Name:______________________________________

Worksheet 04

1 Partial Fractions Review and Examples

The goal is the reverse of finding a common denominator, we want to take a single fraction with a complicated denominator and turn it into a sum of fractions with simpler denominators. Let's consider a simple example with integers, rather than polynomials.

Example Write

$$\frac{3}{10}$$

as the sum of two fractions (each smaller, in absolute value, than 1). (and w/ distinct denominators)

$$\frac{3}{10} = \frac{3}{5 \cdot 2}$$

so, we consider

$$\frac{a}{5} + \frac{b}{2} = \frac{2a+5b}{10}$$

If we set $a=-1$ and $b=1$ we get

$$\frac{3}{10} = \frac{-1}{5} + \frac{1}{2}$$

which was our goal.

Now, lets consider a simple polynomial.

Example Write

$$\frac{3}{x^2+3x+2} \xrightarrow{\text{factors as}} \frac{3}{(x+1)(x+2)}$$

as the sum of two rational functions.

So we consider

$$① \quad \frac{A}{x+1} + \frac{B}{x+2}$$

numerators one degree lower than the factor they are over

$$= \frac{(A+B)x + (2A+B)}{(x+1)(x+2)}.$$

Now, if we solve

$A+B=0$ and $2A+B=3$ $\xRightarrow{\text{then}}$ $A=3$, $B=-3$

Then, plugging back in to ① we get

$$\frac{3}{(x+1)(x+2)} = \frac{3}{x+1} - \frac{3}{x+2}$$

Example Write

$$\frac{x^2}{x^3+x^2+x+1}$$

as the sum of two fractions.

① Factor $\dfrac{x^2}{(x^2+1)(x+1)}$ ② then $= \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+1}$

③ expand $= \dfrac{Ax^2+Ax+Bx+B+Cx^2+C}{(x^2+1)(x+1)}.$

④ Set up equations to match numerators.

$$\left.\begin{aligned} A+C&=1 \\ A+B&=0 \\ B+C&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} A&=\tfrac{1}{2} \\ B&=-\tfrac{1}{2} \\ C&=\tfrac{1}{2} \end{aligned}$$

⑤ Pluging in, we get

$$\frac{x^2}{x^3+x^2+x+1} = \frac{1}{2}\left(\frac{x+1}{x^2+1} + \frac{1}{x+1}\right)$$

Example Write

$$\frac{x^2+x+4}{x^4+2x^2+1}$$

as the sum of two fractions.

Note: we cannot use $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+1}$ to break this rational function up because their common denominator is not that of our starting function.

Step 1

$$\frac{x^2+x+4}{x^4+2x^2+1}=\frac{x^2+x+4}{(x^2+1)^2}$$

Step 2

$$=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{(x^2+1)^2}$$

notice that we choose general forms that are one degree lower than the factor in the denominator, not the denominator itself (you can do either way, but this is easier)

Step 3

$$=\frac{(Ax+B)(x^2+1)}{(x^2+1)(x^2+1)}+\frac{Cx+D}{(x^2+1)^2}$$

here only one of our fractions needed a new factor to get a common denominator

Expanding and comparing numerators gives us

Step 4

$$Ax^3+Ax+Bx^2+B+Cx+D=x^2+x+4$$

So, we solve Step 5

$$\left.\begin{array}{c} A=0 \\ B=1 \\ A+C=1 \\ B+D=4 \end{array}\right\} \Rightarrow \begin{array}{c} A=0 \\ B=1 \\ C=1 \\ D=3 \end{array}$$

Step 6

and, thus, $\frac{x^2+x+4}{x^4+2x^2+1}=\frac{1}{x^2+1}+\frac{x+3}{(x^2+1)^2}$

which is what we wanted.

Notice, the reasons we pick the numerators the way that we do are:

- We've already done polynomial long division, so we know the numerator is of larger degree than the denominator.
- We don't know, ahead of time, how the numerators will combine with the factors in the denominator so we assign a general polynomial of one degree lower.

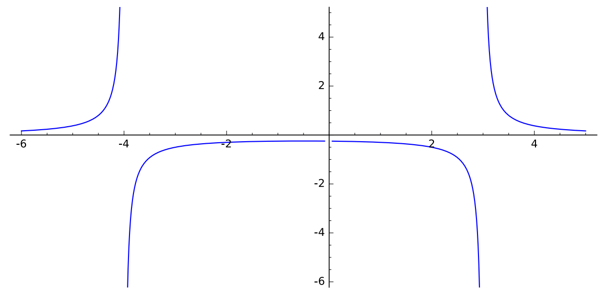
2 Improper Integrals

Improper integrals are, essentially, definite integrals whose bounds of integration are not in the domain of the integrand. They are most commonly encountered when the integrand has vertical asymptotes, or asymptotically approaches the x-axis.

Example The graph of

$$f(x) = \frac{3}{x^2 + x - 12}$$

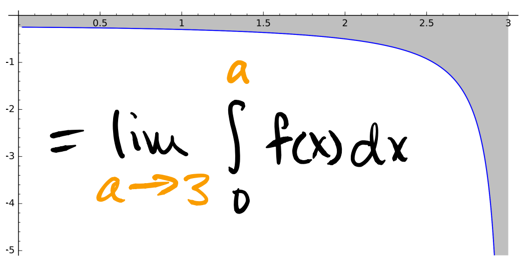
looks like this



We might be interested in computing the definite integral near, or away from, the singularities. That is, we might be interested evaluating any of the following:

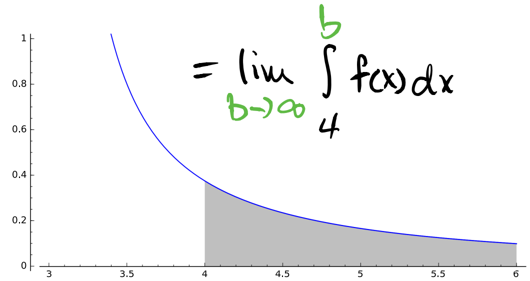
$$\int_0^3 \left(\frac{3}{x^2 + x - 12}\right) dx$$

$$= \lim_{a \to 3} \int_0^a f(x)\,dx$$



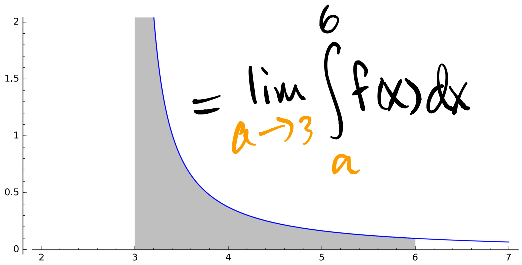
$$\int_4^\infty \left(\frac{3}{x^2 + x - 12}\right) dx$$

$$= \lim_{b \to \infty} \int_4^b f(x)\,dx$$



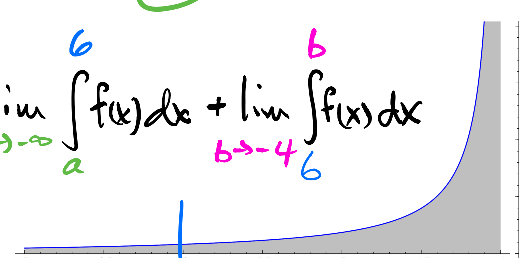
$$\int_3^6 \left(\frac{3}{x^2 + x - 12}\right) dx$$

$$= \lim_{a \to 3} \int_a^6 f(x)\,dx$$



$$\int_{-\infty}^{-4} \left(\frac{3}{x^2 + x - 12}\right) dx$$

$$= \lim_{a \to -\infty} \int_a^6 f(x)\,dx + \lim_{b \to -4} \int_6^b f(x)\,dx$$



← you chose this (here, 6)

A key point to remember is that an improper integral is a definite integral where *exactly one* of the bounds of integration does not lie in the domain of the integrand, which is a continuous function on the rest of the interval defined by these bounds. To formally define such an integral, we replace the problematic bound with a variable, use the properties of limits to make our computations.

Exercise 1. *Write*

$$\frac{4}{21}$$

as the sum of two fractions (each smaller, in absolute value, than 1).

First, we note that $\frac{4}{21} = \frac{4}{7 \cdot 3}$.

Next we set $\frac{A}{7} + \frac{B}{3} = \frac{4}{21}$ and find a common denominator on the left to get $\frac{3A+7B}{21} = \frac{4}{21}$. So, if we set $A=-1$ and $B=1$ we get

$$-\frac{1}{7} + \frac{1}{3} = \frac{4}{21}$$

Exercise 2. *Write*

$$\frac{5}{12}$$

as the sum of three fractions (each with a distinct denominator and smaller, in absolute value, than 1). How many different ways can you write it as the sum of two fractions (with the same restrictions)?

Note that $\frac{5}{12} = \frac{5}{2 \cdot 2 \cdot 3}$

So, we try to find a solution to

$$\frac{5}{12} = \frac{A}{2} + \frac{B}{2^2} + \frac{C}{3}$$

note: this is similar to our polynomial example, before, in that we cannot start w/ $\frac{A}{2} + \frac{B}{2} + \frac{C}{3}$ because then our common denom is 6, and not 12.

So, we get

$$\frac{A \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3} + \frac{B \cdot 3}{2^2 \cdot 3} + \frac{C \cdot 2^2}{3 \cdot 2^2}$$

Note, again, we need only add the missing factors to each term

Now, to match denominators and stay w/in the given restrictions we need to find A, B and C w/

$$6A + 3B + 4C = 5 \quad \text{where}$$

$$0 < |A| < 2$$
$$0 < |B| < 4$$
$$0 < |C| < 3$$

Since we have the fewest options for A, we can start there:

<u>Case 1</u>: If $A = 1$ then we need to find B and C with $6 + 3B + 4C = 5$.

We can see that $B = 1$ and $C = -1$ work, so we get

$$\frac{5}{12} = \frac{1}{2} + \frac{1}{4} - \frac{1}{3}$$

As we have a solution, we are done.

For the second part of the question we might try finding solutions to:

$$\frac{5}{12} = \frac{A}{3} + \frac{B}{4}$$

$$\frac{5}{12} = \frac{A}{6} + \frac{B}{4}$$

Note: the important thing is that our proposed decomposition is chosen so that the common denom. is equal to the one we want.

First, we factor

Exercise 3. *Write*

$$\frac{3}{x^2+x-12} = \frac{3}{(x-3)(x+4)}$$

as the sum of two rational functions.

now, we set $\dfrac{3}{x^2+x-12} = \dfrac{A}{x-3} + \dfrac{B}{x+4}$

we cross multiply and compare numerators to get the system of equations

$$\begin{array}{ll} ① & A+B=0 \\ ② & 4A-3B=3 \end{array} \overset{②+3\cdot ①}{\Longrightarrow} 7A=3 \overset{①}{\Longrightarrow} \begin{array}{l} A=\frac{3}{7} \\ B=-\frac{3}{7} \end{array}$$

So $\dfrac{3}{x^2+x-12} = \dfrac{3}{7}\left(\dfrac{1}{x-3} - \dfrac{1}{x+4}\right)$

Exercise 4. *Write*

$$\frac{x^2-x+4}{x^4+6x^2+9} \overset{\text{factoring}}{=} \frac{x^2-x+4}{(x^2+3)^2}$$

as the sum of two rational functions.

now, we set $\dfrac{x^2-x+4}{x^4+6x^2+9} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{(x^2+3)^2}$

we cross multiply and compare numerators to get

$$Ax^3+3Ax+Bx^2+3B+Cx+D = x^2-x+4$$

which yeilds the system of equations

$$\left.\begin{array}{r} A=0 \\ B=1 \\ 3A+C=-1 \\ 3B+D=4 \end{array}\right] \Longrightarrow \begin{array}{l} A=0 \\ B=1 \\ C=-1 \\ D=1 \end{array}$$

so $\dfrac{x^2-x+4}{x^4+6x^2+9} = \dfrac{1}{x^2+3} + \dfrac{-x+1}{(x^2+3)^2}$

Exercise 5. *Compute the integrals in the example at the beginning of Section 2. (Hint: Use the partial fraction decomposition you came up with in Exercise 3 to get started.)*

Using our work from 3, we get that

$$\int \left(\frac{3}{x^2+x-12}\right)dx = \int \left(\frac{3}{7}\left(\frac{1}{x-3} - \frac{1}{x+4}\right)\right)dx$$

$$= \frac{3}{7}\left(\ln|x-3| - \ln|x+4|\right) + C$$

Now, we can use the FTC to evaluate the desired integrals:

$$\lim_{a\to 3}\int_0^a f(x)\,dx = \lim_{a\to 3}\Big(\big(\ln|a-3| - \ln|a+4|\big) - \big(\ln|3| - \ln|4|\big)\Big)$$

$$\big(\; -\infty \;-\; \ln|7|\big) - \big(\ln|3| - \ln|4|\big)$$

Diverges.

$$\lim_{b\to\infty}\int_4^b f(x)\,dx = \lim_{b\to\infty}\Big(\big(\ln|b-3| - \ln|b+4|\big) - \big(\ln|1| - \ln|8|\big)\Big)$$

$$\big(\;\infty \;-\; \infty\;\big) - \big(\;0\; - \ln|8|\big)$$

WARNING!

$\infty - \infty \neq 0$

(we need to use properties of logarithms to fix this)

$$= \lim_{b\to\infty}\left(\underbrace{\ln\left|\frac{b-3}{b+4}\right|}_{\ln|1|=0} + \ln|8|\right)$$

$$= \ln|8|.$$

$$\lim_{a \to 3} \int_a^6 f(x)\,dx = \quad \ldots$$

(see me if you need help with these)

$$\lim_{a \to -\infty} \int_a^6 f(x)\,dx + \lim_{b \to -4} \int_6^b f(x)\,dx = \quad \ldots$$

Exercise 6. *Compute*

$$\int_0^\infty \frac{x^2 - x + 4}{x^4 + 6x^2 + 9}$$

(Hint: Use the partial fraction decomposition you came up with in Exercise 4 to get started.)

From exercise 4, we get that

$$\int \left(\frac{x^2 - x + 4}{x^4 + 6x^2 + 9}\right) dx = \int \left(\frac{1}{x^2+3} + \frac{-x+1}{(x^2+3)^2}\right) dx$$

$$= \underline{\int \left(\frac{1}{x^2+3}\right) dx} + \underline{\int \left(\frac{-x+1}{(x^2+3)^2}\right) dx}$$

To solve

$$\int \left(\frac{1}{x^2+3}\right) dx$$

we make a substitution by using

$$\tan^2\theta + 1 = \sec^2\theta$$

to set

$$x^2 + 3 = 3\tan^2\theta + 3 = 3\sec^2\theta$$

to get

$$\frac{x}{\sqrt{3}} = \tan\theta^{\star} \quad \text{and thus} \quad \frac{1}{\sqrt{3}}\, dx = \sec^2\theta \, d\theta .$$

So, we have

$$\int \left(\frac{1}{x^2+3}\right) dx = \sqrt{3} \int \frac{\sec^2\theta}{3\sec^2\theta} \, d\theta = \frac{\sqrt{3}}{3}\theta + C$$

from $\star$ we have that $\theta = \arctan\left(\frac{x}{\sqrt{3}}\right)$, so

$$\int \left(\frac{1}{x^2+3}\right) dx = \frac{\sqrt{3}}{3}\left(\arctan\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

To solve

$$\int\left(\frac{-x+1}{(x^2+3)^2}\right)dx,$$ we first re-write as

$$-\int\left(\frac{x}{(x^2+3)^2}\right)dx + \int\left(\frac{1}{(x^2+3)^2}\right)dx$$

$u = x^2+3$

$du = 2x\,dx$

$$= -\frac{1}{2}\int\left(\frac{1}{u^2}\right)du + \int \frac{1}{(x^2+3)^2}\,dx$$

$$= \frac{1}{2}\left(\frac{1}{x^2+3}\right) + \underbrace{\int \frac{1}{(x^2+3)^2}\,dx}_{\text{another decomposition and trig sub later...}}$$

$$= \frac{1}{2}\left(\frac{1}{x^2+3}\right) + \frac{1}{18}\left(\frac{3x}{x^2+3} + \sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

So, finally, we get that

$$\int\left(\frac{x^2-x+4}{x^4+6x^2+9}\right)dx = \int\left(\frac{1}{x^2+3}\right)dx + \int\left(\frac{-x+1}{(x^2+3)^2}\right)dx$$

$$= \frac{x+3}{6x^2+18} + \frac{7\arctan\left(\frac{x}{\sqrt{3}}\right)}{6\sqrt{3}} + C$$

Now, we apply the definition of an improper integral to get

$$\int_0^\infty \left(\frac{x^2-x+4}{x^4+6x^2+9}\right)dx = \lim_{b\to\infty} \int_0^b \left(\frac{x^2-x+4}{x^4+6x^2+9}\right)dx$$

So, by FTC, we got

$$\int_0^\infty \left(\frac{x^2-x+4}{x^4+6x^2+9}\right)dx = \lim_{b\to\infty}\left(\underbrace{\frac{b+3}{6b^2+18}}_{\to 0} + \overbrace{\frac{7\arctan\left(\frac{b}{\sqrt{3}}\right)}{6\sqrt{3}}}^{\to \pi/2}\right)$$

$$-\left(\frac{0+3}{6 0^2+18} + \frac{7\arctan\left(\frac{0}{\sqrt{3}}\right)}{6\sqrt{3}}\right)$$

$$= \frac{7\pi}{12\sqrt{3}} - \frac{3}{18},$$

and we are done.

3 Optional Practice

Exercise 7. *How might you apply your observations about the number of ways you can write*

$$\frac{5}{12}$$

from, Exercise 2 to find a partial fraction decomposition of

$$\frac{x}{(x+1)(x-3)^2}.$$

What if one of the factors were repeated three times?

Exercise 8. *Consider integrals of the following forms:*

i) $\displaystyle\int_0^1 \left(\frac{1}{x^p}\right) dx$

ii) $\displaystyle\int_1^\infty \left(\frac{1}{x^p}\right) dx$

For what values of p does each converge? For what values of p do they diverge?

Exercise 9. *Partial fraction decompositions allow us to compute the integrals of many rational functions. The principal being, if we can break our initial integral into a sum of parts with less complicated denominators, then we can use tools we have to evaluate the integral. The following are general forms of the types of integrands we need to evaluate after completing a partial fraction decomposition. If you are able to find general rules for these forms, then you are able to evaluate entire families of rational functions.*

Find the general rule for each of the following integrals. Pay careful attention to any assumptions you make about the coefficients, and break your solutions into cases if needed.

i) $\displaystyle\int \left(\frac{A}{Bx+E}\right) dx$

ii) $\displaystyle\int \left(\frac{Ax+B}{(Cx+D)^2}\right) dx$

iii) $\displaystyle\int \left(\frac{Ax^2+Bx+C}{(Dx+E)^3}\right) dx$

iv) $\displaystyle\int \left(\frac{a_1x+a_0}{b_2x^2+b_1x+b_0}\right) dx$

v) $\displaystyle\int \left(\frac{a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{(b_2x^2+b_1x+b_0)^2}\right) dx$