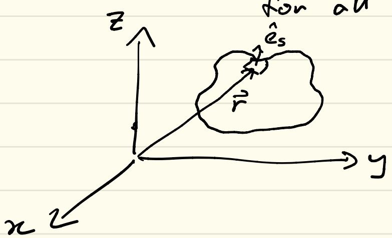
To solve a neutron transport problem, we need
1. differential equation of the system
2. boundary conditions ✓
3. initial conditions ✓

Boundary conditions

- Vacuum boundary condition

$$\varphi(\vec{r}, E, \hat{\Omega}, t) = 0 \quad \text{for } \hat{\Omega}\cdot\hat{e}_s < 0$$

for all $\vec{r}$ on S



- Albedo boundary condition

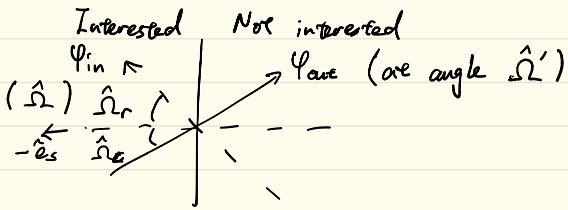
$$\varphi_{in}(\vec{r}, E, \hat{\Omega}, t) = \alpha(E)\,\varphi_{out}(\vec{r}, E, \hat{\Omega}', t)$$

$\alpha(E) = 1$ Reflective BC

$\alpha(E) = 0$ Vacuum BC

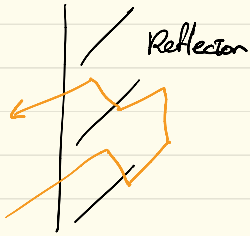
$0 < \alpha(E) < 1$ Some neutrons leave & some reflect back

$$\hat{e}_s\cdot\hat{\Omega} = -\hat{e}_s\cdot\hat{\Omega}'$$
$$(\hat{\Omega}\times\hat{\Omega}')\cdot\hat{e}_s = 0$$



- White boundary condition

All neutrons passing out of the volume V over the surface S will return with an angular distribution that is isotropic

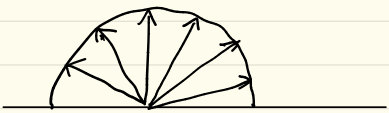


This is appropriate for a thick scattering medium assumed at the boundary.

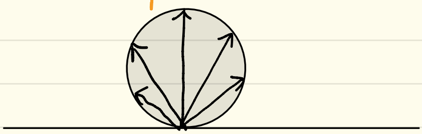
$$\psi(\vec{r}, E, \hat{\Omega}, t) = \int_{\hat{e}_s \cdot \hat{\Omega}' > 0} \frac{|\hat{e}_s \cdot \hat{\Omega}'|}{\pi} \psi(\vec{r}, E, \hat{\Omega}', t)\, d\hat{\Omega}'$$

$\forall \hat{e}_s \cdot \hat{\Omega} < 0$ and $\vec{r}$ on S

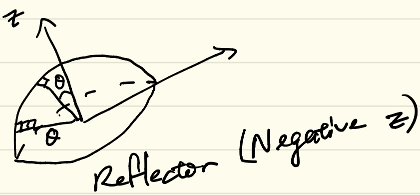
X "Isotropic" $\frac{1}{2\pi}$



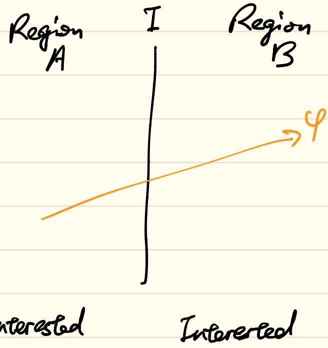
Isotropic $\frac{|\hat{\Omega} \cdot \hat{n}|}{\pi}$



For small θ, there are fewer "unit solid angles" than large θ.

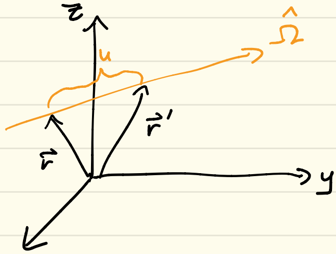


Interface condition



$$\varphi_b(\vec{r}_I, E, \hat{\Omega}, t) = \varphi_a(\vec{r}_I, E, \hat{\Omega}, t)$$

Transport equation in one dimension



Neutron streams some distance u from $\vec{r}$ to $\vec{r}\,'$ along a characteristic path in the direction $\hat{\Omega}$.

$$\frac{\partial\varphi}{\partial u} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial u}$$

$$= \mu\frac{\partial\varphi}{\partial x} + \eta\frac{\partial\varphi}{\partial y} + \xi\frac{\partial\varphi}{\partial z} \qquad = \hat{\Omega}\cdot\nabla\varphi$$

$$\mu = \cos\theta, \quad \eta = \sin\theta\sin\gamma, \quad \xi = \sin\theta\cos\gamma$$

For one dimension in Cartesian geometry,

Integrated by γ from 0 to 2π

$$\left[\ \frac{1}{v}\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial x} + \Sigma_t\varphi = \int_{-1}^{1}\int_0^\infty \Sigma_s(x, E'\to E, \mu'\to\mu)\varphi(x, E', \mu', t)\,dE'\,d\mu' + Q_{ex}(x, E, \mu, t)\right.$$

For isotropic scattering, $\Sigma_s(x, E'\to E)$

normalization factor

$$\int \Sigma_s(x, E'\to E, \mu'\to\mu)\,d\mu = \int_{-1}^{1} c\,\Sigma_s(x, E'\to E)\,d\mu$$

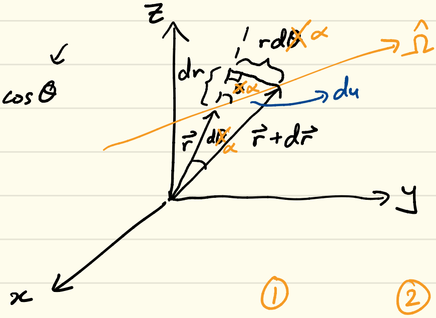
$$= \underline{2c}\,\Sigma_s(x, E'\to E)$$

$$\Rightarrow c = \frac{1}{2}$$

$$\frac{1}{2}\int_{-1}^{1}\int_{0}^{\infty} \Sigma_s(x, E'\to E)\,\varphi(x,E',\mu',t)\,dE'\,d\mu'$$

In spherical geometry

$r\,,\ \mu = \cos\theta$



① $\dfrac{\partial r}{\partial u} = \cos\theta = \mu$

$$-\frac{r\,d\theta}{du} = \sin\theta = \sqrt{1-\mu^2}$$

② $\dfrac{\partial \mu}{\partial u} = \dfrac{d(\cos\theta)}{\partial u}$

$$= \frac{\partial(\cos\theta)}{\partial\theta}\frac{\partial\theta}{\partial u}$$

$$= \frac{\partial(\cos\theta)}{\partial\theta}\frac{\sin\theta}{-r}$$

$$= \frac{\sin^2\theta}{r}$$

$$= \frac{1-\mu^2}{r}$$

$$\Rightarrow \frac{\partial\varphi}{\partial u} = \frac{\partial\varphi}{\partial r}\overset{①}{\frac{\partial r}{\partial u}} + \frac{\partial\varphi}{\partial\mu}\overset{②}{\frac{\partial\mu}{\partial u}}$$

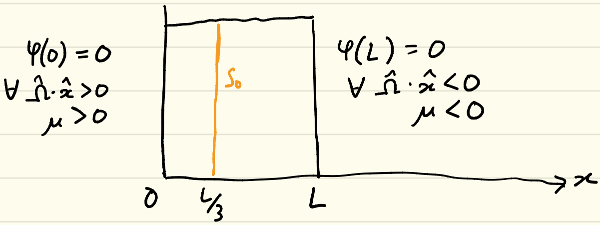
$$= \frac{\partial\varphi}{\partial r}\mu + \frac{\partial\varphi}{\partial\mu}\frac{(1-\mu^2)}{r}$$

Streaming term in spherical geometry

Full form is in B & G 1.7

Example Problem

Solve for the angular flux in a purely absorbing slab with a plane isotropic source at $x = L/3$. Assume vacuum BCs on both sides of the slab.



$$\hat{\Omega}\cdot\nabla\psi(\vec{r},\hat{\Omega}) + \Sigma_t\psi(\vec{r},\hat{\Omega}) = \int_{4\pi} d\hat{\Omega}'\,\cancelto{0}{\Sigma_s}(\vec{r},\hat{\Omega}'\to\hat{\Omega})\psi(\vec{r},\hat{\Omega}') + S(\vec{r},\hat{\Omega})$$

$$\mu\frac{\partial\psi}{\partial x} + \Sigma_a\psi(x,\mu) = S(x,\mu)$$

The integrating factor is $e^{\Sigma_a x/\mu}$

$$\frac{\partial}{\partial x}\left[\psi(x,\mu)\,e^{\Sigma_a x/\mu}\right] = \frac{S(x,\mu)}{\mu}\,e^{\Sigma_a x/\mu}$$

The isotropic plane source can be expressed as

$$S(x,\cancel{\mu}) = \frac{\int_0^1 S_0\,\delta(x-L/3)\,d\mu}{\int_{-1}^1 d\mu} \qquad \mu > 0$$

$$\left(\int_{-1}^0 S_0\,\delta(x-L/3)\,d\mu\right) \qquad \mu < 0$$

$$= \frac{S_0}{2}\,\delta(x-L/3) \quad ✓$$

$$= \begin{cases} \dfrac{S_0}{2} & x = L/3 \\ 0 & x \neq L/3 \end{cases}$$

For $\mu > 0$, we integrate between 0 and x

$$\underset{\mu>0}{\varphi^+(x,\mu)}\, e^{\Sigma_a x/\mu} - \overset{0\ (BC)}{\cancel{\varphi^+(0,\mu)}}\, e^0 = \int_0^x \frac{S(x',\mu)}{\cancel{\mu}} e^{\Sigma_a x'/\mu}\, dx'$$

$$\int_0^x \frac{S_0}{2} \delta\left(x' - \frac{L}{3}\right) e^{\Sigma_a x'/\mu}\, dx'$$

$$\Rightarrow \varphi^+(x,\mu) = \begin{cases} 0 & x < \frac{L}{3} \\ \frac{S_0}{2\mu} e^{-\Sigma_a (x - L/3)/\mu} & x > \frac{L}{3} \end{cases}$$

For $\mu < 0$, we integrate from L to x

$$\varphi^-(x,\mu)\, e^{\Sigma_a x/\mu} - \overset{0}{\cancel{\varphi^-(L,\mu)}}\, e^{\Sigma_a L/\mu} = \int_L^x \frac{S(x',\mu)}{\mu} e^{\Sigma_a x'/\mu}\, dx'$$

$$\Rightarrow \varphi^-(x,\mu) = \begin{cases} 0 & x > \frac{L}{3} \\ \frac{S_0}{2|\mu|} e^{\Sigma_a (x - L/3)/|\mu|} & x < \frac{L}{3} \end{cases}$$