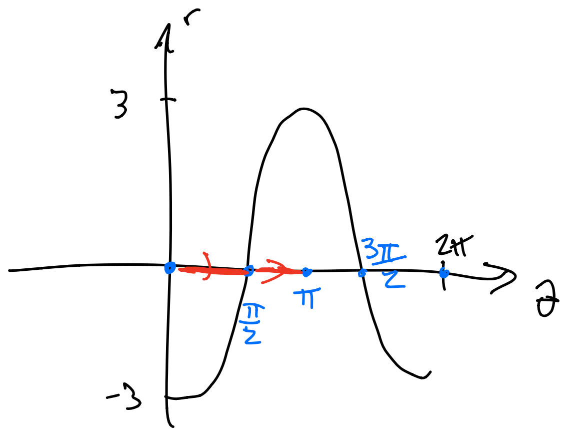
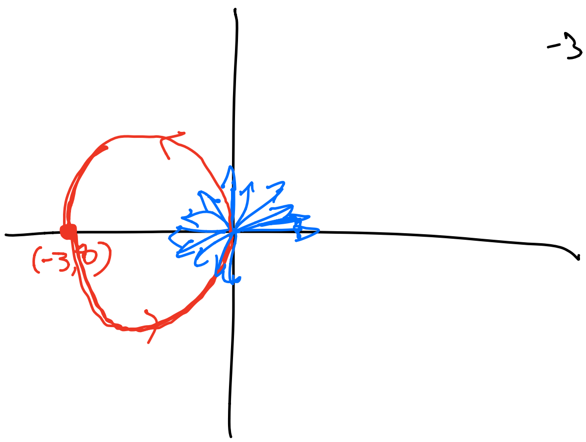
11.3

#32

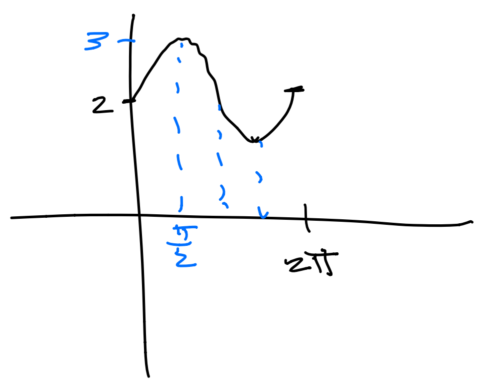
$r = -3\cos\theta$

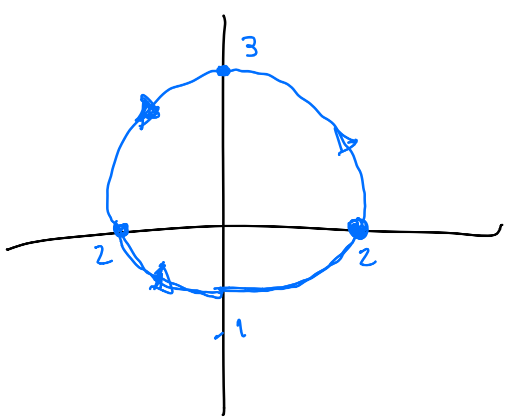




#42

$r = 2 + \sin\theta$



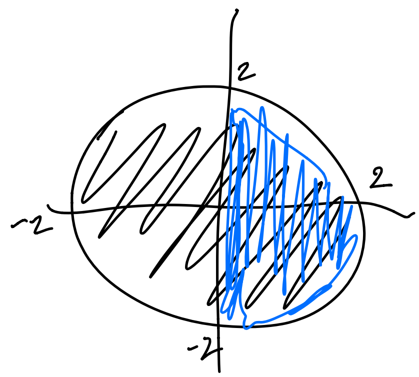


16.4

#10

$$\iint_R \sqrt{4-x^2-y^2}$$

$R = \{(x,y) \mid x^2+y^2 \leq 4,\ x \geq 0 \}$



$R = \{ (r,\theta) \mid 0 \leq r \leq 2 ,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \}$

Name:____________________________________

Worksheet 15

1 Practice

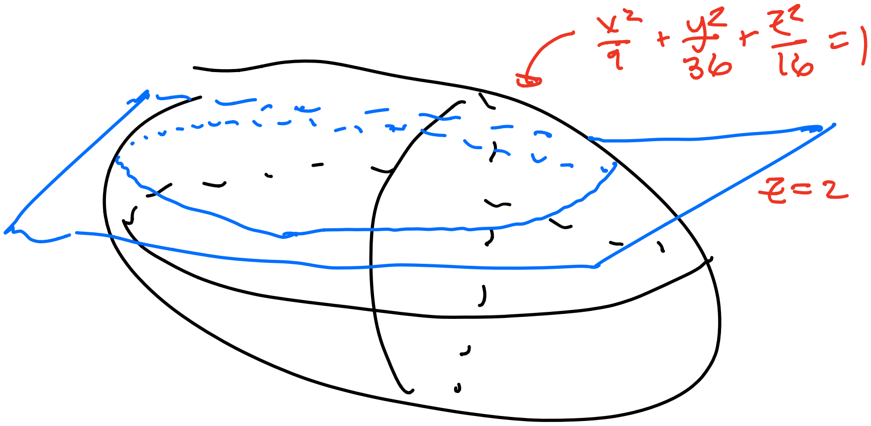
Example Set up an integral, in rectangular coordinates, to compute the volume bounded above by

$$\boxed{\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{16} = 1}$$

and below by

$$z = 2$$

so that we integrate with respect to z, then y, then x.



First, we note that our z values are bounded below by $z=2$ and above by $\frac{z^2}{16} = 1 - \frac{x^2}{9} - \frac{y^2}{36}$

or

$z = \pm 4\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{36}}$

So, $\quad 2 \leq z \leq 4\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{36}}$

(Note: we only need the top half of the ellipsoid, here, so we only need the positive solutions)

Next we see that both x and y have the most freedom when $z=0$, or is as close as it can get to zero. In this case, when $z=2$.

So, we want all x and y that satisfy

$$\frac{x^2}{9}+\frac{y^2}{36}+\frac{(\overset{z=2}{\overbrace{2}})^2}{16}=1.$$

We want to integrate w/ respect to y, next so we solve to get

$$y=\pm 6\sqrt{\frac{3}{4}-\frac{x^2}{9}}$$

and this gives bounds on y in terms of x, so we integrate

$$-6\sqrt{\frac{3}{4}-\frac{x^2}{9}}\le y\le 6\sqrt{\frac{3}{4}-\frac{x^2}{9}}.$$

Finally we see that $y=0$ gives x the most freedom, so we find

$$\frac{x^2}{9}+\frac{\overset{y=0}{0}}{36}+\frac{(\overset{z=2}{2})^2}{16}=1 \Rightarrow x^2=\frac{3}{4}\cdot 9=\frac{27}{4}$$

$$\Rightarrow x=\pm\frac{3}{2}\sqrt{3} \qquad -\frac{3\sqrt{3}}{2}\le x\le\frac{3\sqrt{3}}{2}$$

So, finally, our integral for the volume is

$$V=\int_{-\frac{3\sqrt{3}}{2}}^{\frac{3\sqrt{3}}{2}}\int_{-6\sqrt{\frac{3}{4}-\frac{x^2}{9}}}^{6\sqrt{\frac{3}{4}-\frac{x^2}{9}}}\int_{2}^{4\sqrt{1-\frac{x^2}{9}-\frac{y^2}{36}}} 1\,dz\,dy\,dx.$$

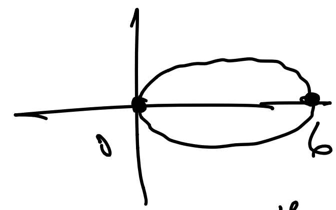
Exercise 1. *Set up an integral, in rectangular coordinates, to compute the volume bounded below*

$$\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$$

and above

$$\frac{y}{6}+\frac{z}{4}=1$$

so that we integrate with respect to z, then x, then y.



$\frac{y}{6}+\frac{z}{4}=1$

1) Find bounds for z in terms of x and y

2) Find bounds for x in terms of y

3) Find numerical bounds for y.

1) z is above $\frac{y}{6}+\frac{z}{4}=1$ so $z \geq 4\left(1-\frac{y}{6}\right)$

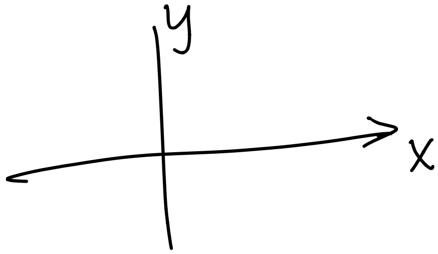
z is below $\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$ so $z \leq 4\sqrt{1-\frac{x^2}{16}-\frac{y^2}{36}}$

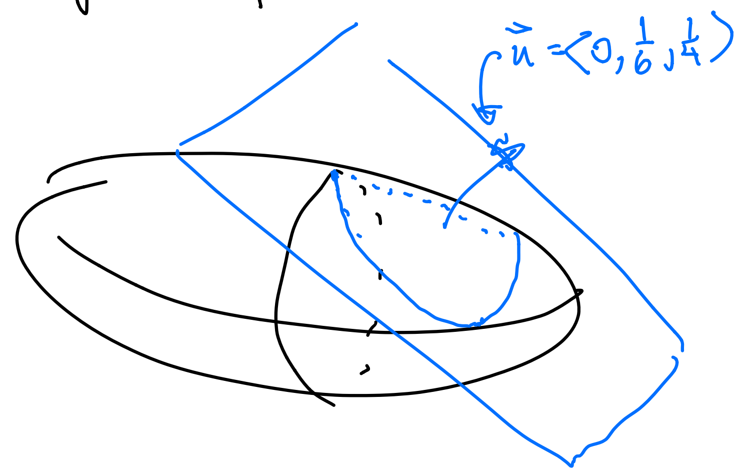
2) plug $z=4\left(1-\frac{y}{6}\right)$ into $\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$

and solve for x to get

$$-2\sqrt{2}\sqrt{1-\frac{(y-3)^2}{9}} \leq x \leq 2\sqrt{2}\sqrt{1-\frac{(y-3)^2}{9}}$$

3) $0 \leq y \leq 6$



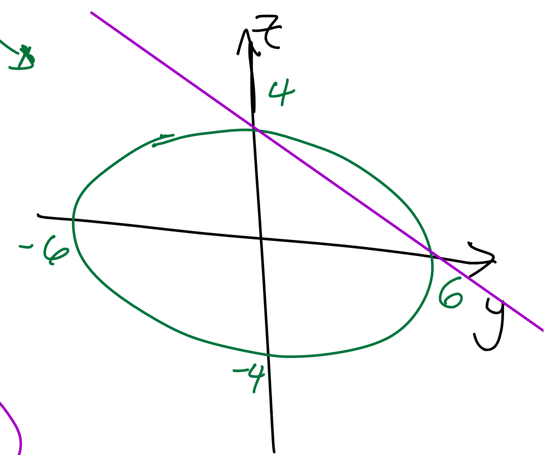


Why $0 \le y \le 6$?

From our first equation $\frac{x^2}{16} + \frac{y^2}{36} + \frac{z^2}{16} = 1$ we know that y and z have the largest ranges when $x = 0$.

Then, we have

$$\frac{y^2}{36} + \frac{z^2}{16} = 1 \quad \text{(an elipse)}$$



and

$$\frac{y}{6} + \frac{z}{4} = 1$$

$$\Rightarrow z = 4 - \frac{2}{3}y \quad \text{(a line)}$$

So, the only values of y that lie in our solid are between 0 and 6.

Example Switching the order of integration can, sometimes, make an impossible integral possible. With this in mind, evaluate the following:

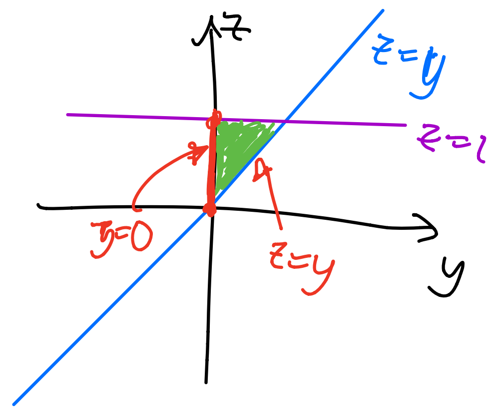
$$\int_0^2 \int_0^1 \int_y^1 \sinh(z^2)\, dz\, dy\, dx$$

$z?$

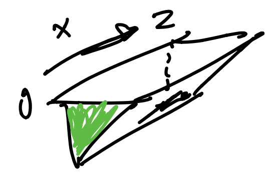
$$R = \begin{cases} y \le z \le 1 \\ 0 \le y \le 1 \\ 0 \le x \le 2 \end{cases}$$

$$\int \sinh(u)\, du = \cosh(u) + C$$

$$\frac{d}{dz} \cosh(z^2) = \sinh(z^2) \cdot 2z$$



$$\int_0^2 \int_0^1 \int_{y=0}^{y=z} \sin(z^2)\, dy\, dz\, dx$$



$$\int_{x=0}^{x=2} \int_{z=0}^{z=1} \left(y \sinh(z^2) \Big|_0^z \right) dz\, dx$$

$$\int_0^2 \int_0^1 z \sinh(z^2)\, dz\, dx$$

$$\int_0^2 \frac{1}{2} \cosh(z^2) \Big|_0^1 dx$$

$$\int_0^2 \frac{1}{2}\cosh(1) - \frac{1}{2}\cosh(0)\ dx$$

$$\frac{1}{2}\cosh(1)x - \frac{1}{2}\cosh(0)x \Big|_0^2$$

$$= \cosh(1) - \cosh(0)$$