Name:____________________________________

Worksheet 03

1 Partial Fractions

1.1 Terminology

Definition A *polynomial* is a function of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Where the a_i are real numbers and n is non-negative integer. The *degree of a polynomial* of this form is n, as long as $a_n \neq 0$.

Definition For $p(x)$ a polynomial. We say that the polynomial $q(x)$ is a *factor* of $p(x)$ whenever there is a polynomial $s(x)$ such that

$$p(x) = q(x) \cdot s(x).$$

Definition For $p(x)$ a polynomial, we say that a factor $q(x)$ of $p(x)$ is a *linear factor* if $q(x)$ is of the form

$$ax + b,$$

and $a \neq 0$.

Definition For $p(x)$ a polynomial, we say that a factor $q(x)$ of $p(x)$ is a *quadratic factor* if $q(x)$ is of the form

$$ax^2 + bx + c,$$

and $a \neq 0$.

Definition A quadratic polynomial is *irreducible* whenever it is not the product of linear factors.

Definition A *rational function* is a function of the form

$$F(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

1.2 Motivation

If we were asked to evaluate

$$\int \frac{18\,x^3 + 15\,x^2 - 4\,x - 4}{2\,x^9 - 17\,x^8 + 43\,x^7 - 124\,x^6 + 345\,x^5 + 60\,x^4 + 1225\,x^3 + 1400\,x^2 + 1625\,x + 625} dx$$

It might be helpful to know that the integrand could be expressed as

$$\frac{3067\,x + 2068}{132300\,(x^2+5)} - \frac{151\,x + 131}{8649\,(x^2+x+1)} + \frac{94\,x + 79}{630\,(x^2+5)^2} - \frac{128}{160083\,(2\,x+1)} - \frac{557141}{104652900\,(x-5)} + \frac{289}{34100\,(x-5)^2},$$

because it is easier to integrate each of these terms individually.

1.3 Undoing Finding A Common Denominator

Essentially, what we need to do is find a way to break a single, complicated, denominator into easier to handle parts. In the context of evaluating the integrals of rational functions, there are two cases we can start with:

1. The degree of the numerator is greater than or equal to that of the denominator.
2. The degree of the numerator is less than that of the denominator.

In the first case, we start with polynomial long division. This will allow us to express our rational function in the form

$$\frac{P(x)}{Q(x)} = d(x) + \frac{r(x)}{Q(x)},$$

where $r(x)$ is a polynomial whose degree is less than the degree of $Q(x)$. Note, $d(x)$ is the *dividend* of this quotient and $r(x)$ is the *remainder.*

Exercise 1. *Evaluate*

$$\int \left(\frac{x^2 + x - 11}{4x - 16} \right) dx$$

Hint: First, use polynomial long division to express the integrand as the sum of its dividend and a quotient of its remainder.

Exercise 2. *Use polynomial long division to express the following rational function as the sum of its dividend and a quotient of its remainder.*

$$f(x) = \frac{x^4 + 5\,x^3 + 7\,x^2 + 14\,x + 6}{x^3 + x^2 + 3\,x}$$

Exercise 3. *Express the following as a single fraction,*

$$\frac{A}{x} + \frac{Bx + C}{x^2 + x + 3}.$$

Exercise 4. *Combine what you found in Exercises 2 and 3 to express the function*

$$f(x) = \frac{x^4 + 5\,x^3 + 7\,x^2 + 14\,x + 6}{x^3 + x^2 + 3\,x}$$

as a sum of terms whose denominators only have one factor.

Exercise 5. *Evaluate*

$$\int \frac{x^4 + 5\,x^3 + 7\,x^2 + 14\,x + 6}{x^3 + x^2 + 3\,x}\,dx$$

Hint: It may be useful, at some point, to "complete the square" before making a substitution.

1.4 A General Strategy

1. Use long division, if possible, to reduce the degree of the numerator of the integrand that you have to deal with.
2. Factor the denominator into its linear and irreducible quadratic factors. Then
   (a) Count all if its linear factors, including repeats. (Example: $(3x+4)^2$ has two linear factors)
   (b) Count all of its irreducible quadratic factors, including repeats.
3. The total number of factors, both quadratic and linear, counting repeats, will be the number of terms in the partial fraction decomposition.
4. To each linear factor, including repeats, assign a constant term, A_i, for $i = 1, 2, 3, \dots$
5. To each quadratic, including repeats, assign a pair of constants and an x, of the form $B_j x + C_j$, for $j = 1, 2, 3, \dots$
6. Write out the general form of the partial fraction decomposition
   (a) The terms you just assigned will be the numerators in the partial fraction decomposition
   (b) The associated factors will be the denominators. If a factor is repeated, then it will show up as a denominator as many times as it was repeated.
   (c) For each repeated factor, raise each appearance in a denominator to the next integer power
7. Express this as a single fraction
8. Solve for each of the constants you assigned, so that the result is equal to your original rational function
9. Evaluate the original integral by evaluating all of the parts

Example To Evaluate

$$\int \frac{14x^2 - 5x + 12}{x(x+1)^2(x^2+1)(x^2+2)^2} dx$$

We complete the steps above:

1. There is no need to do long division because the degree of the denominator is greater than that of the numerator.
2. The denominator is factored.
   (a) It has 3 linear factors: x, $x+1$, and $x+1$
   (b) It has 3 irreducible quadratic factors: x^2+1, x^2+2, and x^2+2
3. So, we know the partial fraction decomposition will have $3+3=6$ terms.
$$\frac{?}{?} + \frac{?}{?} + \frac{?}{?} + \frac{?}{?} + \frac{?}{?} + \frac{?}{?}$$
4. We assign the constant terms A_1, A_2, and A_3 to the linear factors
5. We assign the terms $B_1x + C_1$, $B_2x + C_2$, and $B_3x + C_3$ to the quadratic factors
6. We write out the terms of the partial fraction decomposition as prescribed:
$$\frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} + \frac{B_1x + C_1}{x^2+1} + \frac{B_2x + C_2}{x^2+2} + \frac{B_3x + C_3}{(x^2+2)^2}$$
7. Etc.

1.5 Practice

Exercise 6. *Evaluate*

$$\int \frac{1}{x^2-1}dx$$

First, we note that $x^2-1=(x+1)(x-1)$

So, we use a partial fraction decomposition:

$$\frac{1}{x^2-1}=\frac{a}{x+1}+\frac{b}{x-1}=\frac{a(x-1)+b(x+1)}{(x^2-1)}$$

To make the numerators match, we solve:

$$\textcircled{1}\ a+b=0 \quad \xRightarrow{\textcircled{1}} a=-b \quad \xRightarrow{\textcircled{2}} 2b=1 \Rightarrow b=\frac{1}{2}$$

$$\textcircled{2}\ b-a=1 \qquad \Rightarrow a=-\frac{1}{2}$$

So, we get $\int \frac{1}{x^2-1}dx=-\frac{1}{2}\int \frac{1}{x+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$

$$=\frac{-\ln(x+1)+\ln(x-1)}{2}+C$$

Exercise 7. *Evaluate*

$$\int \frac{x-1}{x^2+3x+2}dx$$

Note $x^2+3x+2=(x+1)(x+2)$

So, we set $\frac{x-1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}=\frac{A(x+2)+B(x+1)}{(x-1)(x+2)}$

Now, if we solve:

$$A+B=1 \qquad \text{we get} \qquad A=-2$$
$$2A+B=-1 \qquad\qquad B=3$$

Thus $\int\frac{x-1}{x^2+3x+2}dx=\int\frac{-2}{x+1}dx+\int\frac{3}{x+2}dx$

$$=-2\ln(x+1)+3\ln(x+2)+C$$

Exercise 8. *Evaluate*

$$\int \frac{1}{(x+a)(x+b)} dx$$

Set $\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} = \frac{(A+B)x + (Ab+Ba)}{(x+a)(x+b)}$

Solving $A+B=0$ $\quad A=-B \Rightarrow -Bb+Ba=1$

$Ab+Ba=1$

$\Rightarrow B(a-b)=1$

$\Rightarrow B = \frac{1}{a-b}$

$\Rightarrow A = \frac{1}{b-a}$

Thus, $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a}\int \frac{1}{x+a}dx + \frac{1}{a-b}\int \frac{1}{x+b} dx$

$= \frac{1}{b-a}\ln(x+a) + \frac{1}{a-b}\ln(x+b)$

Exercise 9. *Evaluate*

$$\int \frac{1}{(x+5)^2(x-1)} dx$$

Start w/ $\frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}$

$\vdots$

2 Discussion Questions

In your groups, consider the following question. Try and come up with examples to test and illustrate your ideas.

Exercise 10. *Suppose a linear factor in the denominator of a rational function is repeated twice, that is,* $(ax+b)^2$ *appears in its factorization. Is it necessary to break this into two parts? Could we achieve the same information if we treated it as we do an irreducible quadratic factor?*

Exercise 11. *The polynomial* $x^3 + x + 1$ *is an irreducible cubic polynomial. How might we modify our approach to if such a factor appeared in your denominator? Could you apply this method to a linear factor that is repeated three times,* $(ax+b)^3$*?*

Exercise 12. *How might we generalize these observations to dealing with denominators containing a degree* n *irreducible factor?*

3 Optional Practice

Hint: $e^{2x}+3e^{x}+2$

$= (e^{x} \pm \_\_)(e^{x} \pm \_\_)$

1. Evaluate

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}dx$$

2. Evaluate

$$\int \frac{\cos(x)}{\sin^2(x)+\sin(x)}dx$$

3. Evaluate

$$\int \left(\frac{(3\,x+2)^2(2\,x-1)}{(x^2+x+1)(x^2+5)^2(2\,x+1)(x-5)^2}\right)dx$$

Note: The most computationally expensive part is already done for you. The integrand is equal to the one in the motivating example at the beginning of this worksheet.