ST2234 (2018/2019 Semester 2) Solutions to Questions in Tutorial 6

<u>Question 1</u>

$f_{X,Y}(x,y)$		x = 1	x = 2	x = 3	$f_Y(y)$
y	1	0.05	0.05	0.10	**0.20**
	2	0.05	0.10	0.35	**0.50**
	3	0	0.20	0.10	**0.30**
$f_X(x)$		**0.10**	**0.35**	**0.55**	1

(a)

x	1	2	3
$f_X(x)$	0.10	0.35	0.55

(b)

y	1	2	3
$f_Y(y)$	0.20	0.50	0.30

(c) $f_{Y|X}(y=3|x=2) = \frac{f_{X,Y}(2,3)}{f_X(2)} = \frac{0.20}{0.35} = \frac{4}{7}$

(d)

y	1	2	3
$f_{Y\|X}(y\|x=2)$	0.05 /0.35 = 1/7	0.10 /0.35 = 2/7	0.20/0.35 = 4/7

(e) X and Y are dependent if $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ for some values of x and y.
$f_{X,Y}(1,1) = 0.05$ and $f_X(1)f_Y(1) = (0.10)(0.20) = 0.02$
$f_{X,Y}(1,1) \neq f_X(1)f_Y(1)$ implies that X and Y are dependent.

<u>Question 2</u>

(a) $$f_{X,Y}(x,y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, & x = 0,1,2,3; \;\; y = 0,1,2; \;\; 1 \le x+y \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\Pr(X=1, Y=1) = f_{X,Y}(1,1) = \frac{\binom{3}{1}\binom{2}{1}\binom{3}{2}}{\binom{8}{4}} = 0.2571$

(c) $\Pr(X+Y \le 2) = f_{X,Y}(0,1) + f_{X,Y}(0,2) + f_{X,Y}(1,0) + f_{X,Y}(1,1) + f_{X,Y}(2,0) = 0.5$

(d) $f_X(x) = \frac{\binom{3}{x}\binom{5}{4-x}}{\binom{8}{4}}$, for $x = 0,1,2,3$ and 0 otherwise.

(e) For $x = 0,1,2,3$, $f_{Y|X}(y|x) = \frac{\binom{2}{y}\binom{3}{4-x-y}}{\binom{5}{4-x}}$ for $y = 0,1,2;\;\; 1 \le x+y \le 4$ and 0 otherwise.

$f_{Y|X}(y|2) = \frac{\binom{2}{y}\binom{3}{2-y}}{\binom{5}{2}}$ for $y = 0,1,2$; and 0 otherwise.

$\Pr(Y=0|X=2) = \frac{1}{10}\binom{2}{0}\binom{3}{2} = \frac{1}{10}(1)(3) = 0.3$

<u>Question 3</u>

(a)

(x,y)	1	2	3	4	5	6
1	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
2	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
3	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
4	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(1, 1)	(1, 0)
5	(0, 1)	(0, 1)	(0, 1)	(1, 1)	(0, 2)	(0, 1)
6	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)

$f_{(X,Y)}(x,y)$		x			$f_Y(y)$
		0	**1**	2	
y	0	16/36 = 4/9	8/36 = 2/9	1/36	**25/36**
	1	8/36 = 2/9	2/36 = 1/18	0	**5/18**
	2	1/36	0	0	**1/36**
$f_X(x)$		**25/36**	**5/18**	**1/36**	1

(b) $\Pr(2X+Y<3) = f_{X,Y}(0,0)+f_{X,Y}(0,1)+f_{X,Y}(0,2)+f_{X,Y}(1,0) = 11/12$

(c) X and Y are dependent if $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ for some values of x and y.
$f_X(0)f_Y(0) = (25/36)(25/36) = 625/1296 \neq f_{X,Y}(0,0) = 4/9$
$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ ➔ X and Y are dependent

<u>Question 4</u>

(a) $k\int_3^5\int_3^5 (x^2+y^2)\,dydx = k\int_3^5 \left[yx^2+\frac{y^3}{3}\right]_3^5 dx = \frac{2}{3}k\int_3^5 3x^2+49\,dx = \frac{2}{3}k[x^3+49x]_3^5 = \frac{392}{3}k$

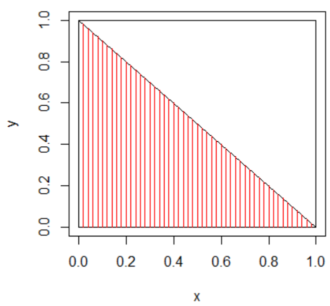
Hence, $k\int_3^5\int_3^5 (x^2+y^2)\,dydx = 1$ implies $\frac{392}{3}k = 1$ or $k = \frac{3}{392}$

(b) $\Pr(3 \le X \le 4 \text{ and } 4 \le Y < 5) = \frac{3}{392}\int_3^4\int_4^5 (x^2+y^2)\,dydx = \frac{3}{392}\int_3^4 \left[yx^2+\frac{y^3}{3}\right]_4^5 dx = \frac{3}{392}\int_3^4 \left(x^2+\frac{61}{3}\right) dx = \frac{1}{392}[x^3+61x]_3^4 = \frac{1}{392}(98) = \frac{1}{4}$

(c) $f_X(x) = \frac{3}{392}\int_3^5 (x^2+y^2)dy = \frac{3}{392}\left(2x^2+\frac{98}{3}\right) = \frac{1}{196}(3x^2+49)$, for $3 \le x \le 5$
$\Pr(3.5 < X < 4) = \frac{1}{196}\int_{3.5}^4 (3x^2+49)\,dx = \frac{1}{196}[x^3+49x]_{3.5}^4 = \frac{1}{196}\frac{365}{8} = \frac{365}{1568} = 0.2328$

<u>Question 5</u>

$$f_{X,Y}(x,y) = \begin{cases} 24xy, & 0 \le x \le 1, \quad 0 \le y \le 1, \quad x+y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$



(a) $f_X(x) = \int_0^{1-x}(24xy)\,dy = [12xy^2]_0^{1-x} = 12x(1-x)^2$, for $0 \le x \le 1$
$f_Y(y) = \int_0^{1-y}(24xy)\,dx = [12x^2y]_0^{1-x} = 12y(1-y)^2$, for $0 \le y \le 1$

(b) $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, for $0 \le x \le 1$ and $0 \le y \le 1$. Hence X and Y are not independent. Alternatively, we may consider a point, let say, $(x,y) = \left(\frac{2}{3},\frac{1}{2}\right)$. Then $f_X\left(\frac{2}{3}\right) = \frac{8}{9}$ and $f_Y\left(\frac{1}{2}\right) = \frac{3}{2}$, while $f_{X,Y}\left(\frac{2}{3},\frac{1}{2}\right) = 0$

(c) For $0 \le x \le 1$, $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}$, for $0 \le y \le 1-x$
$f_{Y|X}\left(y|x=\frac{3}{4}\right) = \frac{2y}{\left(1-\frac{3}{4}\right)^2} = 32y$, for $0 \le y \le \frac{1}{4}$
$\Pr\left(Y < \frac{1}{8}|x=\frac{3}{4}\right) = \int_0^{1/8} f_{Y|X}\left(y|x=\frac{3}{4}\right) dy = \int_0^{1/8}(32y)\,dy = [16y^2]_0^{1/8} = \frac{1}{4}$