NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 11**

1. Determine whether the following are linear transformations. Justify your answer.

   (a) $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-z \\ 1 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   No, since $T(\mathbf{0}) \neq \mathbf{0}$.

   (b) $T_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-z \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   Yes, the standard matrix for T_2 is
$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   (c) $T_3 \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \begin{vmatrix} y & z \\ b & c \end{vmatrix} \\ -\begin{vmatrix} x & z \\ a & c \end{vmatrix} \\ \begin{vmatrix} x & y \\ a & b \end{vmatrix} \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, where a, b, c are in $\mathbb{R}$.

   Yes, the standard matrix for T_3 is
$$T_3\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   (d) $T_4 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\boldsymbol{u}) = \lambda\boldsymbol{u}$ for $\boldsymbol{u} \in \mathbb{R}$, where λ is a fixed scalar.

   Yes, the standard matrix for T_4 is
$$T_4(\boldsymbol{u}) = \lambda I_n \boldsymbol{u}.$$

   (e) $T_5 \colon \mathbb{R}^2 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = xy$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   No. Take $\boldsymbol{u} = (1,0)^T$ and $\boldsymbol{v} = (0,1)^T$, and then $\boldsymbol{u}+\boldsymbol{v} = (1,1)^T$. However,
$$T_5(\boldsymbol{u}+\boldsymbol{v}) = 1 \neq 0 + 0 = T_5(\boldsymbol{u}) + T_5(\boldsymbol{v}).$$

2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T\left(\begin{pmatrix}2\\1\\4\end{pmatrix}\right) = \begin{pmatrix}1\\-3\\-2\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\5\\3\end{pmatrix}\right) = \begin{pmatrix}-4\\2\\-2\end{pmatrix}, \quad T\left(\begin{pmatrix}3\\3\\5\end{pmatrix}\right) = \begin{pmatrix}0\\-2\\-2\end{pmatrix}.$$

Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear tranformation such that

$$S\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x+y\\y+z\\x+z\end{pmatrix}.$$

(a) Find the formula of T.

First, we have

$$\begin{vmatrix}2 & 1 & 3\\1 & 5 & 3\\4 & 3 & 5\end{vmatrix} = -12 \neq 0.$$

Thus the vectors

$$\mathcal{B} = \left\{\begin{pmatrix}2\\1\\4\end{pmatrix}, \quad \begin{pmatrix}1\\5\\3\end{pmatrix}, \quad \begin{pmatrix}3\\3\\5\end{pmatrix}\right\}$$

form a basis for $\mathbb{R}^3$, which determine T uniquely.
Let $(x,y,z)^T$ be any vector in $\mathbb{R}^3$. We need to compute the coordinate of $(x,y,z)^T$ according to the basis $\mathcal{B}$. To do that, we need to get the reduced row-echelon form of the following augmented matrix

$$\left(\begin{array}{ccc|c}2 & 1 & 3 & x\\1 & 5 & 3 & y\\4 & 3 & 5 & z\end{array}\right) \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c}1 & 0 & 0 & -\frac{4x}{3} - \frac{y}{3} + z\\0 & 1 & 0 & -\frac{7x}{12} + \frac{y}{6} + \frac{z}{4}\\0 & 0 & 1 & \frac{1}{12}(17x + 2y - 9z)\end{array}\right).$$

We obtain

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = (-\frac{4x}{3} - \frac{y}{3} + z)\begin{pmatrix}2\\1\\4\end{pmatrix} + (-\frac{7x}{12} + \frac{y}{6} + \frac{z}{4})\begin{pmatrix}1\\5\\3\end{pmatrix} + \frac{1}{12}(17x + 2y - 9z)\begin{pmatrix}3\\3\\5\end{pmatrix}.$$

Then the general formula of T is

$$\begin{aligned}&T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)\\ =&(-\frac{4x}{3} - \frac{y}{3} + z)\begin{pmatrix}1\\-3\\-2\end{pmatrix} + (-\frac{7x}{12} + \frac{y}{6} + \frac{z}{4})\begin{pmatrix}-4\\2\\-2\end{pmatrix} + \frac{1}{12}(17x + 2y - 9z)\begin{pmatrix}0\\-2\\-2\end{pmatrix}\\ =&\begin{pmatrix}x-y\\y-z\\x-z\end{pmatrix}.\end{aligned}$$

(b) Find the standard matrix for T instead of using the formula of T in Part (2a).

We compute the standard matrix following Discussion 7.1.8. Let us summarize Discussion 7.1.8 as follows:

If $T(\boldsymbol{u}_i) = \boldsymbol{v}_i$ for $1 \le i \le 3$ and $\boldsymbol{u}_i, \boldsymbol{v}_i \in \mathbb{R}^3$ and the square matrix $\boldsymbol{B} = (\boldsymbol{u}_1 \ \ \boldsymbol{u}_2 \ \ \boldsymbol{u}_3)$ formed by $\boldsymbol{u}_i$ as columns is invertible, then the standard matrix $\boldsymbol{A}$ of T

$$\boldsymbol{A} = (T(\boldsymbol{e}_1) \ \ T(\boldsymbol{e}_2) \ \ T(\boldsymbol{e}_3)) = (\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3)\boldsymbol{B}^{-1}. \tag{1}$$

By

$$\left(\begin{array}{ccc|c|c|c} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 5 & 3 & 0 & 1 & 0 \\ 4 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -\frac{4}{3} & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & -\frac{7}{12} & \frac{1}{6} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{17}{12} & \frac{1}{6} & -\frac{3}{4} \end{array}\right),$$

applying (1) into our case, we have the standard matrix $\boldsymbol{A}$ of T

$$\boldsymbol{A} = \begin{pmatrix} 1 & -4 & 0 \\ -3 & 2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} & 1 \\ -\frac{7}{12} & \frac{1}{6} & \frac{1}{4} \\ \frac{17}{12} & \frac{1}{6} & -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(c) Find a basis of the range of T.

Recall that the range of T is the column space of $\boldsymbol{A}$. By

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

we have the first and second columns of $\boldsymbol{A}$ is a basis of the column space of $\boldsymbol{A}$. Hence

$$R(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(d) Find a basis of the kernel of T.

By the reduced row-echelon form in (2), we have

$$\operatorname{Ker}(T) = \text{ the nullspace of } \boldsymbol{A} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(e) Use this example to verify the Dimension Theorem for Linear Transformation.

By Parts (c) and (d), we have

$$\dim R(T) + \dim \operatorname{Ker}(T) = 2 + 1 = 3 = \dim \mathbb{R}^3.$$

(f) Find the formula of $T \circ S$ and $S \circ T$.

$$T \circ S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\begin{pmatrix} x+y \\ y+z \\ x+z \end{pmatrix} = \begin{pmatrix} x-z \\ y-x \\ y-z \end{pmatrix}$$
$$S \circ T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = S\begin{pmatrix} x-y \\ y-z \\ x-z \end{pmatrix} = \begin{pmatrix} x-z \\ x+y-2z \\ 2x-y-z \end{pmatrix}.$$

3. A linear operator T on $\mathbb{R}^n$ is called an isometry if $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

(a) If T is an isometry on $\mathbb{R}^n$, show that $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. (Hint: Compute $T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v})$ in two different ways.)

$$\begin{aligned} T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v}) =& (T(\boldsymbol{u}) + T(\boldsymbol{v})) \cdot (T(\boldsymbol{u}) + T(\boldsymbol{v})) \\ =& T(\boldsymbol{u}) \cdot T(\boldsymbol{u}) + 2T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) + T(\boldsymbol{v}) \cdot T(\boldsymbol{v}) \\ =& \|u\|^2 + 2T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) + \|v\|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v}) =& \|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v}) \cdot (\boldsymbol{u}+\boldsymbol{v}) \\ =& \|\boldsymbol{u}\|^2 + 2\boldsymbol{u} \cdot \boldsymbol{v} + \|\boldsymbol{v}\|^2. \end{aligned}$$

Hence, $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$.

(b) Let $\boldsymbol{A}$ be the standard matrix for a linear operator T. Show that T is an isometry if and only if $\boldsymbol{A}$ is an orthogonal matrix. (See also Question 5.32.)

$$\begin{aligned} & T \text{ is an isometry} \qquad \text{(by Part (a))} \\ \iff & T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n \\ \iff & T(\boldsymbol{u})^T T(\boldsymbol{v}) = \boldsymbol{u}^T \boldsymbol{v} \\ \iff & \boldsymbol{u}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} \\ \iff & \boldsymbol{u}^T (\boldsymbol{A}^T \boldsymbol{A} - I_n) \boldsymbol{v} = 0 \\ \iff & \boldsymbol{A}^T \boldsymbol{A} - I_n = \boldsymbol{0}_{n \times n} \qquad \text{(See Remark as below.)} \\ \iff & \boldsymbol{A}^T \boldsymbol{A} = I_n \\ \iff & \boldsymbol{A} \text{ is an orthogonal matrix.} \end{aligned}$$

Remark. For any square matrix $\boldsymbol{B} = (b_{ij})_{n \times n}$ of order n, we have

$$\boldsymbol{e}_i^T \boldsymbol{B} \boldsymbol{e}_j = b_{ij}.$$

So if $\boldsymbol{e}_i^T \boldsymbol{B} \boldsymbol{e}_j = 0$ for all i, j, then $b_{ij} = 0$ for all i, j, that is, $\boldsymbol{B} = \boldsymbol{0}_{n \times n}$.

4. Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $P\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} \qquad \text{for } x \in \mathbb{R}^n.$$

(a) Show that P is a linear transformation by the following fact: Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $T(c\boldsymbol{u}+d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$ for all $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathbb{R}^n$ and $c, d \in \mathbb{R}$, then T is a linear transformations.

For all $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathbb{R}^n$ and $c, d \in \mathbb{R}$, we have

$$\begin{aligned} P(c\boldsymbol{u}+d\boldsymbol{v}) =& (c\boldsymbol{u}+d\boldsymbol{v}) - (\boldsymbol{n}\cdot(c\boldsymbol{u}+d\boldsymbol{v}))\boldsymbol{n} \\ =& (c\boldsymbol{u}+d\boldsymbol{v}) - (c\boldsymbol{n}\cdot\boldsymbol{u} + d\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} \\ =& c(\boldsymbol{u} - (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n} + d(\boldsymbol{v} - (\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} \\ =& cP(\boldsymbol{u}) + dP(\boldsymbol{v}). \end{aligned}$$

Following from the fact, we have T is a linear transformation.

(b) Prove that $P \circ P = P$.

$$\begin{aligned} (P\circ P)(\boldsymbol{x}) =& P(P(\boldsymbol{x})) = P(\boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}) \\ =& (\boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}) - [\boldsymbol{n}\cdot(\boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n})]\boldsymbol{n} \\ =& \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} - [\boldsymbol{n}\cdot\boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}\cdot\boldsymbol{n}]\boldsymbol{n}. \end{aligned}$$

Since n is unit, i.e., $n \cdot n = 1$,

$$(P\circ P)(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} = P(\boldsymbol{x}),$$

which completes the proof.

(c) Show that $\mathrm{Ker}(P) = \mathrm{span}\{\boldsymbol{n}\}$ and the rang $R(P) = \mathrm{span}\{\boldsymbol{n}\}^\perp$. Recall for a subspace W of $\mathbb{R}^n$, $W^\perp = \{\boldsymbol{u}\in\mathbb{R}^n \colon \boldsymbol{u}\cdot\boldsymbol{w} = 0 \text{ for all } \boldsymbol{w}\in W\}$.

By definition, $\boldsymbol{u}$ is in $\mathrm{Ker}(P)$ if and only if $P(\boldsymbol{u}) = 0$, that is, $\boldsymbol{u} = (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n}$.
If $\boldsymbol{u} \in \mathrm{span}\{\boldsymbol{n}\}$, then $\boldsymbol{u} = c\boldsymbol{n}$. Since $\boldsymbol{n}$ is an orthonormal basis of $\mathrm{span}\{\boldsymbol{n}\}$, then $\boldsymbol{u} = (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n}$. Thus $\boldsymbol{u}$ is in $\mathrm{Ker}(P)$.
If $\boldsymbol{u}$ is in $\mathrm{Ker}(P)$, then $\boldsymbol{u} = (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n}$, which is in $\mathrm{span}\{\boldsymbol{n}\}$.
Therefore, $\mathrm{Ker}(P) = \mathrm{span}\{\boldsymbol{n}\}$.
Next, we show that $R(P) = \mathrm{span}\{\boldsymbol{n}\}^\perp$.
For every $u \in \mathbb{R}^n$, we have

$$P(\boldsymbol{u})\cdot\boldsymbol{n} = (\boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n})\cdot\boldsymbol{n} = \boldsymbol{x}\cdot\boldsymbol{n} - \boldsymbol{n}\cdot\boldsymbol{x} = 0,$$

that is, $P(\boldsymbol{u}) \in \mathrm{span}\{\boldsymbol{n}\}^\perp$. Thus, $R(P) \subseteq \mathrm{span}\{\boldsymbol{n}\}^\perp$.
For every vector $\boldsymbol{v} \in \mathrm{span}\{\boldsymbol{n}\}^\perp$ (i.e., $\boldsymbol{v}\cdot\boldsymbol{n} = 0$),

$$P(\boldsymbol{v}) = \boldsymbol{v} - (\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} = \boldsymbol{v}.$$

Then $\boldsymbol{v} \in R(P)$ and $\mathrm{span}\{\boldsymbol{n}\}^\perp \subseteq R(P)$.
Therefore $R(P) = \mathrm{span}\{\boldsymbol{n}\}^\perp$.