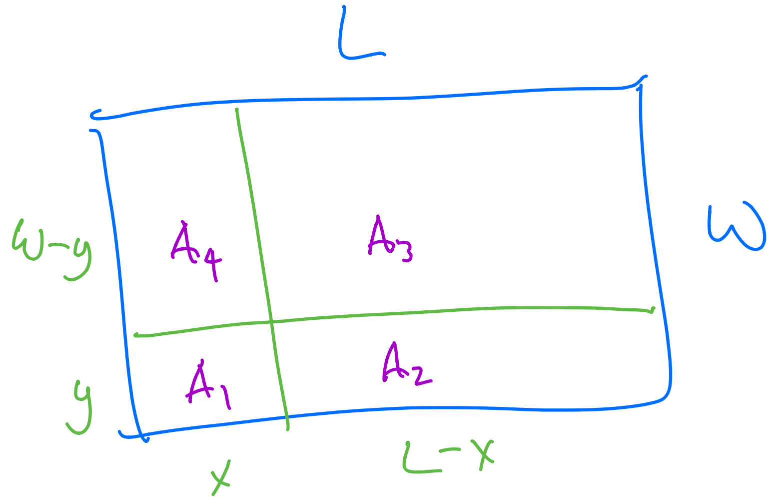
Name:________________________________

Worksheet 13

1 Practice

Exercise 1. *A rectangle with dimensions L and W is cut into four smaller rectangles by two lines, parallel to the sides. Find the minimum and maximum values of the sums of the squares of the areas of the smaller rectangles.*



$A_1 = xy$

$A_2 = (L-x)y$

$A_3 = (L-x)(W-y)$

$A_4 = (W-y)x$

We want to minimize

$$f(x,y) = A_1^2 + A_2^2 + A_3^2 + A_4^2$$

$$= x^2y^2 + (L-x)^2y^2 + (L-x)^2(W-y)^2 + (W-y)^2x^2$$

$$= \left[x^2 + (L-x)^2\right]\left[y^2 + (W-y)^2\right]$$

So, we compute the partial derivatives

$$f_x(x,y) = 2x - 2(L-x)$$

$$f_y(x,y) = 2y - 2(W-y)$$

then check for critical values

$$f_x(x,y) = 0 \iff 2x = 2(L-x) \iff x = L/2$$

$$f_y(x,y) = 0 \iff 2y = 2(W-y) \iff y = W/2$$

Next, we use the second derivative test

$f_{xx} = 4 \quad f_{yy} = 4$

$f_{xy} = 0$

$$\Rightarrow D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 16 > 0 \Rightarrow (L/2, W/2)$$

is a minimum

So, the maximum must occur on the boundary, i.e., when $x = 0$ or L and $y = 0$ or W

Exercise 2. *Suppose f is a differentiable function of a single variable, and consider the surface defined by $z = xf(y/x)$. At which points on this surface do the planes tangent to the surface include the origin?*

Let $g(x,y,z) = x f(\frac{x}{y}) - z$, then we may use ∇g to define the tangent plane to the level surface $g(x,y,z) = 0$.

$$\nabla g = \left\langle f'\left(\frac{x}{y}\right)\frac{x}{y} + f\left(\frac{x}{y}\right), -f'\left(\frac{x}{y}\right)\frac{x^2}{y^2}, -1 \right\rangle$$

So, the tangent plane to the surface at the point (x_0, y_0, z_0) is defined by

$$\nabla g(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$\left(f'\left(\frac{x_0}{y_0}\right)\frac{x_0}{y_0} + f\left(\frac{x_0}{y_0}\right)\right)(x - x_0) + \left(-f'\left(\frac{x_0}{y_0}\right)\frac{x_0^2}{y_0^2}\right)(y - y_0) + (-1)(z - z_0)\Big) = 0$$

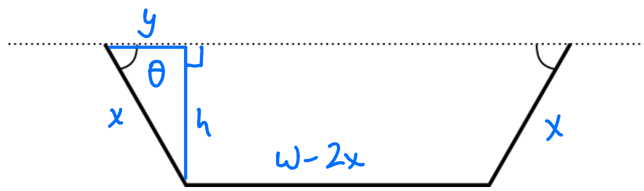
Now, if the tangent plane at this point includes the origin, then $(x,y,z) = (0,0,0)$ is a solution to this equation. So, we are looking for (x_0, y_0, z_0) w/

$$-f'\left(\frac{x_0}{y_0}\right)\frac{x_0^2}{y_0} - x_0 f\left(\frac{x_0}{y_0}\right) + f'\left(\frac{x_0}{y_0}\right)\frac{x_0^2}{y_0} + z_0 = 0$$

or $\quad -x_0 f\left(\frac{x_0}{y_0}\right) + z_0 = 0 \iff x_0 f\left(\frac{x_0}{y_0}\right) - z_0 = 0$

which is true for every point on the surface because this is the equation for the surface. So, every tangent plane to the surface includes the origin.

Exercise 3. *A contractor, bidding on a contract to construct an aqueduct across a valley, hires you as a consultant. The contractor does business with a factory that is able to manufacture rolls of sheet metal, up to w meters wide (they have several options for widths, and they want to know for each width, so they are asking in general). They plan to bend these sheets to form a channel for the water, and want to know how they ought to be bent in order to maximize the amount of water that can travel down the duct.*



A cross section of the duct channel design. The dark line has total length w, and is bent symmetrically.

a) *Assuming the bends are made in a symmetric fashion, where should they be made? How steep should they be (measured relative to the top of the duct)? (Recall: The area of a trapezoid is given by $\frac{1}{2}h(b_1 + b_2)$.)*

b) *Suppose the length of the aqueduct is 2km long, and the contractor who hired you submits their bid with a design that uses sheets of width 3m. A competitor has the ability to use the same types of sheets, and can bend them into semi-circles. If the competitor designs their aqueduct to move the same amount of water, what is the difference in square meters of metal required between the competing designs? If the steel for the duct costs \$60 per square meter, what will the difference be in costs for just this part of the construction?*

a) To maximize the cross section of the duct we want to maximize

$$A = \frac{1}{2}(b_1 + b_2) \cdot h$$ in terms of x and θ, (see diagram)

Note: $\sin\theta = \frac{h}{x} \Rightarrow h = x\sin\theta$

$\cos\theta = \frac{y}{x} \Rightarrow y = x\cos\theta$

So $b_1 = \omega - 2x$

$b_2 = \omega - 2x + 2x\cos\theta$

$h = x\sin\theta$

and

$$A(x,\theta) = \frac{1}{2}\big((\omega - 2x) + (\omega - 2x + 2x\cos\theta)\big) \cdot x\sin\theta$$

$$= \frac{1}{2}(2\omega - 4x + 2x\cos\theta)\, x\sin\theta$$

$$= x^2\cos\theta\sin\theta - 2x^2\sin\theta + \omega x\sin\theta \quad \left(\text{for } \begin{matrix} 0 < x \leq \frac{\omega}{2} \\ 0 < \theta \leq \frac{\pi}{2} \end{matrix}\right)$$

So

$$A_x(x,\theta) = 2x\cos\theta\sin\theta - 4x\sin\theta + \omega\sin\theta$$

$$A_\theta(x,\theta) = x^2(-\sin\theta\sin\theta + \cos\theta\cos\theta) - 2x^2\cos\theta + wx\cos\theta$$
$$= x^2(\cos^2\theta - \sin^2\theta) - 2x^2\cos\theta + wx\cos\theta$$
$$= x^2(2\cos^2\theta - 1) - 2x^2\cos\theta + wx\cos\theta$$

Now,

$A_x(x,\theta) = 0$ when $\sin\theta(2x\cos\theta - 4x + w) = 0$

but $0 < \theta \le \frac{\pi}{2}$, so $\sin\theta \neq 0$ and

$A_x(x_0,\theta) = 0$ when $2x\cos\theta - 4x = -w$

or $\cos\theta = \dfrac{4x-w}{2x}$

If $A_\theta(x,\theta) = 0$, as well, then

$$0 = x^2\left(2\left(\frac{4x-w}{2x}\right)^2 - 1\right) - 2x^2\left(\frac{4x-w}{2x}\right) + wx\left(\frac{4x-w}{2x}\right)$$
$$= x^2\left(2\left(\frac{16x^2 - 8xw + w^2}{4x^2}\right) - 1\right) - x(4x-w) + w\left(\frac{4x-w}{2}\right)$$
$$= 8x^2 - 4xw + \frac{w^2}{2} - x^2 - 4x^2 + wx + 2wx - \frac{w^2}{2}$$
$$= 3x^2 - wx$$
$$= x(3x - w)$$

when $x = \frac{1}{3}w$ and $\cos\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$

We check the boundaries to make sure we have found the global max

$A\left(\frac{w}{2}, \theta\right) = \frac{1}{8}w^2\sin 2\theta$ which is largest when $\theta = \frac{\pi}{4}$

$A\left(x, \frac{\theta}{2}\right) = wx - 2x^2$ for $0 < x \le w/2$ and

we check $x = \frac{w}{2}$ and $x = \frac{w}{4}$ (usual EVT)

Now

$$A\left(\frac{w}{4}, \frac{\pi}{2}\right) = A\left(\frac{w}{2}, \frac{\pi}{4}\right) < A\left(\frac{1}{3}w, \frac{\pi}{3}\right)$$

So, $\left(\frac{1}{3}w, \frac{\pi}{3}\right)$ is the absolute maximum.