Chapter 4

Special Probability Distributions

Overview

- **Discrete Distributions**
  - Discrete uniform distribution
  - Bernoulli and Binomial distributions
  - Negative binomial distribution
  - Poisson distribution
  - Poisson approximation to Binomial distribution

Overview (Continued)

- **Continuous Distributions**
  - Continuous uniform distribution
  - Exponential distribution
  - Normal distribution
  - Normal approximation to Binomial distribution

4.1 Discrete Uniform Distribution

Definition 4.1

- If the random variable X assumes the values $x_1, x_2, \cdots, x_k$, with equal probability,
- then the random variable X is said to have a discrete uniform distribution and the probability function is given by

$$f_X(x) = 1/k, \qquad x = x_1, x_2, \cdots, x_k,$$

and 0 otherwise.

Mean and Variance of Discrete Uniform Distribution

Theorem 4.1

The mean and variance of the **discrete uniform distribution** are

$$\mu = E(X) = \sum x\, f_X(x) = \sum_{i=1}^{k} x_i \frac{1}{k} = \frac{1}{k}\sum_{i=1}^{k} x_i \,,$$

Mean and Variance of Discrete Uniform Distribution (Continued)

Theorem 4.1 (Continued)

$$\sigma^2 = V(X) = \sum (x-\mu)^2 f_X(x) = \frac{1}{k}\sum_{i=1}^{k}(x_i-\mu)^2$$

or

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{k}\left(\sum_{i=1}^{k} x_i^2\right) - \mu^2$$

Example 1

- When a light bulb is selected at random from a box that contains a 40-watt bulb, a 60-watt bulb, a 75-watt bulb, and a 100-watt bulb,
- each element of the sample space $S = \{40, 60, 75, 100\}$ occurs with probability 1/4.
- Therefore we have a discrete uniform distribution with

$$f_X(x) = \frac{1}{4}, \qquad x = 40, 60, 75, 100,$$

and 0 otherwise.

Example 2

- When a die is tossed, each element of the sample space $S = \{1, 2, 3, 4, 5, 6\}$ occurs with probability $1/6$.
- Therefore, we have a uniform distribution with

$$f_X(x) = \frac{1}{6}, \qquad x = 1, 2, \cdots, 6,$$

and 0 otherwise.

$$\mu = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5.$$

Example 3 (Continued)

$$\sigma^2 = \frac{1}{6}[(1-3.5)^2 + \cdots + (6-3.5)^2] = \frac{35}{12}.$$

Alternatively,

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{6}(1^2 + \cdots + 6^2) - 3.5^2 = \frac{35}{12}.$$

4.2 Bernoulli and Binomial Distributions

4.2.1 Bernoulli Distribution

- A **Bernoulli experiment** is a random experiment with only two possible outcomes, say ‘success’ or ‘failure’ (e.g. head or tail, defective or non-defective, boy or girl, yes or no.).

- It is convenient to code the two outcomes 1 and 0.

4.2 Bernoulli and Binomial Distributions

Definition 4.2

- A random variable X is defined to have a Bernoulli distribution if the probability function of X is given by
$$f_X(x) = p^x(1-p)^{1-x}, \quad x = 0, 1;$$
where the parameter p satisfies $0 < p < 1$.
$f_X(x) = 0$ for other X values.
- $(1-p)$ is often denoted by q.
- $\Pr(X = 1) = p$ and $\Pr(X = 0) = 1 - p = q$.

Parameter and Family of Distributions

Remarks:

- Suppose $f_X(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution.
- Such a quantity is called a **parameter** of the distribution.
- p is the **parameter** in the Bernoulli distribution.
- The collection of all probability distributions for different values of the parameter is called a **family** of probability distributions.

Bernoulli Distribution

Theorem 4.2

If X has a Bernoulli distribution, then the mean and variance of X are

$$\mu = E(X) = p,$$

and

$$\sigma^2 = V(X) = p(1-p) = pq.$$

Example 1

- A box contains 4 blue and 6 red balls. If a ball is drawn at random from the box, what is the probability that a blue ball is chosen?

Example 1 (Continued)

<u>Solution</u>

- Let $X = 1$ if a blue ball is chosen and 0 otherwise. Then
- $\Pr(X = 1) = 4/10 = 0.4$ and
  $\Pr(X = 0) = 1 - \Pr(X = 1) = 0.6.$
- That is

$$f_X(x) = \begin{cases} 0.4, & x = 1, \\ 0.6, & x = 0, \end{cases}$$

and 0 otherwise.

Example 2

- A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several times independently so that the probability of success, say p, remains the **same** from trial to trial.
- A box contains 4 blue and 6 red balls. 5 balls are drawn at random from the box, one at a time and **<u>with replacement</u>**.
- Let the draw of a blue ball be considered as a success.
- Then we have a sequence of 5 Bernoulli trials with $p = \Pr(\text{a blue ball}) = 4/10 = 0.4$.

Example 2 (Continued)

- We now consider a sequence of n Bernoulli trials.
- Let X_1 denote the result (0 or 1) of the first trial, X_2 denote the result of the second trial, and so on.
- The result of the n trials is then a sequence $(X_1, X_2, \cdots, X_n)$, where each element X_i is a Bernoulli variable.
- Let X be the number of successes (say, number of 1's) in the sequence of n Bernoulli trials.
- Then $X = X_1 + X_2 + \cdots + X_n$

  e.g. $X = 1 + 1 + 0 + \cdots + 0 + 1$

Example 2 (Continued)

- Consider a particular sequence (outcome), say,

$$\underbrace{SS\cdots S}_{x\ times}\quad \underbrace{FF\cdots F}_{(n-x)\ times}$$

- Its probability is

$$\underbrace{pp\cdots p}_{x\ times}\quad \underbrace{qq\cdots q}_{(n-x)\ times}\quad = p^x q^{n-x}.$$

Example 2 (Continued)

- It is noted that any other sequence with x 'successes' and $n - x$ 'failures', regardless of its order, has the same probability.
- There are altogether ${}_nC_x$ such sequences with x 'successes' and $n - x$ 'failures'.
- Hence the probability that $X = x$ is given by

$$\Pr(X = x) = f_X(x) = \binom{n}{x} p^x (1 - p)^{n-x},$$

for $x = 0, 1, \cdots, n$.

4.2.2 Binomial Distributions

Definition 4.3

- A random variable X is defined to have a **binomial distribution** with two parameters n and p, (i.e. $X \sim B(n, p)$), if the probability function of X is given by

$$\Pr(X = x) = f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} p^x q^{n-x},$$

  for $x = 0, 1, \cdots, n$, where p satisfies $0 < p < 1, q = 1 - p$, and n ranges over the positive integers.

- X is the **number of successes** that occur in n **independent Bernoulli trials**.

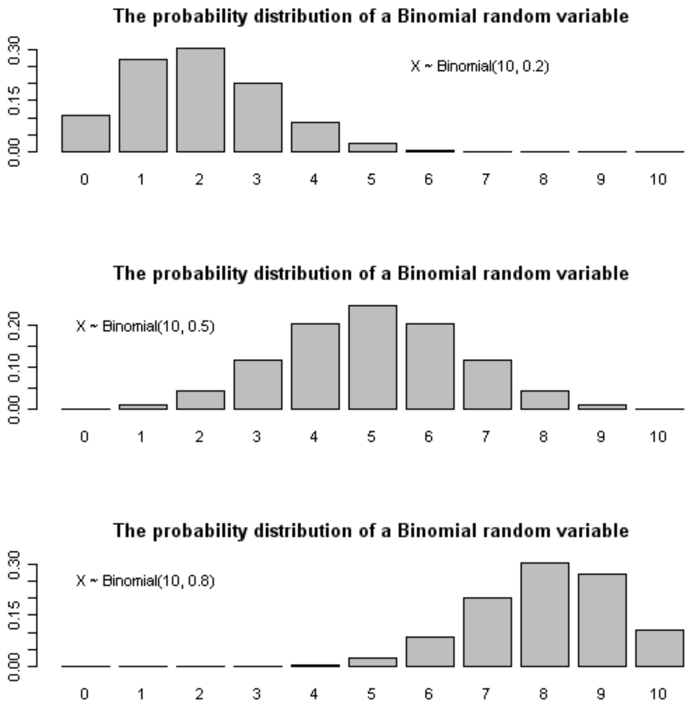
Binomial Distributions (Continued)

Note:

- When $n = 1$, the probability distribution of X becomes
$$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1.$$

- Therefore Bernoulli distribution is a special case of the binomial distribution.

Graphs of Binomial Distributions



Mean and Variance of Binomial Distributions

Theorem 4.3

- If X has a binomial distribution with parameters n and p, (i.e. $X \sim B(n, p)$)
- then the mean and variance of X are

$$\mu = E(X) = np$$

and

$$\sigma^2 = V(X) = np(1 - p) = npq.$$

Conditions for a Binomial Experiment

1. It consists of n repeated Bernoulli trials.
2. Only two possible outcomes: success and failure in **each** trial
3. $\Pr(\text{success}) = p$ is the same constant in each trial.
4. Trials are **independent**.

- The random variable X is the **number of successes** among the $\boldsymbol{n}$ **trials** in a binomial experiment.
- Then $X \sim B(n, p)$.

Example 1

- What is the probability of observing exactly 6 heads when an unbiased coin is flipped 10 independent times?

Solution to Example 1

- Let X denote the number of heads in 10 independent tosses.
- Since there are two outcomes with same probability of success, namely 1/2 for all 10 independent trials, therefore $X \sim B(10, 0.5)$.
- Hence

$$\Pr(X = 6) = {}_{10}C_6 \times 0.5^6 \times (1 - 0.5)^4$$
$$= \frac{10!}{(6! \times 4!)} \left(\frac{1}{2}\right)^{10}$$
$$= 0.2051.$$

Solution to Example 1 (Continued)

- Alternatively, by making use of the table of upper tail of the c.d.f. of a Binomial distribution for $n = 10$ and $p = 0.5$, we have

$$\begin{aligned}\Pr(X = 6) &= \Pr(X \geq 6) - \Pr(X \geq 7) \\ &= 0.3770 - 0.1719 = 0.2051.\end{aligned}$$

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Example 2

- The probability that a child recovers from a certain kind of disease is 0.2.
- If 20 children are known to have contracted this disease, what is the probability that
  (a) at least 8 survive and
  (b) from 2 to 5 survive (inclusive)?

Solution to Example 2

- Let X denote the number of children out of the 20 children recover from the disease.
- Then $X \sim B(20, 0.2)$.

(a) $\Pr(X \geq 8) = 0.0321.$

(b)
$$\begin{aligned}
\Pr(2 \leq X \leq 5) &= \Pr(X \leq 5) - \Pr(X \leq 1) \\
&= 1 - \Pr(X \geq 6) - (1 - \Pr(X \geq 2)) \\
&= \Pr(X \geq 2) - \Pr(X \geq 6) \\
&= 0.9308 - 0.1958 = 0.7350.
\end{aligned}$$

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Example 3

- A man claims to have extrasensory perception (ESP).
- As a test, a fair coin is flipped 10 times, and he is asked to predict the outcome in advance.
- The man gets 7 out of 10 correct.
- What is the probability that he would have done at least this well if he had no ESP?

Solution to Example 3

- If he had no ESP, then the probability that he guesses correctly for each outcome is 0.5.
- Hence the number, X, of correct guesses out of 10 guesses, follows the binomial distribution with parameters $n = 10$ and $p = 1/2$.
- That is $X \sim B(10, 0.5)$.

Solution to Example 3 (Continued)

- Thus, the probability he would have done at least that well if he had no ESP is given by

$$\Pr(X=7) = {}_{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} = 0.1172.$$

- Alternatively,

$$\begin{aligned} \Pr(X=7) &= \Pr(X \geq 7) - \Pr(X \geq 8) \\ &= 0.1719 - 0.0547 \\ &= 0.1172. \end{aligned}$$

Example 4

- Suppose that 20% of all copies of a particular textbook fail a certain binding strength test.
- 20 copies of the textbook were randomly selected.

(a) Find the probability that at most 8 copies fail the test.

(b) Find the probability that exactly 8 copies fail the test.

Solution to Example 4

- Let X denote the number of copies among 20 randomly selected copies that fail the test.
- Then X has a binomial distribution with $n = 20$ and $p = 0.2$.

(a) The probability that at most 8 copies fail the test is

$$\Pr(X \leq 8) = 1 - \Pr(X \geq 9) = 1 - 0.0100 = 0.99.$$

(b) The probability that exactly 8 copies fail the test is

$$\begin{aligned} \Pr(X = 8) &= \Pr(X \geq 8) - \Pr(X \geq 9) \\ &= 0.0321 - 0.0100 = 0.0221. \end{aligned}$$

Example 5

- An electronics manufacturer claims that at most 10% of its power supply units need services during the warranty period.
- To investigate this claim, technicians at a testing laboratory purchase 20 units and subject each one to accelerated testing to simulate use during the warranty period.
- Let p denote the probability that a power supply unit needs repair during the period (the proportion of all such units that need repair).

Example 5 (Continued)

- The laboratory technicians must decide whether the data resulting from the experiment supports that claim the $p \leq 0.10$.
- Let X denote the number of units among 20 sampled that need repair, so $X \sim B(20, p)$. (Why?)
- Consider the decision rule:
  - Reject the claim that $p \leq 0.10$ in favour of the conclusion that $p > 0.10$ if $x \geq 5$, (where x is the observed value of X) and
  - consider the claim plausible if $x \leq 4$.

Example 5 (Continued)

- The probability that the claim is rejected when $p = 0.10$ (an incorrect conclusion) is

$$\Pr(X \geq 5 \text{ when } p = 0.10) = 0.0432$$

- The probability that the claim is not rejected when $p = 0.20$ (a different type of incorrect conclusion) is

$$\begin{aligned} &\Pr(X \leq 4 \text{ when } p = 0.20) \\ &= 1 - \Pr(X \geq 5 \text{ when } p = 0.20) \\ &= 1 - 0.3704 = 0.6296. \end{aligned}$$

Example 5 (Continued)

- The first probability is rather small, but the second is intolerably large.

- When $p = 0.20$, the manufacturer has grossly understated in percentage of units that need service, and the stated decision rule is used, 63% of all samples will result in the manufacturer's claim being judged plausible!

4.3 Negative Binomial Distribution

- Let us consider an experiment where the properties are the same as those listed for a binomial experiment, with the exception that the trials will be repeated until a **fixed** number of successes occur.
- We are interested in the probability of the k-th success occurs on the x-th trials where x is the random variable. (Notice that in Binomial distribution, we are interested in the probability of x successes in n trials)

Negative Binomial Distribution (Continued)

- For example, suppose we want to find the probability that the fifth success occurs in the seventh trials
- In other words, we need to have 4 successes in the first six trials and a success in the seventh trial such as SSFSFSS or SFSSFSS
- The probability of obtaining such sequence is p^5q^2, where p is the probability of a success
- The number of all possible sequences is ${}_6C_4$ (Note: The seventh trial must be a success)

Negative Binomial Distribution (Continued)

- Let X be a random variable represents the number of trials to produce the 5 successes in a sequence of independent Bernoulli trials

- Then

$$\Pr(X = 7) = \binom{6}{4} p^4 q^{6-4} p = \binom{6}{4} p^5 q^2$$

Negative Binomial Distribution (Continued)

- Let X be a random variable represents the number of trials to produce the k successes in a sequence of independent Bernoulli trials
- The random variable X is said to follow a Negative Binomial distribution with parameters k and p (i.e. $NB(k, p)$.)
- The probability function of X is given by

$$\Pr(X = x) = f_X(x) = \binom{x-1}{k-1} p^k q^{x-k},$$

for $x = k, k+1, k+2, \cdots$

Negative Binomial Distribution (Continued)

- If $X \sim NB(k, p)$, then it can be shown that

$$E(X) = \frac{k}{p}$$

and

$$Var(X) = \frac{(1-p)k}{p^2}$$

Example 1

- In an NBA championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

  (a) What is the probability that team A will win the series in 6 games?

  (b) What is the probability that team A will win the series?

Example 1 (Continued)

Solution

- Let X be the number of games that Team A plays to win the series (i.e. Team A wins 4 games in the X games)
- The probability of Team A winning a game is 0.55.
- Therefore X follows a Negative Binomial Distribution with parameters $k = 4$ and $p = 0.55$.
- That is $X \sim NB(4, 0.55)$

Example 1 (Continued)

Solution

(a) $\Pr(X = 6) = \binom{5}{3} 0.55^4 (1 - 0.55)^{6-4} = 0.1853$

(b)
$$\Pr(X \geq 4) = \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$$
$$= \binom{3}{3} 0.55^4 (1 - 0.55)^{4-4} + \binom{4}{3} 0.55^4 (1 - 0.55)^{5-4} + \binom{5}{3} 0.55^4 (1 - 0.55)^{6-4} + \binom{6}{3} 0.55^4 (1 - 0.55)^{7-4}$$
$$= 0.6083.$$

Example 2

- At a “busy time”, a telephone exchange is very near capacity, so callers have difficulty placing their calls.
- It may be of interest to know the number of attempts necessary in order to make a connection.
- Suppose that we let $p = 0.05$ be the probability of connection during a busy time.
- We are interested in knowing the probability that 5 attempts are necessary for a successful call.

Example 2 (Continued)

- The number of trials that are required to have the first success is known to follow a special case of negative binomial distribution called **geometric distribution**

Example 2 (Continued)

Solution

- Let X be the number of attempts are necessary for the first successful call.
- The probability of connecting is 0.05.
- Therefore X follows a Negative Binomial Distribution with parameters $k = 1$ and $p = 0.05$.
- (or X follows a Geometric Distribution with $p = 0.05$)
- That is $X \sim NB(1, 0.05)$ (or $X \sim Geom(0.05)$)

Example 2 (Continued)

Solution

$$\Pr(X = 5) = \binom{5-1}{1-1} 0.05^1 (1 - 0.05)^{5-1}$$

$$= 0.05(0.95)^4 = 0.0407$$

4.4 Poisson Distribution

- Experiments yielding numerical values of a random variable X, the number of successes occurring during a given time interval or in a specified region, are called **Poisson experiments**.
- The given time interval, t, may be of any length, such as a minute, a day, a week, a month, or even a year.

Poisson Distribution (Continued)

- Hence a Poisson experiment might generate observations for the random variable X representing the **number** of telephone calls in an hour received by an office, or the **number** of postponed games due to rain during a football season.
- The specified region could be a line segment, an area, a volume, or perhaps a piece of material.
- In this case, X might represent the number of mushrooms in a plot of land, the number of bacteria in a given culture, or the number of typing errors in a page.

Poisson Experiment

A Poisson experiment is one that possesses the following properties:

1. The number of successes occurring in one time interval or specified region are **independent** of those occurring in any other disjoint time interval or region of space.
2. The probability of a single success occurring during a very short time interval or in a small region is **proportional** to the length of the time interval or the size of the region and does not depend on the number of successes occurring outside this time interval or region.

Poisson Experiment (Continued)

3. The probability of more than one success occurring in such a short time interval or falling in such a small region is negligible.

Poisson Distribution

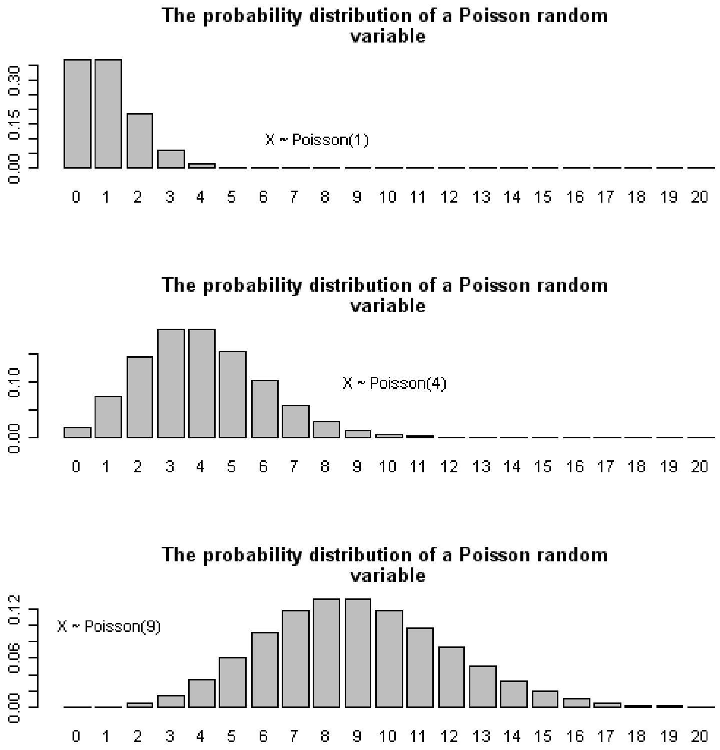
Definition 4.4

- The number of successes X in a Poisson experiment is called a **Poisson** random variable.
- The probability distribution of the Poisson random variable X, is called the Poisson distribution and the probability function is given by

$$f_X(x) = \Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \cdots$$

where λ is the average number of successes occurring in the given time interval or specified region and $e \approx 2.1718281818\cdots$

Poisson Distribution



Mean and Variance of Poisson RV

Theorem 4.4

If X has a **Poisson** distribution with parameter λ, then

$$E(X) = \lambda$$

and

$$V(X) = \lambda.$$

Mean and Variance of Poisson RV (Continued)

Proof

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \quad \text{where } y = x - 1,$$

$$= \lambda$$

Proof (Continued)

- The variance of the Poisson distribution is obtained by first finding

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!}$$

$$= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \quad \text{where } y = x-2,$$

$$= \lambda^2.$$

Proof (Continued)

- The variance of the Poisson distribution is obtained by first finding

$$E[X(X-1)] = \cdots$$

$$= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \quad \text{where } y = x - 2,$$

$$= \lambda^2.$$

- Hence $V(X) = E[X(X-1)] + E(X) - \left(E(X)\right)^2$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Example 1

- The average number of robberies in a day is four in a certain big city.
- What is the probability that six robberies occurring in two days?

<u>Solution</u>

- Let X be the number of robberies in two days.
- Then $X \sim P(\lambda)$ where $\lambda = 2 \times 4 = 8$.
- $\Pr(X = 6) = \dfrac{e^{-8}(8)^6}{6!} = 0.1222.$

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Example 1 (Continued)

- The average number of robberies in a day is four in a certain big city.
- What is the probability that six robberies occurring in two days?

Alternatively, we can make use of the table of c.d.f. of a Poisson distribution.

$$\begin{aligned}\Pr(X=6) &= \Pr(X\geq 6)-\Pr(X\geq 7)\\ &= 0.8086-0.6866=0.1222.\end{aligned}$$

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Example 2

- If the average number of oil tankers arriving each day at a port is known to be 10.

- The facilities at the port can handle at most 15 tankers per day.

- What is the probability that on a given day tankers will have to be sent away?

Solution to Example 2

- Let X be the number of tankers arriving each day.
- Then $X \sim P(\lambda)$, where $\lambda = 10$.

$$\Pr(X > 15) = \sum_{x=16}^{\infty} \frac{e^{-10}10^x}{x!} = 1 - \sum_{x=0}^{15} \frac{e^{-10}10^x}{x!}$$

$$= 1 - e^{-10}\left(1 + 10 + \frac{10^2}{2!} + \cdots + \frac{10^{15}}{15!}\right)$$

$$= 0.0487.$$

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Solution to Example 2 (Continued)

- Alternatively the answer can be read out from the cumulative probability table of Poisson distribution.

$$\Pr(X > 15) = \Pr(X \geq 16) = 0.0487.$$

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Example 3

A can company reports that the number of breakdowns per 8 hour shift on its machine-operated assembly line follows a Poisson distribution, with a mean of 1.5.

(a) What is the probability of exactly two breakdowns during the midnight shift?

(b) What is the probability of fewer than two breakdowns during the afternoon shift?

(c) What is the probability that no breakdowns during three consecutive 8 hour shifts?

(Assume the machine operates independently across shifts.)

Solution to Example 3

Let X be the number of breakdowns per 8 hour shift.

Then $X \sim P(\lambda)$, where $\lambda = E(X) = 1.5$.

(a) The probability of exactly 2 breakdowns during the night shift is

$$\Pr(X = 2) = (1.5)^2 e^{-1.5}/2! = 0.2510.$$

(b) The probability of fewer than 2 breakdowns during the afternoon shift is

$$\begin{aligned}\Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1) \\ &= e^{-1.5} + 1.5\, e^{-1.5} = 0.5578.\end{aligned}$$

Solution to Example 3 (Continued)

(c) To find the probability of no breakdowns during three consecutive 8 hour shifts, we consider first the probability of no breakdown during one 8 hour shift:

$$\Pr(X = 0) = e^{-1.5} = 0.2231.$$

Solution to Example 3 (Continued)

(c) (Continued)

- Next we define a random variable Y to be the number of 8 hour shifts experiencing no breakdowns in the 3 consecutive 8 hour shifts.
- If the machine operates independently across shifts, Y is a binomial random variable with $n = 3$ and $p = 0.2231$ and the desired probability is

$$\Pr(Y = 3) = {}_3C_3\,(0.2231)^3(1 - 0.2231)^0 = 0.0111.$$

Example 4

- As a check on the quality of the wooden doors produced by a company, its owner requested that each door undergoes inspection for defects before leaving the plant.
- The plant's quality control inspector found that on the average 1 square foot of door surface contains 0.5 minor flaw.
- Subsequently, 1 square foot of each door's surface was examined for flaws. The owner decided to have all doors reworked that were found to have two or more minor flaws in the square foot of surface that was inspected.

Example 4 (Continued)

(a) What is the probability that a door will fail inspection and be sent back for reworking?

(b) What is the probability that a door will pass inspection?

Example 4 (Continued)

<u>Solution</u>

Let X be the number of flaws found in 1 square foot of door surface.

Then $X \sim P(\lambda)$, where $\lambda = E(X) = 0.5$.

(a) $\Pr(X \geq 2) = 1 - \Pr(X < 2)$

$$= 1 - \Pr(X = 0) - \Pr(X = 1)$$

$$= 1 - \mathrm{e}^{-0.5} - \frac{(0.5)\mathrm{e}^{-0.5}}{1!} = 0.0902.$$

(b) $\Pr(X < 2) = 1 - 0.0902 = 0.9098.$

4.5 Poisson Approximation to the Binomial Distribution

Theorem 4.5

- Let X be a **Binomial** random variable with parameters n and p. That is

$$\Pr(X = x) = f_X(x) = {}_nC_r\, p^x q^{n-x}, \text{ where } q = 1 - p.$$

- Suppose that $n \to \infty$ and $p \to 0$ in such a way that $\lambda = np$ remains a constant as $n \to \infty$.
- Then X will have approximately a Poisson distribution with parameter np. That is

$$\lim_{\substack{p \to 0 \\ n \to \infty}} \Pr(X = x) = \frac{e^{-np}(np)^x}{x!}$$

Example 1

- The probability, p, of an individual car having an accident at a junction is 0.0001.
- If there are 1000 cars passing through the junction during certain period of a day, what is the probability of two or more accidents occurring during that period?

Example 1 (Continued)

Solution

- Let X denote the number of accidents among the 1000 cars.
- Then $X \sim B(1000, 0.0001)$. Hence

$$\Pr(X \geq 2) = \sum_{x=2}^{1000} \binom{1000}{x} 0.0001^x 0.9999^{1000-x}$$

Example 1 (Continued)

- The evaluation of these numbers is too difficult.
  We shall apply the result of the above theorem.
- Since $n = 1000$ and $p = 0.0001$, hence
$$np = \lambda = 0.1.$$
- Thus
$$\begin{aligned}\Pr(X \geq 2) &= 1 - \Pr(X = 0) - \Pr(X = 1) \\ &\approx 1 - e^{-0.1} - (0.1)e^{-0.1}/1! \\ &= 0.0047.\end{aligned}$$

Example 2

- In a manufacturing process in which glass items are being produced, defects or bubbles occur, occasionally rendering the piece undesirable for marketing.
- It is known that on the average 1 in every 1000 of these items produced has one or more bubbles.
- What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Solution to Example 2

- Let X be the number of items processing bubbles.
- Then $X \sim B(8000, 0.001)$.
- Use the Poisson approximation to the binomial. Put $\lambda = np = 8000 \times 0.001 = 8$, and hence X approximately $\sim P(\lambda)$.
- The (approximate) probability is given by

  $\Pr(X < 7) = 1 - \Pr(X \geq 7) \approx 1 - 0.6866 = 0.3134.$

Note: $\Pr(X \geq 7)$ is obtained from a Binomial table which shows the upper cumulative probability.

Remark

- If p is close to 1, we can still use Poisson distribution to approximate binomial probabilities by interchanging what we have defined to be a success and a failure so that changing p to a value close to zero.

4.6 Continuous Uniform Distribution

Definition 4.5

- A random variable is said to have a **uniform** distribution over the interval $[a, b], -\infty < a < b < \infty$, denoted by $U(a, b)$, if its probability density function is given by

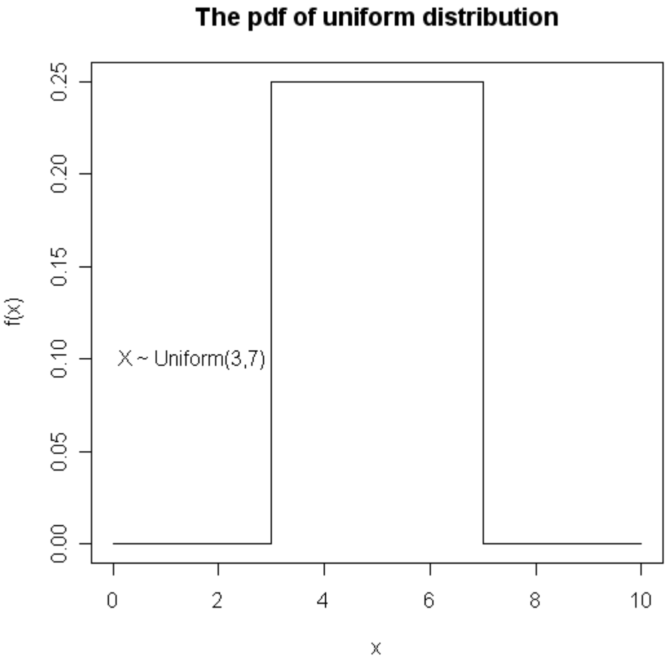
$$f_X(x) = \frac{1}{b-a}, \qquad \text{for } a \le x \le b,$$

and 0 otherwise.

Continuous Uniform Distribution (Continued)

Definition 4.5 (Continued)

- This distribution is also referred to as rectangular distribution because of the rectangular shape of the p.d.f.



Mean and Variance of Cont Uniform RV

Theorem 4.6

If X is uniformly distributed over $[a, b]$, then

$$E(X) = \frac{a+b}{2}, \qquad \text{and} \quad V(X) = \frac{1}{12}(b-a)^2.$$

Proof

$$E(X) = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_{x=a}^b$$

$$= \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2}.$$

Mean and Variance of Cont Uniform RV (Continued)

Proof (Continued)

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_{x=a}^{b}$$

$$= \frac{1}{b-a}\frac{(b^3-a^3)}{3}$$

$$= \frac{a^2+ab+b^2}{3}.$$

Mean and Variance of Cont Uniform RV (Continued)

Proof (Continued)

Hence

$$V(X) = E(X^2) - E(X)^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{1}{12}(a^2 - 2ab + b^2)$$

$$= \frac{(b-a)^2}{12}.$$

Example 1

- A point is chosen at random on the line segment [0, 2].
- What is the probability that the chosen point lies between 1 and 3/2?

Example 1 (Continued)

Solution

- Let X be the position of the point. $X \sim U(0, 2)$.
- That is

$$f_X(x) = \frac{1}{2}, \text{ for } 0 \leq x \leq 2,$$

and 0 otherwise.

$$\Pr\left(1 \leq X \leq \frac{3}{2}\right) = \int_1^{3/2} \frac{1}{2}\, dx = \frac{1}{2}[x]_{x=1}^{3/2} = \frac{1}{4}.$$

Example 2

Find the c.d.f. of a uniformly distributed random variable X between $[a, b]$.

<u>Solution</u>

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$
$$= \cdots$$

Example 2

Solution

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{x} 0\,dt\,, & \text{for } x < a, \\ \displaystyle\int_{-\infty}^{a} 0\,dt + \int_{a}^{x} \frac{1}{b-a}\,dt\,, & \text{for } a \le x \le b, \\ \displaystyle\int_{-\infty}^{a} 0\,dt + \int_{a}^{b} \frac{1}{b-a}dt + \int_{b}^{x} 0\,dt\,, & \text{for } b < x. \end{cases}$$

Example 2 (Continued)

Solution (Continued)

$$F_X(x) = \begin{cases} 0, & \text{for } x < a, \\ \dfrac{x-a}{b-a}, & \text{for } a \leq x \leq b, \\ 1, & \text{for } b < x. \end{cases}$$

4.7 Exponential Distribution

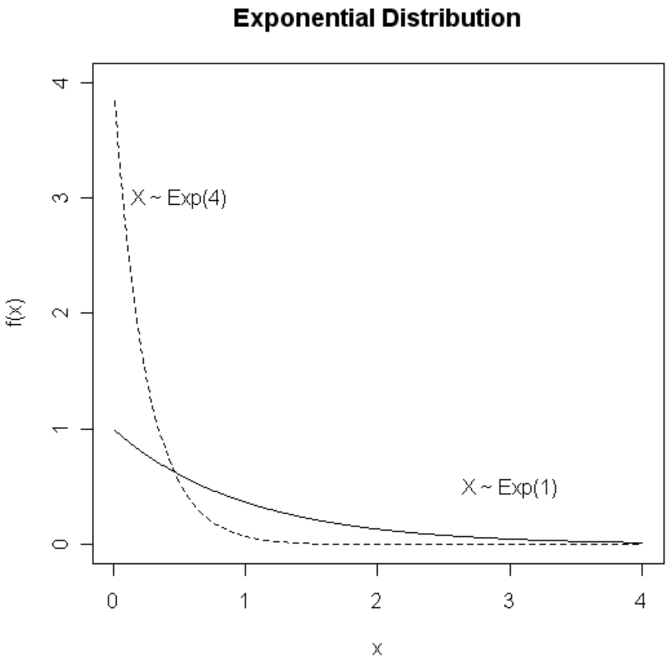
Definition 4.6

- A continuous random variable X assuming all nonnegative values is said to have an exponential distribution with parameter $\alpha > 0$ if its probability density function is given by

  $$f_X(x) = \alpha e^{-\alpha x}, \qquad \text{for } x > 0.$$

  and 0 otherwise.

- Note : $\int_{-\infty}^{\infty} f(x)\, dx = 1$



Mean & Variance of Exponential RV

Theorem 4.7

- If X has an **Exponential** distribution with parameter $\alpha > 0$, then

$$E(X) = \frac{1}{\alpha}$$

and

$$V(X) = \frac{1}{\alpha^2}.$$

Mean & Variance of Exponential RV (Continued)

Proof

$$E(X) = \int_0^\infty x\alpha e^{-\alpha x}\, dx = \int_0^\infty x\, d(-e^{-\alpha x})$$

$$= [-x\, e^{-\alpha x}]_0^\infty - \int_0^\infty (-e^{-\alpha x})dx = \int_0^\infty (e^{-\alpha x})dx$$

$$= \left[-\frac{1}{\alpha}e^{-\alpha x}\right]_0^\infty = \frac{1}{\alpha}.$$

Mean & Variance of Exponential RV (Continued)

Proof (Continued)

$$E(X^2) = \int_0^\infty x^2 \alpha e^{-\alpha x}\, dx = \int_0^\infty x^2 d(-e^{-\alpha x})$$

$$= [-x^2\, e^{-\alpha x}]_0^\infty - \int_0^\infty (-e^{-\alpha x}) d(x^2)$$

$$= \frac{2}{\alpha}\int_0^\infty x\, \alpha e^{-\alpha x} dx = \frac{2}{\alpha}\left(\frac{1}{\alpha}\right) = \frac{2}{\alpha^2}.$$

Hence

$$V(X) = \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha^2}.$$

Remark

- The p.d.f. can be written in the form

$$f_X(x) = \frac{1}{\mu}\, e^{-x/\mu}, \qquad \text{for } x > 0.$$

  and 0 otherwise.

- Then

$$E(X) = \mu \quad \text{and} \quad V(X) = \mu^2.$$

No Memory Property of Exponential Distribution

Theorem 4.8

Suppose that X has an **exponential** distribution with parameter $\alpha > 0$.

Then for any two positive numbers s and t, we have

$$\Pr(X > s + t \mid X > s) = \Pr(X > t)\,.$$

No Memory Property of Exponential Distribution

(Continued)

Theorem 4.8 (Continued)

Proof

$$\Pr(X > s + t \,|X > s) = \frac{\Pr(\{X > s + t\} \cap \{X > s\})}{\Pr(X > s)}$$

$$= \frac{\Pr(X > s + t)}{\Pr(X > s)}$$

$$= \frac{\int_{s+t}^{\infty} \alpha\, e^{-\alpha x}\, dx}{\int_{s}^{\infty} \alpha\, e^{-\alpha x}\, dx}$$

No Memory Property of Exponential Distribution (Continued)

Proof (Continued)

$$\frac{\Pr(X > s + t)}{\Pr(X > s)} = \frac{\int_{s+t}^{\infty} \alpha\, e^{-\alpha x}\, dx}{\int_{s}^{\infty} \alpha\, e^{-\alpha x}\, dx} = \frac{[-e^{-\alpha x}]_{x=s+t}^{\infty}}{[-e^{-\alpha x}]_{x=s}^{\infty}} = \frac{e^{-\alpha(s+t)}}{e^{-\alpha s}}$$

$$= e^{-\alpha t} = \Pr(X > t).$$

Note:

$$\Pr(X > t) = \int_{t}^{\infty} \alpha\, e^{-\alpha x}\, dx = [-e^{-\alpha x}]_{x=t}^{\infty} = e^{-\alpha t}$$

No Memory Property of Exponential Distribution (Continued)

The above theorem states that the **exponential** distribution has ‘**no memory**’ in the following sense:

- Let X denote the life length of a bulb.
- Given that the bulb has lasted s time units

  (i.e. $X > s$),
- then the probability that it will last for the next t units (i.e. $X > s + t$) is the same as the probability that it will last for the first t units as brand new.

Example 1

- Find the c.d.f. of the exponential distribution with parameter α.

<u>Solution</u>

- For $x \geq 0$,

$$F_X(x) = \Pr(X \leq x) = \int_0^x \alpha\, e^{-\alpha t}\, dt = [-e^{-\alpha t}]_0^x = 1 - e^{-\alpha x},$$

  and 0 otherwise.
- Hence

$$\Pr(X > x) = e^{-\alpha x}, \qquad \text{for } x > 0.$$

Example 2

- Suppose that the failure time, T, of a system is exponentially distributed, with a mean of 5 years.
- What is the probability that at least two out of five of these systems are still functioning at the end of 8 years?

<u>Solution</u>

- Since $E(T) = 5$, therefore $\alpha = 1/5$.
- Hence $T \sim Exp(1/5)$.

$$\Pr(T > 8) = \int_8^\infty \frac{1}{5}\, e^{-8/5}\, dx = e^{-8/5} \approx 0.2.$$

Example 2 (Continued)

- Let X represent the number of systems out of the five systems still functioning after 8 years.
- Then, $X \sim B(5, 0.2)$.
- Hence we obtain from the statistical table
$$\Pr(X \geq 2) = 0.2627.$$

Example 3

Suppose that the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an **exponential** distribution with expected response time equal to 5 seconds.

(a) Find the probability that the response time is at most 10 seconds.

(b) Find the probability that the response time is between 5 and 10 seconds.

Solution to Example 3

Since $E(X) = p5$, therefore $X \sim Exp(1/5)$.

(a) $\Pr(X \le 10) = 1 - e^{-10/5} = 1 - e^{-2}$

$= 1 - 0.1353 = 0.8647$.

(b) $\Pr(5 \le X \le 10) = \Pr(X \le 10) - \Pr(X \le 5)$

$= (1 - e^{-10/5}) - (1 - e^{-5/5})$

$= e^{-1} - e^{-2}$

$= 0.3679 - 0.1353$

$= 0.2326$.

Applications of the Exponential Distribution

- The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events such as customers arriving at a service facility or calls coming in to a switchboard.

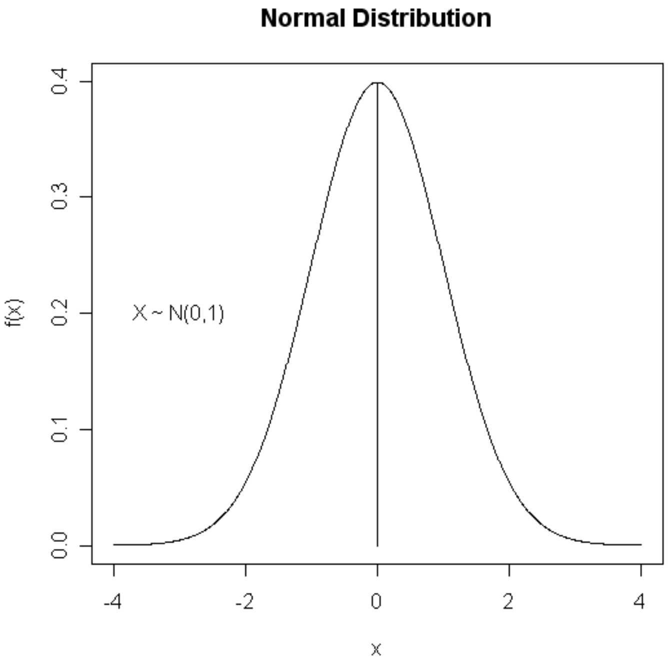
4.8 Normal Distribution

Definition 4.7

- The random variable X assuming all real values, $-\infty < x < \infty$, has a **normal** (or **Gaussian**) distribution if its probability density function is given by
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad -\infty < x < \infty,$$
where $-\infty < \mu < \infty$ and $\sigma > 0$.
- It is denoted by $N(\mu, \sigma^2)$.
- μ and σ are called parameters of the normal distribution.

Properties of the normal distribution

1. The graph of this distribution is of bell-shaped and called the normal curve and it is symmetrical about the vertical line $x = \mu$.

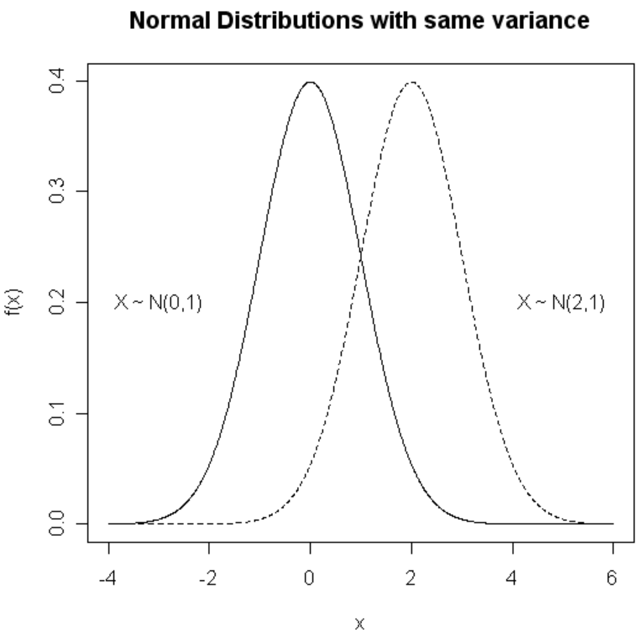


Properties of the normal distribution (Continued)

2. The maximum point occurs at $x = \mu$ and its value is
$$\frac{1}{\sqrt{2\pi}\sigma}$$
3. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
4. The total area under the curve and above the horizontal axis is equal to 1.
$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1.$$

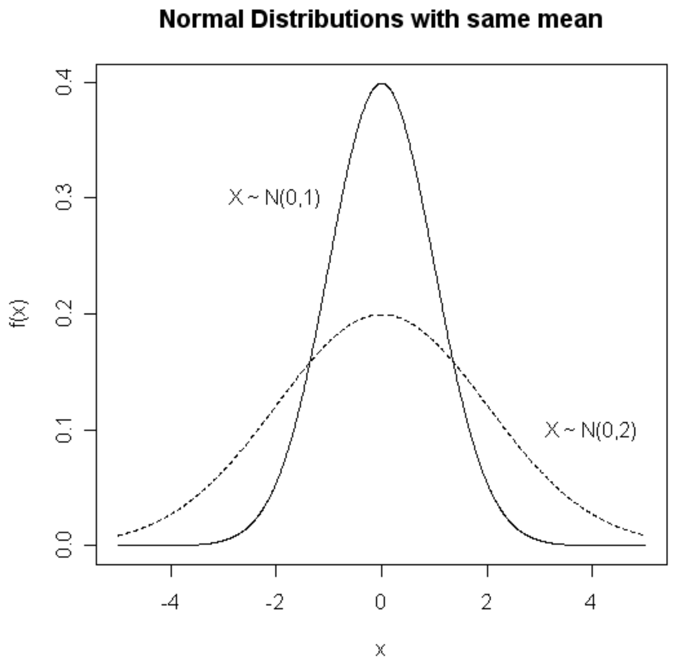
Properties of the normal distribution (Continued)

5. It can be shown that $E(X) = \mu$ and $V(X) = \sigma^2$.
6. Two normal curves are identical in shape if they have the same σ^2. But they are centered at different positions when their means are different.



Properties of the normal distribution (Continued)

7. As σ increases, the curve flattens; and as σ decreases, the curve sharpens.



Properties of the normal distribution (Continued)

8. If X has distribution $N(\mu, \sigma^2)$, and if

$$Z = \frac{(X - \mu)}{\sigma}$$

then Z has the $N(0, 1)$ distribution.

That is, $E(Z) = 0$ and $V(Z) = 1$.

We say that Z has a standardized normal distribution.

That is, the p.d.f. of Z may be written as

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right).$$

Properties of the normal distribution (Continued)

- The importance of the standardized normal distribution is the fact that it is tabulated.
- Whenever X has distribution $N(\mu, \sigma^2)$, we can always simplify the process of evaluating the values of $\Pr(x_1 < X < x_2)$ by using the transformation $Z = (X - \mu)/\sigma$. Hence $x_1 < X < x_2$ is equivalent to
$$(x_1 - \mu)/\sigma < Z < (x_2 - \mu)/\sigma.$$
- Let $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$. Then
$$\Pr(x_1 < X < x_2) = \Pr(z_1 < Z < z_2).$$

Example 1

- Given $X \sim N(50, 10^2)$, find $\Pr(45 < X < 62)$.

<u>Solution</u>

$$
\begin{aligned}
&\Pr(45 < X < 62) \\
&= \Pr[(45 - 50)/10 < Z < (62 - 50/10)] \\
&= \Pr(-0.5 < Z < 1.2) \\
&\Pr(Z < 1.2) - \Pr(Z < -0.5) \\
&= 1 - \Pr(Z \geq 1.2) - \Pr(Z > 0.5) \\
&= 1 - 0.1151 - 0.3085 = 0.5764.
\end{aligned}
$$

Microsoft Excel Worksheet

Statistical Tables

- Any introduction to statistics textbook gives statistical tables that give the values $\Phi(z)$ for a given z, where $\Phi(z)$ is the cumulative distribution function of a standardized Normal random variable Z.
- Or $1 - \Phi(z)$, the upper cumulative probability for a given z
- Thus

$$\Phi(z) = \Pr(Z \leq z).$$
$$1 - \Phi(z) = \Pr(Z > z).$$

- For example, $\Phi(1.2) = \Pr(Z \leq 1.2) = 0.8849.$
- $1 - \Phi(1.2) = \Pr(Z > 1.2) = 0.1151.$

Microsoft Excel Worksheet

Statistical Tables (Continued)

- Some statistical tables give the 100α percentage points, z_α, of a standardized Normal distribution, where

$$\alpha = \Pr(Z \geq z_\alpha) = \int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

- For example, $z_{0.05} = 1.645$, $z_{0.01} = 2.326$
- Since the p.d.f. of Z is symmetrical about 0, therefore

$$\Pr(Z \geq z_\alpha) = \Pr(Z \leq -z_\alpha) = \alpha.$$

- (i.e. $f_Z(-z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(-z)^2}{2}\right) = f_Z(z)$)

Example using Statistical Tables

If $X \sim N(3, 0.5^2)$, find the probability that

(a) $X < 2.3$,

(b) $X > 2.5$ and

(c) $3.5 < X < 4$.

Example using Statistical Tables (Continued)

Solution

(a) $\Pr(X < 2.3) = \Pr(Z < (2.3 - 3)/0.5) = \Pr(Z < -1.4)$

$= \Pr(Z > 1.4) = 0.0808.$

(b) $\Pr(X > 2.5) = \Pr(Z > (2.5 - 3)/0.5) = \Pr(Z > -1)$

$= 1 - \Pr(Z \geq 1) = 1 - 0.1587 = 0.8413.$

(c) $\Pr(3.5 < X < 4) = \Pr((3.5 - 3)/0.5 < Z < (4 - 3)/0.5)$

$= \Pr(Z < 2) - \Pr(Z < 1)$

$= \Pr(Z \geq 1) - \Pr(Z \geq 2)$

$= 0.1587 - 0.02275 = 0.13595.$

Statistical Functions in Microsoft Excel

- "=NORM.DIST(x; μ; σ,;true)" gives $\Pr(X < x)$, where $X \sim N(\mu, \sigma^2)$.

For examples,

- $X \sim N(50, 102)$
- "=NORM.DIST(45;50;10; true)" $= \Pr(X < 45) = 0.308538$.
- "=NORM.DIST(62;50;10; true)" $= \Pr(X < 62) = 0.88493$.

For standard normal, we have

- "=NORM.S.DIST(z)" gives $\Pr(Z < z)$.

For examples

- "=NORM.S.DIST(–0.5)" gives $\Pr(Z < -0.5) = 0.308538$.
- "=NORM.S.DIST(1.2)" gives $\Pr(Z < 1.2) = 0.88493$.

Statistical Functions in Microsoft Excel (Continued)

- "= NORM.INV(probability; μ; σ)" gives the inverse of normal cumulative distribution for the specified mean μ and standard deviation σ.

For example

- $X \sim N(74, 7^2)$
- "=NORM.INV(0.88,;74; 7)" gives 82.225.
- That is $\Pr(X < 82.225) = 0.88$

For standard normal, we have

- "=NORM.S.INV (probability)" gives the inverse of standard normal cumulative distribution.
- For example, "=NORM.S.INV (0.88)" gives 1.175.

Example 1

- On a common test, the average grade was 74 and the standard deviation was 7.
- If 12 % of the class are given A's, and the grades are curved to follow a normal distribution,
- what is the lowest possible A and the highest possible B?

Example 1 (Continued)

<u>Solution</u>

- We want to find x such that $\Pr(X > x) = 0.12$.

  $$\Pr(X > x) = 0.12 \text{ implies } \Pr(Z > (x - 74)/7) = 0.12,$$

  where $Z = (X - 74)/7$.

- From the table, $\Pr(Z > z) = 0.12$ implies $z = 1.175$* (linear interpolation is used).

Microsoft Excel Worksheet

Example 1 (Continued)

- Thus $(x - 74)/7 = 1.175$.
- Therefore $x = 74 + (1.175)7 = 82.225$.
- Hence, the lowest possible A is 83 and the highest possible B is 82.

Example 1 (Continued)

* <u>Linear interpolation</u>

- Let $\Pr(Z > a) = 0.12$.
- From the normal table, we have

  $\Pr(Z \geq 1.17) = 0.121$ and $\Pr(Z \geq 1.18) = 0.119$.
- Hence $\frac{a-1.17}{1.18-1.17} = \frac{0.12-0.121}{0.119-0.121}$

$$\Longrightarrow \quad a = 1.17 + 0.01\left(\frac{-0.001}{-0.002}\right) = 1.175.$$

Example 2

- Refer to Example 1. Find the 60th percentile.

<u>Solution</u>

- The 60% percentile is the X value such that
$$\Pr(X \leq x) = 0.6.$$
- From the statistical table (linear interpolation is used), we have $\Pr(Z < 0.2533) = 0.6$, where $Z \sim N(0,1)$
- Therefore
$$(x - 74)/7 = 0.2533$$
Hence
$$x = 7(0.2533) + 74 = 75.77.$$

Example 3

- Let X be the amount of sugar which a filling machine puts into '500g' packets.
- The actual amount of sugar filled varies from packets to packets.
- Suppose $X \sim N(\mu, 4^2)$.
- If only 2% of the packets contain less than 500g of sugar.
- What must be the mean fill of these packets?

Solution to Example 3

- $\Pr(X < 500) = 0.02$ implies
  $\Pr(Z < (500 - \mu)/4) = 0.02$ or
  $\Pr(Z > -(500 - \mu)/4) = 0.02$,
  where $Z = (X - \mu)/4$.
- From the table, we have $\Pr(Z > 2.0537) = 0.02$.
- Therefore $-(500 - \mu)/4 = 2.0537$.
- Hence $\mu = 508.2$.
- That is, the mean fill of these packets is 508.2 g.

Example 4

- The breakdown voltage of a randomly chosen diode of a particular type is known to be normally distributed.
- What is the probability that a diode's breakdown voltage is within 1 s.d. of its mean value?

Solution

- This question can be answered without knowing either μ or σ^2, as long as the distribution is known to be normal.
- That is, the answer is the same for **any** normal distribution.

Example 4 (Continued)

$$\begin{aligned}
&\Pr(X \text{ is within 1 s.d. of its mean}) \\
&= \Pr(\mu - \sigma \le X \le \mu + \sigma) \\
&= \Pr[(\mu - \sigma - \mu)/\sigma \le Z \le (\mu + \sigma - \mu)/\sigma] \\
&= \Pr(-1 \le Z \le 1) \\
&= \Pr(Z \le 1) - \Pr(Z < -1) \\
&= (1 - 0.1587) - 0.1587 = 0.6826.
\end{aligned}$$

- Similarly, the probability that X is observed to be within 2 s.d. is $\Pr(-2 \le Z \le 2) = 0.9544$ and
- within 3 s.d. is $\Pr(-3 \le Z \le 3) = 0.9974$.

Example 5

- Gauges are used to reject all components in which a certain dimension is not within the specification $1.50 \pm d$.
- It is known that this measurement is **normally** distributed with mean 1.50 and standard deviation 0.2.
- Determine the value of d such that the specifications "covers" 95% of the measurements.

Solution to Example 5

- We want to find d such that

$$\Pr(1.5 - d < X < 1.5 + d) = 0.95.$$

$$\Rightarrow \quad \Pr\left(-\frac{d}{0.2} < Z < \frac{d}{0.2}\right) = 0.95,$$

where

$$Z = \frac{X - 1.5}{0.2}$$

Solution to Example 5 (Continued)

From the table,

- $\Pr(-z < Z < z) = 0.95$ implies $\Pr(Z > z) = 0.025$

  Hence $z = 1.96$.
- Thus

$$\frac{d}{0.2} = 1.96.$$

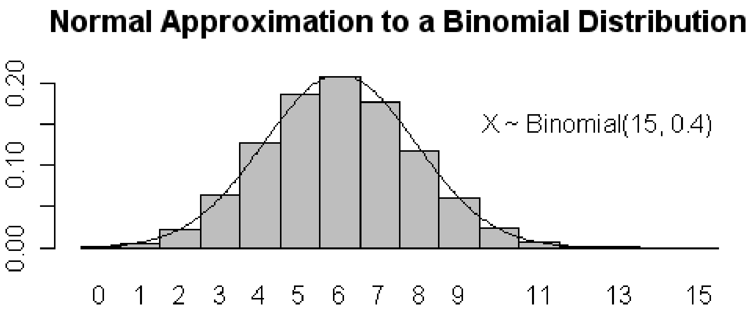
- Therefore $d = 1.96(0.2) = 0.392$.

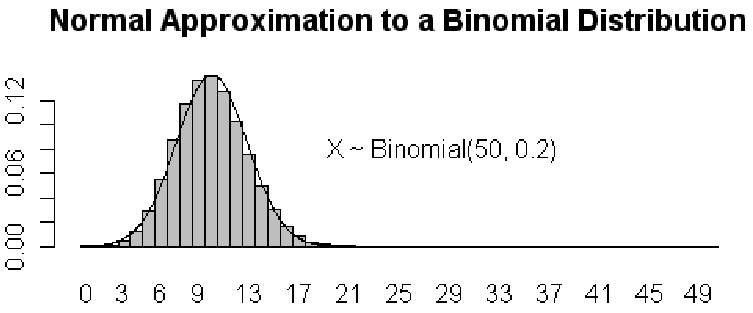
4.9 Normal approximation to the binomial distribution

- When $n \to \infty$ and $p \to 0$, we may use Poisson distribution to approximate a binomial distribution as has been shown in Section 4.5.
- When $n \to \infty$ and $p \to 1/2$, we can also use normal distribution to approximate the binomial distribution. In fact, even when n is small and p is not extremely close to 0 or 1, the approximation is fairly good.
- A good rule of thumb is to use the normal approximation only when

$$np > 5 \qquad \text{and} \qquad nq > 5.$$

Normal approximation to the binomial distribution





Theorem

Theorem

- If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$,
- then as $n \to \infty$,

$$Z = \frac{X - np}{\sqrt{npq}} \text{ is approximately } \sim N(0,1)$$

Example 1

- If $X \sim B(15, 0.4)$, then

$$\Pr(X = 4) = {}_{15}C_4(0.4)^4(0.6)^{11} = 0.1268.$$

- By normal approximation, we may consider

$$Y \sim N(\mu, \sigma^2)$$

with $\mu = np = 6$ and $\sigma^2 = npq = 3.6$.

Example 1 (Continued)

Hence

$$
\begin{aligned}
\Pr(X = 4) &= \Pr(3.5 < X < 4.5) \\
&\approx \Pr(3.5 < Y < 4.5) \\
&= \Pr\left[\frac{3.5 - 6}{\sqrt{3.6}} < Z < \frac{4.5 - 6}{\sqrt{3.6}}\right] \\
&\approx \Pr(-1.32 < Z < -0.79) \\
&= \Pr(Z \geq 0.79) - \Pr(Z \geq 1.32) \\
&= 0.2148 - 0.0934 = 0.1214.
\end{aligned}
$$

Continuity Correction

Note: In the above calculations, we have made the continuity correction to improve the approximation. In general, we have:

(a) $\Pr(X = k) \approx \Pr(k - ½ < X < k + ½)$.

(b) $\Pr(a \leq X \leq b) \approx \Pr(a - ½ < X < b + ½)$.

$\Pr(a < X \leq b) \approx \Pr(a + ½ < X < b + ½)$.

$\Pr(a \leq X < b) \approx \Pr(a - ½ < X < b - ½)$.

$\Pr(a < X < b) \approx \Pr(a + ½ < X < b - ½)$.

(c) $\Pr(X \leq c) = \Pr(0 \leq X \leq c) \approx \Pr(-½ < X < c + ½)$.

(d) $\Pr(X > c) = \Pr(c < X \leq n) \approx \Pr(c + ½ < X < n + ½)$.

Example 2

- A system is made up of 100 components, and each of which has a reliability equal to 0.90.
- These components function independently of one another, and the entire system functions only when at least 80 components function.
- What is the probability that the system functioning?

Solution to Example 2

- Let X be the number of components functioning.
- Then $X \sim B(100, 0.9)$.
- Thus $E(X) = (100)(0.9) = 90$ and

  $V(X) = 100(0.9)(0.1) = 9.$
- The system is functioning if $80 \leq X \leq 100$.

$$\Pr(80 \leq X \leq 100) = \Pr\left(\frac{79.5 - 90}{3} < \frac{X - 90}{3} < \frac{100.5 - 90}{3}\right)$$

$$\approx \Pr(-3.5 < Z < 3.5) \approx \Pr(Z < 3.5) - \Pr(Z < -3.5)$$

$$= 0.9995.$$