

# Partial Differentiation

## Chapter 7. Functions of Several Variables

## 7.1 Introduction

In elementary calculus, we encountered scalar functions of one variable, e.g.  $f = f(x)$ . However, many physical quantities in engineering and science are described in terms of scalar functions of several variables. For example,

(i) Mass density of a lamina can be described by  $\delta(x, y)$ , where  $(x, y)$  are the coordinates of a point on the lamina.

(ii) Pressure in the atmosphere can be described by  $P(x, y, z)$  where  $(x, y, z)$  are the coordinates of a point in the atmosphere.

(iii) Temperature distribution of a heated metal ball can be described by  $T(x, y, z, t)$  where  $(x, y, z)$  are the coordinates of a point in the ball and  $t$  is the time.

## 7.1.1 Functions of Two Variables

A function  $f$  of **two variables** is a rule that assigns to each ordered pair of real numbers  $(x, y)$  a real number denoted by  $f(x, y)$ .

We usually write  $z = f(x, y)$  to indicate that  $z$  is a function of  $x$  and  $y$ . Moreover,  $x, y$  are called the independent variables and  $z$  is called the dependent variable. The set of all ordered pairs  $(x, y)$  such that  $f(x, y)$  can be defined is called the **domain** of  $f$ .

## 7.1.2 Example

(a)  $f(x, y) = x^2y^3.$

This is a function of two variables which is defined for any  $x$  and  $y$ . So the domain of  $f$  is the set of all  $(x, y)$  with  $x, y \in \mathbf{R}$ .



(b)  $f(x, y) = \sqrt{1 - x^2 - y^2}.$

This function is only defined when  $1 - x^2 - y^2 \geq 0$ ,

or equivalently  $x^2 + y^2 \leq 1$ .

So the domain of  $f$  is the set

$$D = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Note that  $D$  represents all the points in the  $xy$  plane lying within (and on) the unit circle.

(c) We can also define  $f$  in “pieces” as a **compound function**. For example

$$f(x, y) = \begin{cases} \sqrt{x - y} & \text{if } x > y, \\ \sqrt{y - x} & \text{if } x < y, \\ 1 & \text{if } x = y. \end{cases}$$

### 7.1.3 Functions of Three or More Variables

We can define functions of **three** variables  $f(x, y, z)$ , **four** variables  $f(x, y, z, w)$ , etc in a similar way.

## 7.2 Geometric Representation

### 7.2.1 Graphs of functions of two variables

The graph of a function  $f(x)$  of one variable is a curve in the  $xy$ -plane, which can be regarded as the set of all points  $(x, y)$  in the  $xy$ -plane such that  $y = f(x)$ .

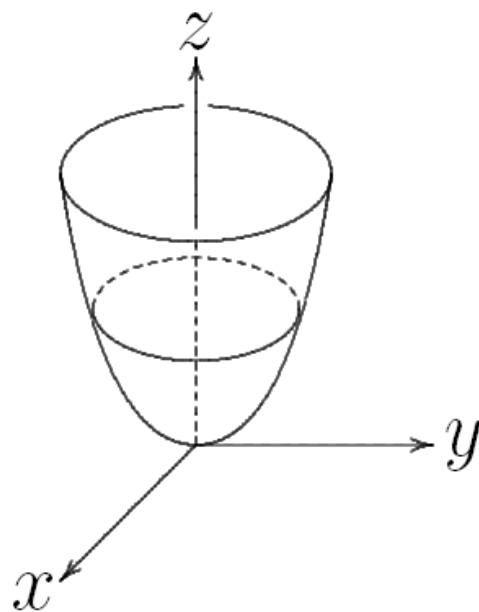
By analogy, we have the **graph** of a function  $f(x, y)$  of two variables is the set of all points  $(x, y, z)$  in the three dimensional  $xyz$ -space such that  $z = f(x, y)$ . This set represents a surface in the  $xyz$ -space.

## 7.2.2 Example

The graph of  $f(x, y) = 5 - 3x - 2y$  is the plane with equation  $z = 5 - 3x - 2y$  (or  $3x + 2y + z = 5$ ).

### 7.2.3 Example

The graph of  $g(x, y) = 8x^2 + 2y^2$  is the paraboloid (see diagram below) with equation  $z = 8x^2 + 2y^2$ .



## 7.3 Partial Derivatives

Let  $f = f(x, y)$  be a function of two variables. Either a change of  $x$  or a change of  $y$  can cause a change of  $f$ . In order to measure the rate of change of  $f$  with respect to the variable  $x$ , we need to fix the variable  $y$ , and vice versa. Note that when we fix one of the variables of  $f(x, y)$ , then it becomes a function of one variable.



### 7.3.1 Example

Let  $f(x, y) = x^2 - 2xy + 3y^3$ . If we fix  $y = 2$ , say, then

$$f(x, 2) = x^2 - 4x + 24$$

is a function in  $x$  alone.

Similarly, if we fix  $x = -1$ , then

$$f(-1, y) = 1 + 2y + 3y^3$$

is a function in  $y$  alone.

## 7.3.2 First order partial derivatives

Let  $f(x, y)$  be a function of two variables. Then the (first order) **partial derivative of  $f$  with respect to  $x$**  at the point  $(a, b)$  is

$$\left. \frac{d}{dx} f(x, b) \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

We say that the partial derivative does not exist at  $(a, b)$  if the limit on the RHS above does not exist.

When the above partial derivative exists, we denote it by  $\left. \frac{\partial f}{\partial x} \right|_{(a,b)}$  or  $f_x(a, b)$ .

# cyrillic alphabet cursive

ah B V G D ye yaw Zhe Z  
*Aa Bb Vv Gg Dd ye ye yaw yaw Zhe Z*

U y K L M N oh P R  
*Uu Uy Kk Ll Mm Nn oh Pp Rr*

S T uh F ch Ts Ch Sh ShCh  
*Cc Tt uh Ff ch Ts Ts Ch Ch Sh ShCh*

hard soft i e yu ya X as in German ich.  
*h h i i e e yu ya X* Vowels in Caps are LONG

Similarly, the (first order) **partial derivative of  $f$  with respect to  $y$**  (instead of  $x$ ) at the point  $(a, b)$  is:

$$\left. \frac{d}{dy} f(a, y) \right|_{y=b} = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

and is denoted by

$$\left. \frac{\partial f}{\partial y} \right|_{(a,b)} \quad \text{or} \quad f_y(a, b).$$

If we let  $z = f(x, y)$ , we also write

$$f_x = \frac{\partial z}{\partial x}, \quad \text{and} \quad f_y = \frac{\partial z}{\partial y}.$$

In practice, when we compute  $f_x(a, b)$  (resp.  $f_y(a, b)$ ), we simply treat the  $y$  (resp.  $x$ ) variable of  $f(x, y)$  as constant and differentiate  $f$  with respect to  $x$  (respectively  $y$ ) before substituting  $x = a$  and  $y = b$ .

### 7.3.3 Example

Let  $f(x, y) = (x^3 + y) \cos(y^2)$ .

Find  $f_x(2, 0)$ , and  $f_y(2, 0)$ .



**Solution:** Treat  $y$  as a constant and compute

$$\begin{aligned} f_x &= \frac{d}{dx} f(x, y) = \frac{d}{dx} (x^3 + y) \cos(y^2) \\ &= 3x^2 \cos(y^2) \end{aligned}$$

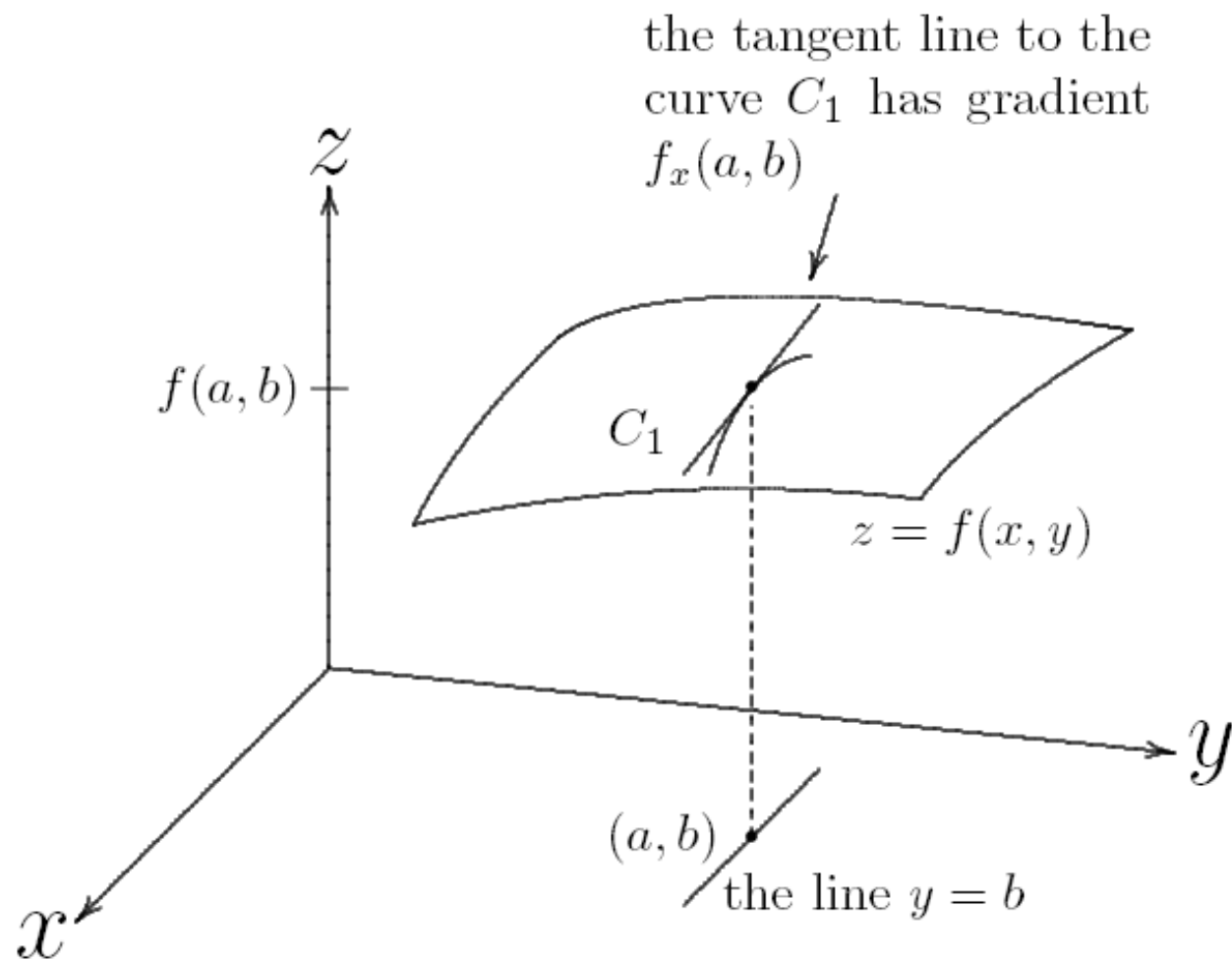
Then  $f_x(2, 0) = 3(2)^2 \cos(0^2) = 12$ .

Treat  $x$  as a constant and compute

$$\begin{aligned} f_y &= \frac{d}{dy} f(x, y) = \frac{d}{dy} (x^3 + y) \cos(y^2) \\ &= \cos(y^2) - (x^3 + y) \sin(y^2) 2y. \end{aligned}$$

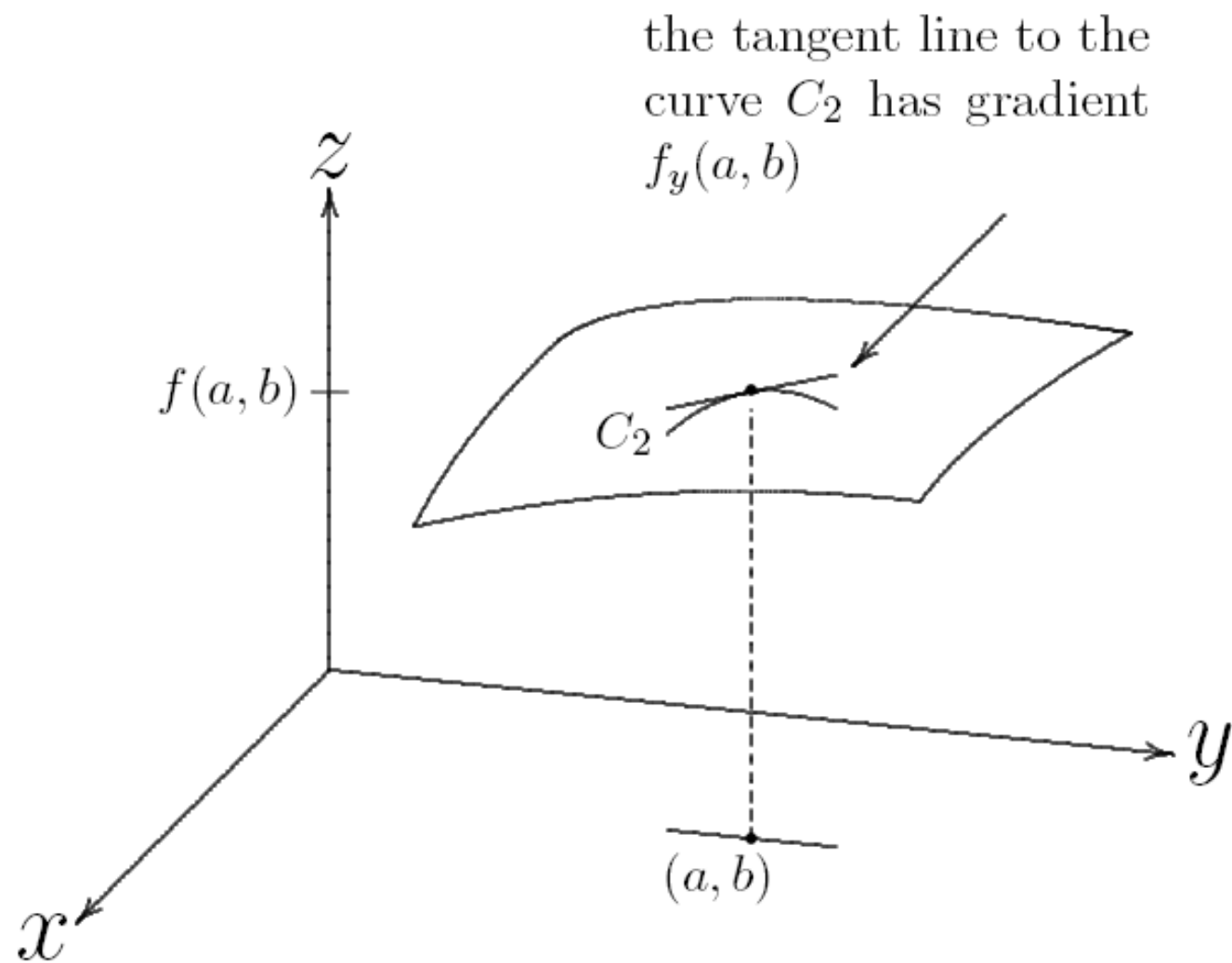
Then  $f_y(2, 0) = \cos(0^2) - (2^3 + 0) \sin(0^2) 2(0) = 1$ .

### 7.3.4 Geometric interpretation



Geometrically,  $f_x(a, b)$  measures the rate of change of  $f$  in the direction of vector  $\mathbf{i}$  at the point  $(a, b)$ . If we consider the line  $y = b$  on the  $xy$ -plane parallel to the  $x$ -axis and passing through the point  $(a, b)$ , the image of this line under  $f$  is a curve  $C_1$  on the surface  $z = f(x, y)$ . Then  $f_x(a, b)$  is just the gradient of the tangent line to  $C_1$  at  $(a, b)$ .

Similarly,  $f_y(a, b)$  is just the gradient of the tangent line at  $(a, b)$  of the curve  $C_2$  traced out as the image of the line  $x = a$  under  $f$ .



### 7.3.5 Higher order partial derivatives

We have seen that the partial derivatives  $f_x$  and  $f_y$  of a function of two variables  $f$  are also functions of two variables. Hence, we can study the partial derivatives of  $f_x$  and  $f_y$ .

The **second order partial derivatives** of  $f$  are:

$$\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial^2 f}{\partial x^2} & f_{xy} &= (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} & f_{yy} &= (f_y)_y = \frac{\partial^2 f}{\partial y^2}. \end{aligned}$$

If  $z = f(x, y)$ , we also have the following notation:

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} \quad f_{xy} = \frac{\partial^2 z}{\partial y \partial x} \quad f_{yx} = \frac{\partial^2 z}{\partial x \partial y} \quad f_{yy} = \frac{\partial^2 z}{\partial y^2}.$$



### 7.3.6 Example

Find the second partial derivatives of

$$f(x, y) = 4x^3 + x^2y^3 - 6y^2.$$

**Solution:** We have  $f_x = 12x^2 + 2xy^3$ , so

$$f_{xx} = 24x + 2y^3, \quad f_{xy} = 6xy^2.$$

We have  $f_y = 3x^2y^2 - 12y$ , so

$$f_{yx} = 6xy^2, \quad f_{yy} = 6x^2y - 12.$$

### 7.3.7 Mixed Derivatives

For most functions in practice, we have

$$f_{xy}(a, b) = f_{yx}(a, b). \quad (1)$$

### 7.3.8 Example

Let  $f(x, y) = xy + \frac{e^y}{y^2 + 1}$ . Find  $f_{yx}$ .

**Solution:** The notation  $f_{yx}$  means differentiating first with respect to  $y$  and then with respect to  $x$ .

This is the same as  $f_{xy}$ , i.e. we can postpone the differentiation with respect to  $y$  and differentiate first with respect to  $x$ :

$$f_x = y \implies f_{xy} = (f_x)_y = 1.$$

This is much easier than to first differentiate with respect to  $y$ !

### 7.3.9 Functions of Three or More Variables

For functions of three or more variables, we have similar definitions and notations for partial derivatives.

For example, for functions of three variables  $f(x, y, z)$ , we fix two of the variables and differentiate with respect to the third one.

$$f_x, \quad f_y, \quad f_z, \quad \text{or} \quad \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}.$$

## 7.4 Chain Rule

Suppose the length  $\ell$ , width  $w$  and height  $h$  of a box change with time. At time  $t_0$ , the dimensions of the box are  $\ell = 2$  m,  $w = 3$  m,  $h = 4$  m, and  $\ell$  and  $w$  are increasing at a rate of  $5 \text{ ms}^{-1}$  while  $h$  is decreasing at a rate of  $6 \text{ ms}^{-1}$ . What is the rate of change of the volume of the box at time  $t_0$ ?

In the above problem, the volume  $V$  of the box is a function of three variables in  $\ell, w, h$ :

$$V = V(\ell, w, h)$$

while these three variables are in turn functions of time  $t$ :  $\ell = \ell(t)$ ,  $w = w(t)$ ,  $h = h(t)$ . Clearly, a change of  $t$  will cause a change of  $V$ .



We say that  $V$  is a *composite* function of  $t$  and write

$$V(t) = V(\ell(t), w(t), h(t)).$$

The rate of change of  $V$  is given by  $\frac{dV}{dt}$ .

Can we express  $\frac{dV}{dt}$  in terms of  $\frac{d\ell}{dt}$ ,  $\frac{dw}{dt}$ , and  $\frac{dh}{dt}$ ?

The answer is given by Chain rule.

### 7.4.1 Chain rule for one independent variable on $f(x, y)$

Suppose  $z = f(x, y)$  is a function of two variables  $x$  and  $y$ , and  $x = x(t)$ ,  $y = y(t)$  are both functions of  $t$ . Then  $z$  is a function of  $t$ :  $z(t) = f(x(t), y(t))$  and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

### 7.4.2 Example

Let  $z = 3xy^2 + x^4y$ , where  $x = \sin 2t$ ,  $y = \cos t$ .

Find  $\frac{dz}{dt}$ .

**Solution:**

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (3y^2 + 4x^3y)(2 \cos 2t) + (6xy + x^4)(-\sin t).\end{aligned}$$

### 7.4.3 Chain rule for two independent variables on $f(x, y)$

Suppose  $z = f(x, y)$  is a function of two variables  $x$  and  $y$ , and  $x = x(s, t)$ ,  $y = y(s, t)$  are both functions of two variables  $s$  and  $t$ . Then  $z$  is a function of  $s$  and  $t$ :  $z(s, t) = f(x(s, t), y(s, t))$  and

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

### 7.4.4 Example

Let  $z = e^{2x} \cos 3y$ , where  $x = st^2$ ,  $y = s^2t$ .

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\ &= (2e^{2x} \cos 3y)t^2 + (-3e^{2x} \sin 3y)(2st).\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\ &= (2e^{2x} \cos 3y)(2st) + (-3e^{2x} \sin 3y)(s^2).\end{aligned}$$

### 7.4.5 Chain rule for one independent variable on $f(x, y, z)$

Chain rules can be extended in a similar way for functions of three or more variables.

For example, suppose  $w = f(x, y, z)$  is a function of three variables  $x, y$  and  $z$ , and  $x = x(t)$ ,  $y = y(t)$ ,

$z = z(t)$  are functions of  $t$ . Then  $w$  is a function of  $t$ :  $w(t) = f(x(t), y(t), z(t))$  and we have

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$



### 7.4.6 Example

Back to the problem at the beginning of this section.

The volume

$$V(\ell, w, h) = \ell \times w \times h.$$

Hence

$$\frac{\partial V}{\partial \ell} = wh, \quad \frac{\partial V}{\partial w} = \ell h, \quad \frac{\partial V}{\partial h} = \ell w.$$

By chain rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \\ &= wh \frac{d\ell}{dt} + \ell h \frac{dw}{dt} + \ell w \frac{dh}{dt} \end{aligned}$$

We are given, at time  $t_0$ ,

$$\ell = 2 \text{ m}, \quad w = 3 \text{ m}, \quad h = 4 \text{ m},$$

$$\text{while } \frac{d\ell}{dt} = 5 \text{ ms}^{-1}, \quad \frac{dw}{dt} = 5 \text{ ms}^{-1} \text{ and } \frac{dh}{dt} = -6 \text{ ms}^{-1}.$$

Hence

$$\frac{dV}{dt} = (3)(4)(5) + (2)(4)(5) + (2)(3)(-6) = 64 \text{ m}^3\text{s}^{-1}.$$

### 7.4.7 Chain rule for two independent variables on $f(x, y, z)$

Now suppose  $w = f(x, y, z)$  is a function of three variables  $x$ ,  $y$  and  $z$ , and  $x = x(s, t)$ ,  $y = y(s, t)$ ,  $z = z(s, t)$  are functions of two variables  $s$  and  $t$ .

Then  $w$  is a function of  $s$  and  $t$ :

$w(s, t) = f(x(s, t), y(s, t), z(s, t))$  and we have

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s} \\ \frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}\end{aligned}$$

## 7.5 Directional Derivatives

### 7.5.1 Extension of Partial Derivatives

We have seen earlier that, given a function  $f(x, y)$ , the partial derivatives give the rates of change of  $f$  with respect to  $x$  and  $y$ , i.e. along the directions of  $x$ - and  $y$ -axes.

A natural question to ask is, what is the rate of change of  $f$  along an arbitrary direction? This gives rise to the notion of directional derivatives.

Let  $f$  be a function of  $x$  and  $y$ .

The **directional derivative** of  $f$  at  $(a, b)$  in the direction of a unit vector  $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$  is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

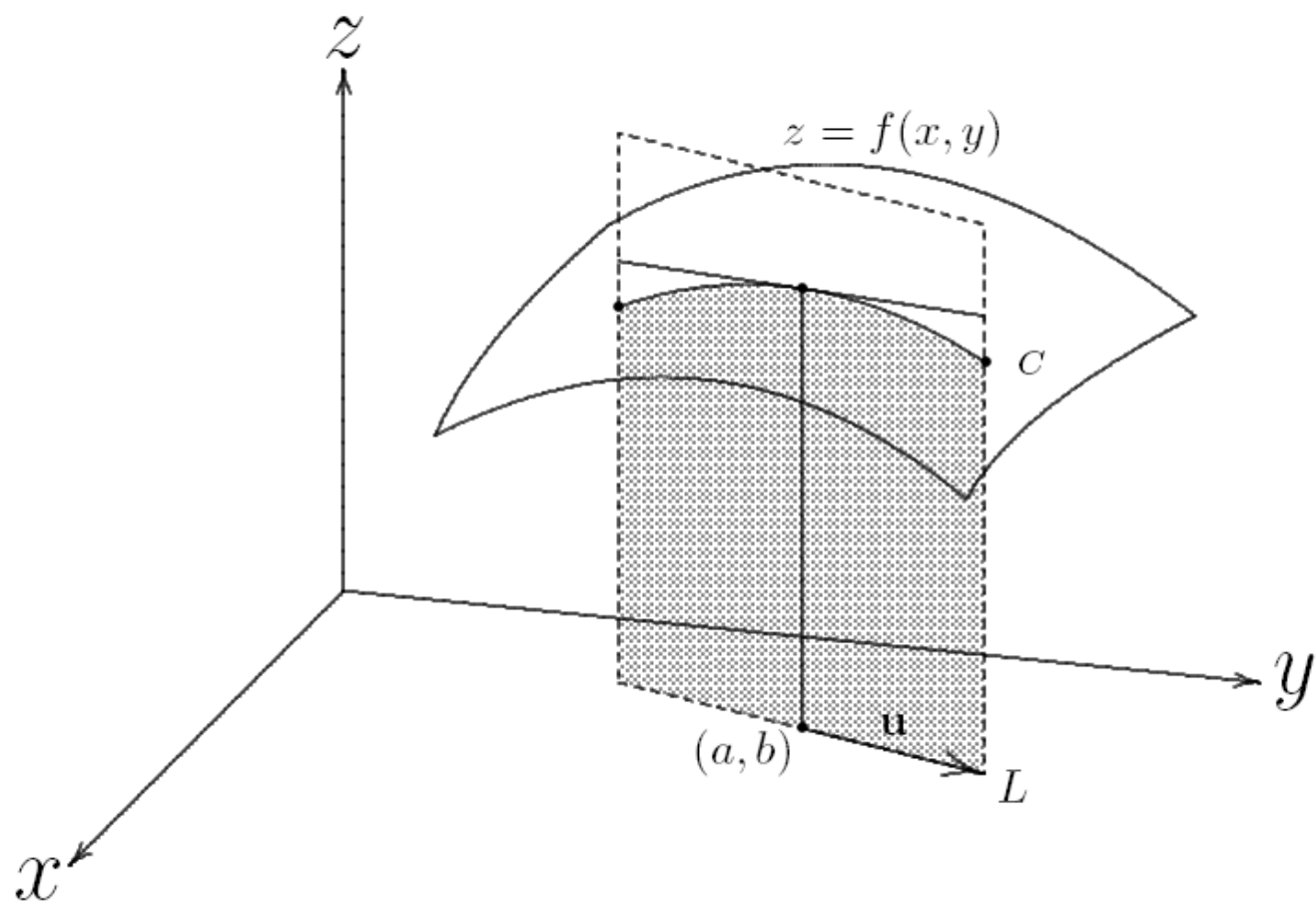
if this limit exists.



Note that  $D_{\mathbf{i}}f(a, b) = f_x(a, b)$  and  $D_{\mathbf{j}}f(a, b) = f_y(a, b)$ , where  $\mathbf{i}$  and  $\mathbf{j}$  are the standard unit vectors of the  $xy$ -plane.

## 7.5.2 Geometrical meaning

Let  $L$  be the line in the  $xy$ -plane passing through the point  $(a, b)$  and parallel to  $\mathbf{u}$ . Then  $L$  traces out a curve  $C$  on the surface represented by  $z = f(x, y)$  as shown in the diagram.



Then  $D_{\mathbf{u}}f(a, b)$  gives the gradient of the tangent line to the curve  $C$  at the point  $(a, b)$ .

### 7.5.3 A formula

Since  $D_{\mathbf{u}}f(a, b)$  is also the rate of change of  $f(x, y)$  at  $(a, b)$  in the direction of  $\mathbf{u}$ , and the coordinates  $x$  and  $y$  refer to points on the line  $L$ :

$$x = a + u_1t, \quad y = b + u_2t, \quad z = 0,$$

it follows that the chain rule can be used, as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} .$$

Thus,

$$D_{\mathbf{u}}f(a, b) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2.$$

### 7.5.4 Example

Let  $f(x, y) = x^2 - 3xy^2 + 2y^3$ . Find  $D_{\mathbf{u}}f(2, 1)$ , where

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

**Solution:** First,  $f_x = 2x - 3y^2$ ,  $f_y = -6xy + 6y^2$ .

Thus  $f_x(2, 1) = 1$  and  $f_y(2, 1) = -6$ .

Therefore,

$$D_{\mathbf{u}}f(2, 1) = (1)\left(\frac{\sqrt{3}}{2}\right) + (-6)\left(\frac{1}{2}\right) = \frac{\sqrt{3} - 6}{2}.$$

## Gradient Vector

In view of the expression of the directional derivative in terms of partial derivatives, it is convenient and useful to introduce the notion of a *gradient vector*.

The **gradient** of  $f(x, y)$  is the vector (function)

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j}.$$



For a given unit vector  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ , we obtain

$$\begin{aligned}\nabla f(a, b) \cdot \mathbf{u} \\&= (f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\&= f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2 \\&= D_{\mathbf{u}}f(a, b).\end{aligned}$$

Thus,

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Noting that  $\nabla f(a, b)$  and  $\mathbf{u}$  are vectors, let  $\theta$  be the angle ( $0 \leq \theta \leq \pi$ ) between them. Then

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = \|\nabla f(a, b)\| \cos \theta.$$

Since  $-1 \leq \cos \theta \leq 1$ , we obtain some useful properties of the above formula when  $\nabla f(a, b) \neq \mathbf{0}$ :

## **Facts.**

- (1) The function  $f$  increases most rapidly in the direction  $\nabla f(a, b)$ .
- (2) The function  $f$  decreases most rapidly in the direction  $-\nabla f(a, b)$ .

**Example.** Let  $f(x, y) = \sqrt{9 - x^2 - y^2}$ . Find the largest possible value of  $D_{\mathbf{u}}f(2, 1)$ .

First compute

$$f_x = \frac{-x}{\sqrt{9 - x^2 - y^2}} , \quad f_y = \frac{-y}{\sqrt{9 - x^2 - y^2}} .$$

$$\begin{aligned}\nabla f(2, 1) &= f_x(2, 1)\mathbf{i} + f_y(2, 1)\mathbf{j} \\ &= -\mathbf{i} - \frac{1}{2}\mathbf{j}.\end{aligned}$$

$$||\nabla f(2, 1)|| = \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}.$$

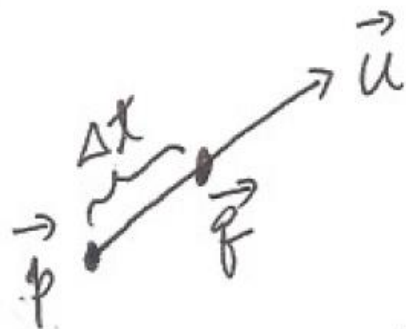
### 7.5.5 Physical meaning

As we mentioned at the beginning of this section, the directional derivative  $D_{\mathbf{u}}f(a, b)$  measures the change in the value  $df$  of a function  $f$  when we move a small distance  $dt$  from the point  $(a, b)$  in the direction of the vector  $\mathbf{u}$ :

$$df = D_{\mathbf{u}}f(a, b) \cdot dt.$$



Suppose the point  $\vec{p}$  moves a small distance  $\Delta t$  along a unit vector  $\vec{u}$  to a new position  $\vec{q}$ .



Then the increment in  $f$  has an approximation:

$$\Delta f \approx \left[ D_{\vec{u}} f(\vec{p}) \right] (\Delta t)$$

### 7.5.6 Example

Let  $f(x, y) = x^2y^3 + 1$ .

Estimate how much the value of  $f$  will change if a point  $Q$  moves 0.1 unit from  $(2, 1)$  towards  $(3, 0)$ .

**Solution:**  $Q$  moves in the direction  $(3 \mathbf{i} + 0 \mathbf{j}) - (2 \mathbf{i} + \mathbf{j}) = \mathbf{i} - \mathbf{j}$ .

The unit vector  $\mathbf{u}$  along this direction is  $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ .

Now  $f_x = 2xy^3$ ,  $f_y = 3x^2y^2$ .

Thus  $f_x(2, 1) = 4$  and  $f_y(2, 1) = 12$ .

Therefore,

$$D_{\mathbf{u}}f(2, 1) = (4)\left(\frac{1}{\sqrt{2}}\right) + (12)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{8}{\sqrt{2}}.$$

So

$$df = D_{\mathbf{u}}f(2, 1) \cdot dt = \left(-\frac{8}{\sqrt{2}}\right)(0.1) \approx -0.57.$$

So the value of  $f$  decreases by approximately 0.57 unit.

### 7.5.7 Functions of Three Variables

We can also define directional derivatives for functions of three variables.

Let  $f$  be a function of  $x$ ,  $y$  and  $z$ . The directional derivative of  $f$  at  $(a, b, c)$  in the direction of a unit

vector  $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$  in the  $xyz$  space is

$$D_{\mathbf{u}}f(a, b, c)$$

$$= \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h}$$

if this limit exists.

Similarly, we have the formula

$$D_{\mathbf{u}}f(a, b, c) = f_x(a, b, c)u_1 + f_y(a, b, c)u_2 + f_z(a, b, c)u_3$$

## 7.6 Maximum and Minimum Values

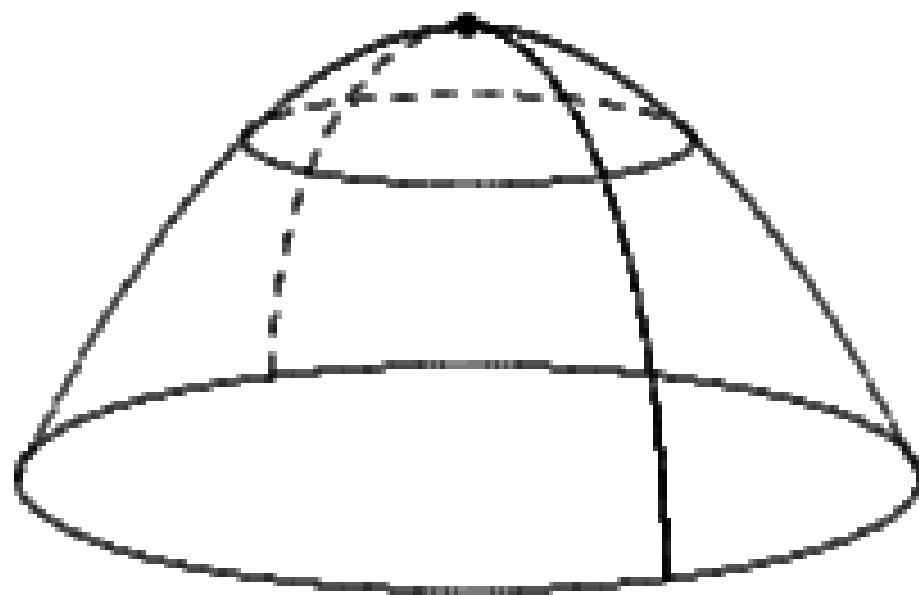
### 7.6.1 Local maximum and minimum

(1)  $f(x, y)$  has a **local maximum** at  $(a, b)$  if

$$f(x, y) \leq f(a, b) \quad \text{for all points } (x, y) \text{ near } (a, b).$$

The number  $f(a, b)$  is called a **local maximum value**.

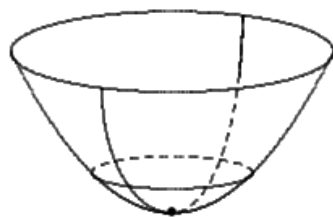




(2)  $f(x, y)$  has a **local minimum** at  $(a, b)$  if

$f(x, y) \geq f(a, b)$  for all points  $(x, y)$  near  $(a, b)$ .

The number  $f(a, b)$  is called a **local minimum value**.



## 7.6.2 Critical Points

A function  $f$  may have a local maximum or minimum at  $(a, b)$  if

- (i)  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ ; or
- (ii)  $f_x(a, b)$  or  $f_y(a, b)$  does not exist.

A point of  $f$  that satisfies (i) or (ii) above is called a *critical point*.

### 7.6.3 Example

Let  $f(x, y) = x^2 + y^2 + 4x - 8y + 24$ . Find the local maxima and local minima of  $f$ , if any.

**Solution:** The partial derivatives exist for any point.

So we solve  $f_x(x, y) = 0$  and  $f_y(x, y) = 0$ .

$$f_x = 2x + 4 = 0 \Rightarrow x = -2$$

$$f_y = 2y - 8 = 0 \Rightarrow y = 4.$$

This gives a solution  $(x, y) = (-2, 4)$ .

Is this point a local maximum, a local minimum, or none?

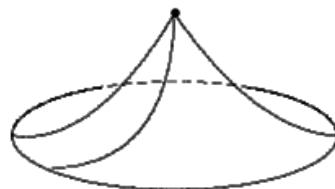
By completing squares, we have

$$f(x, y) = 4 + (x + 2)^2 + (y - 4)^2 \geq 4.$$

Therefore, we conclude that  $(-2, 4)$  is a local minimum of  $f$  with minimum value 4.

### 7.6.4 Example

The following diagram shows the graph of a function  $f(x, y)$  which has a local maximum at a point  $(a, b)$  but  $f_x(a, b)$  and  $f_y(a, b)$  does not exist.



### 7.6.5 Saddle points

Let  $(a, b)$  be a point of  $f$  with  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ . We say  $(a, b)$  is a **saddle point** of  $f$  if there are some directions along which  $f$  has a local maximum at  $(a, b)$  and some directions along which  $f$  has a local minimum at  $(a, b)$ .



### 7.6.6 Example

Find the local maximum or local minimum of  $f(x, y) = 2y^2 - 3x^2$ , if any.

**Solution:** As before, we solve

$$f_x = -6x = 0 \quad \text{and} \quad f_y = 4y = 0.$$

The only solution is  $(x, y) = (0, 0)$ . So this is the only critical point of  $f$ .

However, this point  $(0, 0)$  is neither a local maximum nor a local minimum of  $f$ .

To see this, consider the function  $f$  along the  $x$ -axis which has equation  $y = 0$ . Substituting this equation into  $f(x, y)$ , we have

$$f(x, 0) = -3x^2 \leq 0.$$

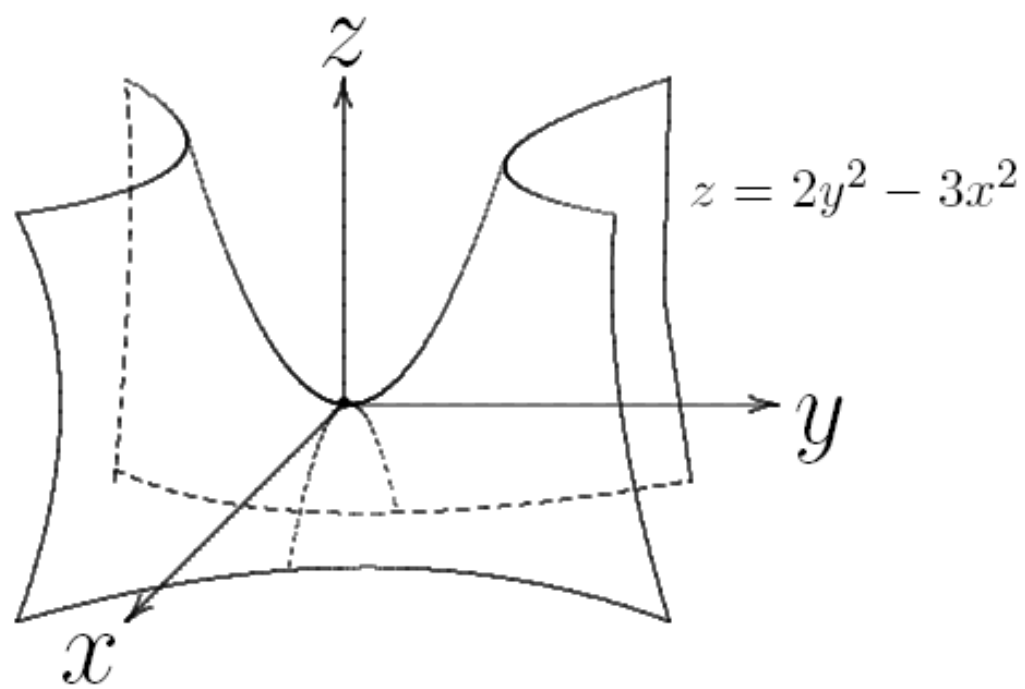
So along  $x$ -axis,  $f$  has a local maximum at  $(0, 0)$ .

On the other hand, if we consider  $f$  along the  $y$ -axis which has equation  $x = 0$ , we have

$$f(0, y) = 2y^2 > 0.$$

So along  $y$ -axis,  $f$  has a local minimum at  $(0, 0)$ .

Therefore  $f$  has a saddle point at  $(0, 0)$ .



## 7.6.7 Second Derivative Test

When the partial derivatives exist, we can determine the type of critical point using the following systematic approach:

Let  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ .

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.$$

(a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f$  has a local minimum at  $(a, b)$ .

(b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f$  has a local maximum at  $(a, b)$ .

(c) If  $D < 0$ , then  $f$  has a saddle point at  $(a, b)$ .

(d) If  $D = 0$ , then no conclusion can be drawn.

### 7.6.8 Example

Find the local maximum, local minimum and saddle points (if any) of  $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ .



$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$$

$$f_x = 3x^2 + 6x = 0 \Rightarrow x=0, x=-2$$

$$f_y = 3y^2 - 6y = 0 \Rightarrow y=0, y=2.$$

$(0, 0), (0, 2), (-2, 0), (-2, 2)$  are all the critical points.

$$f_{xx} = 6x + 6$$

$$f_{xy} = 0$$

$$f_{yx} = 0$$

$$f_{yy} = 6y - 6$$

critical pts	$f_{xx}$	$f_{xy} = f_{yx}$	$f_{yy}$	$f_{xx}f_{yy} - f_{xy}^2$	conclusion
$(0, 0)$	6	0	-6	-36	saddle.
$(0, 2)$	6	0	6	+36	loc. min.
$(-2, 0)$	-6	0	-6	+36	loc. max.
$(-2, 2)$	-6	0	6	-36	saddle.

# More Examples

# Example

Find and classify all the critical points of

$$f(x, y) = 4xy - 2x^2 - y^4 - 81.$$

$$f_x = 0 \Rightarrow 4y - 4x = 0 \Rightarrow x = y \text{ ---- } (1)$$

$$f_y = 0 \Rightarrow 4x - 4y^3 = 0 \Rightarrow x = y^3 \text{ ---- } (2)$$

$$(1) \& (2) \Rightarrow y^3 - y = 0 \Rightarrow y = -1, 0, 1.$$

$\therefore (-1, -1), (0, 0), (1, 1)$  are the critical points.

$$f_{xx} = -4, \quad f_{xy} = 4, \quad f_{yy} = -12y^2$$

critical point	$f_{xx}$	$f_{yy}$	$f_{xy}$	$f_{xx}f_{yy} - (f_{xy})^2$
$(-1, -1)$	$-4$	$-12$	$4$	$+$
$(0, 0)$	$-4$	$0$	$4$	$-$
$(1, 1)$	$-4$	$-12$	$4$	$+$

$\therefore (-1, -1)$  and  $(1, 1)$  are local maximums.

$(0, 0)$  is a saddle point.

# Example

Let  $f(x, y, z)$  be a differentiable function of three variables and

$$f(1, 2, 3) = 2011, \quad \frac{\partial f}{\partial x}(1, 2, 3) = -2, \quad \frac{\partial f}{\partial y}(1, 2, 3) = 3.$$

It was found that if the point  $Q$  moved from  $(1, 2, 3)$  a distance 0.2 unit towards  $(3, 6, 7)$ , the value of  $f$  became 2012. Estimate the value of  $\frac{\partial f}{\partial z}(1, 2, 3)$ .



$$\text{Let } \frac{\partial f}{\partial z}(1, 2, 3) = a$$

$$\vec{u} = \frac{(3, 6, 7) - (1, 2, 3)}{\|(3, 6, 7) - (1, 2, 3)\|} = \frac{(2, 4, 4)}{6} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$0.2\left(\frac{1}{3}(-2) + \frac{2}{3}(3) + \frac{2}{3}a\right) \approx 2012 - 2011 = 1$$

$$-2 + 6 + 2a \approx 15$$

$$a \approx \frac{11}{2} = \underline{\underline{5.5}}$$

# Example

Let  $f$  be the function given by

$$f(x, y, z) = x^2y + z^2.$$

- (i) Find the gradient of  $f$  at the point  $(1, 2, 3)$ . Give **exact values** for your answer.
- (ii) Find the directional derivative of  $f$  at the point  $(1, 2, 3)$  in the direction of the vector that joins the point  $(1, 2, 3)$  to the point  $(-1, 1, 5)$ . Give **exact value** for your answer.

$$(i) \quad \nabla f = 2xy \vec{i} + x^2 \vec{j} + 2z \vec{k}$$

$$\nabla f(1, 2, 3) = \underline{\underline{4 \vec{i} + \vec{j} + 6 \vec{k}}}$$

$$(ii) \quad \vec{u} = \frac{(-1, 1, 5) - (1, 2, 3)}{\|(-1, 1, 5) - (1, 2, 3)\|} = \frac{(-2, -1, 2)}{3} = -\frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$D_{\vec{u}} f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \vec{u}$$

$$= -\frac{8}{3} - \frac{1}{3} + 4 = \underline{\underline{1}}$$

# Example

Suppose  $f(x, y)$  has continuous partial derivatives of all orders. If

$$\nabla f = (xy^2 + kx^2y + x^3) \mathbf{i} + (x^3 + x^2y + y^2) \mathbf{j},$$

find the value of the constant  $k$ .

$$\frac{\partial}{\partial y} (xy^2 + kx^2y + x^3) = \frac{\partial}{\partial x} (x^3 + x^2y + y^2)$$

$$\Rightarrow 2xy + kx^2 = 3x^2 + 2xy$$

$$\Rightarrow \underline{\underline{k = 3.}}$$