NOTES:

Recall: $\sin^2\theta + \cos^2\theta = 1$

$\div \cos^2\theta$: $\tan^2\theta + 1 = \sec^2\theta$.

Rearrange to solve for $(\text{trig})^2$:

① $\cos^2\theta = 1 - \sin^2\theta$

② $\sec^2\theta = 1 + \tan^2\theta$

③ $\tan^2\theta = \sec^2\theta - 1$

$\xRightarrow{\times A}$

① $A\cos^2\theta = A - A\sin^2\theta$, for $A \in \mathbb{R}$.

② $A\sec^2\theta = A + A\tan^2\theta$

③ $A\tan^2\theta = A\sec^2\theta - A$

(*) **STYLE:** ① $\# - f(\theta)$

② $\# + f(\theta)$

③ $f(\theta) - \#$

STRATEGY: Match the function inside the integral with corresponding style, and substitute in the desired function. Simplify your expression so it is easier to apply the differential. Substitute everything back into integral. This is best described by doing a couple of examples.

Example 1:

$$*\int \frac{8x}{\sqrt{9-16x^4}}\,dx$$

Solution:

① **Style:** $\# - f(x)$

$\Rightarrow A\cos^2\theta = A - A\sin^2\theta.$

① Substitution: $9-16x^4 = 9-9\sin^2\theta = 9\cos^2\theta$

$$\Longrightarrow 16x^4 = 9\sin^2\theta$$

$$\Longrightarrow 4x^2 = 3\sin\theta$$

② Apply differential: $d(4x^2) = d(3\sin\theta)$

$$\Longrightarrow 8x\,dx = 3\cos\theta\,d\theta$$

③ Collect and substitute terms in integral:

$$\int \frac{8x}{\sqrt{9-16x^4}}\,dx \qquad 8x\,dx = 3\cos\theta\,d\theta \qquad 9-16x^4 = 9\cos^2\theta$$

$$= \int \frac{3\cos\theta\,d\theta}{\sqrt{9\cos^2\theta}}$$

$$= \int \frac{3\cos\theta}{3\cos\theta}\,d\theta$$

$$= \int d\theta = \theta + C$$

Lastly, you want the answer to be given in the original variable, in this case x, so we have to solve for θ in terms of x. We have several candidates for equations, but one is certainly the easiest:

$$4x^2 = 3\sin\theta$$

$$\Longrightarrow \sin\theta = \frac{4x^3}{3}$$

$$\Longrightarrow \theta = \sin^{-1}\left(\frac{4x^3}{3}\right).$$

So, $\displaystyle\int \frac{8x}{\sqrt{9-16x^4}}\,dx = \sin^{-1}\left(\frac{4x^3}{3}\right) + C$ $\blacksquare$

Example 2:

$$* \int \frac{10x}{16+25x^4}\,dx$$

Solution:

Style: $\# + f(x)$

$\Rightarrow A\sec^2\theta = A + A\tan^2\theta$

① **Substitution:** $16+25x^4 = 16+16\tan^2\theta = 16\sec^2\theta$

$\Rightarrow 25x^4 = 16\tan^2\theta$

$\Rightarrow 5x^2 = 4\tan\theta$

② **Apply differential:** $d(5x^2) = d(4\tan\theta)$

$\Rightarrow 10x\,dx = 4\sec^2\theta\,d\theta$

③ **Collect and substitute terms in integral:**

$$\int \frac{10x}{16+25x^4}\,dx \qquad 10x\,dx = 4\sec^2\theta\,d\theta$$

$$16+25x^4 = 16\sec^2\theta$$

$$= \int \frac{4\sec^2\theta \, d\theta}{16\sec^2\theta}$$

$$= \int \frac{1}{4} \, d\theta$$

$$= \frac{1}{4}\theta + C$$

Again, we solve for θ in terms of x:

$$5x^2 = 4\tan\theta$$

$$\Longrightarrow \tan\theta = \frac{5x^2}{4}$$

$$\Longrightarrow \theta = \tan^{-1}\left(\frac{5x^2}{4}\right)$$

So, $\displaystyle\int \frac{10x}{16+25x^4}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{5x^2}{4}\right) + C$ $\blacksquare$