Name:____________________________________

Worksheet 08

1 Vector Valued Functions

Definition A *vector valued function* is a function whose domain is a set of real numbers, and whose range is a set of vectors.

Usually, we borrow the notation of parametric functions to say that $\vec{r}(t)$ is a vector valued function with *component functions*, $f(t)$, $g(t)$, and $h(t)$, or

$$\vec{r}(t) =< f(t), g(t), h(t) >= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

Example Find a vector valued function that represents the curve of intersection of the surfaces described by:

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ y + z &= 3 \end{aligned}$$

Exercise 1. *Find a vector valued function that represents the curve of intersection of the surfaces described by:*

$$
\begin{aligned}
x^2 + y^2 &= 9 \\
y + z &= 3
\end{aligned}
$$

Exercise 2. *Find a vector valued function that represents the curve of intersection of the surfaces described by:*

$$
\begin{aligned}
x^2 + y^2 &= 4 \\
z &= x^3
\end{aligned}
$$

Exercise 3. *Find a vector valued function that represents the curve of intersection of the surfaces described by:*

$$y = x^2$$
$$x^2 + 4y^2 + 4z^2 = 16 \text{ for } z \geq 0$$

2 Derivatives of Vector Valued Functions

Definition The *derivative of a vector valued function*, $\vec{r}(t)$, is defined as

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h},$$

if this limit[1] exists.

We say that $\vec{r}\,'(t_0)$ describes the *tangent vector* to the curve defined by $\vec{r}(t)$ at the point described by $\vec{r}(t_0)$, provided $\vec{r}\,'(t_0)$ exists and is non-zero.

Theorem 1. *If $\vec{r}(t) =< f(t), g(t), h(t) >= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, where $f(t)$, $g(t)$, and $h(t)$ are differentiable functions, then*

$$\vec{r}\,'(t) =< f'(t), g'(t), h'(t) >= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}.$$

The Product Rule

If $\vec{u}(t)$ is a vector valued function, and $f(t)$ is a real valued function, then Theorem 1 and the product rule easily give us that

$$\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)u\vec{(t)} + f(t)\vec{u}\,'(t). \tag{1}$$

Exercise 4. *Suppose $\vec{u}(t)$ and $\vec{v}(t)$ are vector valued functions. Which of the following is true?:*

$$\frac{d}{dt}\Big[\vec{u}(t) \bullet \vec{v}(t)\Big] = \Big(\vec{u}\,'(t) \bullet \vec{v}(t)\Big) + \Big(\vec{u}(t) \bullet \vec{v}\,'(t)\Big) \tag{2}$$

$$\frac{d}{dt}\Big[\vec{u}(t) \times \vec{v}(t)\Big] = \Big(\vec{u}\,'(t) \times \vec{v}(t)\Big) + \Big(\vec{u}(t) \times \vec{v}\,'(t)\Big) \tag{3}$$

Prove your answer is correct.

[1] Limits of vector valued functions are defined by taking the limits of the component functions.

3 Definite Integrals

Definition The *definite integral of a vector valued function*, $\vec{r}(t) =< f(t), g(t), h(t) >$, is defined as

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}. \tag{4}$$

The Fundamental Theorem of Calculus then extends naturally to

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a), \tag{5}$$

where $\vec{R}(t)$ is any antiderivative of $\vec{r}(t)$.

Exercise 5. *Find a parametric equation describing the tangent line to the curve traced out by the vector valued function*

$$\vec{r}(t) =< t, e^t, 2t - t^2 >,$$

at the point $(0, 1, 0)$.

Exercise 6. *Given the vector valued function*

$$\vec{r}(t) =< t^2, t\sqrt{t-1}, t\sin(\pi t) >,$$

evaluate

$$\int_1^2 \vec{r}(t)dt$$

Challenge

Exercise 7. *If a curve has property that the position vector $\vec{r}(t)$ is always perpendicular to the tangent vector $\vec{r}\,'(t)$, show that the curve lies on a sphere centered at the origin.*