Lecture 01

Linear systems and their solutions

Elementary Row Operations

Learning outcomes for Lecture 01

Section 1.1 Linear systems and their solutions

(1) What is a linear equation in 2 variables? What is a linear equation in n variables?

(2) What is a solution to a linear equation? What is the solution set of a linear equation? What is a general solution for the linear equation?

(3) Geometrical interpretation of

(i) solutions to a linear equation in 2 variables;

(ii) solutions to a linear equation in 3 variables;

Learning outcomes for Lecture 01

Section 1.1 Linear systems and their solutions

(4) What is a linear equation in n variables?

(Solutions, solution set, general solution.)

(5) Consistent and inconsistent linear systems.

(6) All linear systems have either (a) no solution;

(b) exactly one solution or (c) infinitely many solutons.

(Fact 1)

(7) Geometrical discussion of Fact 1 using 2-dimensional and 3-dimensional examples.

Learning outcomes for Lecture 01

Section 1.2 Elementary row operations

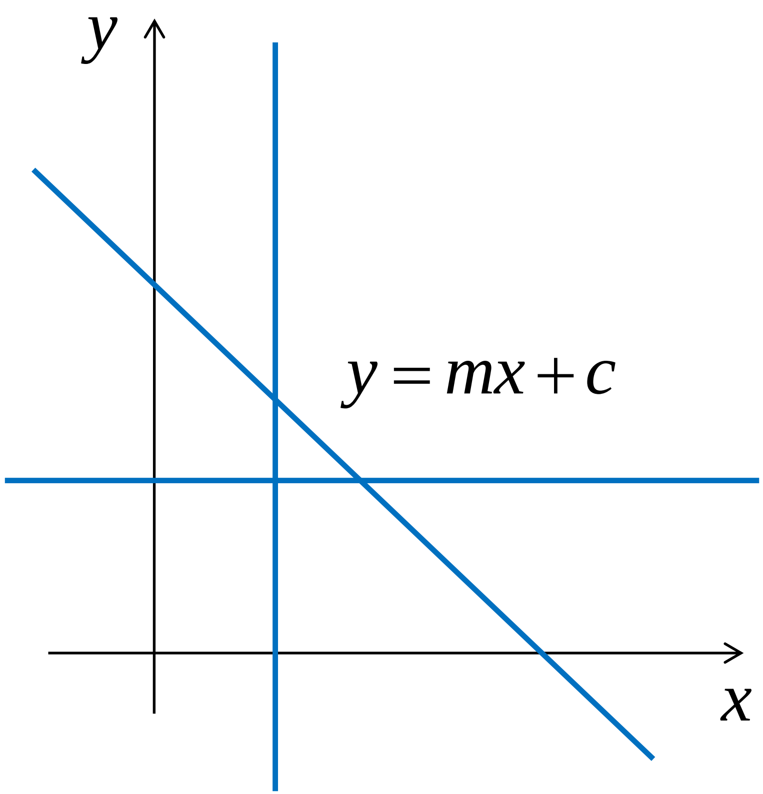
(1) What is an augmented matrix? What do the rows and columns of the augmented matrix represent?

(2) Three types equation manipulation and correspondingly three types of row operations. (ERO).

(3) Row equivalent matrices.

(4) (Theorem 1.2.7) Two linear systems whose augmented matrices are row equivalent will have the same solution set.

Discussion 1.1.1
How do you represent a line?



$$ax + by = c$$

$$by = -ax + c$$

$$y = -\frac{a}{b}x + \frac{c}{b} \quad \text{(if } b \neq 0\text{)}$$

$$x = \frac{c}{a} \quad \text{(if } b = 0, a \neq 0\text{)}$$

$\boxed{ax + by = c}$ a, b not both zero

is a linear equation in variables x and y.

Definition 1.1.2 (Linear equations)

A linear equation in n variables $x_1, x_2, \ldots, x_n$ is

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real constants.

$x_1, x_2, \ldots, x_n$ are also called unknowns.

If $a_1, a_2, \ldots, a_n$ are all zero, we call it a zero equation.

Some linear equations

3 variables x, y, z	$3x+2y-2z=3$
4 variables x_1, x_2, x_3, x_4	$x_1-0x_2-3x_3+4x_4=0$ (or simply $x_1-3x_3+4x_4=0$)
4 variables w, x, y, z	$w-x+y=4z$

These are not linear equations

$$xy = 2$$

$$y = \log_2 x + 3$$

$$x^2 + 2x + 6 = y$$

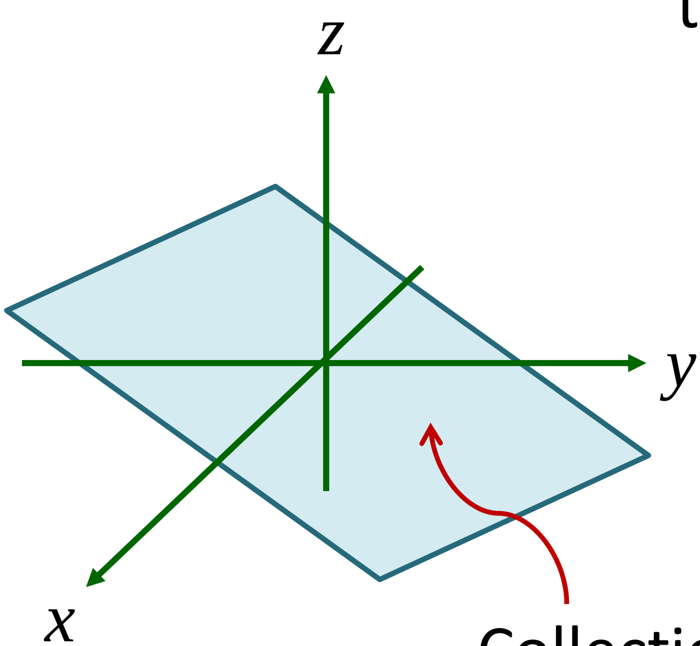
$$\sin x + \cos^2 y = 2.5$$

Example 1.1.3.3 (plane in 3D)

$ax+by+cz=d$ is a linear equation in variables x, y, z.

(a, b, c not all zero)

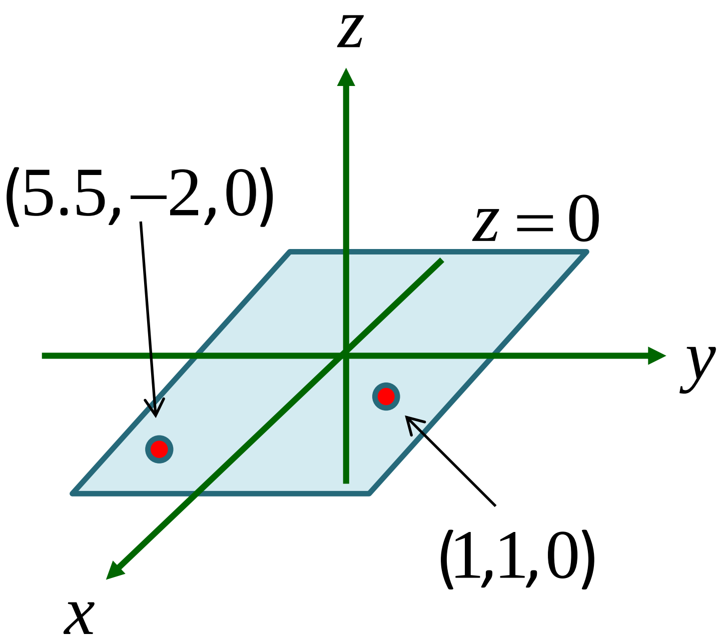
$ax+by+cz=d$ is a plane in the 3D (three dimensional) space.

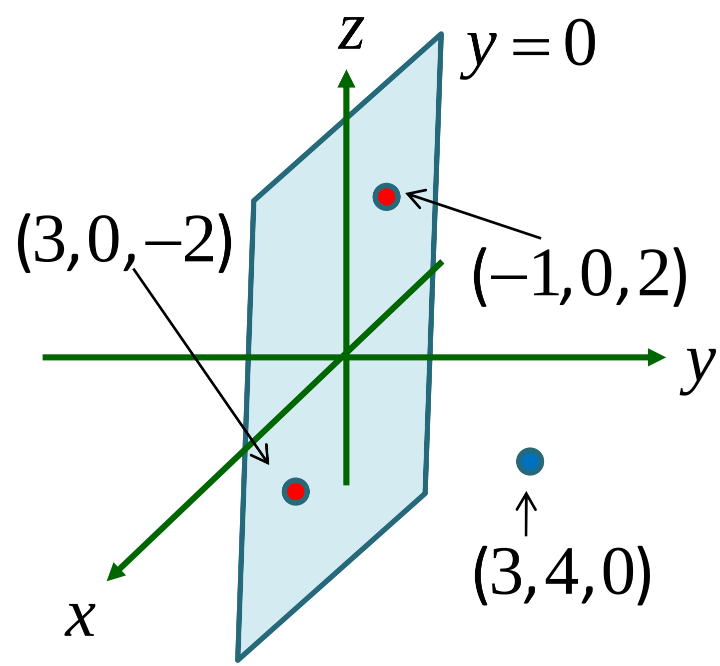


Every point in this space is represented by 3 numbers x, y, z, and is denoted by (x, y, z).

Collection of all points (x, y, z) that satisfies the equation $ax+by+cz=d$.

Example 1.1.3.3 (plane in 3D)





Remember:

$0x+0y+1z=0$ (that is, $z=0$)

is still a linear equation in

3 variables x,y,z.

$(3,4,0)$ does not lie on

the plane, that is,

$x=3, y=4, z=0$

does not satisfy

$0x+1y+0z=0.$

Definition 1.1.4 (Solutions)

Linear equation: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ (*)

Given n real numbers $s_1, s_2, \ldots, s_n$, we say

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

is a solution of the linear equation (*) if the equation is satisfied when we substitute $s_1, s_2, \ldots, s_n$ into (*).

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad (*)$$

Definition 1.1.4
(Solution Set, General Solutions)

Put all solutions of an equation into a set

$\rightarrow$ Solution Set of the equation.

$$\{ \qquad \qquad \qquad \}$$

An expression that gives us all the solutions in the set

$\rightarrow$ General Solution of the equation.

$$\begin{cases} x = \ldots \\ y = \ldots \\ z = \ldots \end{cases}$$

Example 1.1.5*
(2 variables, algebraic)

$$x+2y=2$$

If $x=s$ is any real number, then

$$x=s, y=\frac{1}{2}(2-s)$$

is a solution to the equation.

A general solution to the equation is

$$\begin{cases} x &= s \\ y &= \frac{1}{2}(2-s) \end{cases}$$ where s is an arbitrary parameter

$$\begin{cases} x &= s \\ y &= \frac{1}{2}(2-s),\ s \in \mathbb{R} \end{cases}$$

Example 1.1.5*
(2 variables, algebraic)

$$x+2y=2$$

If $y=t$ is $\boxed{\text{any real number,}}$ then

$x=2-2t, y=t$

is a solution to the equation.

A(nother) general solution to the equation is

$$\begin{cases} x &= 2-2t \\ y &= t \end{cases}$$ where t is an $\boxed{\text{arbitrary parameter}}$

$$\begin{cases} x &= 2-2t \\ y &= t,\ t\in\mathbb{R} \end{cases}$$

General solutions are

not unique!

Example 1.1.5*
(2 variables, algebraic)

$x + 2y = 2$

Solutions include:

$$\begin{cases} x = s \\ y = \frac{1}{2}(2-s), \quad s \in \mathbb{R} \end{cases}$$

$$\begin{cases} x = 1 \\ y = \frac{1}{2} \end{cases}$$

$$\begin{cases} x = 1.4 \\ y = 0.3 \end{cases}$$

$$\begin{cases} x = 2-2t \\ y = t, \quad t \in \mathbb{R} \end{cases}$$

$$\begin{cases} x = 2 \\ y = 0 \end{cases}$$

$$\begin{cases} x = 2.8 \\ y = -0.4 \end{cases}$$

How many solutions are there (in the solution set)?

Infinitely many!

Example 1.1.5*
(3 variables, algebraic)

$$x - 2y + 3z = 1$$

A general solution is: $\begin{cases} x = 1 + 2s - 3t \\ y = s \\ z = t, \quad s, t \in \mathbb{R} \end{cases}$

Can you write down another general solution?

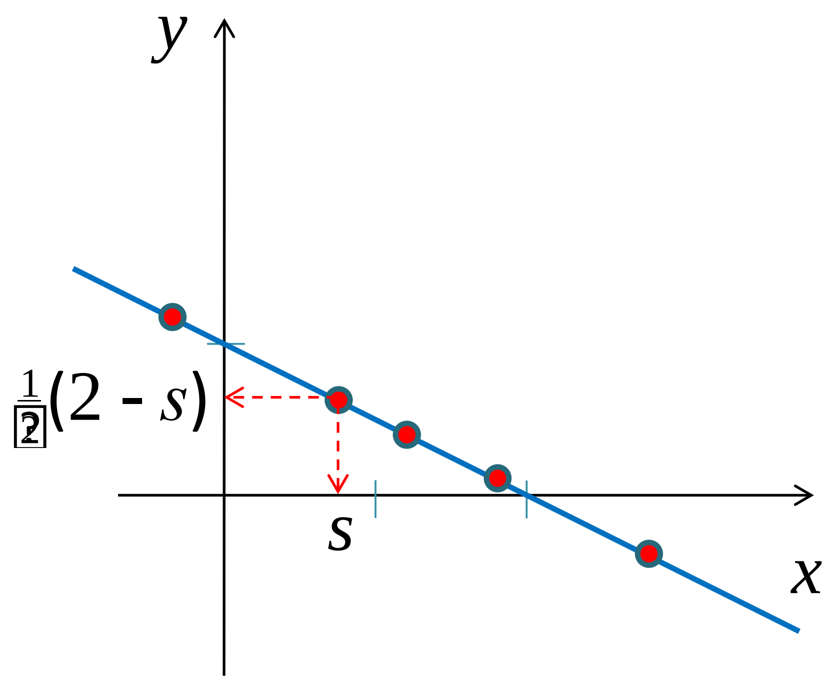
$$x + 2y + 0z = 2$$

A general solution is: $\begin{cases} x = 2 - 2s \\ y = s \\ z = t, \quad s, t \in \mathbb{R} \end{cases}$

Example 1.1.5*
(2 variables, geometric)

$$x + 2y = 2$$

$$\begin{cases} x = s \\ y = \frac{1}{2}(2 - s), \quad s \in \mathbb{R} \end{cases}$$



The solution set of the equation $x + 2y = 2$

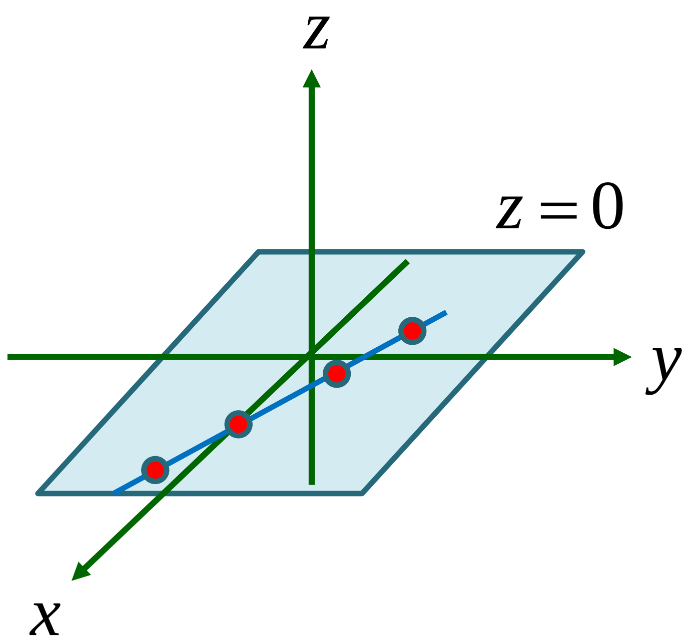
contains all the points $(x, y) = (s, \frac{1}{2}(2 - s))$, $s \in \mathbb{R}$.

These points form the line $x + 2y = 2$.

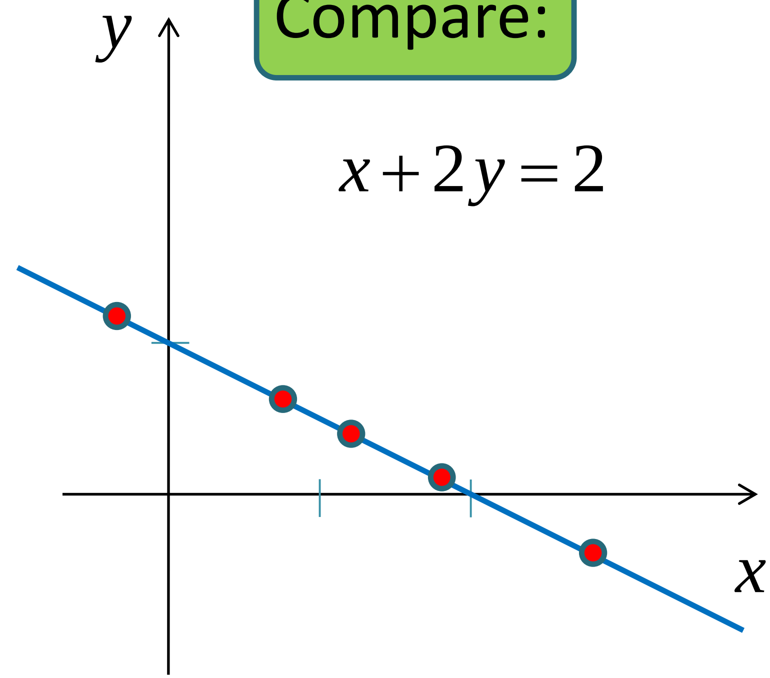
Example 1.1.5*
(3 variables, geometric)

$$x + 2y + 0z = 2$$

$$\begin{cases} x = 2 - 2s \\ y = s \\ z = t, \quad s,t \in \mathbb{R} \end{cases}$$



Compare:

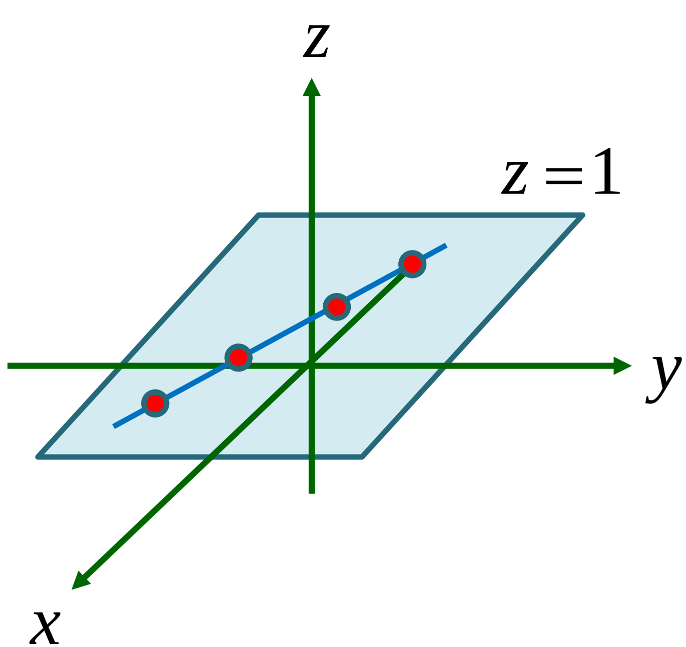


$$\begin{cases} x = 2 - 2t \\ y = t, \quad t \in \mathbb{R} \end{cases}$$

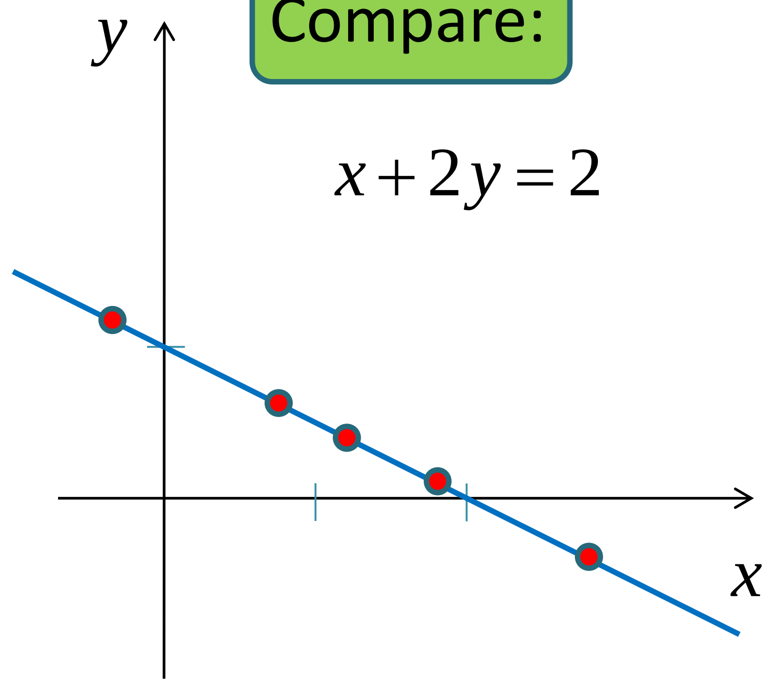
Example 1.1.5*
(3 variables, geometric)

$$x + 2y + 0z = 2$$

$$\begin{cases} x &= 2 - 2s \\ y &= s \\ z &= t, \quad s,t \in \mathbb{R} \end{cases}$$



Compare:



$$\begin{cases} x &= 2 - 2t \\ y &= t, \quad t \in \mathbb{R} \end{cases}$$

Example 1.1.5*
(3 variables, geometric)

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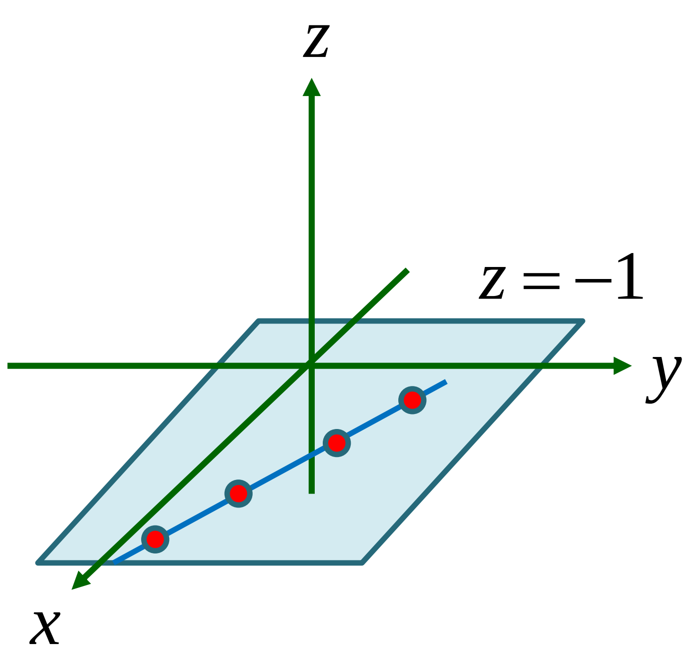

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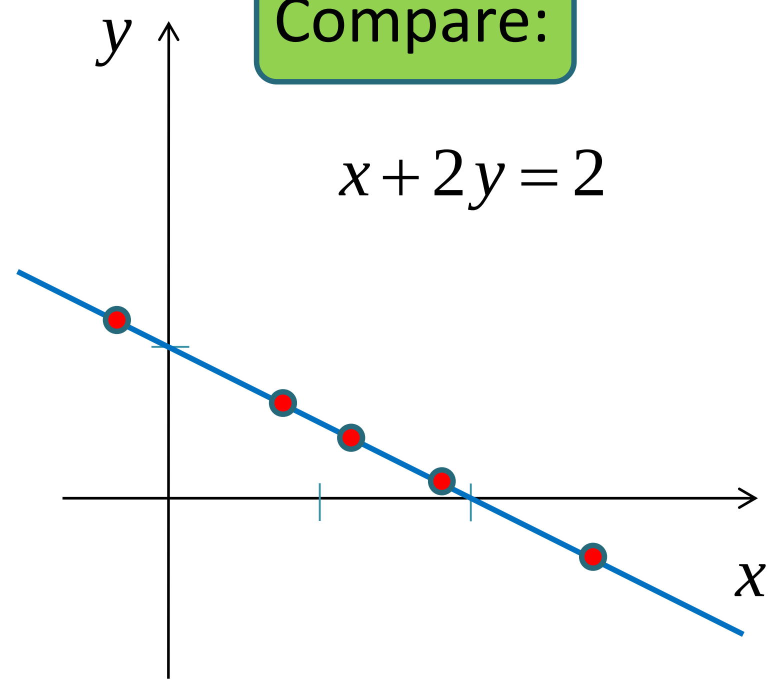
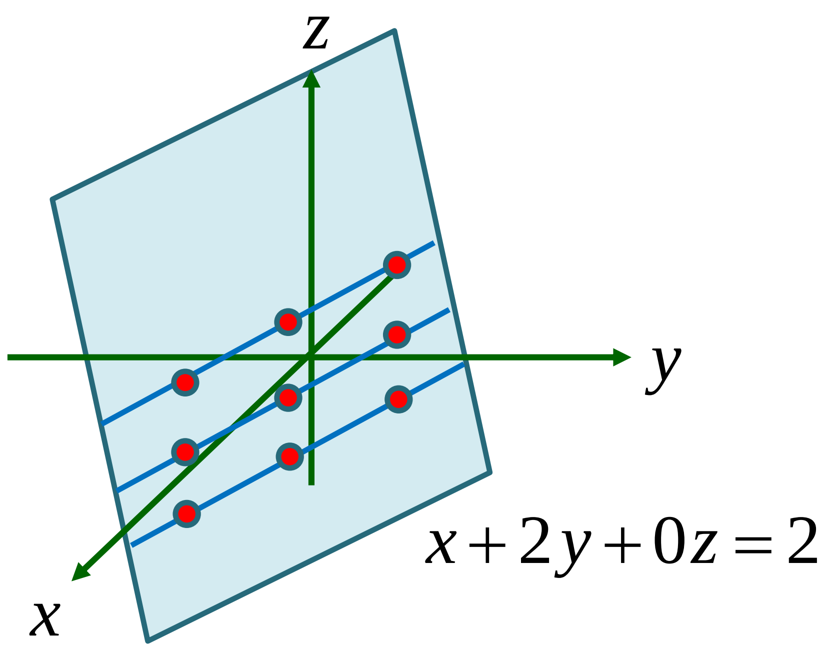


$$\begin{cases} x &= 2 - 2t \\ y &= t, \quad t \in \mathbb{R} \end{cases}$$

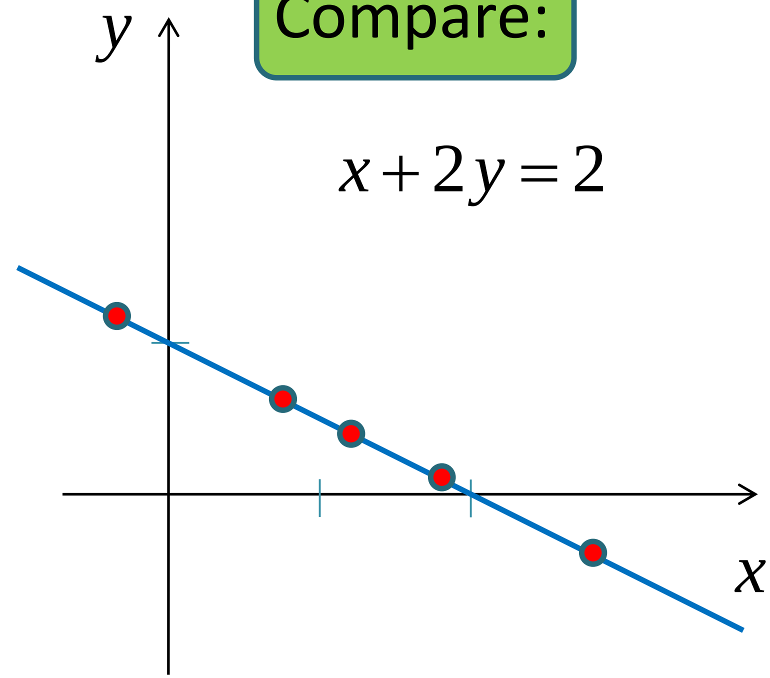
Example 1.1.5*
(3 variables, geometric)

$$x + 2y + 0z = 2$$

$$\begin{cases} x &= & 2-2s \\ y &= & s \\ z &= & t, \quad s,t \in \mathbb{R} \end{cases}$$



Compare:



$$\begin{cases} x &= & 2-2t \\ y &= & t, \quad t \in \mathbb{R} \end{cases}$$

Definition 1.1.6 (Linear systems)

A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations (or linear system).

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{cases}$$

$a_{11}, a_{12}, \ldots, a_{mn}, b_1, b_2, \ldots, b_m$ are real constants.

Definition 1.1.6 (Solutions)

Compare: $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$

one equation: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

Linear system:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{cases}$$

$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$

is a solution if it satisfies every equation in the linear system.

Definition 1.1.6 (Solution Set, General Solutions)

Put all solutions of an equation into a set

$\rightarrow$ Solution Set of the equation.

$$\{ \qquad \qquad \qquad \}$$

An expression that gives us all the solutions in the set

$\rightarrow$ General Solution of the equation.

$$\begin{cases} x = \ldots \\ y = \ldots \\ z = \ldots \end{cases}$$

Example 1.1.7 (solutions)

$$\begin{cases} 4w & - & x & + & 3y & = & -1 \\ 3w & + & x & + & 9y & = & -4 \end{cases}$$

$w=1, x=2, y=-1$ is a solution.

$$\begin{cases} 4(1) & - & 2 & + & 3(-1) & = & -1 \\ 3(1) & + & 2 & + & 9(-1) & = & -4 \end{cases}$$

$w=2, x=3, y=-2$ is not a solution.

$$\begin{cases} 4(2) & - & 3 & + & 3(-2) & = & -1 \\ 3(2) & + & 3 & + & 9(-2) & \neq & -4 \end{cases}$$

Remark 1.1.8
Do we always have solutions?

$$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

I say $x+y$ should be 1.

??

I say $x+y$ should be 2.

Definition 1.1.9 (Consistent, inconsistent)

A linear system that has no solutions is inconsistent.

In this case, the solution set of the linear system is an empty set.

A linear system that has at least one solution is consistent.

In this case, the solution set of the linear system is non empty.

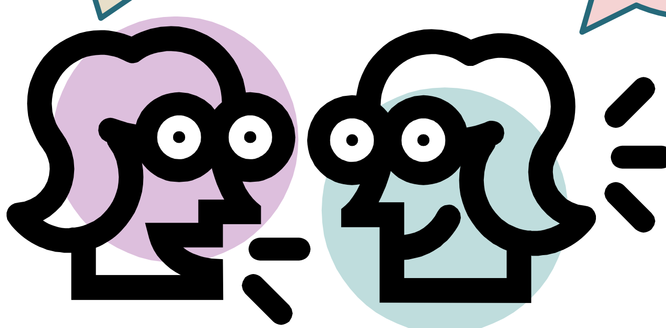
Remark 1.1.10
How many solutions can a linear system have?

It turns out, every linear system has either no solution, exactly one solution or infinitely many solutions.

See Question 2.22 in the textbook.

Is there a linear system with exactly 3 solutions?

No!

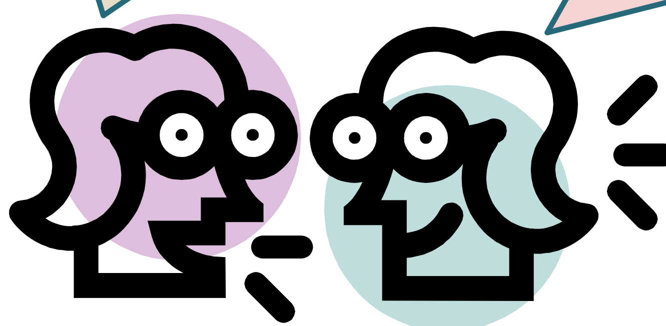


Remark 1.1.10
How many solutions can a linear system have?

It turns out, every linear system has either no solution, exactly one solution or infinitely many solutions.

So what if a linear system has at least 3 solutions?

Then it will have infinitely many!



Remark

If a linear system has exactly one solution, we say that the linear system has a unique solution.

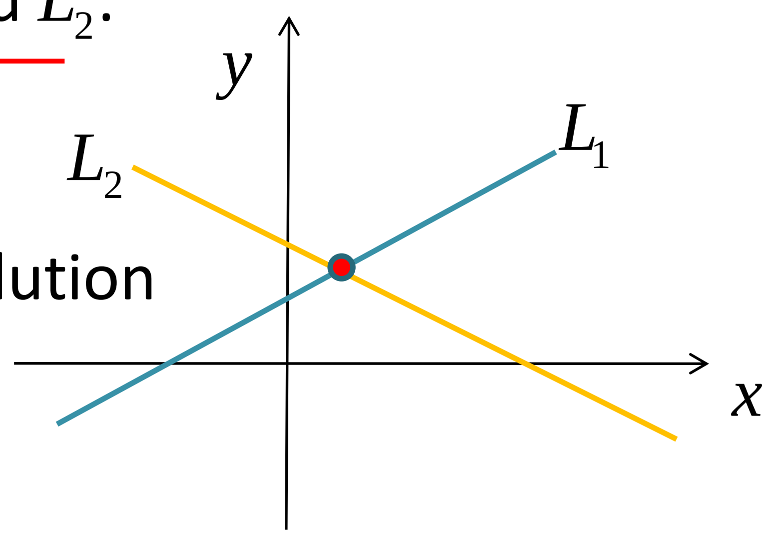
Discussion 1.1.11 (2 variables)

L_1 and L_2 are two lines in the xy plane.

$$\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2 & (L_2) \end{cases}$$

A solution to the linear system is a point (x, y) that lies on both L_1 and L_2.

exactly one solution



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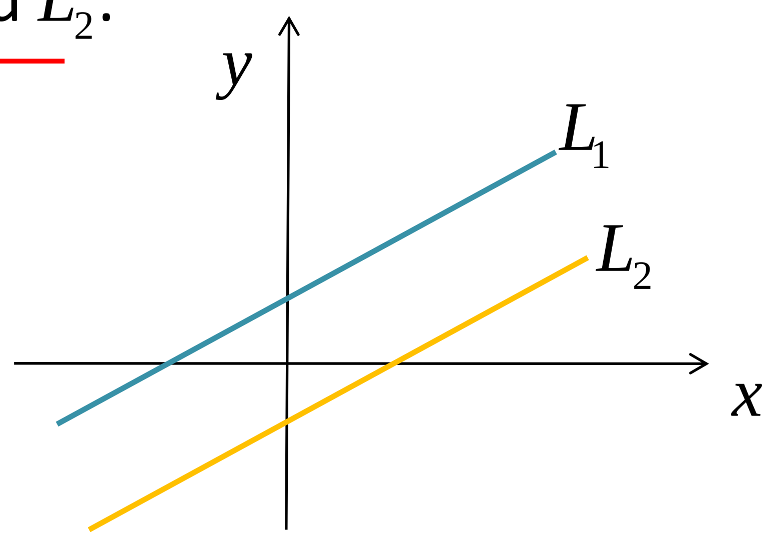
A solution to the linear system is a point (x, y) that lies on both L_1 and L_2.

no solution



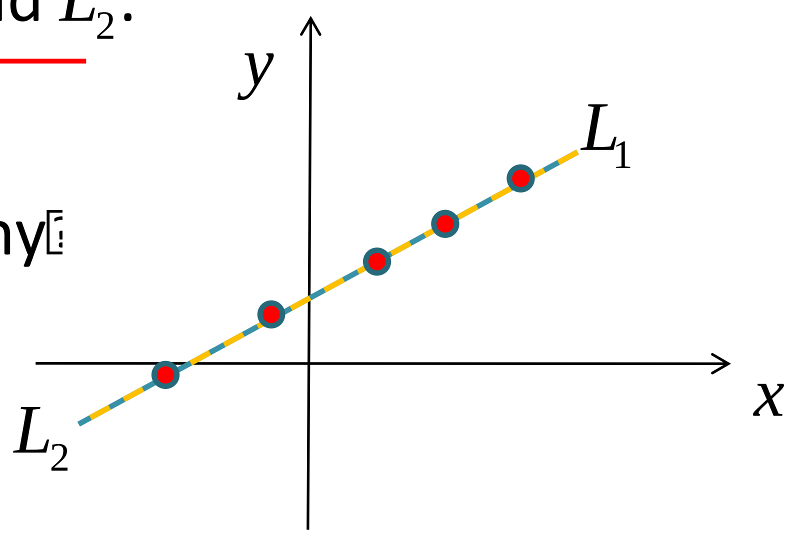
Discussion 1.1.11 (2 variables)

L_1 and L_2 are two lines in the xy plane.

$$\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2 & (L_2) \end{cases}$$

A solution to the linear system is a point (x, y) that lies on both L_1 and L_2.

infinitely many solutions



Chapter 1 Problem 8 (3 planes)

See Discussion 1.1.11 for two planes

p_1, p_2 and p_3 are three planes in the three dimensional space.

$$\begin{cases} a_1x & + & b_1y & + & c_1z & = & d_1 & & (p_1) \\ a_2x & + & b_2y & + & c_2z & = & d_2 & & (p_2) \\ a_3x & + & b_3y & + & c_3z & = & d_3 & & (p_3) \end{cases}$$

A solution to the linear system is a point (x, y, z) that lies on p_1, p_2 and p_3.

No solution?

Exactly one solution?

Infinitely many solutions?

Definition1.2.1 (Augmented matrix)

A linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

can be represented by a rectangular array of numbers:

Definition 1.2.1 (Augmented matrix)

$$\begin{array}{cc} & \text{columns} \\ \text{rows} & \left(\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array}\right) \end{array}$$

is called the $\boxed{\text{augmented matrix}}$ of the linear system.

Note that if the linear system has n variables and m equations, then the augmented matrix will have m rows and $(n+1)$ columns.

Example 1.2.2* (augmented matrix)

The augmented matrix for

$$\begin{cases} 4x + 5y - z = 1 \\ \quad\quad 2y + 2z = 0 \\ 3x - y - 9z = -1 \\ x \quad\quad - 2z = 3 \end{cases}$$

is

Discussion 1.2.3
How will you solve this?

$$\begin{cases} 2x + y = 1 & (1) \\ x - 3y = -2 & (2) \end{cases}$$

multiply (2) by 2

$$\begin{cases} 2x + y = 1 & (1) \\ 2x - 6y = -4 & (3) \end{cases}$$

Subtract (3) from (1)

Add (-1) times of (3) to (1)

$$\begin{cases} 0x + 7y = 5 & (4) \\ 2x - 6y = -4 & (3) \end{cases}$$

$$7y = 5 \Rightarrow \underline{y = \frac{5}{7}}$$

Substitute $y = \frac{5}{7}$ into equation (3) $\Rightarrow \underline{x = \frac{1}{7}}$

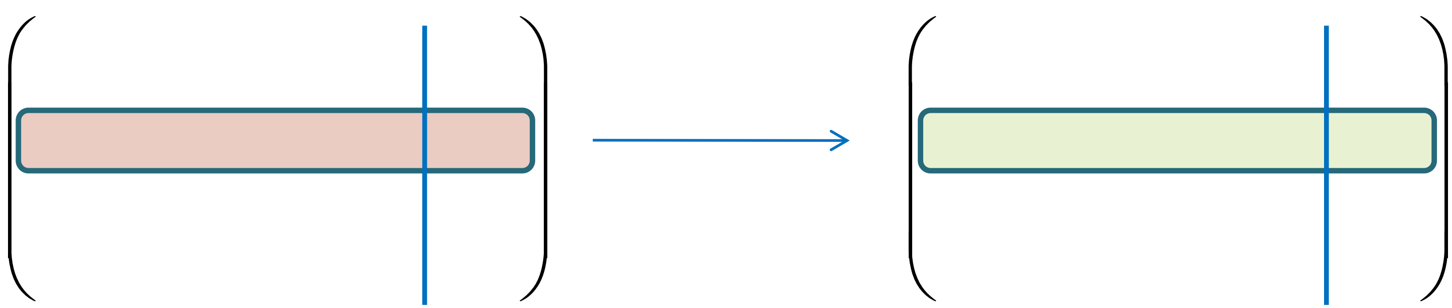
Discussion 1.2.3
In terms of augmented matrix?

What you do to equations in a linear system:

Multiply an equation by a non zero constant

What you do to rows of the augmented matrix:

Multiply a row by a non zero constant



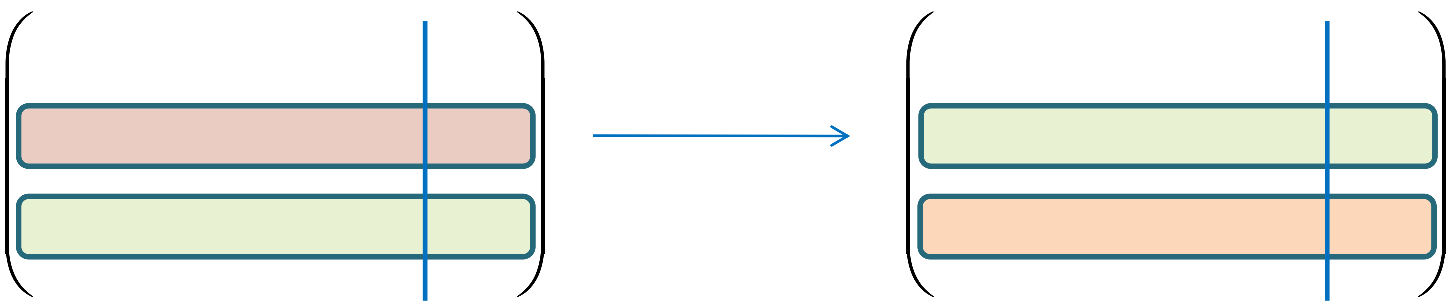
Discussion 1.2.3
In terms of augmented matrix?

What you do to equations in a linear system:

Interchange two equations

What you do to rows of the augmented matrix:

Interchange two rows



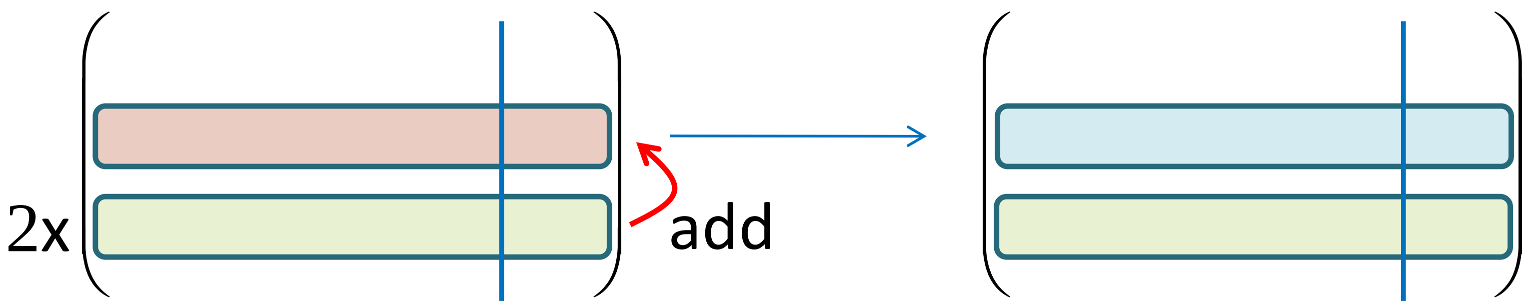
Discussion 1.2.3
In terms of augmented matrix?

What you do to equations in a linear system:

Add a multiple of one equation to another equation

What you do to rows of the augmented matrix:

Add a multiple of one row to another row



Definition 1.2.4
(Elementary Row Operations a.k.a. ERO)

The three operations

1) Multiply a row by a non zero constant
2) Interchanging two rows
3) Adding a multiple of one row to another row

performed on an augmented matrix are called elementary row operations.

Remark: Elementary row operations can be performed on any matrix in general (not just augmented matrices).

Example 1.2.5 (elementary row operations)

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

add -2 times of (1) to (2)

add -2 times of row 1 to row 2

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

Example 1.2.5 (elementary row operations)

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ 3x + 9y \quad = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

add -3 times of (1) to (3) — add -3 times of row 1 to row 3

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad + 6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

Example 1.2.5 (elementary row operations)

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad -4y - 4z = 4 & (4) \\ \quad +6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

add $\frac{6}{4}$ times of (4) to (5)

add $\frac{6}{4}$ times of row 2 to row 3

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad -4y - 4z = 4 & (4) \\ \quad\quad -15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

Example 1.2.5
Can we solve this linear system?

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad -4y - 4z = 4 & (4) \\ \qquad -15z = 9 & (6) \end{cases}$$

From equation (6) $\Rightarrow z = -\frac{3}{5}$

Substitute $z = -\frac{3}{5}$ into equation (4) $\Rightarrow y = -\frac{2}{5}$

Substitute $y = -\frac{2}{5}, z = -\frac{3}{5}$ into equation (1) $\Rightarrow x = \frac{11}{5}$

Example 1.2.5
Can we solve this linear system?

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad\quad - 15z = 9 & (6) \end{cases} \qquad (*)$$

So $x = \frac{11}{5}, y = -\frac{2}{5}, z = -\frac{3}{5}$ is the only solution to (*)

But what has this got to do with the original linear system?

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases}$$

Definition 1.2.6 (Row equivalent)

Two augmented matrices are said to be row equivalent if one can be obtained from the other by a series of elementary row operations.

Remark: The concept of row equivalent matrices can be used for any matrix in general (not just augmented matrices).

Example (row equivalent)

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

add -2 times of row 1 to row 2

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

all row equivalent

add -3 times of row 1 to row 3

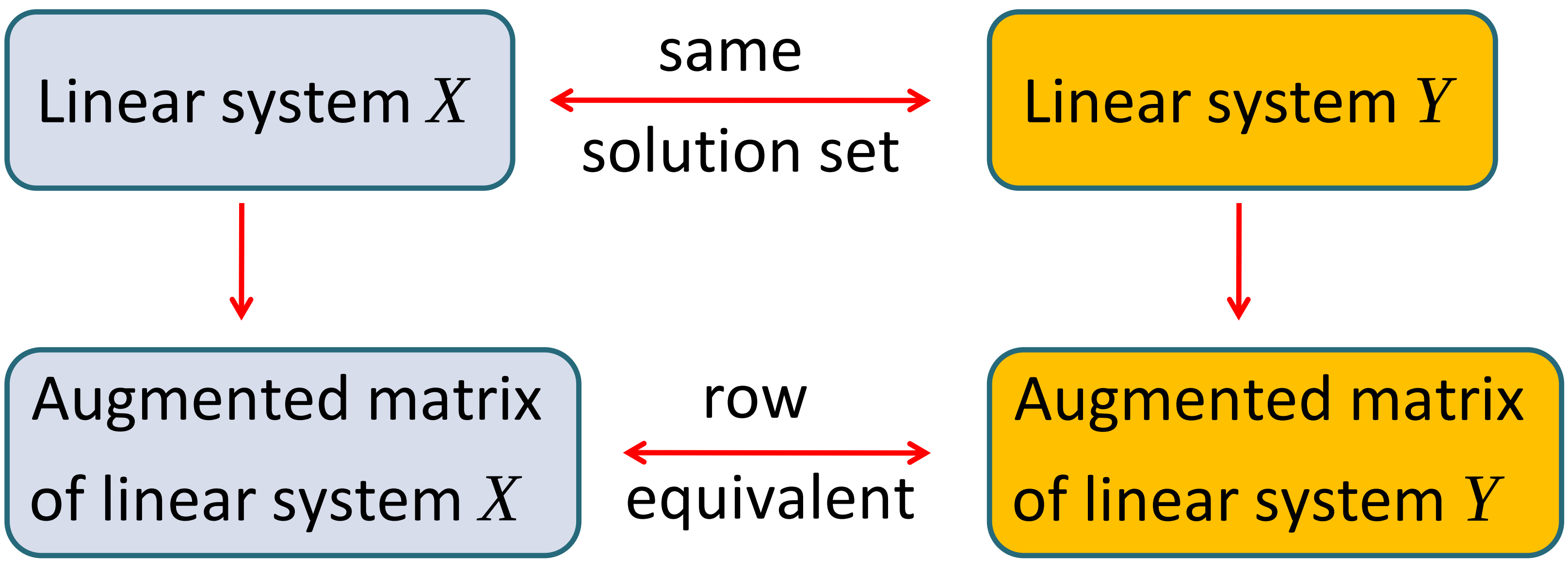
$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

add $\frac{6}{4}$ times of row 2 to row 3

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

Theorem 1.2.7
(row equivalent augmented matrices)

If augmented matrices of two linear systems are row equivalent, then the two linear systems have the same solution set.



Example 1.2.8 (row equivalent augmented matrices)

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases}$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases}$$

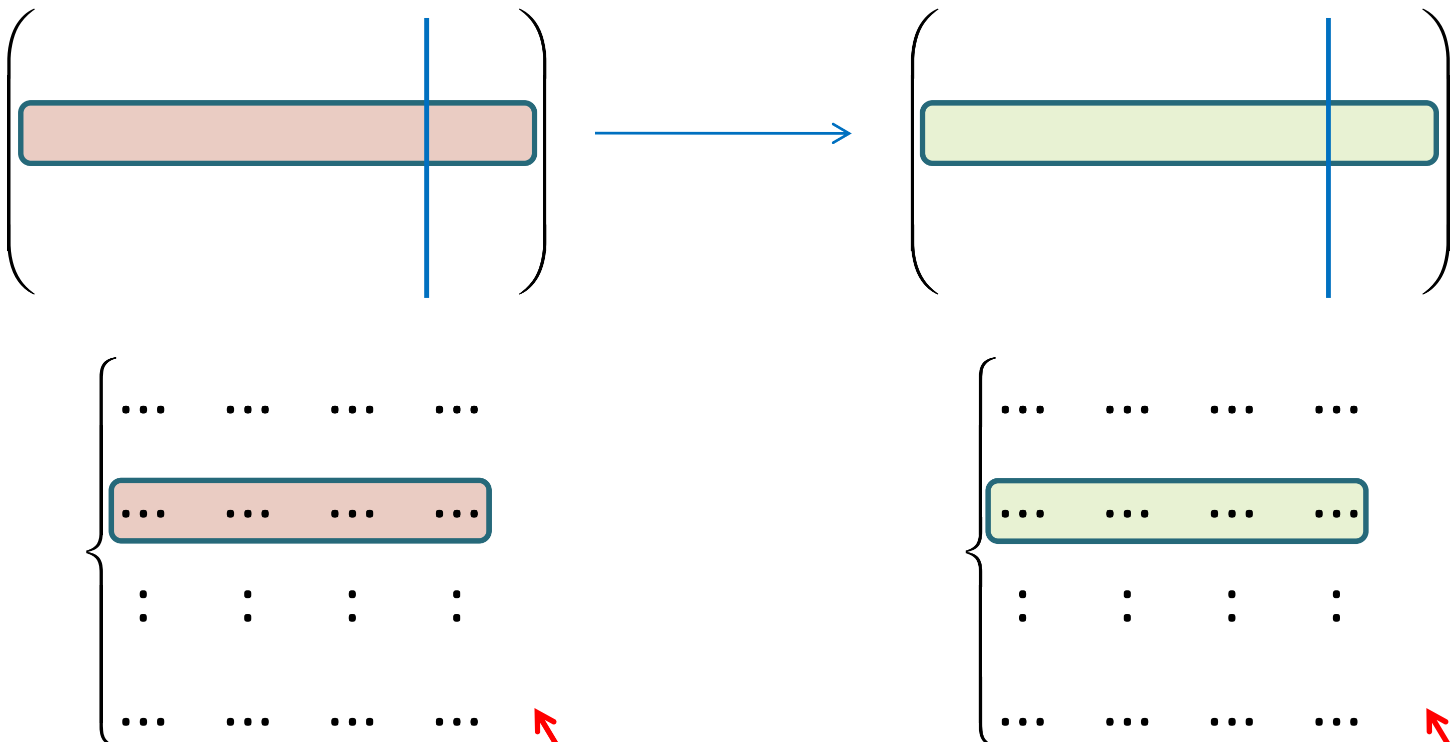
$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ +6y - 9z = 3 & (5) \end{cases}$$

All have the same solution set.

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ -15z = 9 & (6) \end{cases}$$

Remark 1.2.9
(Why is Theorem 1.2.7 true?)

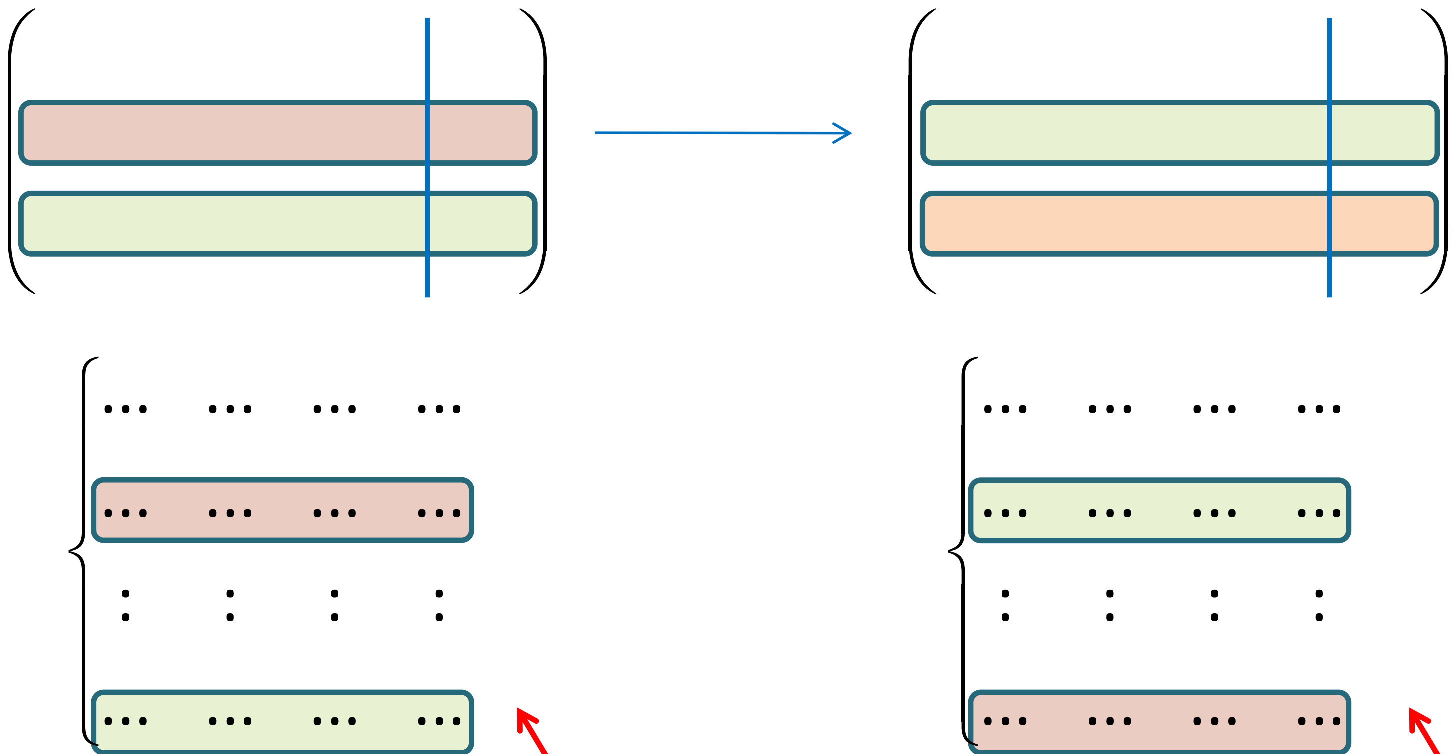
1) Multiply a row by a non zero constant



(…,…,…) is a solution of if and only if it is a solution of

Remark 1.2.9
(Why is Theorem 1.2.7 true?)

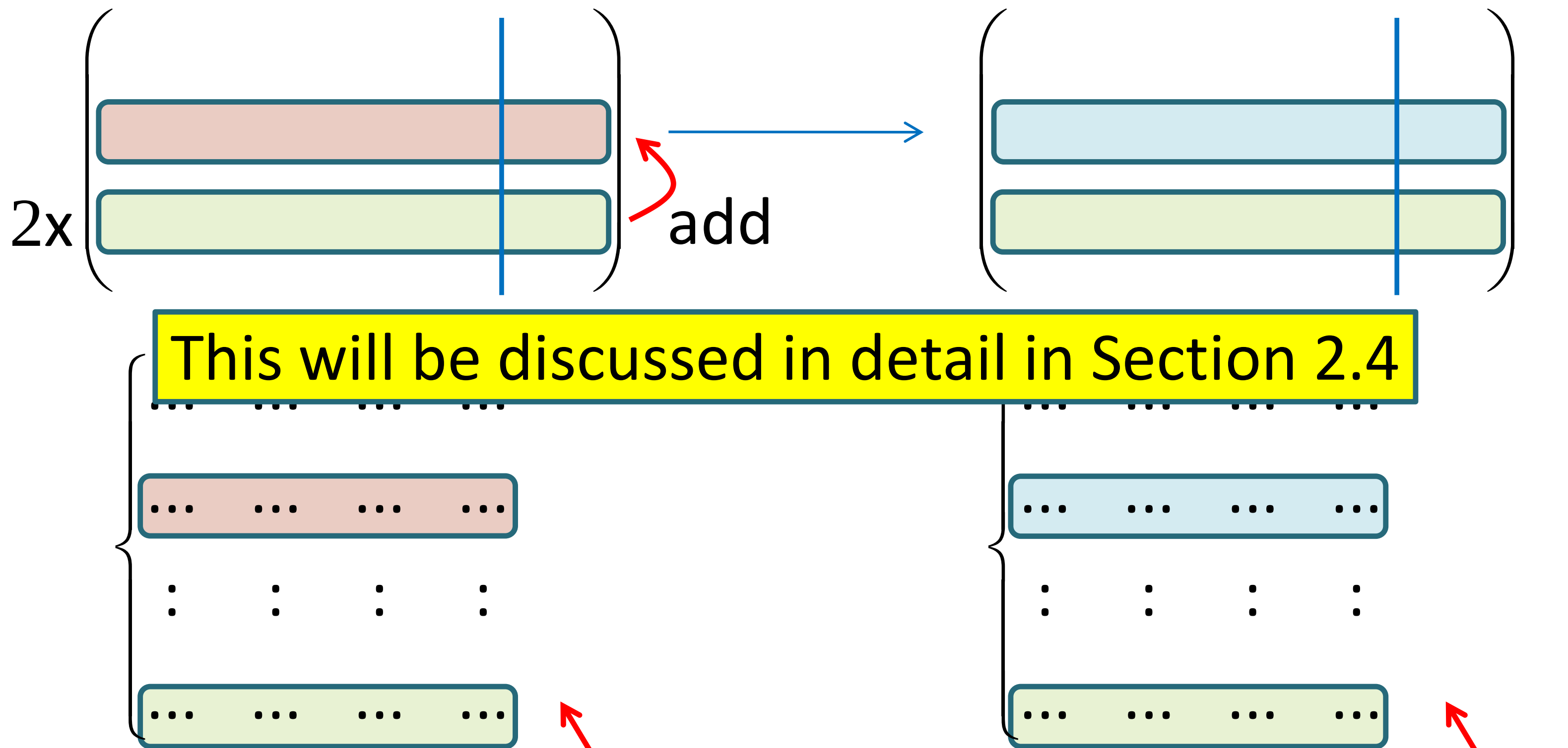
2) Interchanging two rows



(…,…,…) is a solution of if and only if it is a solution of

Remark 1.2.9
(Why is Theorem 1.2.7 true?)

3) Adding a multiple of one row to another row



(…,…,…) is a solution of if and only if it is a solution of

End of Lecture 01

Lecture 02:

Row-echelon forms

Gaussian Elimination (till Example 1.4.7)