# Review 2.3 - 2.5

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September 8, 2018

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# Definition of Inverse Matrix

- To solve $ax = b$ we know if $a \neq 0$ then $x = a^{-1}b$.
- How about $AX = B$? (Matrix Case).
- We want $X = A^{-1}B$. So we need to define $A^{-1}$.

## Definition

Let $A$ be a square matrix of order $n$. Then $A$ is said to be invertible if there exists a square matrix $B$ of order $n$ such that

$$AB = BA = I.$$

Such a matrix $B$ is called an inverse of $A$. A square matrix is called *singular* if it has no inverse.

# Cancellation Law for Matrices

- If $A$ is invertible matrix of order $m$, then

$$AB_1 = AB_2 \quad \Rightarrow \quad B_1 = B_2,$$

  and

$$C_1A = C_2A \quad \Rightarrow \quad C_1 = C_2,$$

  if the matrix multiplications are defined.
- If $A$ is sigular, the cancellation law may not hold.

# Uniqueness of Inverse

## Theorem

*If $B$ and $C$ are inverses of a square matrix $A$, then $B = C$.*

## Proof.

$B$ is inverse of $A$, we have

$$BA = I, \quad AB = I.$$

$C$ is inverse of $A$, we have

$$CA = I, \quad AC = I.$$

So

$$AB = I \quad \Rightarrow \quad CAB = CI \quad \Rightarrow \quad IB = C \quad \Rightarrow \quad B = C.$$

$\square$

# Properties of Inverse Matrix

**Theorem**

*Let $A, B$ be two invertible matrices of the same size and $c$ a nonzero scalar. Then*

1. $cA$ *is invertible and* $(cA)^{-1} = \frac{1}{c}A^{-1}$.
2. $A^T$ *is invertible and* $(A^T)^{-1} = (A^{-1})^T$.
3. $A^{-1}$ *is invertible and* $(A^{-1})^{-1} = A$.
4. $AB$ *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

See Q2.

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# Multiply the $i$th row of $A$ by $k$

- Let $A$ be a $m \times n$ matrix, and define the matrix $E_i(k)$ of order $m$ as

$$E_i(k) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & k & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{1}$$

- Multiply the $i$th row of $A$ by $k \Leftrightarrow$ pre-multiply $E$ to $A$, which is $E_i(k)A$.
- $E_i(k)$ is invertible, the inverse of $E_i(k)$ is simply replace $k$ in (1) by $\frac{1}{k}$, which is $E_i(\frac{1}{k})$.

# Interchange the $i$th row and the $j$th row

- Let $A$ be a $m \times n$ matrix, and define the matrix $E_{ij}$ of order $m$ as

$$E_{ij} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{2}$$

- Interchange the $i$th row and the $j$th row $\Leftrightarrow$ pre-multiply $E_{ij}$ to $A$, which is $E_{ij}A$.
- The matrix $E_{ij}$ is invertible and $E_{ij}^{-1} = E_{ij}$.

# Add $k$ times of $i$th row to the $j$th row $(i < j)$

- Let $A$ be a $m \times n$ matrix, and define the matrix $E_{ij}(k)$ of order $m$ as

$$E_{i,j}(k) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{3}$$

- Add $k$ times of $i$th row to the $j$th row $(i < j)$ $\Leftrightarrow$ pre-multiply $E_{ij}(k)$ to $A$, which is $E_{ij}(k)A$.
- The matrix $E_{ij}(k)$ is invertible and $E_{ij}(k)^{-1} = E_{ij}(-k)$.

# Elementary Matrices

We can find that, $E_i(k), E_{i,j}$ and $E_{ij}(k)$ can be obtained by doing the corresponding elementary row operations to the identity matrix.

**Definition**

A square matrix is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Note: All elementary matrices are invertible and their inverse are also elementary matrices (as indicated by the above).

# Part of Main Theorem on Inverse Matrices

**Thinking**: If $A$ is invertible then for $Ax = b$ we have $x = A^{-1}b$. Since the inverse of matrix $A$ is unique, so $x = A^{-1}b$ is also unique. Now look at the reduced row-echelon form of this linear system:

$$(A|b) \rightarrow \left(\begin{array}{ccc|c} 1 & & 0 & * \\ & \ddots & & \vdots \\ 0 & & 1 & * \end{array}\right)$$

## Theorem

*If $A$ is a square matrix, then the following statements are equivalent:*

1. *$A$ is invertible.*
2. *The linear system $Ax = 0$ has only the trivial solution.*
3. *The reduced row-echelon form of $A$ is an identity matrix.*
4. *$A$ can be expressed as a product of elementary matrices.*

# Practical Method for Computing the Inverse of a Matrix

- Let $A$ be an invertible matrix of order $n$. They by the above theorem, we can perform elementary row operations to reduced $A$ to its reduced row-echelon form, which is the identity matrix. Which says that there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = I.$$

- Post-multiply $A^{-1}$ on the both sides of the above, we get

$$E_k E_{k-1} \cdots E_2 E_1 A A^{-1} = I A^{-1} = A^{-1} \quad \Rightarrow \quad E_k E_{k-1} \cdots E_2 E_1 = A^{-1}.$$

- So we only need to perform elementary row operations to $(A|I)$, then we can get the inverse matrix.

$$E_k E_{k-1} \cdots E_2 E_1 (A|I) = (E_k E_{k-1} \cdots E_2 E_1 A | E_k E_{k-1} \cdots E_2 E_1 I) = (I|A^{-1})$$

- See Q1, Q3 and Q4.

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# Definition of Determinants

## Definition

Let $A = (a_{ij})_{n\times n}$, and $M)ij$ be an $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting the $i$th row and $j$th column. Then the determinant of $A$ is defined as

$$det(A) = \begin{cases} a_11 & if\ n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & if\ n > 1 \end{cases} \tag{4}$$

where

$$A_{ij} = (-1)^{i+j} det(M_{ij}).$$

The number $A_{ij}$ is called the $(i,j)-$cofactor of $A$. The about definition of determinant is called the **cofactor expansion**.

See Q5.