CS1231: Discrete Structures

Tutorial 5

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Quick Review

- ▶ A function f from A to B, is an assignment of exactly one element of B to each element of A.
- A is the domain. B is the codomain. $\{f(a): a \in A\}$ is the range.
- If f(a) = b, then b is the **image** of a; and a is the **preimage** of b.
- A function $f: X \to Y$ is **one-to-one** or **injective** if

$$\forall a, b \in X, f(a) = f(b) \Rightarrow a = b.$$

• A function $f: X \to Y$ is **onto** or **surjective** if

$$\forall y \in Y \exists x \in X, f(x) = y.$$

- ▶ The function f is a **bijection** if it is both 1-1 and onto.
- $\lfloor x \rfloor = n$ if $n \le x < n+1$, $\lceil x \rceil = n$ if $n-1 < x \le n$, where $n \in \mathbb{Z}$.

Menu

Question 1	Question 4	Question 6(c)
Q 0.00000000000000000000000000000000000	Question 5	0 ((1)
Question 2	Question 6(a)	Question 6(d)
Question 3	Question 6(b)	Question 7

- 1. Determine whether $f: \mathbb{Z} \to \mathbb{R}$ is function.
- (a) $f(n) = \pm n$.
- (b) $f(n) = \sqrt{n^2 + 1}$.
- (c) $f(n) = 1/(n^2 4)$.
- (d) $f(n) = \sin n$.

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Answer.

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A function f from \mathbb{Z} to \mathbb{R} , is an assignment of exactly one element (image) of \mathbb{R} to each element of \mathbb{Z} .

- (a) No, image not unique.
- (b) Yes, every $n \in \mathbb{Z}$ has a unique image.
- (c) No. 2 has no image.
- (d) Yes.

- 2. Find the domain and range of these functions.
- (a) The function that assigns TO each nonnegative integer its last digit.
- (b) The function that assigns the next integer TO a positive integer.
- (c) The function that assigns TO a bit string the number of one bits in the string.
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		domain	range
	(a)		
Answer.	(b)		
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3. Find these values:

(a)
$$|1.1| =$$

(b)
$$[1.1] =$$

(c)
$$|-.1| =$$

(d)
$$[-.1] =$$

(e)
$$\left[\frac{1}{2} + \left[\frac{3}{2}\right]\right] = \left[\frac{1}{2} + \right] =$$

(f)
$$\left[\frac{1}{2} + \left[\frac{3}{2}\right]\right] = \left[\frac{1}{2} + \right] =$$

Recall

- 3. Find these values:
- (a) |1.1| = 1
- (b) [1.1] =
- (c) [-.1] =
- (d) [-.1] =
- (e) $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor = \left\lfloor \frac{1}{2} + \right\rfloor =$
- (f) $\begin{bmatrix} \frac{1}{2} + \begin{bmatrix} \frac{3}{2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \end{bmatrix} =$

- 3. Find these values:
- (a) |1.1| = 1
- (b) [1.1] = 2
- (c) [-.1] =
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- (e) $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor = \left\lfloor \frac{1}{2} + \right\rfloor =$
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(1) 12 | 1211 | 12 |

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$(1) |_{\overline{2}} + |_{\overline{2}}| = |_{\overline{2}} + 1|$

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 $[x] = n \text{ if } n \leqslant x < n+1, \ [x] = n \text{ if } n-1 < x \leqslant n, \text{ where } n \in \mathbb{Z}.$

- 4. Determine whether $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto.
- (a) f(m,n) = 2m n
- (b) $f(m,n) = m^2 n^2$.

A function $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is **onto** if $\forall a \in \mathbb{Z} \exists (m,n) \in \mathbb{Z} \times \mathbb{Z} (f(m,n)=a)$. That is, for each $a \in \mathbb{Z}$, there is a **preimage** (m,n) of a.

		(a)	(b)
		$(m,n) \in \mathbb{Z} \times \mathbb{Z}$	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$
	$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
dea.	0		
	1		
	2		
	a		

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(h)

	(a <i>)</i>	(b)
	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$
$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
0	(0,0)	
1		
2		
a		
	$a \in \mathbb{Z}$ 0 1 2 a	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$ $a \in \mathbb{Z} \text{s.t. } 2m-n=a$

(-)

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		$(m,n)\in\mathbb{Z}\times\mathbb{Z}$ s.t. $2m-n=a$	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$
	$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
ldea.	0	(0,0)	
•	1	(0, -1)	
	2		
	a		

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		$(m,n)\in\mathbb{Z}\times\mathbb{Z}$ s.t. $2m-n=a$	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$
	$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
ldea.	0	(0,0)	
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	2	(0, -2)	
	\overline{a}		

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		$(m,n) \in \mathbb{Z} \times \mathbb{Z}$ s.t. $2m-n=a$	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$
	$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
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$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
0	(0,0)	(0,0)
1	(0, -1)	
2	(0, -2)	
a	(0, -a)	
	0 1 2	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$ $a \in \mathbb{Z} \text{s.t. } 2m - n = a$ $0 (0,0)$ $1 (0,-1)$ $2 (0,-2)$

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$a \in \mathbb{Z}$	s.t. 2m - n = a	s.t. $m^2 - n^2 = a$
0	(0,0)	(0,0)
1	(0, -1)	(1,0)
2	(0, -2)	?
a	(0, -a)	
	0 1 2	$(m,n) \in \mathbb{Z} \times \mathbb{Z}$ $a \in \mathbb{Z} \text{s.t. } 2m - n = a$ $0 (0,0)$ $1 (0,-1)$ $2 (0,-2)$

Observation

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = \ , m - n = \\ \therefore \ m = \ , n = \\ \mathsf{Case} \ 2. \ m + n = \ , m - n = \\ \therefore \ m = \ , n = \\ \mathsf{Case} \ 3. \ m + n = \ , m - n = \\ \therefore \ m = \ , n = \\ \mathsf{Case} \ 4. \ m + n = \ , m - n = \\ \therefore \ m = \ , n = \end{cases}$$

Case 2.
$$m+n=$$
 , $m-n=$
Case 3. $m+n=$, $m-n=$
Case 4. $m+n=$, $m-n=$
Case 4. $m+n=$, $m-n=$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = \\ \therefore \ m = &, n = \\ \mathsf{Case} \ 2. \ m + n = &, m - n = \\ \therefore \ m = &, n = \\ \mathsf{Case} \ 3. \ m + n = &, m - n = \\ \therefore \ m = &, n = \\ \mathsf{Case} \ 4. \ m + n = &, m - n = \\ \therefore \ m = &, n = \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore m = &, n = \\ \mathsf{Case} \ 2. \ m + n = &, m - n = \\ \therefore m = &, n = \\ \mathsf{Case} \ 3. \ m + n = &, m - n = \\ \therefore m = &, n = \\ \mathsf{Case} \ 4. \ m + n = &, m - n = \\ \therefore m = &, n = \end{cases}$$

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$$\therefore m = , n =$$
Case 2. $m + n = -1, m - n =$
 $\therefore m = , n =$
Case 3. $m + n = , m - n =$
 $\therefore m = , n =$
Case 4. $m + n = , m - n =$
 $\therefore m = , n =$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore m = -n, n = \\ \mathsf{Case} \ 2. \ m + n = -1, m - n = -2 \\ \therefore m = -n, n = \\ \mathsf{Case} \ 3. \ m + n = -n, m - n = \\ \therefore m = -n, n = \\ \mathsf{Case} \ 4. \ m + n = -n, m - n = \\ \therefore m = -n, n = \\ \cdots m = -n, n = -n = \\ \cdots m = -n, n = -n = -n \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

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$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore \ m = 1.5, n = \\ \mathsf{Case} \ 2. \ m + n = -1, m - n = -2 \\ \therefore \ m = \quad , n = \\ \mathsf{Case} \ 3. \ m + n = 2, m - n = 1 \\ \therefore \ m = \quad , n = \\ \mathsf{Case} \ 4. \ m + n = -2, m - n = -1 \\ \therefore \ m = \quad , n = \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore m = 1.5, n = -0.5 \\ \mathsf{Case} \ 2. \ m + n = -1, m - n = -2 \\ \therefore m = &, n = \\ \mathsf{Case} \ 3. \ m + n = 2, m - n = 1 \\ \therefore m = &, n = \\ \mathsf{Case} \ 4. \ m + n = -2, m - n = -1 \\ \therefore m = &, n = \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore m = 1.5, n = -0.5 \\ \mathsf{Case} \ 2. \ m + n = -1, m - n = -2 \\ \therefore m = -1.5, n = \\ \mathsf{Case} \ 3. \ m + n = 2, m - n = 1 \\ \therefore m = -n, n = \\ \mathsf{Case} \ 4. \ m + n = -2, m - n = -1 \\ \therefore m = -n, n = -1 \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

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$$m^2 - n^2 = (m - n)(m + n).$$

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$$m^2 - n^2 = (m - n)(m + n) = 2 \begin{cases} \mathsf{Case} \ 1. \ m + n = 1, m - n = 2 \\ \therefore m = 1.5, n = -0.5 \\ \mathsf{Case} \ 2. \ m + n = -1, m - n = -2 \\ \therefore m = -1.5, n = 0.5 \\ \mathsf{Case} \ 3. \ m + n = 2, m - n = 1 \\ \therefore m = 1.5, n = 0.5 \\ \mathsf{Case} \ 4. \ m + n = -2, m - n = -1 \\ \therefore m = -1.5, n = 0.5 \end{cases}$$

$$m^2 - n^2 = (m - n)(m + n).$$

$$m^2-n^2=(m-n)(m+n)=2 \begin{cases} \mathsf{Case}\ 1.\ m+n=1, m-n=2 \\ \therefore\ m=1.5, n=-0.5 \\ \mathsf{Case}\ 2.\ m+n=-1, m-n=-2 \\ \therefore\ m=-1.5, n=0.5 \\ \mathsf{Case}\ 3.\ m+n=2, m-n=1 \\ \therefore\ m=1.5, n=0.5 \\ \mathsf{Case}\ 4.\ m+n=-2, m-n=-1 \\ \therefore\ m=-1.5, n=-0.5 \end{cases}$$

Answer.

(a) Onto: Any $a \in \mathbb{Z}$ has (0, -a) as a preimage.

Answer.

- (a) Onto: Any $a \in \mathbb{Z}$ has (0, -a) as a preimage.
- (b) 2 has no preimage. Suppose there exists (m,n) such that $f(m,n)=m^2-n^2=2$. Then (m+n)(m-n)=2. This means (i) m-n=1 and m+n=2 or (ii) m-n=-1 and
- means (i) m-n=1 and m+n=2 or (ii) m-n=-1 and m+n=-2. From (i) we get 2m=3, a contradiction. From (ii) we get 2m=-3, again a contradiction.

- 5. Determine which of the following are bijections. For those that are bijections, find the inverse functions.
- (a) $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = -3x^2 + 7$.
- (b) $f: \mathbb{R} \{-2\} \to \mathbb{R} \{1\}$ where f(x) = (x+1)/(x+2).

- A function is a bijection if it is 1-1 and onto.
- \triangle A function f is **1-1** if $f(x) = f(y) \rightarrow x = y$
- \triangle A function f is **onto** if every one in the range has a preimage.

	1-1	onto
(a)		
(b)		

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	1-1	onto
(a)	No.	
(b)		

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	1-1	onto
(a)	No. $f(1) = f(-1)$	
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	1-1	onto
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	1-1	onto
(a)	No. $f(1) = f(-1)$	No. 9 has no preimgae
(b)		

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	1-1	onto
(a)	No. $f(1) = f(-1)$	No. 9 has no preimgae
(b)	Yes.	

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	1-1	onto
(a)	No. $f(1) = f(-1)$	No. 9 has no preimgae
(b)	Yes.	Yes.

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
 where $f(x) = \frac{x+1}{x+2}$.

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

 \Rightarrow

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{1}{x+2} = -1$$

$$\begin{array}{ccc} x+2 & x+2 \\ & \end{array}$$

$$x+2$$
 $x+2$ $x+2$ Onto:

Onto:
$$Pick u \in$$

$$f(x) = f(y)$$
 Pick $y \in$ We need to

$$= f(y)$$

$$-\frac{1}{x+2} = 1 - \frac{1}{y+2}$$
We need to find

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t.

$$f(x) = y$$

$$f(x) = y$$

$$\Rightarrow y =$$

$$\Rightarrow y = 0$$

$$\Rightarrow x + 2 = \frac{1}{2}$$

$$\Rightarrow y = \\ \Rightarrow x + 2 = -\\ \Rightarrow x =$$

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
 where $f(x) = \frac{x+1}{x+2}$.

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{1}{x+2} = \frac{1}{x+2} = 1 - \frac{1}{x+2}$$

$$x+2$$
 $x+2$ $x+2$

1-1: Onto:
$$f(x) = f(y)$$
 Pick $y \in$

$$f(x) = f(y)$$
 Pick $y \in$
 $\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$ We need to find x s.t.

$$1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t. $f(x) = y$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$f(x) = y$$

$$\frac{1}{y+2} = \frac{1}{y+2} \qquad f(x) = y$$

$$\Rightarrow y =$$

$$\Rightarrow y = \\ \Rightarrow x + 2 = \frac{1}{}$$

$$\Rightarrow x + 2 = \frac{1}{}$$

$$\Rightarrow x =$$

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
 where $f(x) = \frac{x+1}{x+2}$.

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{x+1}{x+2} = \frac{x}{x+2}$$

$$f(x) = x + 2 \qquad x + 2 \qquad x + 2$$

1-1: Onto:
$$f(x) = f(y)$$
 Pick $y \in$

$$y = f(y)$$
 Pick $y \in 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$ We need to find x s.t.

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t.
$$f(x) = y$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$f(x) = y$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x+2 = y+2$$

$$f(x) = y$$

$$\Rightarrow y =$$

$$x + 2 = y + 2 \qquad \Rightarrow y = \\ \Rightarrow x + 2 = \frac{1}{-1}$$

$$\Rightarrow x + 2 = \frac{1}{2}$$

$$\Rightarrow x = \frac{1}{2}$$

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
 where $f(x) = \frac{x+1}{x+2}$.

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) =$$

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

- 1-1: Onto:
- f(x) = f(y)
 - Pick $y \in$
 - f(x) = y
- $\Rightarrow \frac{1}{r+2} = \frac{1}{n+2}$ $\Rightarrow y =$ $\Rightarrow x + 2 = y + 2$
- $\Rightarrow x + 2 = \frac{1}{x}$ $\Rightarrow x = y$ $\Rightarrow x =$
- $\Rightarrow 1 \frac{1}{r+2} = 1 \frac{1}{n+2}$ We need to find x s.t.

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
 where $f(x) = \frac{x+1}{x+2}$.

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{1}{x+2} = \frac{1}{x+2} = 1 - \frac{1}{x+2}$$

$$x+2$$
 $x+2$ $x+2$ 1-1:

1-1: Onto:
$$f(x) = f(y)$$
 Pick $y \in \mathbb{R} - \{1\}$

$$f(x) = f(y)$$

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
PICK $y \in \mathbb{R} - \{1\}$
We need to find x s.t.

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t. $\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$ $f(x) = y$

$$\frac{1}{x+2} = \frac{1}{y+2} \qquad f(x) = y$$

$$x+2 = y+2 \qquad \Rightarrow y = y = y$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x + 2 = y + 2$$

$$\Rightarrow x = y$$

$$\Rightarrow x = y$$

$$f(x) = y$$

$$\Rightarrow y = y$$

$$\Rightarrow x + 2 = \frac{1}{y+2}$$

$$\Rightarrow x + 2 - y + 2$$

$$\Rightarrow x = y$$

$$\Rightarrow x + 2 = \frac{1}{2}$$

$$\Rightarrow x = \frac{1}{2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$f(x) = y$$

(b)
$$f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$$
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$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

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$$x+2$$
 $x+2$ $x+2$ 1-1:

1-1: Onto:
$$f(x) = f(y)$$
 Pick $y \in \mathbb{R} - \{1\}$

$$f(x) = f(y)$$
 Pick $y \in \mathbb{R} - \{1\}$ $\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$ We need to find x s.t.

$$1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t. $f(x) = y$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x+2 - y+2$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

$$\frac{1}{x+2} = \frac{1}{y+2}
x+2 = y+2$$

$$f(x) = y
\Rightarrow y = 1 - \frac{1}{x+2}$$

 $\Rightarrow x =$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x+2 = y+2$$

$$\Rightarrow x = y$$

$$f(x) = y$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

$$\Rightarrow x+2 = \frac{1}{y+2}$$

(b) $f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$ where $f(x) = \frac{x+1}{x+2}$.

 $\Rightarrow x = y$

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{1}{x+2} = \frac{1}{x+2} = 1 - \frac{1}{x+2}$$

$$x+z$$
 $x+z$ $x+z$

$$f(x) = f(y)$$
 Pick $y \in \mathbb{R} - \{1\}$
 $\Rightarrow 1 - \frac{1}{x} = 1 - \frac{1}{x}$ We need to find x s.t.

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t.

$$\Rightarrow 1 - \frac{1}{x+2} \equiv 1 - \frac{1}{y+2}$$
 where to find x s.t.
$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$f(x) = y$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x + 2 = y + 2$$

$$\Rightarrow x = y$$

$$f(x) = y$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

$$\Rightarrow x + 2 = \frac{1}{1-y}$$

$$\frac{1}{x+2} = \frac{1}{y+2}$$

$$\frac{1}{x+2} = \frac{1}{y+2}$$

$$x+2 = y+2$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

 $\Rightarrow x =$

(b) $f: \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$ where $f(x) = \frac{x+1}{x+2}$.

 $\Rightarrow x = y$

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

$$f(x) = \frac{1}{x+2} = \frac{1}{x+2} = 1 - \frac{1}{x+2}$$

$$x + 2 \qquad x + 2 \qquad x + 2$$
1-1:
$$f(x) - f(y)$$
Pick $y \in \mathbb{R} - \{1\}$

$$f(x) = f(y)$$
 Pick $y \in \mathbb{R} - \{1\}$ $\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{x+2}$ We need to find x s.t.

$$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$
We need to find x s.t.
$$f(x) = y$$

$$1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$$
 We need to find x s.t. $f(x) = y$

$$f(x) = y$$

$$x + 2 = y + 2$$

$$x = y$$

$$f(x) = y$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

$$\Rightarrow x + 2 = \frac{1}{1-y}$$

 $\Rightarrow x = \frac{2y-1}{1-y}$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow \frac{1}{x+2} = \frac{1}{y+2}$$

$$\Rightarrow x+2 = y+2$$

$$\Rightarrow y = 1 - \frac{1}{x+2}$$

Answer.

- (a) Not 1-1: f(1) = f(-1).
- (b) 1-1: f(a) = f(b) implies (a+1)/(a+2) = (b+1)/(b+2). This yields a = b. Onto: Let $y \in \mathbb{R} \{1\}$. Set f(x) = y. Solving, we get x = (2y-1)/(1-y). Thus (2y-1)/(1-y) is a preimage. The above calculations also show that $f^{-1}(x) = (2x-1)/(1-x)$.

6. Given a set S, the characteristic function of $A\subseteq S$, $k_A:S\to\mathbb{Z}$ is defined as

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A|=\sum_{x\in S}k_A(x).$ Prove that $\forall A,B\subseteq S$ and $\forall x\in S$:

(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B(x)$.

(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B$ Idea.

$$k_{A \cap B}(x) = \begin{cases} i.\epsilon \\ i.\epsilon \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

6. Given a set S, the characteristic function of $A\subseteq S$, $k_A:S\to\mathbb{Z}$ is defined as

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A| = \sum_{x \in S} k_A(x)$. Prove that $\forall A, B \subseteq S$ and $\forall x \in S$:

(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B(x)$.

(a) $\kappa_{A \cap B}(x) = \kappa_A(x)$. Idea.

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A| = \sum_{x \in S} k_A(x)$. Prove that $\forall A, B \subseteq S$ and $\forall x \in S$:

(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B(x)$.

Idea.

$$1 \quad \text{if } x \in A \cap B \text{ i.e.}$$

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \end{cases}$$

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \\ & \text{i.e.} \end{cases}$$

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \\ & \text{i.e.} \end{cases}$$

if $x \in A$ and $x \in B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B(x)$.

Idea.

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \\ 0 & \text{i.e.} \end{cases}$$

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A + B \text{ i.} \\ 0 & \text{i.} \end{cases}$$

if $x \in A$ and $x \in B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A| = \sum_{x \in S} k_A(x)$. Prove that $\forall A, B \subseteq S$ and $\forall x \in S$:

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Idea.

$$k_{A \sim B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \end{cases}$$

 $k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \\ 0 & \text{if } x \notin A \cap B \text{ i.e.} \end{cases}$

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e.} \\ 0 & \text{if } x \notin A \cap B \text{ i.e.} \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A| = \sum_{x \in S} k_A(x)$. Prove that $\forall A, B \subseteq S$ and $\forall x \in S$:

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ldea.

$$k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e. } x \in A \text{ and } x \in B \\ 0 & \text{if } x \notin A \cap B \text{ i.e.} \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$\int if x \in A \text{ and } x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$= \left\{ \begin{array}{c} \text{if } x \notin A \text{ and } x \in B \\ \text{if } x \in A \text{ and } x \notin B \\ \text{if } x \notin A \text{ and } x \notin B \end{array} \right.$$

if $x \in A$ and $x \in B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

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$$= \begin{cases} 1 \times 1 = k_A(x) \times k_B(x) & \text{if } x \in A \text{ and } x \in B \\ & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$= \begin{cases} 1 \times 1 = k_A(x) \times k_B(x) & \text{if } x \in A \text{ and } x \in B \\ 0 \times 1 = k_A(x) \times k_B(x) & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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Idea. $k_{A \cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \text{ i.e. } x \in A \text{ and } x \in B \\ 0 & \text{if } x \notin A \cap B \text{ i.e. } x \notin A \text{ or } x \notin B \end{cases}$

$$= \begin{cases} 1\times 1 = k_A(x)\times k_B(x) & \text{if } x\in A \text{ and } x\in B\\ 0\times 1 = k_A(x)\times k_B(x) & \text{if } x\notin A \text{ and } x\in B\\ 1\times 0 = k_A(x)\times k_B(x) & \text{if } x\in A \text{ and } x\notin B\\ 0\times 0 = k_A(x)\times k_B(x) & \text{if } x\notin A \text{ and } x\notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

$$n_{A \cup B}(x) = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \end{pmatrix}$$
 i.e

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

$$\kappa_{\overline{A} \cup \overline{A}}$$

$$\kappa_{\overline{A \cup B}}(x) = \left\{ \right.$$
 i.e.

- $k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x)$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e.} \\ & \text{i.e.} \end{cases}$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in A \cup B \text{ i.e.} \\ & \text{i.e.} \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$\begin{cases}
1 & \text{if } x \in \overline{A \cup B} \text{ i.e.}
\end{cases}$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e.} \\ 0 & \text{i.e.} \end{cases}$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in A \cup B \text{ i.e.} \\ 0 & \text{i.e.} \end{cases}$$

- $k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A},\overline{B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e.} \end{cases}$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e.} \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e.} \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \\ 0 & \text{i.e. } x \notin A \end{cases} \quad x \notin B$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e.} \end{cases} \quad x \notin B$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

ldea.

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e.} \end{cases}$$

if
$$x \notin A$$
 and $x \notin B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A+B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e. } x \in A \quad x \in B \end{cases}$$

if
$$x \notin A$$
 and $x \notin B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$k_{\overline{A+B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

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if
$$x \notin A$$
 and $x \notin B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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$$k_{\overline{A+B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

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$$\text{if } x \notin A \text{ and } x \notin B$$

$$\text{if } x \in A \text{ and } x \notin B$$

$$\text{if } x \notin A \text{ and } x \in B$$

$$\text{if } x \notin A \text{ and } x \in B \\ \text{if } x \in A \text{ and } x \in B \\$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A+B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e. } x \in A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \notin B \\ & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \end{cases}$$

$$\text{if } x \in A \text{ and } x \in B$$

$$\text{if } x \in A \text{ and } x \in B$$

$$\text{if } x \in A \text{ and } x \notin B$$

$$\text{if } x \notin A \text{ and } x \in B$$

$$\text{if } x \in A \text{ and } x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e. } x \in A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \notin B \\ (1-1)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$\begin{cases} (1-1)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e. } x \in A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \notin B \\ (1-1)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \notin B \\ (1-0)(1-1) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$\begin{cases} (1-0)(1-1) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(b)
$$k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)).$$

$$k_{\overline{A \cup B}}(x) = \begin{cases} 1 & \text{if } x \in \overline{A \cup B} \text{ i.e. } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin \overline{A \cup B} \text{ i.e. } x \in A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \notin B \\ (1-1)(1-0) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \notin B \\ (1-0)(1-1) = (1-k_A(x))(1-k_B(x)) & \text{if } x \notin A \text{ and } x \in B \\ (1-1)(1-1) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$\begin{cases} (1-1)(1-1) = (1-k_A(x))(1-k_B(x)) & \text{if } x \in A \text{ and } x \\ k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$\text{dea.}$$

$$k \cdot p(x) = \begin{cases} i. \end{cases}$$

$$k_{A-B}(x) =$$

$$k_{A-B}(x) = \begin{cases} & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

$$k_{A-B}(x) = \begin{cases} & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

$$k_{A-B}(x) = \begin{cases} & \text{i.e.} \end{cases}$$

- $k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$(c) \ \kappa_{A-B}(x) = (1 - \kappa_B(x))\kappa_A(x)$$
Idea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{i.} \\ & \text{i.} \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e.} \end{cases}$$

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e.} \\ & \text{i.e.} \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

Idea.
$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A-B \text{ i.e.} \\ 0 & \text{i.e.} \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$k_{A-B}(x) = \int 1$$
 if $x \in A - B$ i.e.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e.} \\ 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

ldea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \\ 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases} \quad x \notin B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases}$$

$$k_{A-B}(x) = \begin{cases} 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases}$$

if
$$x \in A$$
 and $x \notin B$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

ldea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A & x \in B \end{cases}$$

if
$$x \in A$$
 and $x \notin B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c)
$$k_{A-B}(x) = (1 - k_B(x))k_A(x)$$
.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

if
$$x \in A$$
 and $x \notin B$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

$$= \left\{ \begin{array}{c} \text{if } x \in A \text{ and } x \notin B \\ \text{if } x \notin A \text{ and } x \notin B \\ \text{if } x \notin A \text{ and } x \in B \\ \text{if } x \in A \text{ and } x \in B \end{array} \right.$$

$$\lim_{k_A(x) = 0} x \in A \text{ and } x \in A$$

$$\lim_{k_B(x) = 0} x \in A \text{ if } x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$
 $k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

Idea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

 $= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ (1-0) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$= (1 - k_B(x))k_A(x) \quad \text{if } x \notin A \text{ and } x \notin B$$

$$\text{if } x \notin A \text{ and } x \in B$$

and
$$x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

Idea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ (1-0) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ (1-0) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \notin B \\ (1-1) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$(1-1) \cdot 0 = (1-\kappa_B(x))\kappa_A(x) \quad \text{if } x \notin A \text{ and } x \in B$$

$$\text{if } x \in A \text{ and } x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

Idea.

$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e. } x \notin A \text{ or } x \in B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ (1-0) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

$$= \begin{cases} (1-0) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ (1-0) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \notin B \\ (1-1) \cdot 0 = (1-k_B(x))k_A(x) & \text{if } x \notin A \text{ and } x \in B \\ (1-1) \cdot 1 = (1-k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \in B \end{cases}$$

$$(1-1) \cdot 0 = (1-k_B(x))k_A(x) \quad \text{if } x \notin A \text{ and } x \in B$$

$$(1-1) \cdot 1 = (1-k_B(x))k_A(x) \quad \text{if } x \in A \text{ and } x \in B$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S}) \ \text{(by (b))} \\ &= |S| - (\sum_{x \in S}) \ \text{(by (b))} \\ &= |S| - (\sum_{x \in S}) \ \text{(by (a))} \end{aligned}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (a)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (a)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x))(1 - k_B(x))(1 - k_B(x)) \\ &$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ &= |S| - () \end{aligned}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)}) \\ &= |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x)) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)) \\ &= |S| - ($$

$$\begin{aligned} (*) & |A| = \sum_{x \in S} k_A(x) \\ (a) & k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ (b) & k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ (d) & \text{Deduce from (b) that } |A \cup B| = |A| + |B| - |A \cap B|. \\ \text{Idea.} \\ & |A \cup B| = |S| - |\overline{A \cup B}| \text{ (by definition of complement)} \\ & = |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)}) \\ & = |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x)) \\ & = |S| - (|S| -) \text{ (by (*))} \end{aligned}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)}) \\ &= |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x)) \\ &= |S| - (|S| - |A| -) \ \text{(by (*))} \end{aligned}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \end{aligned}$$
 Idea.
$$|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)}$$

$$= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)})$$

$$= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)})$$

$$= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x)))$$

$$= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)})$$

$$= |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x))$$

$$= |S| - (|S| - |A| - |B| +) \ \text{(by (*))}$$

$$= |S| - (|S| - |A| - |B| +) \ \text{(by (*))}$$

$$\begin{aligned} (*) & |A| = \sum_{x \in S} k_A(x) \\ (a) & k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ (b) & k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ (d) & \text{Deduce from (b) that } |A \cup B| = |A| + |B| - |A \cap B|. \\ \text{Idea.} \\ & |A \cup B| = |S| - |\overline{A \cup B}| \text{ (by definition of complement)} \\ & = |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ & = |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)}) \\ & = |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x)) \\ & = |S| - (|S| - |A| - |B| + |A \cap B|) \text{ (by (*))} \\ & = |S| - (|S| - |A| - |B| + |A \cap B|) \text{ (by (*))} \end{aligned}$$

$$\begin{aligned} &(*) \ |A| = \sum_{x \in S} k_A(x) \\ &(a) k_{A \cap B}(x) = k_A(x) \cdot k_B(x). \\ &(b) k_{\overline{A \cup B}}(x) = (1 - k_A(x))(1 - k_B(x)). \\ &(d) \ \ \text{Deduce from (b) that } \ |A \cup B| = |A| + |B| - |A \cap B|. \\ &\text{Idea.} \\ &|A \cup B| = |S| - |\overline{A \cup B}| \ \text{(by definition of complement)} \\ &= |S| - (\sum_{x \in S} k_{\overline{A \cup B}}(x))(\text{by (*)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x))(1 - k_B(x)))(\text{by (b)}) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_A(x)k_B(x))) \\ &= |S| - (\sum_{x \in S} (1 - k_A(x) - k_B(x) + k_{A \cap B}(x)))(\text{by (a)}) \\ &= |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A \cap B}(x)) \\ &= |S| - (|S| - |A| - |B| + |A \cap B|) \ \text{(by (*))} \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

Recall

$$|x| = n$$
 if $n \le x < n+1$

► ≤ 3*x* <

Recall

$$|x| = n$$
 if $n \le x < n+1$

▶ $3n \le 3x <$

Recall

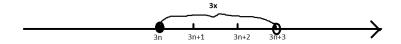
$$|x| = n$$
 if $n \le x < n+1$

▶ $3n \le 3x < 3n + 3$

Recall

$$|x| = n \text{ if } n \leq x < n+1$$

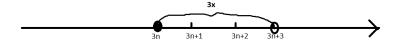
▶ $3n \le 3x < 3n + 3$



Recall

$$|x| = n \text{ if } n \leq x < n+1$$

▶ $3n \le 3x < 3n + 3$



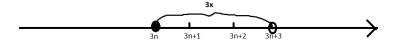
• Case 1. $3n \le 3x < 3n + 1$

7. Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

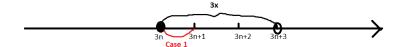
Recall

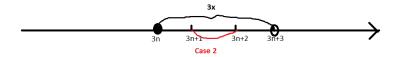
$$|x| = n$$
 if $n \le x < n+1$

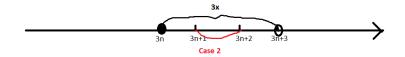
▶ $3n \le 3x < 3n + 3$



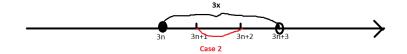
• Case 1. $3n \le 3x < 3n + 1$



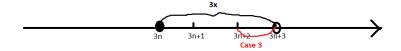




• Case 3. $3n + 2 \le 3x < 3n + 3$



• Case 3. $3n + 2 \le 3x < 3n + 3$



► Case 1.
$$\leq x <$$
 $\Leftrightarrow 3n \leq 3x < 3n + 1$

► $\lfloor 3x \rfloor = : 3n \leq 3x < 3n + 1$

► $\lfloor x \rfloor = : \leq x <$.

► $\lfloor x + 1/3 \rfloor = : \leq x + 1/3 <$

► $\lfloor x + 2/3 \rfloor = : \leq x + 2/3 <$.

► Case 2. $\leq x < \Leftrightarrow 3n + 1 \leq 3x < 3n + 2$

► $\lfloor 3x \rfloor = : 3n + 1 \leq 3x < 3n + 2$

► $\lfloor x \rfloor = : \leq x <$.

► $\lfloor x + 1/3 \rfloor = : \leq x <$.

• Case 3. $\leqslant x < \Leftrightarrow 3n+2 \leqslant 3x < 3n+3$.

 $\leq x + 2/3 <$

 $|x+2/3| = : \le x+2/3 <$

► [3x] = : $3n + 2 \le 3x < 3n + 3$ ► [x] = : $\le x <$. ► |x + 1/3| = : $\le x + 1/3 <$

```
• Case 1. n \le x < \Leftrightarrow 3n \le 3x < 3n + 1
    |3x| = 3n \le 3x < 3n + 1
   |x| = : n \leq x <
   [x+1/3] = : \le x+1/3 <
   [x+2/3] = : \le x+2/3 < .
• Case 2. \leqslant x < \Leftrightarrow 3n+1 \leqslant 3x < 3n+2.
   |3x| = 3n + 1 \le 3x < 3n + 2
   |x| = : \leqslant x <
   [x+1/3] = : \le x+1/3 < .
```

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : \leq x < .$ |x+1/3| = : $\leq x+1/3 <$

|x + 2/3| = :

 $\leq x + 2/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = : 3n \le 3x < 3n + 1$

► $[x] = : n \le x < n + 1/3 < ...$

► $[x + 1/3] = : \le x + 1/3 < ...$

► $[x + 2/3] = : \le x + 2/3 < ...$

► Case 2. $\le x < ...$

⇒ $3n + 1 \le 3x < 3n + 2...$

► $[3x] = : 3n + 1 \le 3x < 3n + 2...$

► $[x] = : \le x < ...$

 $[x+1/3] = : \le x+1/3 < .$

• Case 3. $\leqslant x < \Leftrightarrow 3n+2 \leqslant 3x < 3n+3$.

 $\leq x + 2/3 <$

 $|x+2/3| = : \le x+2/3 <$

► [3x] = : $3n + 2 \le 3x < 3n + 3$ ► [x] = : $\le x <$. ► |x + 1/3| = : $\le x + 1/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
    |3x| = 3n \le 3x < 3n + 1
    |x| = : n \le x < n + 1/3 <
    |x + 1/3| = : \le x + 1/3 <
    [x+2/3] = : \le x+2/3 < .
    |3x| = 3n + 1 \le 3x < 3n + 2
    |x| = : < n + 1/3 \le x <
```

• Case 2.
$$n + 1/3 \le x < \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x <$$

$$[3x] = : 3n + 1 \le 3x < 3n + 2$$

 $[x] = : < n + 1/3 \le x < .$

$$[x] = : < n + 1/3 \le x <$$

 $[x + 1/3] = : \le x + 1/3 <$

$$[x] = : < n + 1/3 \le x < .$$

$$[x + 1/3] = : \le x + 1/3 < .$$

$$|x + 1/3| = |x + 1/3| = |x + 1/3| = |x + 2/3| = |x + 2/3| < |x$$

Case 3.
$$\leqslant x < \Leftrightarrow 3n+2 \leqslant 3x < 3n+3$$
.

$$|3x| = : 3n+2 \le 3x < 3n+3$$

$$|x| = : \le x < .$$

$$|x| = : \qquad \leq x < .$$

$$|x + 1/3| = : \qquad \leq x + 1/3 < .$$

• Case 3.
$$\leqslant x < \Leftrightarrow 3n+2 \leqslant 3x < 3n$$

 $|x + 2/3| = : \le x + 2/3 <$

► [3x] = : $3n + 2 \le 3x < 3n + 3$ ► [x] = : $\le x <$. ► |x + 1/3| = : $\le x + 1/3 <$

|x + 2/3| =:

• Case 3. $\leqslant x < \Leftrightarrow 3n+2 \leqslant 3x < 3n+3$.

 $\leq x + 2/3 <$

```
► Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1

► [3x] = : 3n \le 3x < 3n + 1

► [x] = : n \le x < n + 1/3 < ...

► [x + 1/3] = : \le x + 1/3 < ...

► [x + 2/3] = : \le x + 2/3 < ...

► Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2...

► [3x] = : 3n + 1 \le 3x < 3n + 2...

► [x] = : < n + 1/3 \le x < n + 2/3 < ...
```

 $[x+1/3] = : \le x+1/3 < .$

• Case 3. $n + 2/3 \le x < \Leftrightarrow 3n + 2 \le 3x < 3n + 3$.

 $\leq x + 2/3 <$

 $|x+2/3| = : \le x+2/3 <$

 $\begin{array}{l} \bullet \ [3x] = & : 3n+2 \leqslant 3x < 3n+3 \\ \bullet \ [x] = : & < n+2/3 \leqslant x < \\ \bullet \ [x+1/3] = : & \leqslant x+1/3 < \end{array}$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$[x + 1/3] = : \qquad \leq x + 1/3 < .$$

$$[x+1/3] = : \qquad \leq x+1/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

|x + 2/3| = :

 $\leq x + 2/3 <$

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x \le n+2/3 \le$$

$$[3x] = : 3n+1 \le 3x < 3n+2$$

$$[x] = : < n+1/3 \le x < n+2/3 < .$$

$$|x| = : < n + 1/3 \le x < n + 2/3 < .$$

$$|x + 1/3| = : \le x + 1/3 < .$$

$$[x + 1/3] = : \qquad \leq x + 1/3 < .$$

$$[x + 1/3] = : \qquad \leq x + 1/3 < .$$

$$(x + 1/3) = : \le x + 1/3 < .$$

 $(x + 2/3) = : \le x + 2/3 < .$

►
$$[x + 2/3] = : \le x + 2/3 < .$$
► Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$|x + 1/3| = : < x + 1/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

$$|x| = 3x < 3n + 2$$

$$|x| = 3x < 3n + 2$$

$$|x| = 3x < 3n + 2$$

Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

∴ $[3x] = 3n$: $3n \le 3x < 3n + 1$

∴ $[x] = : n \le x < n + 1/3 < n + 1$.

∴ $[x + 1/3] = : \le x + 1/3 <$

∴ $[x + 2/3] = : \le x + 2/3 <$.

Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

$$|3x| = 3n+1 \leqslant 3x < 3n+2$$

$$[3x] = : 3n+1 \le 3x < 3n+2 [x] = : < n+1/3 \le x < n+2/3 < .$$

$$|x| = : < n + 1/3 \le x < n + 2/3 < .$$

 $|x + 1/3| = : \le x + 1/3 < .$

$$[x+1/3] = : \le x+1/3 < .$$
 $[x+2/3] = : \le x+2/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$[x + 1/3] = : \le x + 1/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 \le x$$

$$|x| = 3x < 3n + 2$$

$$|x| = 3x < 3n + 2$$

$$|x| = 3x < n + 2/3 < 3x < n + 2/3 < 3x < n + 1/2 < 1x < n + 1/2$$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

 $|x + 1/3| = : < x + 1/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

 $[x + 1/3] = : \le x + 1/3 < .$

$$|x + 1/3| = : \le x + 1/3 < .$$

$$|x + 2/3| = : \le x + 2/3 < .$$

$$|x + 2/3| = : \le x + 2/3 < .$$

►
$$[x + 2/3] =$$
 : $≤ x + 2/3 <$.
► Case 3. $n + 2/3 ≤ x < n + 1 \Leftrightarrow 3n + 2 ≤ 3x < 3n + 3.$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$[x + 1/3] = : \le x + 1/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 <$$

$$|3x| = 3n+1 \leqslant 3x < 3n+2$$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
    |3x| = 3n: 3n \le 3x < 3n + 1
    |x| = n: n \le x < n + 1/3 < n + 1.
    |x+1/3| = : < n+1/3 \le x+1/3 <
    |x+2/3| = : \le x+2/3 < .
• Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.
    |3x| = 3n + 1 \le 3x < 3n + 2
```

$$|3x| = 3n + 1 \le 3x < 3n + 2$$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 < .$$

$$\begin{bmatrix} x \end{bmatrix} = : < n + 1/3 \le x < n + 2/3 < .$$

 $|x + 1/3| = : \le x + 1/3 < .$

$$[x + 1/3] = : \leq x + 1/3 < .$$

$$\begin{array}{cccc} [x+1/3] & : & \leq x+1/3 < \\ & [x+2/3] & : & \leq x+2/3 < \\ & . \end{array}$$

►
$$[x + 2/3] =$$
 : $\leq x + 2/3 <$.
 • Case 3. $n + 2/3 \leq x < n + 1 \Leftrightarrow 3n + 2 \leq 3x < 3n + 3$.

$$[3x] = 3n + 1 \le 3x < 3n + 2$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

|x + 2/3| = :

 $\leq x + 2/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
    |3x| = 3n: 3n \le 3x < 3n + 1
    |x| = n: n \le x < n + 1/3 < n + 1.
    |x + 1/3| = : < n + 1/3 \le x + 1/3 < n + 2/3 <
    |x+2/3| = : \le x+2/3 < .
```

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

$$[3x] = : 3n+1 \le 3x < 3n+2$$

$$[x] = : < n+1/3 \le x < n+2/3 < .$$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 < .$$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

 $[x + 1/3] = : \le x + 1/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < [x + 1/3] = : \le x + 1/3 < .$$

$$|x + 1/3| = |x + 1/3| < |x + 1/3| < |x + 2/3| = |x + 2/3| < |x$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$[x+1/3] = : \le x+1/3 < .$$

$$|x+2/3| = : \le x+2/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

```
Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1

↓ [3x] = 3n: 3n \le 3x < 3n + 1

↓ [x] = n: n \le x < n + 1/3 < n + 1.

↓ [x + 1/3] = : n < n + 1/3 \le x + 1/3 < n + 2/3 <

↓ [x + 2/3] = : \le x + 2/3 <.

Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.

↓ [3x] = : 3n + 1 \le 3x < 3n + 2
```

• $[x] = : < n + 1/3 \le x < n + 2/3 < .$ • $|x + 1/3| = : n \le x + 1/3 < .$

• Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$.

 $\leq x + 2/3 <$

 $|x+2/3| = : \le x+2/3 <$

• $[3x] = : 3n + 2 \le 3x < 3n + 3$ • $[x] = : < n + 2/3 \le x < n + 1.$ • $[x + 1/3] = : \le x + 1/3 < x < 3n + 3$

$$\begin{array}{c} \bullet \quad \text{Case } 1. \ n \leqslant x < n + 1/3 \Leftrightarrow 3n \leqslant 3x < 3n + 1 \\ \bullet \quad \lfloor 3x \rfloor = 3n \colon 3n \leqslant 3x < 3n + 1 \\ \bullet \quad \lfloor x \rfloor = n \colon n \leqslant x < n + 1/3 < n + 1. \\ \bullet \quad \lfloor x + 1/3 \rfloor = \quad \colon n < n + 1/3 \leqslant x + 1/3 < n + 2/3 < n + 1. \\ \bullet \quad \lfloor x + 2/3 \rfloor = \quad \colon \quad \leqslant x + 2/3 < \quad . \\ \bullet \quad \text{Case } 2. \ n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2. \end{array}$$

$$|3x| = 3n+1 \leqslant 3x \leqslant 3n+2$$

$$\begin{aligned}
[3x] &= : 3n+1 \leqslant 3x < 3n+2 \\
[x] &= : < n+1/3 \leqslant x < n+2/3 < .
\end{aligned}$$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

 $[x + 1/3] = : n \le x + 1/3 < .$

$$[x] = x < n + 1/3 \le x < n + 2/3 <$$

$$[x + 1/3] = x = n$$

$$\le x + 1/3 < x < n + 2/3 < x < n + 2/3 < x < n < 2/3$$

$$|x + 2/3| = : \le x + 2/3 < .$$
Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

►
$$[3x] =$$
 : $3n + 1 \le 3x < 3n + 2$
► $[x] =$: $< n + 1/3 \le x < n + 2/3 <$.

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

$$\begin{array}{l} \bullet \quad \underline{\mathsf{Case} \ 1. \ n \leqslant x < n + 1/3 \Leftrightarrow 3n \leqslant 3x < 3n + 1} \\ \bullet \quad [3x] = 3n: \ 3n \leqslant 3x < 3n + 1 \\ \bullet \quad [x] = n: \ n \leqslant x < n + 1/3 < n + 1. \\ \bullet \quad [x + 1/3] = n: \ n < n + 1/3 \leqslant x + 1/3 < n + 2/3 < n + 1. \\ \bullet \quad [x + 2/3] = \quad : \qquad \qquad \leqslant x + 2/3 < . \\ \bullet \quad \underline{\mathsf{Case} \ 2. \ n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2.} \end{array}$$

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

 $|3x| = 3n + 1 \le 3x < 3n + 2$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 < .$$

$$[x] = : < n + 1/3 \le x < n + 2/3 <$$
 $[x + 1/3] = : n \le x + 1/3 <$.

$$[x+1/3] = : n \qquad \leqslant x+1/3 < \dots$$

$$[x+1/3] = : n \qquad \leqslant x+1/3 < \dots$$

$$[x + 2/3] = : \le x + 2/3 < .$$
• Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

$$|x| = : < n + 1/3 \le x < n + 2/3 < .$$

$$|x + 1/3| = : n \le x + 1/3 < .$$

$$|x + 2/3| = : \le x + 2/3 < .$$

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x$$

 $|3x| = 3n + 1 \le 3x < 3n + 2$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

Case 2.
$$n + 1/3 \le x < n + 2/3 \le 3n + 1 \le 3x$$

► Case 1.
$$n \leqslant x < n + 1/3 \Leftrightarrow 3n \leqslant 3x < 3n + 1$$

► $[3x] = 3n$: $3n \leqslant 3x < 3n + 1$

► $[x] = n$: $n \leqslant x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \leqslant x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = : < n + 2/3 \leqslant x + 2/3 <$.

► Case 2. $n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

 $|3x| = 3n + 1 \le 3x < 3n + 2$

$$[3x] = : 3n + 1 \le 3x < 3n + 2$$

 $[x] = : < n + 1/3 \le x < n + 2/3 < ...$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$
 $[x + 1/3] = : n = x + 1/3 < x < n + 1/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

 $[x + 1/3] = : n \le x + 1/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < [x + 1/3] = : n \le x + 1/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
     |3x| = 3n: 3n \le 3x < 3n + 1
     |x| = n: n \le x < n + 1/3 < n + 1.
     |x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.
     |x + 2/3| = : < n + 2/3 \le x + 2/3 < n + 1.
• Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.
```

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

 $|3x| = 3n + 1 \le 3x < 3n + 2$

$$[3x] = : 3n + 1 \le 3x < 3n + 2$$

 $[x] = : < n + 1/3 \le x < n + 2/3 <$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$[x + 1/3] = : n \le x + 1/3 < .$$

►
$$[x + 2/3] = : \le x + 2/3 < .$$
► Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

$$|3x| = : 3n+1 \le 3x < 3n+2$$

$$|x| = : < n+1/3 \le x < n+2/3 < .$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
     |3x| = 3n: 3n \le 3x < 3n + 1
     |x| = n: n \le x < n + 1/3 < n + 1.
     |x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.
     |x + 2/3| = : n < n + 2/3 \le x + 2/3 < n + 1.
• Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.
```

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

|x + 2/3| = :

$$[3x] = : 3n+1 \le 3x < 3n+2$$

$$[x] = : < n+1/3 \le x < n+2/3 < .$$

$$|x| = : < n + 1/3 \le x < n + 2/3 <$$

•
$$[x + 1/3] = : n$$
 $\leq x + 1/3 < .$
• $[x + 2/3] = : \leq x + 2/3 < .$
• Case $3, n + 2/3 \leq x < n + 1 \Leftrightarrow 3n + 2 \leq 3x < 3n + 3.$

 $\leq x + 2/3 <$

$$[x+1/3] = : n \qquad \leq x+1/3 <$$

$$[x+2/3] = : \leq x+2/3 <$$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$[x + 1/3] = : n \le x + 1/3 < .$$

$$[x + 2/3] = : < x + 2/3 < .$$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
     |3x| = 3n: 3n \le 3x < 3n + 1
     |x| = n: n \le x < n + 1/3 < n + 1.
     |x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.
     |x + 2/3| = n: n < n + 2/3 \le x + 2/3 < n + 1.
• Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.
```

$$|3x| = 3n+1 \le 3x < 3n+2$$

$$[3x] = : 3n+1 \le 3x < 3n+2$$

$$[x] = : < n+1/3 \le x < n+2/3 < .$$

$$|x| = : < n + 1/3 \le x < n + 2/3 <$$

$$\lfloor x \rfloor = : < n + 1/3 \le x < n + 2/3 < .$$

 $\lfloor x + 1/3 \rfloor = : n \le x + 1/3 < .$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$[x + 1/3] = : n \le x + 1/3 < .$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

$$|3x| = 3n + 2 \le 3x < 3n + 3$$

$$[3x] = : 3n+1 \le 3x < 3n+2$$

$$[x] = : < n+1/3 \le x < n+2/3 < ...$$

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
     |3x| = 3n: 3n \le 3x < 3n + 1
     |x| = n: n \le x < n + 1/3 < n + 1.
     |x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.
     |x + 2/3| = n: n < n + 2/3 \le x + 2/3 < n + 1.
• Case 2. n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2.
```

$$\frac{\mathsf{Case}\ 2.\ n+1/3\leqslant x < n+2/3 \Leftrightarrow 3n+1\leqslant 3x < 3n+2.}{ [3x] = 3n+1:\ 3n+1\leqslant 3x < 3n+2}$$

$$[3x] = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$[x] = : < n + 1/3 \le x < n + 2/3 < .$$

$$|x| = 3n + 1, \ 3n + 1 \le 3x \le 3n + 2$$

$$|x| = \frac{1}{2} < n + 1/3 \le x < n + 2/3 < \frac{1}{2}$$

$$\lfloor x \rfloor = : < n + 1/3 \le x < n + 2/3 < .$$

 $\lfloor x + 1/3 \rfloor = : n \le x + 1/3 < .$

$$|x| - 1/3 \le x < n + 2/3 < 1$$

$$|x + 1/3| = 1$$

$$|x + 1/3| = 1$$

$$|x + 1/3| < 1$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

• $|3x| = 3n + 2 \le 3x < 3n + 3$

$$[x + 2/3] = : \leq x + 2/3 < .$$
• Case 3. $n + 2/3 \leq x < n + 1 \Leftrightarrow 3n + 2 \leq 3x < 3n + 3$.

$$|x| = : < n + 1/3 \le x < n + 2/3 < .$$

$$|x + 1/3| = : n \le x + 1/3 < .$$

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

► Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $[x] = : n < n + 1/3 \le x < n + 2/3 <$

$$[x] = 3n + 1, \ 3n + 1 \le 3x < 3n + 2$$

$$[x] = : n < n + 1/3 \le x < n + 2/3 <$$

$$|x| = : n < n + 1/3 \le x < n + 2/3 < .$$

$$|x + 1/3| = : n \le x + 1/3 < .$$

$$[x] = : n < n + 1/3 \le x < n + 2/3 < .$$

 $[x + 1/3] = : n \le x + 1/3 < .$

$$|x + 2/3| = |x + 2/3| \le |x$$

$$|x + 2/3| = : \leqslant x + 2/3 < .$$
• Case 3. $n + 2/3 \leqslant x < n + 1 \Leftrightarrow 3n + 2 \leqslant 3x < 3n + 3.$
• $|3x| = : 3n + 2 \leqslant 3x < 3n + 3.$

$$[3x] = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$[x] = : n < n + 1/3 \le x < n + 2/3 < .$$

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

► Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.
► $|3x| = 3n + 1$: $3n + 1 \le 3x < 3n + 2$

$$[3x] = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$[x] = : n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$|x| = : n < n + 1/3 \le x < n + 2/3 < n + 1.$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $|x + 1/3| = : n \le x + 1/3 < .$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$

$$|x + 1/3| = |x + 1/3| < |x$$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = : n \le x + 1/3 < .$$

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $|x| = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[3x] = 3n + 1: 3n + 1 \le 3x < 3n + 2$$
$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $|x + 1/3| = n < n + 2/3 \le x + 1/3 < n < n$.

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $[x + 1/3] = : n < n + 2/3 \le x + 1/3 < .$

$$[x + 2/3] = : \leq x + 2/3 < .$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

$$|3x| = : 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = : n < n + 2/3 \le x + 1/3 < .$$

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n + 1$

► $\lfloor x \rfloor = n$: $n \le x < n + 1/3 < n + 1$.

► $\lfloor x + 1/3 \rfloor = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $\lfloor x + 2/3 \rfloor = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x] = n: \ n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x + 1/3] = n: \ n < n + 2/3 < m + 1/3 < m + 1.$$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x + 1/3] = : n < n + 2/3 \le x + 1/3 < n + 1.$$

►
$$[x+2/3] = : \le x+2/3 < .$$
► Case 3. $n+2/3 \le x < n+1 \Leftrightarrow 3n+2 \le 3x < 3n+3.$

$$|x + 1/3| = : n < n + 2/3 \le x + 1/3 < n + 1.$$

$$|x + 2/3| = : \le x + 2/3 < .$$
• Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

 $|3x| = 3n + 2 \le 3x < 3n + 3$ $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

▶ Case 1.
$$n \le x < n+1/3 \Leftrightarrow 3n \le 3x < 3n+1$$

▶ $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n+1$

▶ $\lfloor x \rfloor = n$: $n \le x < n+1/3 < n+1$.

▶ $\lfloor x+1/3 \rfloor = n$: $n < n+1/3 \le x+1/3 < n+2/3 < n+1$.

•
$$\lfloor x + 2/3 \rfloor = n$$
: $n < n + 2/3 \leqslant x + 2/3 < n + 1$.
• Case 2. $n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

• $[3x] = 3n + 1$: $3n + 1 \le 3x < 3n + 2$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $|x| = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

•
$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
• $[x + 1/3] = n$: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$x + 1/3$$
] = n : $n < n + 2/3 \le x + 1/3 < n + 1$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

 $|3x| = : 3n + 2 \le 3x < 3n + 3$
 $|x| = : < n + 2/3 \le x < n + 1$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
$$|3x| = : 3n + 2 \le 3x < 3n + 3$$

$$[x+2/3] = : \le x+2/3 <$$
Case 3 $n+2/3 \le x \le n+1 \Leftrightarrow 3n+2 \le 3x \le 3$

 $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n + 1$

► $\lfloor x \rfloor = n$: $n \le x < n + 1/3 < n + 1$.

► $\lfloor x + 1/3 \rfloor = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $\lfloor x + 2/3 \rfloor = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$\frac{|3x| = 3n + 1}{3n + 1} \le \frac{3x}{3n + 2}$$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$x = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $x + 1/3 = n$: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$(x + 1/3) = n$$
: $n < n + 2/3 \le x + 1/3 < n + 1$.
 $(x + 2/3) = x + 1 \le x + 2/3 < x + 1/3 < n + 1$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$[x + 1/3] = n \cdot n < n + 2/3 \le x + 1/3 < n + 1 .$$

$$[x + 2/3] = n + 1 \le x + 2/3 < n + 1 \Leftrightarrow 3n + 2 < 3x < 3n + 3 .$$

$$|3x| = : 3n + 2 \le 3x < 3n + 3$$

$$|x| = : < n + 2/3 \le x < n + 1.$$

► Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
► $|3x| = 3n + 2 \le 3x < 3n + 3$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
• $|3x| = 3n + 2 \le 3x < 3n + 3$

Case 3.
$$n + 2/3 \leqslant x < n + 1 \Leftrightarrow 3n + 2 \leqslant 3x < 3$$

 $|x+1/3| = : \le x+1/3 <$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n + 1$

► $\lfloor x \rfloor = n$: $n \le x < n + 1/3 < n + 1$.

► $\lfloor x + 1/3 \rfloor = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $\lfloor x + 2/3 \rfloor = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

$$\quad \text{Case 2. } n+1/3 \leqslant x < n+2/3 \Leftrightarrow 3n+1 \leqslant 3x < 3n+2.$$

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[3x] = 3n + 1; \ 3n + 1 \le 3x < 3n + 2$$

$$[x] = n; \ n < n + 1/3 \le x < n + 2/3 < n + 1.$$

•
$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
• $[x + 1/3] = n$: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$[x+2/3] = : n+1 \le x+2/3 < n+4/3 < .$$
 Case 3. $n+2/3 \le x < n+1 \Leftrightarrow 3n+2 \le 3x < 3n+3$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

 $|x| = : < n + 2/3 \le x < n + 1.$ $|x+1/3| = : \le x+1/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $|3x| = 3n$: $3n \le 3x < 3n + 1$

► $|x| = n$: $n \le x < n + 1/3 < n + 1$.

$$[x+1/3] = n: n < n+1/3 \le x+1/3 < n+2/3 < n+1.$$

$$|x+2/3| = n: n < n+2/3 \le x+2/3 < n+1.$$

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = 3n+1: 3n+1 \le 3x < 3n+2$$

$$|3x| = 3n + 1; \ 3n + 1 \le 3x < 3n + 2$$

$$|x| = n; \ n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $[x + 1/3] = n$: $n < n + 2/3 < x + 1/3 < n + 1$

$$[x + 2/3] = : n + 1 \le x + 2/3 < n + 4/3 < n + 2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$|x + 2/3| = : n + 1 \le x + 2/3 < n + 4/3 < n + 2.$$
Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3.$

$$[3x] = : 3n + 2 \le 3x < 3n + 3$$

$$[x] = : < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = : \le x + 1/3 < .$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
 $|3x| = 3n + 2 \le 3x < 3n + 3$

$$[x+1/3] = n. \ n < n+2/3 \le x+1/3 < n+1.$$

$$[x+2/3] = n+1 \le x+2/3 < n+4/3$$

$$[x+1/3] = n: \ n < n+2/3 \le x+1/3 < n+1.$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $|3x| = 3n$: $3n \le 3x < 3n + 1$

► $|x| = n$: $n \le x < n + 1/3 < n + 1$.

$$[x+1/3] = n: \ n < n+1/3 \le x+1/3 < n+2/3 < n+1.$$

$$|x+2/3| = n: \ n < n+2/3 \le x+2/3 < n+1.$$

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$\frac{\text{Case 2: } n+1/3 \leqslant x < n+2/3 \Leftrightarrow 3n+1 \leqslant 3x < 3n+2.}{ [3x] = 3n+1: 3n+1 \leqslant 3x < 3n+2}$$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $|x| = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n: \ n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x + 1/3] = n: \ n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

►
$$[x + 2/3] = n + 1$$
: $n + 1 \le x + 2/3 < n + 4/3 < n + 2$
► Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$

Case 3.
$$n + 2/3 \leqslant x < n + 1 \Leftrightarrow 3n + 2 \leqslant 3x < 3n$$

$$|3x| = 3n+2 \leqslant 3x < 3n+3$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

 $|3x| = 3n + 2 \le 3x < 3n + 3$

$$[3x] = : 3n + 2 \le 3x < 3n + 3$$

$$[x] = : < n + 2/3 \le x < n + 1.$$

$$|3x| = 3n+2 \leqslant 3x \leqslant 3n+3$$

$$[3x] = : 3n + 2 \le 3x < 3n + 3$$

$$[3x] = : 3n + 2 \le 3x < 3n + 3$$

$$[x] = : < n + 2/3 \le x < n + 1.$$

$$|x| = : < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = : \le x + 1/3 <$$

$$[x] = \frac{1}{x^2 + 1/3} = \frac{1$$

$$|x| = 1$$
 $|x + 1/3| = 1$ $|x + 1/3| = 1$ $|x + 1/3| = 1$

Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

$$[3x] = 3n: 3n \le 3x < 3n + 1$$

$$|x| = n: n \le x < n + 1/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.$$

•
$$\lfloor x + 2/3 \rfloor = n$$
: $n < n + 2/3 \le x + 2/3 < n + 1$.
• Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.
 $|3x| = 3n + 1$: $3n + 1 \le 3x < 3n + 2$

$$\begin{bmatrix} 3x \end{bmatrix} = 3n + 1; \ 3n + 1 \le 3x < 3n + 2$$
$$|x| = n; \ n < n + 1/3 \le x < n + 2/3 < n + 1$$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x + 2/3] = n + 1: n + 1 \le x + 2/3 < n + 4/3 < n + 2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$[x+2/3] = n+1: \ n+1 \leqslant x+2/3 < n+4/3 < n+2.$$
 Case 3. $n+2/3 \leqslant x < n+1 \Leftrightarrow 3n+2 \leqslant 3x < 3n+3.$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$[x] = : < n + 2/3 \le x < n + 1.$$

$$[x] = : < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = : \le x + 1/3 < .$$

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = \frac{1}{2} < n + \frac{2}{3} < x < n + 1$$

Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

$$[3x] = 3n: 3n \le 3x < 3n + 1$$

$$|x| = n: n \le x < n + 1/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.$$

•
$$\lfloor x + 2/3 \rfloor = n$$
: $n < n + 2/3 \leqslant x + 2/3 < n + 1$.
• Case 2. $n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2$.

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $|x| = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$|x| = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
 $|x + 1/3| = n$: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

e 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$|x| = : n < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = : \le x + 1/3 < .$$

► Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
► $[3x] = 3n + 2$: $3n + 2 \le 3x < 3n + 3$

$$[x+1/3] = n: \ n < n+2/3 \le x+1/3 < n+1.$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $|3x| = 3n$: $3n \le 3x < 3n + 1$

► $|x| = n$: $n \le x < n + 1/3 < n + 1$.

$$[x+1/3] = n: n < n+1/3 \le x+1/3 < n+2/3 < n+1.$$

$$|x+2/3| = n: n < n+2/3 \le x+2/3 < n+1.$$

• Case 2.
$$n+1/3 \leqslant x < n+2/3 \Leftrightarrow 3n+1 \leqslant 3x < 3n+2$$
.

$$|3x| = 3n+1: 3n+1 \le 3x < 3n+2$$

$$|x| = n: n < n+1/3 < x < n+2/3 < n+1$$

$$|x| = 3n + 1. \ 3n + 1 \le 3x < 3n + 2$$

$$|x| = n: \ n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x+1/3] = n: \ n < n+2/3 \le x+1/3 < n+1.$$

•
$$[x + 2/3] = n + 1$$
: $n + 1 \le x + 2/3 < n + 4/3 < n + 2$.
se 3 $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
• $[3x] = 3n + 2$: $3n + 2 \le 3x < 3n + 3$

•
$$[3x] = 3n + 2$$
: $3n + 2 \le 3x < 3n + 3$
• $[x] = n$: $n < n + 2/3 \le x < n + 1$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$[x] = n: n < n + 2/3 \le x < n + 1.$$

$$[x] = n: n < n + 2/3 \le x < n + 1.$$

$$[x + 1/3] = : \le x + 1/3 < .$$

$$|x + 2/3| = :$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n + 1$

► $\lfloor x \rfloor = n$: $n \le x < n + 1/3 < n + 1$.

► $\lfloor x + 1/3 \rfloor = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

•
$$\lfloor x + 2/3 \rfloor = n$$
: $n < n + 2/3 \leqslant x + 2/3 < n + 1$.
• Case 2. $n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.
• $[3x] = 3n + 1$: $3n + 1 \le 3x < 3n + 2$

$$\begin{bmatrix} 3x \end{bmatrix} = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $|x| = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

•
$$[x + 1/3] = n$$
: $n < n + 2/3 \le x + 1/3 < n + 1$.
• $[x + 2/3] = n + 1$: $n + 1 \le x + 2/3 < n + 4/3 < n + 2$.

$$\lim_{n \to \infty} \frac{1}{n} \left(\frac{x + 2}{3} \right) = n + 1, \ n + 1 \leqslant x + 2/3 \leqslant n + 4/3 \leqslant n + 2.$$

$$\lim_{n \to \infty} 3n + 2/3 \leqslant x < n + 1 \Leftrightarrow 3n + 2 \leqslant 3x < 3n + 3.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
 $|3x| = 3n + 2$: $3n + 2 \le 3x < 3n + 3$

•
$$[x + 2/3] = n + 1$$
: $n + 1 \le x + 2/3 < n + 4/3 < n + 2$.
• Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$.

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$

|x| = n: $n < n + 2/3 \le x < n + 1$. $|x+1/3| = n+1 \le x+1/3 <$

```
• Case 1. n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1
     |3x| = 3n: 3n \le 3x < 3n + 1
     |x| = n: n \le x < n + 1/3 < n + 1.
     |x + 1/3| = n: n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1.
     |x + 2/3| = n: n < n + 2/3 \le x + 2/3 < n + 1.
```

• Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

• Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

|x + 2/3| =

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

 $|x+1/3| = n+1 \le x+1/3 < n+4/3 < n+1/3 < n+1/$

 $\leq x + 2/3 <$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

|x| = n: $n \le x < n + 1/3 < n + 1$. |x + 1/3| = n: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

$$[x+2/3] = n: n < n+2/3 \le x+2/3 < n+1.$$
Case 2 $n+1/3 \le x \le n+2/3 \Leftrightarrow 3n+1 \le 3x \le 3n+2$

• Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

$$[3x] = 3n+1: 3n+1 \le 3x < 3n+2$$

|x| = n: $n < n + 1/3 \le x < n + 2/3 < n + 1$. |x + 1/3| = n: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$[x + 2/3] = n + 1: n + 1 \le x + 2/3 < n + 4/3 < n + 2.$$

• Case 3. $n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$.

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$[x] = n: n < n + 2/3 \le x < n + 1.$$

|x + 1/3| = : $n + 1 \le x + 1/3 < n + 4/3 < n + 2$.

$$\quad \underline{ \mathsf{Case 1.} \ n \leqslant x < n + 1/3 \Leftrightarrow 3n \leqslant 3x < 3n + 1 }$$

$$|3x| = 3n$$
: $3n \le 3x < 3n + 1$

$$|x| = n$$
: $n \le x < n + 1/3 < n + 1$.

$$[x+1/3] = n: n < n+1/3 \le x+1/3 < n+2/3 < n+1.$$

$$|x+2/3| = n: n < n+2/3 \le x+2/3 < n+1.$$

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

• Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

•
$$[3x] = 3n + 2$$
: $3n + 2 \le 3x < 3n + 3$
• $[x] = n$: $n < n + 2/3 \le x < n + 1$.

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = n + 1: n + 1 \le x + 1/3 < n + 4/3 < n + 2.$$

$$[x+1/3] = n+1: n+1 \le x+1/3 < n+4/3 < n+2.$$

$$|x+1/3| = n+1: n+1 \le x+1/3 < n+4/3 < n+2.$$

$$|x+2/3| = :$$

• Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x <$$

► Case 1.
$$n \le x < n+1/3 \Leftrightarrow 3n \le 3x < 3n+1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n+1$

► $\lfloor x \rfloor = n$: $n \le x < n+1/3 < n+1$.

► $\lfloor x+1/3 \rfloor = n$: $n < n+1/3 \le x+1/3 < n+2/3 < n+1$.

► $\lfloor x+2/3 \rfloor = n$: $n < n+2/3 \le x+2/3 < n+1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$\frac{|3x| = 3n + 1}{3n + 1} \le \frac{x}{x} < \frac{n + 2}{3} \le \frac{3n + 1}{3} \le \frac{3n + 2}{3}$$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$[x] = n: n < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = n + 1: n + 1 \le x + 1/3 < n + 4/3 < n + 2.$$

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < x+2/3 < x+4/3 <$$

 $< n + 4/3 \le x + 2/3 <$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

► Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.
$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

►
$$[3x] = 3n + 2$$
: $3n + 2 \le 3x < 3n + 3$
► $[x] = n$: $n < n + 2/3 \le x < n + 1$.
► $[x + 1/3] = n + 1$: $n + 1 \le x + 1/3 < n + 4/3 < n + 2$.

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

Case 3.
$$n + 2/3 \leqslant x < n + 1 \Leftrightarrow 3n + 2 \leqslant 3x < 3n$$

 $< n + 4/3 \le x + 2/3 < n + 5/3 <$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$|x + 2/3| = n + 1: n + 1 \le x + 2/3 < n + 4/3 < n + 4/$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

► Case 2. $n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$.

Lase 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$

 $|3x| = 3n + 1$: $3n + 1 \le 3x < 3n + 2$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$

$$|3x| = 3n + 1: 3n + 1 \le 3x < 3n + 2$$

$$|x| = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x+1/3] = n: n < n+2/3 \le x+1/3 < n+1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

e 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

Asse 5.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

• $[3x] = 3n + 2$: $3n + 2 \le 3x < 3n + 3$
• $[x] = n$: $n < n + 2/3 \le x < n + 1$.

|x+1/3| = n+1: $n+1 \le x+1/3 < n+4/3 < n+2$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.
$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$[x+1/3] = n \cdot n < n+2/3 \le x+1/3 < n+1.$$

$$[x+2/3] = n+1 : n+1 \le x+2/3 < n+4/3 < n+4/3 < n+1/3 < n+1/3$$

 $n+1 < n+4/3 \le x+2/3 < n+5/3 <$

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$[x + 1/3] = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $\lfloor 3x \rfloor = 3n$: $3n \le 3x < 3n + 1$

► $\lfloor x \rfloor = n$: $n \le x < n + 1/3 < n + 1$.

► $\lfloor x + 1/3 \rfloor = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $\lfloor x + 2/3 \rfloor = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$|3x| = 3n + 1; \ 3n + 1 \le 3x < 3n + 2$$

$$|x| = n; \ n < n + 1/3 < x < n + 2/3 < n + 1$$

•
$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
• $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

•
$$[x] = n$$
: $n < n + 1/3 \le x < n + 2/3 < n + 1$.
• $[x + 1/3] = n$: $n < n + 2/3 \le x + 1/3 < n + 1$.

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

• Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$[3x] = 3n + 2$$
: $3n + 2 \le 3x < 3n + 3$
 $|x| = n$: $n < n + 2/3 \le x < n + 1$.

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

$$[3x] = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$[x+1/3] = n+1: n+1 \le x+1/3 < n+4/3$$

$$|x+2/3| = :$$

 $n+1 < n+4/3 \le x+2/3 < n+5/3 < n+2$.

$$[x+1/3] = n+1: n+1 \le x+1/3 < n+4/3 < n+2.$$

$$|x+2/3| =$$

$$|x + 1/3| = n + 1, \ n + 1 \le x + 1/3 < n + 4/3$$

$$|x + 2/3| =$$

$$[x] = n: n < n + 2/3 \le x < n + 1.$$

► Case 1.
$$n \le x < n + 1/3 \Leftrightarrow 3n \le 3x < 3n + 1$$

► $[3x] = 3n$: $3n \le 3x < 3n + 1$

► $[x] = n$: $n \le x < n + 1/3 < n + 1$.

► $[x + 1/3] = n$: $n < n + 1/3 \le x + 1/3 < n + 2/3 < n + 1$.

► $[x + 2/3] = n$: $n < n + 2/3 \le x + 2/3 < n + 1$.

• Case 2.
$$n + 1/3 \le x < n + 2/3 \Leftrightarrow 3n + 1 \le 3x < 3n + 2$$
.

$$[3x] = 3n+1: \ 3n+1 \le 3x < 3n+2$$

$$[3x] = 3n + 1$$
: $3n + 1 \le 3x < 3n + 2$
 $[x] = n$: $n < n + 1/3 \le x < n + 2/3 < n + 1$.

$$[x] = n: n < n + 1/3 \le x < n + 2/3 < n + 1.$$

$$|x + 1/3| = n: n < n + 2/3 \le x + 1/3 < n + 1.$$

$$[x+2/3] = n+1: n+1 \le x+2/3 < n+4/3 < n+2.$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$
.

$$|3x| = 3n + 2: 3n + 2 \le 3x < 3n + 3$$

Case 3.
$$n + 2/3 \le x < n + 1 \Leftrightarrow 3n + 2 \le 3x < 3n + 3$$

 $|3x| = 3n + 2$; $|3n + 2| \le 3x < 3n + 3$

•
$$[3x] = 3n + 2$$
: $3n + 2 \le 3x < 3n + 3$
• $[x] = n$: $n < n + 2/3 \le x < n + 1$.

$$|x| = n: n < n + 2/3 \le x < n + 1.$$

$$|x + 1/3| = n + 1: n + 1 \le x + 1/3 < n + 4/3$$

$$[x] = n: \ n < n + 2/3 \le x < n + 1.$$

$$[x + 1/3] = n + 1: \ n + 1 \le x + 1/3 < n + 4/3 < n + 2.$$

$$[x+1/3] = n+1: n+1 \le x+1/3 < n+4/3$$

$$[x+2/3] = n+1:$$

$$[x+1/3] = n+1: n+1 \le x+1/3 < n+4/3$$

$$[x+2/3] = n+1:$$

$$n+1 < n+4/3 \le x+2/3 < n+5/3 < n+2.$$

$$[x + 1/3] = n + 1: n + 1 \le x + 1/3 < n + 4/3$$

Answer.

Case (i)
$$n \le x < n + \frac{1}{3}$$
: $[3x] = 3n$,

$$[x] = [x + \frac{1}{3}] = [x + \frac{2}{3}] = n.$$

Case (ii)
$$n + \frac{1}{2} \le x < n + \frac{2}{2}$$
: $|3x| =$

Case (ii)
$$n + \frac{1}{3} \le x < n + \frac{2}{3}$$
: $\lfloor 3x \rfloor = 3n + 1$, $\lfloor x \rfloor = \lfloor x + \frac{1}{3} \rfloor = n$, $|x + \frac{2}{3}| = n + 1$.

Case (iii)
$$n + \frac{2}{3} \le x < n + 1$$
: $\lfloor 3x \rfloor = 3n + 2$, $\lfloor x \rfloor = n$, $\lfloor x + \frac{1}{3} \rfloor = \lfloor x + \frac{2}{3} \rfloor = n + 1$.