Name:____________________________________

Worksheet 11

1 Review

1.1 Vectors

Key ideas:

- *Vectors spaces* are a pair of sets, (V, F), where the elements of V are vectors, and the elements of F are scalars.
- Vector spaces come with two new operations: *scalar multiplication*, and *inner products* (or *dot products*).
- In *real vector spaces* the vectors are arrows with direction and magnitude, and the scalars are real numbers. So, $V = \mathbb{R}^n$ and $F = \mathbb{R}$, but we usually just refer to this vector space as $\mathbb{R}^n$ and assume the field of scalars is the real numbers unless otherwise specified.
- To describe real vectors, we use the representative that starts are the origin and points to a specific point in $\mathbb{R}^n$. For example, the "arrow" that points from $(0, 0, 0, 0)$ to $(1, -2, -3, 5)$ is one representative of the vector $< 1, -2, -3, 5 >$ in $\mathbb{R}^4$.
- In $\mathbb{R}^3$ we have another operation, the *cross product*.

1.2 Equations of Lines and Planes

Key ideas:

- Vectors allow us to generalize the idea of slope, a concept that only applied to lines in $\mathbb{R}^2$, to any dimension.
- Given a starting position (a *point* in $\mathbb{R}^n$) and a direction (a *vector* in $\mathbb{R}^n$) we can
  - write a vector valued function tracing the line through that point and along that direction, or
  - write a vector equation representing the plane containing that point, and whose normal vector points in that direction.
- We say that lines are parallel if they each contain representatives of the same vectors.
- We say that planes are parallel if they have normal vectors in common.
- If lines intersect, we can compute the angle between them by computing the angle between non-zero vectors with representatives that lie along those lines (taking one from each line).
- If planes intersect, we compute the angle between them by computing the angle between their normal vectors.

1.3 Tangent Lines

Key idea:

- If a vector valued function traces a curve in space, the derivative of that function, at a given point, returns the direction of the tangent line, as a vector, there. This is the generalization of the derivative of a real valued function returning the slope of the tangent line at each point of the graph of that function.

1.4 Functions of Several Variables

Key ideas:

- A function is a rule that assigns to each element in its domain a single element in its co-domain. A function of several variables is one whose domain has more than one dimension.
- We usually denote these functions by

  $$f : \mathbb{R}^n \to \mathbb{R}$$

  or

  $$(x_1, x_2, \ldots, x_n) \to f(x_1, x_2, \ldots, x_n).$$

- The graph of a function of n variable is a subset of $\mathbb{R}^{n+1}$, and is made up of all the tuples of a the form

  $$(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

  where $(x_1, x_2, \ldots, x_n)$ is in the domain of f.
- It can be helpful to study *cross sections* (fixing some input(s)) or the *level sets* (fixing the output) when trying to understand how any particular example behaves.

1.5 Partial Derivatives

Key ideas:

- If we want to understand the rate of change of a function of several variables, with respect to one of its independent variables, then we treat all of the other independent variables as constants and differentiate just as we would if the function were a function of that single variable.
- Given $f : \mathbb{R}^n \to \mathbb{R}$, the partial derivative with respect to the k-th independent variable is a new function, $f_{x_k} : \mathbb{R}^n \to \mathbb{R}$. If we chose any point, $(a_1, a_2, \ldots, a_n)$, in the domain of f, then

  $$f_{x_k}(a_1, a_2, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_n)$$

  can be interpreted as the slope of the function of one variable defined by

  $$F(x_k) = f(a_1, a_2, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_n).$$

1.6 Tangent Planes and Total Differentials

Key ideas:

- Tangent planes are the generalization of tangent lines from two dimensions into three.
- The graph of $f(x, y) = z$ is a surface in $\mathbb{R}^3$, and if we fix a point, $(a, b, f(a, b))$, on that surface, then we can use f to find the equation of the tangent plane at that point.
- When we fix one variable as constant, then $f(x, b) = g(x)$ (or $f(a, y) = h(y)$) can be treated as a function of a single variable, whose graph lives in the plane $y = b$ (or $x = a$). The slope of the tangent lines to these graphs can be used to find vectors with representatives that lie in these planes, and are tangent to the graph of f. If we fix a particular point, then the tangent vectors that we find in each of these planes are orthogonal to one another, and we can compute their cross product to find a normal vector for the tangent plane at this point.
- For a function of two variables, $f(x, y) = z$, the total differential, df or dz, is given by $dz = f_x(x, y)dx + f_y(x, y)dy$

2 Practice

For the exercises in this section, consider the following function of several variables:

$$f(x, y) = x^2 + 2x - 4y^2 + 4y \tag{1}$$

Exercise 1. *Sketch and label the cross section of the graph of f by the plane $x = 1$.*

Exercise 2. *Sketch and label the cross section of the graph of f by the plane $y = -2$.*

Exercise 3. *Find an equation for the tangent line to the cross section you found in Exercise 1, when* $y = -2$.

Exercise 4. *Find an equation for the tangent line to the cross section you found in Exercise 2, when* $x = 1$.

Exercise 5. *Find a vector in* $\mathbb{R}^3$ *that describes the direction of the tangent line you found in Exercise 3. Then, write a vector equation for this tangent line in* $\mathbb{R}^3$. *Check your answer in at least two points.*

Exercise 6. *Find a vector in* $\mathbb{R}^3$ *that describes the direction of the tangent line you found in Exercise 4. Then, write a vector equation for this tangent line in* $\mathbb{R}^3$. *Check your answer in at least two points.*

Exercise 7. *Using the vectors you found in Exercises 5 and 6, find an equation for the tangent plane to the graph of f at the point $(1, -2, -21)$.*

Exercise 8. *Compute the total differential of f.*

3 The Chain Rule

Recall, for functions of a single variable, we had the following result:

Theorem 1 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then*

$$\frac{d}{dx}(f \circ g)(x) = \left(\left(\frac{d}{dx}f\right) \circ g\right)(x) \cdot \left(\frac{d}{dx}g(x)\right),$$

or, in Leibniz notation, if $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

3.1 Composition of Functions of Several Variables

Before we can generalize the idea of the chain rule to higher variables, we have to figure out what it means to compose functions of several variables. Lets start by thinking about the single variable situation, and build from there.

Example Suppose we define a function, $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = 2x^2 + 1.$$

Then, we define another function, $g : \mathbb{R} \to \mathbb{R}$, by setting

$$g(t) = t - 1.$$

We define the composition of these functions as

$$(f \circ g)(t) = f(g(t)),$$

which is a new function, in the single variable t. This direct replacement of x with the function $g(t)$ works because the outputs of g coincide with the allowable values of x in the function f.

Exercise 9. *Suppose we have two functions of several variables:*

$$\begin{aligned} f &: \mathbb{R}^n \to \mathbb{R} \\ g &: \mathbb{R}^m \to \mathbb{R} \end{aligned}$$

For which values of n and m can we define a composition of these functions as

$$(f \circ g)(x_1, x_2, \ldots, x_m) = f(g(x_1, x_2, \ldots, x_m))?$$

Why don't other values work?

Exercise 10. *Suppose*

$$\begin{aligned} f(x,y) &= x^2 + 2xy + y^2 \\ g(t) &= t + 1 \\ h(t) &= 2t^2 \end{aligned}$$

Compute the following:

1. $f(g(1), h(1)) =$
2. $f(h(1), g(1)) =$

If we say that $F(t) = f(g(t), h(t))$, is F a function of several variables?

Exercise 11. *Suppose*

$$\begin{aligned} f(x,y) &= x^2 + 2xy + y^2 \\ g(t,s) &= st + s^2 \\ h(t,s) &= t^2 - s^2 \end{aligned}$$

Compute the following:

1. $f(g(1,2), h(3,4)) =$
2. $f(h(2,4), g(3,1)) =$

If we say that $G(t,s) = f(g(t,s), h(t,s))$, is G a function of several variables?

Exercise 12. *Suppose*

$$\begin{aligned} f(x) &= x^2 + 2x \\ g(u,v) &= uv + u^2 \end{aligned}$$

Compute the following:

1. $f(g(3,-1)) =$

If we say that $H(u,v) = f(g(u,v))$, is H a function of several variables?

3.2 The General Chain Rule

Whenever we have a function, in any number of variables, we may replace each variable with a new function, also in any number of variables, as long as the outputs of these new functions make sense as values for the variable that they replace. After this replacement we have a new function and a new set of independent variables. For each independent variable, we can compute a partial derivative.

Theorem 2 (The Chain Rule (General Version))**.** *Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function in the n independent variables $x_1, x_2, \ldots, x_n$. Further suppose that, for $i = 1, 2, \ldots, n$, $f_i : \mathbb{R}^m \to \mathbb{R}$ is a differentiable function in the m independent variables $t_1, t_2, \ldots, t_m$.*

If we replace each of the n independent variables in F with the corresponding component function*, $x_i = f_i(t_1, t_2, \ldots, t_m)$, then we have changed F into a function in m variables:*

$$F\left(f_1(t_1, t_2, \ldots, t_m), f_2(t_1, t_2, \ldots, t_m), \ldots, f_n(t_1, t_2, \ldots, t_m)\right)$$

Now, if we compute the partial derivative of F with respect to t_j we get

$$\frac{\partial F}{\partial t_j} = \frac{\partial F}{\partial x_1}\frac{\partial f_1}{\partial t_j} + \frac{\partial F}{\partial x_2}\frac{\partial f_2}{\partial t_j} + \cdots + \frac{\partial F}{\partial x_n}\frac{\partial f_n}{\partial t_j}.$$

Exercise 13. *Find all first partial derivatives of F from Exercise 10. How many second partial derivatives does F have?*

Exercise 14. *Find all first partial derivatives of G from Exercise 11. How many second partial derivatives does G have?*

Exercise 15. *Find all first partial derivatives of H from Exercise 12. How many second partial derivatives does H have?*

3.3 Implicit Differentiation

If $F(x, y)$ is a function of two variables, sometimes the equation

$$F(x, y) = 0 \tag{2}$$

allows use to, implicitly, define y as a function of x. That is, we can find a function, f, with $f(x) = y$. Then, we can re-write (2) as

$$F(x, f(x)) = 0. \tag{3}$$

If possible, we may apply the Chain Rule to both sides of (3) to get

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Exercise 16. *Use (4) to write an expression for dy/dx in terms of F_x and F_y.*

Exercise 17. *If $\sqrt{xy} = 1 + x^2 y$, find dy/dx.*

Exercise 18. *Suppose that we can use the equation $G(x, y, z) = 0$, to express z as a function of x and y. That is, we can find a function, g, with $g(x, y) = z$.*

1. *Use the Chain Rule to differentiate both sides of $G(x, y, z) = 0$ with respect to x*
2. *Use the Chain Rule to differentiate both sides of $G(x, y, z) = 0$ with respect to y*
3. *Express $\partial z/\partial x$ in terms of F_x and F_z.*
4. *Express $\partial z/\partial y$ in terms of F_y and F_z.*

Exercise 19. *If $xyz = \cos(x + y + z)$, find $\partial z/\partial x$ and $\partial z/\partial y$.*