Review of Section 1.4 - Section 1.5

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A linear system is inconsistent – has no solution

- If the last column of a row-echelon form of the augmented matrix is a pivot column, i.e. there is a row with nonzero last entry but zero elsewhere.
- See the following row-echelon form

$$\left(\begin{array}{cccccccc|c} & \otimes & * & & & & & & * \\ & & & \otimes & * & & & & * \\ & & 0 & & & \ddots & & & * \\ & & & & & & \otimes & * & * \\ 0 & & \cdots & & & \cdots & & {\color{red}0} & {\color{red}\otimes} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1}$$

A linear system has only one solution

- A consistent linear system has only solution if except the last column, every column of a row-echelon form of the augmented matrix is a pivot column.
- See the following row-echelon form

$$\left(\begin{array}{ccccc|c}
\otimes & & & & & * \\
 & \otimes & & & & * \\
0 & & \ddots & & & * \\
 & & & \otimes & * & * \\
 & & & & \otimes & * \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{2}$$

A linear system has infinitely many solutions

- A consistent linear system has infinitely many solutions if apart from the last column, a row-echelon form of the augmented matrix has at least one more non-pivot column.
- See the following row-echelon form

$$\left(\begin{array}{cc|c|ccccc|c} & \otimes & * & & & & & & * \\ & & & \otimes & * & & & & * \\ & & 0 & & & \ddots & & & * \\ & & & & & & \otimes & * & * \\ 0 & & \cdots & & & \cdots & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad (3)$$

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Transform Non-linear Equations to a Linear System

We now consider the following non-linear equations:

$$\begin{cases} a_{11}f_1(x_1) + a_{12}f_2(x_2) & + \cdots + & a_{1n}f_n(x_n) = b_1, \\ & \vdots & \\ a_{m1}f_1(x_1) + a_{m2}f_2(x_2) & + \cdots + & a_{mn}f_n(x_n) = b_m. \end{cases} \tag{4}$$

Now if we let $y_1 = f_1(x_1), y_2 = f_2(x_2), \cdots, y_n = f_n(x_n)$. Then we have a linear system

$$\begin{cases} a_{11}y_1 + a_{12}y_2 & + \cdots + & a_{1n}y_n = b_1, \\ & \vdots & \\ a_{m1}y_1 + a_{m2}y_2 & + \cdots + & a_{mn}y_n = b_m. \end{cases} \tag{5}$$

Solve this new linear system, we get the solution $y_1 = s_1, \cdots, y_n = s_n$. So if $s_i \in Range(f_i), \forall i = 1, \cdots, n$, then we have $x_i = f_i^{-1}(s_i), \forall i = 1, \cdots, n$.

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1. General idea: follow the steps of Guassian Elimination to reduce the augmented matrix to row-ehelon form.
2. When we need to divide a number that depends on the unknown coefficients, split into two cases, denominator is zero or nonzero.
3. Pay attention to the last nonzeros row, make sure we do not have a row with nonzero lat entry but zero elsewhere in the row echelon form.

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Homogenous Linear System

A system of linear equations is said to be **homogeneous** if it has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 & + \cdots + & a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 & + \cdots + & a_{2n}x_n = 0, \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 & + \cdots + & a_{mn}x_n = 0. \end{cases} \tag{6}$$

Note that

1. $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to the homogeneous system and it is called the **trivial solution**.
2. A homogeneuos system of linear equations has either only the trivial solution or infinitely many solutions in addition to the trivial solution.
3. A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.