Midterm Exam 3

MAT133 - Calculus II

5/5/2017

Instructions

Do not start until you are told to do so.

Please, turn off your phone and secure it in your bag. Leave your bags, pencil cases, calculators, notes, books, jackets, hats, food, and other belongings at the front of the classroom. You are permitted to have a transparent drink bottle, pens, pencils, and erasers at your desk. Please, plan to stay in the classroom for the entire duration of the exam.

You will have 80 minutes to complete this exam. Read all instructions carefully. Your responses to all items on this exam must be your own. No outside references, notes, calculators, or other aides are permitted. As it is crowded, please, refrain from glancing at the papers of those around you, and take care that your work is protected. A reference sheet, and pages for scratch work can be found attached to the end of the exam, you may detach these pages if you like. Do not detach any other pages from the exam.

The exam sections are weighted as follows:

- 32 points - Concept Check
- 48 points - Essential Skills
- 15 points - Advanced Skills
- 5 points - True/False Statements
- 5 points - Bonus

Within each section, all problems are weighted the same.

If you find yourself unable to finish a question, do your best to describe your attempts and reasoning. Partial credit may be awarded for demonstrating meaningful effort towards a solution.

Raise your hand if you have any questions, or require clarification of any instructions, during the exam. Good luck!

Clearly print your name in the box below. Do not write your name in any other location unless you are submitting page(s), not attached to the rest of your exam, containing work that you want scored.

Name: Key

2 Essential Skills (48 points)

Complete **all** problems in this section. Your responses in this section should be precise and detailed, with justification given for all steps that rely on results from calculus. Clearly indicated your final answer to each exercise. Correct answers with insufficient work or justification may not receive full credit.

2.1

Find a Cartesian equation for the plane tangent to the surface $z = 4x^2 - y^3 + 2xy$, at the point $(1, 2, 0)$.

Set $f(x,y,z) = 4x^2 - y^3 + 2xy - z$

the normal vector to $f(x,y,z) = 0$ at

$(1,2,0)$ is $\nabla f(1,2,0)$.

Now, $\nabla f(x,y,z) = \langle 8x + 2y, -3y^2 + 2x, -1 \rangle$

So $\nabla f(1,2,0) = \langle 12, -10, -1 \rangle$, and the tangent plane has vector equation

$$\langle 12, -10, -1 \rangle \cdot \langle x-1, y-2, z \rangle = 0$$

which yields the Cartesian equation

$$12(x-1) - 10(y-2) - z = 0$$

or

$$12x - 10y - z + 8 = 0.$$

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.2

Given

$$f(x,y) = x^2 + y^2 + x^2y + 4$$

Determine the absolute minimum and maximum values of f on the set

$$D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$$

$f_x = 2x + 2xy$

$f_y = 2y + x^2$

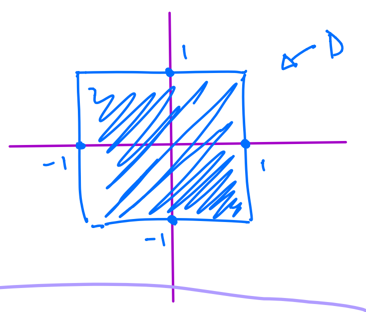
outside of D

$2x(2+y) = 0 \iff x = 0$ or $y = -2$

and

$2y + x^2 = 0$ when $y = 0$ (because x must be zero already)

So $(0,0)$ is a critical point with critical value $f(0,0) = 4$.



On the boundary of D we can fix $x = \pm 1$ and vary $-1 \le y \le -1$, or fix $y = \pm 1$ and vary $0 \le x \le 1$

Case 1: $x = -1$ $\quad -1 \le y \le 1$

$f(-1, y) = y^2 + y + 5$

$\frac{d}{dy} f(-1,y) = 2y + 1$

$\Rightarrow f(-1,y)$ has a local min at $y = -\frac{1}{2}$

$f(-1,-1) = 5$

$f(-1, -\frac{1}{2}) = \frac{19}{4}$

$f(-1, 1) = 7$

Case 2: $x = 1$, $-1 \le y \le 1$

Same as Case 1.

Case 3 $\quad -1 \le x \le 1 \quad y = -1$

$f(x,-1) = 5$, a constant

check critical point and endpoints

Case 4: $-1 \le x \le 1 \quad y = 1$ $\quad \frac{d}{dx} f(x,1) = 4x \Rightarrow$

$f(x,1) = 2x^2 + 5$ $\quad f(x,1)$ has a local min at $x = 0$

$f(-1,1) = 7 \quad f(0,1) = 5 \quad f(1,1) = 7$

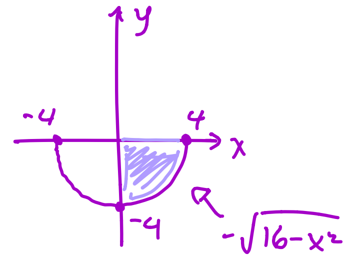
So, the minimum of $f(x,y)$ on D is $f(0,0) = 4$ and the maximum is $f(-1,1) = f(1,1) = 7$

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.3

a) Using polar coordinates, write an iterated integral equivalent to the following:

$$\int_0^4 \int_{-\sqrt{16-x^2}}^{0} \overbrace{\left(x^3 + xy^2\right)}^{(x^2+y^2)x}\, dy\, dx$$



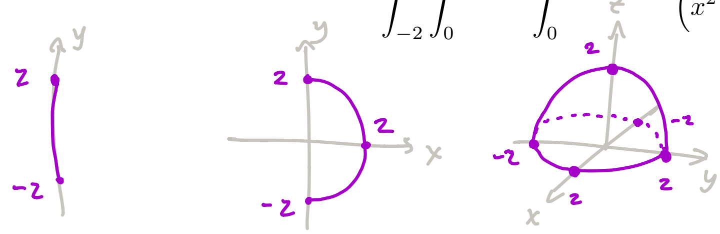
$$\int_{-\frac{\pi}{2}}^{0} \int_0^4 r^2(r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} \int_0^4 r^4 \cos\theta\, dr\, d\theta$$

$$\left(= \int_{\frac{3\pi}{2}}^{2\pi} \int_0^4 r^4 \cos\theta\, dr\, d\theta \right)$$

b) Using spherical coordinates, write an iterated integral equivalent to the following:

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \left(x^2\sqrt{x^2+y^2+z^2}\right) dz\, dx\, dy$$



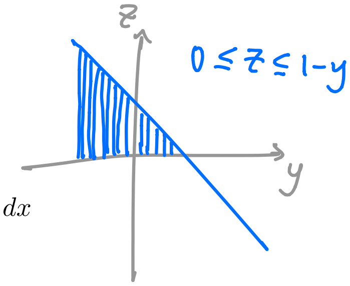
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \left((\rho\sin\phi\cos\theta)^2\rho\right) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

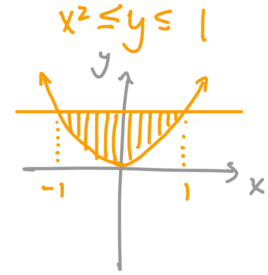
☐ Put an “X” in this box if work to be scored, for this problem, is located on a **scratch page**.

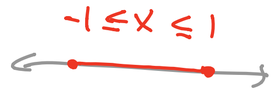
2.4

Given the iterated integral

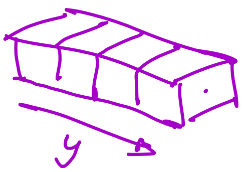
$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$$

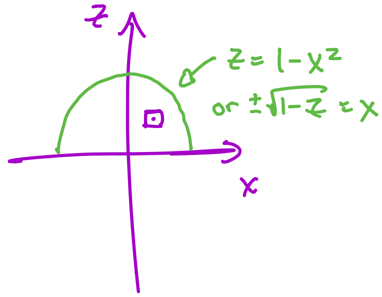


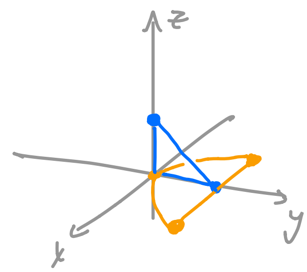




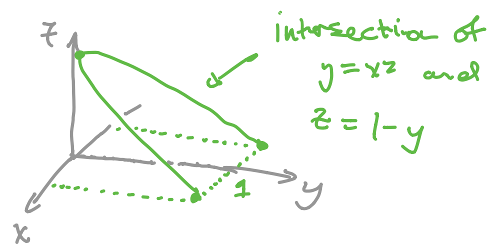
a) Write an equivalent iterated integral in the order $dy\,dx\,dz$





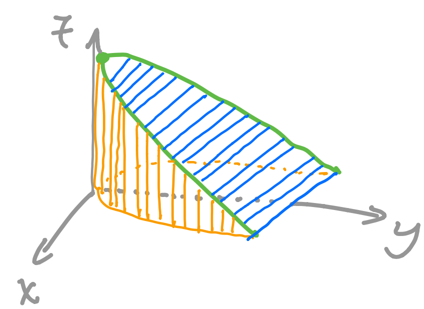


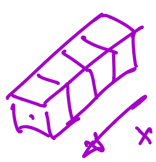
$$\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} f(x,y,z)\,dy\,dx\,dz$$



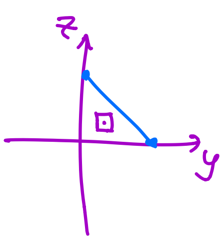
$$= \int_0^1\int_0^{\sqrt{1-z}}\int_{x^2}^{1-z} f(x,y,z)\,dy\,dx\,dz + \int_0^1\int_{-\sqrt{1-z}}^{0}\int_{x^2}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

b) Write an equivalent iterated integral in the order $dx\,dz\,dy$





$$\int_0^1\int_0^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy$$



$$= 2\int_0^1\int_0^{1-y}\int_0^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-y}\int_0^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_0^{1-y}\int_{-\sqrt{y}}^{0} f(x,y,z)\,dx\,dz\,dy$$

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

3 Advanced Skills (15 points)

Complete exactly **one of the two** problems in this section. Your response in this section should be precise and detailed, with justification given for all steps that rely on results from calculus. Clearly indicated your final answer. Correct answers with insufficient work or justification may not receive full credit.

3.1

If n is a positive integer and a function f, with continuous second order partial derivatives, satisfies the equation

$$f(tx, ty) = t^n f(x, y) \tag{1}$$

then it is called *homogeneous of degree* n.

a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

(Hint: Differentiate (1) with respect to t)

c) If f is homogeneous of degree n, show that

$$f_x(tx, ty) = t^{n-1} f_x(x, y).$$

a) $f(tx,ty) = (tx)^2(ty) + 2(t)(ty)^2 + 5(ty)^3$

$= t^3(x^2y + y^2 + 5y)$

$= t^3 \cdot f(x,y)$ ✓

b) If f is homogeneous of degreen n, then $f(tx,ty) = t^n f(x,t)$ (def)

So,

$$\frac{d}{dt}\left(f(tx,ty)\right) = \frac{d}{dt}\left(t^n f(x,y)\right)$$

$$\Rightarrow \frac{\partial f}{\partial x}\left(\frac{d}{dt}(tx)\right) + \frac{\partial f}{\partial y}\left(\frac{d}{dt}(ty)\right) = n(t^{n-1}) \cdot f(x,y) + t^n \frac{d}{dt} f(x,y) \quad \text{(chain rule)}$$

$$\Rightarrow x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = n(t^{n-1}) f(x,y)$$

Note: this is really $x \cdot \frac{\partial}{\partial x} f(tx,ty) + y \cdot \frac{\partial}{\partial y} f(tx,ty) = n(t^{n-1}) f(x,y)$

, so we set $t=1$ and we have the desired result.

[X] Put an "X" in this box if work, for this problem, continues on the back of **this page**.

c) As f is homogeneous of degree n, we know that

$$f(tx,ty) = t^n f(x,y)$$

So

$$\frac{\partial}{\partial x}\left(f(tx,ty)\right) = \frac{\partial}{\partial x}\left(t^n f(x,y)\right)$$

$$\Rightarrow f_x(tx,ty)\cdot\frac{\partial}{\partial x}(tx) + f_y(tx,ty)\cdot\frac{\partial}{\partial x}(ty) = t^n\left(f_x(x,y)\right) \quad \left(\text{Chain rule.}\right)$$

$$\Rightarrow f_x(tx,ty)t = t^n\left(f_x(x,y)\right) \quad \left(\frac{\partial}{\partial x}(xt)=1,\ \frac{\partial}{\partial x}(ty)=0\right)$$

$$\Rightarrow f_x(tx,ty) = t^{n-1}\left(f_x(x,y)\right) \quad (\text{divide by } t)$$

$\blacksquare$

3.2

Let n be an integer, let r and R be real numbers with $0 < r < R$.

a) Evaluate $I = \iint_D \left(\frac{1}{(x^2+y^2)^{n/2}} \right) dA$. Where D is the region bounded between circles, centered at the origin, of radius r and R. (Note: There may be more than one answer, depending on the value of n.)

b) For what values of n does $\lim_{r \to 0^+} I$ exist?

c) Evaluate $J = \iiint_E \left(\frac{1}{(x^2+y^2+z^2)^{n/2}} \right) dV$. Where E is the region bounded between spheres, centered at the origin, of radius r and R.

d) For what values of n does $\lim_{r \to 0^+} J$ exist?

a) $$I = \int_0^{2\pi} \int_r^R \frac{1}{(t^2)^{n/2}} \cdot t \, dt \, d\partial$$

$$= \int_0^{2\pi} \int_r^R t^{(1-n)} \, dt \, d\partial = 2\pi \int_r^R t^{1-n} \, dt = \begin{cases} 2\pi\left(\frac{1}{2-n} t^{2-n} \right)\Big|_r^R & \text{for } n \neq 2 \\ \ln(t) & \text{for } n = 2 \end{cases}$$

$$= \begin{cases} \frac{2\pi}{2-n}\left(R^{2-n} - r^{2-n} \right) & \text{for } n \neq 2 \\ 2\pi\left(\ln(R/r) \right) & \text{for } n = 2 \end{cases}$$

b) $\lim_{r \to 0^+} I$ exists as long as $2-n > 0$, or $n < 2$.

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c) $$J = \int_0^{2\pi}\int_0^{\pi}\int_r^{R} \frac{1}{(\rho^2)^{n/2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \int_0^{\pi} \sin(\phi)\, d\phi \cdot \int_r^{R} \rho^{2-n}\, d\rho$$

$$= \begin{cases} \dfrac{4\pi}{3-n} \cdot \rho^{3-n} \Big|_r^R & \text{for } n \neq 3 \\ 4\pi \ln(\rho) \Big|_r^R & \text{for } n = 3 \end{cases} = \begin{cases} \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{for } n \neq 3 \\ 4\pi \ln(R/r) & \text{for } n = 3 \end{cases}$$

d) $\lim\limits_{r \to 0^+} J$ exists when $3-n>0$ or $n<3$.

4 True/False (5 points)

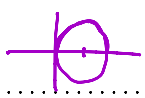
Determine the truth value of the following statements.
Clearly mark them as **TRUE** or **FALSE** in the space provided. Be aware that a statement is only true if it is *always* true. That is to say, if there is *even one example* that makes a statement false, then statement is false. You do not need to provide a proof, or counterexample, to justify your answers. No partial credit will be awarded.

1. If $f(x, y) = z$ is differentiable on all of $\mathbb{R}^2$, and has two local maxima, then it must also have at least one local minimum.

← two local maxima
no local minima

TRUE FALSE

2. $\{(x, y) \mid (x-1)^2 + y^2 \leq 1\} = \{(r, \theta) \mid r \leq 2\cos(\theta)\}$



TRUE FALSE

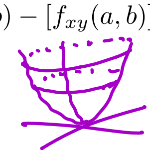
3. $\displaystyle\int_5^{10}\int_3^6 \frac{x^2+y^3}{xy^2+x^2y}\,dx\,dy = \int_3^6\int_5^{10} \frac{x^2+y^3}{xy^2+x^2y}\,dy\,dx$

Fubini

TRUE FALSE

4. If $f(x, y)$ has a critical point at (a, b), and $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$, then $f(a, b)$ is neither a local minimum nor a local maximum.

$f(x,y) = x^4 + y^4$ and $(0,0)$



TRUE FALSE

5. The volume of the solid that lies above $z = \sqrt{x^2+y^2}$ and below $z = x^2 + y^2 + z^2$ is given by $V = \displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos(\phi)} \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta.$

see example 16.8.4

TRUE FALSE