Name:______________________________________

Worksheet 12

1 Directional Derivatives

1.1 Definitions and Basic Properties

To see how directional derivatives are a generalization of partial derivatives, compare the definitions:

Definition For a function, $f : \mathbb{R}^n \to \mathbb{R}$, we define the *partial derivative*, with respect to the i-th independent variable, at the point $(a_1, a_2, \ldots, a_n)$, by

$$f_{x_i}(a_1, a_2, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots a_n) - f(a_1, a_2, \ldots a_n)}{h}, \tag{1}$$

whenever this limit exists.

Definition For a function, $f : \mathbb{R}^n \to \mathbb{R}$, we define the *directional derivative*, in the direction of the unit vector $\vec{u} =< b_1, b_2, \ldots, b_n >$, at the point $(a_1, a_2, \ldots, a_n)$, by

$$D_{\vec{u}} f(a_1, a_2, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1 + hb_1, a_2 + hb_2, \ldots, a_n + hb_n) - f(a_1, a_2, \ldots a_n)}{h}, \tag{2}$$

whenever this limit exists.

Notice that we can pick the unit vector $\vec{u} =< 0, 0, \ldots, 0, 1, 0, \ldots, 0 >$, with a 1 in the i-th component, and the partial derivative becomes a special case of the directional derivative. When interpreting the directional derivative, we say that: $D_{\vec{u}} f(a_1, a_2, \ldots, a_n)$ is the instantaneous rate of change of the dependent variable with respect to changes in the independent variables, in the direction of $\vec{u}$, at the point $(a_1, a_2, \ldots, a_n)$.

Exercise 1. *Let $z = f(x, y)$ be a function of two variables. Let $\vec{u} =< a, b >$ be a unit vector, and let the point (x_0, y_0) be some point in the domain of f. Define a new function, g, in the single variable h, by*

$$g(h) = f(x_0 + ha, y_0 + hb).$$

Using the definition of derivative, and the definition of partial derivative, verify that

$$g'(0) = D_{\vec{u}} f(x_0, y_0). \tag{3}$$

Exercise 2. *Considering the functions defined in Exercise 1, use the chain rule to compute* $g'(h)$*.*

Exercise 3. *How can we use the expression found in Exercise 2 to conclude the following?:*

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \tag{4}$$

The argument outlined by Exercises 1, 2, and 3, can be generalized to higher dimensions to give use the following theorem:

Theorem 1. *If we are given a function, $f : \mathbb{R}^n \to \mathbb{R}$, and a unit vector $\vec{u} =< b_1, b_2, \ldots, b_n >$, then*

$$D_{\vec{u}} f = f_{x_1} b_1 + f_{x_2} b_2 + \cdots + f_{x_n} b_n.$$

Now, in order to take advantage of our vector operations, we define a new object:

Definition For a function, $f : \mathbb{R}^n \to \mathbb{R}$, we define the *gradient* of f, denoted ∇f, by

$$\nabla f =< f_{x_1}, f_{x_2}, \ldots, f_{x_n} > .$$

So, we can re-write the conclusion of Theorem 1 as

$$D_{\vec{u}} f = \nabla f \bullet \vec{u}. \tag{5}$$

1.2 Practice

Exercise 4. *Consider the function $f(x, y) = \sin(2x+3y)$, the point $P(-6, 4)$, and vector $\vec{u} =< \sqrt{3}/2, -1/2 >$. (Note: The given vector is not necessarily a unit vector, proceed accordingly.)*

1. *Evaluate ∇f at P.*
2. *Evaluate $D_{\vec{u}} f$ at P.*

Exercise 5. *Consider the function* $f(x, y, z) = \sqrt{x + yz}$*, the point* $P(1, 3, 1)$*, and vector* $\vec{u} = \frac{1}{7} < 2, 3, 6 >$.

1. *Evaluate* ∇f *at* P.
2. *Evaluate* $D_{\vec{u}} f$ *at* P.

1.3 More Properties and Applications

1.3.1 Maximum Rates of Change

Combining equation (5), and Theorem 6.2.2 (from our reference sheets), we get a new theorem:

Theorem 2. *If f is a differentiable function, and $\vec{u}$ is a unit vector then the maximum value of $D_{\vec{u}}f$ at a given point is $|\nabla f|$ at that point, and it is achieved when $\vec{u}$ is parallel to ∇f.*

Exercise 6. *Find the maximum rate of change in $f(x, y) = y^2/x$ at the point $P(2, 4)$, and the direction in which it occurs.*

Exercise 7. *Find the maximum rate of change in $f(x, y) = \sin(xy)$ at the point $P(1, 0)$, and the direction in which it occurs.*

1.3.2 Tangent Planes to Level Surfaces

One type of surface that we encounter when studying functions of several variables, are level surfaces of functions in three variables.

For the exercises in this section, suppose that S is a level surface defined by

$$F(x, y, z) = k, \tag{6}$$

and $P(x_0, y_0, z_0)$ is a point on S. Further, suppose that the space curve defined by $\vec{r}(t) =< x(t), y(t), z(t) >$ lies on S and passes through P when $t = t_0$.

Exercise 8. *Complete the statement:*

$$F\left(x(t), y(t), z(t)\right) = \tag{7}$$

Exercise 9. *Use the chain rule to differentiate both sides of (7), then express your answer in terms of ∇F and $\vec{r}\,'(t)$. What does this tell you about the relationship between tangent vectors to $\vec{r}(t)$ and the gradient of F?*

Exercise 10. *Find an equation for the plane with normal vector $\nabla F(x_0, y_0, z_0)$, and containing the point P. Why can we say that this plane is tangent to S? (Hint: What curves are we allowed to use for $\vec{r}(t)$?)*

2 Maxima and Minima

Surfaces in three dimensions, like curves in two, have maxima and minima. And, just as we could use horizontal tangent lines or points where the derivative was undefined to find candidate points when looking for these extrema, we can use horizontal tangent planes (parallel to the xy-plane) or points where partial derivatives are undefined.

Definition Given a function $z = f(x, y)$, we say that (a, b) is a *critical point* of f if

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. $f_x(a, b)$ or $f_y(a, b)$ is undefined.

Exercise 11. *Find the critical points of* $f(x, y) = x^4 + y^4 - 4xy + 1$.

Finding critical points gives us points at which the tangent planes to a surface are parallel to the xy-plane. However, we need to go further to understand the behavior of the surface near these points. There are three types of behavior, in particular, that we can use our current tools to identify:

1. *Local maxima*, or points on the surface that have a greater z-values than all of points on the surface, nearby.

2. *Local minima*, or points on the surface that have a lesser z-values than all of points on the surface, nearby.

3. *Saddle points*, or points where the graph of the surface crosses a horizontal tangent plane.

To make these classifications, we turn to the Second Derivative Test for functions of two variables:

Theorem 3 (The Second Derivative Test)**.** *Suppose that $z = f(x,y)$ has continuous second partial derivatives in a neighborhood of the point (a,b), and that (a,b) is a critical point of f. Define*

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2.$$

We, then, have the following:

1. *If $D(a,b) > 0$, and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*

2. *If $D(a,b) > 0$, and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*

3. *If $D(a,b) < 0$, then $f(a,b)$ is a saddle.*

Exercise 12. *Use the Second Derivative Test to classify the critical points you found in Exercise 11, or explain why you cannot.*

3 Challenge

Exercise 13. *Let $\vec{u}$ and $\vec{v}$ be non-parallel unit vectors, and let $z = f(x, y)$. If you are given $D_{\vec{u}}f(x_0, y_0)$, and $D_{\vec{v}}f(x_0, y_0)$, is it possible to find $\nabla f(x_0, y_0)$? If so, how would you do this? If not, why not?*

Exercise 14. *Find an equation for the plane that passes through the point $P(2, 3, 4)$ and cuts off the smallest volume in the first octant.*