CS1231: Discrete Structures

Tutorial 5

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Quick Review

- A **function** f from A to B, is an assignment of exactly one elementof B to each element of A.
- A is the **domain**. B is the **codomain**. $\{f(a) : a \in A\}$ is the **range**.
- If $f(a) = b$, then b is the **image** of a; and a is the **preimage** of b.
- A function $f : X \to Y$ is **one-to-one** or **injective** if
$$\forall a, b \in X, f(a) = f(b) \Rightarrow a = b.$$
- A function $f : X \to Y$ is **onto** or **surjective** if
$$\forall y \in Y \exists x \in X, f(x) = y.$$
- The function f is a **bijection** if it is both 1-1 and onto.
- $\lfloor x \rfloor = n$ if $n \leqslant x < n + 1$, $\lceil x \rceil = n$ if $n - 1 < x \leqslant n$, where $n \in \mathbb{Z}$.

Menu

Question 1

Question 2

Question 3

Question 4

Question 5

Question 6(a)

Question 6(b)

Question 6(c)

Question 6(d)

Question 7

1. Determine whether $f : \mathbb{Z} \to \mathbb{R}$ is function.

(a) $f(n) = \pm n$.

(b) $f(n) = \sqrt{n^2 + 1}$.

(c) $f(n) = 1/(n^2 - 4)$.

(d) $f(n) = \sin n$.

Recall

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2. Find the domain and range of these functions.

(a) The function that assigns TO each nonnegative integer its last digit.

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(c) The function that assigns TO a bit string the number of one bits in the string.

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✍ A **function** is an assignment of exactly one element of range TO each element of domain.

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(a) $f(m, n) = 2m - n$

(b) $f(m, n) = m^2 - n^2$.

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Observation

$$m^2 - n^2 = (m-n)(m+n).$$

$$m^2 - n^2 = (m-n)(m+n) = 2 \begin{cases} \text{Case 1. } m+n = \quad , m-n = \\ \therefore m = \qquad , n = \\ \text{Case 2. } m+n = \qquad\quad , m-n = \\ \therefore m = \qquad\qquad , n = \\ \text{Case 3. } m+n = \quad , m-n = \\ \therefore m = \qquad , n = \\ \text{Case 4. } m+n = \qquad\quad , m-n = \\ \therefore m = \qquad\qquad , n = \end{cases}$$

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(a) Onto: Any $a \in \mathbb{Z}$ has $(0, -a)$ as a preimage.

(b) 2 has no preimage. Suppose there exists (m, n) such that $f(m, n) = m^2 - n^2 = 2$. Then $(m + n)(m - n) = 2$. This means (i) $m - n = 1$ and $m + n = 2$ or (ii) $m - n = -1$ and $m + n = -2$. From (i) we get $2m = 3$, a contradiction. From (ii) we get $2m = -3$, again a contradiction.

5. Determine which of the following are bijections. For those that are bijections, find the inverse functions.

(a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = -3x^2 + 7$.

(b) $f : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$ where $f(x) = (x + 1)/(x + 2)$.

Recall

- ✍ A function is **a bijection** if it is 1-1 and onto.
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Observation:

$$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2}$$

1-1:

$f(x) = f(y)$

$\Rightarrow 1 - \frac{1}{x+2} = 1 - \frac{1}{y+2}$

$\Rightarrow$

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Onto:

Pick $y \in$

We need to find x s.t.

$f(x) = y$

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Answer.

(a) Not 1-1: $f(1) = f(-1)$.

(b) 1-1: $f(a) = f(b)$ implies $(a+1)/(a+2) = (b+1)/(b+2)$. This yields $a = b$. Onto: Let $y \in \mathbb{R} - \{1\}$. Set $f(x) = y$. Solving, we get $x = (2y-1)/(1-y)$. Thus $(2y-1)/(1-y)$ is a preimage. The above calculations also show that $f^{-1}(x) = (2x-1)/(1-x)$.

6. Given a set S, the characteristic function of $A \subseteq S$, $k_A : S \to \mathbb{Z}$ is defined as

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

It is easy to see that $|A| = \sum_{x \in S} k_A(x)$. Prove that $\forall A, B \subseteq S$ and $\forall x \in S$:

(a) $k_{A \cap B}(x) = k_A(x) \cdot k_B(x)$.

Idea.

$$k_{A \cap B}(x) = \begin{cases} & \text{i.e.} \\ & \text{i.e.} \end{cases}$$

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Idea.

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$$= \begin{cases} \\ \\ \\ \end{cases}$$

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$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \qquad\qquad x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases}$$

$$= \Bigg\{$$

$$k_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad k_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

(c) $k_{A-B}(x) = (1 - k_B(x))k_A(x)$.

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$$k_{A-B}(x) = \begin{cases} 1 & \text{if } x \in A - B \text{ i.e. } x \in A \text{ and } x \notin B \\ 0 & \text{if } x \notin A - B \text{ i.e.} \end{cases}$$

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$$= \begin{cases} (1-0) \cdot 1 = (1 - k_B(x))k_A(x) & \text{if } x \in A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \notin B \\ & \text{if } x \notin A \text{ and } x \in B \\ & \text{if } x \in A \text{ and } x \in B \end{cases}$$

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(*) $|A| = \sum_{x \in S} k_A(x)$
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(d) Deduce from (b) that $|A \cup B| = |A| + |B| - |A \cap B|$.

Idea.

$$
\begin{aligned}
|A \cup B| &= |S| - \qquad\qquad \text{(by definition of complement)} \\
&= |S| - (\qquad\qquad\qquad)\text{(by (*))} \\
&= |S| - (\qquad\qquad\qquad\qquad\qquad)\text{(by (b))} \\
&= |S| - (\qquad\qquad\qquad\qquad\qquad\qquad) \\
&= |S| - (\qquad (\qquad\qquad\qquad\qquad\qquad))\text{(by (a))} \\
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&= |S| - (\sum_{x\in S} \qquad\qquad\qquad\qquad\qquad\qquad)\text{(by (b))}\\
&= |S| - (\sum_{x\in S} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad)\\
&= |S| - (\sum_{x\in S} (\qquad\qquad\qquad\qquad\qquad\qquad\qquad))\text{(by (a))}\\
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&= |S| - (\sum_{x \in S} \qquad\qquad\qquad\qquad\qquad\qquad) \\
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&= |S| - \Big(\sum_{x\in S} (1-k_A(x))(1-k_B(x))\Big)\text{(by (b))}\\
&= |S| - \Big(\sum_{x\in S} (1-k_A(x)-k_B(x)+k_A(x)k_B(x))\Big)\\
&= |S| - \Big(\sum_{x\in S} (1-k_A(x)-k_B(x)+ \qquad\qquad)\Big)\text{(by (a))}\\
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&= |S| - (\sum_{x \in S} 1 - \sum_{x \in S} k_A(x) - \sum_{x \in S} k_B(x) + \sum_{x \in S} k_{A\cap B}(x)) \\
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&= |S| - (\sum_{x\in S} (1-k_A(x)-k_B(x)+k_A(x)k_B(x)))\\
&= |S| - (\sum_{x\in S} (1-k_A(x)-k_B(x)+k_{A\cap B}(x)))\text{(by (a))}\\
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&= |S| - (|S| - |A| - |B| + |A\cap B|) \text{ (by (*))}\\
&= |A| + |B| - |A\cap B|
\end{aligned}
$$

7. Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Recall

✍ $\lfloor x \rfloor = n$ if $n \leqslant x < n+1$

- $\leqslant 3x <$

7. Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Recall

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- $3n \leqslant 3x < 3n + 3$

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Recall

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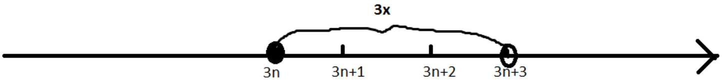
- $3n \leqslant 3x < 3n + 3$



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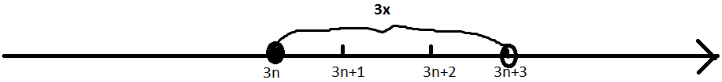
- $3n \leqslant 3x < 3n + 3$



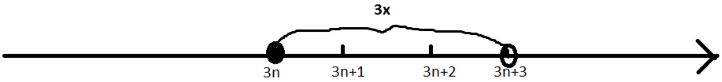
- Case 1. $3n \leqslant 3x < 3n + 1$

7. Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

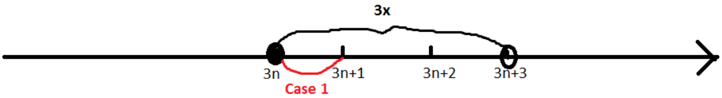
Recall

✍ $\lfloor x \rfloor = n$ if $n \leqslant x < n+1$

- $3n \leqslant 3x < 3n+3$



- Case 1. $3n \leqslant 3x < 3n+1$



- Case 2. $3n + 1 \leqslant 3x < 3n + 2$

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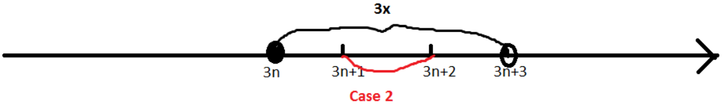
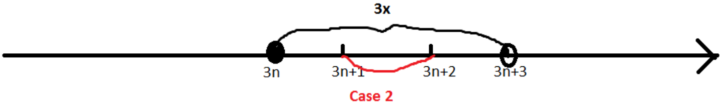
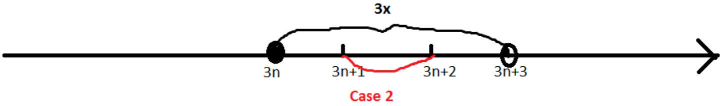


- Case 2. $3n + 1 \leqslant 3x < 3n + 2$

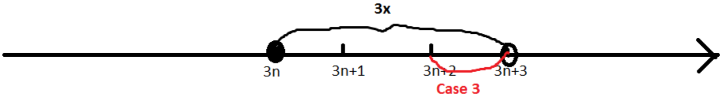


- Case 3. $3n + 2 \leqslant 3x < 3n + 3$

- Case 2. $3n + 1 \leqslant 3x < 3n + 2$



- Case 3. $3n + 2 \leqslant 3x < 3n + 3$



- Case 1. $\leqslant x <$ $\Leftrightarrow 3n \leqslant 3x < 3n+1$
  - $\lfloor 3x \rfloor =$: $3n \leqslant 3x < 3n+1$
  - $\lfloor x \rfloor =$: $\leqslant x <$.
  - $\lfloor x + 1/3 \rfloor =$: $\leqslant x + 1/3 <$.
  - $\lfloor x + 2/3 \rfloor =$: $\leqslant x + 2/3 <$.
- Case 2. $\leqslant x <$ $\Leftrightarrow 3n+1 \leqslant 3x < 3n+2.$
  - $\lfloor 3x \rfloor =$: $3n+1 \leqslant 3x < 3n+2$
  - $\lfloor x \rfloor =$: $\leqslant x <$.
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  - $\lfloor x \rfloor =$: $n \leqslant x <$.
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- Case 1. $n \leqslant x < n + 1/3 \Leftrightarrow 3n \leqslant 3x < 3n + 1$
  - $\lfloor 3x \rfloor =$: $3n \leqslant 3x < 3n + 1$
  - $\lfloor x \rfloor =$: $n \leqslant x < n + 1/3 <$.
  - $\lfloor x + 1/3 \rfloor =$: $\leqslant x + 1/3 <$.
  - $\lfloor x + 2/3 \rfloor =$: $\leqslant x + 2/3 <$.
- Case 2. $\leqslant x <$ $\Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2.$
  - $\lfloor 3x \rfloor =$: $3n + 1 \leqslant 3x < 3n + 2$
  - $\lfloor x \rfloor =$: $\leqslant x <$.
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  - $\lfloor 3x \rfloor = \quad : \; 3n \leqslant 3x < 3n + 1$
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  - $\lfloor x + 1/3 \rfloor = \quad : \qquad \leqslant x + 1/3 < \qquad .$
  - $\lfloor x + 2/3 \rfloor = \quad : \qquad \leqslant x + 2/3 < \qquad .$
- <u>Case 2. $n + 1/3 \leqslant x < n + 2/3 \Leftrightarrow 3n + 1 \leqslant 3x < 3n + 2.$</u>
  - $\lfloor 3x \rfloor = \quad : \; 3n + 1 \leqslant 3x < 3n + 2$
  - $\lfloor x \rfloor = \quad : \quad < n + 1/3 \leqslant x < n + 2/3 < \qquad .$
  - $\lfloor x + 1/3 \rfloor = \quad : \qquad \leqslant x + 1/3 < \qquad .$
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- <u>Case 3. $n + 2/3 \leqslant x < \qquad \Leftrightarrow 3n + 2 \leqslant 3x < 3n + 3.$</u>
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  - $\lfloor 3x \rfloor = 3n$: $3n \leqslant 3x < 3n + 1$
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  - $\lfloor x \rfloor =$: $< n + 2/3 \leqslant x < n + 1.$
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Answer.

Case (i) $n \leqslant x < n + \frac{1}{3}$: $\lfloor 3x \rfloor = 3n$,
$\lfloor x \rfloor = \lfloor x + \frac{1}{3} \rfloor = \lfloor x + \frac{2}{3} \rfloor = n$.

Case (ii) $n + \frac{1}{3} \leqslant x < n + \frac{2}{3}$: $\lfloor 3x \rfloor = 3n + 1$, $\lfloor x \rfloor = \lfloor x + \frac{1}{3} \rfloor = n$,
$\lfloor x + \frac{2}{3} \rfloor = n + 1$.

Case (iii) $n + \frac{2}{3} \leqslant x < n + 1$: $\lfloor 3x \rfloor = 3n + 2$, $\lfloor x \rfloor = n$,
$\lfloor x + \frac{1}{3} \rfloor = \lfloor x + \frac{2}{3} \rfloor = n + 1$.