Worksheet 09

1 Projectile Motion

Exercise 1. Given a projectile with initial velocity $||\vec{v}(0)|| = v_0$, that is launched with an angle of elevation α , from the surface of a planet with gravitational constant g, derive a vector valued function describing the position vector with respect to time.

2 Kepler's First Law of Planetary Motion

First we recall two of Newton's Laws:

Law (Second Law of Motion).

$$\vec{F} = m\vec{a} \tag{1}$$

Where \vec{F} is force and m is mass.

Law (Law of Universal Gravitation Between a Planet and a Star).

$$\vec{F} = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\vec{u}.$$
 (2)

Where \vec{F} is the gravitational force on the planet, m and M are the masses of the planet and the star, respectively, \vec{r} is the position vector of the planet, and $\vec{u} = \frac{1}{r}\vec{r}$, where $r = ||\vec{r}||$.

Law (Kepler's First Law of Planetary Motion). A planet revolves around a star with an elliptical orbit with the star at one focus.

The proof of this law happens in two parts:

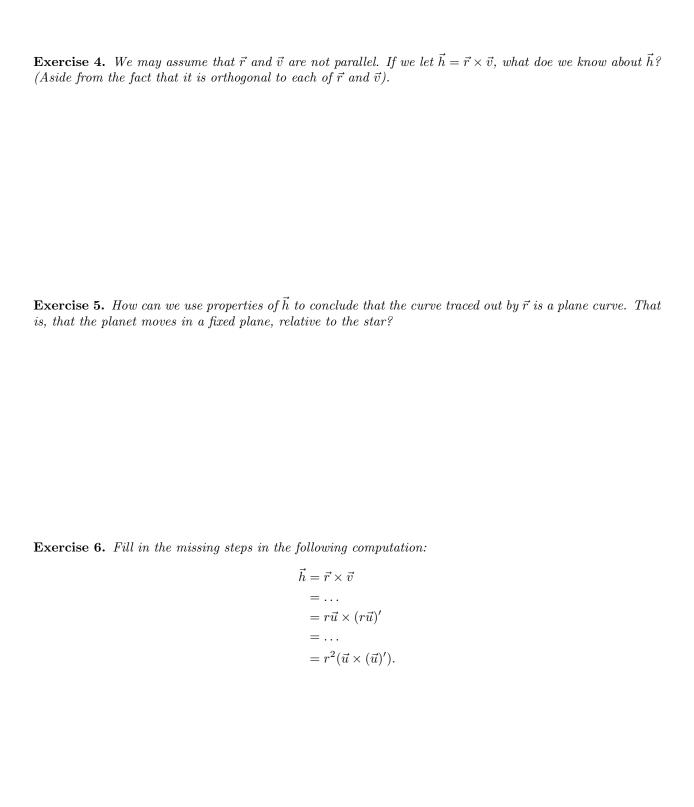
- 1. We show that the planet moves around the star in a fixed plane, relative to the star.
- 2. We show that the path it traces in this plane is an ellipse.

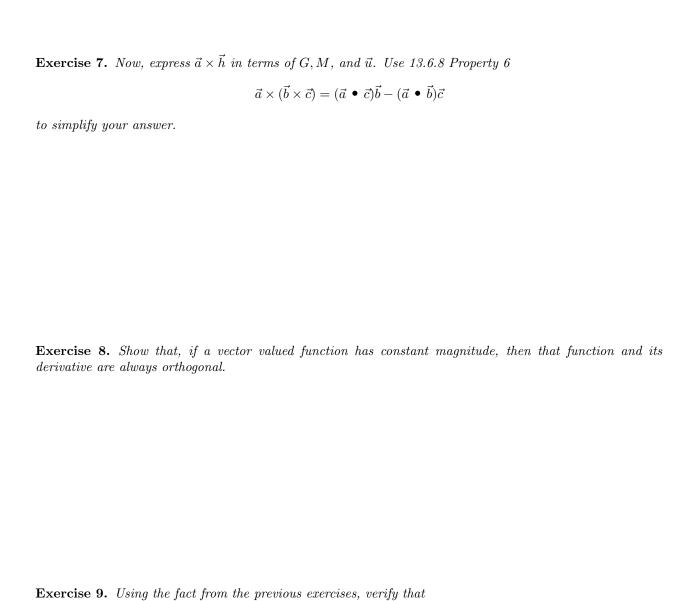
Exercise 2. Using equations (1) and (2), conclude that the position and acceleration vectors are parallel. What does this tell us about $\vec{r} \times \vec{a}$?

Exercise 3. Compute

$$\frac{d}{dt} \left(\vec{r} \times \vec{v} \right),\,$$

where \vec{v} is the velocity vector of the planet.





 $\vec{a} \times \vec{h} = GM(\vec{u})'$

 $(\vec{v} \times \vec{h})' = GM(\vec{u})'$

and, thus,

Note, that integrating both sides of the equation from Exercise 9 gives us

$$\vec{v} \times \vec{h} = GM\vec{u} + \vec{C} \tag{3}$$

where \vec{C} is a constant vector.

Until now, we haven't chosen our coordinate axes. To make life a bit more convenient, we will just a range our planet-star system so that the vector \vec{h} points in the standard z-axis direction. That is, we ensure that \vec{h} and \hat{k} are parallel.

Exercise 10. In which plane does the constant vector from equation (3) lie?

Exercise 11. If θ is the angle between \vec{r} and \vec{C} , then (r,θ) are the polar coordinates of the planet. Verify that

$$\vec{r} \bullet (\vec{v} \times \vec{h}) = GMr + rc\cos(\theta),$$

where $c = ||\vec{c}||$.

Now, we have that

$$r = \frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{GM + c\cos(\theta)} = \frac{1}{GM} \frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{1 + d\cos(\theta)}$$
(4)

where d = c/GM.

Exercise 12. Simplify the numerator of equation (4). (Denote $||\vec{h}|| = h$.)

Exercise 13. Let $f = h^2/d$. Simplify the equation from the previous exercise so that it is in terms of r, d, f and θ , only. Compare the result to Theorem 11.6.6 in your book.