NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 9**

1. (a) In $\mathbb{R}^2$, find the distance from the point $(1,5)$ to the line $x-y=0$.
   (b) In $\mathbb{R}^3$, find the distance from the point $(1,0,-2)$ to the plane $2x+y-2z=0$.
   (c) In $\mathbb{R}^3$, find the distance from the point $(1,0,-2)$ to the line

$$L=\{(t,2t,2t) \mid t\in\mathbb{R}\}.$$

   (a) The line is spanned by $(1,1)$. The projection of $(1,5)$ onto the line is $\frac{(1,5)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1) = (3,3)$. So the distance from $(1,5)$ to the line is $d((1,5),(3,3)) = ||(1,5)-(3,3)|| = ||(-2,2)|| = \sqrt{8}$.

   (b) The standard method is first to find the projection $\boldsymbol{p}$ of $\boldsymbol{w}$ onto the plane $2x+y-2z=0$. Then the distance from $\boldsymbol{w}$ to the plane is $d(\boldsymbol{w},\boldsymbol{p})$. However, the computation is quite tedious. In the following, we present an alternative method:

   The distance from the point $\boldsymbol{w}=(1,0,-2)$ to the plane $2x+y-2z=0$ is equal to the length of the projection of $\boldsymbol{w}$ onto the line perpendicular to the plane, i.e. the line spanned by $\boldsymbol{u}=(2,1,-2)$. So the distance is

$$\left\|\frac{\boldsymbol{w}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u}\right\| = \frac{|\boldsymbol{w}\cdot\boldsymbol{u}|}{\boldsymbol{u}\cdot\boldsymbol{u}}||\boldsymbol{u}|| = 2.$$

   (c) The line is spanned by $(1,2,2)$. The projection of $(1,0,-2)$ onto the line is $\frac{(1,0,-2)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)}(1,2,2) = (-\frac{1}{3},-\frac{2}{3},-\frac{2}{3})$. So the distance from $(1,0,-2)$ to the line is $d((1,0,-2),(-\frac{1}{3},-\frac{2}{3},-\frac{2}{3})) = ||(1,0,-2)-(-\frac{1}{3},-\frac{2}{3},-\frac{2}{3})|| = 2$.

2. Let $V=\text{span}\{\boldsymbol{v_1}=(1,0,1),\boldsymbol{v_2}=(0,1,-2)\}$.
   (a) Is $\{\boldsymbol{v_1},\boldsymbol{v_2}\}$ a basis for V? Justify your answer.
   (b) Use Gram-Schmidt Process to find an orthonormal basis for V.
   (c) Compute the projection of $\boldsymbol{w}=(1,1,1)$ onto V using
      (i) Theorem 5.2.15 (Orthogonal projection); and
      (ii) Theorem 5.3.8 together with Theorem 5.3.10 (Least Squares solution).

   (a) Yes, since $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are linearly independent vectors.
   (b) Applying the Gram-Schmidt Process to $\{\boldsymbol{v_1},\boldsymbol{v_2}\}$, we obtain an orthonormal basis $\{\boldsymbol{u_1},\boldsymbol{u_2}\}$ for V where $\boldsymbol{u_1}=\frac{1}{\sqrt{2}}(1,0,1)$ and $\boldsymbol{u_2}=\frac{1}{\sqrt{3}}(1,1,-1)$.
   (c) (i) The projection of $\boldsymbol{w}$ onto V is $(\boldsymbol{w}\cdot\boldsymbol{u_1})\boldsymbol{u_1}+(\boldsymbol{w}\cdot\boldsymbol{u_2})\boldsymbol{u_2}=(\frac{4}{3},\frac{1}{3},\frac{2}{3})$.

(ii) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}$. The least squares solution to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$, the projection of $\boldsymbol{w}$ onto V is $(\frac{4}{3}, \frac{1}{3}, \frac{2}{3})$.

3. A series of experiments were performed to investigate the relationship between two physical quantities x and y. The results of the experiments are shown in the table below.

x	0	1	2	3
y	3	2	4	4

(a) Find a least squares solution $\boldsymbol{x} = (\hat{a}, \hat{b})$ if it is believed that x and y are related linearly, that is, $y = ax + b$.

(b) Find a least squares solution $\boldsymbol{x} = (\hat{a}, \hat{b}, \hat{c})$ if it is believed that x and y are related by the quadratic polynomial $y = ax^2 + bx + c$.

(a) We find a least squares solution to

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 4 \end{pmatrix}.$$

Solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$,

$$\left(\begin{array}{cc|c} 14 & 6 & 22 \\ 6 & 4 & 13 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{2} \end{array}\right)$$

So a least squares solution is $(\hat{a}, \hat{b}) = (\frac{1}{2}, \frac{5}{2})$.

(b) We find a least squares solution to

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 4 \end{pmatrix}.$$

Solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$,

$$\left(\begin{array}{ccc|c} 98 & 36 & 14 & 54 \\ 36 & 14 & 6 & 22 \\ 14 & 6 & 4 & 13 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{4} \end{array}\right).$$

So a least squares solution is $(\hat{a}, \hat{b}, \hat{c}) = (\frac{1}{4}, -\frac{1}{4}, \frac{11}{4})$.

4. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}\}$ be a basis for $\mathbb{R}^n$.

   (a) For any vector $\boldsymbol{x} \in \mathbb{R}^n$, show that $||\boldsymbol{x}|| = ||\boldsymbol{Ax}||$.

   (b) For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{Ax}, \boldsymbol{Ay})$.

   (c) For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the same as the angle between $\boldsymbol{Ax}$ and $\boldsymbol{Ay}$.

   (d) Show that $T = \{\boldsymbol{Au_1}, \boldsymbol{Au_2}, \cdots, \boldsymbol{Au_n}\}$ is also a basis for $\mathbb{R}^n$.

   (e) If S is an orthogonal basis, show that T is also an orthogonal basis.

   (f) If S is orthonormal, is T orthonormal?

   (a) $||\boldsymbol{Ax}||^2 = (\boldsymbol{Ax})^T(\boldsymbol{Ax}) = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{x}^T\boldsymbol{x} = ||\boldsymbol{x}||^2$. Since both $||\boldsymbol{x}||$ and $||\boldsymbol{Ax}||$ are nonnegative, we have $||\boldsymbol{Ax}|| = ||\boldsymbol{x}||$.

   (b) $d(\boldsymbol{Ax}, \boldsymbol{Ay}) = ||\boldsymbol{Ax} - \boldsymbol{Ay}|| = ||\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{y})|| = ||\boldsymbol{x} - \boldsymbol{y}|| = d(\boldsymbol{x}, \boldsymbol{y})$

   (c) $(\boldsymbol{Ax}) \cdot (\boldsymbol{Ay}) = (\boldsymbol{Ax})^T\boldsymbol{Ay} = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Ay} = \boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}$. So

$$\begin{aligned}\text{the angle between } \boldsymbol{x} \text{ and } \boldsymbol{y} &= \cos^{-1}\left(\frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||\,||\boldsymbol{y}||}\right) \\ &= \cos^{-1}\left(\frac{(\boldsymbol{Ax}) \cdot (\boldsymbol{Ay})}{||\boldsymbol{Ax}||\,||\boldsymbol{Ay}||}\right) \\ &= \text{the angle between } \boldsymbol{Ax} \text{ and } \boldsymbol{Ay}.\end{aligned}$$

   (d) It suffices to show that T is a linearly independent set.

$$\begin{aligned} & c_1\boldsymbol{Au_1} + c_2\boldsymbol{Au_2} + \cdots + c_n\boldsymbol{Au_n} = \boldsymbol{0} \\ \Rightarrow\ & \boldsymbol{A}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}) = \boldsymbol{0} \\ \Rightarrow\ & c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n} = \boldsymbol{0} \quad (\text{since } \boldsymbol{A} \text{ is invertible}) \\ \Rightarrow\ & c_1 = c_2 = \cdots = c_n = 0 \quad (\text{since } S \text{ is a basis})\end{aligned}$$

   Thus T is a linearly independent set.

   (e) If S is an orthogonal basis, then $\boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0$ for all $i \neq j$. By part (c), we have shown that $\boldsymbol{Au_i} \cdot \boldsymbol{Au_j} = \boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0$, so T is also an orthogonal set. As T is a basis for $\mathbb{R}^n$ (by part (d)), we conclude that T is an orthogonal basis for $\mathbb{R}^n$.

   (f) By part (a), we have shown that for all $i = 1, \cdots, n$, $||\boldsymbol{u_i}|| = ||\boldsymbol{Au_i}||$. So if S is an orthonormal set, each $\boldsymbol{u_i}$ is a unit vector and thus each $\boldsymbol{Au_i}$ is also a unit vector. Thus, T is an orthonormal set.

5. Let

$$\boldsymbol{v_1} = (1, 1, 1, -1), \qquad \boldsymbol{v_2} = (1, 1, 3, 5).$$

   It is easy to see that $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is an orthogonal set. Extend this set to an orthogonal basis for $\mathbb{R}^4$.

We first extend the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ to a basis for $\mathbb{R}^4$.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

So $\{\boldsymbol{v_1}, \boldsymbol{v_2}, (0,1,0,0), (0,0,0,1)\}$ is a basis for $\mathbb{R}^4$. By Gram-Schmidt process, let $\boldsymbol{u_3} = (0,1,0,0)$ and $\boldsymbol{u_4} = (0,0,0,1)$. We will find $\boldsymbol{v_3}$ and $\boldsymbol{v_4}$.

$$\begin{aligned} \boldsymbol{v_3} &= \boldsymbol{u_3} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2} \\ &= (-\frac{5}{18}, \frac{13}{18}, -\frac{1}{3}, \frac{1}{9}) \\ \boldsymbol{v_4} &= \boldsymbol{u_4} - \frac{\boldsymbol{u_4} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_4} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2} - \frac{\boldsymbol{u_4} \cdot \boldsymbol{v_3}}{\boldsymbol{v_3} \cdot \boldsymbol{v_3}}\boldsymbol{v_3} \\ &= (\frac{2}{13}, 0, -\frac{3}{26}, \frac{1}{26}) \end{aligned}$$

Now $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ is an orthogonal basis for $\mathbb{R}^4$.