Evaluate

$$\int_0^{2\sqrt{3}} \left(\frac{x^3}{\sqrt{16-x^2}}\right) dx.$$

We notice that the function in the denominator includes a number minus a function, which reminds us of the identity

$$1-\sin^2(\theta)=\cos^2(\theta). \tag{1}$$

So, we start by multiplying both sides by 16, to make our constants match, and we consider the equations

$$16\cos^2(\theta)=16-16\sin^2(\theta)=16-x^2.$$

This gives us that

$$\begin{aligned} -16\sin^2(\theta) &= -x^2 && \text{which simplifies to} \\ 4\sin(\theta) &= x && \text{then, applying the differential operator, we get} \\ 4\cos(\theta)d\theta &= dx. \end{aligned} \tag{2}$$

Now, we make our substitution to get

$$\begin{aligned} \int_{x=0}^{x=2\sqrt{3}} \left(\frac{x^3}{\sqrt{16-x^2}}\right) dx &= \int_{\theta=0}^{\theta=\pi/3} \left(\frac{(4\sin(\theta))^3}{\sqrt{16\cos^2(\theta)}}\right) 4\cos(\theta)d\theta \qquad (3) \\ &= 64\int_0^{\pi/3} (\sin^3(\theta))d\theta \\ &= 64\int_0^{\pi/3} (\sin^2(\theta)\sin(\theta))d\theta \\ &= 64\int_0^{\pi/3} ((1-\cos^2(\theta))\sin(\theta))d\theta && \text{Using equation (1).} \end{aligned}$$

Here, we make the substitution $u=\cos(\theta)$, which gives us $du=-\sin(\theta)d\theta$, to get

$$\begin{aligned} 64\int_0^{\pi/3} ((1-\cos^2(\theta))\sin(\theta))d\theta &= -64\int_{u=1}^{u=1/2} ((1-u^2)du \qquad (4) \\ &= -64\left(u-\frac{1}{3}u^3\right)\Bigg|_1^{1/2} \qquad (5) \\ &= -64\left(\cos(\theta)-\frac{1}{3}\cos^3(\theta)\right)\Bigg|_0^{\pi/3} \\ &= -64\left(\left(\frac{1}{2}-\frac{1}{24}\right)-\left(1-\frac{1}{3}\right)\right)=\frac{40}{3}. \end{aligned}$$

Note, the change in the bounds for the integral in equation (3) comes from our changing variable from x to θ. You can use arcsine function and equation (2) to find the proper bounds. A similar switch happens in equation (4), due to the substitution $u=\cos(\theta)$.

It is perfectly okay to finish our computation using the bounds in equation (5). As long as we are careful about the bounds of integration when we make substitutions, we do not have to go all the way back to the original variables.