Name:____________________________________

Worksheet 09

1 Projectile Motion

Exercise 1. *Given a projectile with initial velocity* $||\vec{v}(0)|| = v_0$*, that is launched with an angle of elevation* α*, from the surface of a planet with gravitational constant* g*, derive a vector valued function describing the position vector with respect to time.*

2 Kepler's First Law of Planetary Motion

First we recall two of Newton's Laws:

Law (Second Law of Motion)**.**

$$\vec{F} = m\vec{a} \tag{1}$$

Where $\vec{F}$ is force and m is mass.

Law (Law of Universal Gravitation Between a Planet and a Star)**.**

$$\vec{F} = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\vec{u}. \tag{2}$$

Where $\vec{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the star, respectively, $\vec{r}$ is the position vector of the planet, and $\vec{u} = \frac{1}{r}\vec{r}$, where $r = ||\vec{r}||$.

Law (Kepler's First Law of Planetary Motion)**.** *A planet revolves around a star with an elliptical orbit with the star at one focus.*

The proof of this law happens in two parts:

1. We show that the planet moves around the star in a fixed plane, relative to the star.

2. We show that the path it traces in this plane is an ellipse.

Exercise 2. *Using equations (1) and (2), conclude that the position and acceleration vectors are parallel. What does this tell us about $\vec{r} \times \vec{a}$?*

Exercise 3. *Compute*

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right),$$

where $\vec{v}$ is the velocity vector of the planet.

Exercise 4. *We may assume that $\vec{r}$ and $\vec{v}$ are not parallel. If we let $\vec{h} = \vec{r} \times \vec{v}$, what doe we know about $\vec{h}$? (Aside from the fact that it is orthogonal to each of $\vec{r}$ and $\vec{v}$).*

Exercise 5. *How can we use properties of $\vec{h}$ to conclude that the curve traced out by $\vec{r}$ is a plane curve. That is, that the planet moves in a fixed plane, relative to the star?*

Exercise 6. *Fill in the missing steps in the following computation:*

$$\begin{aligned}\vec{h} &= \vec{r} \times \vec{v} \\ &= \dots \\ &= r\vec{u} \times (r\vec{u})' \\ &= \dots \\ &= r^2(\vec{u} \times (\vec{u})').\end{aligned}$$

Exercise 7. *Now, express* $\vec{a} \times \vec{h}$ *in terms of* $G, M,$ *and* $\vec{u}$*. Use 13.6.8 Property 6*

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

to simplify your answer.

Exercise 8. *Show that, if a vector valued function has constant magnitude, then that function and its derivative are always orthogonal.*

Exercise 9. *Using the fact from the previous exercises, verify that*

$$\vec{a} \times \vec{h} = GM(\vec{u})'$$

and, thus,

$$(\vec{v} \times \vec{h})' = GM(\vec{u})'$$

.

Note, that integrating both sides of the equation from Exercise 9 gives us

$$\vec{v} \times \vec{h} = GM\vec{u} + \vec{C} \tag{3}$$

where $\vec{C}$ is a constant vector.

Until now, we haven't chosen our coordinate axes. To make life a bit more convenient, we will just arange our planet-star system so that the vector $\vec{h}$ points in the standard z-axis direction. That is, we ensure that $\vec{h}$ and $\hat{k}$ are parallel.

Exercise 10. *In which plane does the constant vector from equation (3) lie?*

Exercise 11. *If θ is the angle between $\vec{r}$ and $\vec{C}$, then (r, θ) are the polar coordinates of the planet. Verify that*

$$\vec{r} \bullet (\vec{v} \times \vec{h}) = GMr + rc\cos(\theta),$$

where $c = ||\vec{c}||$.

Now, we have that

$$r = \frac{\vec{r} \bullet (\vec{v} \times \vec{h})}{GM + c\cos(\theta)} = \frac{1}{GM} \frac{\vec{r} \bullet (\vec{v} \times \vec{h})}{1 + d\cos(\theta)} \tag{4}$$

where $d = c/GM$.

Exercise 12. *Simplify the numerator of equation (4). (Denote $||\vec{h}|| = h$.)*

Exercise 13. *Let $f = h^2/d$. Simplify the equation from the previous exercise so that it is in terms of r, d, f and θ, only. Compare the result to Theorem 11.6.6 in your book.*