NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 7**

1. Consider the set $V = \{(x, y, z) \mid ax + by + cz = 0\} \subseteq \mathbb{R}^3$.

   (a) Describe the set V geometrically. Is V a subspace of $\mathbb{R}^3$?

   (b) If V contains $\boldsymbol{v_1} = (1, -4, 6)$ and $\boldsymbol{v_2} = (0, 2, -4)$, use Gaussian Elimination to find a, b, c.

   (c) Is $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ a basis for V? Justify your answer.

   (d) Show that $T = \{\boldsymbol{u_1}, \boldsymbol{u_2}\}$, where $\boldsymbol{u_1} = (1, 1, 0)$, $\boldsymbol{u_2} = (1, 5, -8)$, is also a basis for V.

   (e) Find the transition from T to S.

   (f) Is it possible to compute $(\boldsymbol{v})_S$ for the vector $\boldsymbol{v} = (1, 1, 2)$? Justify your answer.

2. (a) Suppose $\boldsymbol{P}$ is the transition matrix from S to T, where $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, $T = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ are bases for $\mathbb{R}^2$. If

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{v_2} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \boldsymbol{P} = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix},$$

   find $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$.

   (b) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ where

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \boldsymbol{v_1} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

   (i) Verify that S and T are both bases for $\mathbb{R}^3$.

   (ii) Find the transition matrix from T to S.

   (c) Suppose $\boldsymbol{Q}$ is the transition matrix from S to T, where $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, $T = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ are bases for $\mathbb{R}^2$. If

$$\boldsymbol{v_1} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \boldsymbol{w_1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix},$$

   find $\boldsymbol{v_2}$ and $\boldsymbol{w_2}$.

3. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$ where

$$\boldsymbol{u_1} = (0, 1, 0, 0), \;\; \boldsymbol{u_2} = (-1, 0, 2, -3), \;\; \boldsymbol{u_3} = (0, 1, 0, 0)$$

$$\boldsymbol{u_4} = (1, 1, -2, 3), \;\; \boldsymbol{u_5} = (1, 6, 2, 0), \;\; \boldsymbol{u_6} = (0, 7, 0, 2).$$

(a) By finding a row-echelon form of $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 3 \\ 1 & 6 & 2 & 0 \\ 0 & 7 & 0 & 2 \end{pmatrix}$, find a basis for $V = \text{span}(S)$.

(b) Find another basis T for $V = \text{span}(S)$ such that T is a subset of S.

4. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{pmatrix}.$$

(a) Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$. Which are the non pivot columns of $\boldsymbol{R}$? Write each of the non pivot columns of $\boldsymbol{R}$ as a linear combination of the pivot columns of $\boldsymbol{R}$.

(b) Which columns of $\boldsymbol{A}$ corresponds to the pivot columns of $\boldsymbol{R}$? Recall that these columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$. Write each of the remaining columns of $\boldsymbol{A}$ as a linear combination of these basis vectors.

(c) What do you observe when comparing the answers in (a) and (b)?

5. Prove Theorem 4.1.11 (from the textbook):

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Prove the following statements.

(a) A given set of columns of $\boldsymbol{A}$ is linearly independent if and only if the set of corresponding columns of $\boldsymbol{B}$ is linearly independent.

(b) A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the set of corresponding colmuns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.