Generalized Partial derivatives to get Directional derivatives

Defined the gradient of f.

Used the gradient to find the direction of greatest change.

Used the gradient to generate tangent planes to level surfaces.

ex

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$

what is the direction of the greatest rate of change at the point $(1,2,-2)$

$$\nabla f = \langle f_x , f_y , f_z \rangle$$

$$= \langle (x^2+y^2+z^2)^{-\frac{1}{2}} \cdot x , (x^2+y^2+z^2)^{-\frac{1}{2}} \cdot y , (x^2+y^2+z^2)^{-\frac{1}{2}} \cdot z \rangle$$

$$\nabla f(1,2,-2) = \langle \frac{1}{3} \cdot 1 , \frac{1}{3} \cdot 2 , \frac{1}{3} \cdot (-2) \rangle$$

$$= \langle \frac{1}{3} , \frac{2}{3} , -\frac{2}{3} \rangle$$

Suppose $yz = \ln(x+z)$ defines a surface.

find the tangent plane @ $(0,0,1)$

Let $f(x,y,z) = yz - \ln(x+z)$

$f(x,y,z) = 0$ is a level surface of f

$$\nabla f = \left\langle -\frac{1}{x+z},\ z,\ y - \frac{1}{x+z} \right\rangle$$

$$\vec{n} = \nabla f(0,0,1) = \langle -1, 1, -1 \rangle$$

$(x_0, y_0, z_0) = (0,0,1)$

$$\boxed{\langle -1, 1, -1 \rangle \cdot \langle \underset{x-x_0}{x},\ \underset{y-y_0}{y},\ \underset{z-z_0}{z-1} \rangle = 0}$$ expand

$$\vec{n} \cdot \langle \vec{x} - \vec{x}_0 \rangle = 0$$

$D_{\vec{u}} f$

$\vec{u} = \langle a, b \rangle$ is a unit vector

$D_{\vec{u}} f = f_x a + f_y b$

$D_{\vec{u}} f(c,d)$ is the rate of change in z as we head in the $\vec{u}$ direction at the point (c,d)

Crit points in higher dimensions

Check where $f_x = f_y = 0$

To see if these are min/max's or saddle

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - [f_{xy}]^2$$

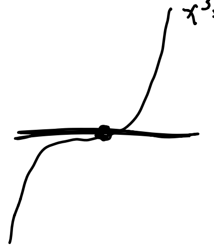
$D > 0$ and $f_{xx} > 0$ then we are at a local min

$D > 0$ and $f_{xx} < 0$ then we are at a local max

$D < 0$ then we are at a saddle

$D = 0$ we know nothing.

$x^3 = y$



$x^2 - y^2 = z$

extreme values

ex $f(x) = 2x^2 + 1$ on $0 \leq x \leq 4$

find all local and global extrema

1 - find crit points
2 - plug those and the endpoints into f
3 - rank them.

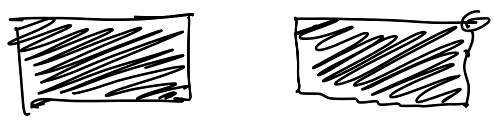
In functions of 2-variables

The Extreme Value Theorem:

If f is continuous on a closed region R, then f has an absolute max and an absolute min in R.

A closed region is one that contains all of its boundary points

ex



closed

not closed

Ex

find the absolute min and max of

$$f(x,y) = x^4 + y^4 - 4xy + 2$$

on the region

$$R = \{(x,y) \mid 0 \le x \le 3, \text{ and } 0 \le y \le 2\}$$

all points (x,y) such that ________

Find crit pts.

$$f_x = 4x^3 - 4y$$

$$f_y = 4y^3 - 4x$$

want $f_x=0$ and $f_y=0$

$$\left.\begin{array}{ll} f_x=0 \text{ when } x^3=y \\ f_y=0 \text{ when } y^3=x \end{array}\right\} \Rightarrow (x^3)^3=x$$

$$\Rightarrow x(x^8-1)=0$$

when $x=0$ or $x=\pm 1$

Crit points are

$(0,0)$ ← in R

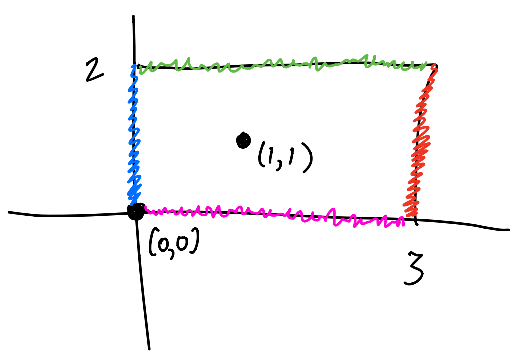
$(1,1)$ ← in R

$(-1,-1)$ ← not in R

$$f(x,y)=x^4+y^4-4xy+2$$

on the region

$$R=\{(x,y)\mid 0\le x\le 3, \text{ and } 0\le y\le 2\}$$



- $f(0,y)=y^4+2 \quad 0\le y\le 2$
- $f(3,y)$
- $f(x,0)$
- $f(x,2)$

$f(x,y) = 2x^3 + y^4 \qquad D = \{(x,y) \mid x^2+y^2 \leq 1\}$

$f_x = 6x^2 \qquad f_x = 0$ when $x=0$

$f_y = 4y^3 \qquad f_y = 0$ wh $y=0$

$\Rightarrow$ only crit point is $(0,0)$

the boundary of D is the curve $x^2+y^2=1$
$\Rightarrow y^2 = 1-x^2$

$f(x,y) = 2x^3 + y^4$

let $g(x) = 2x^3 + (1-x^2)^2 \qquad -1 \leq x \leq 1$

Lagrange multipliers

Q: When will $f(x,y,z)$ be the largest, subject to the constraint $g(x,y,z) = k$?

1) Suppose $f(x,y,z)$ has an extreme value at (x_0,y_0,z_0) on the surface S (level surface defined by $g(x,y,z)=k$)

2) Suppose $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve on S w/ $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$.

3) Define $h(t) = f(x(t), y(t), z(t))$ (same as from $\vec{r}(t)$)

$h'(t_0) = 0$ (because $f(x_0,y_0,z_0)$ is an extreme value of f on S and as $\vec{r}(t)$)

lives on S, it is an extreme value of $h(t)$.

$$h'(t_0) = 0$$
$$= \nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0)$$

4) from WS12

$$\nabla g(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$$

so $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

some scalar

Strategy:

① find all x, y, z and λ w/

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

② plug all of the points back into f to find max and min.