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NPRE 560 Oleksandr Yardas Fall 2024
HW 4 Due 2024.11.18

- Show your work.
- This work must be submitted online as a `.pdf` through Canvas.
- Work completed with LaTeX or Jupyter earns 1 extra point. Submit source file (e.g. `.tex` or `.ipynb`) along with the `.pdf` file.
- If this work is completed with the aid of a numerical program (such as Python, Wolfram Alpha, or MATLAB) all scripts and data must be submitted in addition to the `.pdf`.
- If you work with anyone else, document what you worked on together.

Worked with Nathan Ryan and Anthony Boyd

1. (Hetrick 6-3, 6-4, 6-5, 6-6, 6-7, 6-8) Below, G(s)H(s) is the open-loop transfer function. For each, plot the stable region for the closed-loop system in the $K(\alpha)$ plane.

   Poles of the closed-loop transfer function, $Y(s)$, characterize the stability of the system. If the poles of $Y(s)$ are in the left-hand half of the complex plane. An equivalent condition is that the roots of the characteristic equation $D(s) = P_G(s)P_H(s) + D_G(s)D_H(s)$ are all negative and real. This in turn, is equivalent to the condition than all of the Routh numbers of the characteristic polynomial are positive, that is, all of the numbers in first column of the Routh array are positive. If $D(s)$ has the form

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \dots$$

   then the Routh array is given by

$$\begin{array}{ccc} a_n & a_{n-2} & \dots \\ a_{n-1} & a_{n-3} & \dots \\ b_1 & \dots & \end{array}$$

   where the Routh numbers are the first column of this array. The last Routh number is $a_0$, which is appended to the end of the first column. $b_1$ is given by

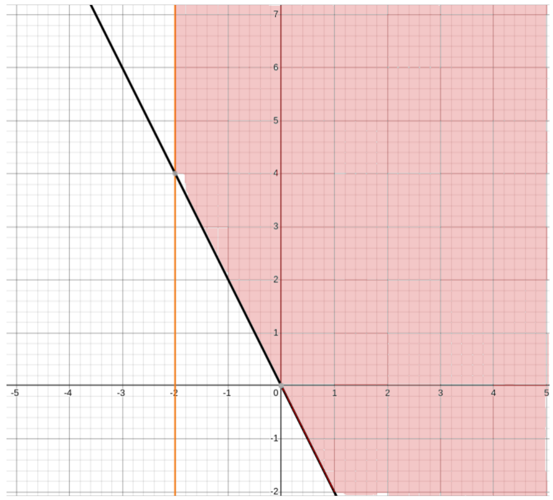
$$b_1 = a_{n-2} - \frac{a_n a_{n-3}}{a_{n-1}}$$

   In practice, we form $D(s)$ by taking the sum of the numerator and the denominator of $G(s)H(s)$ $G(s)H(s) = \frac{K}{(s+2)(s+\alpha)}$

   (a) (5 points) **Solution:** $D(s) = s^2 + (2+\alpha)s + 2\alpha + K$. We don't need to calculate $b_1$ since $a_{n-3} = 0$. So our Routh array is

$$\begin{array}{c} 1 \\ 2+\alpha \\ 2\alpha + K \end{array}$$

So the region of stability is for $\alpha > -2$ and $K > -2\alpha$:

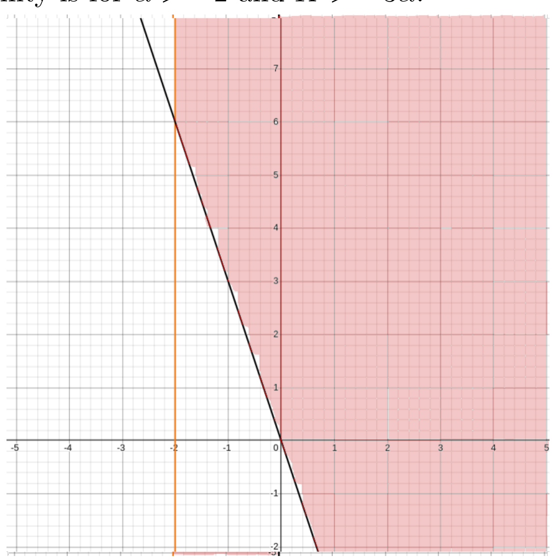


(b) (5 points) $G(s)H(s) = \frac{K+\alpha}{(s+2)(s+\alpha)}$

**Solution:** $D(s) = s^2 + (2+\alpha)s + 3\alpha + K$. Again, $a_{n-3} = 0$ so we can immediately get out Routh numbers:

$$\begin{array}{c} 1 \\ 2+\alpha \\ 3\alpha + K \end{array}$$

So the region of stability is for $\alpha > -2$ and $K > -3\alpha$:

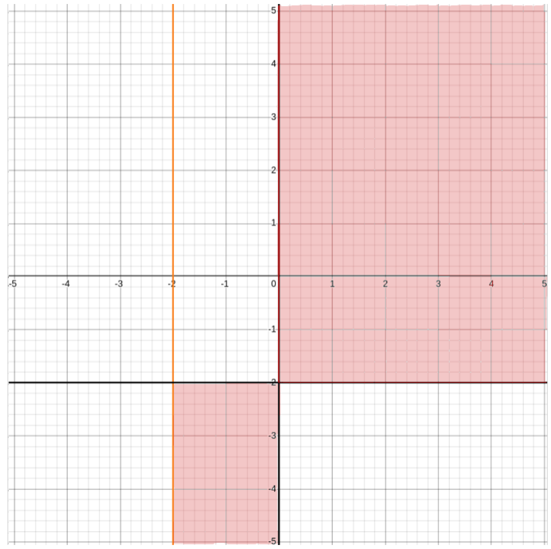


(c) (5 points) $G(s)H(s) = \frac{K\alpha}{(s+2)(s+\alpha)}$

**Solution:** $D(s) = s^2 + (2+\alpha)s + \alpha(2+K)$. Again, $a_{n-3} = 0$ so we can immediately get out Routh numbers:

$$\begin{array}{c} 1 \\ 2+\alpha \\ \alpha(K+2) \end{array}$$

So the region of stability is for $\alpha > -2$ and $K > -2$



(d) (5 points) $G(s)H(s) = \frac{K+\alpha}{(s+2)^2(s+\alpha)}$

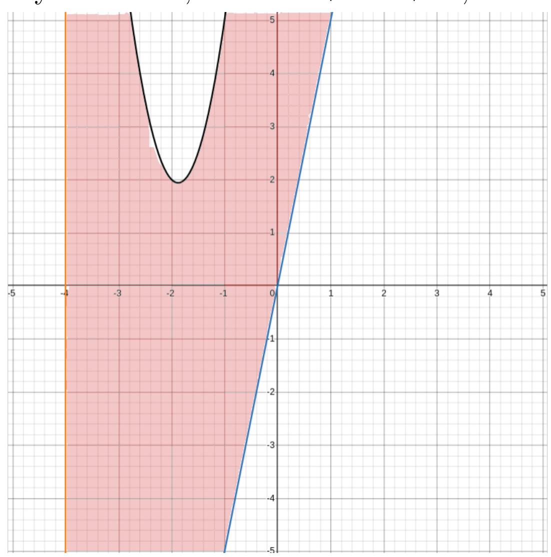
**Solution:** $D(s) = s^3 + (4+\alpha)s^2 + 4(1+\alpha)s + 5\alpha + K$, so $b_1 = 4(1+\alpha) - \frac{5\alpha+K}{4+\alpha}$ The Routh array is then

$$\begin{array}{cc} 1 & 4(1+\alpha) \\ 4+\alpha & 5\alpha+K \\ 4(1+\alpha) - \frac{5\alpha+K}{4+\alpha} & 0 \\ 5\alpha+K & \end{array}$$

So the region of stability is $\alpha > -4$, $K < 4\alpha^2 + 15\alpha + 16$, and $K > 5\alpha$

slightly off

-2

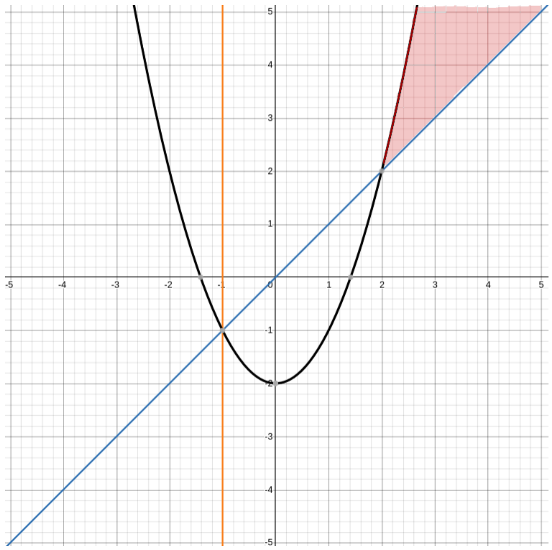


(e) (10 points) $G(s)H(s) = \frac{K+\alpha}{(s-1)(s+2)(s+\alpha)}$

**Solution:** $D(s) = s^3 + (alpha+1)s^2 + (\alpha-2)s - \alpha + K$, so $b_1 = \alpha - 2 - \frac{K-\alpha}{\alpha+1}$. The Routh array is then

$$\begin{array}{cc} 1 & \alpha - 2 \\ 1+\alpha & K-\alpha \\ \alpha - 2 - \frac{K-\alpha}{\alpha+1} & 0 \\ K-\alpha & \end{array}$$

So the region of stability is $\alpha > -1$, $K > \alpha$, and $K < \alpha^2 - 2$.

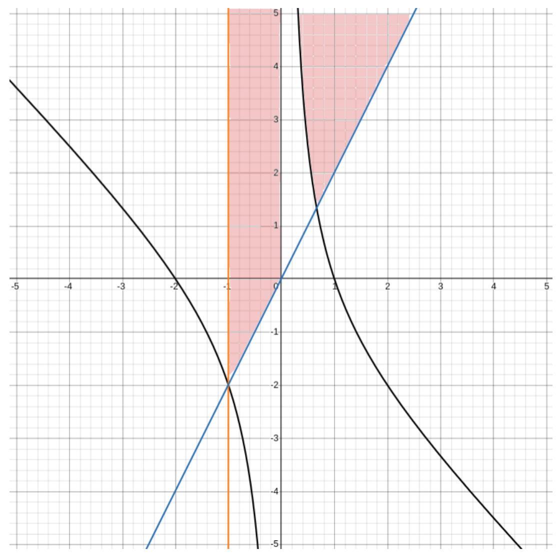



(f) (10 points) $G(s)H(s) = \frac{K(s+1)}{(s-1)(s+2)(s+\alpha)}$

**Solution:** $D(s) = s^3 + (\alpha+1)s^2 + (K+\alpha-2)s - 2\alpha + K$, so $b_1 = K + \alpha - 2 - \frac{K-2\alpha}{1+\alpha}$. The Routh array is then

$$\begin{array}{cc} 1 & K+\alpha-2 \\ \alpha+1 & K-2\alpha \\ K+\alpha-2-\frac{K-2\alpha}{1+\alpha} & 0 \\ K-2\alpha & \end{array}$$

So the region of stability is $\alpha > -1$, $K > 2\alpha$, and $K > \frac{2}{\alpha} - \alpha - 1$



2. (20 points) (D&H 6-26) Explicitly perform the Laplace transform inversion of the zero power transfer function, $Z(s)$ to obtain the impulse response function $\mathscr{Z}(t)$.

**Solution:** The zero power transfer function is

$$Z(s) = \frac{1}{s}\left[\Lambda + \sum_{j=1}^{6} \frac{\beta_j}{s+\lambda_j}\right]^{-1}$$

The impulse response function is

$$\mathscr{Z}(t) = \frac{1}{\Lambda} + \sum_{j=2}^{7} \frac{e^{s_j t}}{s_j\left[\Lambda + \sum_{i=1}^{6} \frac{\beta_i \lambda_i}{(s_j+\lambda_i)^2}\right]}$$

To get $\mathscr{Z}(t)$ from $Z(s)$, we need to perform an inverse Laplace transform on $Z(s)$. The inverse Laplace transform has the form:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{st} g(s) ds$$

...

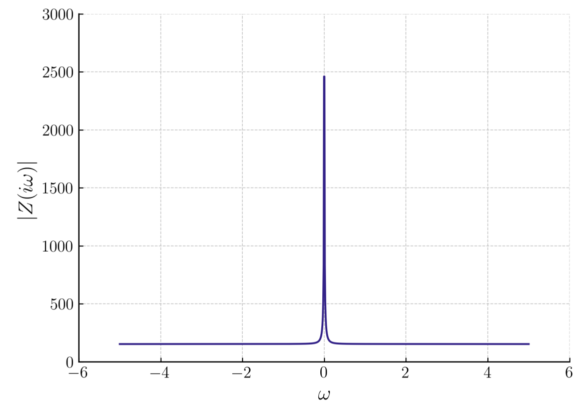
3. (30 points) (D&H 6-23) Calculate the amplitude and phase angle for the zero power transfer function and plot these as functions of frequency. Assume one group of delayed neutrons with $\beta = 0.0065$, $\lambda = 0.08 s^{-1}$, and $\Lambda = 10^{-4} s$.

**Solution:** The one-group zero-power transfer function is

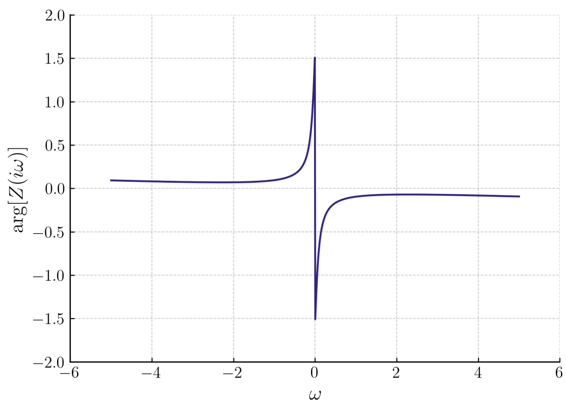
$$Z(s) = \frac{1}{s}\left[\Lambda + \frac{\beta}{s+\lambda}\right]^{-1}$$

From D&H 6.II.D.3, we have that the amplitude is $|Z(i\omega)| = \sqrt{\mathrm{Re}^2[Z(i\omega)] + \mathrm{Im}^2[Z(i\omega)]}$, and the phase angle is given by $\arctan(\frac{\mathrm{Im}[Z(i\omega)]}{\mathrm{Re}[Z(i\omega)]})$. In the context of complex numbers, this is sometimes called the arg function or the $\arctan 2$ function.

Plots of the magnitude and phase angle are below:




This makes more sense in rads.




4. (10 points) (D&H 6-32) In a prompt critical reactor excursion, a large amount of reactivity (measured above prompt critical) $\rho_0$ is instantaneously inserted in an equillibrium reactor at $t = 0$. Assume that:

   - the effect of delayed neutrons is negligible on the time scales under consideration.
   - the reactor shuts itself down by thermal expansion of the core in such a way that negative reactivity is "added" proportional to the total heat energy generated up to time $t$. That is, $\rho = \rho_0 - \gamma \int_0^t P(t')dt'$.

Find the power level $P(t)$ where $t$ is measured from the time of reactivity insertion, and is measured in units of the prompt neutron lifetime. [This is known as the Fuchs-Hansen model of reactor excursion.]

**SOLUTION:** (The `solution` environment was cutting off my answer, so I'm doing it out of the environment) In the prompt case neglecting delayed neutrons, our point reactor kinetic equation is simply

$$\frac{dP}{dt} = \frac{\rho(t)}{\Lambda}P(t)$$

Expanding $\rho(t) = \rho_0 - \gamma \int_0^t P(t')dt'$, we get

$$\frac{dP}{dt} = \left[\frac{\rho_0}{\Lambda} - \frac{\gamma}{\Lambda}\int_0^t P(t')dt'\right]P(t) \tag{1}$$

Now, letting $a \equiv \frac{p_0}{\Lambda}$ and $b \equiv \frac{\gamma}{\Lambda}$, define $y(t) = a - b\int_0^t P(t')dt'$. We can then rewrite Equation 1 as

$$\frac{dP}{dt} = P(t)y(t) \tag{2}$$

Now, notice that $\frac{dy}{dt} = -bP(t)$, and $\frac{d^2y}{dt^2} = -b\frac{dP}{dt}$. Substituting in Equation 2 into the $\frac{d^2y}{dt^2}$, and using $\frac{dy}{dt} = -bP(t)$, we get

$$\frac{d^2y}{dt^2} = y\frac{dy}{dt}$$

Integrating both sides with resepect to $t$ yields

$$\frac{dy}{dt} = \int y\frac{dy}{dt}dt$$

Using integration by parts on the RHS, we get $\int y\frac{dy}{dt}dt = \frac{1}{2}y^2 + C$, where $C$ is some constant. Since the constant is arbitrary, we can redefine it as anything we want. Suppose we redefine the constant as $-\frac{1}{2}c^2$. We then have that

$$\frac{dy}{dt} = \frac{1}{2}(y^2 - c^2) \tag{3}$$

to find $c$, note that $\left.\frac{dy}{dt}\right|_{t=0} = -bP_0 = \frac{1}{2}(y(0)^2 - c^2)$. Based on the definition of $y(t)$, we have that $y(0) = a$, so we are left with the equality $-bP_0 = \frac{1}{2}(a^2 - c^2)$. Solving for $c$ yields

$$c = \sqrt{a^2 + 2bP_0}$$

Now, we have $y$ in a form that we cannot solve directly. Let's reformulate $y$ in terms of another function $x(t)$:

$$y(t) = \frac{1}{x(t)} + c \tag{4}$$

Note that $x(0) = \frac{1}{a-c}$ Substituing this definition into Equation 3, we get

$$-\frac{1}{x^2(t)}\frac{dx}{dt} = \frac{1}{2}(\frac{1}{x^2} + 2cx(t))$$

Rearranging terms yields

$$\frac{dx}{dt} + cx = -\frac{1}{2}$$

The homogenous part of the solution is $u(0)e^{-ct}$. The non-homogenous part takes some guesswork but assuming a solution of the form $A(1 - e^{-ct})$ yields $A = -\frac{1}{2c}$, so we have

$$x(t) = \frac{1}{a - c} e^{-ct} - \frac{1}{2c}(1 - e^{-ct})$$

Which we can rearrange to

$$x(t) = -\frac{1}{2c}(1 + \frac{c + a}{c - a} e^{-ct})$$

Interting this into Equation 4, we get

$$y(t) = -2c(1 + \frac{c + a}{c - a} e^{-ct})^{-1} + c$$

Substituting this into $\frac{dy}{dt} = -bP(t)$, we get

$$\frac{-2c^2 Ae^{-ct}}{(1 + Ae^{-ct})^2} = -bP(t)$$

where $A \equiv \frac{c+a}{c-a}$. Solving for $P$, we get

$$P(t) = \frac{2c^2 Ae^{-ct}}{b(1 + Ae^{-ct})^2}$$