Name:____________________________________

Worksheet 10

1 Warm-Up

Exercise 1. *Determine the domain of the following functions of more than one variable. Sketch the domain, or describe it geometrically, if possible.*

a) $r(x, y) = \dfrac{\sqrt{x^2 + y^2}}{xy + 1}$

$xy \neq -1$

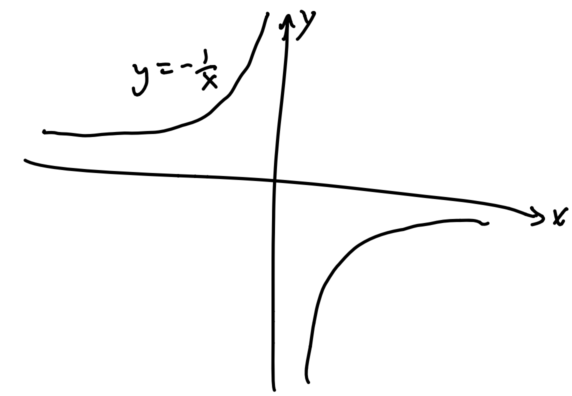
or

$x \neq -\frac{1}{y}$

or

$y \neq -\frac{1}{x}$

$D = \mathbb{R}^2 -$

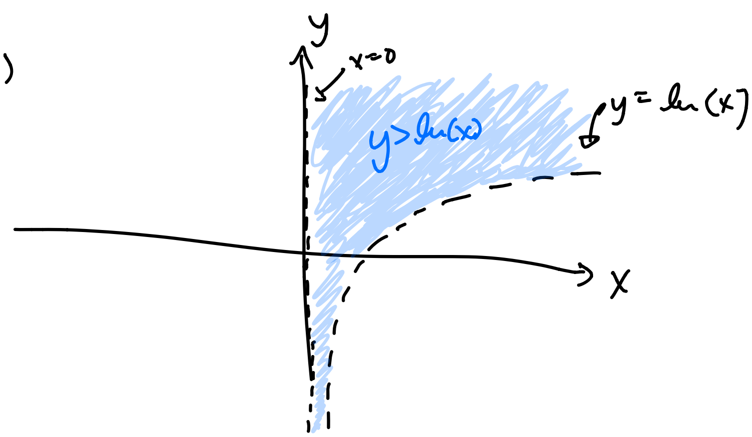


b) $q(x, y) = \ln(y - \ln(x))$

$y - \ln(x) > 0$

and $x > 0$

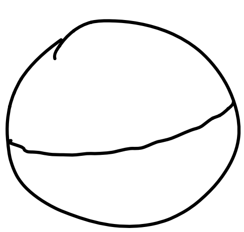
$\Longrightarrow y > \ln(x)$, $x > 0$



c) $p(x, y, z) = \dfrac{\sqrt{16 - x^2 - y^2 - z^2}}{xyz}$

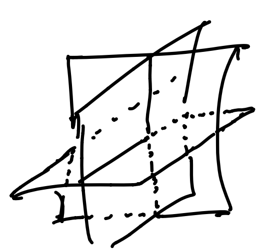
$x^2 + y^2 + z^2 \leq 16$ ⟵ a solid ball centered at the origin of radius 4.

$x \neq 0$, $y \neq 0$, $z \neq 0$ $\Longrightarrow$ no points on any of the $x=0$ $y=0$ or $z=0$ planes



solid

$-$



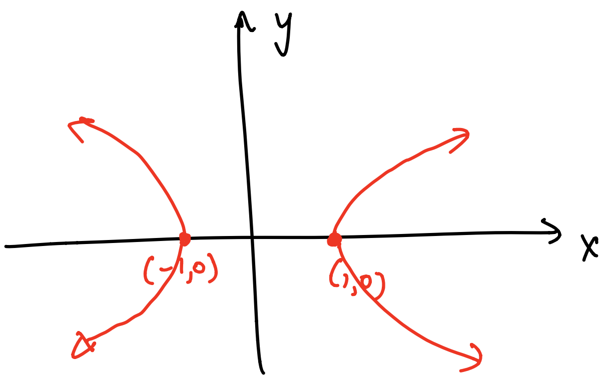
2 Functions of Several Variables and Partial Derivatives

For the remainder of this section, consider the following function:

$$z = f(x, y) = x^2 - y^2 \tag{1}$$

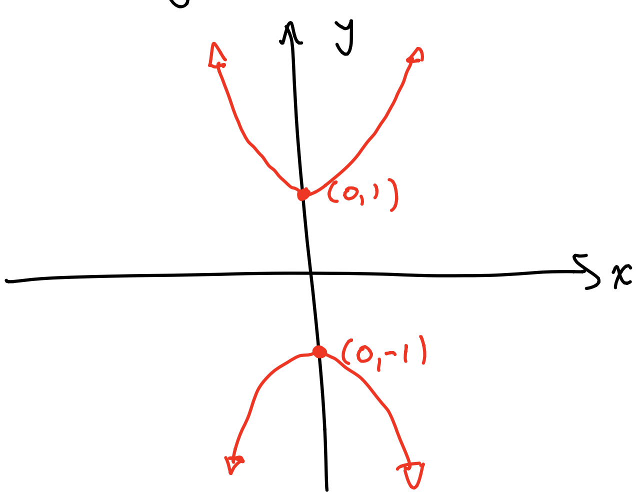
Exercise 2. *Sketch the level set associated with $z = 1$. (Recall, the level set of a function associated with any constant is the set of all inputs which the function send to that constant.)*

$1 = x^2 - y^2$



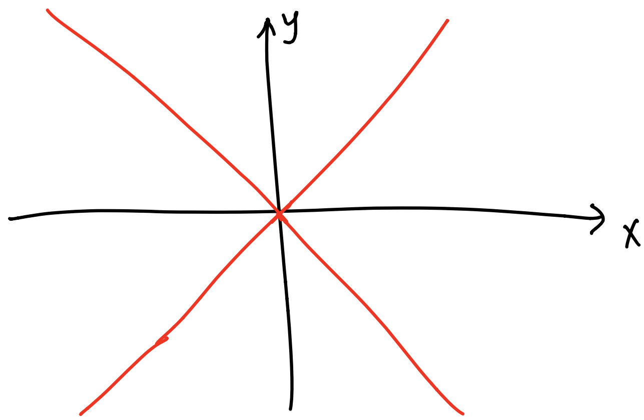
Exercise 3. *Sketch the level set associated with $z = -1$.*

$-1 = x^2 - y^2$

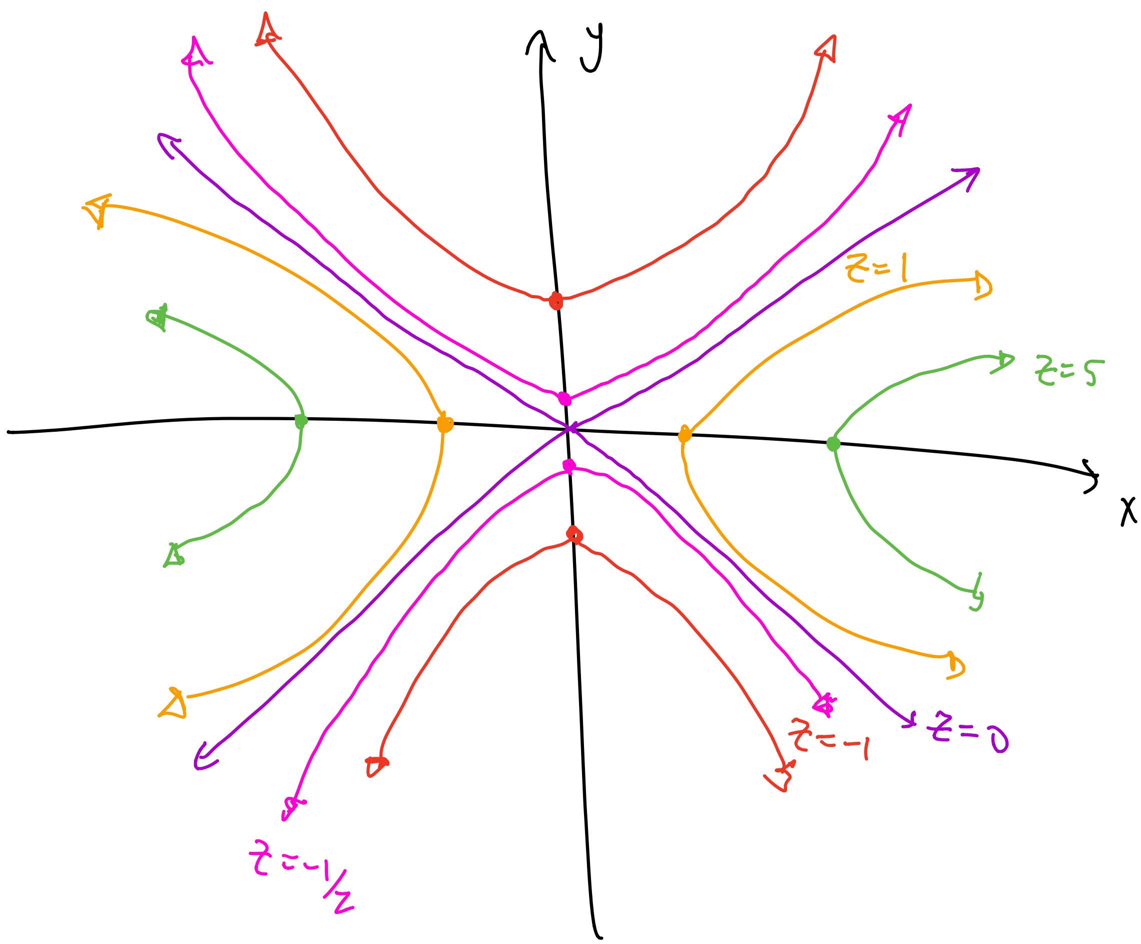


Exercise 4. *Sketch the level set associated with $z = 0$.*

$$0 = x^2 - y^2 \Rightarrow x^2 = y^2$$

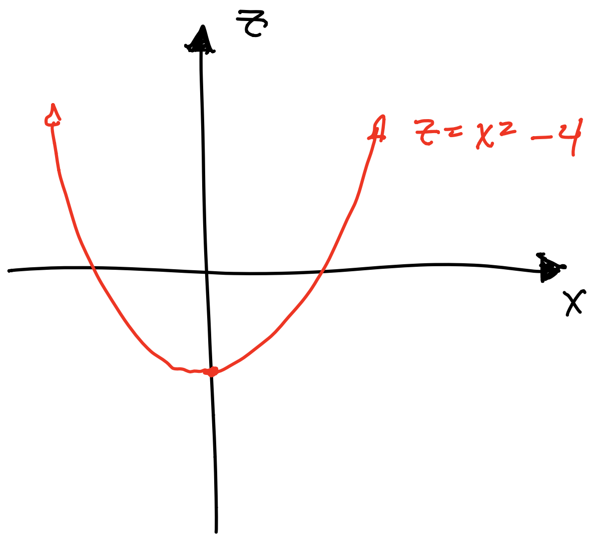


Exercise 5. *Combine your answers to Exercises 2,3, and 4 on a single set of axes, along with the level sets associated to $z = -1/2$, and $z = 5$. Label each contour curve with the z-value to which it is associated.*



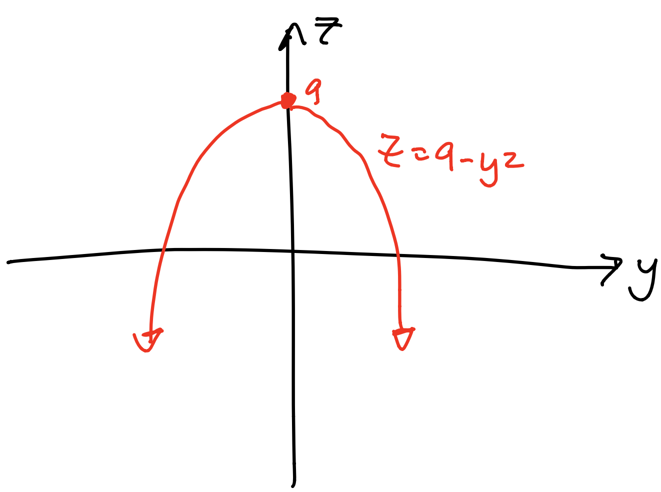
Exercise 6. *Suppose we define $g(x) = f(x, b)$, where $b = 2$. In which plane does the graph of g lie? Sketch this graph of g in this plane, with the axes labeled appropriately.*

$g(x) = f(x,2) = x^2 - 4$, lies in the $y=2$ paine



Exercise 7. *Suppose we define $h(y) = f(a, y)$, where $a = -3$. In which plane does the graph of h lie? Sketch the graph of h in this plane, with the axes labeled appropriately.*

$h(y) = f(-3, y) = 9 - y^2$ lies in the $x=-3$ plane



Exercise 8. *At what point do the graphs of g and h intersect? Find the slope of the tangent line to g at this point of intersection, in the plane where the graph of g lies. Then, find the slope of the tangent line to h at this point of intersection, in the plane where the graph of h lies.*

g lies on the $y=2$ plane $\quad g(-3)=5$

h lies on the $x=-3$ plane $\quad h(2)=5$

so the curves intersect at the point $(-3,2,5)$

the tangent line to g at $x=-3$ has slope $g'(-3)=-6$

the tangent line to h at $y=2$ has slope $h'(2)=4$

(Note: the slope of g is $-6=\frac{\Delta z}{\Delta x}$ in the $y=2$ plane

and the slope of h is $4=\frac{\Delta z}{\Delta y}$ in the $x=-3$ plane)

Note: the computations you've just done (to get the slopes of the tangent lines) gives you the value of the partial derivatives of f with respect to x and then to y, at the point of intersection. To compute partial derivatives, in general, we simply treat as constant all of the independent variables of our function, except for the one with respect to which we are differentiating.

Example If $W(x,y,z) = 2xy + yz^3 + 1$, we can compute partial derivatives with respect to any of the independent variables and the results will be

$$\begin{aligned} W_x(x,y,z) &= 2y, \\ W_y(x,y,z) &= 2x + z^3, \\ W_z(x,y,z) &= 3z^2y. \end{aligned}$$

The subscript denotes which variable we are differentiating with respect to.

Formally, we have the following definition:

Definition Given a function of several variables,

$$f(x_1, x_2, \ldots, x_n),$$

We define the derivative of f with respect to x_i, for any $i =, 1, 2 \ldots, n$, by

$$f_{x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{h}$$

whenever this limit exists.

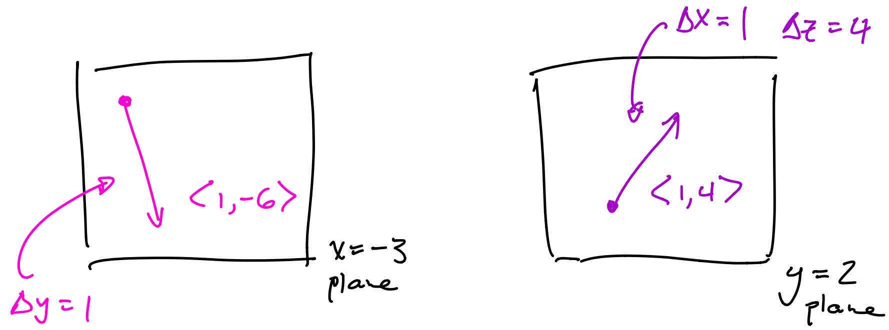
Exercise 9. *Using the general rule of treating as constant any variables other than the one you are differentiating with respect to, verify the following:*

$$\begin{aligned} f_x(-3,2) &= g'(-3) \\ f_y(-3,2) &= h'(2) \end{aligned}$$

$$f(x,y) = x^2 - y^2$$

$$\left.\begin{aligned} f_x(x,y) &= 2x \\ f_y(x,y) &= -2y \end{aligned}\right\} \Rightarrow \begin{aligned} f_x(-3,2) &= -6 = g'(-3) \checkmark \\ f_y(-3,2) &= 4 = h'(2) \checkmark \end{aligned}$$

Exercise 10. *Find vectors in $\mathbb{R}^3$ that are parallel to the tangent lines you found in Exercise 8.*



to take these into 3-dimensions we need to ensure they stay parallel to the planes in which they lie. Thus, they cannot have a non-zero entry in the corresponding component.

$\langle 1,-6\rangle$ becomes $\langle 0,1,-6\rangle$, so it remains parallel to a line w/ a fixed x component.

$\langle 1,4\rangle$ becomes $\langle 1,0,4\rangle$, so it remains parallel to a line w/ a fixed y-component.

Exercise 11. *The vectors that you found in Exercise 10 can be used to determine the plane tangent to the graph of $f(x, y)$ at the point where g and h intersect. Find the equation of this tangent plane.*

As each of the vectors we just found is tangent to the graph of $f(x,y)$ at the point $(-3,2,5)$ we can use the cross product to find a normal vector:

$$\vec{n} = \langle 0,1,-6\rangle \times \langle 1,0,4\rangle$$

$$= \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 0 & -6 \\ 1 & 4 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\hat{k}$$

$$= \langle 4,-6,-1\rangle$$

and then $\langle 4,-6,-1\rangle \cdot \langle x+3, y-2, z-5\rangle = 0$ gives an equation of the tangent plane.

3 Practice

Exercise 12. *Compute the partial derivatives with respect to* x, y *and* z *of*

$$f(x, y) = xz - xyz\tan(xz)$$

Mixed partial derivatives are the result of taking partial derivatives of partial derivatives. For example $(f_x)_y(x, y, z) = f_{xy}(x, y, z)$. Note: the convention is to take the partial derivatives in the order they appear from left to right.

Exercise 13. *Given the function*

$$f(x, y) = x^3y^5 + 2x^4y$$

compute the following:

a) f_{xy}

b) f_{yx}

c) f_{xx}

d) f_{yy}

e) f_{xyx}

f) f_{yxy}

Exercise 14. *Find the equation of the tangent plane to*

$$f(x, y) = x^2 + xy + 3y^2$$

at the point $(1, 1, 5)$.

Exercise 15. *Given a function*

$$f(x, y) = z$$

Develop an equation for the tangent plane to the graph of f *at the point* $(a, b, f(a, b))$ *using the tools we have developed involving vectors and partial derivatives.*

Exercise 16. *Tangent lines give us an approximation of a curve at a given point. Similarly, tangent planes give us an approximation of a surface at a given point. Explain how you could use the equation of the plane tangent to*

$$f(x, y) = \frac{xy\sin(x - y)}{1 + x^2 + y^2}$$

at the point $(1, 1, 0)$ *to approximate the value of* $f(3, 1)$*. Would this be a good approximation? Explain.*