CS1231: Discrete Structures

Tutorial 10

Li Wei

Department of Mathematics
National University of Singapore

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Quick Review

- Product Rule; Sum Rule.
- $|A \cup B| = |A| + |B| - |A \cap B|$;
  $|A \cup B \cup C| =$
  $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.
- Probability of an event E is $P(E) = |E|/|S|$ (S is the sample space)
- The Binomial Theorem $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$
- Graphs: Definitions
- Handshaking Theorem: Let $G = (V, E)$ be a graph. Then

  $$\sum_{v \in V} \deg(v) = 2|E|.$$

  The **degree** of a vertex v in G, $\deg(v)$, is the number of edges incident with v, with each loop counted as 2.

Menu

Question 1

Question 2

Question 3

Question 4

Question 5

Question 6

Question 7

Question 8

Question 9

1. Determine the number of integers between 1 and 2019 inclusive, which are multiples of 6 or 7 or 9 but not multiples of 12.

Idea.

(1) Let U be the set of all integers between 1 and 2019 inclusive.
(2) Let A_6 be all integers in U which are multiples of 6.
(3) Let A_7 be all integers in U which are multiples of 7.
(4) Let A_9 be all integers in U which are multiples of 9.
(5) Let A_{12} be all integers in U which are multiples of 12.
(6) What is the set of integers between 1 and 2019 inclusive, which are multiples of 6 or 7 or 9 but not multiples of 12?

(7) Note that

$$
\begin{aligned}
& A_6 \cup A_7 \cup A_9 \\
=& (A_6 \cup A_7 \cup A_9) \cap U \\
=& (A_6 \cup A_7 \cup A_9) \cap (A_{12} \cup \overline{A_{12}}) \\
=& \\
\therefore & |A_6 \cup A_7 \cup A_9| \\
=&
\end{aligned}
$$

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=& ((A_6 \cup A_7 \cup A_9) \cap A_{12}) \cup ((A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}) \\
\therefore & |A_6 \cup A_7 \cup A_9| \\
=& |(A_6 \cup A_7 \cup A_9) \cap A_{12}| + |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|
\end{aligned}$$

$|A_6 \cup A_7 \cup A_9|$

1. $|A_6 \cup A_7 \cup A_9| =$

2. ; ;
.

.

.

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7. $|A_6 \cup A_7 \cup A_9| =$

$|A_6 \cup A_7 \cup A_9|$

1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| =$; $|A_7| =$; $|A_9| =$.
3. $A_6 \cap A_7$: multiples of
   $|A_6 \cap A_7| =$.
4. $A_6 \cap A_9$: multiples of
   $|A_6 \cap A_9| =$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

$|A_6 \cup A_7 \cup A_9|$

1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| =$;
   $|A_9| =$.
3. $A_6 \cap A_7$: multiples of
   $|A_6 \cap A_7| =$.
4. $A_6 \cap A_9$: multiples of
   $|A_6 \cap A_9| =$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$;
   $|A_9| =$.
3. $A_6 \cap A_7$: multiples of
   $|A_6 \cap A_7| =$.
4. $A_6 \cap A_9$: multiples of
   $|A_6 \cap A_9| =$.
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   $|A_7 \cap A_9| =$.
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   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$;
   $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of
   $|A_6 \cap A_7| =$.
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   $|A_6 \cap A_9| =$.
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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$; $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of 42.
   $|A_6 \cap A_7| =$.
4. $A_6 \cap A_9$: multiples of
   $|A_6 \cap A_9| =$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$; $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of 42. $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of
   $|A_6 \cap A_9| =$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$; $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of 42.
   $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of 18.
   $|A_6 \cap A_9| =$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$; $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of 42. $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of 18. $|A_6 \cap A_9| = \lfloor 2019/18 \rfloor = 112$.
5. $A_7 \cap A_9$: multiples of
   $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
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3. $A_6 \cap A_7$: multiples of 42. $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of 18. $|A_6 \cap A_9| = \lfloor 2019/18 \rfloor = 112$.
5. $A_7 \cap A_9$: multiples of 63. $|A_7 \cap A_9| =$.
6. $A_6 \cap A_7 \cap A_9$: multiples of $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$; $|A_9| = \lfloor 2019/9 \rfloor = 224$.
3. $A_6 \cap A_7$: multiples of 42. $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of 18. $|A_6 \cap A_9| = \lfloor 2019/18 \rfloor = 112$.
5. $A_7 \cap A_9$: multiples of 63. $|A_7 \cap A_9| = \lfloor 2019/63 \rfloor = 32$.
6. $A_6 \cap A_7 \cap A_9$: multiples of
   $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

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1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
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4. $A_6 \cap A_9$: multiples of 18. $|A_6 \cap A_9| = \lfloor 2019/18 \rfloor = 112$.
5. $A_7 \cap A_9$: multiples of 63. $|A_7 \cap A_9| = \lfloor 2019/63 \rfloor = 32$.
6. $A_6 \cap A_7 \cap A_9$: multiples of 126. $|A_6 \cap A_7 \cap A_9| =$.
7. $|A_6 \cup A_7 \cup A_9| =$

$|A_6 \cup A_7 \cup A_9|$

1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
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5. $A_7 \cap A_9$: multiples of 63. $|A_7 \cap A_9| = \lfloor 2019/63 \rfloor = 32$.
6. $A_6 \cap A_7 \cap A_9$: multiples of 126. $|A_6 \cap A_7 \cap A_9| = \lfloor 2019/126 \rfloor = 16$.
7. $|A_6 \cup A_7 \cup A_9| =$

$|A_6 \cup A_7 \cup A_9|$

1. $|A_6 \cup A_7 \cup A_9| = |A_6| + |A_7| + |A_9| - |A_6 \cap A_7| - |A_6 \cap A_9| - |A_7 \cap A_9| + |A_6 \cap A_7 \cap A_9|$.
2. $|A_6| = \lfloor 2019/6 \rfloor = 336$; $|A_7| = \lfloor 2019/7 \rfloor = 288$;
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   $|A_6 \cap A_7| = \lfloor 2019/42 \rfloor = 48$.
4. $A_6 \cap A_9$: multiples of 18.
   $|A_6 \cap A_9| = \lfloor 2019/18 \rfloor = 112$.
5. $A_7 \cap A_9$: multiples of 63.
   $|A_7 \cap A_9| = \lfloor 2019/63 \rfloor = 32$.
6. $A_6 \cap A_7 \cap A_9$: multiples of 126.
   $|A_6 \cap A_7 \cap A_9| = \lfloor 2019/126 \rfloor = 16$.
7. $|A_6 \cup A_7 \cup A_9| = 336 + 288 + 224 - 48 - 112 - 32 + 16 = 672$

$|(A_6 \cup A_7 \cup A_9) \cap A_{12}|$

- $A_{12} \subseteq A_6 \Rightarrow A_{12} \subseteq A_6 \cup A_7 \cup A_9 \Rightarrow$
  $(A_6 \cup A_7 \cup A_9) \cap A_{12} = \qquad .$
-
- $\therefore |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|$
  $= |A_6 \cup A_7 \cup A_9| - |(A_6 \cup A_7 \cup A_9) \cap A_{12}| =$

$|(A_6 \cup A_7 \cup A_9) \cap A_{12}|$

- $A_{12} \subseteq A_6 \Rightarrow A_{12} \subseteq A_6 \cup A_7 \cup A_9 \Rightarrow (A_6 \cup A_7 \cup A_9) \cap A_{12} = A_{12}$.
- $|(A_6 \cup A_7 \cup A_9) \cap A_{12}| =$
- $\therefore |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|$
  $= |A_6 \cup A_7 \cup A_9| - |(A_6 \cup A_7 \cup A_9) \cap A_{12}| =$

$|(A_6 \cup A_7 \cup A_9) \cap A_{12}|$

- $A_{12} \subseteq A_6 \Rightarrow A_{12} \subseteq A_6 \cup A_7 \cup A_9 \Rightarrow$
  $(A_6 \cup A_7 \cup A_9) \cap A_{12} = A_{12}$.
- $|(A_6 \cup A_7 \cup A_9) \cap A_{12}| = |A_{12}| =$
- $\therefore |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|$
  $= |A_6 \cup A_7 \cup A_9| - |(A_6 \cup A_7 \cup A_9) \cap A_{12}| =$

$|(A_6 \cup A_7 \cup A_9) \cap A_{12}|$

- $A_{12} \subseteq A_6 \Rightarrow A_{12} \subseteq A_6 \cup A_7 \cup A_9 \Rightarrow$
  $(A_6 \cup A_7 \cup A_9) \cap A_{12} = A_{12}$.
- $|(A_6 \cup A_7 \cup A_9) \cap A_{12}| = |A_{12}| = \lfloor 2019/12 \rfloor = 168$
- $\therefore |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|$
  $= |A_6 \cup A_7 \cup A_9| - |(A_6 \cup A_7 \cup A_9) \cap A_{12}| =$

$|(A_6 \cup A_7 \cup A_9) \cap A_{12}|$

- $A_{12} \subseteq A_6 \Rightarrow A_{12} \subseteq A_6 \cup A_7 \cup A_9 \Rightarrow$ $(A_6 \cup A_7 \cup A_9) \cap A_{12} = A_{12}$.
- $|(A_6 \cup A_7 \cup A_9) \cap A_{12}| = |A_{12}| = \lfloor 2019/12 \rfloor = 168$
- $\therefore |(A_6 \cup A_7 \cup A_9) \cap \overline{A_{12}}|$ $= |A_6 \cup A_7 \cup A_9| - |(A_6 \cup A_7 \cup A_9) \cap A_{12}| = 672 - 168 = 504$

2. What is the probability that a die never comes up an even number when it is rolled six times?
(1) Find the number of all possible outcomes $|S|$.

(2) Let E be the set of all outcomes in which never comes up an even number. Then what is $|E|$?

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6^6

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$|E| = 3^6$

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6^6

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$|E| = 3^6$

$$P(E) = \frac{3^6}{6^6} = \frac{1}{2^6} = \frac{1}{64}$$

3. Find the coefficient of a^5b^7 in the expansion of $(a-2b)^{12}$.

Idea.

1. a can come from any 5 factors of $(a-2b)$.

3. Find the coefficient of a^5b^7 in the expansion of $(a-2b)^{12}$.

Idea.

1. a can come from any 5 factors of $(a-2b)$. $\binom{12}{5}$
2. b can come from the rest factors of $(a-2b)$.

3. Find the coefficient of a^5b^7 in the expansion of $(a-2b)^{12}$.

Idea.

1. a can come from any 5 factors of $(a-2b)$. $\binom{12}{5}$
2. b can come from the rest factors of $(a-2b)$. $\binom{7}{7}(-2)^7$
3. Ans.

3. Find the coefficient of a^5b^7 in the expansion of $(a-2b)^{12}$.

Idea.

1. a can come from any 5 factors of $(a-2b)$. $\binom{12}{5}$
2. b can come from the rest factors of $(a-2b)$. $\binom{7}{7}(-2)^7$
3. Ans. $\binom{12}{5}\binom{7}{7}(-2)^7 = -101376$.

4. Find the coefficient of $a^5b^2c^8$ in the expansion of $(a+b+c)^{15}$.
Idea.

1. a can come from any 5 factors of $(a+b+c)$.

4. Find the coefficient of $a^5b^2c^8$ in the expansion of $(a+b+c)^{15}$.

Idea.

1. a can come from any 5 factors of $(a+b+c)$. $\binom{15}{5}$
2. b can come from any 2 of the rest factors of $(a+b+c)$.

4. Find the coefficient of $a^5b^2c^8$ in the expansion of $(a+b+c)^{15}$.
Idea.

1. a can come from any 5 factors of $(a+b+c)$. $\binom{15}{5}$
2. b can come from any 2 of the rest factors of $(a+b+c)$. $\binom{10}{2}$
3. c can come from the remaining factors of $(a+b+c)$.

4. Find the coefficient of $a^5b^2c^8$ in the expansion of $(a+b+c)^{15}$.

Idea.

1. a can come from any 5 factors of $(a+b+c)$. $\binom{15}{5}$
2. b can come from any 2 of the rest factors of $(a+b+c)$. $\binom{10}{2}$
3. c can come from the remaining factors of $(a+b+c)$. $\binom{8}{8}$
4. Ans.

4. Find the coefficient of $a^5b^2c^8$ in the expansion of $(a+b+c)^{15}$.

Idea.

1. a can come from any 5 factors of $(a+b+c)$. $\binom{15}{5}$
2. b can come from any 2 of the rest factors of $(a+b+c)$. $\binom{10}{2}$
3. c can come from the remaining factors of $(a+b+c)$. $\binom{8}{8}$
4. Ans. $\binom{15}{5}\binom{10}{2}\binom{8}{8} = 135135.$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

5. Express each of the following in close form, i.e., in a single expression.

$$\sum_{i=0}^{n} \binom{n}{i} 4^i; \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{2n-2k} 2^{2k}$$

Idea.

$$\sum_{i=0}^{n} \binom{n}{i} 4^i = \sum_{i=0}^{n} \binom{n}{i} 4^i (\quad)^{n-i} =$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{2n-2k} 2^{2k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\qquad\qquad\qquad)^k (\quad)^{n-k} =$$

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$$\sum_{i=0}^{n} \binom{n}{i} 4^i = \sum_{i=0}^{n} \binom{n}{i} 4^i (1)^{n-i} = (4+1)^n = 5^n$$

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$$= \sum_{k=0}^{n} \binom{n}{k} (-1 \times 2^2)^k (\quad)^{n-k} =$$

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$$= \sum_{k=0}^{n} \binom{n}{k} (-1 \times 2^2)^k (3^2)^{n-k} = (-1 \times 2^2 + 3^2)^n = 5^n$$

6. There are 7 students $a_1, \ldots a_7$ in a graph theory class. Students are told to divide themselves into several groups for project work with unrestricted group size. The following pairs of students cannot work together:

$$(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_2, a_6), (a_3, a_6),$$
$$(a_4, a_6), (a_4, a_7), (a_5, a_6), (a_5, a_7)$$

Describe a graph G that models these relations between the students. Use G to find the **minimum** number of groups needed so that any of the above pair of students are not in the same group.
Idea.

- Vertices:
- Edges:

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Idea.

- Vertices: $a_1, a_2, a_3, a_4, a_5, a_6, a_7$
- Edges:

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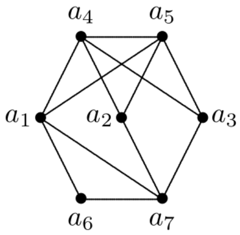
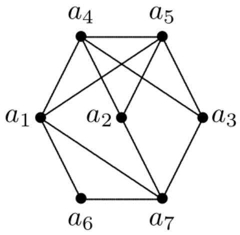
Idea.

- Vertices: $a_1, a_2, a_3, a_4, a_5, a_6, a_7$
- Edges: Can work together



Use G to find the **minimum** number of working groups.
Notice, a_1 cannot work with a_2 or a_3;
a_2 cannot work with a_1 or a_3;
a_3 cannot work with a_1 or a_2.
Thus, at least   working groups.



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Notice, a_1 cannot work with a_2 or a_3;
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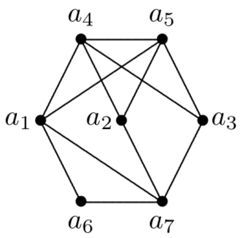
Thus, at least 3 working groups.
Working with a_1: a_1,
Working with a_2:
Working with a_3:



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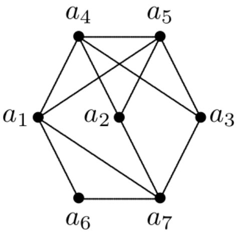
Working with a_1: a_1, a_4,
Working with a_2:
Working with a_3:



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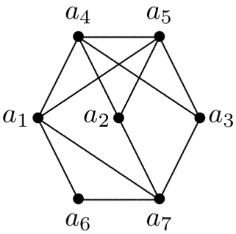
Working with a_1: a_1, a_4, a_5,
Working with a_2:
Working with a_3:



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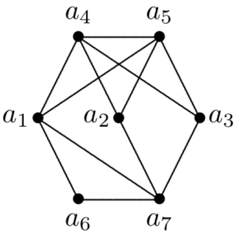
Working with a_1: a_1, a_4, a_5, a_6,
Working with a_2:
Working with a_3:



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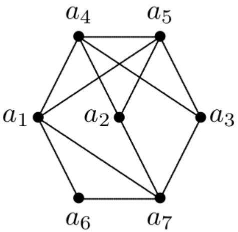

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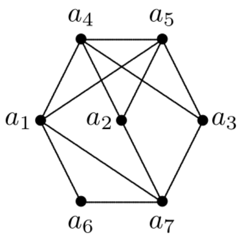
Working with a_2: a_2
Working with a_3:



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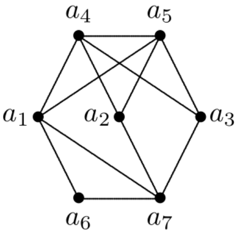
Thus, at least 3 working groups.
Working with a_1: a_1, a_4, a_5, a_6, a_7
Working with a_2: a_2
Working with a_3: a_3

Answer.

Let G be the graph whose vertices are the 7 students and two vertices are adjacent iff they are not one of the pairs. Any group of students must correspond to subgraph that is complete. The largest complete subgraph is a K_3. Thus the largest possible group size is 3 and so we need at least 3 groups. This is indeed possible. For example: $\{a_2, a_4, a_5\}, \{a_1, a_6\}, \{a_3, a_7\}$.



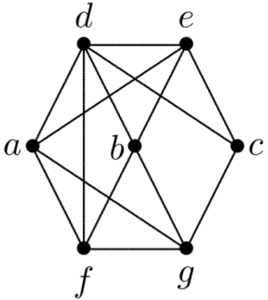
7. In the following graph:

(i) Find a simple circuit of length 8

(ii) Find the largest value of n such that C_n is a subgraph.

(iii) Find all the neighbors of b.

(iv) Find two different paths of length 3 from c to e.



Recall

Let G be a graph and $n \in \mathbb{Z}$.

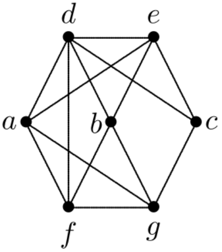
- ✍ Two distinct vertices in G are **neighbors** if they are joined by an edge.
- ✍ The **cycle** C_n, consists of n vertices: $v_1, v_2, \ldots, v_n$ and n edges: $v_1v_2, v_2v_3, \ldots, v_nv_1$.
- ✍ A **path of length** n from vertex u to vertex v is an alternating sequence of vertices and edges of G: $v_0e_1v_1e_2\ldots v_{n-1}e_nv_n$ where $u = v_0$, $v = v_n$, and each e_i is incident to v_{i-1} and v_i for $i = 1, \ldots, n$
- ✍ A path is a **circuit** if $u = v$ and $n > 0$.
- ✍ A path or circuit is **simple** if the edges it traverses are pairwise distinct.

(i) Find a simple circuit of length 8. (Simple circuit: Starting point = ending point; no repeat edges)

(ii) Find the largest value of n such that C_n is a subgraph. (C_n: starting point = ending point; no other repeat vertices)

(iii) Find all the neighbors of b.(Neighbor: connected by edges)

(iv) Find two different paths of length 3 from c to e.



(i) Find a simple circuit of length 8. (Simple circuit: Starting point $=$ ending point; no repeat edges)
$adfbgcdea$

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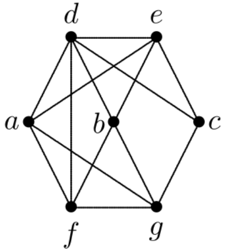

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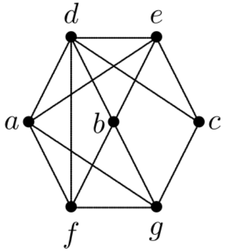

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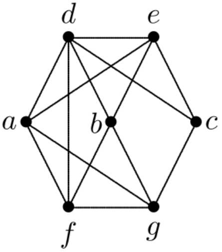
f, g, d, e

(iv) Find two different paths of length 3 from c to e.



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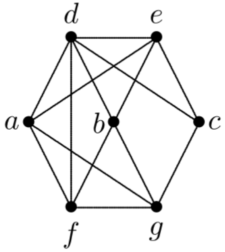
f, g, d, e

(iv) Find two different paths of length 3 from c to e.
$cgae$, $cece$



8. Either draw the graph with the specified properties or explain why such a graph does not exists:

(i) 4 vertices, degrees $1, 1, 1$ and 4.

(ii) 4 vertices, degrees $1, 2, 3$ and 4.

Recall (Handshaking Theorem)

Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

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Answer. (i) No. Degree sum must be even.

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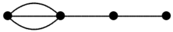
Answer. (i) No. Degree sum must be even.

(ii) Yes. 

9. Let G be a graph with 50 edges, 9 vertices of degree 2, 10 vertices of degree 6. The degrees of the other vertices are either 3 or 5. How many vertices does the graph have?

Recall (Handshaking Theorem)

Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

# Vertices	Degrees
9	2
10	6
	3
	5

Then $\sum_{v \in V} \deg(v) =$
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# Vertices	Degrees
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10	6
a	3
	5

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# Vertices	Degrees
9	2
10	6
a	3
b	5

Then $\sum_{v \in V} \deg(v) = 9 \times 2 + 10 \times 6 + 3a + 5b$

$2|E| =$

$a = \quad , b =$

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b	5

Then $\sum_{v \in V} \deg(v) = 9 \times 2 + 10 \times 6 + 3a + 5b$
$2|E| = 2 \times 50 = 100$
$a = 4, b =$

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Recall (Handshaking Theorem)

Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

# Vertices	Degrees
9	2
10	6
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Then $\sum_{v \in V} \deg(v) = 9 \times 2 + 10 \times 6 + 3a + 5b$
$2|E| = 2 \times 50 = 100$
$a = 4, b = 2$

Answer.

Let a and b be the number of vertices of degrees 3 and 5, respectively. Then

$$9 \times 2 + 3a + 5b + 10 \times 6 = 100.$$

Therefore $3a + 5b = 22$. The only solution is $a = 4$ and $b = 2$. Thus the answer is 25.