Review of 6.1 - 6.3

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Eigenvalues and Eigenvectors

Definition

Let A be a **square** matrix of order n. A **nonzero** column vector $u \in \mathbb{R}^n$ is called an *eigenvector* of A if

$$Au = \lambda u$$

for some scalar λ. The scalar λ is called an *eigenvalue* of A and u is said to be an eigenvector of A associated with the eigenvalue λ

Note: u is assumed to be nonzero column vector.

How to compute eigenvalues?

1. Remark 6.1.5:

$$
\begin{aligned}
& \lambda \text{ is an eigenvalue of } A \\
\Leftrightarrow\ & Au = \lambda u, \quad u \neq 0 \\
\Leftrightarrow\ & \lambda u - Au = 0, \quad u \neq 0 \\
\Leftrightarrow\ & (\lambda I - A)u = 0, \quad u \neq 0 \\
\Leftrightarrow\ & \text{ the linear system } (\lambda I - A)u = 0 \text{ has non-trivial solutions} \\
\Leftrightarrow\ & \det(\lambda I - A) = 0
\end{aligned}
$$

2. Note that $\det(\lambda I - A)$ is a polynomial of degree at most n. (WHY?, consider co-factor expansion of $\lambda I - A$ along the first row)
3. $\det(\lambda I - A)$ is called the characteristic polynomial of A.
4. All possible eigenvalues of A is exactly all the roots of characteristic polynomial of A.

Eigenspace of A

Definition

The **solution space** of the linear system $(\lambda I - A)x = 0$ is called the eigenspace of A associated with the eigenvalue λ and is denoted by E_λ.

Note:

1. First we need to compute the eigenvalues of A as in the last slide.
2. Then recalled the point that shows us how to compute the solution space of a homogeneous linear system.
3. Apply the skill in 2 to the homogeneous linear system $(\lambda I - A)x = 0$.

How to compute all the eigenvectors of A?

1. If u is an eigenvector of A associated with λ, then for any scalar $\alpha \neq 0$, αu is also an eigenvector of A associated with λ.
2. If u and v are two eigenvectors of A associated with λ, then for any scalar $\alpha \neq 0$ and $\beta \neq 0$, $\alpha u + \beta v$ is also an igenvector of A associated with λ.
3. As indicated by the concept of eigenspace, we have that the set of all eigenvectors of A associated with λ is a vector space, this vector space is exactly the solution space of the homogeneous linear system $(\lambda I - A)x = 0$.
4. So we can find a basis $\{u_1, \cdots, u_k\}$ for E_λ, which is exactly the basis for the solution space of the homogeneous linear system $(\lambda I - A)x = 0$.
5. Now each eigenvector of A associated with λ can be written as a linear combination of the basis $\{u_1, \cdots, u_k\}$ for E_λ.

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Diagonalization

Definition

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Here the matrix P is said to diagonalize A.

Theorem (6.2.3)

Let A be a square matrix of order n. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Determine if A is diagonalizable and find P.

Given a square matrix A of order n, we want to determine whether A is diagonalizable. Also, if A is diagonalizable, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

1. Find all distinct eigenvalues $\lambda_1, \cdots, \lambda_k$. (can be obtained by solving the characteristic equation of A.)
2. For each eigenvalue λ_i find a basis S_{λ_i} for the eigenspace E_{λ_i}.
3. Let $S = S_{\lambda_1} \cup \cdots \cup S_{\lambda_k}$. Then if $|S| < n$, A is not diagonalizable. If $|S| = n$, say $S = \{u_1, \cdots, u_n\}$, then

$$P = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

   is an invertible matrix that diagonalizes A. And the $(i,i)-$entry of $P^{-1}AP$ is the eigenvalue corresponding to the eigenvector u_i.

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Orthogonal Diagonalization

Definition

A square matrix A is called *orthogonal diagonalizable* if there exits an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

Question: When exactly a matrix is diagonalizable?

Theorem (6.3.4)

A square matrix is orthogonally diagonalizable if and only if it is symmetric.

This Theorem tells us that a symmetric matrix of order n always have n linear independent eigenvectors. (WHY?)

Find P?

Given a symmetric matrix A of order n, we want to find an orthogonal matrix P such that P^TAP is a diagonal matrix.

1. Find all distinct eigenvalues $\lambda_1, \cdots, \lambda_k$.
2. For each eigenvalue λ_i, first find a basis S_{λ_i} for the eigenspace E_{λ_i} and then use the Gram-Schmidt Process to transfer S_{λ_i} to an orthonormal basis T_{λ_i}.
3. Let $T = T_{\lambda_1} \cup \cdots T_{\lambda_k}$, say $T = \{v_1, \cdots, v_n\}$. Then

$$P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

is an orthogonal matrix that diagonalizes A. And the $(i, i)-$entry of P^TAP is the eigenvalue corresponding to the eigenvector v_i.