Answers/Solutions of Exercise 3 (Version: September 30, 2018)

1. $\boldsymbol{u} = (1,\ \sqrt{3})$, $\boldsymbol{v} = (-\sqrt{3},\ -1)$, $\boldsymbol{u} + \boldsymbol{v} = (1 - \sqrt{3},\ -1 + \sqrt{3})$,
   $3\boldsymbol{u} - 2\boldsymbol{v} = (3 + 2\sqrt{3},\ 2 + 3\sqrt{3})$.

2. (a) Substituting $(x, y) = (1, 2)$ and $(2, -1)$ into the equation $ax + by = c$, we has a system of linear equations

$$\begin{cases} a + 2b - c = 0 \\ 2a - \ \ b - c = 0 \end{cases}$$

which implies $a = \frac{3}{5}c$ and $b = \frac{1}{5}c$. In set notation, the line is

$$\{(x, y) \mid 3x + y = 5\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{5-t}{3}, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

(b) Substituting $(x, y, z) = (0, 1, -1)$, $(1, -1, 0)$ and $(0, 2, 0)$ into the equation $ax + by + cz = d$, we has a system of linear equations

$$\begin{cases} \quad\ \ b - c - d = 0 \\ a - \ b \qquad - d = 0 \\ \quad\ 2b \qquad - d = 0 \end{cases}$$

which implies $a = \frac{3}{2}d$, $b = \frac{1}{2}d$ and $c = -\frac{1}{2}d$. In set notation, the plane is

$$\{(x, y, z) \mid 3x + y - z = 2\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{2-s+t}{3}, s, t) \mid s, t \in \mathbb{R}\} \text{ (explicit)}.$$

(c) In explicit form, the line is

$$\{(1, -1, 0) + t(-1, 2, -1) \mid t \in \mathbb{R}\} = \{(1 - t, -1 + 2t, -t) \mid t \in \mathbb{R}\}.$$

To find the implicit form, we need to find two non-parallel planes containing the two points $(0, 1, -1)$ and $(1, -1, 0)$. The intersection of the two planes will give us the required line. Substituting $(0, 1, -1)$ and $(1, -1, 0)$ into $ax + by + cz = d$ we has a system of linear equations

$$\begin{cases} \qquad b - c - d = 0 \\ a - b \qquad - d = 0 \end{cases}$$

We obtain $a = c + 2d$ and $b = c + d$. There are infinitely many such planes. For example, we can write the line implicitly as

$$\{(x, y, z) \mid x + y + z = 0 \text{ and } 2x + y = 1\}.$$

3. $A = B = C = F$ and A, D, E are all different.

4. (a) U and V contains the origin but W does not.

   (b) $$\begin{cases} 2x - \ \ y + 3z = 0 \\ 3x + 2y - \ \ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{5}{7}t \\ y = \frac{11}{7}t \\ z = t \end{cases} \text{where } t \in \mathbb{R}$$

   So $U \cap V = \{\, (-\frac{5}{7}t,\ \frac{11}{7}t,\ t) \mid t \in \mathbb{R}\,\}$.

   $$\begin{cases} 3x + 2y - \ \ z = 0 \\ x - 3y - 2z = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{11}(2 + 7t) \\ y = \frac{1}{11}(-3 - 5t) \\ z = t \end{cases} \text{where } t \in \mathbb{R}$$

   So $V \cap W = \{\, (\frac{2+7t}{11},\ \frac{-3-5t}{11},\ t) \mid t \in \mathbb{R}\,\}$.

5. (a) A is a line joining the points $(1, 1, 1)$ and $(2, 3, 4)$.

   (b) Let $B = \{(x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0\}$. Since $x + y - z = 1$ and $x - 2y + z = 0$ are two non-parallel planes, B is the line of intersection of the two planes. To show that $A = B$, it suffices to show that the line A lies on both planes. This is true because $(1 + t) + (1 + 2t) - (1 + 3t) = 1$ and $(1 + t) - 2(1 + 2t) + (1 + 3t) = 0$ for all $t \in \mathbb{R}$.

   (c) For example, $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

6. Since

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} - 0 + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix} - 0 = a + b - d - c,$$

$V = \{(a, b, c, d) \mid a + b - d - c = 0\} = \{(x, y, z, w) \mid x + y - z - w = 0\} = T$.

On the other hand, $S \neq T$ because $(1, -1, 0, 0) \in T$ but $(1, -1, 0, 0) \notin S$.

7. (a) For example, $P = \{(1 + s - t,\ s, t) \mid s, t \in \mathbb{R}\}$.

   (b) A lies in P because $a - a + 1 = 1$. Since both B and C pass through $(0, 0, 0)$ and $(0, 0, 0) \notin P$, B and C does not lies in P.

   (c) B intersects P at one point, $(1, 0, 0)$.

   (d) The plane $x - y + z = 0$ contains C but not A and B.

(e) No. By Discussion 1.4.11, the solution set of a consistent nonzero linear system in three variables represents a point, a line or a plane in $\mathbb{R}^3$. Suppose we have a nonzero linear system whose solution set contains both B and C. Then the solution set must be a plane. However, the plane containing both B and C is the xz-plane which does not contain A. So the solution set cannot contain A.

8. $(2,3,-7,3)$, $(0,0,0,0)$ and $(-4,6,-13,4)$ are vectors in $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}$ while $(1,1,1,1)$ is not.

9. S_4 and S_6 span $\mathbb{R}^3$ while S_1, S_2, S_3 and S_5 do not span $\mathbb{R}^3$.

10. (a) Since $(1,1,0)$ and $(5,2,3)$ satisfy the equation $x-y-z=0$, $(1,1,0),(5,2,3)\in V$ and hence $\text{span}(S)\subseteq V$.

    Note that a general solution of $x-y-z=0$ is $x=s+t$, $y=s$, $z=t$ where $s,t\in\mathbb{R}$. Let $(s+t,s,t)$ be any vector in V. Consider the following equation:

$$a(1,1,0)+b(5,2,3)=(s+t,s,t) \quad\Leftrightarrow\quad \begin{cases} a+5b=s+t \\ a+2b=s \\ \quad\;\, 3b=t. \end{cases}$$

    Since

$$\left(\begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array}\right),$$

    the system is consistent for all $s,t\in\mathbb{R}$. So $V\subseteq\text{span}(S)$.

    We have shown that $\text{span}\{(1,1,0),\,(5,2,3)\}=V$.

    (b) Since

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

    by Discussion 3.2.5, $\text{span}\{(1,1,0),\,(5,2,3),\,(0,0,1)\}=\mathbb{R}^3$.

11. (a) $$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -2 \\ -5 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{array}\right)$$

    Since $\boldsymbol{u_2}\notin\text{span}\{\boldsymbol{v_1},\boldsymbol{v_2}\}$, $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}\neq\text{span}\{\boldsymbol{v_1},\boldsymbol{v_2}\}$.

(b) $\left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussiann}} \left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -8 & 8 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

$\left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

So $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

12. (a) $\left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Since $\boldsymbol{u_2} \notin \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \not\subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 2 & 2 & 6 & 2 & 1 & 4 & 1 & -1 \\ 0 & 5 & 9 & -1 & 0 & 0 & 3 & 6 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & 6 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 & 2 & 5 & 12 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \neq \mathbb{R}^4$.

13. By Example 3.2.8.2, S_1 does not span $\mathbb{R}^3$.

Since $\boldsymbol{w} - \boldsymbol{u} = -(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{v} - \boldsymbol{w})$, $\text{span}(S_2) = \text{span}\{\boldsymbol{u} - \boldsymbol{v},\ \boldsymbol{v} - \boldsymbol{w}\}$ and by Example 3.2.8.2, S_2 does not span $\mathbb{R}^3$.

S_3 spans $\mathbb{R}^3$: Since $\text{span}(S_3) \subseteq \mathbb{R}^3$, we only need to show $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \text{span}(S_3)$. Note that

$$\begin{aligned} \boldsymbol{u} &= \tfrac{1}{2}[(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})], \\ \boldsymbol{v} &= \tfrac{1}{2}[-(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})], \\ \boldsymbol{w} &= \tfrac{1}{2}[-(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})]. \end{aligned}$$

By Theorem 3.2.10, $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \text{span}(S_3)$. Hence $\text{span}(S_3)$ spans $\mathbb{R}^3$.

Using the same argument as for S_3, we can show that both S_4 and S_5 also span $\mathbb{R}^3$.

14. (a) True. Let $\boldsymbol{u} = (u)$ for $u \neq 0$. Then for any $(c) \in \mathbb{R}^1$, $(c) = \frac{c}{u}\boldsymbol{u}$.
    (b) False. For example, let $\boldsymbol{u} = (1, 1)$, $\boldsymbol{v} = (2, 2)$.
    (c) False. For example, let $S_1 = \{(1, 0), (0, 1)\}$, $S_2 = \{(1, 0), (0, 2)\}$.
    (d) False. For example, let $S_1 = \{(1, 0)\}$, $S_2 = \{(0, 1)\}$.

15. (a) Yes. See Remark 3.3.3.1.
    (b) No. It does not contain the zero vector.
    (c) No. $(1, 1, 1)$ belongs to the set but $2(1, 1, 1)$ does not.
    (d) No. $(0, 0, 1)$ belongs to the set but $\frac{1}{2}(0, 0, 1)$ does not.
    (e) Yes. It is $\text{span}\{(0, 0, 1)\}$.
    (f) No. It does not contain the zero vector.
    (g) No. $(1, 1, 0)$ and $(0, 0, 1)$ belong to the set but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ does not.
    (h) No. $(3, 2, 1)$ belongs to the set but $-(3, 2, 1)$ does not.
    (i) Yes. It is a solution set of a homogeneous linear system.
    (j) Yes. It is $\text{span}\{(1, 0, 0),\ (0, 1, 1)\}$.
    (k) No. $(1, 1, 1)$ and $(2, 2, 4)$ belong to the set but $(1, 1, 1) + (2, 2, 4) = (3, 3, 5)$ does not.

16. (a) Yes. It is a solution set of a homogeneous linear system.
    (b) No. $(1, 0, 0, 1)$ and $(0, 2, 0, 1)$ belong to the set but $(1, 0, 0, 1) + (0, 2, 0, 1) = (1, 2, 0, 2)$ does not.
    (c) No. $(1, 1, -1, -1)$ and $(0, 4, 0, 2)$ belong to the set but $(1, 1, -1, -1) + (0, 4, 0, 2) = (1, 5, -1, 1)$ does not.
    (d) Yes. It is $\text{span}\{(0, 1, 0, 0),\ (0, 0, 0, 1)\}$.
    (e) No. $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$ belong to the set but $(1, 0, 0, 0) + (0, 0, 1, 0) = (1, 0, 1, 0)$ does not.
    (f) No. It does not contain the zero vector.
    (g) Yes. It is a solution set of a homogeneous linear system.
    (h) No. $(1, 0, 0, -1)$ and $(0, 0, 4, 1)$ belong to the set but $(1, 0, 0, -1) + (0, 0, 4, 1) = (1, 0, 4, 0)$ does not.

17. (a) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (b) e.g. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. (c) e.g. $\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$. (d) Not possible.

18. (a) $W + \boldsymbol{v}$ is the line $x + y = 2$ in $\mathbb{R}^2$.

    (b) $W + \boldsymbol{v}$ is the line $\{(0, 0, 1) + c(1, 1, 1) \mid c \in \mathbb{R}\}$ in $\mathbb{R}^3$.

    (c) $W + \boldsymbol{v}$ is the plane $x + y + z = 1$ in $\mathbb{R}^3$.

19. $U \cap V$ is a subspace of $\mathbb{R}^3$ because it is a line in $\mathbb{R}^3$ passing through the origin. $V \cap W$ is not a subspace since it does not contain the origin.