Review of Section 1.1 - Section 1.4

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Linear Equation

- (Linear Equation). A linear equation in n variable $x_1, \cdots, x_n$ has the form
$$a_1x_1 + \cdots + a_nx_n = b. \tag{1}$$
- Kown how to identify what it means to be linear.
- n number $s_1, \cdots, s_n$ is called a solution to the above linear equation if (1) satisfied when we substitute the values into the equation accordingly.
- (Geometrical Interpretation).
  - In xy-plane, $ax + by = c$(a,b not all zeros) is a line on the plane.
  - In xyz-space, $ax + by + cz = d$(a,b,c not all zeros) is a plane in the space.
  - See Q3 (a) and (b).

Linear System

- A finite set of linear equations in n variable $x_1, \cdots, x_n$ is called a linear system.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

- n number $s_1, \cdots, s_n$ is called a solution to the above linear system if 'it' is a solution to every linear equation in this system.
- When $m = 2, n = 2$, two lines in xy-plane; no solutions, one solution and infinitely many solutions.
- When $m = 2, n = 3$, two planes in xyz-space; no solutions and infinitely many solutions (two cases: Intersection to form a line or two planes are equal). See Q3 (c) and (d).
- See Q2 and Q5.

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Augmented Matrix and Elementary Row Operations

A linear system of the form (2) can be represented by a rectangular array of numbers which we call it the augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right). \tag{3}$$

Three basic techniques for solving a linear system (also called elementary row operations):

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

Row Equivalent Augmented Matrix

Two augmented matrices are said to be row requivalent if one can be obtained from the other by a series of elementary row operations.

Theorem (1.2.7, Page 8)

If augmented matrices of two systems of linear equations are row equivalent, then the two systems have the same set of solutions.

To prove the theorem, we only need to check that every elementary row operation applied to an augmented matrix will not change the solution set of the corresponding linear system. **(Noted that, we have proven the theorem in Q5.)**

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(Reduced) Row-Echelon Forms

An augmented matrix is said to be in **row-echelon form** if it has the following porperties (1) and (2):

1. Rows that consist of entirely of zeros are grouped together at the bottom of the matrix.
2. In any two successive rows that do not consist entirely of zores, the first nonzeor number (which is called the leading entry or pivot point) in the lower row occurs farther to the right than the the first nonzeros number in the higher row.
   Furthermore, if (3) and (4) hold,
3. The leading number of every nonzero row is 1;
4. In each pivot column, except the pivot point, all other entries are zero.
   Then we called this augmented matrix in **reduced row-echelon form**.

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Solve the Linear System

As we have seen in Theorem 1.2.7 (Page 8), the row operations do not changed the solutions to the system. Thus, by solving the system corresponding to the augmented matrix in a row-echelon form or the reduced row-echelon form, which is much easier, we can find the solution to the original system easily.

In order to reduce the augmented matrix to a row-echelon form or the reduced row-echelon form, we need to use Gaussian Elimination or Guass-Jordan Elimination.

Guassian Elimination (Algorithm 1.4.2)

The following procedures reduce an any augmented matrix to a row-echelon form by using elementary row operations.

- (Step 1). Locate the leftmost column that does not consist entirely of zeros.
- (Step 2). Interchange the top row with another row, if necessary, to bring a nonzeros entry to the top of the column found in Step 1.
- (Step 3). For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.
- (Step 4). Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until entire matrix is in row-echelon form.

Gauss-Jordan Elimination (Algorithm 1.4.3)

Given an augmented matrix, use Algorithm 1.4.2 to reduce it to a row-echelon form. Then follow the follwing procedures to reduce it to a reduced row-echelon form.

- (Step 5). Multiply a suitable constant to each row so that all the leading entries becomes 1.
- (Step 6). Begining with the last nonzero row and working upward, add suitable multiples of each row to the rows above introduce zeros above the leading entries.

Question:

- Are these two algorithms always stop after finite number of elementary row operations?
- Is row-echelon form of an augmented matrix unique? How about the reduced row-echelon form?

Useful Notation

When doing elemetary row operations, we adopt the following notations:

1. kR_i means "multiply the ith row by the constant k".
2. $R_i \leftrightarrow R_j$ means "interchange the ith and jth rows".
3. $R_j + kR_i$ means "add k times of the ith row to the jth row".

We will use the above algorithms and notations to solve Q4 (c), while other questions in Q4 can be solved in a similar way.