Review of 3.6 - 4.1

Liang Ling
E0220121@...

MATH@NUS

October 16, 2018

Table of Contents

Main Theorem on Invertible Matricex

Theorem (3.6.11)

Let A be an $n \times n$ matrices. The following statements are equivalent.

1. *A is invertible.*
2. *The linear system $Ax = 0$ has only the trivial solution.*
3. *The reduced row-echelon form of A is an identity matrix.*
4. *A can be expressed as a product of elementary matrices.*
5. $\det(A) \neq 0$.
6. *The rows of A form a basis for $\mathbb{R}^n$.*
7. *The columns of A form a basis for $\mathbb{R}^n$.*

A simple application of Theorem 3.6.11

- Question: We want to verify that whether a set of n vectors $S = \{u_1, \cdots, u_n\} \subset \mathbb{R}^n$ is a basis for $\mathbb{R}^n$, how to do it?
- Let $u_i = (a_{i1}, \cdots, a_{in}) \in \mathbb{R}^n, 1 \leq i \leq n$, and let

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Now if $\det(A) \neq 0$ then S is a basis for $\mathbb{R}^n$, otherwise it is not a basis for $\mathbb{R}^n$. (By 7 of Theorem 3.6.11.)

Table of Contents

Column Coordinate vector

- $S = \{u_1, \cdots, u_k\}$, is a basis for V, and let $v \in V$, then

$$v = c_1 u_1 + \cdots + c_k u_k, \quad (v)_S = (c_1, \cdots, c_k) \in \mathbb{R}^k.$$

- Sometimes, it is more convenient to write the coordinate vector in the form of column vector.

$$[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Transition Matrices

Let $S = \{u_1, \cdots, u_k\}$, $T = \{v_1 \cdots .v_k\}$ be two bases for V. We want to find out the relation between $[w]_S$ and $[w]_T$.

- Since T is a basis, we can have $[u_1]_T, \cdots, [u_k]_T$. Which means

$$u_i^T = (v_1^T \; v_2^T \; \cdots \; v_k^T)[u_i]_T, \quad 1 \le i \le k.$$

- Then

$$\begin{aligned} w^T &= (u_1^T \; u_2^T \; \cdots \; u_k^T)[w]_S \\ &= (v_1^T \; v_2^T \; \cdots \; v_k^T)([u_1]_T \, [u_2]_T \; \cdots \; [u_k]_T)[w]_S \\ &= (v_1^T \; v_2^T \; \cdots \; v_k^T)[w]_T. \end{aligned}$$

- So $[w]_T = ([u_1]_T \, [u_2]_T \; \cdots \; [u_k]_T)[w]_S$.

Transition Matrices — Continued.

Let $P = ([u_1]_T \, [u_2]_T \, \cdots \, [u_k]_T)$, then

$$[w]_T = P[w]_S.$$

Definition

Let $S = \{u_1, \cdots, u_k\}$, $T = \{v_1 \cdots .v_k\}$ be two bases for a vector space V. The $k \times k$ square matrix $P = ([u_1]_T \, [u_2]_T \, \cdots \, [u_k]_T)$ is called the **transition matrix** from S to T.

Theorem (3.7.5)

Let S and T be two bases of a vector space and let P be the transition matrix from S to T. Then

1. *P is invertible; and*
2. *P^{-1} is the transition matrix from T to S.*

Table of Contents

Row Space and Column Space

Let A be a $m \times n$ matrix, we can write A as

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

Where r_i is the *ith* row of A and c_j is the *jth* column of A.

- The row space of $A = span\{r_1, r_2, \cdots, r_m\} \subset \mathbb{R}^n$.
- The column space of $A = span\{c_1, c_2, \cdots, c_n\} \subset \mathbb{R}^m$.

Find a basis for $V = span(S)$ — method 1.

1. Let $S = \{u_1, u_2, \cdots, u_k\} \subset \mathbb{R}^n$, and

$$A = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}.$$

2. So a basis for $V = span(S)$ is equal to a basis of the row space of matrix A.
3. Use Gauss-Jordan Algorithm to get the reduced row-echelon R form of A.
4. The set of non-zero rows in R is a basis for the row space of A. (See Remark 4.1.9)

Find a basis for $V = span(S)$ — method 2.

1. Let $S = \{u_1, u_2, \cdots, u_k\} \subset \mathbb{R}^n$, and
$$A = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_k^T \end{bmatrix}.$$
2. So a basis for $V = span(S)$ is equal to a basis of the column space of matrix A.
3. Use Gauss-Jordan Algorithm to get the reduced row-echelon R form of A.
4. The set of pivot columns in R is a basis for the column space of R. (See Example 4.1.12)
5. If we want a basis for V such that it is a subset of S, then the columns in A that correspond to the pivot columns in R is a basis for V. (See Theorem 4.1.11 and Example 4.1.12)