NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1101R Linear Algebra I

2018-2019 (Semester 1) **Tutorial 4**

1. Let $\boldsymbol{A}$ be an invertible matrix of order 4. Its reduced row-echelon form of $\boldsymbol{A}$ is obtained by the following sequence of elementary row operations:

$$\boldsymbol{A} \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{R_1 - 2R_2} \xrightarrow{R_2 - 4R_4} \xrightarrow{R_3 \leftrightarrow R_4} \xrightarrow{R_1/2} \xrightarrow{R_4/5} \boldsymbol{I}.$$

Find $\det(3\boldsymbol{A}^T)$ and $\det(2\boldsymbol{A}^{-1})$.

Answer. First, we compute $\det \boldsymbol{A}$. Following from the row operations, we have

$$E_6E_5E_4E_3E_2E_1\boldsymbol{A} = I.$$

Then $\det(\boldsymbol{A}) = \prod_{i=1}^{6} \det(E_i)^{-1}$. As $\det(E_1) = \det(E_4) = -1$, $\det(E_2) = \det(E_3) = 1$, $\det(E_5) = \frac{1}{2}$, $\det(E_6) = \frac{1}{5}$, we have $\det(\boldsymbol{A}) = (-1) \times (-1) \times 2 \times 5 = 10$.

Hence

$$\det(3\boldsymbol{A}^T) = 3^4 \det(\boldsymbol{A}^T) = 81 \det(\boldsymbol{A}) = 810,$$
$$\det(2\boldsymbol{A}^{-1}) = 2^4 \det(\boldsymbol{A}^{-1}) = 16 \det(\boldsymbol{A})^{-1} = \frac{8}{5}.$$

2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n.

(a) Show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent then $\boldsymbol{A}$ and $\boldsymbol{B}$ are simultaneously singular or invertible, i.e., $\det(\boldsymbol{A}) = 0$ if and only if $\det(\boldsymbol{B}) = 0$.

(b) If $\det(\boldsymbol{A}) = \det(\boldsymbol{B}) = 0$, are $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent? If the answer is no, please construct a counter-example. If the answer is yes, please prove your answer.

Ans. Part (a). Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent, $\boldsymbol{B} = E_kE_{k-1}\cdots E_1\boldsymbol{A}$ for some elementary matrices. Then $\det(\boldsymbol{B}) = \prod_{i=1}^{k} \det(E_i)\det(\boldsymbol{A})$. Since E_i are invertible, $\prod_{i=1}^{k} \det(E_i)$ is nonzero. Hence $\det(\boldsymbol{A}) = 0$ if and only if $\det(\boldsymbol{B}) = 0$. That is, $\boldsymbol{A}$ and $\boldsymbol{B}$ are simultaneously singular or invertible.

Part (b). No. For example, $\boldsymbol{A} = \boldsymbol{0}_{2\times 2}$ and $\boldsymbol{B} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We have $\det(\boldsymbol{A}) = (\boldsymbol{B}) = 0$, but they are not rwo equivalent.

3. Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 \\ 10 & 0 & 3 \\ -6 & 5 & -5 \end{pmatrix}.$$

(a) Find the adjoint of $\boldsymbol{A}$.

Answer.

$$\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} \begin{vmatrix} 0 & 3 \\ 5 & -5 \end{vmatrix} & -\begin{vmatrix} 10 & 3 \\ -6 & -5 \end{vmatrix} & \begin{vmatrix} 10 & 0 \\ -6 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -6 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -6 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 10 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 10 & 0 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} -15 & 32 & 50 \\ 5 & -11 & -17 \\ 6 & -13 & -20 \end{pmatrix}^T = \begin{pmatrix} -15 & 5 & 6 \\ 32 & -11 & -13 \\ 50 & -17 & -20 \end{pmatrix}.$$

(b) Compute the inverse of $\boldsymbol{A}$.

Answer.

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A}) = -\begin{pmatrix} -15 & 5 & 6 \\ 32 & -11 & -13 \\ 50 & -17 & -20 \end{pmatrix} = \begin{pmatrix} 15 & -5 & -6 \\ -32 & 11 & 13 \\ -50 & 17 & 20 \end{pmatrix}.$$

(c) Solve for y by using Cramer's Rule:

$$\begin{cases} x + 2y - z = 14 \\ 10x + 3z = 27 \\ -6x + 5y - 5z = 12 \end{cases}$$

Answer.

$$y = \frac{\begin{vmatrix} 1 & 14 & -1 \\ 10 & 27 & 3 \\ -6 & 12 & -5 \end{vmatrix}}{\det(\boldsymbol{A})} = \frac{-5}{-1} = 5.$$

(d) Let $\boldsymbol{u} = (1, 2, -1), \boldsymbol{v} = (10, 0, 3), \boldsymbol{w} = (-6, 5, -5)$. Show that $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbb{R}^3$

(Tips for tutors: students may follow the method outlined in Example 3.2.4.5 in thr textbook rather than to think of it in terms of the invertibility of A (or A^T). Similarly for 3(e), students may go through solving for the coefficients directly like in Example 3.2.2.)

Answer. For every $\boldsymbol{x} \in \mathbb{R}^3$, write $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$, which is a linear system of variables a, b, and c. We need to show that this linear system is consisten, i.e., every vector $\boldsymbol{x}$ in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.
Let us write the matrix form of this linear system:

$$\begin{pmatrix} 1 & 10 & -6 \\ 2 & 0 & 5 \\ -1 & 3 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{x}, i.e., \boldsymbol{A}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{x}.$$

Since $\boldsymbol{A}$ is invertible, $\boldsymbol{A}^T$ is invertible. Thus

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\boldsymbol{A}^T)^{-1}\boldsymbol{x}.$$

That is, every vector $\boldsymbol{x}$ in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(e) Write $(14, 27, 12)$ as a linear combination of $\boldsymbol{u} = (1, 2, -1), \boldsymbol{v} = (10, 0, 3), \boldsymbol{w} = (-6, 5, -5)$.

Answer. Write $(14, 27, 12) = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$,
As $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$, by Part (b), we have

$$(\boldsymbol{A}^T)^{-1} = \begin{pmatrix} 15 & -32 & -50 \\ -5 & 11 & 17 \\ -6 & 13 & 20 \end{pmatrix}.$$

By Part (d), we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\boldsymbol{A}^T)^{-1} \begin{pmatrix} 14 \\ 27 \\ 12 \end{pmatrix} = \begin{pmatrix} 15 & -32 & -50 \\ -5 & 11 & 17 \\ -6 & 13 & 20 \end{pmatrix} \begin{pmatrix} 14 \\ 27 \\ 12 \end{pmatrix} = \begin{pmatrix} -1254 \\ 431 \\ 507 \end{pmatrix}.$$

4. Let $\boldsymbol{A}$, $\boldsymbol{D}$ and $\boldsymbol{P}$ be square matrices of the same size such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. (Here $\boldsymbol{P}$ is invertible.)

(a) Show that $\det(\boldsymbol{A}) = \det(\boldsymbol{D})$.

Answer.

$$\begin{aligned} \det(\boldsymbol{A}) &= \det(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}) \\ &= \det(\boldsymbol{P}) \det(\boldsymbol{D}) \det(\boldsymbol{P}^{-1}) \\ &= \det(\boldsymbol{P}) \det(\boldsymbol{D}) \det(\boldsymbol{P})^{-1} \\ &= \det(\boldsymbol{D}). \end{aligned}$$

(b) If $\boldsymbol{A}$ is invertible, show that $\boldsymbol{A}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$ for all integers n.

Answer. First, we use the mathematical induction to prove it for $n \geq 0$. When $n = 0$, by definition, $\boldsymbol{A}^0 = \boldsymbol{I} = \boldsymbol{D}^0$. By $\boldsymbol{P}\boldsymbol{I}\boldsymbol{P}^{-1} = \boldsymbol{I}$ we have $\boldsymbol{A}^0 = \boldsymbol{I} = \boldsymbol{P}\boldsymbol{D}^0\boldsymbol{P}^{-1}$.

Assume that the statement is true for $n = k$. For $n = k+1$,

$$\begin{aligned}\boldsymbol{A}^{k+1} =& \boldsymbol{A}^k\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \text{ (by assumption)}\\ =& \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{I}\boldsymbol{D}\boldsymbol{P}^{-1}\\ =& \boldsymbol{P}\boldsymbol{D}^{k+1}\boldsymbol{P}^{-1}.\end{aligned}$$

By induction, we have $\boldsymbol{A}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$ for all nonnegative integers n.

As

$$\boldsymbol{A}^{-1} = (\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1})^{-1} = (\boldsymbol{P}^{-1})^{-1}\boldsymbol{D}^{-1}\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}^{-1}\boldsymbol{P}^{-1},$$

$$\boldsymbol{A}^{-n} = (\boldsymbol{A}^{-1})^n = \boldsymbol{P}(\boldsymbol{D}^{-1})^n\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{D}^{-n}\boldsymbol{P}^{-1}.$$

Hence $\boldsymbol{A}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$ for all integers n.

5. Determine the values of a and b so that the following three points are in the same line

$$(1, 2, -1), \qquad (10, 0, 3), \qquad (a, b, -5).$$

Answer. Recall that a linear in $\mathbb{R}^3$ is represented explicitly in set notation by

$$\{(1, 2, -1) + t(9, -2, 4) \mid t \in \mathbb{R}\}.$$

The point $(a, b, -5)$ lies in this line if and only if there exists a real number t so that $(1, 2, -1) + t(9, -2, 4) = (a, b, -5)$, that is,

$$\left\{\begin{array}{rcrcr} 1 & + & 9t & = & a\\ 2 & - & 2t & = & b\\ -1 & + & 4t & = & -5\end{array}\right.$$

The linear system is consistent if and only if $a = -8$ and $b = 4$.