbf{e}_{1:t}))$$

  for some function f

Forward Recursive Estimation

- View calculation as two parts:
  - Current state distribution is projected forward from t to t + 1
  - It is then updated using new evidence $\mathbf{e}_{t+1}$,

$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\
&= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \\
&= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})
\end{aligned}
$$

where α is a normalizing factor

- Question: Can you explain each step of the derivation?

Forward Recursive Estimation

- View calculation as two parts:
  - Current state distribution is projected forward from t to t + 1
  - It is then updated using new evidence $\mathbf{e}_{t+1}$,

Step 1: Dividing up the evidence
Step 2: Use Baye's Rule
Step 3: by the Markov property of evidence

$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\
&= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \\
&= \alpha \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{Sensor model}} \underbrace{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})}_{\text{One step prediction}}
\end{aligned}
$$

where α is a normalizing factor

- Question: Can you explain each step of the derivation?

Forward Recursive Estimation

- Obtain one step prediction for the next state by conditioning on the current state $\mathbf{X}_t$:

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

- Note: within the summation:
  - 1st term – transition model
  - 2nd term – current state distribution

Forward Recursive Estimation

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Recursive formulation

- Note: within the summation:
  - 1st term – transition model
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Forward Process

- Think of:
  - Filtered estimate $P(\mathbf{X}_t | \mathbf{e}_{1:t})$ as a "message" $\mathbf{f}_{1:t}$ propagated forward along the sequence, modified by each transition and updated by each new observation
- The process is:

$$\mathbf{f}_{1:t+1} = \alpha \text{FORWARD}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$$

where FORWARD implements the update

$$\mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t})$$

- For discrete state variables:
  - time and space for each update are constant (independent of t) (Why is this important?)