Name:__________________________________

Worksheet 06

1 Recap

Until recently, we studied real, two dimensional, space. We understood points, and lines, and functions, and addition and multiplication. Now, we are studying vector spaces. They look similar, but they have new properties.

Definition A *vector space* is a set, whose elements we call *vectors*, along with a field of *scalars*, where the following operations are well defined:

- vector addition (we can add two vectors to get a new vector: $\vec{u} + \vec{v} = \vec{w}$)
- scalar multiplication (we can multiply a vector by a scalar to get a new vector: $c\vec{r} = \vec{s}$)

We are going to focus on a special collection of vector spaces, where our vectors are represented by tuples with entries in $\mathbb{R}$, and our field of scalars is $\mathbb{R}$, and we define scalar multiplication as multiplication in each component of our vectors.

Notation If we are looking at vectors of the form

$$\vec{v} = < a_1, a_2, \dots, a_n >,$$

where each a_i is a real numbers then we say we our vectors are from $\mathbb{R}^n$. Then, if we have a real scalar, c, we define scalar multiplication by

$$c\vec{v} = < ca_1, ca_2, \dots, ca_n >,$$

where standard real number multiplication is applied in each component.

In this class of vector spaces, there is an *inner product* called the *dot product*:

Definition The dot product, $\bullet$, is an inner product between vectors in $\mathbb{R}^n$, defined by

$$< a_1, a_2, \dots, a_n > \bullet < b_1, b_2, \dots, b_n > = a_1b_1 + a_2b_2 + \dots + a_nb_n.$$

Note that this is fundamentally different from binary operators that we are used to, like $+$ or $\times$, in that it takes two objects of one type (vectors) and returns a single object of another type (a scalar).

Theorem 1. *The* norm *or* magnitude *of a vector, $\vec{v}$, denoted $||\vec{v}||$, is given by*

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}.$$

Theorem 2. *If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, then*

$$\vec{a} \bullet \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta).$$

2 In the Spacial Case of Three Dimensions

In three dimensions we can define another special operation between vectors:

Definition If $\vec{v} = < a_1, a_2, a_3 >$ and $\vec{w} = < b_1, b_2, b_3 >$ then the *cross-product*, $\times$, between $\vec{v}$ and $\vec{w}$ is defined by:

$$\vec{v} \times \vec{w} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

Notice, this is an operation between vectors that results in a vector (not a scalar).

Exercise 1. *Is*

$$\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$$

in general?

Exercise 2. *Pick two vectors in* $\mathbb{R}^3$, $\vec{v}$ *and* $\vec{w}$. *Compute*

$$(\vec{v} \times \vec{w}) \bullet \vec{v}$$

and

$$(\vec{v} \times \vec{w}) \bullet \vec{w}$$

Exercise 3. *Do you get the same answer if you chose any two vectors? Prove your answer.*

Exercise 4. *Which of the following are true?(Justify your answers with a proof or by giving a counterexample)*

i) $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$

ii) $(c\vec{w}) \times \vec{v} = c(\vec{w} \times \vec{v})$

iii) $\vec{w} \times (\vec{v} + \vec{u}) = \vec{w} \times \vec{v} + \vec{w} \times \vec{u}$

iv) $\vec{u} \bullet (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \bullet \vec{w}$

Exercise 5. *Show that, if* $0 \leq \theta \leq \pi$ *is the angle between* $\vec{a}$ *and* $\vec{b}$, *then*

$$||a \times b|| = ||a|| \cdot ||b|| \cdot \sin(\theta).$$

Exercise 6. *Finish this statement, using the cross product:*
Two non-zero vectors are parallel if and only if ...