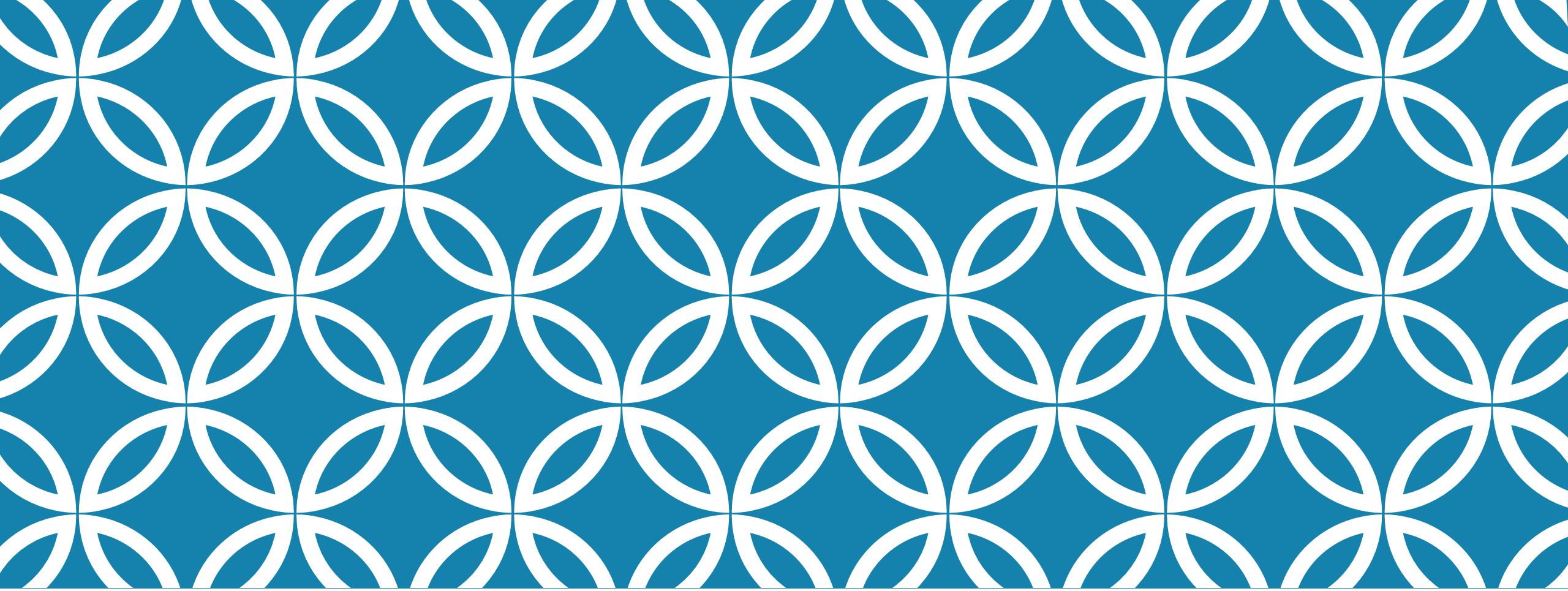

LECTURE 18: BLOOM FILTERS

Harold Soh
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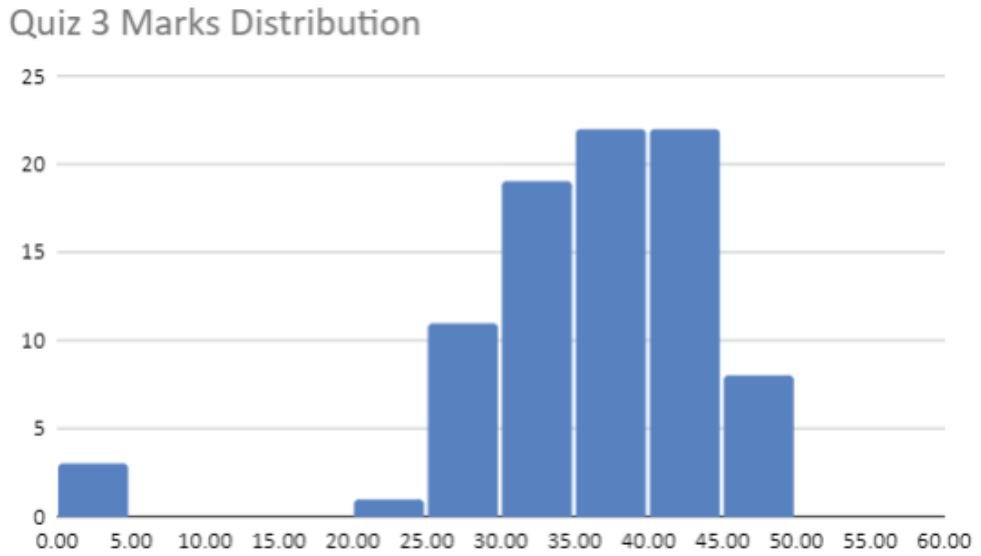
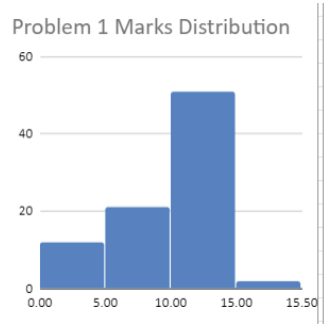
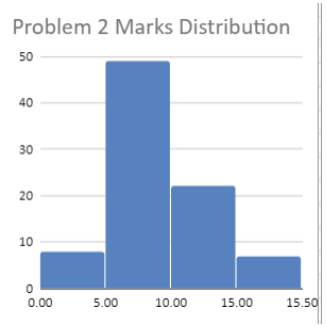
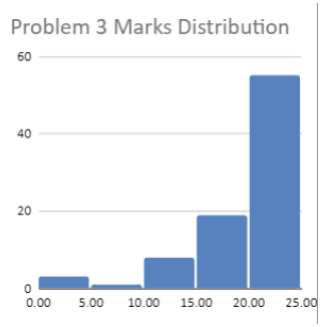
ADMINISTRATIVE ISSUES: QUIZ 3

- Finished grading Quiz 3
- Will be returned on Friday.
- More discussion tomorrow.









QUESTIONS?

Poll Everywhere

https://bit.ly/2LvG9bq

HASHING SUMMARY

Symbol Tables are *pervasive*

Hash tables are fast, efficient data structures for implementing symbol tables

- Under optimistic assumption, provably so.
- In the real world, often so.
- But not perfect!

Beats BSTs:

- for searching.
- but: gave up "order" operations.

CUCKOO HASHING

Did you know?

- Cuckoos lay eggs in other birds nests.
- When the cuckoo bird hatches, it pushes eggs/chicks out of the nest.

What a neat idea!

- Open addressing policy!
- Described by Rasmus Pagh and Flemming Friche Rodler in 2001.

CUCKOO HASHING: PERFORMANCE

Insertions seem to be quite complicated…

But: it takes expected $O(1)$ amortized time!

Analysis requires (a little) graph theory

To be continued in Week 12 …

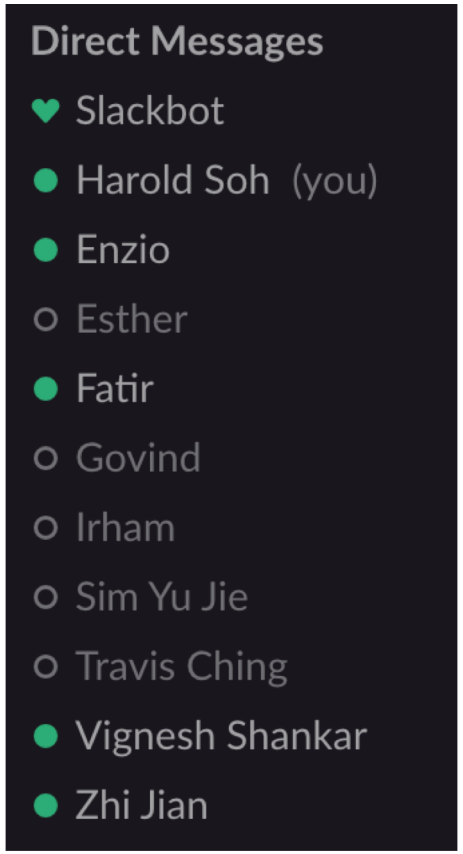
THE **NEXT 2 DAYS**

- Bloom Filters
- Tomorrow: Cuckoo Hashing

PROBLEM: CHECKING WHO IS ONLINE

I have a list of friends:

- Ayush
- Eldon
- Enzio
- Esther
- Fatir
- Govind
- Irham
- Ryan
- Si Jie
- Travis
- Vignesh
- Zhi Jian
- …



How can I quickly check who is online?

OTHER RELATED PROBLEMS

Spam filtering

- Have a list of valid emails and bad emails

Caching (avoiding spurious caching)

- Check if the a website is requested more than once

Preventing Denial-of-Service attacks

- Check if the request comes from a list of bad IP addresses

ONE SOLUTION?

Use a hash table!

Problem: size of the hash table can get large since we store the *keys*.

- Uses too much memory!

What if we had not enough memory?

0	0
1	0
2	www.gmail.com
3	www.apple.com
4	0
5	0
6	www.microsoft.com
7	0
8	www.nytimes.com
9	0

BLOOM FILTER

Space-efficient *Probabilistic* Data-Structure

A query returns:

- **True:** *Possibly* in set
- **False:** *Definitely* not in set

Elements can be added but not removed

- But we will deal with this later.

The more items, the larger the probability of **false positives**

- Saying True when the item is not in the set

FROM LECTURE 5: CORRECTNESS & EFFICIENCY

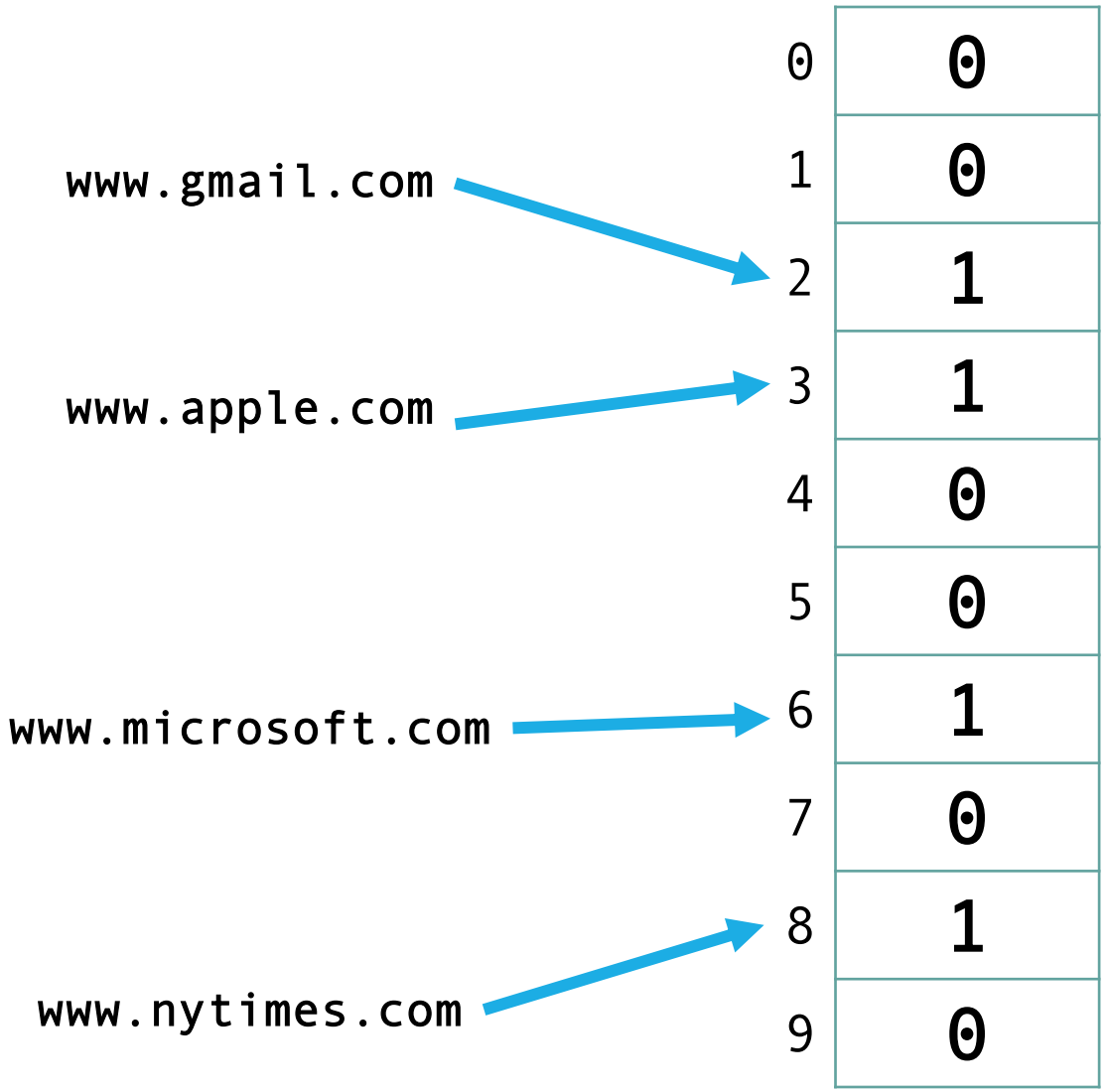
	Inefficient	Efficient
Incorrect	Slow & Wrong	Very fast… but wrong (sometimes useful)
Correct	Correct but slow (sometimes useful)	**We want this!**

IDEA: STORE LESS!

Only set k bits per item in a table of size m.

To start, let k = 1

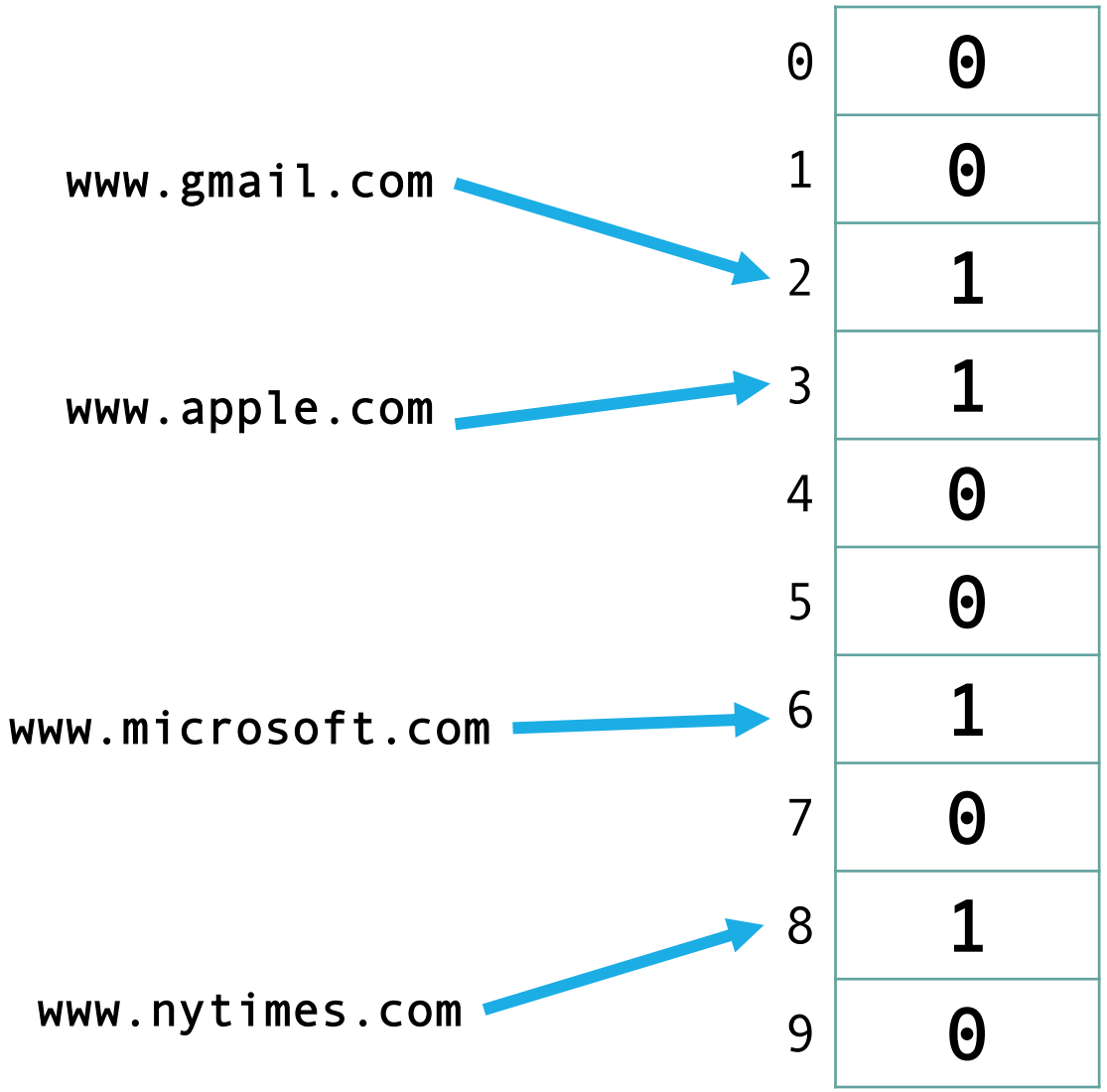
- For each item, we only set 1 bit!



IDEA: STORE LESS!

```
insert(key)
    h = hash(key)
    m_table[h] = 1

lookup(key)
    h = hash(key)
    return (m_table[h] == 1)
```

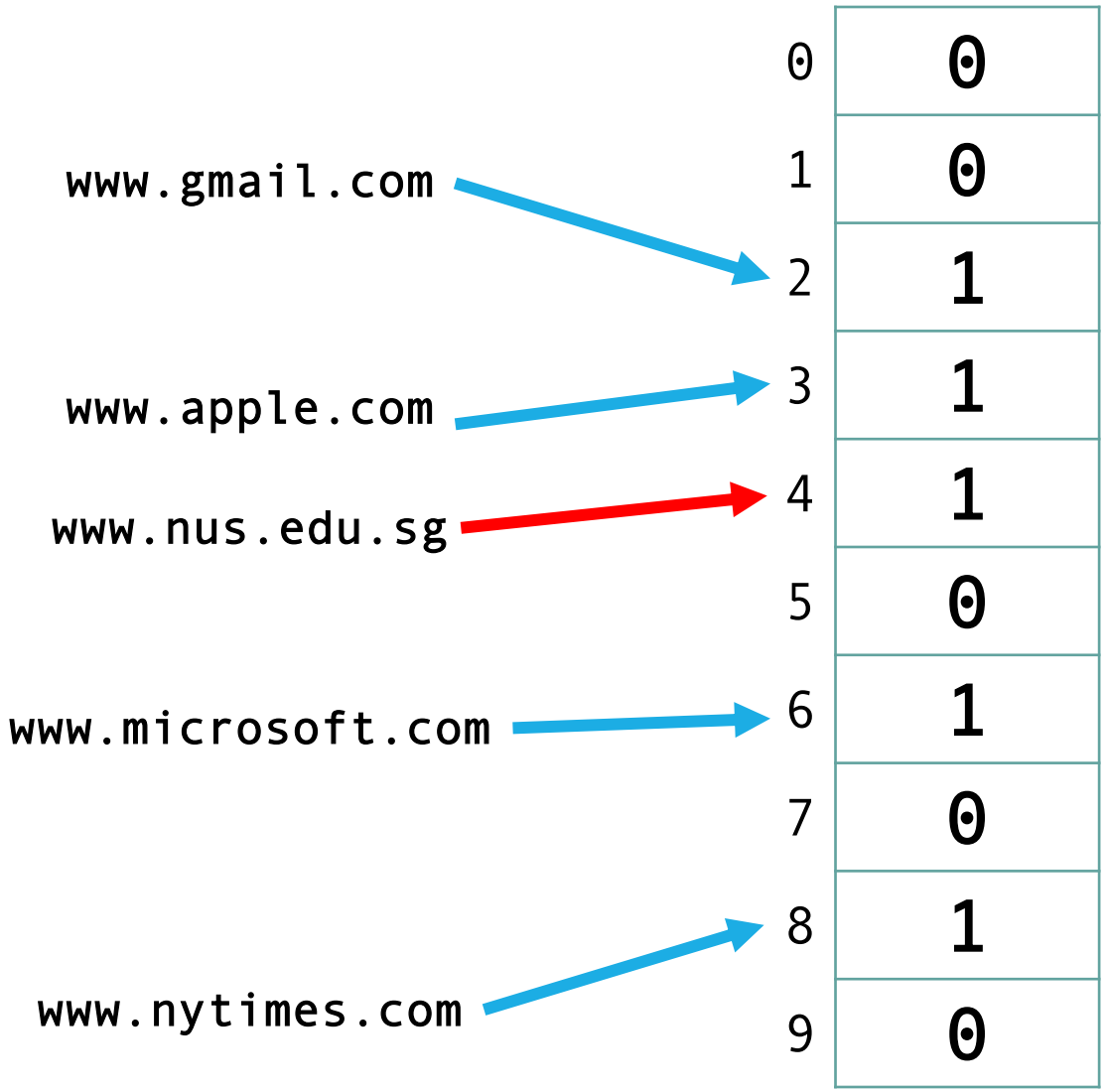


INSERTIONS

```
insert(key)
    h = hash(key)
    m_table[h] = 1
```

Insert www.nus.edu.sg

hash("www.nus.edu.sg") = 4



LOOKUPS

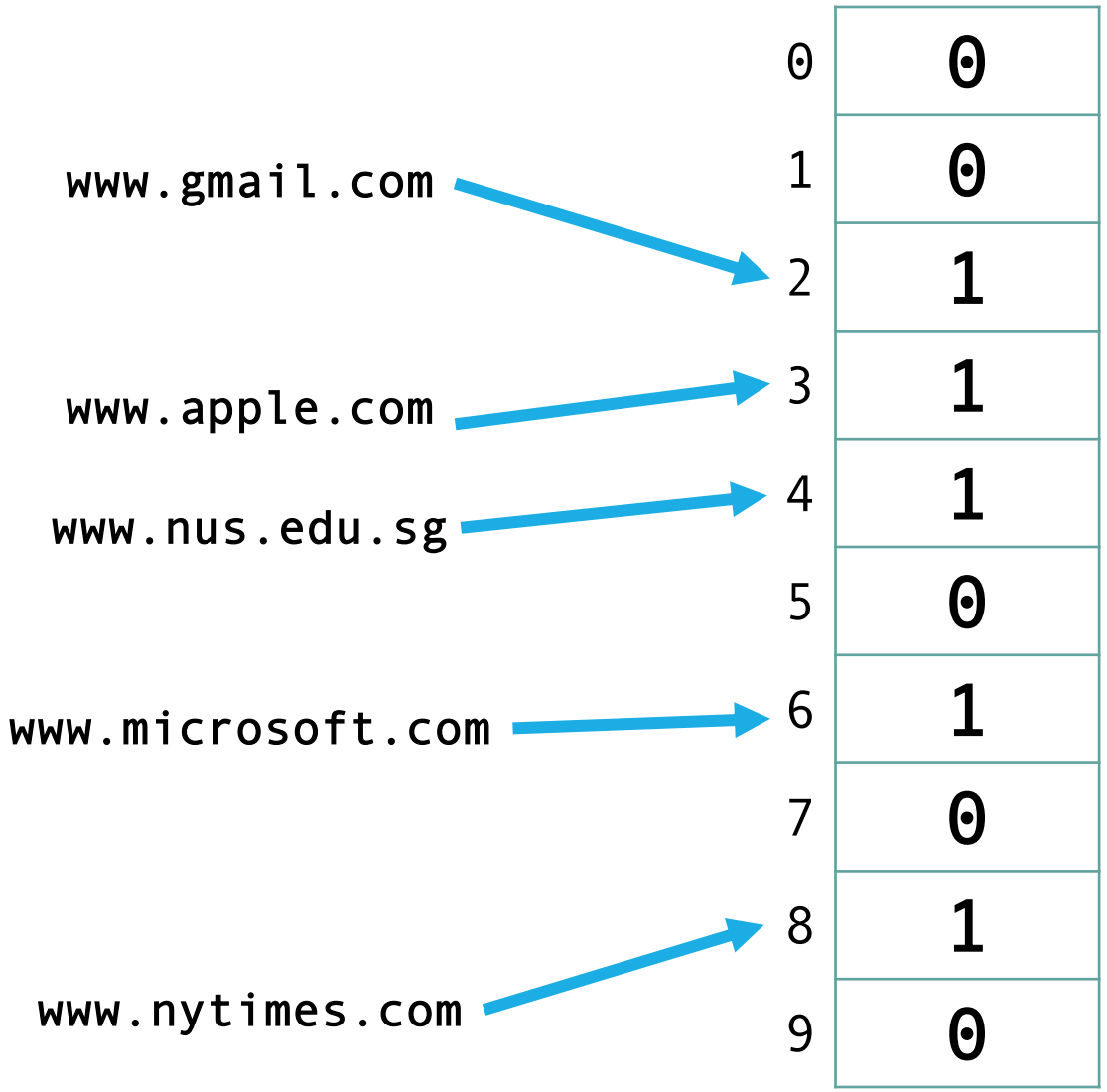
```
lookup(key)
    h = hash(key)
    return (m_table[h] == 1)
```

lookup www.nus.edu.sg

hash("www.nus.edu.sg") = 4

If key is in the table, will always return `true`.

No false negatives!



LOOKUPS

Returns `true` even ntu is not in the set!
FALSE POSITIVE due to collisions

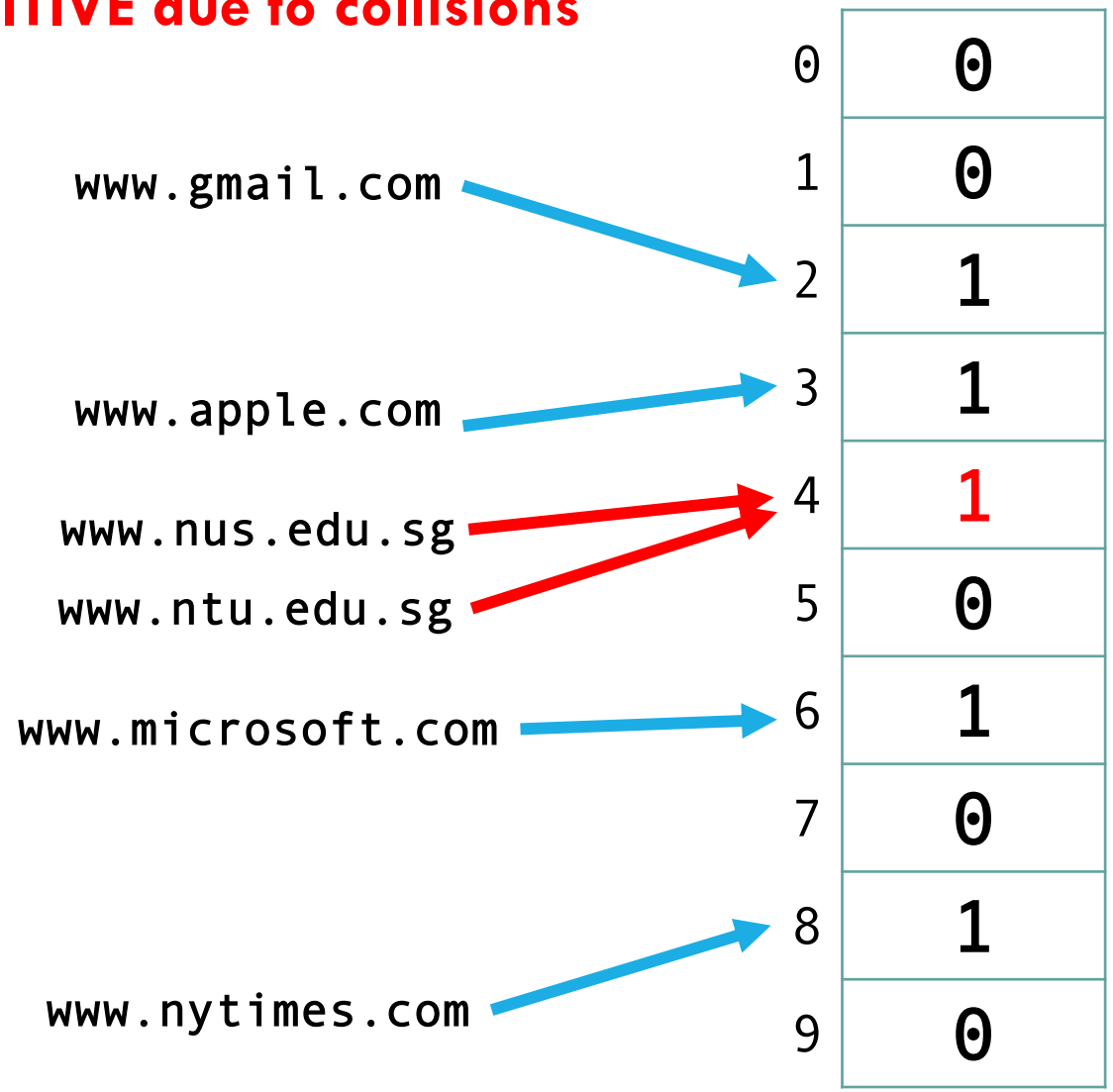
```
lookup(key)
    h = hash(key)
    return (m_table[h] == 1)
```

lookup www.nus.edu.sg

hash("www.nus.edu.sg") = 4

lookup www.ntu.edu.sg

hash("www.ntu.edu.sg") = 4



SPAM FILTERING

Given a Bloom Filter, would you store the good emails or bad emails?

A. Good Emails
B. Bad Emails
C. Not sure

SPAM FILTERING

Why? I would rather get an occasional spam message than put a real email (e.g., from my wife or boss) in the spam folder!

Given a Bloom Filter, would you store the good emails or bad emails?

A. Good Emails

B. Bad Emails

C. Not sure

PROBABILITY OF A FALSE POSITIVE?

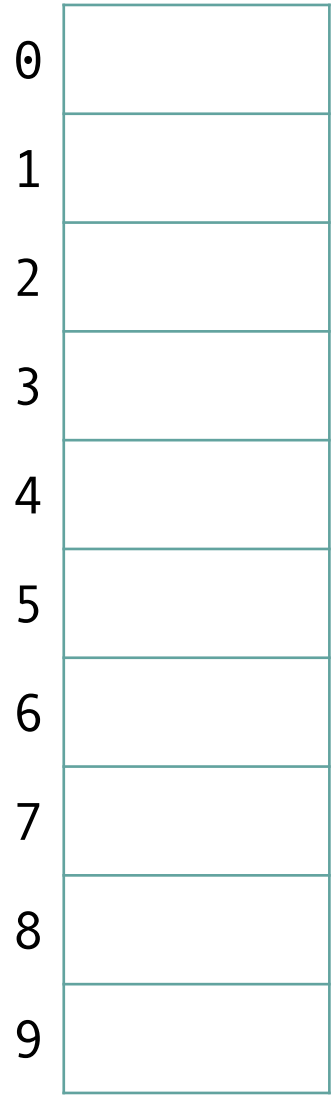
Table size is m, we store n elements

Set $k = 1$ bit per item

Test an item **not** in the set.

$$p(\text{false positive}) = 1 - p(\text{hits a cell with } 0)$$

$$p(\text{hits cell with } 0) = \underbrace{\left(1 - \frac{1}{m}\right)^n}_{\text{Elements are independently hashed}} \approx e^{-\frac{n}{m}}$$



THE NEXT 2 DAYS

- Probability Review
- Bloom Filters
- Cuckoo Hashing

PROBABILITY REVIEW

What do we **mean** when we say:

- "*the probability of getting an even number when rolling a die is ½*"
- "*the probability that I have the rare fatal disease is 90%*"

PROBLEM: YOU'RE NOT FEELING WELL...

You're not feeling well and go to the doctor.

You take a blood test.

Test comes back **positive** for *rare*, *fatal* disease.

Should you:

A. Skip CS2040S and start planning your demise?
B. Not worry.
C. Take the test again (and again) until it comes back negative.
D. Ask for more information.

PROBLEM: YOU'RE NOT FEELING WELL…

You're not feeling well and go to the doctor.

You take a blood test.

Test comes back **positive** for *rare*, *fatal* disease.

- Disease affects 0.1% of the population.
- Test correct identifies 99% of the people who have the disease.
- If you do not have the disease, test may come back positive 2% of the time.

What to do now?

Let's figure out the probability you actually have the disease!

PROBABILITY SPACE

A probability space (Ω, E, P) models a process consisting of outcomes that occur randomly.

Consists of three parts:

- Outcome or sample space Ω
- Event space E
- Probability function $P: E \to \mathbb{R}$

OUTCOME SPACE

Outcome space is a space of possible outcomes or "elementary outcomes", denoted by Ω.

Example: Outcomes of a dice roll, $\Omega = \{1,2,3,4,5,6\}$.

EVENT SPACE

Event space $E \subseteq 2^{\Omega}$ is a subset of the power set of Ω, it is the set of measurable events to which we assign probabilities.

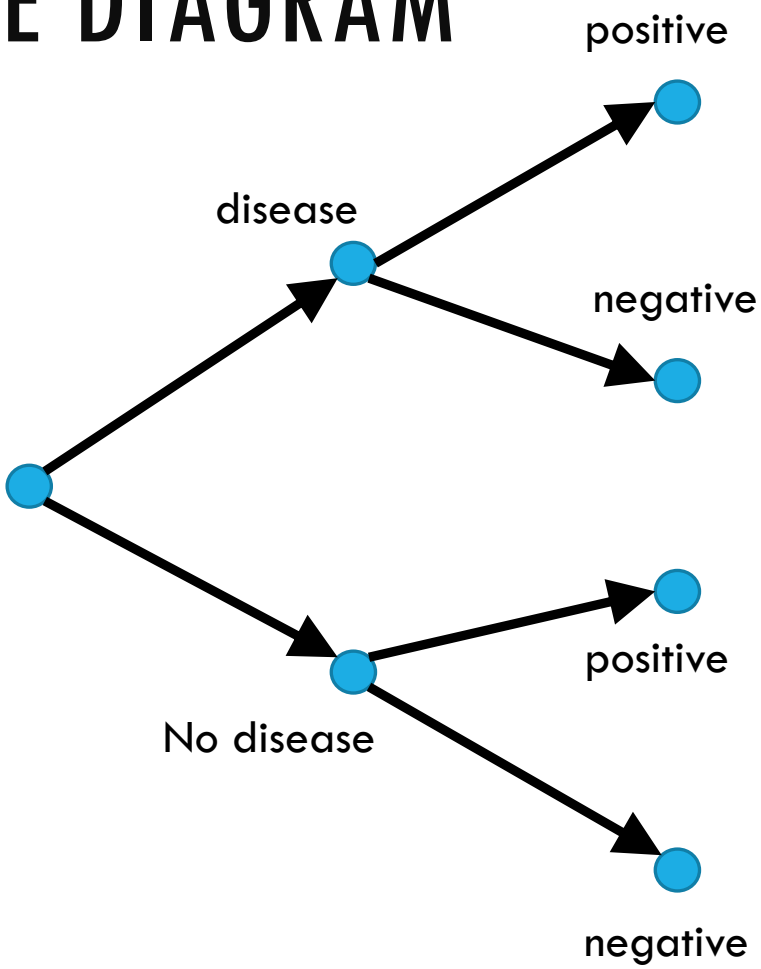
Example: The event space on whether a dice roll is odd or even, $E = \{\emptyset, \{1,3,5\}, \{2,4,6\}, \Omega\}$.

EVENT SPACE

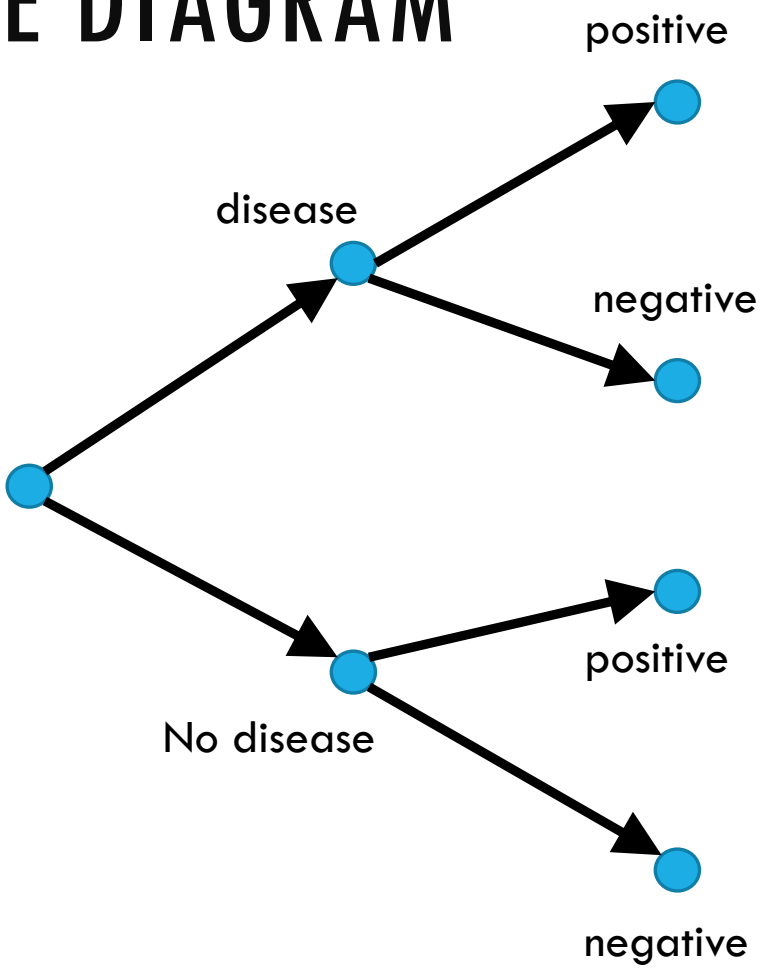
Event space must satisfy three basic properties:

1. It contains the empty event $\emptyset$, and the trivial event Ω.
2. It is closed under union, i.e. if $\alpha, \beta \in E$, then so is $\alpha \cup \beta$.
3. It is closed under complement, i.e. if $\alpha \in E$, then so is $\Omega - \alpha$.

TREE DIAGRAM



TREE DIAGRAM



Outcome Space Ω

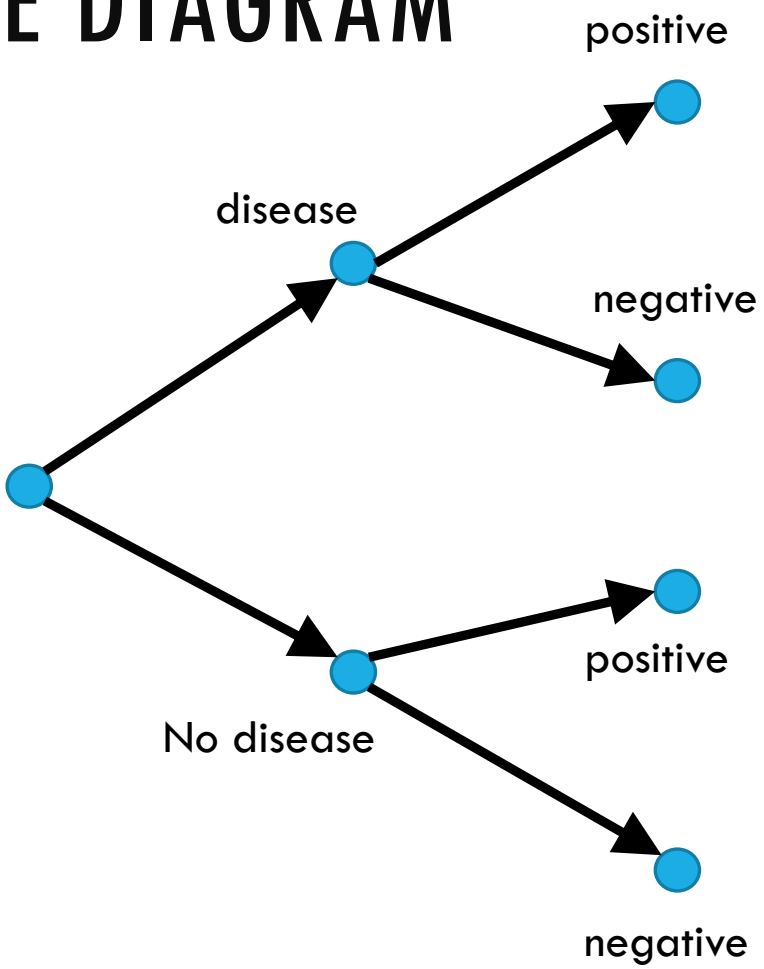
(disease, positive)

(disease, negative)

(no disease, positive)

(no disease, negative)

TREE DIAGRAM



Outcome Space Ω

(disease, positive)

(disease, negative)

(no disease, positive)

(no disease, negative)

Example Event Spaces:

Test is positive or negative

Event Space E

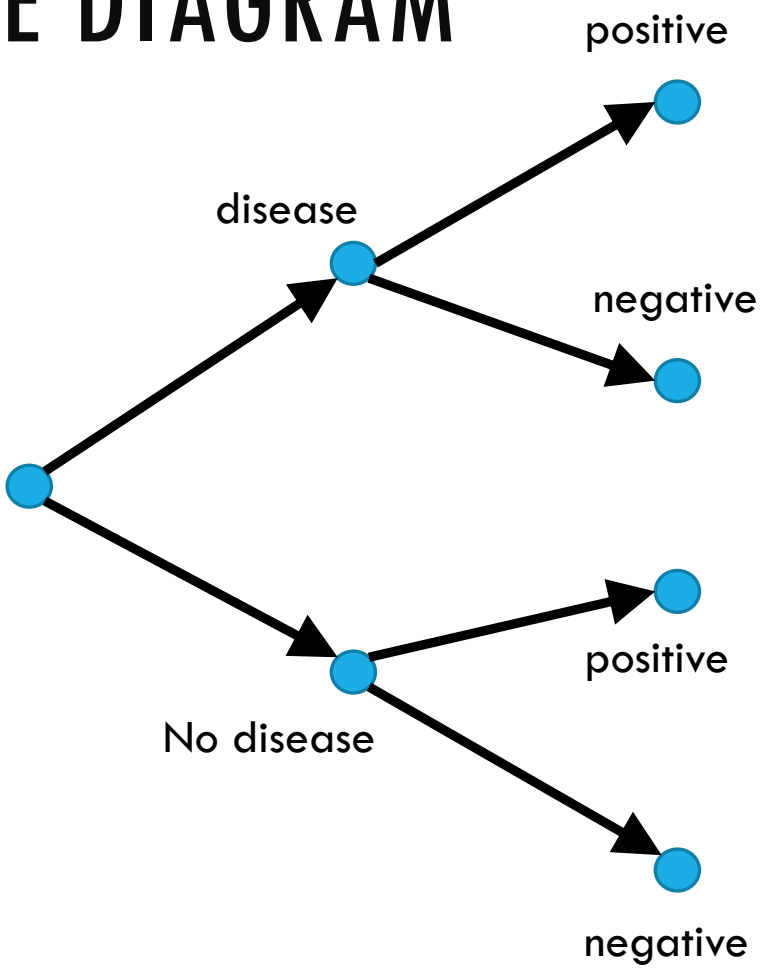
$\emptyset$

{(disease, positive),
(no disease, positive)}

{(disease, negative),
(no disease, negative)}

Ω

TREE DIAGRAM



Outcome Space Ω

(disease, positive)

(disease, negative)

(no disease, positive)

(no disease, negative)

Example Event Spaces:

Test is positive or negative

Event Space E

$\emptyset$

{(disease, positive),
(no disease, positive)}

{(disease, negative),
(no disease, negative)}

Ω

Disease or no disease

Event Space E

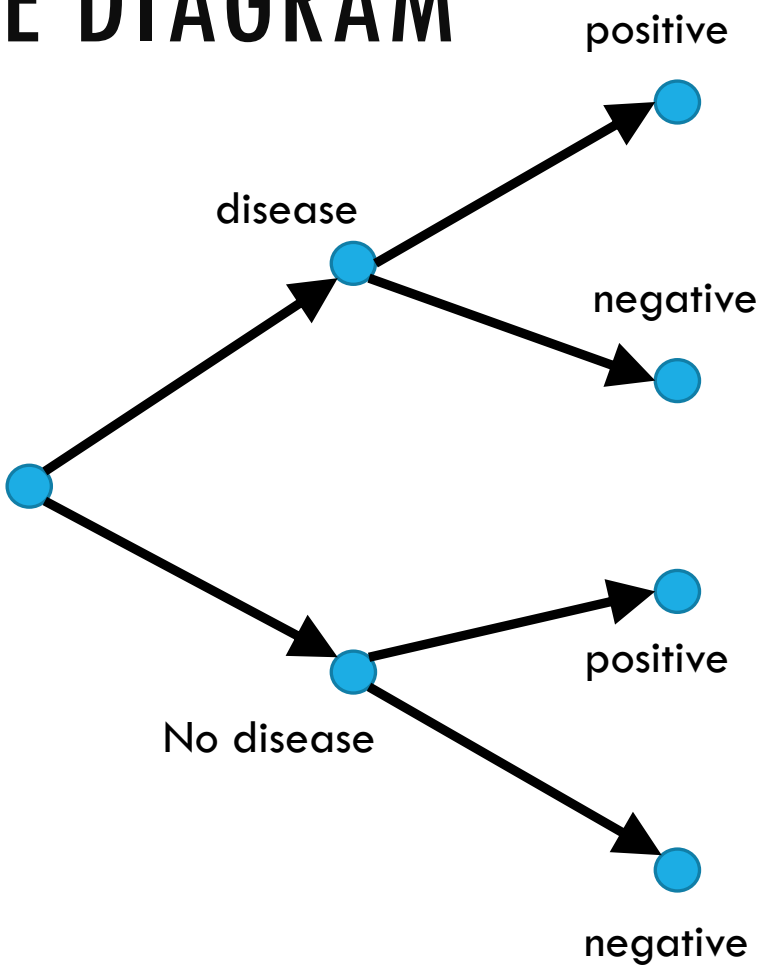
$\emptyset$

{(disease, positive),
(disease, negative)}

{(no disease, positive),
(no disease, negative)}

Ω

TREE DIAGRAM



Outcome Space Ω

(disease, positive)

(disease, negative)

(no disease, positive)

(no disease, negative)

Example Event Spaces:

Event Space E

$\emptyset$

{(disease, positive)}

{(disease, negative)}

{(no disease, positive)}

{(no disease, negative)}

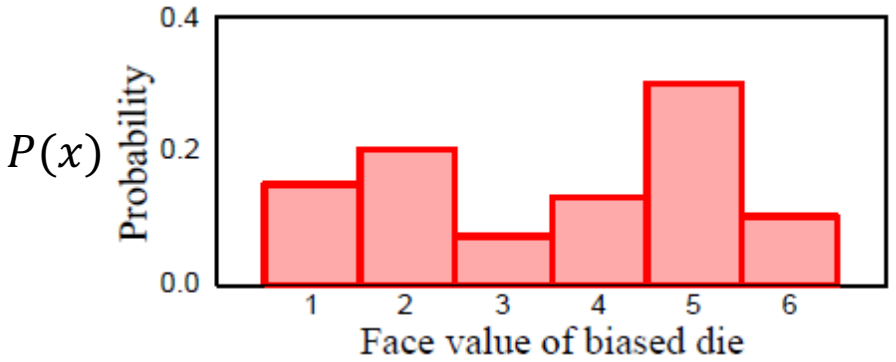
Ω

PROBABILITY DISTRIBUTIONS

A probability distribution P over (Ω, E) is a mapping from events in E to real values that satisfies the following conditions, i.e. axioms of probability:

1. Non-negativity, i.e. $P(\alpha) \geq 0, \ \forall \ \alpha \in E$.
2. Probability of all outcomes sums to 1, i.e. $P(\Omega) = 1$.
3. Mutually disjoint events: If $\alpha, \beta \in E$ and $\alpha \cap \beta = \emptyset$, then $P(\alpha \cup \beta) = P(\alpha) + P(\beta)$.

Discrete: Probability mass function, $P(x)$



$$Val(X) = \{1,2,3,4,5,6\}$$

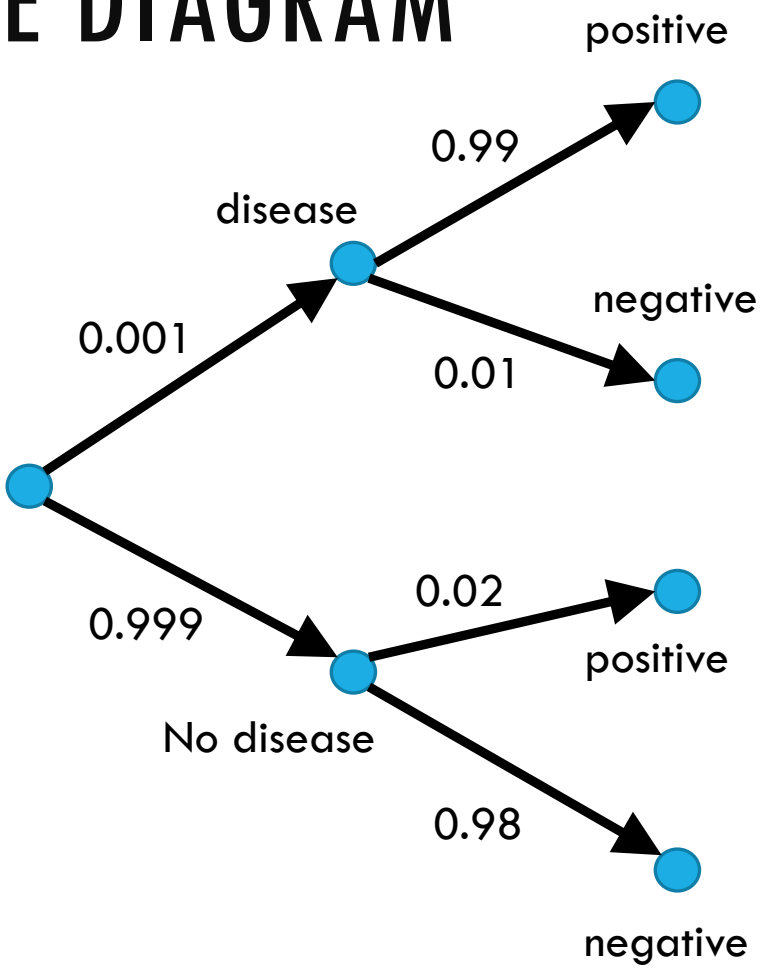
$$\sum_{i=1}^{K} P\left(X = x^i\right) = 1$$

$$0 \leq P\left(X = x^i\right) \leq 1, \ \forall\, i = 1, \ldots K$$

$$K = |Val(X)|$$

Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

TREE DIAGRAM



Event Space E

$\emptyset$

{(disease, positive)}

0.001×0.99

{(disease, negative)}

0.001×0.01

{(no disease, positive)}

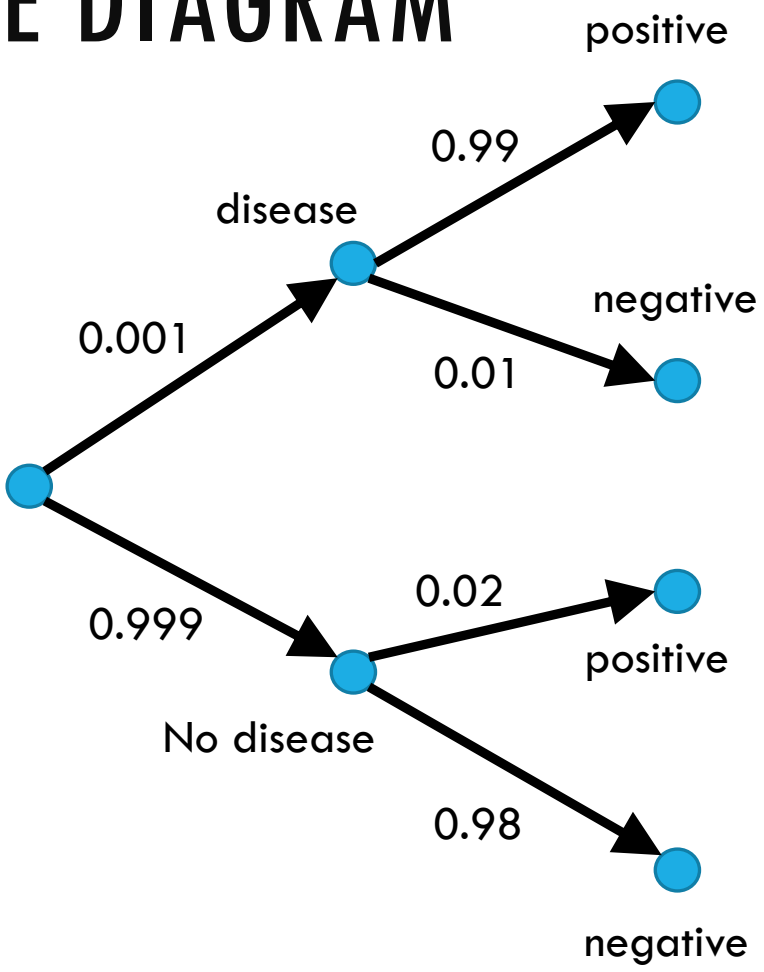
0.999×0.02

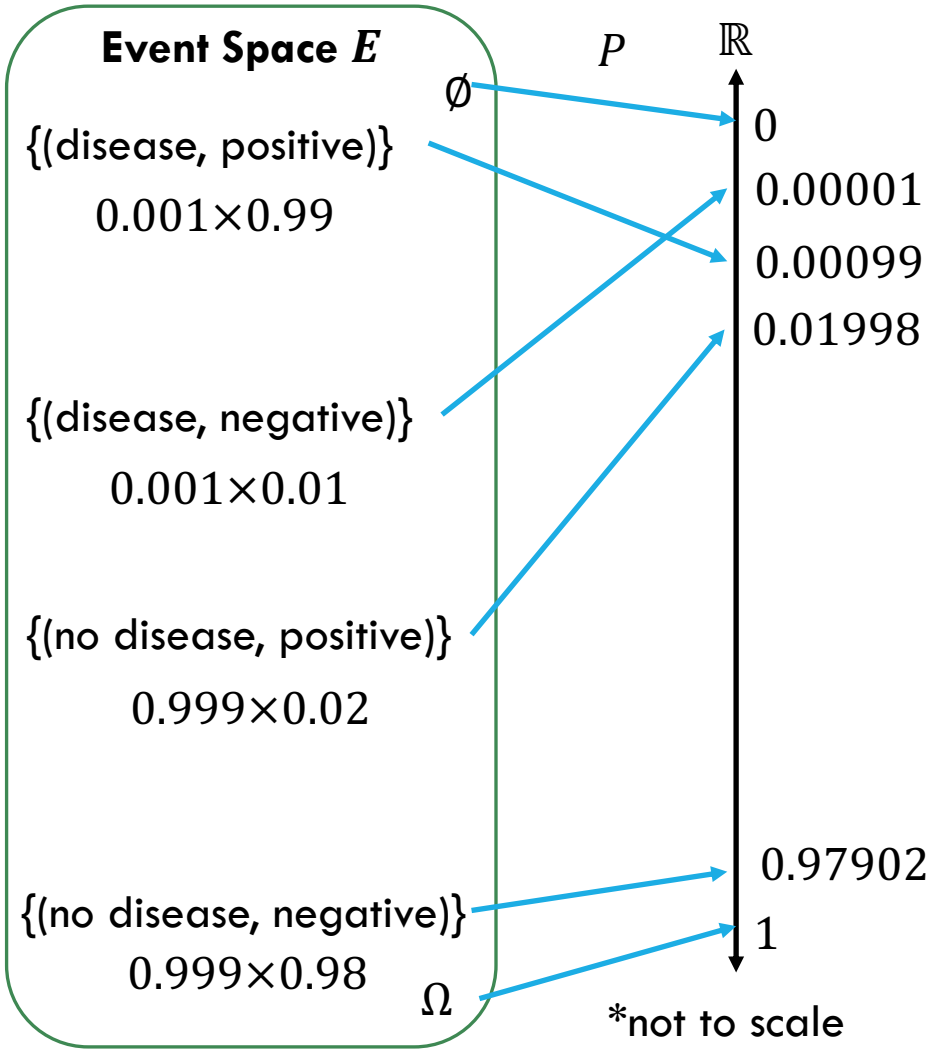
{(no disease, negative)}

0.999×0.98

Ω

TREE DIAGRAM





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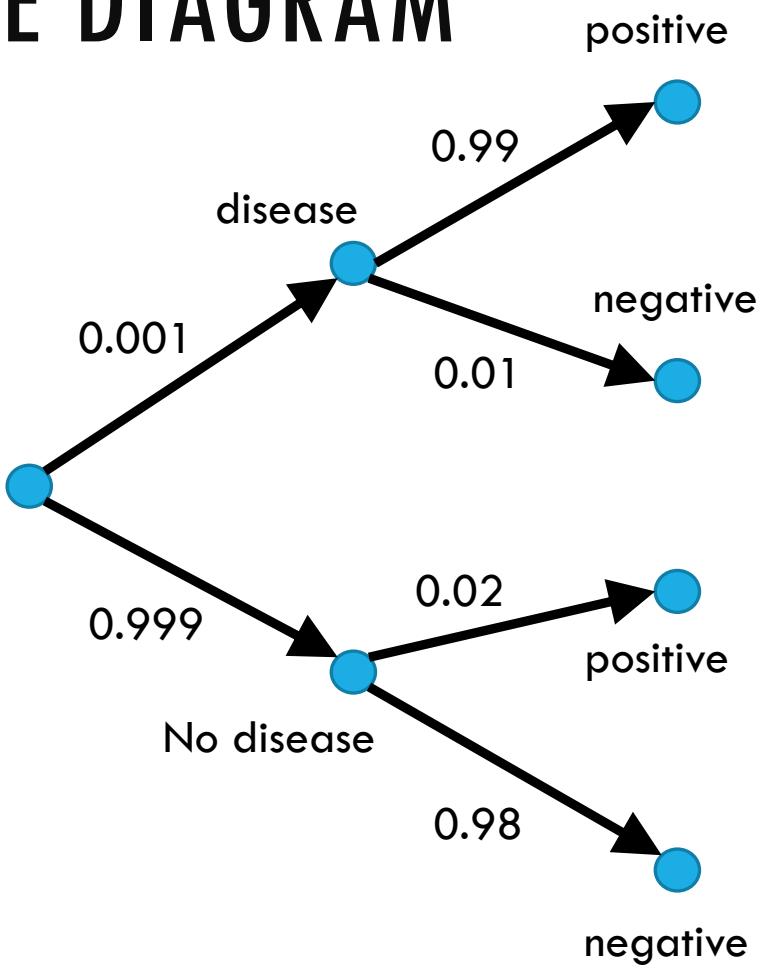

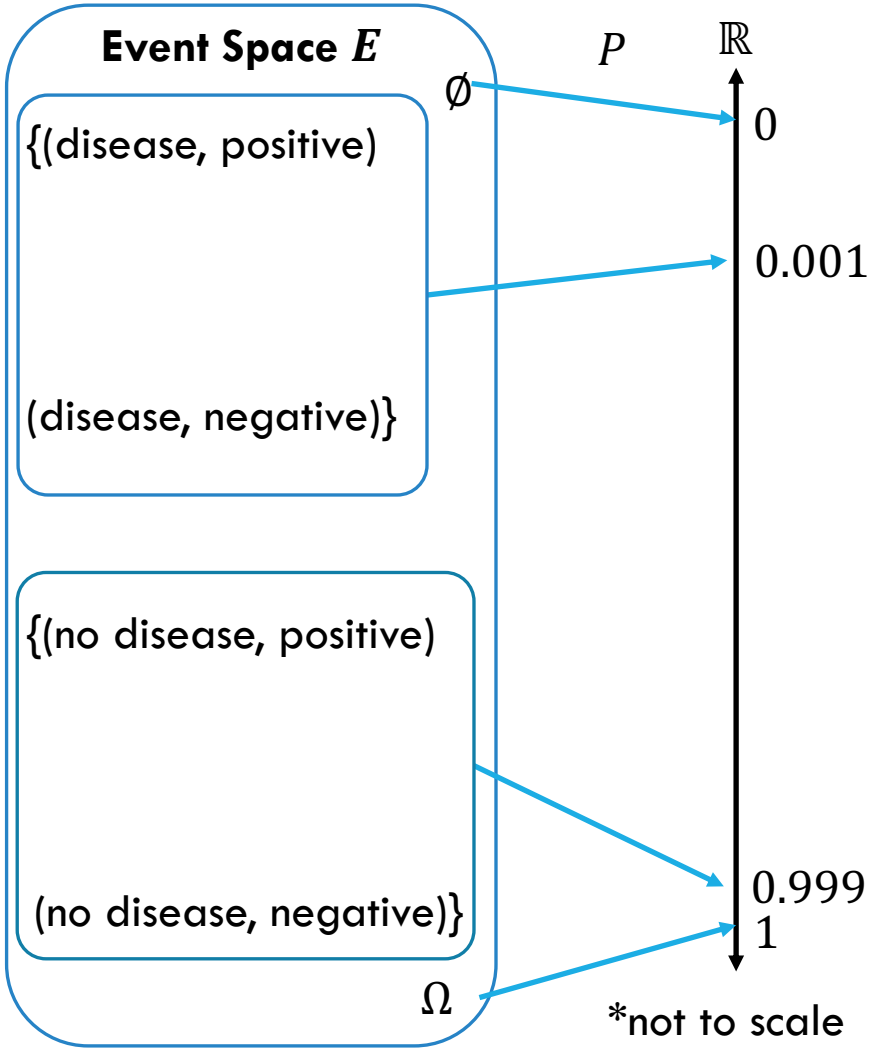

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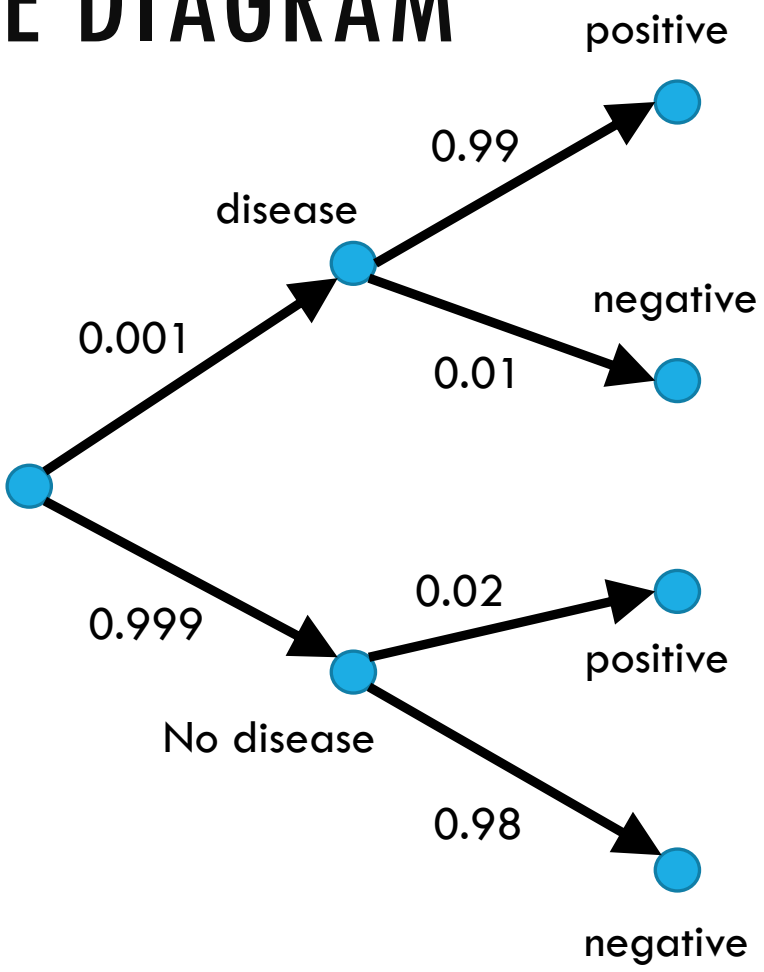
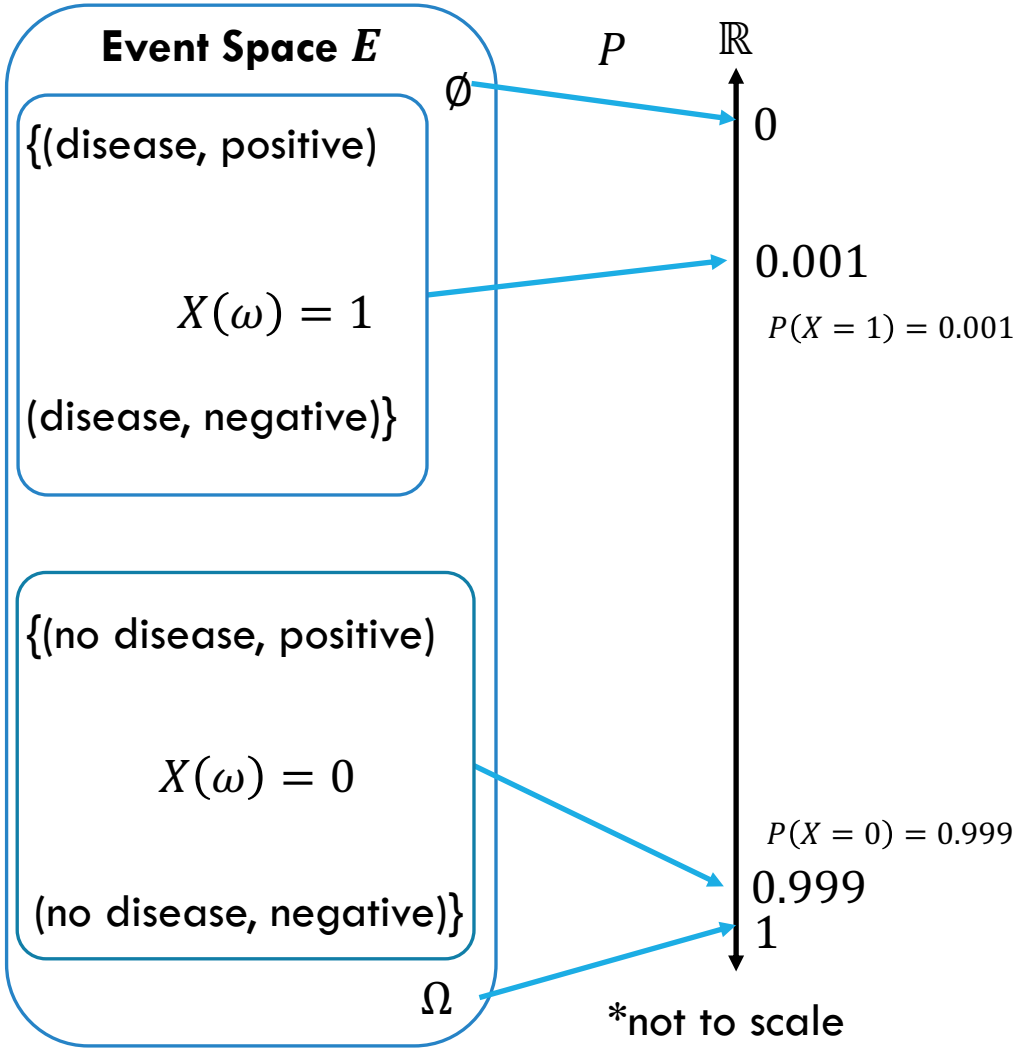




RANDOM VARIABLES

A random variable, denoted as X (upper case), is the machinery for discussing attributes and their values in different outcomes.

More formally, it is a function $X: \Omega \rightarrow S$ that maps a set of possible **outcomes** Ω to a space S

- S usually a subset of $\mathbb{R}$, but can be other sets.

RANDOM VARIABLES

The set of values that a random variable X can take is denoted as $Val(X)$.

A lower case letter, e.g. x, is used to refer to a generic value of a random variable X, a.k.a. realization of the random variable.

Example: We write $P(X = x) \geq 0$ for all $x \in Val(X)$.

$P(x)$ is often used as a shorthand notation for $P(X = x)$.

We use the notation x^i to represent a specific value of X.

INDICATOR RANDOM VARIABLES

Indicator random variable maps every **outcome** to either 0 or 1.

For example: whether you have the disease

$$\Omega = \{(d,\oplus), (\neg d,\oplus), (d,\ominus), (\neg d,\ominus)\}$$

$$X(\omega) = \begin{cases} 1 \text{ if } (d,\oplus) \\ 1 \text{ if } (d,\ominus) \\ 0 \text{ if } (\neg d,\oplus) \\ 0 \text{ if } (\neg d,\ominus) \end{cases}$$

$X = I_\alpha(\omega)$ where $\alpha \in E$ is the event where you have the disease

INDICATOR RANDOM VARIABLES

For example: 3 independent, unbiased coin tosses.

$$\Omega = \{ \text{HHH}, \text{TTT}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH} \}$$

$$X(\omega) = \begin{cases} 1 \text{ if HHH or TTT} \\ 0 \text{ otherwise} \end{cases}$$

$X = I_\alpha(\omega)$ where $\alpha \in E$ is the event where all three coins match.

RANDOM VARIABLES: EXAMPLES

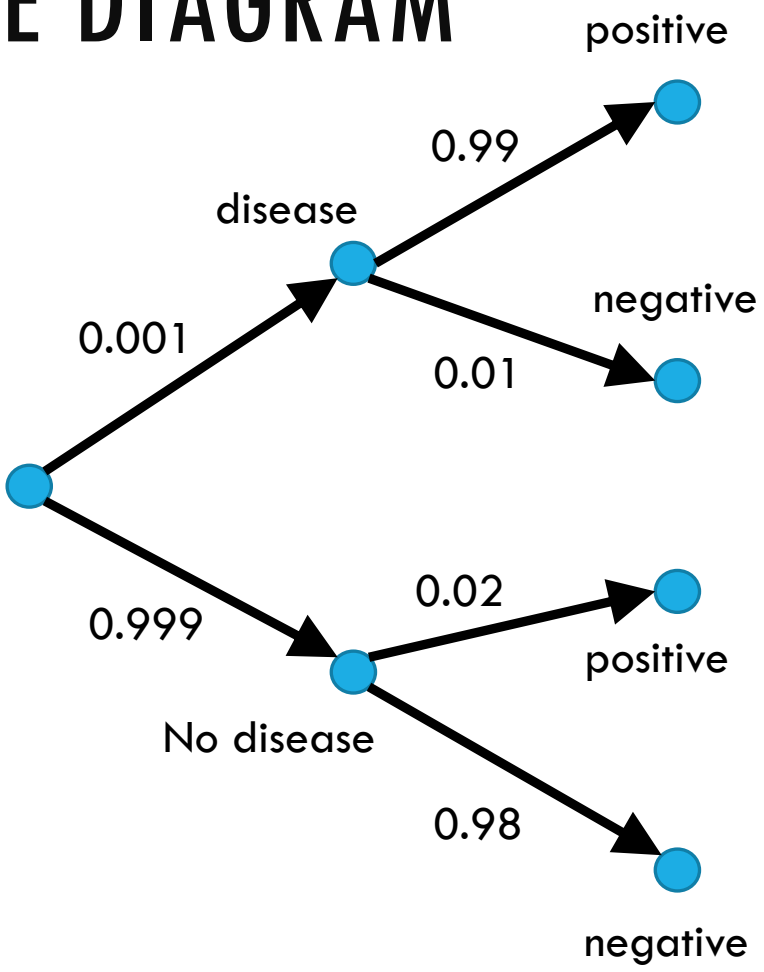
Random variables with discrete values

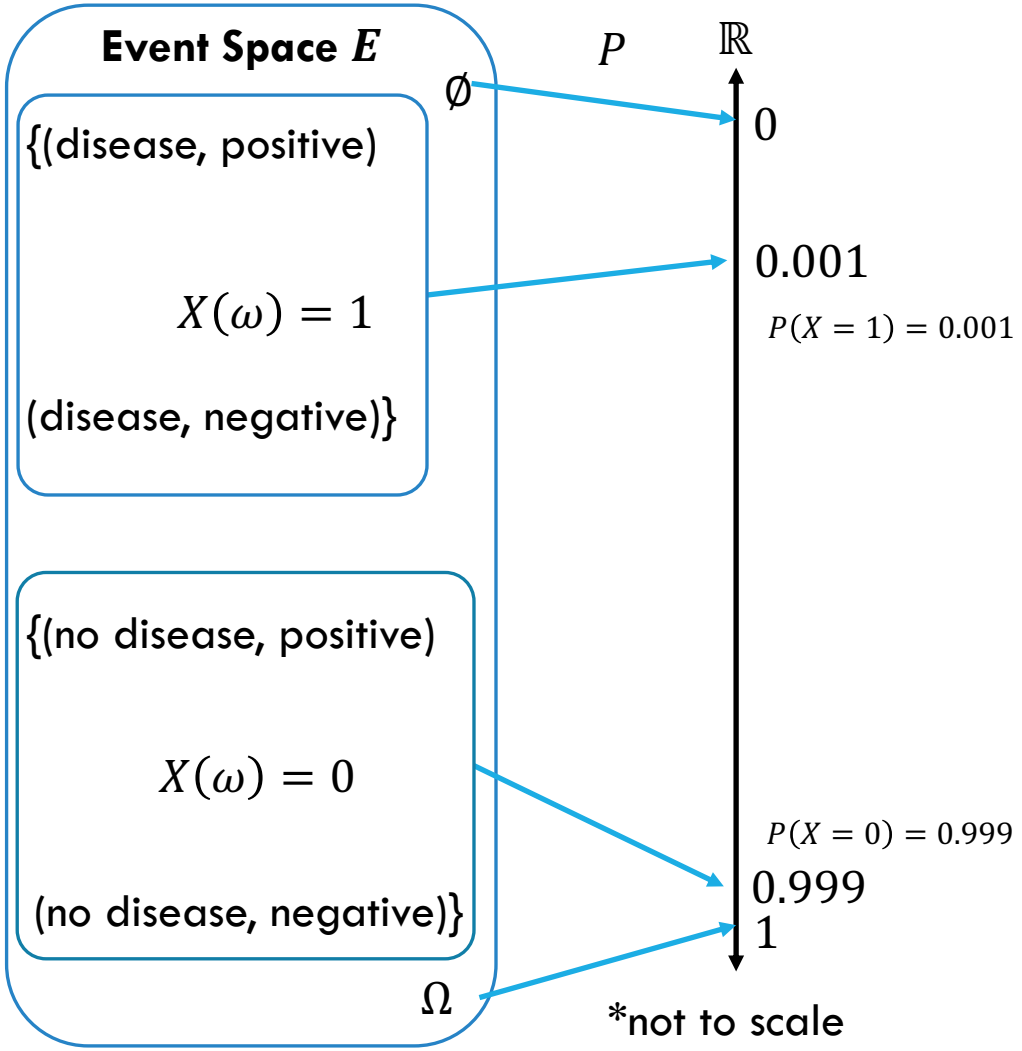
- Rolling a six-faced die: $Val(X) = \{1,2, \dots, 6\}$
- Weather conditions: $Val(X) = \{$"rain", "cloud", "snow", "sun", "wind"$\}$
- Number of people on the next train: $Val(X) = \mathbb{Z}_{\geq 0}$

Continuous random variables

- Time taken to finish an exam: $Val(X) = [1,2]$ hours
- Height of a tree: $Val(X) = \mathbb{R}_{>0}$
- Ambient Temperature: $Val(X) = \mathbb{R}$

TREE DIAGRAM



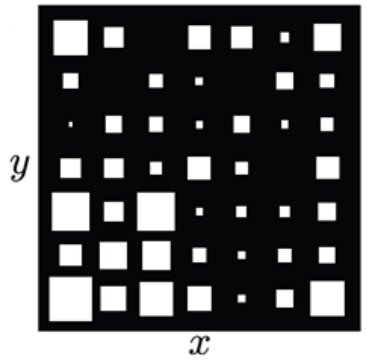


JOINT PROBABILITY

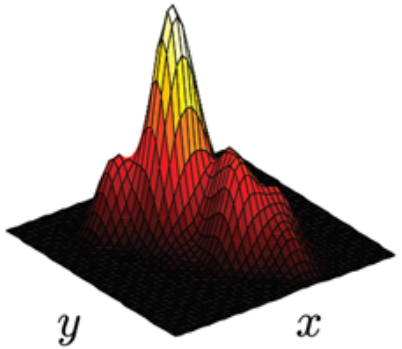
Consider all combination of events of two random variables X and Y.

Some combinations of outcomes are more likely than others.

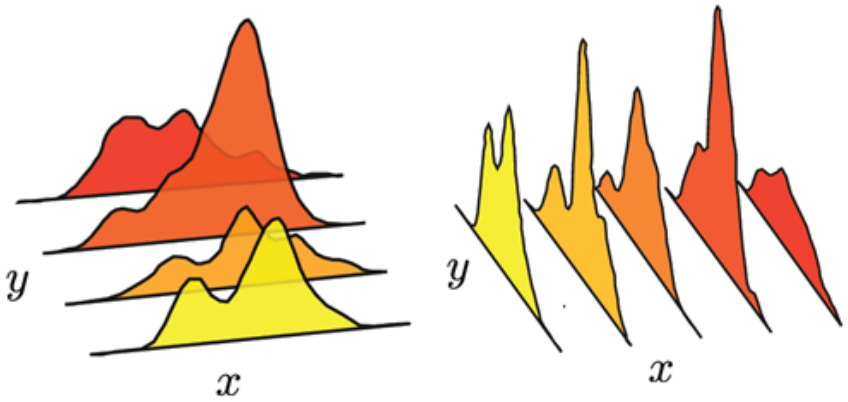
Discrete



Continuous



Discrete-Continuous



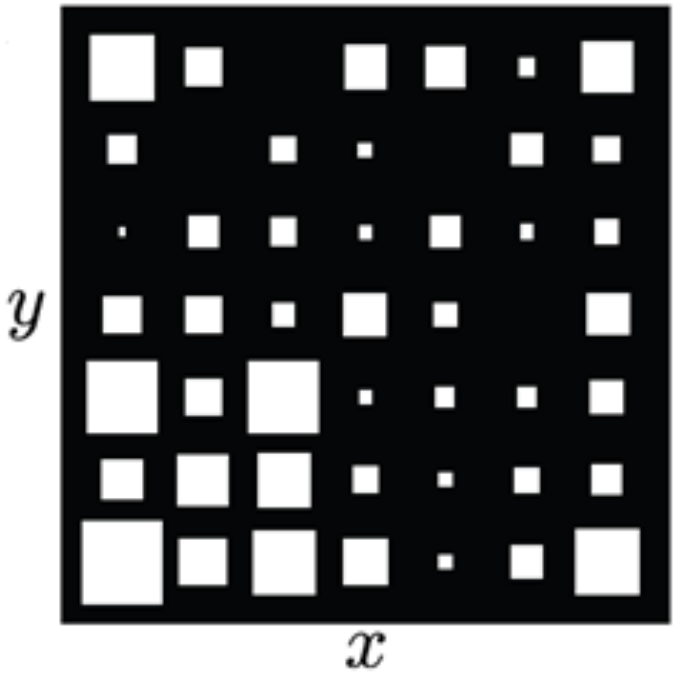
Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

JOINT PROBABILITY

This is captured in the joint probability distribution $p(x, y)$.

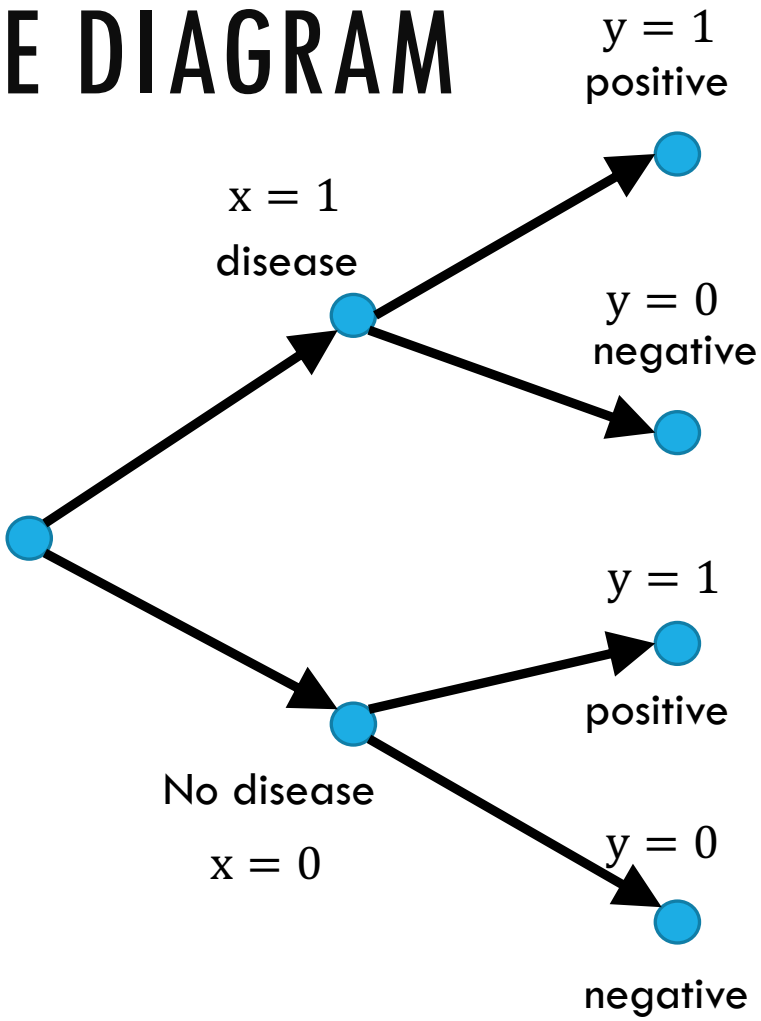
Read as "probability of X and Y".

Can be more than two random variables, i.e. $p(a, b, c, \dots)$.



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

TREE DIAGRAM



We can now have:

$$p(x, y) = P(X = x \text{ and } Y = y)$$

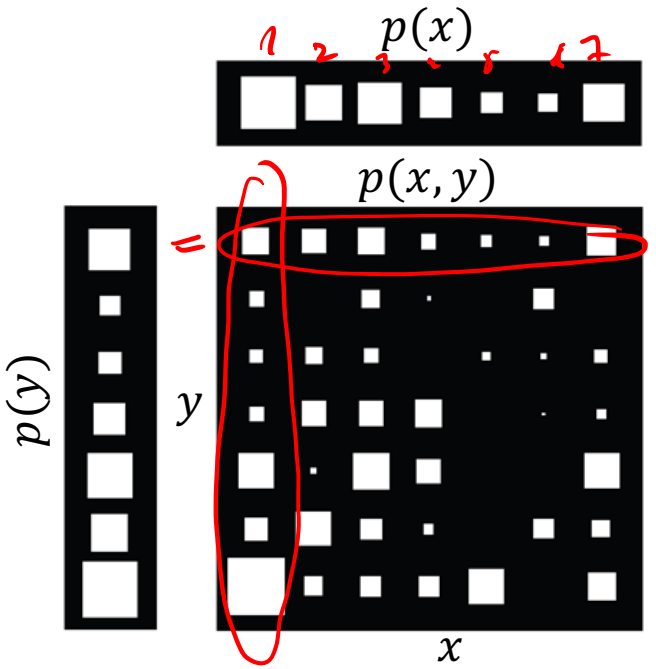
MARGINALIZATION

Recover probability distribution of any variable in a joint distribution by integrating (or summing) over all other variables.

Also known as the "sum rule" of probability.

$$p(x) = \sum_{y} p(x, y)$$

$$p(y) = \sum_{x} p(x, y)$$

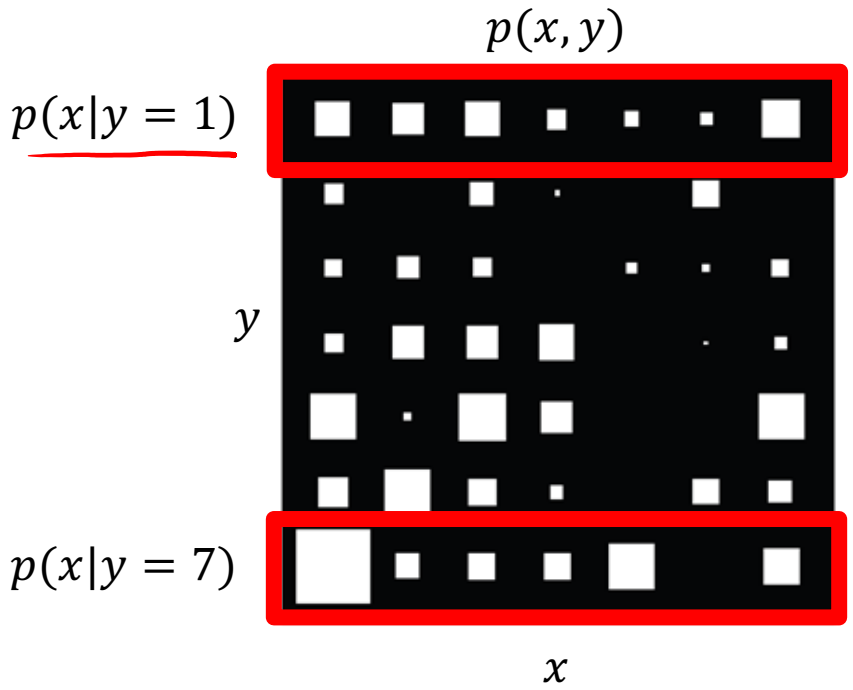


Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

CONDITIONAL PROBABILITY

$p(x|Y = y^*)$: "probability of X given Y $= y^*$".

Relative propensity of the random variable X to take different outcomes given that the random variable Y is fixed to value y^*.



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

PROBABILITY: CONDITIONAL PROBABILITY

$$P(x|Y=y^*) = \frac{p(x,Y=y^*)}{\sum_x p(x,Y=y^*)} = \frac{p(x,Y=y^*)}{p(Y=y^*)}$$

Usually written in compact form:

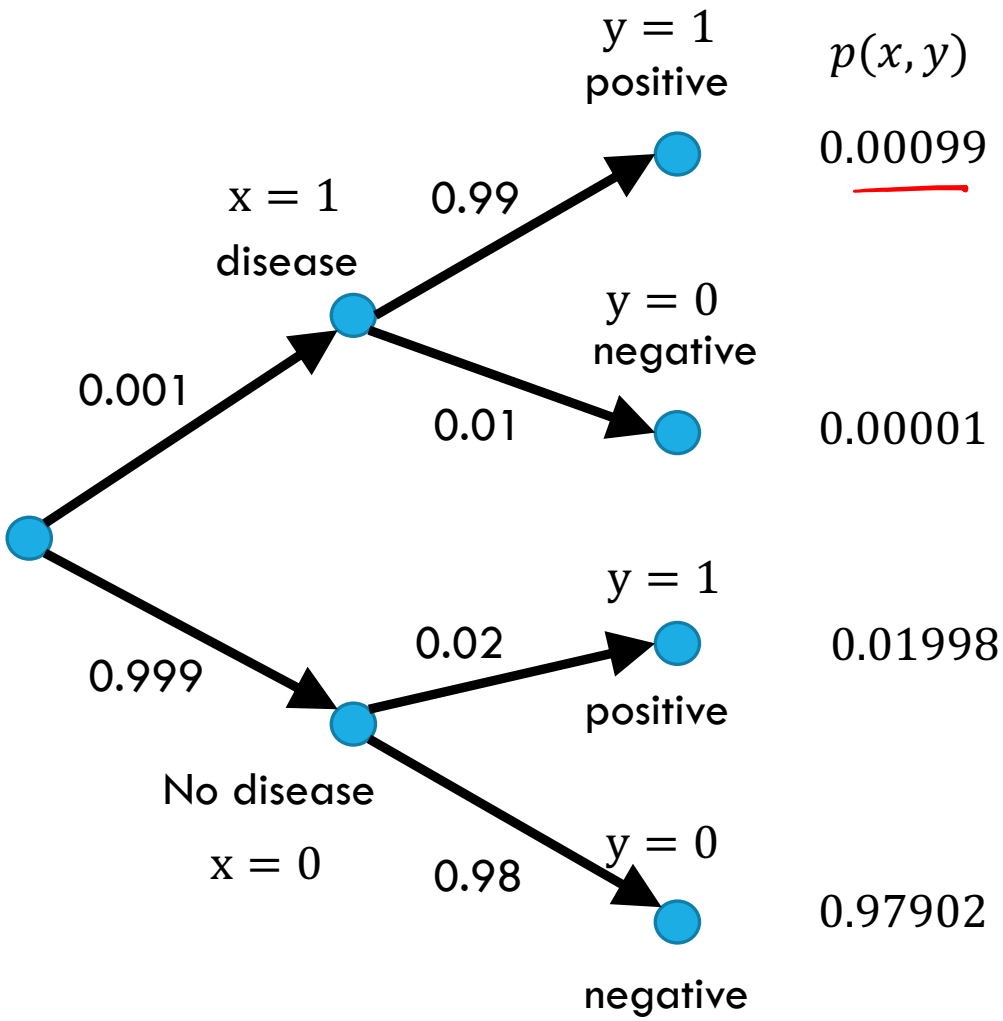
$$p(x|y) = \frac{p(x,y)}{p(y)}$$

Which can be re-arranged to give:

$$p(x,y) = p(x|y)p(y)$$
$$p(x,y) = p(y|x)p(x)$$

known as "chain rule" or "product rule" of probability.

WHAT IS THE PROBABILITY I HAVE THE DISEASE GIVEN THE TEST IS POSITIVE?



$p(x, y)$

0.00099

0.00001

0.01998

0.97902

We want: $p(X = 1 \mid Y = 1)$

$$p(x|Y = 1) = \frac{p(x, Y = 1)}{p(Y = 1)} = \frac{p(x, Y = 1)}{\sum_x p(x, Y = 1)}$$

sum rule

$$p(Y = 1) = 0.00099 + 0.01998 = 0.02097$$

$$p(X = 1|Y = 1) = 0.00099/0.02097 = 0.0472 < 5\%$$

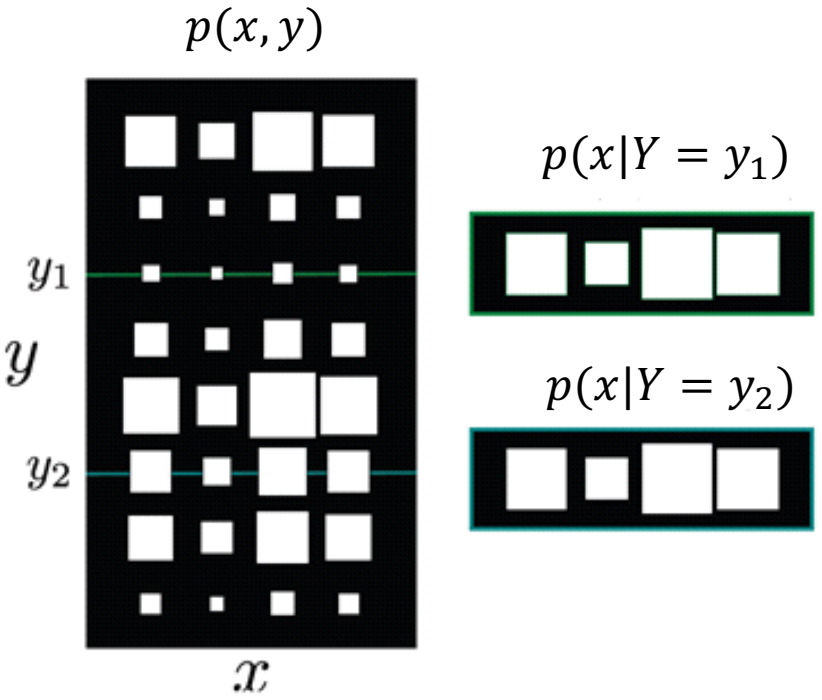
INDEPENDENCE

The independence of X and Y means that every conditional distribution is the same.

The value of Y tells us nothing about X and vice-versa.

$$p(x|y) = p(x)$$

$$p(y|x) = p(y)$$

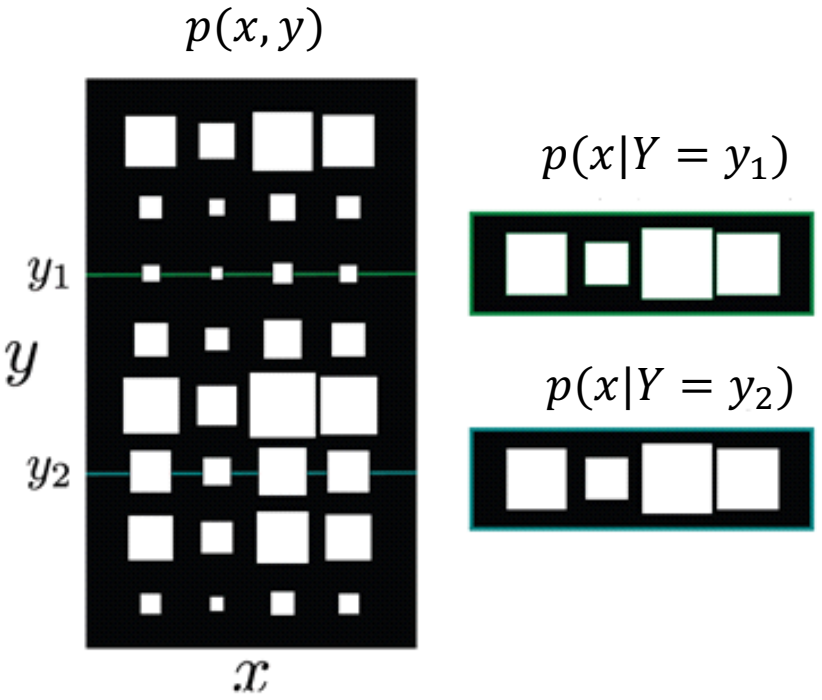


Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

INDEPENDENCE

When variables are independent, the joint factorizes into a product of the marginals:

$$p(x,y) = p(x|y)p(y)$$
$$= p(x)p(y)$$



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

SUMMARY: SUM AND PRODUCT RULES

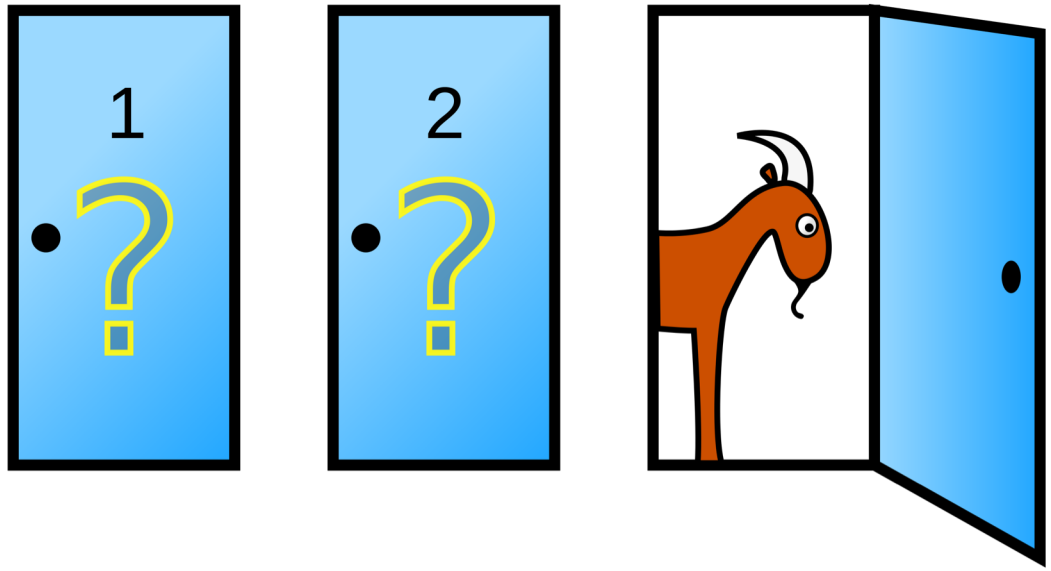
Sum rule:

$$p(x) = \sum_{y} p(x, y)$$

Product/Chain rule:

$$p(x, y) = p(x|y)p(y)$$

YOU'VE SEEN THIS PROBLEM BEFORE, RIGHT?



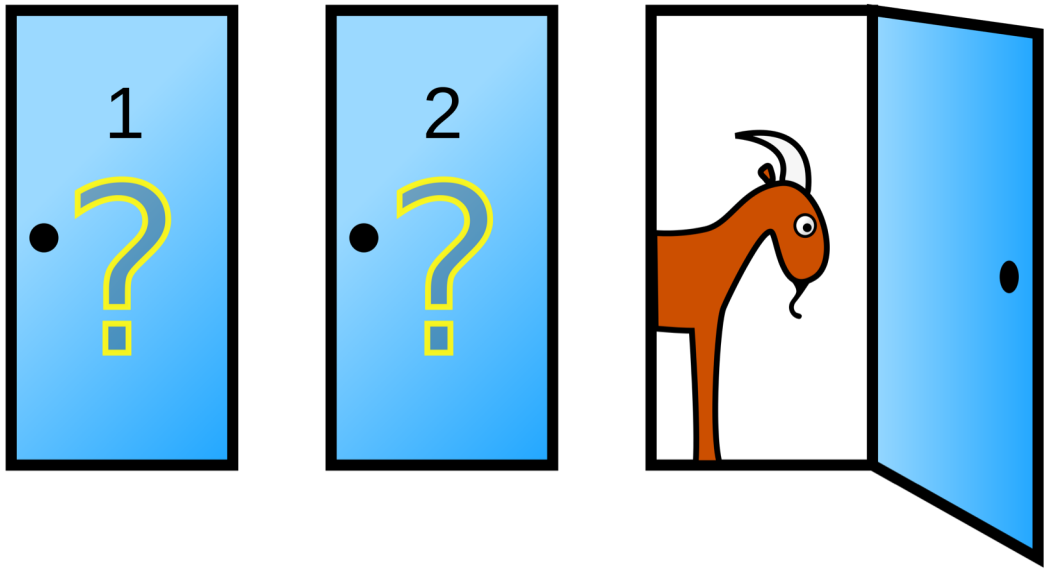
Should you switch?

A. Yes

B. No

You're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say Door 1, and the host, who knows what's behind the doors, opens another door, say Door 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

YOU'VE SEEN THIS PROBLEM BEFORE, RIGHT?



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A. Yes

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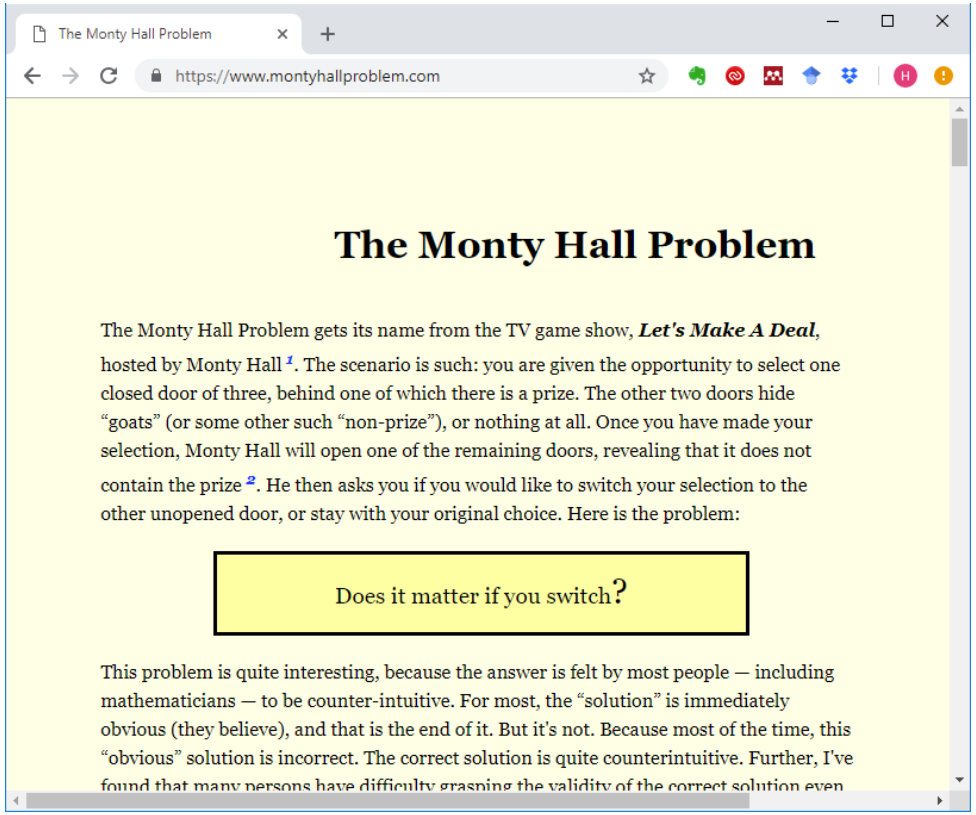
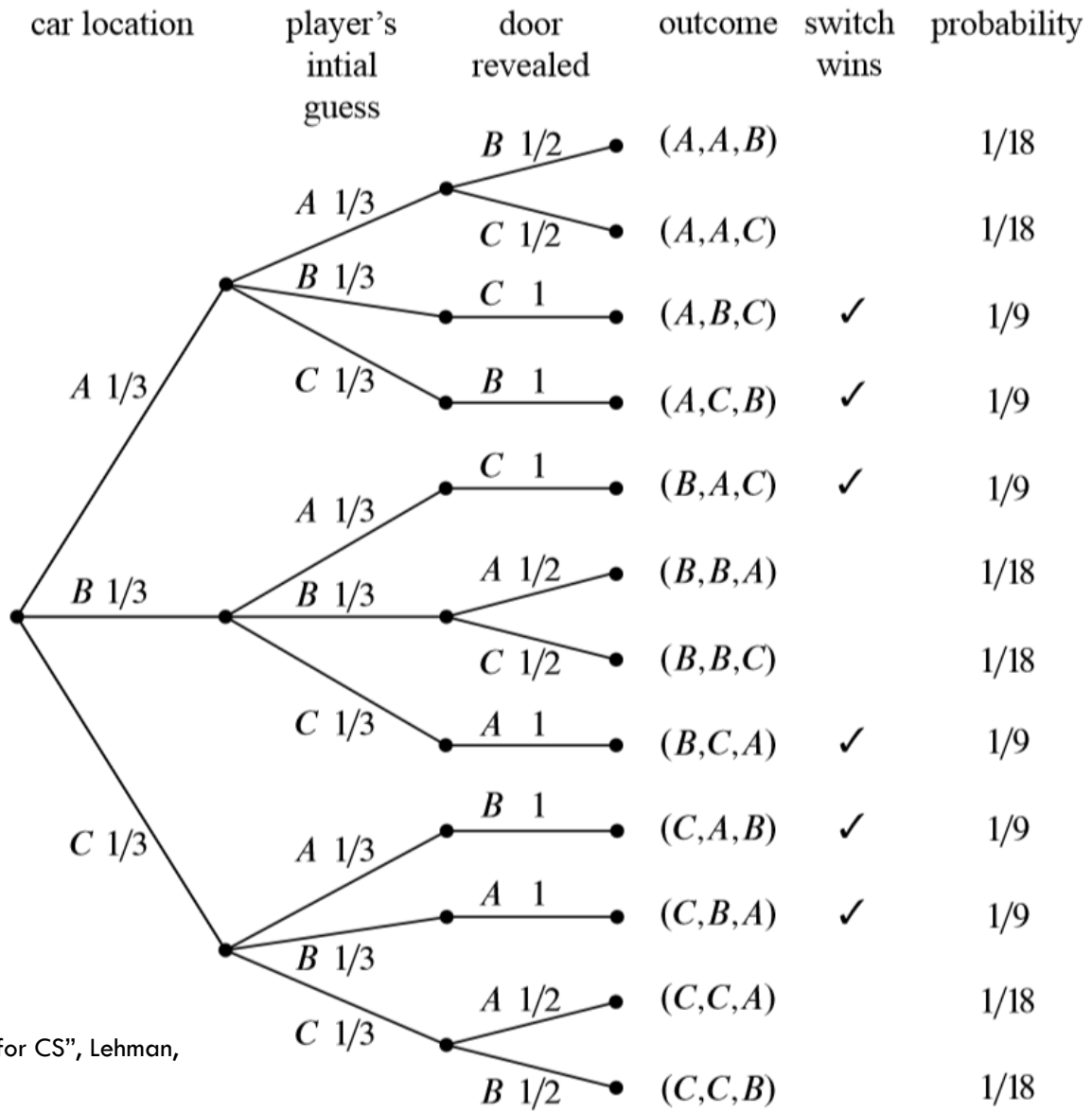
You're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say Door 1, and the host, who knows what's behind the doors, opens another door, say Door 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

MONTY-HALL PROBLEM





Adapted from "Math for CS", Lehman, Leighton and Meyer

$p(\text{switch wins}) = 6 \times \frac{1}{9} = \frac{2}{3}$

Should switch!

2 NUMBERS GAME

Team 1:

- Pick 2 **different** numbers between 0 and 10.
- Write each number on a piece of paper each.
- Turn the papers face down.

Team 2:

- Objective is to pick the **larger number**.
- Pick one of the pieces of paper.
- Have a peek at the number.
- **Decide:** do you keep this number or *switch*?

Question: Can Team 2 win more than 50% of the time?

TEAM 2 STRATEGY

Guess a number $Z \in [0, 10)$

Take a peek at one of the numbers, lets call this x

If $x \leq Z$, switch, else stick with x

STRATEGY ANALYSIS

Let $l < h$ be the numbers chosen by Team 1.

3 possible cases:

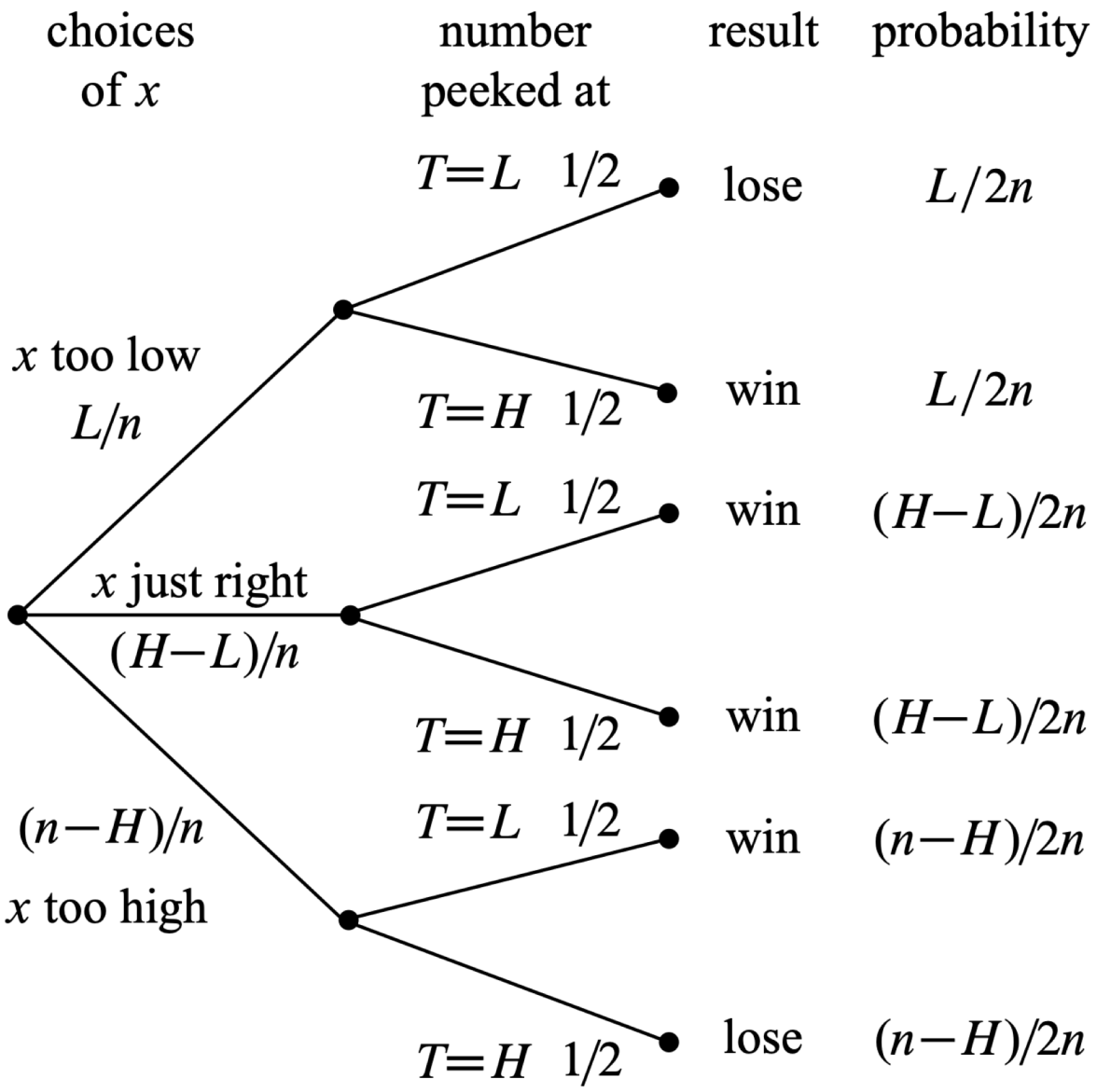
- **Case** M: $l \leq Z < h$:
  - Team 2 wins always!
  - $p(\text{win}|M) = 1$ and $p(M) \geq \frac{1}{10}$
- **Case** H: $h \leq Z$:
  - Team 2 switch. Only wins if picked l
  - $p(\text{win}|H) = \frac{1}{2}$
- **Case** L: $Z < l$:
  - Team 2 switch. Only wins if picked h
  - $p(\text{win}|L) = \frac{1}{2}$

$$p(win) \geq \left(1 \times \frac{1}{10}\right) + \frac{1}{2}\left(1 - \frac{1}{10}\right)$$

$$p(win) \geq \frac{11}{20}$$

More complete derivation in:
MIT Math for CS, Chapter 18.3.3
https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-spring-2015/readings/MIT6_042JS15_Session31.pdf

IN GENERAL



More complete derivation in: MIT Math for CS, Chapter 18.3.3

https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-spring-2015/readings/MIT6_042JS15_Session31.pdf

QUESTIONS?

Poll Everywhere

https://bit.ly/2LvG9bq

THE NEXT 2 DAYS

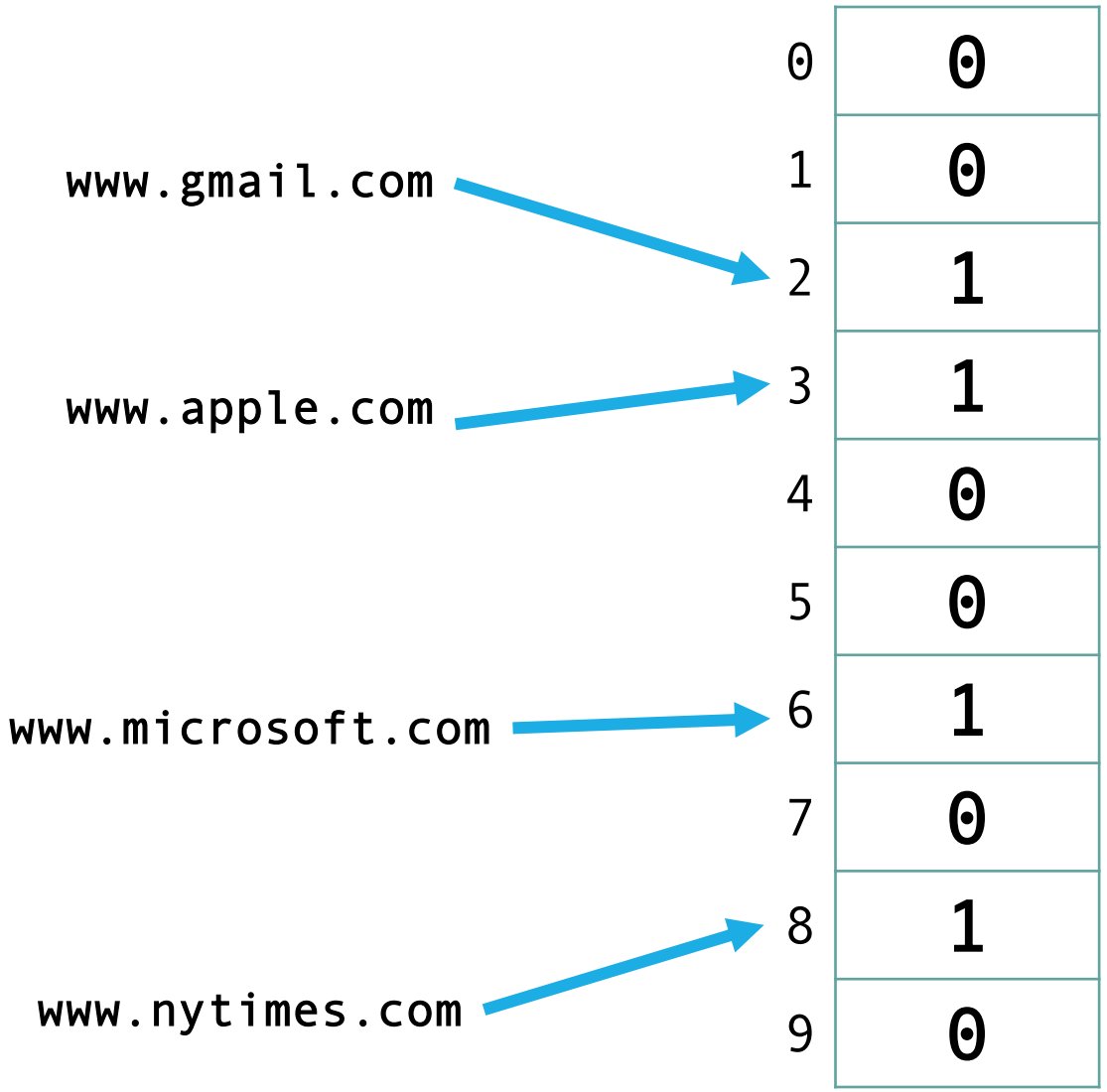
- Probability Review
- Bloom Filters
- Cuckoo Hashing

IDEA: STORE LESS!

Only set k bits per item in a table of size m.

To start, let k = 1

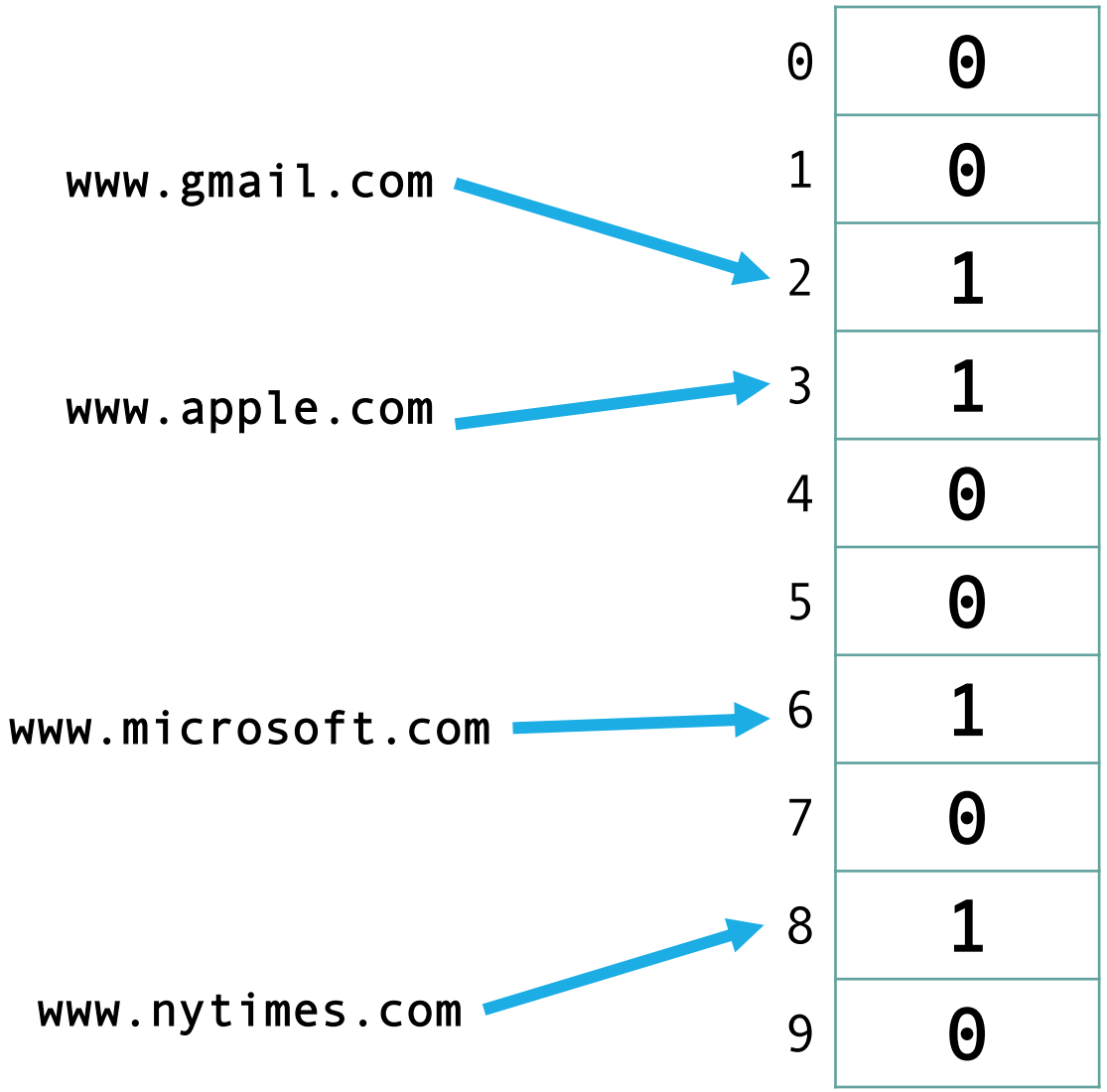
- For each item, we only set 1 bit!



IDEA: STORE LESS!

```
insert(key)
    h = hash(key)
    m_table[h] = 1

lookup(key)
    h = hash(key)
    return (m_table[h] == 1)
```



PROBABILITY OF A FALSE POSITIVE?

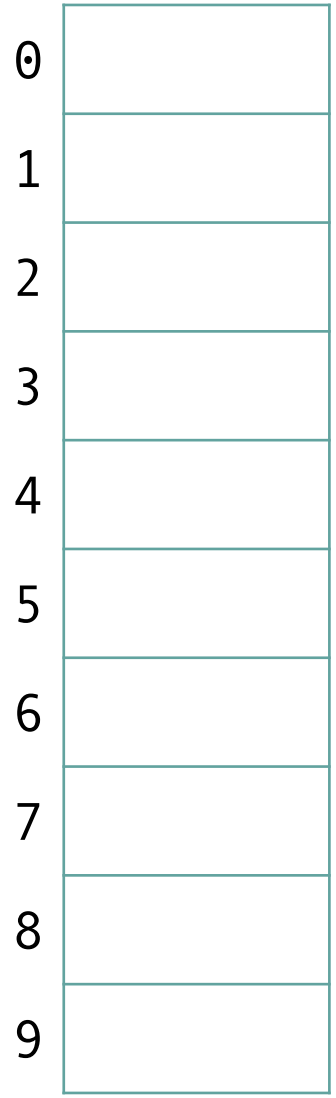
Table size is m, we store n elements

Set $k = 1$ bit per item

Test an item **not** in the set.

$$p(\text{false positive}) = 1 - p(\text{hits a cell with } 0)$$

$$p(\text{hits cell with } 0) = \underbrace{\left(1 - \frac{1}{m}\right)^n}_{\text{Elements are independently hashed}} \approx e^{-\frac{n}{m}}$$



PROBABILITY OF A FALSE POSITIVE?

Test an item **not** in the set.

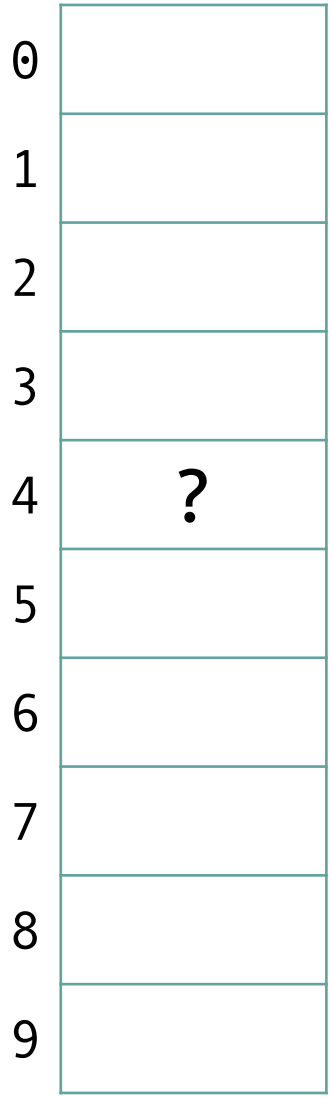
Let's say the item hits cell 4.

What is the probability that the cell is a 0?

- No previous insertion has set it to 1

Say $n = 1$ (only 1 item inserted in the table)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)$ (Simple uniform hashing assumption)



PROBABILITY OF A FALSE POSITIVE?

Say $n = 1$ (only 1 item inserted in the table)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)$ (Simple uniform hashing assumption)

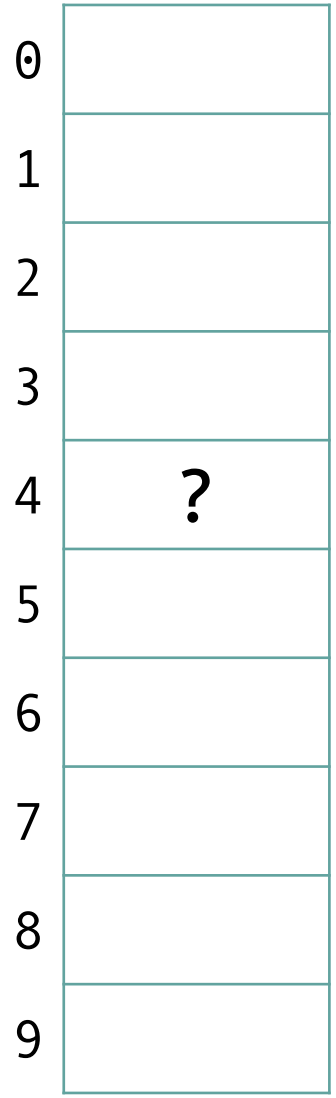
Say n = 2 (2 items inserted)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^2$ (Simple uniform hashing assumption)

…

For n items inserted

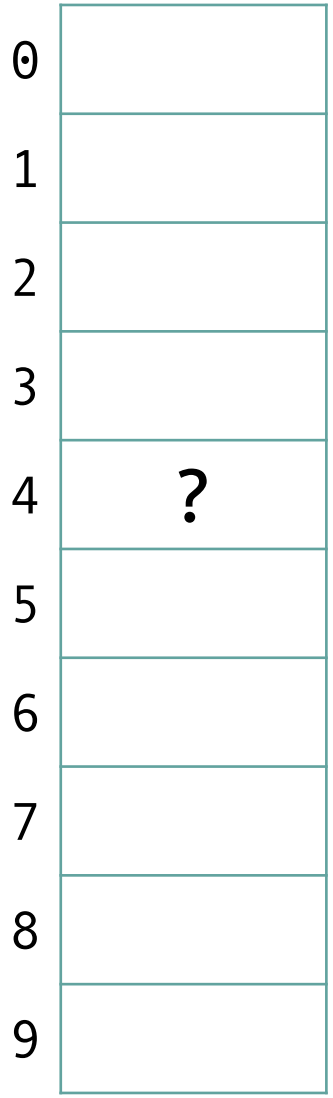
- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^n$ (Simple uniform hashing assumption)



PROBABILITY OF A FALSE POSITIVE?

For n items inserted

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^n \approx e^{-\frac{n}{m}}$
- Note:
  - $\lim_{n\to\infty} \left(1 - \frac{1}{m}\right)^n = e^{-\frac{n}{m}}$



PROBABILITY OF A FALSE POSITIVE?

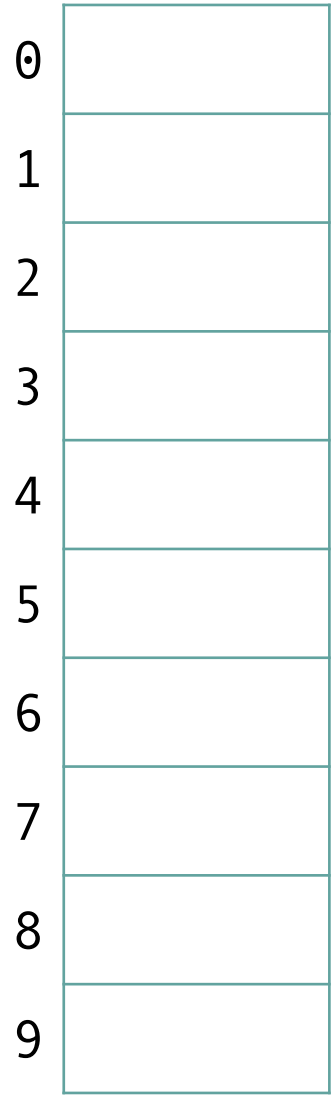
Table size is m, we store n elements

Set $k = 1$ bit per item

Test an item **not** in the set.

$$p(\text{false positive}) = 1 - p(\text{hits a cell with } 0)$$

$$p(\text{hits cell with } 0) = \underbrace{\left(1 - \frac{1}{m}\right)^n}_{\text{Elements are independently hashed}} \approx e^{-\frac{n}{m}}$$



PROBABILITY OF A FALSE POSITIVE?

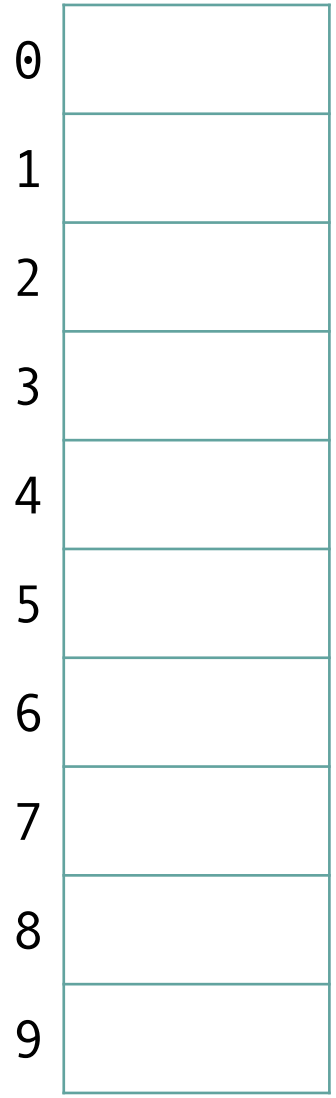
Table size is m, we store n elements

Set $k = 1$ bit per item

Test an item **not** in the set.

$$p(\text{false positive}) = 1 - p(\text{hits a cell with } 0)$$

$$p(\text{false positive}) \approx 1 - e^{-\frac{n}{m}}$$



LET'S SET P(FALSE POSITIVE)!

Given n items, say we want $\hat{p} = 10\% = 0.1$

Then,

- $p(\text{false positive}) = 1 - e^{-\frac{n}{m}} \leq \hat{p}$
- $\frac{n}{m} \leq \ln\left(\frac{1}{1-\hat{p}}\right)$
- $m \geq \frac{n}{\ln\left(\frac{1}{1-\hat{p}}\right)}$
- So, $m \geq 9.49n$

If you wanted the derivation:

$$1 - e^{-\frac{n}{m}} \leq \hat{p}$$
$$e^{-\frac{n}{m}} \geq 1 - \hat{p}$$
$$-\frac{n}{m} \geq \ln(1 - \hat{p})$$
$$\frac{n}{m} \leq \ln\left(\frac{1}{1-\hat{p}}\right)$$
$$m \geq \frac{n}{\ln\left(\frac{1}{1-\hat{p}}\right)}$$

LET'S SET P(FALSE POSITIVE)!

Given n items, say we want $\hat{p} = 1\% = 0.01$

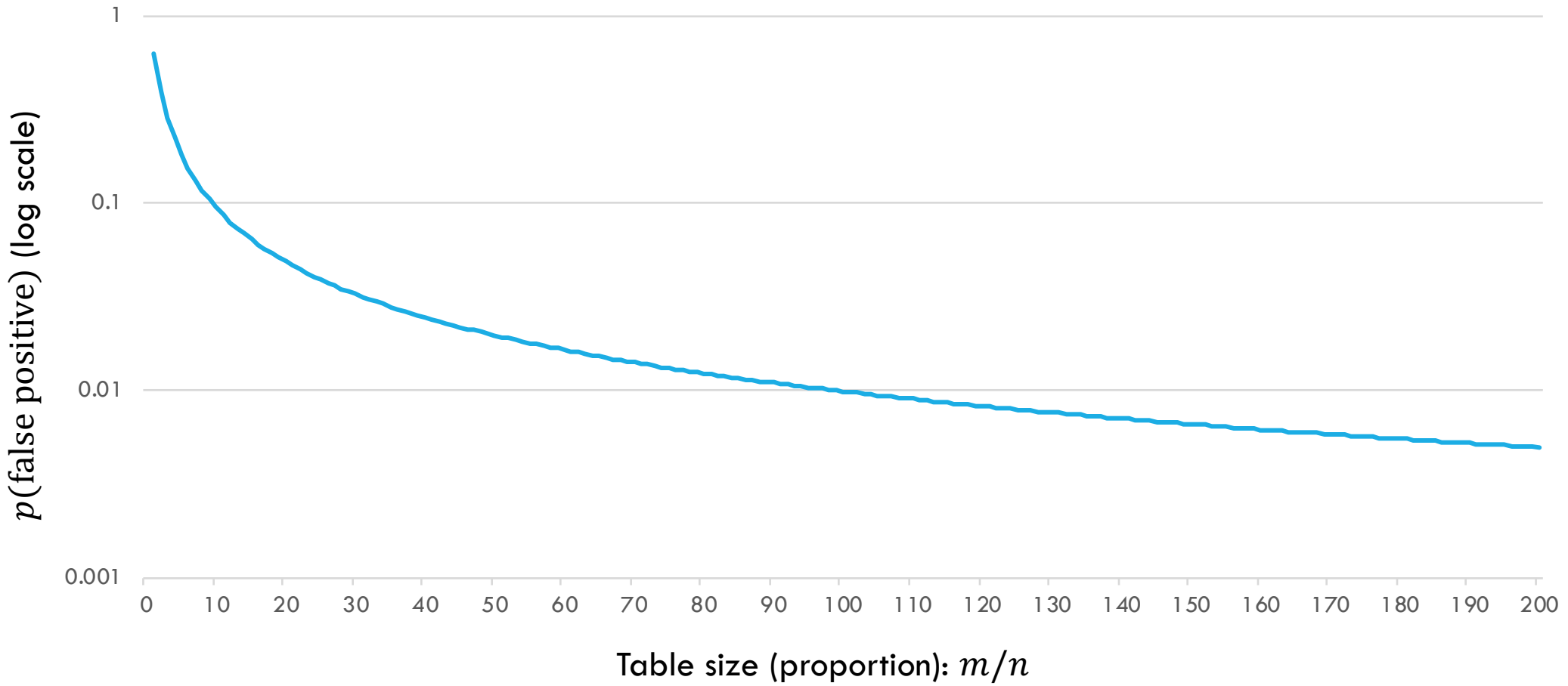
Then,

- $p(\text{false positive}) = 1 - e^{-\frac{n}{m}} \leq \hat{p}$
- $\frac{n}{m} \leq \ln\left(\frac{1}{1-\hat{p}}\right)$
- $m \geq \frac{n}{\ln\left(\frac{1}{1-\hat{p}}\right)}$
- So, $m \geq 99.5n$

If you wanted the derivation:

$$1 - e^{-\frac{n}{m}} \leq \hat{p}$$
$$e^{-\frac{n}{m}} \geq 1 - \hat{p}$$
$$-\frac{n}{m} \geq \log(1 - \hat{p})$$
$$\frac{n}{m} \leq \log\left(\frac{1}{1-\hat{p}}\right)$$
$$m \geq \frac{n}{\log\left(\frac{1}{1-\hat{p}}\right)}$$

PROBABILITY OF FALSE POSITIVE



SUMMARY SO FAR

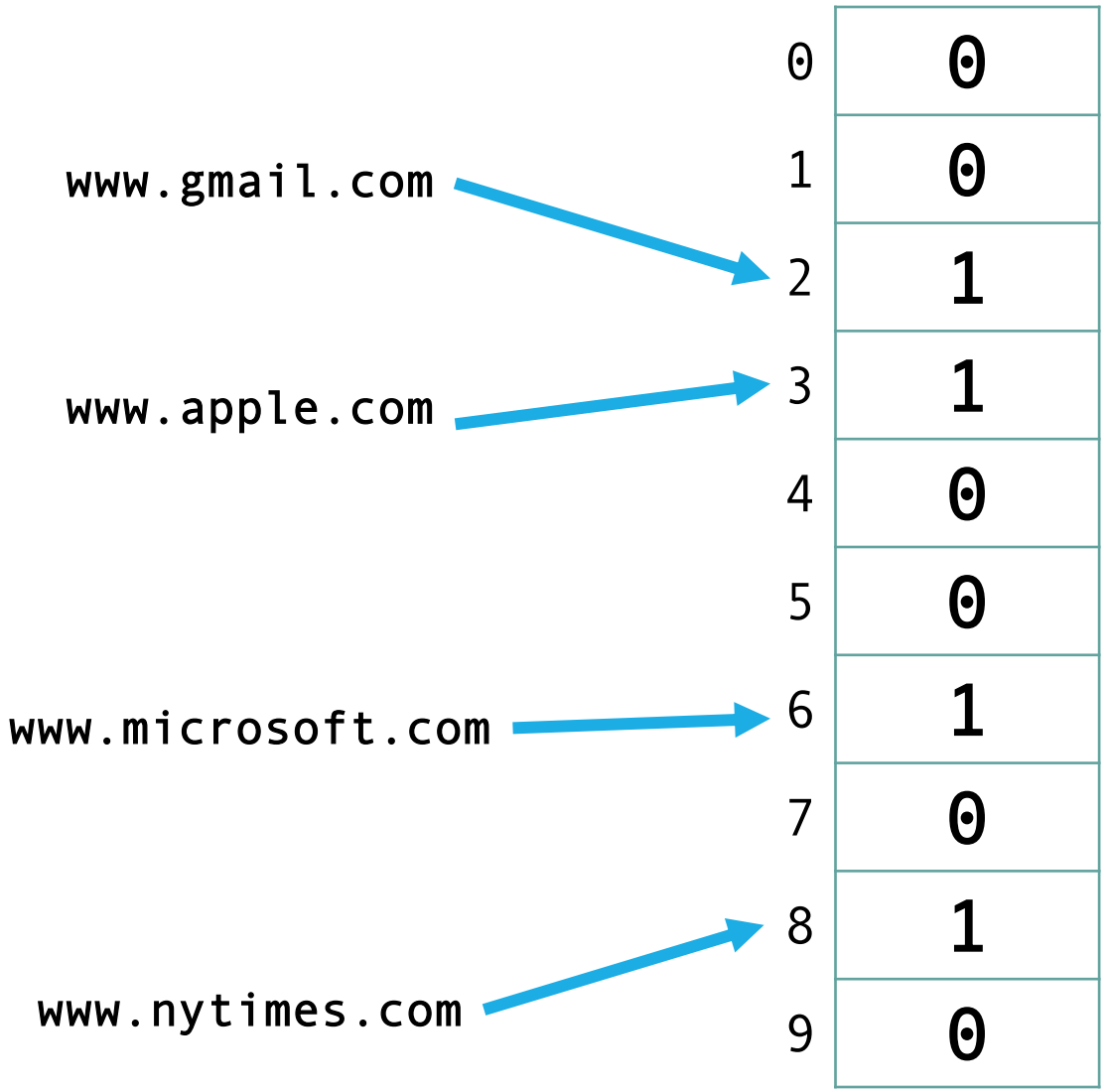
Only set k bits per item in a table of size m.

To start, let k = 1

- For each item, we only set 1 bit!

Reduced space but chance of false positives.

Can we do better?



BLOOM FILTER: SET K BITS PER ITEM

Use k hash functions

E.g., for $k = 2$, have h_1 and h_2

Insert x:

- Set the cells at <u>both</u> $h_1(x)$ and $h_2(x)$ to be 1

Lookup x:

- Return True only if cells at <u>both</u> $h_1(x)$ and $h_2(x)$ are 1

BLOOM FILTER: SET K BITS PER ITEM

Use k hash functions

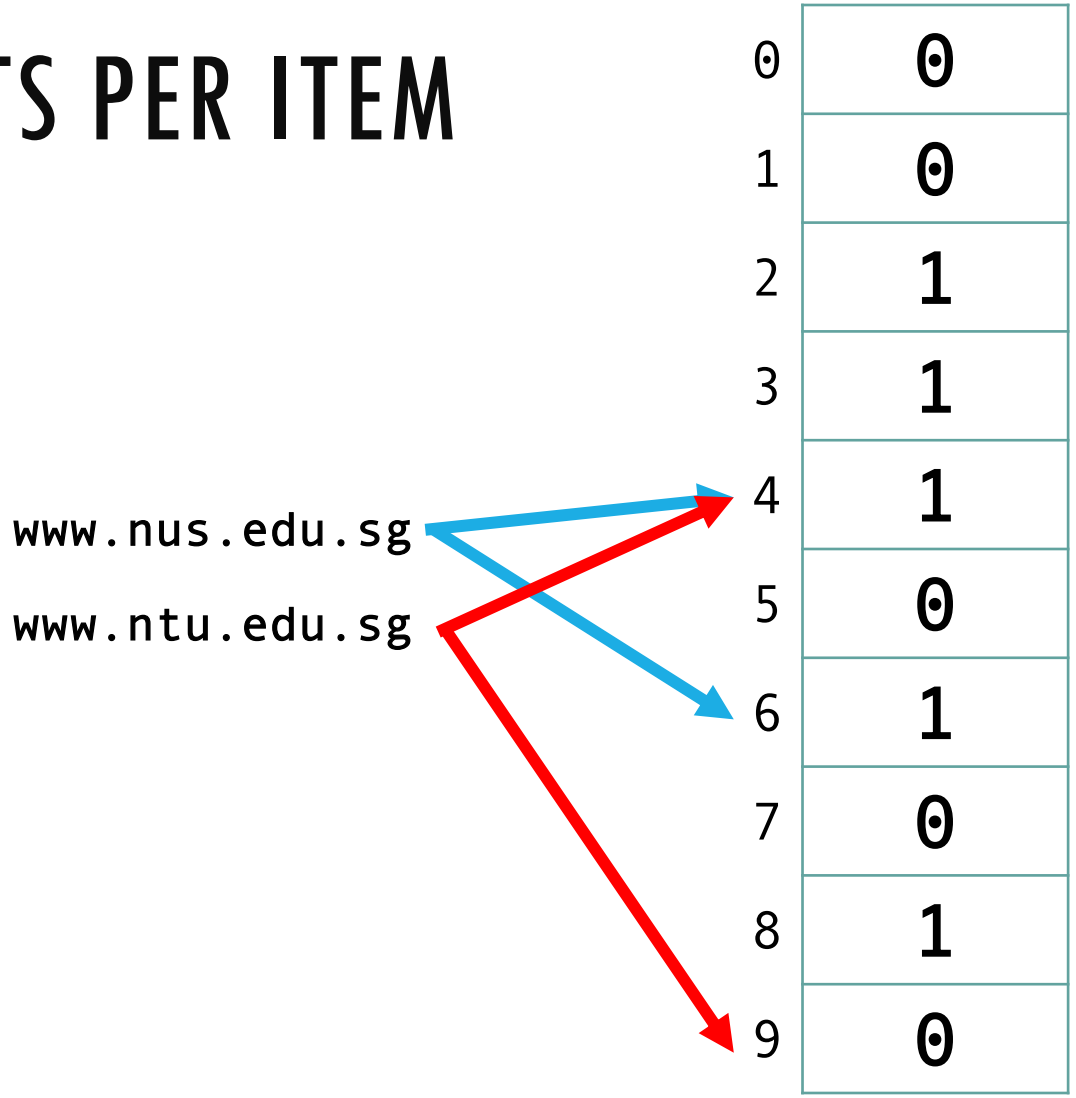
E.g., for $k = 2$, have h_1 and h_2

Insert x:

- Set the cells at <u>both</u> $h_1(x)$ and $h_2(x)$ to be 1

Lookup x:

- Return True only if cells at <u>both</u> $h_1(x)$ and $h_2(x)$ are 1

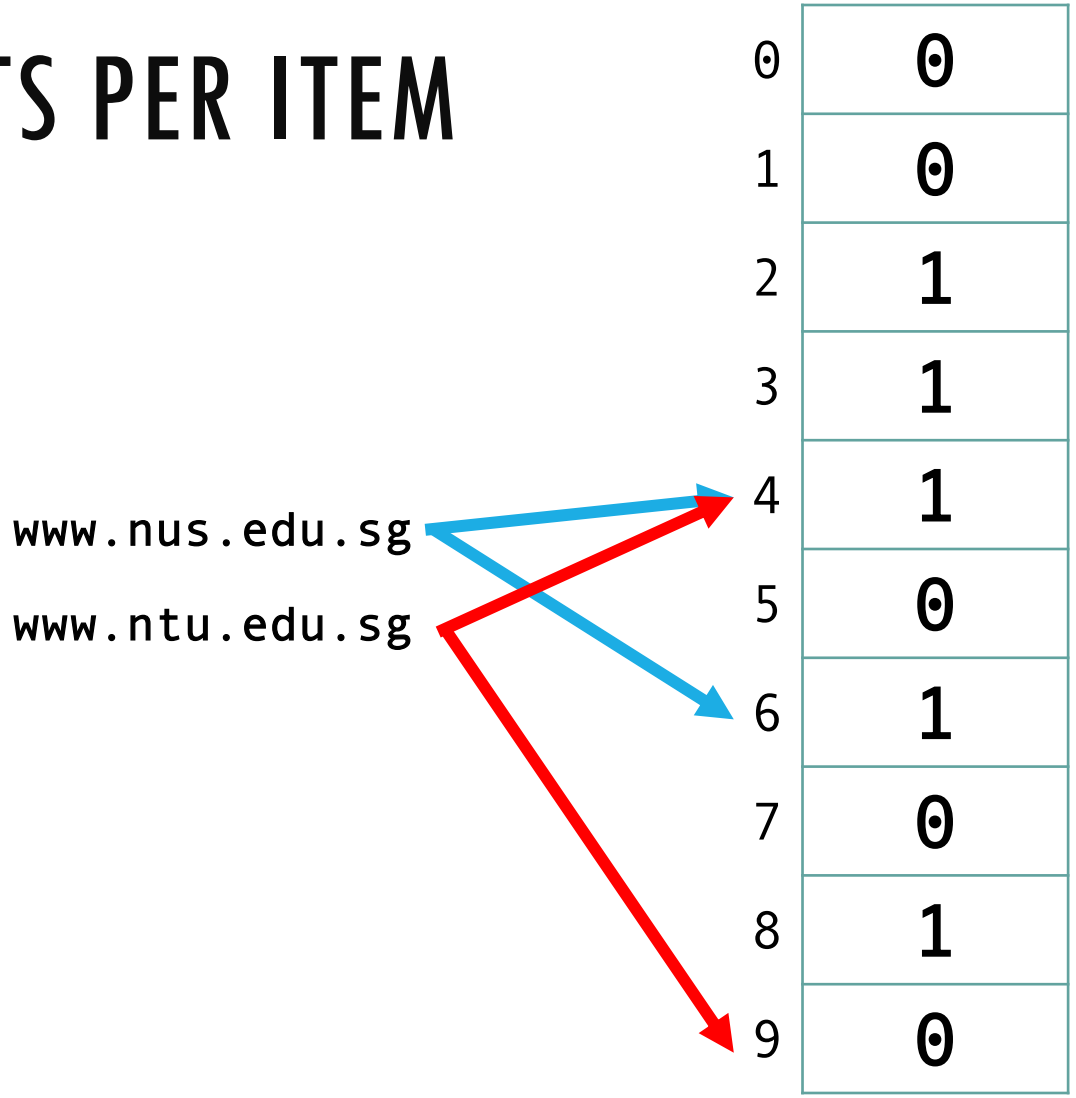


BLOOM FILTER: SET K BITS PER ITEM

Trade-offs:

- Require k collisions for a false positive
- Each item takes more "space"

What is the probability of a false positive?



RECALL: PROB. OF A FALSE POSITIVE

Table size is m, we store n elements

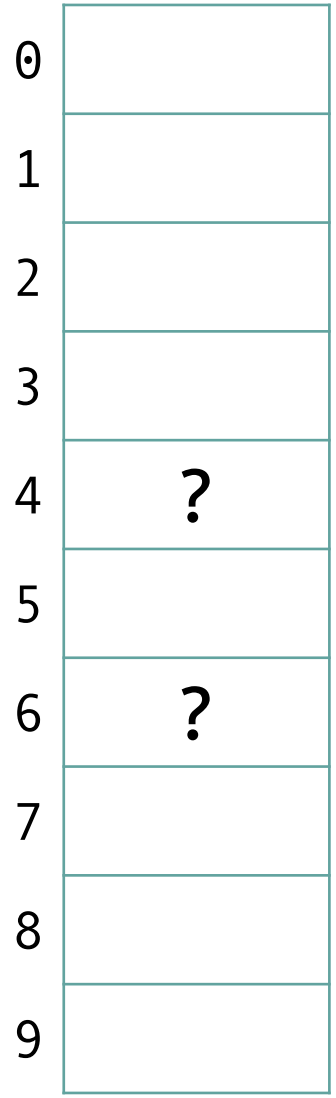
Set k bits per item

Test an item **not** in the set.

$$p(\text{false positive}) = p(\text{all cells are } 1)$$

$$p(\text{false positive}) = \underbrace{\big(1 - p(\text{cell is } 0)\big)^k}$$

Not strictly correct! Assumes independence for probabilities of each bit being set.



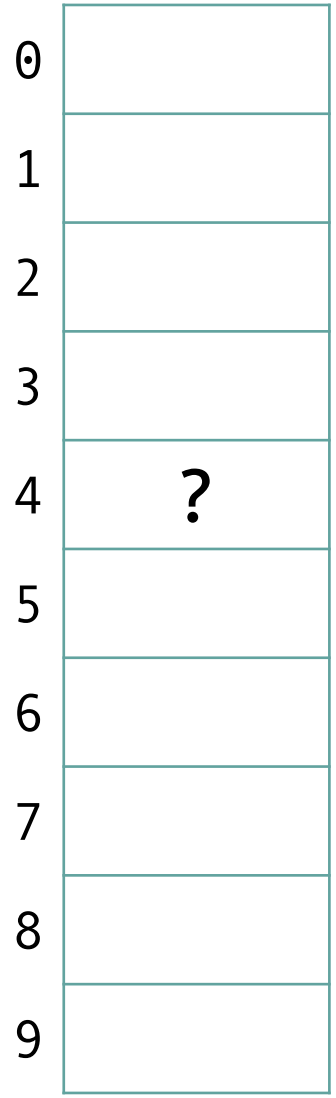
PROBABILITY OF A FALSE POSITIVE?

What is the probability that a cell is a 0?

- No previous insertion has set it to 1
- None of the previous k hash functions (per insert) set it to 1

Say $n = 1$ (only 1 item inserted in the table)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^k$ (Simple uniform hashing assumption)



PROBABILITY OF A FALSE POSITIVE?

Say $n = 1$ (only 1 item inserted in the table)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^{k}$ (Simple uniform hashing assumption)

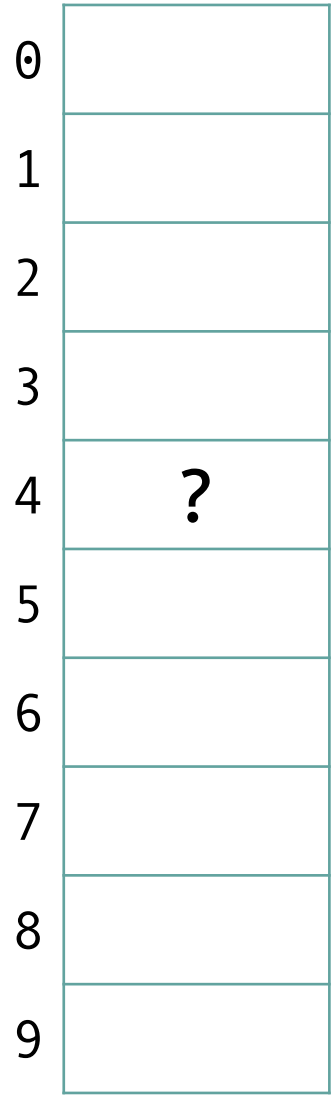
Say $n = 2$ (2 items inserted)

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^{2k}$ (Simple uniform hashing assumption)

…

For n items inserted

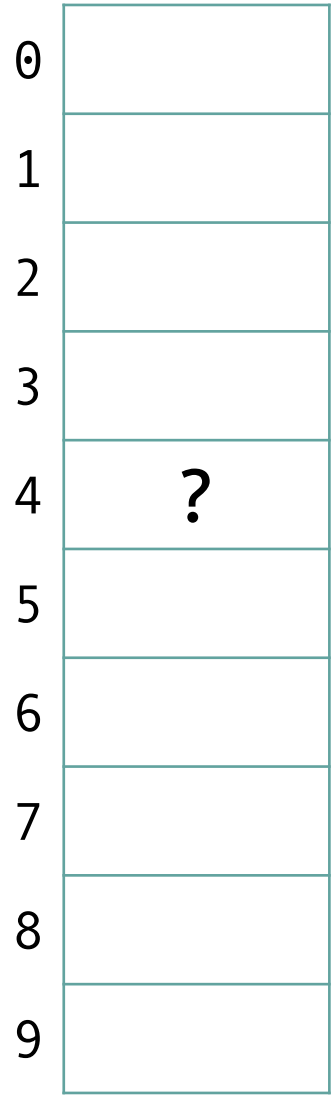
- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^{nk}$ (Simple uniform hashing assumption)



PROBABILITY OF A FALSE POSITIVE?

For n items inserted

- Then $p(\text{cell is } 0) = \left(1 - \frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$



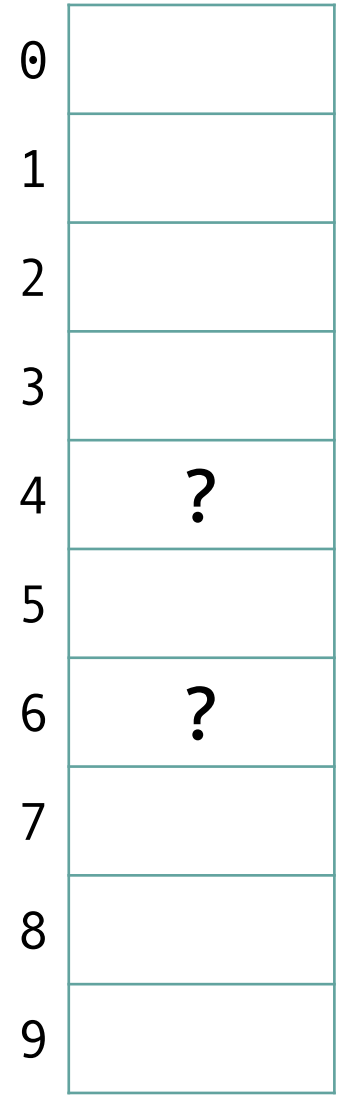
RECALL: PROB. OF A FALSE POSITIVE

Table size is m, we store n elements

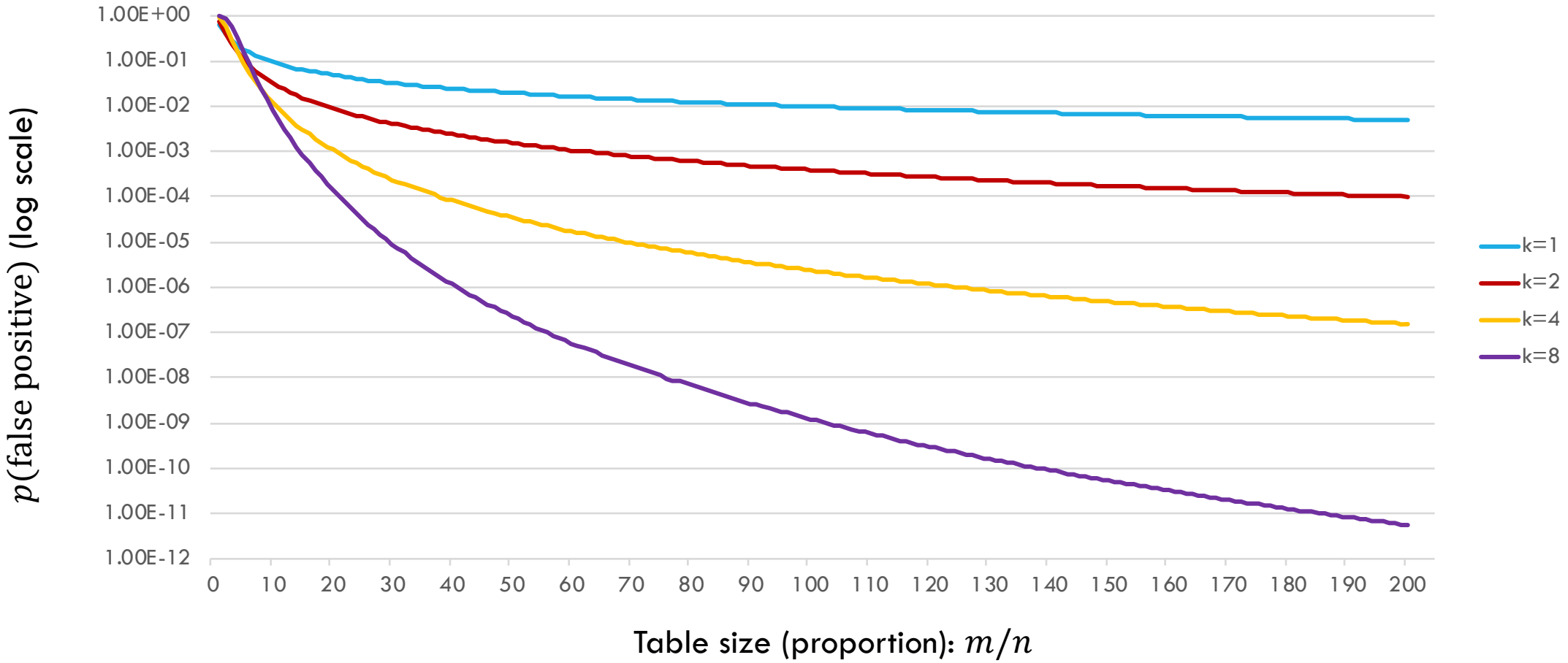
Set k bits per item

Test an item **not** in the set.

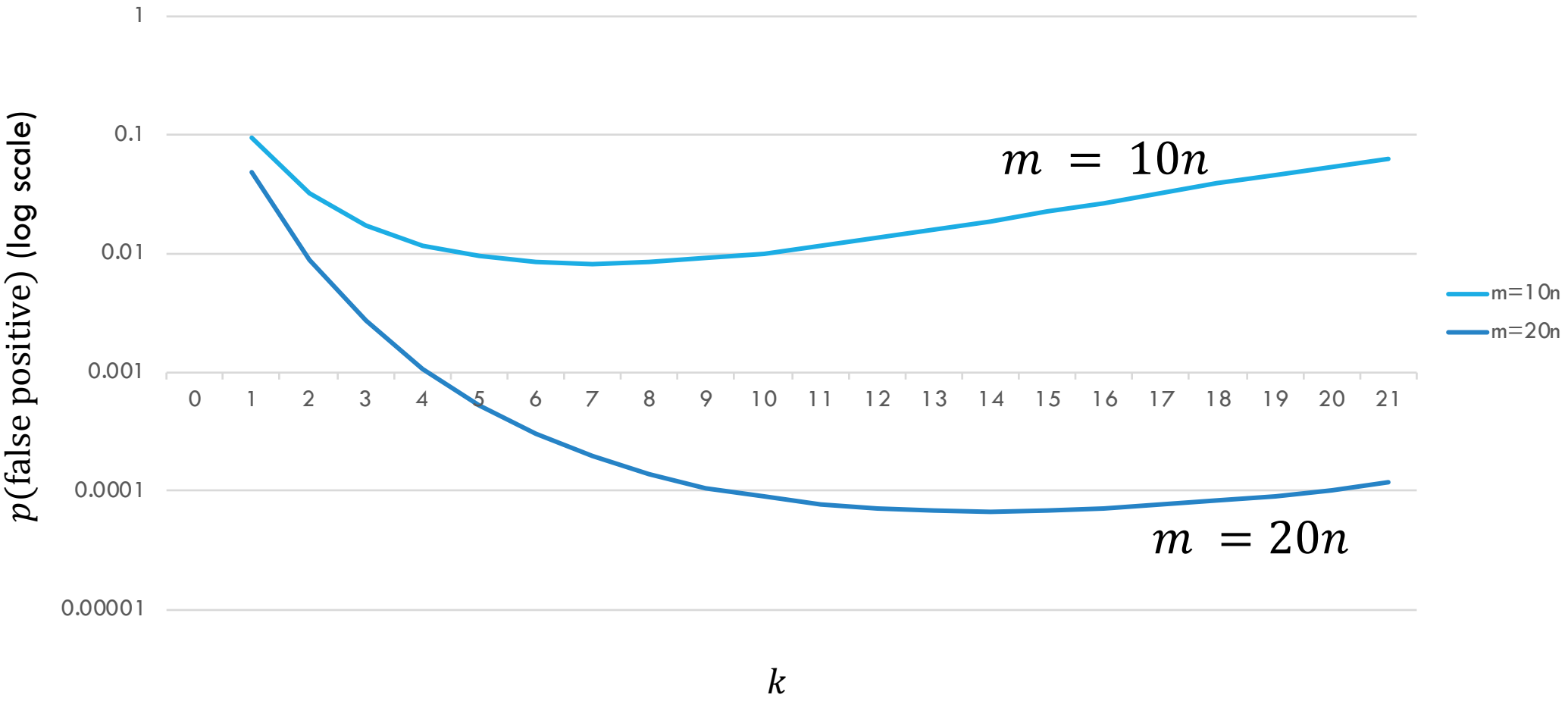
$$p(\text{false positive}) = p(\text{all cells are 1})$$
$$p(\text{false positive}) = \left(1 - p(\text{cell is 0})\right)^k$$
$$p(\text{false positive}) \approx \left(1 - e^{-\frac{kn}{m}}\right)^k$$



PROBABILITY OF FALSE POSITIVE



PROBABILITY OF FALSE POSITIVE VS K



OPTIMAL VALUE FOR K?

Probability of false positive:

- $p(\text{false positive}) \approx \left(1 - e^{-\frac{kn}{m}}\right)^k$

Choose k that minimizes $p(\text{false positive})$:

$$k = \frac{m}{n} \ln 2$$

SET DESIRED P(FALSE POSITIVE) AND N

Given some n items, and a desired false positive rate $\hat{p}$, can we find an optimal m?

Approximate solution:

- Use $k = \frac{m}{n} \ln 2$
- Substitute back into: $p = \left(1 - e^{-\left(\frac{m}{n} \ln 2 \frac{n}{m}\right)}\right)^{\frac{m}{n} \ln 2}$
- Simplify: $\ln p = -\frac{m}{n} (\ln 2)^2$
- So, $m = -\frac{n \log_2 p}{(\ln 2)^2}$

The size of Bloom filter is proportionate to the number of items n and the target false probability

SUMMARY: BLOOM FILTER

Use k hash functions

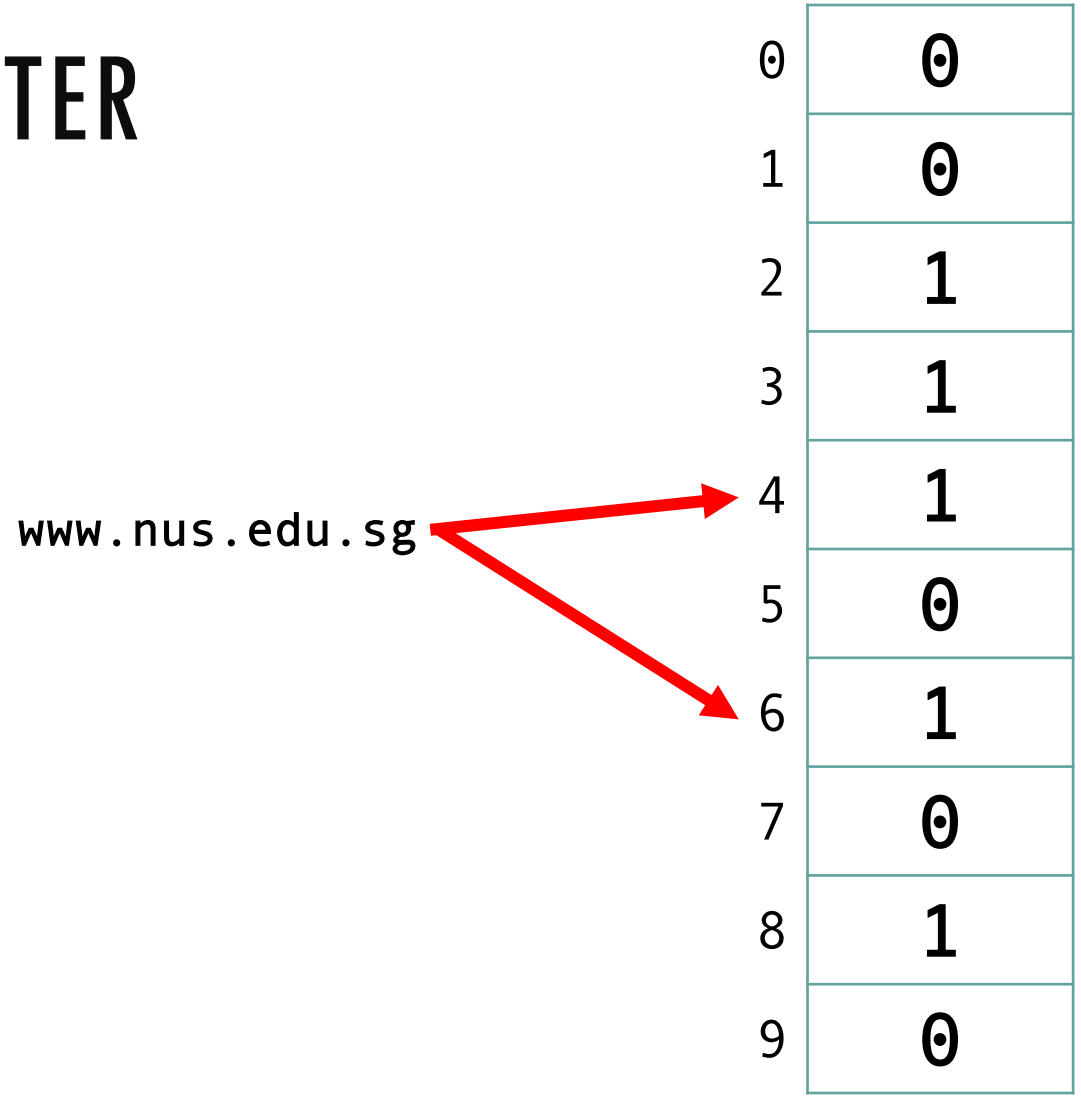
E.g., for $k = 2$, have h_1 and h_2

Insert x:

- Set the cells at <u>both</u> $h_1(x)$ and $h_2(x)$ to be 1

Lookup x:

- Return True only if cells at <u>both</u> $h_1(x)$ and $h_2(x)$ are 1



SUMMARY: BLOOM FILTER

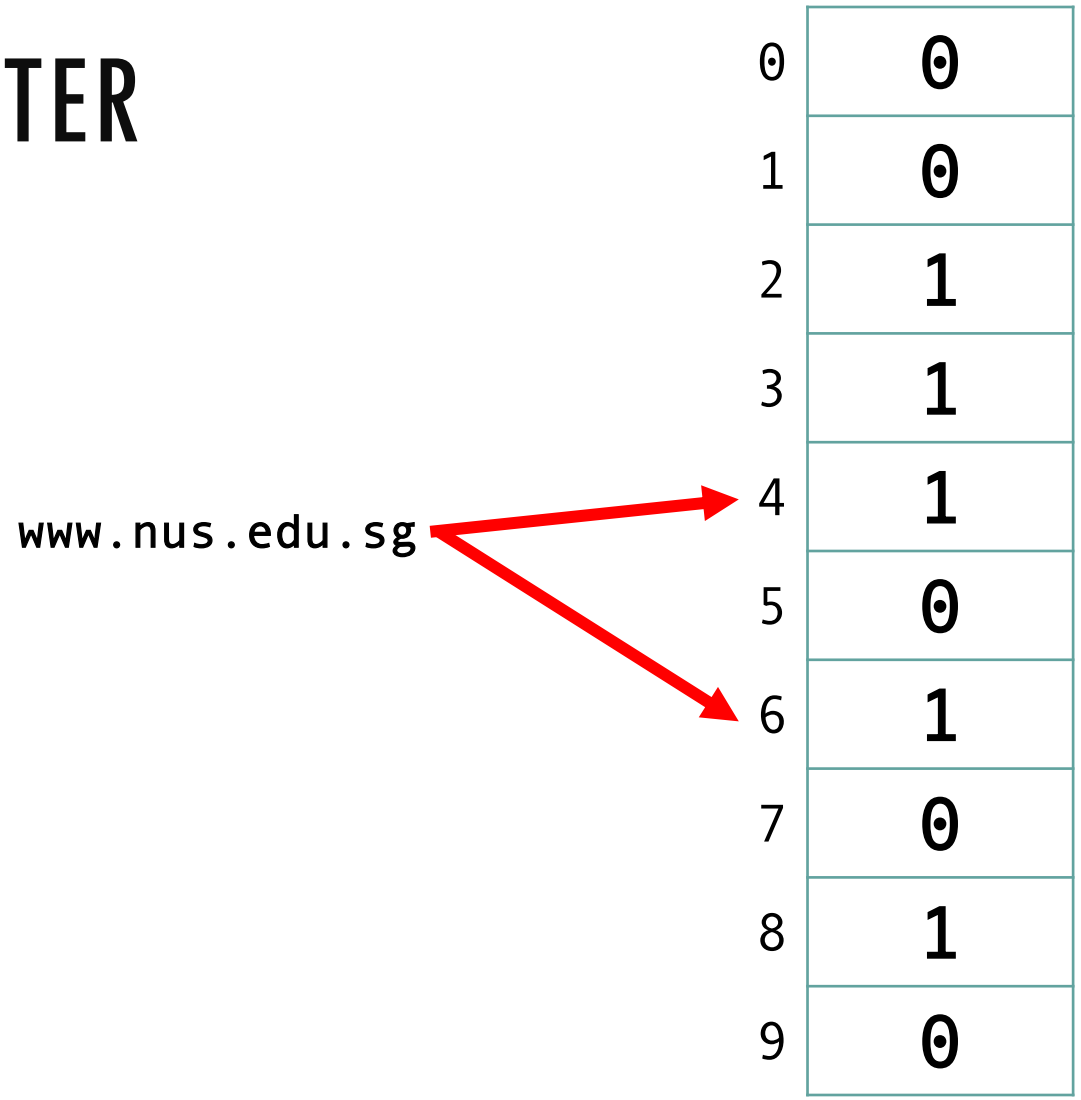
Use k hash functions

Saves space

But has probability of false positives.

Further reading: How about?

- **Deletions:** Can be handled using counters.
- **Intersections** and **Unions**.



QUESTIONS?

Poll Everywhere

https://bit.ly/2LvG9bq

CUCKOO HASHING

Did you know?

- Cuckoos lay eggs in other birds nests.
- When the cuckoo bird hatches, it pushes eggs/chicks out of the nest.

What a neat idea!

- Open addressing policy!
- Described by Rasmus Pagh and Flemming Friche Rodler in 2001.

CUCKOO HASHING: PERFORMANCE

Insertions seem to be quite complicated…

But: it takes expected $O(1)$ amortized time!

Analysis requires (a little) graph theory

To be continued in Week 12 … tomorrow!