NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 1**

1. For each of the following, determine if such a linear system exists that has the following as its general solution. If such a linear system exist, find an example. If it does not, explain why.

$$\begin{cases} x &= 5 - \frac{3s}{2} \\ y &= 2 + \frac{s}{2} \\ z &= s, \quad s \in \mathbb{R} \end{cases}$$

   (a) A linear system with 1 equation and 3 unknowns.

   (b) A linear system with 2 equations and 3 unknowns.

   (c) A linear system with 3 equations and 3 unknowns.

   (a) Not possible. Such a linear system would have two arbitrary parameters.

   (b)
$$\begin{cases} x + \frac{3z}{2} = 5 \\ y - \frac{z}{2} = 2 \end{cases}$$

   (c)
$$\begin{cases} x + \frac{3z}{2} = 5 \\ y - \frac{z}{2} = 2 \\ x + y + z = 7 \end{cases}$$

2. Consider the system of linear equations

$$\begin{cases} 3x_1 + 4x_2 - 5x_3 = -8 \\ x_1 - 2x_2 + x_3 = 2 \end{cases}$$

   (a) For any real number t, verify that $x_1 = \frac{1}{5}(-4+3t), x_2 = \frac{1}{5}(-7+4t), x_3 = t$ is a solution to the linear system.

   (b) Write down two particular solutions to the system.

   (a) Substitute $x_1 = \frac{1}{5}(-4+3t), x_2 = \frac{1}{5}(-7+4t), x_3 = t$ into both equations in the system:

$$\frac{3}{5}(-4+3t) + \frac{4}{5}(-7+4t) - 5t = -8; \quad \frac{1}{5}(-4+3t) - \frac{2}{5}(-7+4t) + t = 2.$$

   Since both equations are satisfied, they are indeed solutions to the system.

   (b) (When $t = 0$) $x_1 = -\frac{4}{5}, x_2 = -\frac{7}{5}, x_3 = 0$ and (when $t = 1$) $x_1 = -\frac{1}{5}, x_2 = -\frac{3}{5}, x_3 = 1$.

3. (a) Give a geometrical interpretation for the linear equation $x + y - z = 2$.

(b) Give a geometrical interpretatino for the linear equation $x - y = 1$ in (i) the xy-plane; and (ii) the xyz-space.

(c) Solve the linear system

$$\begin{cases} x + y - z = 2 \\ x - y = 1 \end{cases}$$

(d) Give a geometrical interpretation for the solution obtained in (c).

(a) It is a plane in the xyz-space.

(b) (i) It is a line in the xy-plane; (ii) It is a plane in the xyz-space.

(c) A general solution to the linear system is

$$\begin{cases} x = \frac{3}{2} + \frac{t}{2} \\ y = \frac{1}{2} + \frac{t}{2} \\ z = t, \quad t \in \mathbb{R} \end{cases}$$

(d) It is a line in the xyz-space.

4. Solve the following linear systems first by using Gaussian Elimination, and then again by using Gauss-Jordan Elimination.

(a)

$$\begin{cases} x_1 + x_2 = 7 \\ 2x_1 + 4x_2 = 18 \end{cases}$$

(b)

$$\begin{cases} x + y - 2z = 1 \\ 2x - 3y + z = -8 \\ 3x + y + 4z = 7 \end{cases}$$

(c)

$$\begin{cases} u + 3v + x + 5y = 2 \\ 2u + 7v + 9x + 2y = 4 \\ 4u + 13v + 11x + 12y = 8 \end{cases}$$

(a) By Gaussian elimination, we have the following row-echelon form

$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 2 & 4 \end{array}\right)$$ which gives, by back substitution, $x_2 = 2$, $x_1 = 5$.

By Gauss-Jordan elimination, we have the reduced row-echelon form,

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$ which gives the same answer $x_2 = 2$, $x_1 = 5$.

(b) By Gaussian elimination, we have the following row-echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right) \text{ which gives, by back substitution, } z = 1, y = 3, x = 0.$$

By Gauss-Jordan elimination, we have the reduced row-echelon form,

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \text{ which gives the same answer } z = 1, y = 3, x = 0.$$

(c) By Gaussian elimination, we have the following row-echelon form

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

. By back substitution, let $x = s$, $y = t$, then $v = -7s + 8t$, $u = 2 - s - 5t - 3(-7s + 8t) = 2 + 20s - 29t$. By Gauss-Jordan elimination, we have the reduced row-echelon form,

$$\left(\begin{array}{cccc|c} 1 & 0 & -20 & 29 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which gives the same general solution

$$\begin{cases} u & = & 2 + 20s - 29t \\ v & = & -7s + 8t \\ x & = & s \\ y & = & t, \quad s, t \in \mathbb{R} \end{cases}$$

5. Consider a linear system with m equations and n unknowns $x_1, x_2, \cdots, x_n$. Denote this linear system by (1).

Suppose one of the equations in (1) is multiplied by a nonzero constant k. Denote the resulting linear system by (2). Show that

$$x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$$

is a solution to (1) if and only if it is a solution to (2).

Repeat the question when (2) is obtained from (1) by adding k times of one equation in (1) to another equation in (1).

Remark: By completing this question, we have essentially proven **Theorem 1.2.7** introduced during lecture.

If $x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$ is a solution to (1), then it satisfies every equation in (1). Note that all except one equation in (2) are identical to (1), so $x_1 = c_1, x_2 =$

$c_2, \cdots, x_n = c_n$ immediately satisfies all except one equation in (2). Let the two corresponding equations in (1) and (2) (that are different) be

$$a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = b_r \quad \text{in (1); and}$$

$$ka_{r1}x_1 + ka_{r2}x_2 + \cdots + ka_{rn}x_n = kb_r \quad \text{in (2)}$$

It is now clear that if

$$a_{r1}c_1 + a_{r2}c_2 + \cdots + a_{rn}c_b = b_r$$

then

$$ka_{r_1}c_1 + ka_{r2}c_2 + \cdots + ka_{rn}c_n = kb_r,$$

that is $x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$ also satisfies the last equation in (2), thus it is also a solution to (2). The converse can be shown in a similar way.

Suppose now that k times the i-th equation in (1) is added to the j-th equation in (1). More precisely, we add

$$ka_{i1}x_1 + ka_{i2}x_2 + \cdots + ka_{in}x_n = kb_i$$

to

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j.$$

Every equation in (2) is identical to their corresponding equations in (1), except the j-th equation in (2), which is now

$$(a_{j1} + ka_{i1})x_1 + (a_{j2} + ka_{i2})x_2 + \cdots + (a_{jn} + ka_{in})x_n = kb_i + b_j.$$

Since $x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$ is a solution to (1), we have

$$ka_{i1}c_1 + ka_{i2}c_2 + \cdots + ka_{in}c_n = kb_i$$

and

$$a_{j1}c_1 + a_{j2}c_2 + \cdots + a_{jn}c_n = b_j.$$

This implies

$$(ka_{i1} + a_{j1})c_1 + (ka_{i2} + a_{j2})c_2 + \cdots + (ka_{in} + a_{jn})c_n = kb_i + b_j$$

which shows that $x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$ satisfies the last equation in (2) and thus is also a solution to (2). The converse can be shown in a similar way.