Name:____________________________________

Worksheet 05

1 New Object - Vectors

Definition A *vector space* is a set, whose elements we call *vectors*, along with a field of *scalars*, where the following operations are well defined:

- vector addition (we can add two vectors to get a new vector: $\vec{u} + \vec{v} = \vec{w}$)
- scalar multiplication (we can multiply a vector by a scalar to get a new vector: $c\vec{r} = \vec{s}$)

Example We can take elements of $\mathbb{R}^4$ to be our set of vectors, and let $\mathbb{C}$ be our field of scalars. Usually, we would denote elements of $\mathbb{R}^4$ as tuples, like: (a_1, a_2, a_3, a_4). In our textbook, to emphasize that these elements are new objects, we write them as $< a_1, a_2, a_3, a_4 >$. The standard way to define the required operations as follows:

- vector addition: $< a_1, a_2, a_3, a_4 > + < b_1, b_2, b_3, b_4 >=< a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 >$
- scalar multiplication: $(r + is) < a_1, a_2, a_3, a_4 >=< (r + is)a_1, (r + is)a_2, (r + is)a_3, (r + is)a_4 >$

2 Operations Between Vectors

Vectors spaces only require that we be able to add two vectors, and multiply vectors by scalars. However, now that we have new objects, it is sometimes interesting to see what other operations we can define.

Exercise 1. *In $\mathbb{R}^2$ we define the operation of standard vector addition by:*

$$< a_1, a_2 > + < b_1, b_2 >=< a_1 + b_1, a_2 + b_2 >$$

Define three new operations, $\heartsuit, \spadesuit$, and $\diamondsuit$ between vectors of $\mathbb{R}^2$. (Note: Define at least one operation whose output is a scalar, and at least one whose output is a vector.)

- $< a_1, a_2 > \heartsuit < b_1, b_2 >=$
- $< a_1, a_2 > \spadesuit < b_1, b_2 >=$
- $< a_1, a_2 > \diamondsuit < b_1, b_2 >=$

Definition In a vector space, V, with scalar field F, an *inner product* is a function

$$P : V \times V \to F$$

that satisfies the following:

1. $P(\vec{v} + c\vec{w}, \vec{z}) = P(\vec{v}, \vec{z}) + cP(\vec{w}, \vec{z})$ for all vectors $\vec{v}, \vec{w}$, and $\vec{z}$ in V, and any scalar, c, in F.
2. $P(\vec{v}, \vec{v}) \geq 0$ for all $\vec{v}$ in V (and $P(\vec{v}, \vec{v}) = 0$ if and only if $\vec{v} = \vec{0}$).
3. $P(\vec{v}, \vec{w}) = P(\vec{w}, \vec{v})$
   (Technically, if F is a field where "conjugate" make sense, the right hand side should be $\overline{P(\vec{w}, \vec{v})}$.)

Exercise 2. *Check if any of the operations you defined in Exercise 1 are inner products. If not, explain why and give an example.*

Exercise 3. *Consider the operation,* $\bullet$ *, between vectors in* $\mathbb{R}^3$ *defined by*

$$< a_1, a_2, a_3 > \bullet < b_1, b_2, b_3 >= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Is $\bullet$ *an inner product? Check all three conditions.*

The operation defined in Exercise 3 is, indeed an inner product on the vector space $(\mathbb{R}^3, \mathbb{R})$. An, since vector spaces of this form are especially popular, we give this inner product a special name.

Definition The *dot product* between vectors in $\mathbb{R}^n$ is defined by

$$< a_1, a_2, \ldots, a_n > \bullet < b_1, b_2, \ldots, b_n >= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Definition If (V, F) is a vector space with inner product, P, we define the *norm* of a vector, $\vec{v}$ by

$$||\vec{v}|| = \sqrt{P(\vec{v}, \vec{v})}.$$

Exercise 4. *Compute the following, with the dot product as the inner product.*

i) $|| < 1, 2, 1 > ||$

ii) $|| < a, b > ||$

iii) $|| < a, b > - < x, y > ||$

Exercise 5. *What is the physical interpretation of the norm of a vector?*

Exercise 6. *Let* $\vec{v} =< a, b >$ *and* $\vec{w} =< c, d >$ *be vectors in* $\mathbb{R}^2$*. Show that*

$$\vec{v} \bullet \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \cos(\theta)$$

where θ *is the angle between* $\vec{v}$ *and* $\vec{w}$ *if they start at the same point.*

- *Hint 1: Draw a diagram illustrating* $\vec{v}$ *and* $\vec{w}$*, with* θ *labeled between them. Then, on the same diagram, draw a representative of* $\vec{v} - \vec{w}$*. Then, label* $||\vec{v}||$ *and* $||\vec{w}||$*, and* $||\vec{v} - \vec{w}||$*.*
- *Apply the law of cosines to your diagram. (*$c^2 = a^2 + b^2 - 2ab\cos(C)$ *for a triangle with sides of length* a, b *and* c*, and* C *the angle opposite side the side of length* c*.)*