Name:__________________________________

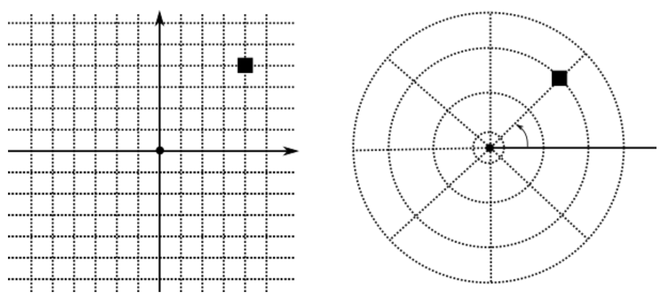
Worksheet 14

1 Coordinate Systems

Coordinate systems give us a way to describe the location of points in space. That is, they give us a way to give directions from a starting point (the origin) to any other point in the space we are describing (a plane, for example). There are two common coordinate systems for the plane: rectangular, and polar.

Roughly:

- Rectangular coordinates are the system with which we are most familiar. We define an origin, then an x-axis and a y-axis. Then, we are given an x-coordinate, and a y-coordinate, as a coordinate pair, (x_0, y_0). To get to that point on the plane, we start at the origin and walk x_0 units in the x-direction then y_0 units in the y-direction.
- In polar coordinates we define an origin, then a polar axis (the ray starting at the origin and going the positive x direction). Every point in the plane lies on some circle of radius r, centered at the origin. If we draw a ray from the origin to a specific point on that circle, that line, that ray forms an angle, θ, with the polar axis (in the counterclockwise direction). Thus, we can describe the location of any specific point with the associated radius and angle as a pair, (r_0, θ_0).

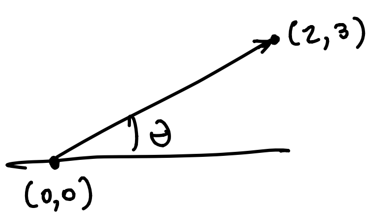


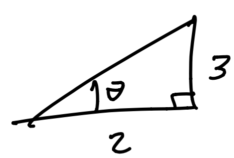
Left: A rectangular coordinate grid. Right: A polar coordinate grid.

Exercise 1. *The point $P(x, y) = (2, 3)$, lies on a circle centered at the origin. What is the radius of that circle? Sketch this circle and label P.*

$r = \sqrt{2^2 + 3^2} = \sqrt{13}$

Exercise 2. *Consider the ray from the origin to the point $P(x, y) = (2, 3)$. What angle does this ray form with the positive x-axis? Sketch this ray and label the angle.*





$\tan\theta = \frac{3}{2}$

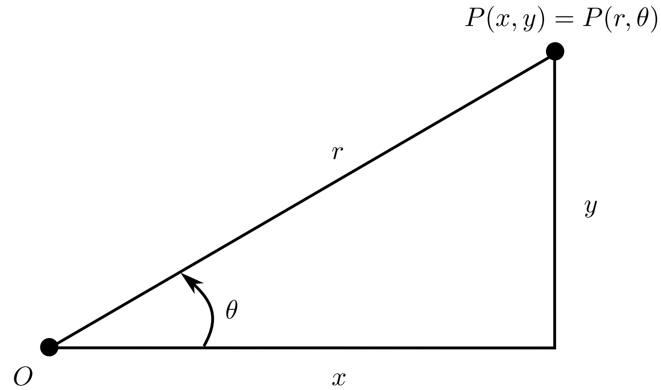
$\theta = \arctan\left(\frac{3}{2}\right)$

$\approx .98$ radians

or

56.3°

Exercises 1 and 2 show how to convert the point $(2,3)$ from rectangular to polar coordinates. We can generalize this approach to any point in the first quadrant.



Using the above diagram we can move back and forth between polar and rectangular coordinates in the first quadrant

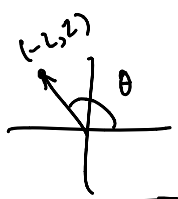
- Given polar coordinates r and θ, we can recover the x and y coordinates with

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

- Given rectangular coordinates x and y, we can find r and θ using

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

Note: These equations, though derived in the first quadrant, are valid for all points in the plane. Additionally, we do not restrict the possible values for r or θ, so a given point may have many polar representations.



Exercise 3. *Consider the point with rectangular coordinates $(-2,2)$. Which of the following polar coordinate pairs describe this point?*

- $(\sqrt{8}, 3\pi/4)$ ✓
- $(8, -5\pi/4)$ ✗
- $(4, 3\pi/4)$ ✗
- $(\sqrt{4}, 11\pi/4)$ ✗
- $(-\sqrt{8}, -3\pi/4)$ ✗
- $(\sqrt{8}, -5\pi/4)$ ✓
- $(-\sqrt{8}, -\pi/4)$ ✓
- $(\sqrt{8}, 11\pi/4)$ ✓

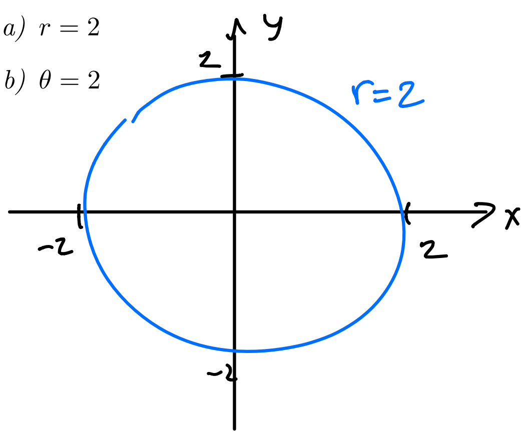
$\sqrt{(-2)^2+2^2}$
$= \sqrt{8}$

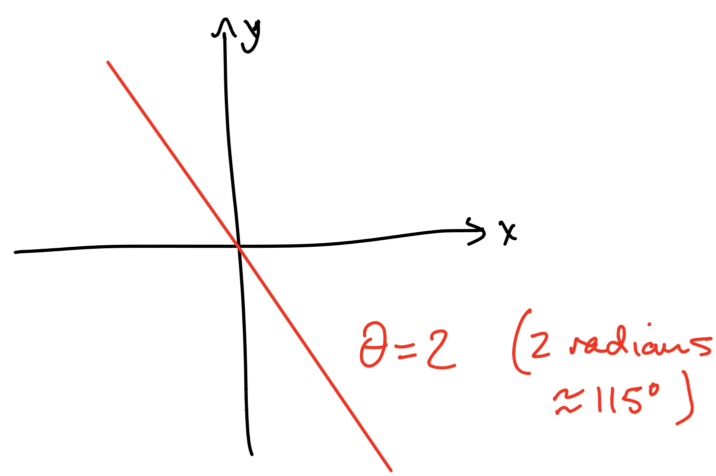
Definition The *graph of a polar equation* $F(r,\theta) = c$ consists of all points that have at least one polar representation that satisfies the equation.

Exercise 4. *Sketch the graphs of the following polar equations:*

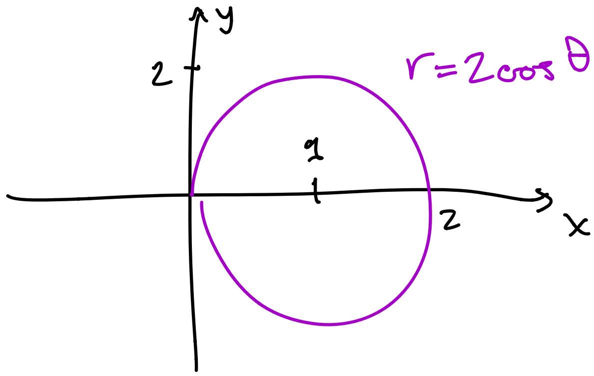
a) $r = 2$

b) $\theta = 2$





Exercise 5. *Sketch the graph of the polar equation $r - 2\cos(\theta) = 0$.*



Note: for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$,
$-1 \leq \cos\theta \leq 0$, resulting in $-2 \leq r \leq 0$.
Thus, there are no points that satisfy the equation in the 2nd or 3rd quadrants.

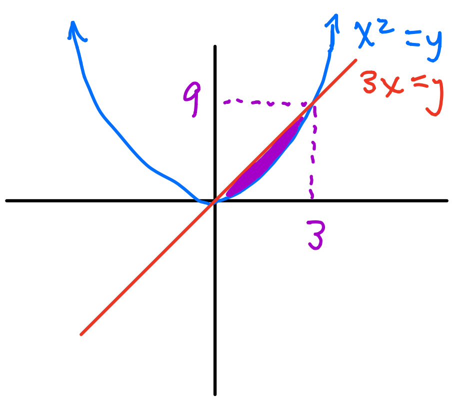
2 Describing Regions

We now have, essentially, three types of regions that we can describe in the plane:

Definition Given a region, D, in the plane, we say that it is:

- A Type I region - if it can be described by $D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
  Read: "D equals the set of all (x, y) pairs that satisfy $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$."
- A Type II region - if it can be described by $D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$
- A polar region - if it can be described by $D = \{(r, \theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$

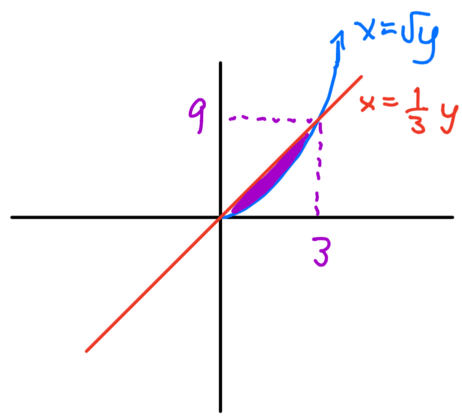
Exercise 6. *Describe the region in the plane bounded by $y = x^2$ and $y = 3x$ as a Type I region.*



$D = \{ (x,y) \mid 0 \leq x \leq 3,\ x^2 \leq y \leq 3x \}$

Exercise 7. *Describe the region in the plane bounded by $y = x^2$ and $y = 3x$ as a Type II region.*

$D = \{ (x,y) \mid 0 \leq y \leq 9,\ \frac{1}{3}y \leq x \leq \sqrt{y} \}$



Exercise 8. *Describe the region bounded by a circle of radius 3, centered at the origin, as:*

a) *The union of two Type I regions.*

b) *The union of two Type II regions.*

c) *A polar region.*

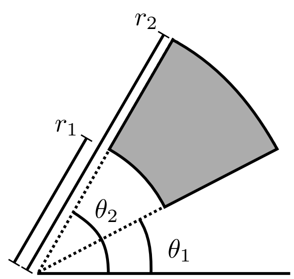
a) $\{(x,y) \mid -3 \leq x \leq 3,\ 0 \leq y \leq \sqrt{9-x^2}\} \cup \{(x,y) \mid -3 \leq x \leq 3,\ 0 \leq y \leq -\sqrt{9-x^2}\}$

b) $\{(x,y) \mid -3 \leq y \leq 3,\ 0 \leq x \leq \sqrt{9-y^2}\} \cup \{(x,y) \mid -3 \leq y \leq 3,\ 0 \leq x \leq -\sqrt{9-y^2}\}$

c) $\{(r,\theta) \mid 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$

(note: $\{(r,\theta) \mid -3 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$ also describes the region but, in practice, we avoid having two descriptions for so many points)

Exercise 9. *One special type of polar region is a* polar rectangle, *like the one pictured below.*



a) *Describe this as a polar region.*

b) *Describe the area of such a region in terms of r_1, r_2, θ_1 and θ_2. (Recall: The area of a sector of a circle is given by $A = \frac{1}{2}r^2\theta$.)*

$$D = \{(r,\theta) \mid r_1 \leq r \leq r_2,\ \theta_1 \leq \theta \leq \theta_2\}$$

$$A = \frac{1}{2}r_2^2(\theta_2 - \theta_1) - \frac{1}{2}r_1^2(\theta_2 - \theta_1)$$

$$= \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

Exercise 10. *We define the center of the polar rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ to be the point that has polar coordinates $r^* = \frac{1}{2}(r_1 + r_2)$ and $\theta^* = \frac{1}{2}(\theta_1 + \theta_2)$. We can also define $\Delta r = r_2 - r_1$ and $\Delta\theta = \theta_2 - \theta_1$. Write an expression for the area of this polar rectangle in terms of $r^*, \Delta r$, and $\Delta\theta$.*

$$A = \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

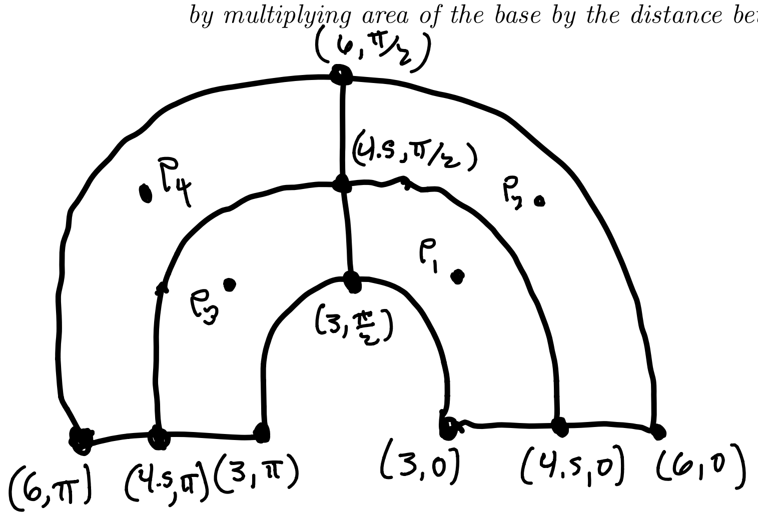
$$= \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta \quad \text{(difference of squares and def of } \Delta\theta.)$$

$$= r^* \Delta r \Delta\theta \quad \text{(def of } r^* \text{ and } \Delta r)$$

Exercise 11. *Suppose $F(r, \theta)$ is a function in polar coordinates whose graph lies above the standard xy-plane. Let $R = [3, 6] \times [0, \pi]$ be a polar rectangle in the standard xy-plane.*

a) Sketch and label R, divided into four smaller polar rectangles.

b) Write a sum, with four terms, that would the volume bounded below $F(r, \theta)$, and above R, using the centers of each of the smaller rectangles to determine the heights of four smaller volumes.(Note: In general, the volume of solid formed by identical bases in parallel planes, with sides made up of parallel lines, is computed by multiplying area of the base by the distance between the planes in which the bases lie.)



$P_1 = (3.75, \frac{\pi}{4})$ $\quad$ $P_3 = (3.75, \frac{3\pi}{4})$

$P_2 = (5.25, \pi/4)$ $\quad$ $P_4 = (5.25, \frac{3\pi}{4})$

$$V \approx F(3.75, \pi/4) \cdot \frac{1}{2}((4.5)^2 - 3^2)(\frac{\pi}{2})$$
$$+ F(5.25, \pi/4) \cdot \frac{1}{2}(6^2 - (4.5)^2)(\frac{\pi}{2})$$
$$+ F(3.75, 3\pi/4) \cdot \frac{1}{2}((4.5)^2 - 3^2)(\frac{\pi}{2})$$
$$+ F(5.25, 3\pi/4) \cdot \frac{1}{2}(6^2 - (4.5)^2)(\frac{\pi}{2})$$

Definition Suppose $F(x, y)$ is a continuous on a polar rectangle $R = [r_1, r_2] \times [\theta_1, \theta_2]$, with $0 \le r_1 \le r_2$ and $0 \le (\theta_2 - \theta_1) \le 2\pi$. Then we define

$$\iint_R F(x,y)dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} F(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta \tag{1}$$
$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} F(r_i^*\cos(\theta_j^*), r_i^*\sin(\theta_j^*))r_i^*\Delta r\Delta\theta.$$

Take a moment to compare this definition to your observations over the past few exercises. In particular, note the appearance of an r, in the integrand, when using polar coordinates.

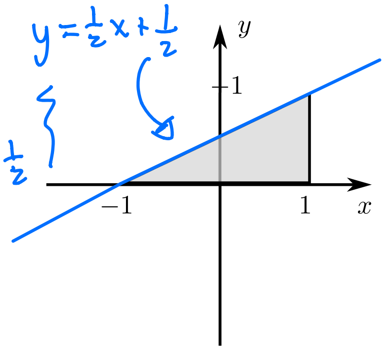
3 Practice

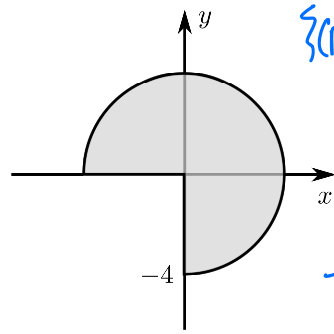
Exercise 12. *Given the following regions:*

$y = \frac{1}{2}x + \frac{1}{2}$

$\{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le \frac{1}{2}x + \frac{1}{2}\}$

$\int_{-1}^{1}\int_{0}^{\frac{1}{2}x+\frac{1}{2}} f(x,y)\,dy\,dx$



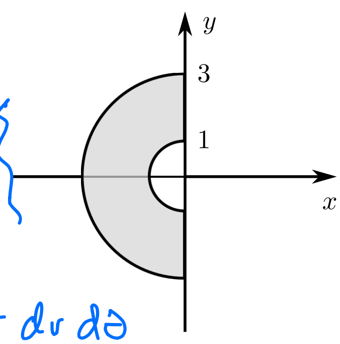


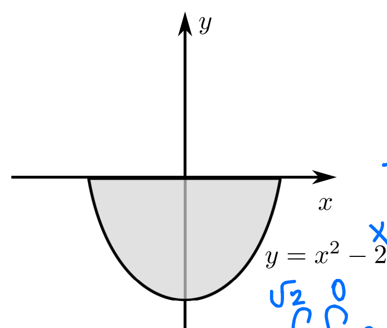
$\{(r,\theta) \mid 0 \le r \le 4,\ -\frac{\pi}{2} \le \theta \le \pi\}$

$\int_{-\frac{\pi}{2}}^{\pi}\int_{0}^{4} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

$\{(r,\theta) \mid 1 \le r \le 3,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{1}^{3} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$





$\{(x,y) \mid -\sqrt{2} \le x \le \sqrt{2},\ x^2 - 2 \le y \le 0\}$

$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-2}^{0} f(x,y)\,dy\,dx$

a) Describe each as a Type I, Type II, or polar region (using set notation).

b) Given a continuous function, $f(x, y)$, set up the integral of f over each region.

Exercise 13. *Find the volume of the solid that lying below the plane $3x+2y+z=12$ and above the rectangle $R=[0,1]\times[-2,3]$.*

$$\int_{-2}^{3}\int_{0}^{1}(12-3x-2y)\,dx\,dy = \ldots$$

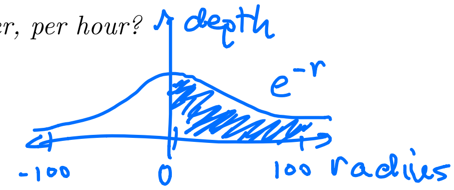
Exercise 14. *Find the volume of the solid lying below the surface $x^2-y^2-z+4=0$ and above the rectangle $R=[-1,1]\times[0,2]$.*

$$\int_{0}^{2}\int_{-1}^{1}(4-x^2+y^2)\,dx\,dy = \ldots$$

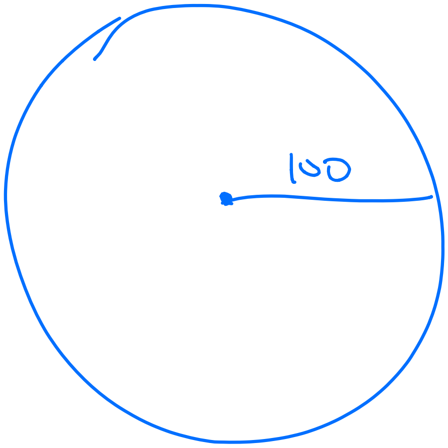
Exercise 15. *A farmer uses a sprinkler that distributes water over a circular region, 200 feet in diameter. At a distance of r feet from the sprinkler head, water is supplied e^{-r} feet deep into the ground, per hour.*

a) *What is the total volume of the soil that is supplied with water, per hour?*

b) *What is the average volume of soil, per square foot, that is supplied with water, per hour?*



a)



$$V = \int_0^{2\pi} \int_0^{100} (e^{-r})\, r \, dr \, d\theta$$

$= \ldots$

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b)

$$\text{avg} = \frac{V}{\pi(100)^2}$$

Exercise 16. *Consider the sum*

$$\int_1^{1/\sqrt{2}} \int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_1^{\sqrt{2}} \int_0^{x} xy\, dy\, dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy\, dy\, dx$$

a) Use polar coordinates to combine this sum into one double integral.

b) Evaluate the integral.

a) $\displaystyle\int_0^{\pi/4}\int_1^2 r^3 \cos\theta \sin\theta \, dr\, d\theta$

b) $\dfrac{15}{16}$