Part 2

Ordinary Differential Equations

1.1 Introduction

A *differential* equation is an equation that contains one or more derivatives of a differentiable function. [In this course we deal only with ordinary DEs, NOT partial DEs.]

The *order* of a d.e. is the order of the equation's highest order derivative; and a d.e. is *linear* if it can be put in the form

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y(x) = F,$$

where a_i, $0 \leq i \leq n$, and F are all functions of x.

For example, $y' = 5y$ and $xy' - \sin x = 0$ are first order linear d.e.; $(y''')^2 + (y'')^5 - y' = e^x$ is third order, nonlinear.

We observe that in general, a d.e. has many solutions, e.g. $y = \sin x + c$, c an arbitrary constant, is a solution of $y' = \cos x$.

Such solutions containing arbitrary constants are called *general solution* of a given d.e.. Any solution obtained from the general solution by giving specific values to the arbitrary constants is called a *particular solution* of that d.e. e.g. $y = \sin x + 1$ is a particular solution of $y' = \cos x$.

Basically, differential equations are solved using integration, and it is clear that there will be as many integrations as the order of the DE. Therefore, THE GENERAL SOLUTION OF AN nth-ORDER DE WILL HAVE n ARBITRARY CONSTANTS.

1.2 Separable equations

A first order d.e. is *separable* if it can be written in the form $M(x) - N(y)y' = 0$ or equivalently, $M(x)dx = N(y)dy$. When we write the d.e. in this form, we say that we have *separated the variables*, because everything involving x is on one side, and everything involving y is on the other.

We can solve such a d.e. by integrating w.r.t. x:

$$\int M(x)dx = \int N(y)dy + c.$$

Example 1. Solve $y' = (1 + y^2)e^x$.

Solution. We separate the variables to obtain

$$e^x dx = \frac{1}{1 + y^2} dy.$$

Integrating w.r.t. x gives

$$e^x = \tan^{-1} y + c,$$

or

$$\tan^{-1} y = e^x - c,$$

or

$$y = \tan(e^x - c).$$

Example 2. Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?

Example 2

Let y = amount of substance in mg at time t in year.

Then $$\begin{cases} \dfrac{dy}{dt} = -ky \\ y(0) = 2 \end{cases}$$

where k is a positive constant.

$$\frac{dy}{y} = -k\,dt$$

$$\int \frac{dy}{y} = \int -k\,dt$$

$$\ln|y| = -kt + C$$

$$|y| = e^{-kt+C}$$

$$= e^{C} e^{-kt}$$

$$\therefore y = e^c e^{-kt} \text{ or } y = -e^c e^{-kt}$$

$$\therefore y = (\text{constant})\, e^{-kt}$$

i.e. $y = Ae^{-kt}$

where A is a constant.

$$y(0) = 2 \Rightarrow 2 = Ae^{-k(0)} = A$$

$$\underline{\underline{y = 2e^{-kt}}}$$

Q. How to find k?

Ans. Of course, the value of k will depend on the substance.

Usually we can calculate k by looking up the half-life of the substance in a chemistry table.

For example, the half-life = T years.

From the solution in Example 2, we know that

$$y = Ae^{-kt}$$

$$\therefore \frac{1}{2}A = Ae^{-kT}$$

$$-\ln 2 = -kT$$

$$k = \frac{\ln 2}{T}$$

Example 3. A copper ball is heated to 100°C. At $t = 0$ it is placed in water which is maintained at 30°C. At the end of 3 mins the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball is 31°C.

Physical information: Experiments show that the rate of change dT/dt of the temperature T of the ball w.r.t. time is proportional to the difference between T and the temp T_0 of the surrounding medium. Also, heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

Example 3

Let T = temperature of the ball at time t.

Then

$$\begin{cases} \dfrac{dT}{dt} = k(T-30) \\ T(0) = 100 \\ T(3) = 70 \end{cases}$$

$$\therefore \quad \frac{dT}{T-30} = k\,dt$$

$$\ln|T-30| = kt + c$$

$$T - 30 = Ae^{kt}$$

$$T(0) = 100 \Rightarrow 100 - 30 = Ae^{k(0)}$$

$$\Rightarrow 70 = A$$

$\therefore \quad T = 30 + 70e^{kt}$

$T(3) = 70 \Rightarrow 70 = 30 + 70e^{3k}$

$\Rightarrow 4 = 7e^{3k}$

$\Rightarrow k = \frac{1}{3}\left(\ln \frac{4}{7}\right) = \underline{\frac{\ln 4 - \ln 7}{3}}$

$$\therefore \quad T = 30 + 70\, e^{(\ln 4 - \ln 7)\,t/3}$$

$$\therefore \quad T = 31 \Rightarrow 1 = 70\, e^{(\ln 4 - \ln 7)\,t/3}$$

$$\Rightarrow \frac{(\ln 4 - \ln 7)}{3}\,t = \ln \frac{1}{70}$$

$$= -\ln 70$$

$$\Rightarrow t = \frac{3 \ln 70}{\ln 7 - \ln 4}$$

$$\approx \underline{\underline{22.78 \text{ min.}}}$$

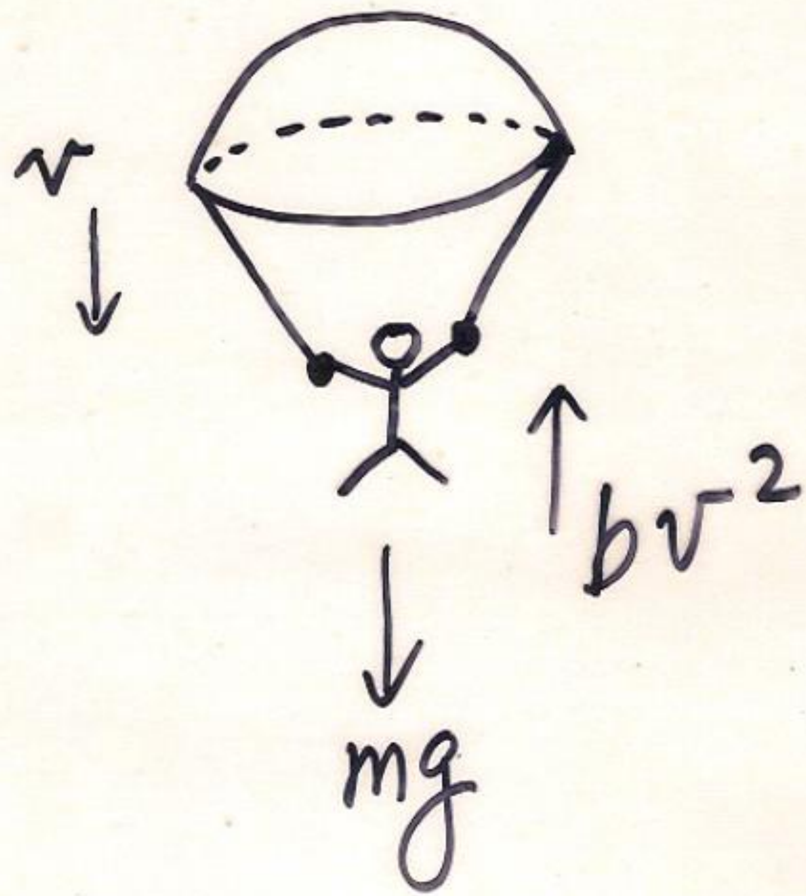
Example 4. Suppose that a sky diver falls from rest toward the earth and the parachute opens at an instant $t = 0$, when sky diver's speed is $v(0) = v_0 = 10\,\mathrm{m/s}$. Find the speed of the sky diver at any later time t.

Physical assumptions and laws:

weight of the man + equipment = 712N,

air resistance = bv^2, where $b = 30\,\mathrm{kg/m}$.

Example 4



Newton's second law

$$\Rightarrow m\frac{dv}{dt} = mg - bv^2$$

$$\frac{dv}{dt} = g - \frac{b}{m}v^2$$

$$\frac{dv}{dt} = -\frac{b}{m}\left(v^2 - \frac{mg}{b}\right)$$

Define $k = \sqrt{\frac{mg}{b}}$ (called the terminal velocity)

$$\therefore \frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

$$\frac{dv}{v^2 - k^2} = -\frac{b}{m}\,dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{b}{m}\,dt$$

$$\left(\frac{1}{v-k} - \frac{1}{v+k}\right) dv = -\frac{2kb}{m} dt$$

$$\ln|v-k| - \ln|v+k| = -\frac{2kb}{m} t + C$$

$$\ln\left|\frac{v-k}{v+k}\right| = -\frac{2kb}{m} t + C$$

$$\frac{v-k}{v+k} = A e^{-\frac{2kb}{m} t}$$

$$v - k = (v + k) A e^{-\frac{2kb}{m} t}$$

$$v\left(1 - A e^{-\frac{2kb}{m} t}\right) = k\left(1 + A e^{-\frac{2kb}{m} t}\right)$$

$$v = \left(\frac{1 + A e^{-\frac{2kb}{m} t}}{1 - A e^{-\frac{2kb}{m} t}}\right) k$$

Note: $\lim_{t \to \infty} v = k$

$= \text{terminal velocity.}$

In example 4, we have

$$v(0) = 10 \text{ m/s}$$

$$mg = 712 \text{ N}$$

$$b = 30 \text{ kg/m}$$

$$\therefore\ k = \sqrt{\frac{mg}{b}}$$

$$= \sqrt{\frac{712}{30}} \approx 4.87 \text{ m/s.}$$

$$10 = v(0) = \left(\frac{1+A}{1-A}\right)k$$

$$\Rightarrow A \approx 0.345$$

$$\therefore\ v = \left(\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}}\right)(4.87)$$

e.g. 2 $$\frac{dy}{dt} = ky$$

e.g. 3 $$\frac{dT}{dt} = k(T - 30)$$

e.g. 4 $$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

Autonomous equations i.e. t does not appear on the R.H.S.

Reduction to separable form

Certain first order d.e. are not separable but can be made separable by a simple change of variable. This holds for equations of the form

$$y' = g(\frac{y}{x}) \tag{1}$$

where g is any function of $\dfrac{y}{x}$. We set $\dfrac{y}{x} = v$, then $y = vx$ and $y' = v + xv'$. Thus (1) becomes $v + xv' = g(v)$, which is separable. Namely, $\dfrac{dv}{g(v) - v} = \dfrac{dx}{x}$. We can now solve for v, hence obtain y.

Example 6.

(a) Solve $2xyy' - y^2 + x^2 = 0$. $[x^2 + y^2 = cx]$

Example 6 (a)

$$2xyy' - y^2 + x^2 = 0$$

$$\Rightarrow \quad y' = \frac{-x^2 + y^2}{2xy} = \frac{-1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

Let $v = \dfrac{y}{x} \Longrightarrow y = xv$

$$\therefore \ y' = v + xv'$$

$$\therefore \ v + xv' = \frac{-1+v^2}{2v}$$

$$xv' = \frac{-1+v^2}{2v} - v = \frac{-1-v^2}{2v}$$

$$\frac{2v\,dv}{1+v^2} = -\frac{dx}{x}$$

$$\ln|1+v^2| = -\ln|x| + c$$

$$1+v^2 = A_1 e^{-\ln|x|} = A_1 e^{\ln\frac{1}{|x|}} = \frac{A_1}{|x|}$$

$$1+\frac{y^2}{x^2} = \frac{A}{x}$$

$$\underline{\underline{x^2+y^2 = Ax}}$$

(b) Solve the initial value problem $y' = \dfrac{y}{x} + \dfrac{2x^3 \cos x^2}{y}$, $y(\sqrt{\pi}\,) = 0$. $\left[y = x\sqrt{(2\sin x^2)}\right]$

Example 6 (b)

$$\begin{cases} y' = \dfrac{y}{x} + \dfrac{2x^3 \cos x^2}{y} \\ y(\sqrt{\pi}) = 0 \end{cases}$$

Let $v = \dfrac{y}{x} \Rightarrow y = xv$

$\Rightarrow y' = v + xv'$

$$\therefore \quad v + xv' = v + \frac{2x^2 \cos x^2}{v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{2x^2 \cos x^2}{v}$$

$$v\,dv = 2x \cos x^2\,dx$$

$$\frac{1}{2}v^2 = \sin x^2 + C$$

$$y^2 = 2x^2(\sin x^2 + C)$$

$$y(\sqrt{\pi}) = 0 \Rightarrow 0 = 2\pi(\sin\pi + C)$$

$$\Rightarrow C = 0$$

$$\therefore \; y^2 = 2x^2 \sin x^2$$

A d.e. of the form $y' = f(ax + by + c)$, where f is continuous and $b \neq 0$ (if $b = 0$, the equation is separable) can be solved by setting $u = ax + by + c$.

Example 7. $(2x-4y+5)y'+x-2y+3=0.$

Set $x-2y=u$, we have

$$(2u+5)\frac{1}{2}(1-u')+u+3=0,$$

$$(2u+5)u'=4u+11.$$

Separating variables and integrating :

$$\left(1 - \frac{1}{4u+11}\right) du = 2dx.$$

Thus $\qquad u - \frac{1}{4}\ell n|4u+11| = 2x + c_1,$

or $\qquad 4x + 8y + \ell n|4x - 8y + 11| = c.$

1.3 Linear First Order ODEs

A d.e. which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

where P and Q are functions of x, is called a linear first order d.e. Relation (1) above is the standard form of such a d.e.

Method to solve

$$y' + Py = Q \quad \text{----} \quad (1)$$

Step 1: Multiply (1) by R which will be determined later.

$$\therefore Ry' + RPy = RQ \quad \text{----} \quad (2)$$

Step 2: Set the L.H.S. of ② equal to $(Ry)'$.

$$\therefore \; (Ry)' = Ry' + RPy$$

$$\Rightarrow R'y + Ry' = Ry' + RPy$$

$$\Rightarrow R'y = RPy$$

$$\Rightarrow \quad R' = RP$$

$$\Rightarrow \quad \frac{dR}{dx} = RP$$

$$\therefore \quad \frac{dR}{R} = P\,dx$$

$$\therefore \quad \ln|R| = \int P\,dx + C$$

$$\therefore \quad R = Ae^{\int P\,dx}$$

Any choice of $A \neq 0$ will do,
so we take $A = 1$ for simplicity

$$\boxed{R = e^{\int P\,dx}} \quad \text{-----} \ (3)$$

R is called an <u>integrating factor</u>

of (1).

Step 3 : With R given by 3,

$$
\begin{aligned}
2 &\Rightarrow (Ry)' = RQ \\
&\Rightarrow d(Ry) = RQ\,dx \\
&\Rightarrow Ry = \int RQ\,dx \\
&\Rightarrow \boxed{y = \frac{1}{R}\int RQ\,dx} \quad \text{----} \ 4
\end{aligned}
$$

Summary

To solve $y' + Py = Q$:

First: find $R = e^{\int P dx}$.

Second: Write down the answer

$$y = \frac{1}{R} \int RQ \, dx$$

Example 8. Solve

(i) $xy' - 3y = x^2,\ x > 0.$

(ii) $y' - y = e^{2x}.$

Example 8 (i)

$$xy' - 3y = x^2, \quad x > 0$$

$$\therefore \quad y' - \frac{3}{x}y = x$$

i.e. $P = -\frac{3}{x}$, $Q = x$.

$$\therefore R = e^{\int P dx} = e^{\int -\frac{3}{x} dx}$$

$$= e^{-3\ln x} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

$$\therefore \quad y = \frac{1}{R}\int RQ\,dx$$

$$= \frac{1}{1/x^3}\int \frac{1}{x^3}\,x\,dx$$

$$= x^3 \int \frac{1}{x^2}\,dx$$

$$= x^3\left(-\frac{1}{x} + C\right)$$

$$\therefore \quad \underline{\underline{y = -x^2 + Cx^3}}$$

Example 8 (ii)

$$y' - y = e^{2x}$$

$$\therefore P = -1, \quad Q = e^{2x}$$

$$\therefore R = e^{\int P dx} = e^{\int -dx} = e^{-x}$$

$$\therefore \quad y = \frac{1}{R}\int RQ\,dx$$

$$= \frac{1}{e^{-x}}\int e^{-x}e^{2x}\,dx$$

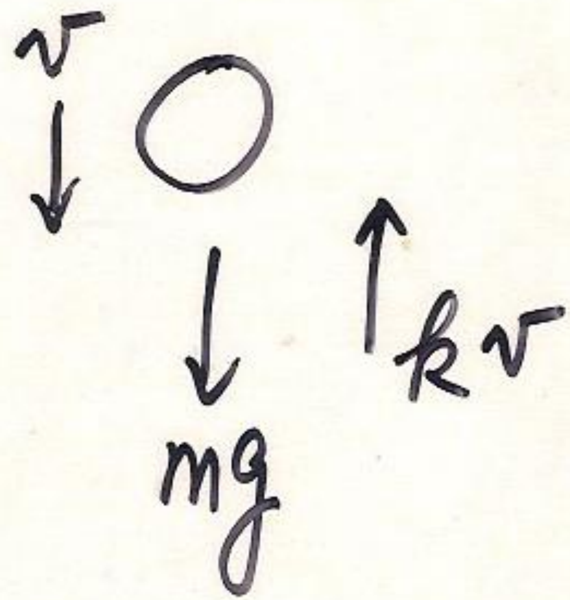
$$= e^{x}\int e^{x}\,dx$$

$$= e^{x}(e^{x}+c)$$

$$\therefore \quad y = e^{2x} + ce^{x}$$

Example 9. Consider an object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. The goal is to find the position $x(t)$ and velocity $v(t)$ at any time t.

Example 9



$$
\begin{cases}
m\dfrac{dv}{dt} = mg - kv \\
v(0) = 0 \\
x(0) = 0
\end{cases}
$$

where v = velocity at time t.

x = vertical distance from starting point at time t.

$$\therefore \quad \frac{dv}{dt} + \frac{k}{m} v = g$$

$$P = \frac{k}{m}, \quad Q = g$$

$$R = e^{\int P\,dt} = e^{\int \frac{k}{m} dt} = e^{\frac{k}{m} t}$$

$$v = \frac{1}{R}\int RQ\,dt$$

$$= \frac{1}{e^{\frac{k}{m}t}}\int e^{\frac{k}{m}t} g\,dt$$

$$= e^{-\frac{k}{m}t}\left(\frac{mg}{k}e^{\frac{k}{m}t} + c_1\right)$$

$$= \frac{mg}{k} + c_1 e^{-\frac{k}{m}t}$$

$$v(0) = 0 \Rightarrow 0 = \frac{mg}{k} + C_1$$

$$\Rightarrow C_1 = -\frac{mg}{k}$$

$$\therefore \quad v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$\frac{dx}{dt} = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$dx = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) dt$$

$$x = \frac{mg}{k}\left(t + \frac{m}{k} e^{-\frac{k}{m}t}\right) + C_2$$

$$x(0)=0 \Rightarrow 0 = \frac{m^2 g}{k^2} + C_2$$

$$\Rightarrow C_2 = -\frac{m^2 g}{k^2}$$

$$\therefore \quad \underline{\underline{x = \frac{mg}{k}t + \frac{m^2 g}{k^2}\left(e^{-\frac{k}{m}t} - 1\right)}}$$

Example 10. At time $t = 0$ a tank contains 20 lbs of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/min and the well stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.

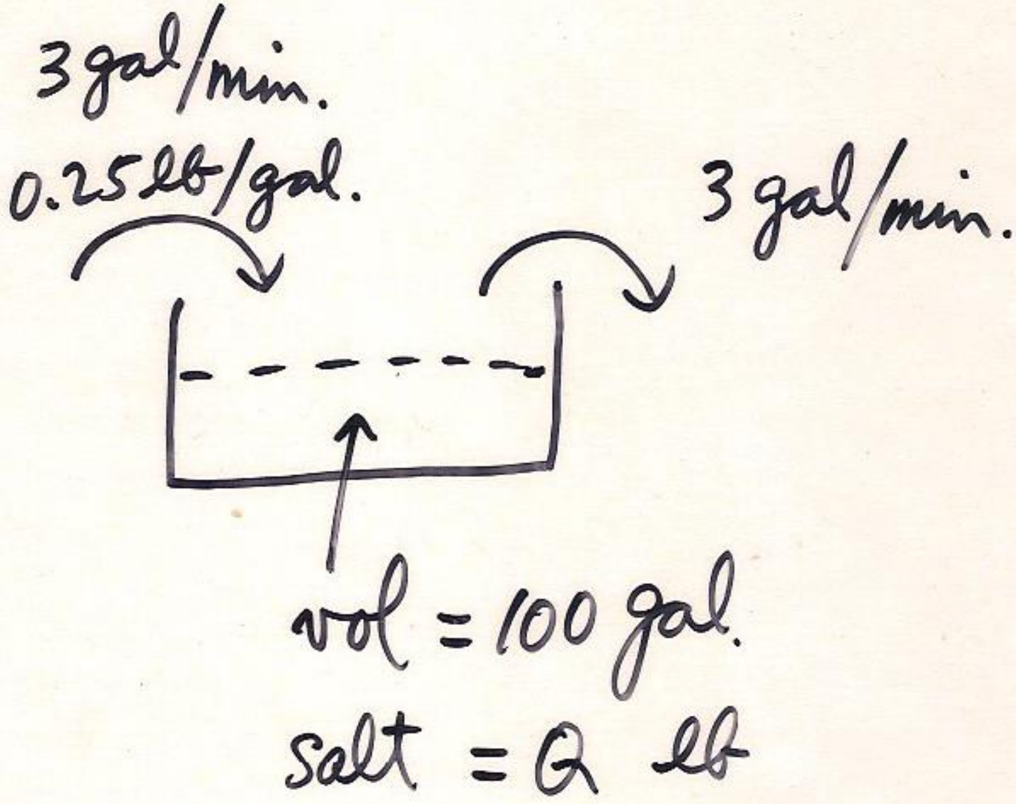
Example 10

$t = 0$

vol = 100 gal.

salt = 20 lb.

t



Note: density of salt solution at t is $\dfrac{Q}{100}$ lb/gal.

Suppose at $t+\Delta t$, there is $Q+\Delta Q$ lb of salt.

$$\Delta Q = (\text{salt input}) - (\text{salt output})$$

$$= 3 \times 0.25 \times \Delta t - 3 \times \frac{Q}{100} \times \Delta t$$

$$\frac{\Delta Q}{\Delta t} = 0.75 - \frac{3Q}{100}$$

$$\Delta t \to 0 \Rightarrow \frac{dQ}{dt} = 0.75 - 0.03Q$$

$$\frac{dQ}{dt} + 0.03Q = 0.75$$

$$R = e^{\int 0.03\,dt} = e^{0.03t}$$

$$Q = \frac{1}{R} \int R(0.75)\,dt$$

$$= e^{-0.03t} \int 0.75\, e^{0.03t}\,dt$$

$$= e^{-0.03t} \left(25 e^{0.03t} + c \right)$$

$$Q(0) = 20 \Rightarrow 20 = 25 + C$$

$$\Rightarrow C = -5$$

$$\therefore Q = 25 - 5e^{-0.03t}$$

Note that

$\lim_{t \to \infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration remains constant at 25 lbs/100 gal.

Example 11. In Example 2 in Section 1.2, we saw that radioactive substances typically decay at a rate proportional to the amount present. Sometimes the product of a radioactive decay is itself a radioactive substance which in turn de-

decays (at a different rate). An interesting example of this is provided by *Uranium-Thorium dating*, which is a method used by palaeontologists to determine how old certain fossils [especially ancient corals] are. Corals filter the sea-water in which they live. Sea-water contains a tiny amount of a certain kind of Uranium [Uranium 234] and the corals absorb this into their bodies. Uranium 234 decays, with a half-life of 245000 years, into Thorium 230, which itself decays with a half-life of 75000 years. Thorium is not found in sea-water; so when the coral dies, it has a certain amount of Uranium in it but no Thorium [because the lifetime of a coral polyp is negligible compared with 245000 years]. It is possible to measure the ratio of the amounts of Uranium and Thorium in any given sample.

From this ratio we want to work out the age of the sample [the time when it died]. This is important if we want to know whether global warming is causing corals to die now. [Maybe they die off regularly over long periods of time and the current deaths have nothing to do with global warming.]

Let U(t) be the amount of Uranium in a particular sample of ancient coral and let T(t) be the amount of Thorium. Because each decay of one Uranium atom produces one Thorium atom, Thorium atoms are being born at exactly the same rate at which Uranium atoms die: so we have

$$U \xrightarrow{k_U U} T \longrightarrow k_T T$$

$$k_U \neq k_T$$

both k_U and k_T are +ve.

$$
\begin{cases}
\dfrac{dU}{dt} = -k_U U & \text{------ (4)} \\[2ex]
\dfrac{dT}{dt} = k_U U - k_T T & \text{---- (5)} \\[2ex]
U(0) = U_0 & \\[1ex]
T(0) = 0 &
\end{cases}
$$

$$④ \Rightarrow \frac{dv}{v} = -k_v\, dt$$

$$\Rightarrow v = v_0 e^{-k_v t} \quad \text{-----} ⑥$$

$\because$ half-life of U is 245000 years

$$\therefore \frac{U_0}{2} = U_0 e^{-k_U (245000)}$$

$$\therefore -\ln 2 = -k_U (245000)$$

$$\therefore k_U = \frac{\ln 2}{245000}$$

In a similar way, we find that

$$k_T = \frac{\ln 2}{75000}$$

5 and 6

$$\Rightarrow \quad \frac{dT}{dt} + k_T T = k_v v_0 e^{-k_v t}$$

Integrating factor is

$$R = e^{\int k_T \, dt} = e^{k_T t}$$

$$\therefore \quad T = \frac{1}{e^{k_T t}} \int e^{k_T t}\, k_v v_0\, e^{-k_v t}\, dt$$

$$= e^{-k_T t} \left\{ \frac{k_v v_0}{k_T - k_v} e^{(k_T - k_v)t} + c \right\}$$

$$T(0) = 0 \Rightarrow 0 = \frac{k_v v_0}{k_T - k_v} + C$$

$$\therefore C = -\frac{k_v v_0}{k_T - k_v}$$

$$\therefore T = \frac{k_v v_0}{k_T - k_v}\left(e^{-k_v t} - e^{-k_T t}\right) \quad \cdots\cdots 7$$

$$\frac{⑦}{⑥} \Rightarrow \frac{T}{U} = \frac{k_U}{k_T - k_U}\left(1 - e^{-(k_T - k_U)t}\right)$$

Note:

1). $\because k_T > k_U$

$\therefore e^{(k_T - k_U)t} > 1$, when $t > 0$

$\therefore e^{-(k_T - k_U)t} < 1$ when $t > 0$

$\therefore t > 0 \Rightarrow \frac{T}{U}$ is +ve.

2). Observe that $\lim\limits_{t \to \infty} T = \lim\limits_{t \to \infty} \upsilon = 0$,

but $\lim\limits_{t \to \infty} \dfrac{T}{\upsilon} = \dfrac{k_\upsilon}{k_T - k_\upsilon} \neq 0$.

3). By measuring $\frac{T}{U}$ at the present time, we can calculate t which gives the age of the sample.

Reduction to linear form

Certain nonlinear d.e.s can be reduced to a linear form. The most important class of such equations are the Bernoulli equations of the form $y' + p(x)y = q(x)y^n$ where n is any real number.

Bernoulli equations

$$\frac{dy}{dx} + Py = Qy^n, \quad n \neq 0, 1.$$

Let $\boxed{z = y^{1-n}}$

$$\frac{dz}{dx} = (1-n)\, y^{-n}\, \frac{dy}{dx} = \frac{1-n}{y^n} \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{y^n}{1-n} \frac{dz}{dx}$$

$$\frac{y^n}{1-n} \frac{dz}{dx} + Py = Q y^n$$

$$\frac{dz}{dx} + (1-n) P y^{1-n} = Q(1-n)$$

$$\underbrace{\frac{dz}{dx} + (1-n) P z = Q(1-n)}_{\text{linear in } z}$$

Examples. To solve

(i) $y' - Ay = -By^2$, A, B constants.

$$y' - Ay = -By^2$$

Let $z = y^{1-2} = y^{-1}$

$\therefore \quad z' = -y^{-2}\, y'$

$\therefore \quad -y^2 z' - Ay = -By^2$

$$z' + A y^{-1} = B$$

$$\therefore \quad z' + A z = B$$

$$R = e^{\int A\,dx} = e^{Ax}$$

$$z = e^{-Ax} \int e^{Ax} B \, dx$$

$$= e^{-Ax} \left(\frac{B}{A} e^{Ax} + c \right)$$

$$\therefore \quad \frac{1}{y} = \frac{B}{A} + c e^{-Ax}$$

$$\therefore \quad y = \frac{1}{\frac{B}{A} + C e^{-Ax}}$$

(ii) $y' + y = x^2y^2$. $[y(Ae^x + x^2 + 2x + 2) = 1]$

$$y' + y = x^2y^2$$

Let $z = y^{1-2} = y^{-1}$

$$\therefore z' = -y^{-2}y'$$

$$\therefore -y^2z' + y = x^2y^2$$

$$z' - {y^{-1}}' = -x^2$$

$$z' - z = -x^2$$

$$R = e^{\int -dx} = e^{-x}$$

$$z = e^{x} \int e^{-x}(-x^2)\,dx$$

$$= e^{x} \left\{ \int x^2 \, d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int 2xe^{-x}dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int (-2x)\, d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} - 2\int e^{-x}dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} + 2e^{-x} + c \right\}$$

$$\therefore \frac{1}{y} = ce^x + x^2 + 2x + 2$$

$$\therefore y = \frac{1}{ce^x + x^2 + 2x + 2}$$

More Examples

Example

A certain substance grows at a rate proportional to the amount present. Initially there are 250 gm of this substance and it grows to 800 gm after 7 hours. How long from the start does it take for the substance to grow to 1600 gm?

$$\frac{dx}{dt} = kx \Rightarrow \frac{dx}{x} = k\,dt \Rightarrow \ln|x| = kt + C \Rightarrow x = Ae^{kt}$$

$$x(0) = 250 \Rightarrow A = 250 \Rightarrow x = 250e^{kt}$$

$$x(7) = 800 \Rightarrow 800 = 250e^{7k} \Rightarrow k = \frac{1}{7}\ln\frac{800}{250}$$

$$1600 = 250e^{kt} \Rightarrow kt = \ln\frac{1600}{250} \Rightarrow t = \frac{\ln 1600 - \ln 250}{k}$$

$$\Rightarrow t = \frac{7(\ln 1600 - \ln 250)}{\ln 800 - \ln 250} \approx \underline{\underline{11.2}} \text{ hours}$$

Example

A body was found at the Canteen. You are a member of the CSI team and you arrived at the crime scene at 8am. Immediately upon arrival, you took the temperature of the victim and found that it was 80^oF. At 9am you took the temperature of the victim again and found that it was 75^oF. You estimated that the victim's temperature was 98.6^oF just before death and that the temperature at the Canteen stayed approximately constant at 70^oF. What is your estimate on the time of death?

Set time $t=0$ at 8 am, t measured in hours.

$$\frac{dT}{dt} = k(T-70)$$

$$\frac{dT}{T-70} = k\,dt$$

$$\ln|T-70| = kt + C$$

$$T - 70 = Ae^{kt}$$

$$T(0) = 80 \Rightarrow A = 80 - 70 = 10$$

$$\therefore T - 70 = 10e^{kt}$$

$$T(1) = 75 \Rightarrow 5 = 10e^{k} \Rightarrow k = -\ln 2$$

$$\therefore T - 70 = 10e^{(-\ln 2)t}$$

$$T(\tau) = 98.6 \Rightarrow 28.6 = 10e^{(-\ln 2)\tau}$$

$$\Rightarrow 2.86 = e^{(-\ln 2)\tau}$$

$$\Rightarrow \tau = -\frac{\ln 2.86}{\ln 2}$$

$$\approx -1.5 \text{ hours}$$

$\therefore$ Time of death $\approx \underline{\underline{6:30 \text{ am}}}$

Example

A fossilized bone is found to contain 40% of the original amount of Carbon-14. We know that the half-life of Carbon-14 is 5600 years. Then the estimated age of the fossil to the nearest 100 years is equal to

$$\frac{dQ}{dt} = kQ \Rightarrow Q = Q_0 e^{kt}$$

$$\frac{1}{2} Q_0 = Q_0 e^{5600k} \Rightarrow k = \frac{-\ln 2}{5600}$$

$$(0.4) Q_0 = Q_0 e^{kt} \Rightarrow t = \frac{\ln 0.4}{k} = \frac{5600 \ln 0.4}{-\ln 2}$$

$$\simeq 7402.8$$

$$\simeq \underline{\underline{7400}}$$

Example

The Jurong Lake has a volume of 700000 m^3. At time $t = 0$, the government starts a water cleaning process so that only fresh clean water flows into the lake. After 5 years, it is found that the pollution in the lake is reduced by 50%. If fresh water flows into the lake at a rate of r cubic metres per year and lake water flows out to the sea at the same rate, what is the value of r correct to the nearest thousands?

$$\frac{dQ}{dt} = -\frac{rQ}{700000} \Rightarrow Q = Q_0 e^{-\frac{rt}{700000}}$$

$$\frac{1}{2}Q_0 = Q_0 e^{-\frac{5r}{700000}}$$

$$\Rightarrow r = \frac{700000 \ln 2}{5}$$

$$\approx 97040.6$$

$$\approx \underline{\underline{97000}}$$

Example

Juliet was standing directly below Romeo's balcony. The moment that Romeo stuck his head out of the balcony, Juliet threw a stone vertically upwards at him at a velocity of u m/s. To her delight, 0.46 seconds after she threw the stone, the stone hit Romeo on the face on its way up. Luckily for Romeo (and to the disappointment of Juliet), at the moment of impact the velocity of the stone was zero. Find the value of u correct to one decimal place, based on the following assumptions: the stone's mass is 0.3 kg, the gravitational constant g equals to 10 m/s^2 and the value of the air resistance at any time equals to $0.3v^2$ Newtons where v is the value of the velocity of the stone at that time measured in m/s.

$v\uparrow \bigcirc \downarrow 0.3v^2$

$\downarrow$

$mg = (0.3)(10) = 3$

$$0.3\frac{dv}{dt} = -3 - 0.3v^2$$

$$\frac{dv}{dt} = -10 - v^2 \Rightarrow \frac{dv}{v^2+10} = -dt$$

$v(0) = u, \quad v(0.46) = 0$

$$\int_u^0 \frac{dv}{v^2+10} = \int_0^{0.46} -dt = -0.46$$

$$\therefore \; 0.46 = \int_0^u \frac{dv}{v^2+10} = \frac{1}{\sqrt{10}} \tan^{-1}\left(\frac{u}{\sqrt{10}}\right)$$

$$u = \sqrt{10}\tan\left(0.46\sqrt{10}\right)$$

$$\approx \underline{\underline{27.10\cdots}}$$

Example

Let y be a solution of the differential equation $\frac{1}{x}y' - 2y = \frac{1}{x}e^{x^2}$, $x > 0$, such that $y\left(1\right) = 2e$. Then $y\left(2\right) =$

$$\frac{1}{x}y' - 2y = \frac{1}{x}e^{x^2} \Rightarrow y' - 2xy = e^{x^2}$$

Integrating factor $= e^{\int -2x\,dx} = e^{-x^2}$

$$y = e^{x^2}\int e^{-x^2}e^{x^2}\,dx = e^{x^2}(x+c)$$

$$y(1) = 2e \Rightarrow 2e = e(1+c) \Rightarrow c = 1$$

$$\therefore y = xe^{x^2} + e^{x^2}$$

$$y(2) = 2e^4 + e^4 = \underline{\underline{3e^4}}$$

An application to population growth

MALTHUS MODEL OF POPULATION

The total population of a country is clearly a function of time, N(t) [NOTE: N may be measured in millions, so values of N less than 1 are meaningful!]. Given the population now, can we predict what it will be in the future?

Suppose that B is a function giving the PER CAPITA BIRTH-RATE in a given society, *ie* B is the number of babies born per second, divided by the total population of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B could depend on time (people might gradually come to realise that large families are no fun, etc...) and it could depend on N

But SUPPOSE YOU DON'T BELIEVE THESE THINGS: suppose you think that people will always have as many kids as they can, no matter what. Then B is constant. Now just as

$$\text{Distance} = \text{Speed} \times \text{Time}$$

when SPEED IS CONSTANT, so also we have

$$\#\text{babies born in time } \delta t = BN\delta t$$

Similarly let D be the death rate per capita; again, it could be a function of t (better medicine, fewer smokers) or N (overcrowding leads to famine/disease) but if we assume that it is constant, then

$$\#\text{deaths in time } \delta t = DN\delta t$$

So the change in N, δN, during δt is

$$\delta N = \#birth - \#deaths$$

PROVIDED there is no emigration or immigration. Thus,

$$\delta N = (B - D)N\delta t$$

and so $\frac{\delta N}{\delta t} = (B - D)N$ or in the limit as $\delta t \rightarrow$

0,

$$\frac{dN}{dt} = (B - D)N = kN \tag{1}$$

if $k = B - D$.

This model of society was put forward by THOMAS MALTHUS in 1798. Clearly Malthus was assuming a socially STATIC society in which human reproductive behaviour never changes with time or overcrowding, poverty etc... What does Malthus' model predict? Suppose that the population NOW is $\hat{N}$, and let $t = 0$ NOW.

From $\frac{dN}{dt} = kN$ we have $\int \frac{dN}{N} = \int k dt = k \int dt = kt + c$

so $ln(N) = kt + c$ and thus $N(t) = Ae^{kt}$.

Since $\hat{N} = N(0) = A$, we get:

$$N(t) = \hat{N}e^{kt} \tag{2}$$

with graphs as shown on figure 1.

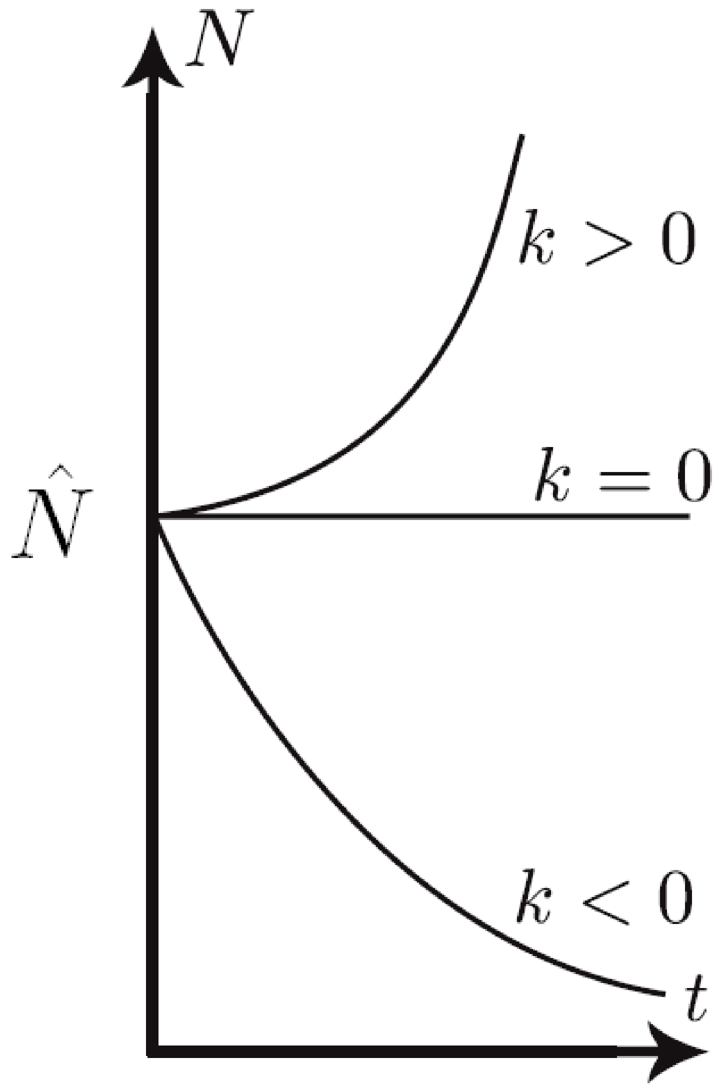


Figure 1: Graphs of $N(t)$, for different values of k

The population collapses if $k < 0$ (more deaths than births per capita), remains stable if (and only if) $k = 0$, and it EXPLODES if $k > 0$ (more births than deaths). Malthus observed that the population of Europe was increasing, so he predicted a catastrophic POPULATION EXPLOSION; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen (in Europe). So Malthus' model is wrong: many millions went to the US, many millions died in wars.

Second, the "static society" assumption has turned out to be wrong in many societies, with B and D both declining as time passed after WW2.

Summary: The Malthus model of population is based on the idea that per capita birth and death rated are independent of time and N. It leads to exponential growth or decay of N.

IMPROVING ON MALTHUS

Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong

- obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N... but WHICH function? Well, surely the SIMPLEST POSSIBLE CHOICE (Remember: always go for the SIMPLE model before trying a complicated one!) is

$$\boxed{D = sN, \quad \begin{array}{c}\text{(LOGISTIC)}\\ \text{ASSUMPTION}\\ s = \text{constant}\end{array}} \tag{3}$$

This represents the idea that, in a world with FINITE RESOURCES, large N will eventually cause starvation and disease and so increase D.

Remark: In modelling, it is often useful to take note of $\boxed{\text{units}}$. Units of D are (#dead people) / second / (total # people) $= (\text{sec})^{-1}$.

Units of N are # (ie no units). So if $D = sN$, units of s must be $(\text{sec})^{-1}$.

As before, let $\hat{N}$ be the value of N at $t = 0$. We have to solve

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

with the condition $N(0) = \hat{N}$

Now we want to solve

$$\frac{dN}{dt} = BN - SN^2, \quad N(0) = \hat{N}.$$

Rewrite the equation as

$$\frac{dN}{dt} - BN = -SN^2$$

We can think of it as a Bernoulli Equation.

Let $z = N^{1-2} = \frac{1}{N}$

$\therefore \; dz = -\frac{1}{N^2}\,dN$

$$\therefore \; \frac{-N^2 dz}{dt} - BN = -SN^2$$

$$\frac{dz}{dt} + \beta \frac{1}{N} = S$$

$$\frac{dz}{dt} + \beta z = S$$

a linear equation in z.

Integrating factor

$$r = e^{\int B\,dt} = e^{Bt}$$

$$\therefore\ z = e^{-Bt}\int s e^{Bt}\,dt$$

$$= e^{-Bt}\left\{\frac{s}{B}e^{Bt} + c\right\}$$

$$\therefore\ \frac{1}{N} = \frac{s}{B} + ce^{-Bt}$$

Let $N_\infty = \frac{B}{S}$ = carrying capacity

$$\therefore \frac{1}{N} = \frac{1}{N_\infty} + Ce^{-Bt}$$

$$N(0) = \hat{N} \Rightarrow \frac{1}{\hat{N}} = \frac{1}{N_\infty} + C$$

$$\Rightarrow C = \frac{1}{\hat{N}} - \frac{1}{N_\infty}$$

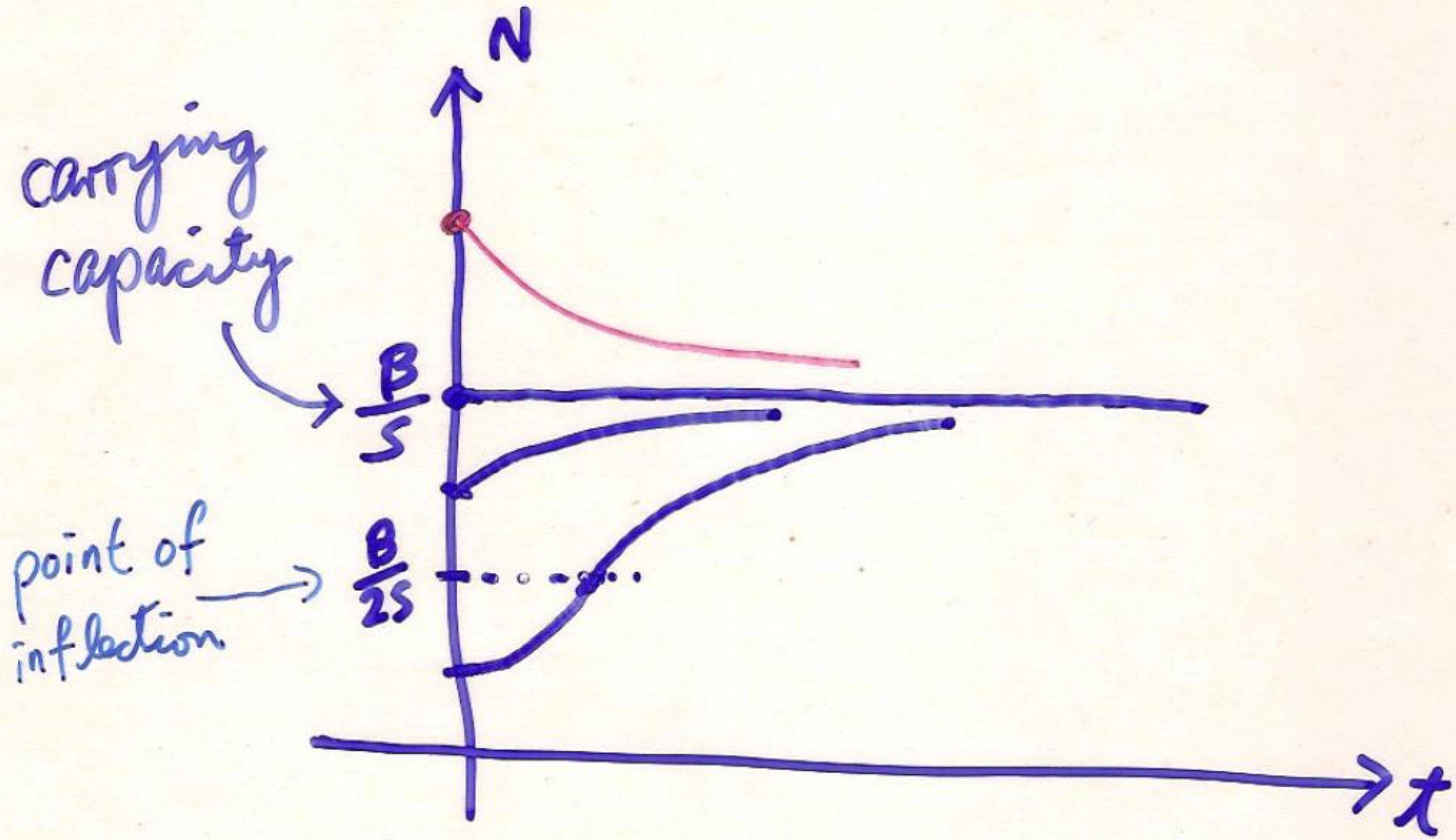
$$\frac{1}{N} = \frac{1}{N_\infty} + \left(\frac{1}{\hat{N}} - \frac{1}{N_\infty}\right) e^{-Bt}$$

$$= \frac{1}{\hat{N} N_\infty} \left\{ \hat{N} + N_\infty e^{-Bt} - \hat{N} e^{-Bt} \right\}$$

$$\therefore \quad N = \frac{\hat{N} N_\infty}{\hat{N} + (N_\infty - \hat{N}) e^{-Bt}}$$

$$\text{i.e.} \quad \boxed{N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}}}$$

Observe: $\lim_{t \to \infty} N = N_\infty$ $(\because B > 0)$


N
carrying capacity
B/S
point of inflection
B/2S
t

Example

The growth of rabbits in your rabbit farm followed a logistic population model with a birth rate per capita of 10 rabbits per rabbit per year. You observed that their number had approached to a logistic equilibrium population of 2500 rabbits. One day your friend Dr. Good visited your farm and suggested that you try to mix some of his latest invention of Vitamin X into your rabbit feed to boost the reproduction rate. You followed his suggestion and after a long period of time, observed that the rabbit population had reached a new logistic equilibrium of 3000 rabbits. If the new rabbit birth rate per capita after Vitamin X was introduced was B rabbits per rabbit per year, what is the value of B ?

$$\frac{10}{S} = 2500 \Rightarrow S = \frac{10}{2500} = \frac{1}{250}$$

$$\frac{B}{S} = 3000 \Rightarrow B = 3000/250$$

$$= \underline{\underline{12}}$$