Name:____________________________________

Worksheet 09

1 Projectile Motion

Exercise 1. *Given a projectile with initial velocity $||\vec{v}(0)|| = v_0$, that is launched with an angle of elevation α, from the surface of a planet with gravitational constant g, derive a vector valued function describing the position vector with respect to time.*

Starting at $\vec{a} = -g\hat{j}$, we integrate to get

Note: gravity is pulling the projectile in the $-\hat{j}$ direction w/ magnitude g.

$$\vec{v}(t) = \int_{t_0=0}^{t} \vec{a}(u)\,du = \int_{t_0=0}^{t} -g\hat{j}\,du = -g\hat{j}\cdot u\Big|_{t_0=0}^{t} + C$$

$$= \vec{a}\cdot t + C$$

So, as $\vec{v}(0) = \vec{v}_0$, we have

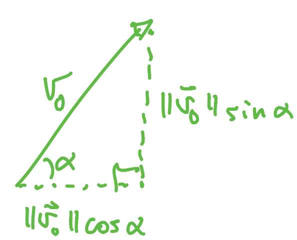
$$\vec{v}(t) = \vec{a}\cdot t + \vec{v}_0$$

Now

$$\vec{r}(t) = \int_{t_0=0}^{t} (\vec{a}\cdot u + \vec{v}_0)\,du = \vec{a}\frac{u^2}{2} + \vec{v}_0 u\Big|_{t_0}^{t} + D$$

$$= \frac{1}{2}\vec{a}t^2 + \vec{v}_0\cdot t + D$$

and $\vec{r}(0) = \vec{0}$, so $\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t$



Now, $\vec{v}_0 = ||\vec{v}_0||\cos(\alpha)\hat{i} + ||\vec{v}_0||\sin(\alpha)\hat{j}$

So $\vec{r}(t) = \left\langle ||\vec{v}_0||\cos(\alpha)\cdot t, -\frac{1}{2}gt^2 + ||\vec{v}_0||\sin(\alpha)t \right\rangle$

2 Kepler's First Law of Planetary Motion

First we recall two of Newton's Laws:

Law (Second Law of Motion)**.**

$$\vec{F} = m\vec{a} \tag{1}$$

Where $\vec{F}$ is force and m is mass.

Law (Law of Universal Gravitation Between a Planet and a Star)**.**

$$\vec{F} = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\vec{u}. \tag{2}$$

Where $\vec{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the star, respectively, $\vec{r}$ is the position vector of the planet, and $\vec{u} = \frac{1}{r}\vec{r}$, where $r = ||\vec{r}||$.

Law (Kepler's First Law of Planetary Motion)**.** *A planet revolves around a star with an elliptical orbit with the star at one focus.*

The proof of this law happens in two parts:

1. We show that the planet moves around the star in a fixed plane, relative to the star.
2. We show that the path it traces in this plane is an ellipse.

Exercise 2. *Using equations (1) and (2), conclude that the position and acceleration vectors are parallel. What does this tell us about $\vec{r} \times \vec{a}$?*

Setting (1) = (2) we get

$$m\vec{a} = -\frac{GMm}{r^3}\vec{r} \implies \vec{a} = \frac{-GM}{r^3}\vec{r}$$

so these vectors are scalar multiples of each other, thus, they are parallel. So, by Corollary 13.4.7, $\vec{a} \times \vec{r} = \vec{0}$.

Exercise 3. *Compute*

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right),$$

where $\vec{v}$ is the velocity vector of the planet.

By Theorem 14.2.3 part 5

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r}\,' \times \vec{v} + \vec{r} \times \vec{v}\,'$$

$$= \underbrace{\vec{v} \times \vec{v}}_{\substack{= \\ 0}} + \underbrace{\vec{r} \times \vec{a}}_{\substack{= \\ 0}} = \vec{0}$$

again by Cor. 13.4.7

by exercise 2

Exercise 4. *We may assume that $\vec{r}$ and $\vec{v}$ are not parallel. If we let $\vec{h} = \vec{r} \times \vec{v}$, what doe we know about $\vec{h}$? (Aside from the fact that it is orthogonal to each of $\vec{r}$ and $\vec{v}$).*

from Ex 3, we can conclude its component functions are constant.

Exercise 5. *How can we use properties of $\vec{h}$ to conclude that the curve traced out by $\vec{r}$ is a plane curve. That is, that the planet moves in a fixed plane, relative to the star?*

As $\vec{r}(t)$ and $\vec{h}(t)$ are orthogonal for all values of t, and we can assume that $\vec{r}(t)$ and $\vec{v}(t)$ are not parallel then $\vec{r}(t)$ and $\vec{v}(t)$ are always in the same plane, perpendicular to $\vec{h}(t)$, containing the origin.

Exercise 6. *Fill in the missing steps in the following computation:*

$$
\begin{aligned}
\vec{h} &= \vec{r} \times \vec{v} \\
1 \quad &= \dots \\
&= r\vec{u} \times (r\vec{u})' \\
2 \quad &= \dots \\
&= r^2(\vec{u} \times (\vec{u})').
\end{aligned}
$$

For 1 we note that $\vec{v} = \vec{r}\,'$ and $\vec{r} = r\vec{u}$

so $\vec{r} \times \vec{v} = (r\vec{u}) \times (r\vec{u})'$

For 2 we apply Theorem 14.2.3 part 3, because r is still a function of t. So

$$
\begin{aligned}
(r\vec{u}) \times (r\vec{u})' &= (r\vec{u}) \times (r'\vec{u} + r\vec{u}\,') \\
&= (r\vec{u} \times r'\vec{u}) + (r\vec{u} \times r\vec{u}\,') && \text{by 13.4.8 part 3} \\
&= r \cdot r'(\vec{u} \times \vec{u}) + r^2(\vec{u} \times \vec{u}\,') && \text{by 13.4.8 part 2} \\
&= r^2(\vec{u} \times \vec{u}\,') && \text{by Cor 13.4.7}
\end{aligned}
$$

Exercise 7. *Now, express $\vec{a} \times \vec{h}$ in terms of G, M, and $\vec{u}$. Use 13.6.8 Property 6*

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

to simplify your answer.

from (1) and (2) we have $\vec{a} = \frac{-GM}{r^2}\vec{u}$, from Ex 6 we have $\vec{h} = r^2(\vec{u} \times \vec{u}')$

So $\vec{a} \times \vec{h} = -\frac{GM}{r^2}\vec{u} \times (r^2(\vec{u} \times \vec{u}')) = -GM(\vec{u} \times (\vec{u} \times \vec{u}'))$

$$= -GM((\vec{u} \bullet \vec{u}')\vec{u} - (\vec{u} \bullet \vec{u})\vec{u}')$$

$$= -GM((\vec{u} \bullet \vec{u}')\vec{u} - (\|\vec{u}\|^2)\vec{u}') \quad (13.3.2.1)$$

$$= -GM((\vec{u} \bullet \vec{u}')\vec{u} - \vec{u}') \quad (\text{as } \|\vec{u}\| = 1)$$

Exercise 8. *Show that, if a vector valued function has constant magnitude, then that function and its derivative are always orthogonal.*

Given $\|\vec{v}(t)\| = C$, then $\vec{v} \bullet \vec{v} = C^2$

So, $\underline{\frac{d}{dt}(\vec{v} \bullet \vec{v}) = 0}$ and $\frac{d}{dt}(\vec{v} \bullet \vec{v}) = \frac{d}{dt}\vec{v} \bullet \vec{v} + \vec{v} \bullet \frac{d}{dt}\vec{v}$ by 14.2.3.4

$$= \underline{2\left(\left(\frac{d}{dt}\vec{v}\right) \bullet \vec{v}\right)} \quad \text{by 13.3.2.2}$$

equating these gives us that $\vec{v} \bullet \frac{d}{dt}\vec{v} = 0$, so a vector valued function and its derivatie are always orthogonal.

Exercise 9. *Using the fact from the previous exercises, verify that*

$$\vec{a} \times \vec{h} = GM(\vec{u})'$$

and, thus,

$$(\vec{v} \times \vec{h})' = GM(\vec{u})'$$

In the last line of Ex 7 we conclude that

$$\vec{a} \times \vec{h} = -GM((\vec{u} \bullet \vec{u}')\vec{u} - \vec{u}')$$

So, by Ex 9, we get

$$\vec{a} \times \vec{h} = GM\vec{u}'$$

and $(\vec{v} \times \vec{h})' = \vec{v} \times \vec{h}' + \vec{v}' \times \vec{h} = \vec{v} \times \vec{h}' + \vec{a} \times \vec{h}$

$$= \vec{a} \times \vec{h} \quad (\text{as } \vec{h} \text{ is constant so } \vec{h}' \text{ is zero})$$

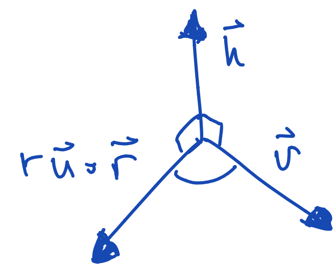
Note, that integrating both sides of the equation from Exercise 9 gives us

$$\vec{v} \times \vec{h} = GM\vec{u} + \vec{C} \tag{3}$$

where $\vec{C}$ is a constant vector.

Until now, we haven't chosen our coordinate axes. To make life a bit more convenient, we will just arange our planet-star system so that the vector $\vec{h}$ points in the standard z-axis direction. That is, we ensure that $\vec{h}$ and $\hat{k}$ are parallel.

Exercise 10. *In which plane does the constant vector from equation (3) lie?*

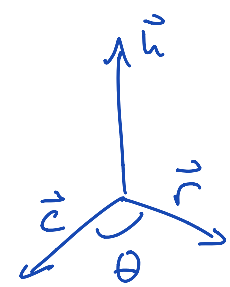


$\vec{v} \times \vec{h}$ and $\vec{u} = \frac{1}{r}\vec{r}$ are both orthogonal to $\vec{h}$, so $\vec{c}$ must lie in the xy-plane as well, for (3) to hold.

Exercise 11. *If θ is the angle between $\vec{r}$ and $\vec{C}$, then (r, θ) are the polar coordinates of the planet. Verify that*

$$\vec{r} \bullet (\vec{v} \times \vec{h}) = GMr + rc\cos(\theta),$$

where $c = ||\vec{c}||$.



$\vec{r} \bullet (\vec{v} \times \vec{h})$

$= \vec{r} \bullet (GM\vec{u} + \vec{c})$ by (3)

$= GM\vec{r} \bullet \vec{u} + \vec{r} \bullet \vec{c}$ by 13.3.2.3

$= (GMr)\vec{u} \bullet \vec{u} + ||\vec{r}|| \cdot ||\vec{c}|| \cos\theta$ by 13.3.2.4 and $\vec{u} = \frac{1}{r}\vec{r}$ and 13.3.3

$= GMr^2 + r.c \cdot \cos\theta$

Now, we have that

$$r = \frac{\vec{r} \bullet (\vec{v} \times \vec{h})}{GM + c\cos(\theta)} = \frac{1}{GM} \frac{\vec{r} \bullet (\vec{v} \times \vec{h})}{1 + d\cos(\theta)} \tag{4}$$

where $d = c/GM$.

Exercise 12. *Simplify the numerator of equation (4). (Denote $||\vec{h}|| = h$.)*

$$\vec{r} \bullet (\vec{v} \times \vec{h})$$

$$= (\vec{r} \times \vec{v}) \bullet \vec{h} \quad \text{by } 13.4.8.5$$

$$= \vec{h} \bullet \vec{h} \quad \text{by def of } \vec{h}$$

$$= ||\vec{h}||^2 \quad \text{by } 13.3.2.1$$

$$= h^2$$

Exercise 13. *Let $f = h^2/d$. Simplify the equation from the previous exercise so that it is in terms of r, d, f and θ, only. Compare the result to Theorem 11.6.6 in your book.*

$$r = \frac{fd}{1 + d\cos(\theta)}$$ is the polar coordinate expression for an elipse