2 Practice

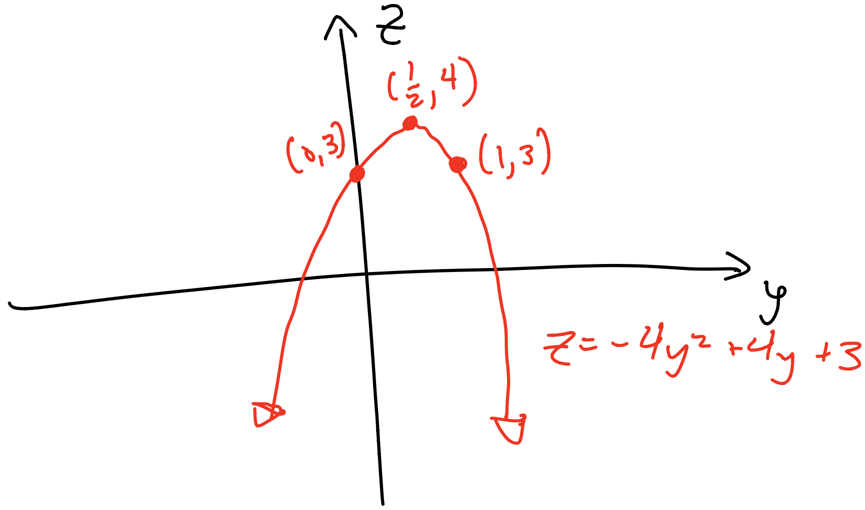
For the exercises in this section, consider the following function of several variables:

$$f(x,y) = x^2 + 2x - 4y^2 + 4y \tag{1}$$

Exercise 1. *Sketch and label the cross section of the graph of f by the plane $x = 1$.*

On the plane $x=1$ we have $f(1,y) = -4y^2 + 4y + 3$

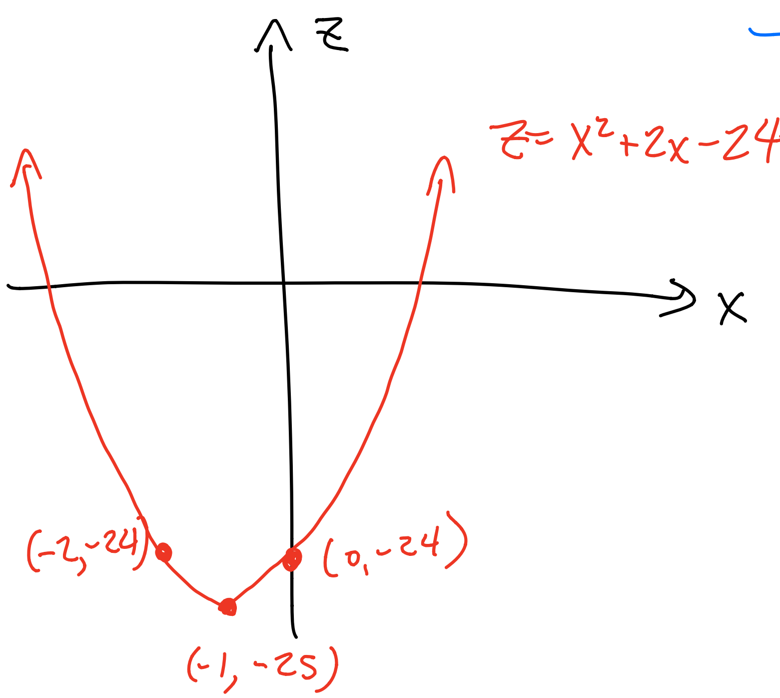
which is a parabola:



note the axes are in terms of y and z, as we have fixed $x=1$.

Exercise 2. *Sketch and label the cross section of the graph of f by the plane $y = -2$.*

On the plane $y=-2$ we have $f(x,-2) = x^2 + 2x - 24$



note the axes are in terms of x and z, as we have fixed $y = -2$.

Exercise 3. *Find an equation for the tangent line to the cross section you found in Exercise 1, when $y = -2$.*

$f(1,y) = -4y^2 + 4y + 3 \Rightarrow f_y(1,y) = -8y + 4$

$\Rightarrow f_y(1,-2) = 20$

So the tangent line goes through the point $(-2, f(1,-2) = -21)$ and has slope 20.

note, again, everything is in terms of y and z

$z + 21 = 20(y+2) \Rightarrow z = 20y + 19$

Exercise 4. *Find an equation for the tangent line to the cross section you found in Exercise 2, when $x = 1$.*

$f(x,-2) = x^2 + 2x - 24 \Rightarrow f_x(x,-2) = 2x + 2 \Rightarrow f_x(1,-2) = 4$

So the tangent line goes through $(1, f(1,2) = 21)$ and has slope 4.

note, again, everything is in terms of x and z

$z + 21 = 4(x-1) \Rightarrow z = 4x - 25$

Exercise 5. *Find a vector in $\mathbb{R}^3$ that describes the direction of the tangent line you found in Exercise 3. Then, write a vector equation for this tangent line in $\mathbb{R}^3$. Check your answer in at least two points.*

The slope of the tangent line was $\frac{\Delta z}{\Delta y} = 20$.

The line lies in the $x=1$ plane, so every point on the line has the same x-value, so the direction vector in $\mathbb{R}^3$ must be zero in the x-component.

Thus, our vector equation is $\langle 1,-2,21\rangle + t\langle 0,1,20\rangle$

Exercise 6. *Find a vector in $\mathbb{R}^3$ that describes the direction of the tangent line you found in Exercise 4. Then, write a vector equation for this tangent line in $\mathbb{R}^3$. Check your answer in at least two points.*

The slope of the tangent line was $\frac{\Delta z}{\Delta x} = 4$.

The line lies in the $y=-2$ plane, so every point on the line has the same y-value, so the direction vector in $\mathbb{R}^3$ must be zero in the y-component.

Thus, our vector equation is $\langle 1,-2,21\rangle + t\langle 1,0,4\rangle$

Exercise 7. *Using the vectors you found in Exercises 5 and 6, find an equation for the tangent plane to the graph of f at the point $(1, -2, -21)$.*

The tangent vectors we found can be used to find a normal vector for the tangent plane by computing their cross product.

$$\vec{n} = \langle 1, 0, 4 \rangle \times \langle 0, 1, 20 \rangle$$

$$= \begin{vmatrix} 0 & 4 \\ 1 & 20 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 4 \\ 0 & 20 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k}$$

$$= \langle 4, -20, 1 \rangle$$

$$\langle 4, -20, 1 \rangle \cdot \langle x-1, y+2, z+21 \rangle = 0$$

gives the equation of the tangent plane

Exercise 8. *Compute the total differential of f.*

$$f(x,y) = x^2 - 2x - 4y^2 + 4y$$

$$f_x(x,y) = 2x - 2$$

$$f_y(x,y) = -8y + 4$$

$$df = (2x-2)\,dx + (-8y+4)\,dy$$

3 The Chain Rule

Recall, for functions of a single variable, we had the following result:

Theorem 1 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then*

$$\frac{d}{dx}(f \circ g)(x) = \left(\left(\frac{d}{dx}f\right) \circ g\right)(x) \cdot \left(\frac{d}{dx}g(x)\right),$$

or, in Leibniz notation, if $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

3.1 Composition of Functions of Several Variables

Before we can generalize the idea of the chain rule to higher variables, we have to figure out what it means to compose functions of several variables. Lets start by thinking about the single variable situation, and build from there.

Example Suppose we define a function, $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = 2x^2 + 1.$$

Then, we define another function, $g : \mathbb{R} \to \mathbb{R}$, by setting

$$g(t) = t - 1.$$

We define the composition of these functions as

$$(f \circ g)(t) = f(g(t)),$$

which is a new function, in the single variable t. This direct replacement of x with the function $g(t)$ works because the outputs of g coincide with the allowable values of x in the function f.

Exercise 9. *Suppose we have two functions of several variables:*

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$g : \mathbb{R}^m \to \mathbb{R}$$

For which values of n and m can we define a composition of these functions as

$$(f \circ g)(x_1, x_2, \ldots, x_m) = f(g(x_1, x_2, \ldots, x_m))?$$

Why don't other values work?

The composition only makes sense if $n=1$ because the output of g is in $\mathbb{R}$, so if f expects an input from $\mathbb{R}^2$, or $\mathbb{R}^3$, etc, then it will not know what to do with something from $\mathbb{R}$.

Exercise 10. *Suppose*

$$\begin{aligned} f(x,y) &= x^2 + 2xy + y^2 \\ g(t) &= t + 1 \\ h(t) &= 2t^2 \end{aligned}$$

Compute the following:

1. $f(g(1), h(1)) =$

2. $f(h(1), g(1)) =$

If we say that $F(t) = f(g(t), h(t))$*, is* F *a function of several variables?*

Exercise 11. *Suppose*

$$\begin{aligned} f(x,y) &= x^2 + 2xy + y^2 \\ g(t,s) &= st + s^2 \\ h(t,s) &= t^2 - s^2 \end{aligned}$$

Compute the following:

1. $f(g(1,2), h(3,4)) =$

2. $f(h(2,4), g(3,1)) =$

If we say that $G(t,s) = f(g(t,s), h(t,s))$*, is* G *a function of several variables?*

Exercise 12. *Suppose*

$$\begin{aligned} f(x) &= x^2 + 2x \\ g(u,v) &= uv + u^2 \end{aligned}$$

Compute the following:

1. $f(g(3,-1)) =$

If we say that $H(u,v) = f(g(u,v))$*, is* H *a function of several variables?*

3.2 The General Chain Rule

Whenever we have a function, in any number of variables, we may replace each variable with a new function, also in any number of variables, as long as the outputs of these new functions make sense as values for the variable that they replace. After this replacement we have a new function and a new set of independent variables. For each independent variable, we can compute a partial derivative.

Theorem 2 (The Chain Rule (General Version))**.** *Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function in the n independent variables $x_1, x_2, \ldots, x_n$. Further suppose that, for $i = 1, 2, \ldots, n$, $f_i : \mathbb{R}^m \to \mathbb{R}$ is a differentiable function in the m independent variables $t_1, t_2, \ldots, t_m$.*

If we replace each of the n independent variables in F with the corresponding component function*, $x_i = f_i(t_1, t_2, \ldots, t_m)$, then we have changed F into a function in m variables:*

$$F\left(f_1(t_1, t_2, \ldots, t_m), f_2(t_1, t_2, \ldots, t_m), \ldots, f_n(t_1, t_2, \ldots, t_m)\right)$$

Now, if we compute the partial derivative of F with respect to t_j we get

$$\frac{\partial F}{\partial t_j} = \frac{\partial F}{\partial x_1}\frac{\partial f_1}{\partial t_j} + \frac{\partial F}{\partial x_2}\frac{\partial f_2}{\partial t_j} + \cdots + \frac{\partial F}{\partial x_n}\frac{\partial f_n}{\partial t_j}.$$

Exercise 13. *Find all first partial derivatives of F from Exercise 10. How many second partial derivatives does F have?*

Exercise 14. *Find all first partial derivatives of G from Exercise 11. How many second partial derivatives does G have?*

Exercise 15. *Find all first partial derivatives of H from Exercise 12. How many second partial derivatives does H have?*

3.3 Implicit Differentiation

If $F(x,y)$ is a function of two variables, sometimes the equation

$$F(x,y) = 0 \tag{2}$$

allows use to, implicitly, define y as a function of x. That is, we can find a function, f, with $f(x) = y$. Then, we can re-write (2) as

$$F(x, f(x)) = 0. \tag{3}$$

If possible, we may apply the Chain Rule to both sides of (3) to get

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Exercise 16. *Use (4) to write an expression for dy/dx in terms of F_x and F_y.*

Exercise 17. *If $\sqrt{xy} = 1 + x^2y$, find dy/dx.*

Exercise 18. *Suppose that we can use the equation $G(x, y, z) = 0$, to express z as a function of x and y. That is, we can find a function, g, with $g(x, y) = z$.*

1. *Use the Chain Rule to differentiate both sides of $G(x, y, z) = 0$ with respect to x*
2. *Use the Chain Rule to differentiate both sides of $G(x, y, z) = 0$ with respect to y*
3. *Express $\partial z/\partial x$ in terms of F_x and F_z.*
4. *Express $\partial z/\partial y$ in terms of F_y and F_z.*

Exercise 19. *If $xyz = \cos(x + y + z)$, find $\partial z/\partial x$ and $\partial z/\partial y$.*