Lets consider our $P_3$ equations

$n=0$

$$\frac{d\phi_1(x)}{dx} + \Sigma_0(x)\,\phi_0(x) = Q_0(x)$$

0 from isotropic source

$n=1$

$$\frac{2\,d\phi_2(x)}{dx} + \frac{d\phi_0(x)}{dx} + 3\Sigma_1(x)\,\phi_1(x) = 3\cancel{Q_1}(x)$$

$n=2$

$$\frac{3\,d\phi_3(x)}{dx} + \frac{2\,d\phi_1(x)}{dx} + 5\Sigma_2(x)\,\phi_2(x) = 5\cancel{Q_2}(x)$$

$n=3$

$$\cancel{\frac{4\,d\phi_4(x)}{dx}} + \frac{3\,d\phi_2(x)}{dx} + 7\Sigma_3(x)\,\phi_3(x) = 7\cancel{Q_3}(x)$$

0 from closure condition

* we are going to solve for $\phi_1$ and $\phi_3$ from $n=1$ and $n=3$ equations and substitute them in to our other equations.

$$-\frac{2 d^2\phi_2(x)}{3\Sigma_1(x)\,dx^2} - \frac{1}{3\Sigma_1}\frac{d^2\phi_0(x)}{dx^2} + \Sigma_0\phi_0(x) = Q_0(x)$$

$$\underbrace{-\frac{9 d^2\phi_2(x)}{7\Sigma_3\,dx^2}}_{\frac{d}{dx}\frac{9}{7\Sigma_3(x)}\frac{d\phi_2(x)}{dx}} - \frac{4 d^2\phi_0(x)}{3\Sigma_1\,dx^2} + 5\Sigma_2\phi_2(x) = 0$$

if our materials in our problem change we may want to not collapse our derivative term.

we chose our discretization such that the $\Sigma$ does not vary <u>within a cell</u>, so it is a constant inside a cell.

check this. There should be a $\phi_0$ term in the eq. somewhere

We can step back further...

$$\frac{d\phi_1(x)}{dx} + \Sigma_0 \phi_0(x) = Q_0(x)$$

$$\alpha_1 \frac{d\phi_2}{dx} + \beta_1 \frac{d\phi_0(x)}{dx} + \Sigma_1 \phi_1(x) = Q_1$$

$$\alpha_{N-1} \frac{d\phi_N}{dx} + \beta_{N-1} \frac{d\phi_{N-2}(x)}{dx} + \Sigma_{N-1}\phi_{N-1}(x) = Q_{N-1}$$

$$0 + \beta_N \frac{d\phi_{N-1}(x)}{dx} + \Sigma_N \phi_N(x) = Q_N$$

from closure condition.

$$\overline{\overline{\Phi}}_e(x) = \begin{bmatrix} \phi_0(x) \\ \phi_2(x) \\ \vdots \\ \phi_{N-1}(x) \end{bmatrix} \qquad \overline{\Phi}_0(x) = \begin{bmatrix} \phi_1 \\ \phi_3 \\ \vdots \\ \phi_N(x) \end{bmatrix}$$

$$(1) \quad \overline{\overline{A}}_1 \frac{d}{dx} \overline{\Phi}_0(x) + \overline{\overline{B}}_1 \overline{\Phi}_e(x) = \overline{Q}_e(x)$$

$$(2) \quad \overline{\overline{A}}_2 \frac{d}{dx} \overline{\Phi}_e(x) + \overline{\overline{B}}_2 \overline{\Phi}_0(x) = \overline{Q}_0(x)$$

$\bar{\bar{B}}_1$ and $\bar{\bar{B}}_2$ are nonsingular and diagonal

$\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$ are tridiagonal

$B_1$ and $B_2$ are functions of $x$

$A_1$ and $A_2$ are coefficient matrices which are <u>not</u> functions of $x$.

Solving for the odd harmonics from eq 2

$$\bar{\Phi}_o(x) = -\bar{\bar{B}}_2^{-1} \bar{\bar{A}}_2 \frac{d}{dx} \bar{\Phi}_e + \bar{\bar{B}}_2^{-1} Q_o$$

and then inserting into equation 1:

$$-\frac{d}{dx}\left(\bar{\bar{A}}_1 \bar{\bar{B}}_2^{-1} \bar{\bar{A}}_2\right) \frac{d}{dx} \bar{\Phi}_e(x) + \bar{\bar{B}} \bar{\Phi}_e(x) = \bar{Q}_e \frac{d}{dx} \left(\bar{\bar{A}}_1 \bar{\bar{B}}^{-1} Q_B\right)$$

The $P_3$ equations in this matrix form look like:

$$-\frac{d}{dx}\left[\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} \frac{1}{3\Sigma_1} & \frac{2}{3\Sigma_1} \\ 0 & \frac{3}{7\Sigma_3} \end{bmatrix} \frac{d}{dx}\begin{bmatrix} \phi_0(x) \\ \phi_2(x) \end{bmatrix}\right] + \begin{bmatrix} \Sigma_0 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} \phi_0(x) \\ \phi_2(x) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Q_0(x) \\ Q_2(x) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -5 & -5 \end{bmatrix} \frac{d}{dx}\begin{bmatrix} \frac{1}{\Sigma_1} & 0 \\ 0 & \frac{1}{\Sigma_3} \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_3 \end{bmatrix}$$

multiplying out these and separating into equations:

$$-\frac{d}{dx}\left[\frac{1}{3\Sigma_1}\frac{d}{dx}\left(\phi_0(x)-2\phi_2(x)\right)\right]+\Sigma_0\phi_0(x)=Q_0(x)-\frac{d}{dx}\frac{Q_1}{\Sigma_1}$$

and

$$-\frac{d}{dx}\left[\frac{2}{15}\Sigma_1\frac{d}{dx}\phi_0(x)+\left(\frac{4}{15\Sigma_1}+\frac{9}{35\Sigma_3}\right)\frac{d}{dx}\phi_2(x)\right]$$

$$+\Sigma_2\phi_2(x)=Q_2(x)\frac{d}{dx}\frac{Q_1(x)}{\Sigma_1}-\frac{3}{5}\frac{d}{dx}\frac{Q_3(x)}{\Sigma_3}$$

If we say $F_0=\phi_0+2\phi_2$

and $F_1=\phi_2$

Our equations become:

$$-\frac{d}{dx}\left[\frac{1}{3\Sigma_1}\frac{d}{dx}F_0(x)\right]+\Sigma_0F_0(x)=Q_0(x)+2\Sigma_0F_1 \tag{I}$$

$$-\frac{d}{dx}\left[\frac{2}{15\Sigma_1}\frac{d}{dx}F_0(x)+\frac{9}{35\Sigma_3}\frac{d}{dx}F_1(x)\right]+\Sigma_2F_1(x)=0 \tag{II}$$

$\Downarrow$ rearranging

$$-\frac{d}{dx}\left[\frac{1}{3\Sigma_1}\frac{dF_0(x)}{dx}\right]+\Sigma_0F_0(x)=Q_0(x)+2\Sigma_0F_1(x) \tag{I}$$

$$-\frac{d}{dx}\left[\frac{9}{35}\Sigma_3\frac{d}{dx}F_1(x)\right]+\left(\frac{4}{5}\Sigma_0+\Sigma_2\right)F_1(x)=\frac{2}{5}\Sigma_0F_0(x)-\frac{2}{5}Q_0(x) \tag{II}$$

Recall we used $F_1$ and $F_2$ to simplify these equations:

$$F_0 = \Phi_0 + 2\Phi_2$$
$$F_1 = \Phi_2$$

we can iteratively solve these equations by assuming an initial value of $F_1(x)$ (typically this is 0), then using it to solve for $F_0$, and then $F_1$.

Lets see what that looks like with our equations. Let $n$ be the iteration number:

$$-\frac{d}{dx}\left[\frac{1}{3\Sigma_1}\frac{d}{dx}F_0^n(x)\right] + \Sigma_0 F_0^n(x) = Q_0(x) + 2\Sigma_0 F_1^{n-1}(x)$$

So in the first iteration we use our initial value of $F_1(x)$ (the 0th iteration, or n-1) to solve for $F_0(x)$ (in the 1st iteration, or n).

↓

Then we use this value of $F_0^n(x)$ to solve for $F_1^n$ in our next equation:

$$-\frac{d}{dx}\left[\frac{9}{35\Sigma_3}\frac{d}{dx}F_1^n(x)\right] + \left(\frac{4}{5}\Sigma_0 + \Sigma_2\right)F_1^n(x) = \frac{2}{5}\Sigma_0 F_0^n(x) - \frac{2}{5}Q_0(x)$$

In steps:

1. assume an initial value for $f_1^{\,0}(x) = 0$
2. solve for $f_0^{\,1}(x)$
3. solve for $f_1^{\,1}(x)$
4. Check against a convergence criteria

$$f_0^{\,n} - f_0^{\,n-1} < \varepsilon$$ where $\varepsilon$ is some threshold

or $$\frac{f_0^{\,n} - f_0^{\,n-1}}{f_0^{\,n-1}} < \eta$$ other threshold

$$f_1^{\,n} - f_1^{\,n-1} < \varepsilon$$

5. if not converged, iterate again
   if converged, stop.

$$\left.\begin{aligned} F_0(x) &= \Phi_0(x) + 2\Phi_2(x) \\ F_1(x) &= \Phi_2(x) \end{aligned}\right\}$$ these $\Phi(x)$ are vectors for spatial differenced solution.