NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 11**

1. Determine whether the following are linear transformations. Justify your answer.

   (a) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x+y \\ y-z \\ 1 \end{pmatrix}\right)$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   (b) $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x+y \\ y-z \\ 0 \end{pmatrix}\right)$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

   (c) $T_3\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_3\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \begin{vmatrix} y & z \\ b & c \end{vmatrix} \\ -\begin{vmatrix} x & z \\ a & c \end{vmatrix} \\ \begin{vmatrix} x & y \\ a & b \end{vmatrix} \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, where a, b, c are in $\mathbb{R}$.

   (d) $T_4\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\boldsymbol{u}) = \lambda \boldsymbol{u}$ for $\boldsymbol{u} \in \mathbb{R}$, where λ is a fixed scalar.

   (e) $T_5\colon \mathbb{R}^2 \to \mathbb{R}$ such that $T_5\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = xy$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T\left(\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}\right) = \left(\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}\right), \quad T\left(\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}\right) = \left(\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}\right), \quad T\left(\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}\right).$$

   Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear tranformation such that

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left(\begin{pmatrix} x+y \\ y+z \\ x+z \end{pmatrix}\right).$$

   (a) Find the formula of T.

   (b) Find the standard matrix for T instead of using the formula of T in Part (2a).

   (c) Find a basis of the range of T.

   (d) Find a basis of the kernel of T.

   (e) Use this example to verify the Dimension Theorem for Linear Transformation.

   (f) Find the formula of $T \circ S$ and $S \circ T$.

3. A linear operator T on $\mathbb{R}^n$ is called an isometry if $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

(a) If T is an isometry on $\mathbb{R}^n$, show that $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. (Hint: Compute $T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v})$ in two different ways.)

(b) Let $\boldsymbol{A}$ be the standard matrix for a linear operator T. Show that T is an isometry if and only if $\boldsymbol{A}$ is an orthogonal matrix. (See also Question 5.32.)

4. Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $P\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \qquad \text{for } x \in \mathbb{R}^n.$$

(a) Show that P is a linear transformation by the following fact: Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $T(c\boldsymbol{u}+d\boldsymbol{v}) = cT(\boldsymbol{u})+dT(\boldsymbol{v})$ for all $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathbb{R}^n$ and $c, d \in \mathbb{R}$, then T is a linear transformations.

(b) Prove that $P \circ P = P$.

(c) Show that $\mathrm{Ker}(T) = \mathrm{span}\{\boldsymbol{n}\}$ and the rang $R(T) = \mathrm{span}\{\boldsymbol{n}\}^{\perp}$. Recall for a subspace W of $\mathbb{R}^n$, $W^{\perp} = \{\boldsymbol{u} \in \mathbb{R}^n \colon \boldsymbol{u} \cdot \boldsymbol{w} = 0 \text{ for all } \boldsymbol{w} \in W\}$.