Tracing the same curve in different ways

Suppose $\vec{r}(t)$ traces a particular space curve.

Then $\vec{r}(f(t))$ traces the same curve, as long as the range of f is the domain of $\vec{r}(t)$.

However, the curve will not be traced at the same pace.

ex

$$\vec{r}(t) = \langle t, 2t, 3t \rangle$$

this traces a line through the origin, we can make a table of points on this line by choosing values of t.

t	$x(t)$	$y(t)$	$z(t)$
-2	-2	-4	-6
-1	-1	-2	-3
0	0	0	0
1	1	2	3
2	2	4	6

Now, we can trace the same curve at a different pace if we say $f(t) = 2t$ and consider $\vec{r}(f(t))$.

Now we have the table of values

t	$x(f(t))$	$y(f(t))$	$z(f(t))$
-2	-4	-8	-12
-1	-2	-4	-6
0	0	0	0
1	2	4	6
2	4	8	12

If we compare both tables, we see that the same line is traced, but at a different pace.

t	$x(t)$	$y(t)$	$z(t)$
-2	-2	-4	-6
-1	-1	-2	-3
0	0	0	0
1	1	2	3
2	2	4	6

4 units of time

in the first table, we traverse the line from $(-2,-4,-6)$ to $(2,4,6)$ in 4 units of time

t	$x(f(t))$	$y(f(t))$	$z(f(t))$
-2	-4	-8	-12
-1	-2	-4	-6
0	0	0	0
1	2	4	6
2	4	8	12

2 units of time

in the second table, the same trip takes only 2 units of time

Note how these observations relate to our discussion of Ch14 review T/F #1 and T/F #12

More on T/F #1

If we still needed to convince ourselves that $\langle t^3, 2t^3, 3t^3\rangle$ defined a line, we can look at three arbitrary points and use our tools to determine they are on the same line.

$P(t_1^3, 2t_1^3, 3t_1^3)$
$Q(t_2^3, 2t_2^3, 3t_2^3)$
$R(t_3^3, 2t_3^3, 3t_3^3)$

$$\overrightarrow{PQ} = \langle t_2^3 - t_1^3, 2(t_2^3 - t_1^3), 3(t_2^3 - t_1^3)\rangle$$
$$= (t_2^3 - t_1^3)\langle 1, 2, 3\rangle$$

$$\overrightarrow{QR} = \langle t_3^3 - t_2^3, 2(t_3^3 - t_2^3), 3(t_3^3 - t_2^3)\rangle$$
$$= (t_3^3 - t_2^3)\langle 1, 2, 3\rangle$$

Now, $\overrightarrow{PQ} \times \overrightarrow{QR} = (t_2^3 - t_1^3)\langle 1,2,3\rangle \times (t_3^3 - t_2^3)\langle 1,2,3\rangle$

$$\overset{6.3.1.2}{=} (t_2^3 - t_1^2)(t_3^3 - t_2^3)\left(\langle 1,2,3\rangle \times \langle 1,2,3\rangle\right)$$

$$\overset{6.3.4}{=} \vec{0}$$

So $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel, and they are in the direction of the line containing P and Q and the line containing Q and R, respectively. So P, Q, and R are colinear.