NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
**MA1101R Linear Algebra I**

**2018-2019 (Semester 1)** **Tutorial 6**

1. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\} \subset \mathbb{R}^n$.

   (a) Show that if $S$ is linearly independent then any non-empty subset $T$ of $S$ is linearly independent.

   Let $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ be a subset of $S$. Since $S$ is linearly independent, by Theorem 3.4.4, each vector $\boldsymbol{v}_i \in T$ is not a linear combination of other vectors in $S$. Thus $\boldsymbol{v}_i$ is also not a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_r$. Therefore, $T$ is linearly independent.

   (b) If any non-empty proper subset of $S$ is linearly independent, is $S$ linearly independent? Justify your answer.

   No. For example, consider $S = \{(1,0),(2,0)\}$, which has only two non-empty proper subsets $\{(1,0)\}$ and $\{(2,0)\}$. Those two vectors are non-zero, so each subset is linearly independent. But $S$ is linearly dependent.

2. Let $S = \{\boldsymbol{u}_1 = (2,-1,1), \boldsymbol{u}_2 = (-1,2,3), \boldsymbol{u}_3 = (2,1,-2), \boldsymbol{u}_4 = (1,2,-9)\}$ and $V = \text{span}(S)$.

   (a) Is $S$ a basis of $V$? Justify.

   No. Since $|S| = 4 > 3$, $S$ is linear dependent. Hence $S$ is not a basis of $V$.

   (b) Find a basis of $V$.

   Let us consider the augmented matrix $(\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4 \mid \boldsymbol{0})$:

$$\left(\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ -1 & 2 & 1 & 2 & 0 \\ 1 & 3 & -2 & -9 & 0 \end{array}\right) \xrightarrow{\text{Gauss-Jordan}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right).$$

   Hence $\boldsymbol{u}_4 = 2\boldsymbol{u}_1 + \boldsymbol{u}_2 - 2\boldsymbol{u}_3$ and we also have $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is linearly independent.
   Then $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a a basis of $V$.

   (c) Determine the dimension of $V$.

   By Part (b), we have $\dim(V) = 3$.

3. (a) Let $U$ be a subspace of $V$. Show that if $\dim(U) = \dim(V)$ then $U = V$.

   Suppose that $U \neq V$. By Theorem 3.6.9, $\dim(U) < \dim(V)$ contradicts with $\dim(U) = \dim(V)$. Hence $U = V$.

(b) For the vector space V in Question 2, show that $V = \mathbb{R}^3$. (This question provides us an alternative way to determined whether $V = \mathbb{R}^n$ or not.)

Since $\dim(V) = 3 = \dim(\mathbb{R}^3)$ and $V \subseteq \mathbb{R}^3$, we have $V = \mathbb{R}^3$.

4. Let $V$ be the solution space of the following homogeneous linear system:

$$\begin{cases} x_1 + x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \\ x_1 + x_2 + 2x_3 - x_4 + 2x_5 = 0 \end{cases}$$

(a) Find a basis $S$ of $V$;

First, we give a general solution of this homogeneous linear system:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & -2 & 4 & 0 \\ 1 & 1 & 2 & -1 & 2 & 0 \end{array}\right) \xrightarrow{\text{Gauss-Jordan}} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right).$$

Thus $V = \{(-r - s + 2t, r, s - 2t, s, t) \colon r, s, t \in \mathbb{R}\}$. Since every solution is of form

$$(-1, 1, 0, 0, 0)r + (-1, 0, 1, 1, 0)s + (2, 0, -2, 0, 1)t,$$

$V$ has a basis $\{(-1, 1, 0, 0, 0), (-1, 0, 1, 1, 0), (2, 0, -2, 0, 1)\}$.

(b) Determine the dimension of $V$;

$\dim(V) = 3$.

(c) Find the coordinate vector of $\boldsymbol{u} = (-1, 1, 0, 2, 1)$ relative to the basis $S$ found in part (a);

After solving the vector equation:

$$(-1, 1, 0, 2, 1) = c_1(-1, 1, 0, 0, 0) + c_2(-1, 0, 1, 1, 0) + c_3(2, 0, -2, 0, 1),$$

we get $(c_1, c_2, c_3) = (1, 2, 1)$. Thus $(\boldsymbol{u})_S = (1, 2, 1)$

(d) Find a vector $\boldsymbol{v}$ such that $(\boldsymbol{v})_S = (3, 2, 1)$ (relative to the basis $S$ obtained in part (a).)

Following the order of the basis in the answer of Part (a), by $\boldsymbol{v}_S = (3, 2, 1)$,

$$\boldsymbol{v} = 3(-1, 1, 0, 0, 0) + 2(-1, 0, 1, 1, 0) + (2, 0, -2, 0, 1) = (-3, 3, 0, 2, 1).$$

Remark. If you have a different basis of follow a different order of $S$, you may find a different $\boldsymbol{v}$, which are also correct.

5. Find a subspace $W$ of $\mathbb{R}^5$ such that $W$ contains the solution space $V$ in Question 4 and $\dim(W) = 4$.

Let $\boldsymbol{u}$ be a vector such that $\boldsymbol{u}$ is not a linear combination of $S$. By Theorem 3.4.10, $S \cup \{\boldsymbol{u}\}$ is linearly independent. Take $W = \operatorname{span}(S \cup \{\boldsymbol{u}\})$. Then $\dim(W) = 4$ and $V \subset W$.

Assume $\boldsymbol{u} = (a_1, a_2, a_3, a_4, a_5)$.

$$\begin{pmatrix} -1 & -1 & 2 & a_1 \\ 1 & 0 & 0 & a_2 \\ 0 & 1 & -2 & a_3 \\ 0 & 1 & 0 & a_4 \\ 0 & 0 & 1 & a_5 \end{pmatrix} \xrightarrow{\text{Gauss Elimination}} \begin{pmatrix} -1 & -1 & 2 & a_1 \\ 0 & -1 & 2 & a_1 + a_2 \\ 0 & 0 & 1 & a_5 \\ 0 & 0 & 0 & a_1 + a_2 + a_4 - 2a_5 \\ 0 & 0 & 0 & a_1 + a_2 + a_3 \end{pmatrix}.$$

For instance, we can choose $\boldsymbol{u} = (1, 0, 0, 0, 0)$. Then

$$W = \operatorname{span}\{(-1, 1, 0, 0, 0),\ (-1, 0, 1, 1, 0),\ (2, 0, -2, 0, 1),\ (1, 0, 0, 0, 0)\},$$

which is a desired subspace.