**National University of Singapore**
**Department of Mathematics**

**Semester 1, 2018/2019** **MA1101R Linear Algebra I** **Homework 1**

## Instruction

(a) This homework set consists of 3 pages and 10 questions.

(b) Do all the problems and submit on Sept. 3 (Monday) for SL1 group or on Sept. 4 (Tuesday) for SL2 group during lecture.

(c) Use A4 size writing paper. Write your full name, student number and tutorial group clearly on the first page of your answer scripts.

(d) Indicate the question numbers clearly (you do not need to copy the questions in your answer sheets).

(e) Show your steps of your working how the answers are derived, unless the questions state otherwise.

(f) Late Submission will not be accepted.

(g) **Warning**: If you are found to have copied answers from your friend(s), both you and your friend(s) will be penalized.

## Problem Set (covering Lectures 1–4).

1. (a) Find a linear equation in the variables $x, y$ and $z$ that has the general solution

$$\begin{cases} x = t \\ y = 2 - t + 3s \\ z = s \end{cases}$$

where $t, s$ are parameters.

**Answer.** Substituting $t = x$ and $s = t$ into $y = 2 - t + 3s$, we obtain a linear equation in the variables $x, y$ and $z$, that is,

$$x + y - 3z = 2.$$

(b) Express the general solution of the equation in part (a) in two other different ways.

**Answer.** **1st way.** We may choose $x = t$ and $y = s$ as arbitrary parameters and solve for $z$, which equals $\frac{1}{3}(2 - t - s)$. Then we obtain a general solution

$$\begin{cases} x = t \\ y = s \\ z = \frac{1}{3}(-2 + t + s) \end{cases}$$

**2nd way.** We may choose $y = t$ and $z = s$ as arbitrary parameters and solve for $x$, which equals $2 - t + 3s$. Then we obtain a general solution

$$\begin{cases} x = 2 - t + 3s \\ y = t \\ z = s \end{cases}$$

(c) Write down a linear system with two equations such that the system has the same general solution as in part (a).

**Answer.** Note that the linear equation in part (a) is not unique. By multiplying an arbitrary nonzero constant, we may obtain a new linear equation whose solution is the same. For instance, $2x + 2y - 6z = 4$. Now, we may align them together and get a desired linear system with two equations whose solution set is the same:

$$\begin{cases} x + y - 3z = 2 \\ 2x + 2y - 6z = 4 \end{cases}$$

2. A row-echelon form of the augmented matrix for a linear system is given by

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{array}\right)$$

(i) Find a general solution of the linear system.
(Use $x_1, x_2, x_3, x_4, x_5$ for the variables of the system.)

**Answer.** We use the back-substitution method to write down a general solution. In this row-echelon form, we have 5 unknowns and 3 pivot columns, so we need 2 arbitrary parameters in a general solution.
By the 3rd row, we have $\frac{1}{2}x_5 = 1$ and solve for $x_5$, that is, $x_5 = 2$. Substitute $x_5 = 2$ into the 2nd row, which is $x_3 - \frac{7}{2}x_5 = -6$. Solve for $x_3 = 1$. To solve for $x_1$, we need arbitrary parameters for $x_2$ and $x_4$ respectively. Let $x_2 = s$ and $x_4 = t$. Substitute $x_5 = 2$, $x_4 = t$, $x_3 = 1$, and $x_2 = s$ into $x_1 + 2x_2 - 5x_3 + 3x_4 + 6x_5 = 14$. We may get $x_1 = 7 - 3t - 2s$. In sum

$$\begin{cases} x_1 = 7 - 3t - 2s \\ x_2 = s \\ x_3 = 1 \\ x_4 = t \\ x_5 = 2 \end{cases}$$

(ii) Find a specific solution of the linear system with $x_2 = 1$ and $x_4 = 0$.

**Answer.** Plugging $x_2 = 1$ and $x_4 = 0$ into the general solution in part (i), which means $t = 0$ and $s = 1$, we have

$$\begin{cases} x_1 = 5 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 0 \\ x_5 = 2 \end{cases}$$

3. (a) Use Gauss-Jordan elimination to reduce the augmented matrix of the following linear system to reduced row-echelon form. Write down the elementary row operation that you use in each step clearly.

$$\begin{cases} \quad\quad -2y + 3z = 1 \\ 3x + 6y - 3z = -2 \\ 6x + 6y + 3z = -1 \end{cases}$$

**Answer.** First, we write down its augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & -1 \end{array}\right)$$

Next, let us perform Gauss elimination:

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & -1 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 3 \end{array}\right)$$

$$\xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then we may continue the Gauss-Jordan elimination:

$$\left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2/(-2)]{R_1/3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

(b) Write down a general solution of the linear system in part (a).

**Answer.** Substitute $z$ with parameter: $z = t$. Then the general solution is given by

$$\begin{cases} x = \frac{1}{3} - 2t \\ y = -\frac{1}{2} + \frac{3}{2}t \\ z = t \end{cases}$$

4. Consider

$$(a)\ \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (b)\ \left(\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$(c)\ \left(\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad (d)\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 5 \end{array}\right).$$

(a) In each part determine whether the matrix is in row-echelon form, reduced row-echelon form, both, or neither.

**Answer.** (a), (b), and (c) are in row-echelon from but not in reduced row-echelon form.

(d) is not in row-echelon from, so it is not in reduced row-echelon form.

(b) For each part, indicate whether the corresponding linear system is consistent. If the system has a unique solution, find it.

**Answer.** (a) is inconsistent; (b), (c) and (d) are consistent and have a unique solution.

$$(b)\ \begin{cases} x &= 4 \\ y &= 4 \\ z &= 1 \end{cases} \qquad (c)\ \begin{cases} x &= 4 \\ y &= 2 \\ z &= 2 \end{cases} \qquad (d)\ \begin{cases} x &= 4 \\ y &= 5 \\ z &= 3 \end{cases}$$

5. Consider the following linear system for $x_1$, $x_2$ and $x_3$, where $a$ and $b$ are constants.

$$\begin{cases} x_1 + x_2 + 3x_3 = 2 \\ x_1 + 2x_2 + 4x_3 = 3 \\ x_1 + 3x_2 + ax_3 = b \end{cases}$$

(a) Write down its augmented matrix.

**Answer.**

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right).$$

(b) Use Gaussian elimination to reduce the augmented matrix to a row echelon form.

**Answer.**

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right)$$

(c) Write down the condition in terms of $a$ and $b$ for the system to be inconsistent (i.e. have no solution).

**Answer.** The system is inconsistent if and only if $a - 5 = 0$ and $b - 4 \neq 0$, that is, $a = 5$ and $b \neq 4$.

(d) Write down the condition in terms of $a$ and $b$ for the system to have infinitely many solutions.

**Answer.** The system has infinitely many solutions if and only if $a - 5 = 0$ and $b - 4 = 0$, that is, $a = 5$ and $b = 4$.

(e) Is it possible for the above system to have exactly one solution? Why?

**Answer.** Yes. The system has exactly one solution if and only if $a - 5 \neq 0$, that is, $a \neq 5$.

6. Consider a linear system for $x_1$, $x_2$ and $x_3$

$$\begin{cases} 2x_1 - x_2 = \lambda x_1 \\ 2x_1 - x_2 + x_3 = \lambda x_2 \\ -2x_1 + 2x_2 + x_3 = \lambda x_3 \end{cases}$$

(a) Is it possible for the system to be inconsistent? Explain.

**Answer.** First, we write down its augmented matrix:

$$\left(\begin{array}{ccc|c} 2-\lambda & -1 & 0 & 0 \\ 2 & -1-\lambda & 1 & 0 \\ -2 & 2 & 1-\lambda & 0 \end{array}\right).$$

Since this linear system is homogeneous, it always has the trivial solution. Thus, it is impossible to be inconsistent.

(b) For what values of $\lambda$ will the system have infinitely many solutions?

**Answer.** Let us perform Gauss Elimination:

$$\left(\begin{array}{ccc|c} 2-\lambda & -1 & 0 & 0 \\ 2 & -1-\lambda & 1 & 0 \\ -2 & 2 & 1-\lambda & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 2 & -1-\lambda & 1 & 0 \\ 2-\lambda & -1 & 0 & 0 \\ -2 & 2 & 1-\lambda & 0 \end{array}\right)$$

$$\xrightarrow[R_3+R_1]{R_2-\frac{2-\lambda}{2}R_1} \left(\begin{array}{ccc|c} 2 & -1-\lambda & 1 & 0 \\ 0 & -\frac{1}{2}(\lambda^2-\lambda) & \frac{\lambda-2}{2} & 0 \\ 0 & 1-\lambda & 2-\lambda & 0 \end{array}\right).$$

Now, we consider two cases: $\lambda = 1$ or $\lambda \neq 1$.

Case 1 $\lambda = 1$. We plug in $\lambda = 1$ and the augmented matrix becomes

$$\left(\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3+2R_2} \left(\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Under the assumption $\lambda = 1$, the system has infinitely many solutions.

Case 2 $\lambda \neq 1$. We may continue the Gauss elimination:

$$\left(\begin{array}{ccc|c} 2 & -1-\lambda & 1 & 0 \\ 0 & -\frac{1}{2}(\lambda^2-\lambda) & \frac{\lambda-2}{2} & 0 \\ 0 & 1-\lambda & 2-\lambda & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 2 & -1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2-\lambda & 0 \\ 0 & -\frac{1}{2}(\lambda^2-\lambda) & \frac{\lambda-2}{2} & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{\lambda}{2}R_3} \left(\begin{array}{ccc|c} 2 & -1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2-\lambda & 0 \\ 0 & 0 & \frac{(\lambda-2)(\lambda+1)}{2} & 0 \end{array}\right).$$

The system has infinitely many solutions if and only if $\frac{(\lambda-2)(\lambda+1)}{2} = 0$, that is, $\lambda = 2$ or $\lambda = -1$, which still satisfies the condition $\lambda \neq 1$.

Overall, the system has infinitely many solutions if and only if

$$\lambda = -1, \qquad \lambda = 2, \qquad \text{or } \lambda = 1.$$

(c) Solve the system when it has infinitely many solutions.

**Answer.** Following part (b), if $\lambda = -1$, its row-echelon form

$$\left(\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ implies } \begin{cases} x_1 &= t \\ x_2 &= t \\ x_3 &= 0 \end{cases};$$

If $\lambda = 2$, its row-echelon form

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ implies } \begin{cases} x_1 &= -\frac{t}{2} \\ x_2 &= 0 \\ x_3 &= t \end{cases};$$

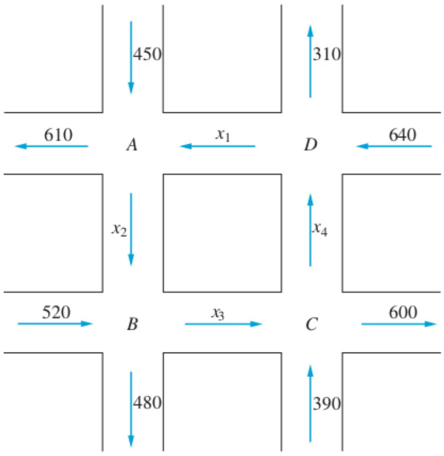
If $\lambda = 1$, its row-echelon form

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ implies } \begin{cases} x_1 &= -\frac{t}{2} \\ x_2 &= -\frac{3t}{2} \\ x_3 &= t \end{cases}.$$

7. In the downtown section of a certain city, two sets of one-way streets intersect as shown in the following. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram. Determine the amount of traffic between each of the four intersections.

(Hint: Set up a linear system with four equations and four variables. If the system has infinitely many solutions, please write a general solution.)

**Answer.** First, let us set up a linear system. In each intersection, the sum of traffic entering equals the sum of traffic leaving, which gives us a linear equation. Here are




four intersections, so we may have 4 linear equations in variables $x_1$, $x_2$, $x_3$ and $x-4$:

$$\begin{cases} 450 + x_1 = 610 + x_2 & \text{Intersection A} \\ 520 + x_1 = 480 + x_2 & \text{Intersection B} \\ 390 + x_1 = 600 + x_2 & \text{Intersection C} \\ 640 + x_1 = 310 + x_2 & \text{Intersection D} \end{cases}$$

Secondly, we write down its augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right).$$

Thirdly, let us perform Gauss-Jordan elimination:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right) \xrightarrow{R_4+R_1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & -1 & 0 & 1 & -170 \end{array}\right)$$

$$\xrightarrow{R_4+R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & -1 & 1 & -210 \end{array}\right) \xrightarrow{R_4+R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Finally, we obtain

$$\begin{cases} x_1 = 330 + t \\ x_2 = 170 + t \\ x_3 = 210 + t \\ x_4 = t \end{cases}$$

The system has infinitely many solutions.

Remark. In this question, the given average volumes may not determine the values of $x_1$, $x_2$, $x_3$, and $x_4$. For instance, it is possible that there are a certain number of cars which driving around the middle block. The given volumes can not determine how many such cars.

8. Solve the following system of nonlinear equations for the unknown angles $\alpha$, $\beta$, and $\gamma$, where $\alpha \in [0, 2\pi]$, $\beta \in [0, 2\pi]$ and $\gamma \in [0, \pi)$.

$$\begin{cases} 2\sin\alpha & - & \cos\beta & + & 3\tan\gamma & = & 3 \\ 4\sin\alpha & + & 2\cos\beta & - & 2\tan\gamma & = & 2 \\ 6\sin\alpha & - & 3\cos\beta & + & \tan\gamma & = & 9 \end{cases}$$

(Hint: There is no need to rewrite the equations before solving the system.)

**Answer.** Using Gauss-Jordan elimination, we may obtain the reduced row-echelon form of the augmented matrix as follows:

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right) \xrightarrow{GJE} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Solving the linear system, we have

$$\begin{cases} \sin\alpha & = & 1 \\ \cos\beta & = & -1 \\ \tan\gamma & = & 0 \end{cases} \Rightarrow \begin{cases} \alpha & = & \frac{\pi}{2} \\ \beta & = & \pi \\ \gamma & = & 0 \end{cases}$$

9. Show that if $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, then $\boldsymbol{Ax} = \boldsymbol{b}$ has at most one solution.

(Hint: Use *Proof by Contradiction*; Or consider the row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{0})$ and $(\boldsymbol{A} \mid \boldsymbol{b})$, and prove the statement directly.)

**Answer.** Suppose that $\boldsymbol{Ax} = \boldsymbol{b}$ has more than one solution. Let us assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are two distinct solutions of $\boldsymbol{Ax} = \boldsymbol{b}$. That is, $\boldsymbol{Ax_1} = \boldsymbol{b}$, $\boldsymbol{Ax_2} = \boldsymbol{b}$ and $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$.

Let us consider $\boldsymbol{Ax_1} - \boldsymbol{Ax_2}$. By the distributive law, $\boldsymbol{Ax_1} - \boldsymbol{Ax_2} = \boldsymbol{A(x_1 - x_2)}$. Since $\boldsymbol{Ax_1} = \boldsymbol{Ax_2} = \boldsymbol{b}$, we have $\boldsymbol{Ax_1} - \boldsymbol{Ax_1} = \boldsymbol{0}$. Hence $\boldsymbol{A(x_1 - x_2)} = \boldsymbol{0}$, that is, $x_1 - x_2$ is a solution of $\boldsymbol{Ax} = \boldsymbol{0}$. However, $x_1 - x_2 \neq \boldsymbol{0}$ contradicts with that $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

Therefore, $\boldsymbol{Ax} = \boldsymbol{b}$ has at most one solution.

10. If

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix}$$

evaluate the following:

$$(a)\ 2A - 3B \qquad (b)\ AB \qquad (c)\ BA.$$

**Answer.**

$$2A{-}3B = \begin{pmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}, \quad BA = \begin{pmatrix} 5 & 5 & 8 \\ -10 & -1 & -9 \\ 15 & 4 & 6 \end{pmatrix}.$$