National University of Singapore
School of Computing
CS3243 Introduction to AI

Tutorial 6: First-Order Logic

Issued: March 16, 2020 Due: Week 10 In Class

Important Instructions:

- *Your solutions for this tutorial must be TYPE-WRITTEN.*
- *Make TWO copies of your solutions: one for you and one to be SUBMITTED TO THE TUTOR IN CLASS. Your submission in your respective tutorial class will be used to indicate your CLASS ATTENDANCE. Late submission will NOT be entertained.*
- *YOUR SOLUTION TO QUESTION* 2 *will be GRADED for this tutorial.*
- *You may discuss the content of the questions with your classmates. But everyone should work out and write up ALL the solutions by yourself.*

1. (Modified Question 8.26 from AIMA 3rd edition) Represent the following sentences in first-order logic, using a consistent vocabulary defined as follows:

   $Took(x, y, z)$: is true if student x took subject y in semester z
   $Score(x, y, z)$: is true if student x obtains score z in subject y
   $Passed(x, y)$: is true if student x passed subject y
   $Buys(x, p)$: is true if person x buys policy p
   $IsSmart(x)$: is true if person x is smart
   $IsExpensive(x)$: is true if x is expensive
   $Sells(x, y, p)$: is true if person x sells policy p to person y
   $IsInsured(x)$: is true if person x is insured
   $IsBarber(x)$: is true if x is a barber
   $Shaves(x, y)$: is true if person x shaves person y

   (a) Some students took French in Spring 2001.
   (b) Every student who takes French passes it.
   (c) Only one student took Greek in Spring 2001.
   (d) The best score in Greek is always higher than the best score in French.

(e) Everyone who buys a policy is smart.

(f) No person buys an expensive policy.

(g) There is an agent who sells policies only to those people who are not insured.

(h) There is a barber who shaves all men in town who do not shave themselves.

(i) There is a barber who shaves all men in town who does not shave himself.

2. Recall that a CNF formula ϕ is defined over a set of variables $X = \{x_1, \dots, x_n\}$; a truth assignment assigns true/false (or equivalently 1 or 0) to every variable $x_i \in X$. Thus, it is useful to think of ϕ as a mapping from $\{0,1\}^n$ to $\{0,1\}$. We say that an assignment $\vec{a} \in \{0,1\}^n$ satisfies ϕ if $\phi(\vec{a}) = 1$. A k-CNF formula is one where each clause contains at most k literals. For example,

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee x_4)$$

is a 3-CNF formula, since the size of each clause is no more than 3.

Show that every CNF formula can be converted to a 3-CNF formula. More formally, suppose that we are given a k-CNF formula ϕ with $k \geq 3$, over n variables $X = \{x_1, \dots, x_n\}$. You need to show that there exists some 3-CNF formula ϕ', whose domain is X and additional variables $Y = \{y_1, \dots, y_m\}$, such that for every truth assignment $\vec{a} \in \{0,1\}^n$, there exists a truth assignment to $y_1, \dots, y_m$, say $\vec{b} \in \{0,1\}^m$ such that $\phi(\vec{a}) = 1$ if and only if $\phi'(\vec{a}; \vec{b}) = 1$.

Hint: We have seen that the resolution algorithm can iteratively reduce the size of clauses e.g.

$$\frac{(x_1 \vee a) \wedge (x_2 \vee x_3 \vee \dots \vee x_m \vee \neg a)}{(x_1 \vee x_2 \vee x_3 \vee \dots x_m)}.$$

The trick in this question is to run the algorithm 'in the other direction' by adding dummy variables (whose value will be determined later as in the resolution algorithm), in order to reduce the size of the clauses.

3. Show that there exists a polynomial time algorithm for deciding whether a 2-CNF formula is satisfiable.

Hint: Note that any clause containing exactly two variables can be written in terms of a conditional, i.e.:

$$x \Rightarrow y \qquad\qquad x \Rightarrow \neg y$$
$$\neg x \Rightarrow y \qquad\qquad \neg x \Rightarrow \neg y$$

Next, look at the directed graph whose nodes are variables (and their negations), and think what happens if there is some cycle containing a variable x and its negation $\neg x$.