Midterm Exam 2

MAT133 - Calculus II

3/17/2017

Instructions

Do not start until you are told to do so.

Please, turn off your phone and secure it in your bag. Leave your bags, pencil cases, calculators, notes, books, jackets, hats, food, and other belongings at the front of the classroom. You are permitted to have a transparent drink bottle, pens, pencils, and erasers at your desk. Please, plan to stay in the classroom for the entire duration of the exam.

You will have 80 minutes to complete this exam. Read all instructions carefully. Your responses to all items on this exam must be your own. No outside references, notes, calculators, or other aides are permitted. As it is crowded, please, refrain from glancing at the papers of those around you, and take care that your work is protected. A reference sheet, and pages for scratch work can be found attached to the end of the exam, you may detach these pages if you like. Do not detach any other pages from the exam.

The exam sections are weighted as follows:

- 36 points - Concept Check
- 40 points - Essential Skills
- 8 points - Intermediate Skills
- 6 points - Advanced Skills
- 10 points - True/False Statements
- 5 points - Bonus

Within each section, all problems are weighted the same.

If you find yourself unable to finish a question, do your best to describe your attempts and reasoning. Partial credit may be awarded for demonstrating meaningful effort towards a solution.

Raise your hand if you have any questions, or require clarification of any instructions, during the exam. Good luck!

Clearly print your name in the box below. Do not write your name in any other location unless you are submitting page(s), not attached to the rest of your exam, containing work that you want scored.

Name: Key

2 Essential Skills (40 points)

Your responses in this section should be precise and detailed, with justification given for all steps that rely on results from calculus. Clearly indicated your final answer to each exercise. Correct answers with insufficient work or justification may not receive full credit.

2.1

For each of the following sets, determine if all three points lie on the same straight line in space.

Set 1- $A(1,2,3), B(0,5,-1), C(2,-1,7)$

Set 2- $D(4,1,3), E(3,-1,0), F(5,3,3)$

Set 1

$\ell(t) = \langle 1,2,3\rangle + t\langle 1,-3,4\rangle$ is a line

$\ell(0)$ points to A
$\ell(-1)$ " " B
$\ell(1)$ " " C
$\Rightarrow$ so all are on the same line

Set 2

$\ell_1(t) = \langle 4,1,3\rangle + t\langle 1,2,3\rangle$ is a line

$\ell_1(0) = \langle 4,1,3\rangle$
$\ell_1(-1) = \langle 3,-1,0\rangle$
$\Rightarrow$ so D and E are on this line

$\ell_2(t) = \langle 3,-1,0\rangle + t\langle -2,-4,-3\rangle$ is a line

$\ell_2(0) = \langle 3,-1,0\rangle$
$\ell_2(-1) = \langle 5,3,3\rangle$
$\Rightarrow$ so E and F are on this line

$$\langle 1,2,3\rangle \times \langle -2,-4,-3\rangle = \left\langle \begin{vmatrix} 2 & 3 \\ -4 & -3 \end{vmatrix}, -, - \right\rangle$$

$$= \langle 6, -, - \rangle \neq \vec{0}$$

by 6.3.4, ℓ_1 and ℓ_2 are not parallel and D, E, F cannot be on the same line.

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.2

For each of the following pairs of vectors, determine whether they are orthogonal, parallel, or neither.

a) $\vec{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ and $\vec{v} = -b\hat{\imath} + a\hat{\jmath}$

b) $\vec{u} =< -3, 9, 6 >$ and $\vec{v} =< 4, -12, -8 >$

c) $\vec{u} =< 1, -1, 2 >$ and $\vec{v} =< 2, -1, 1 >$

a) $\vec{u} = \langle a, b, c \rangle$

$\vec{v} = \langle -b, a, 0 \rangle$

$\vec{u} \cdot \vec{v} = -ab + ab + 0 = 0$

$\vec{u} \times \vec{v} = \begin{vmatrix} b & c \\ a & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a & c \\ -b & c \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \hat{k}$

$= -ac\hat{\imath} - (ac + bc)\hat{\jmath} + (a^2 + b^2)\hat{k} \neq \vec{0}.$

so $\vec{u}, \vec{v}$ are orthogonal and not parallel by 6.2.3 and 6.3.4.

b) $\vec{u} \cdot \vec{v} = -12 - 108 - 48 \neq 0$

$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} 9 & 6 \\ -12 & -8 \end{vmatrix}, -\begin{vmatrix} -3 & 6 \\ 4 & -8 \end{vmatrix}, \begin{vmatrix} -3 & 9 \\ 4 & -12 \end{vmatrix} \right\rangle$

$= \langle 0, 0, 0 \rangle$

so $\vec{u}, \vec{v}$ are not orthogonal and are parallel by 6.2.3 and 6.3.4.

c) $\vec{u} \cdot \vec{v} = 2 + 1 + 2 = 5 \neq 0$

$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}, -, - \right\rangle$

$= \langle 1, -, - \rangle \neq \vec{0}$

So $\vec{u}, \vec{v}$ are neither orthogonal nor parallel, by 6.2.3 and 6.3.4

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.3

Given vectors $\vec{a}, \vec{b}$, and $\vec{c}$, in $\mathbb{R}^3$, prove that

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b}).$$

$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$ by 6.3.1.3

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$ by 6.3.1.4

$= -\vec{b} \times \vec{a} + \vec{a} \times \vec{b}$ by 6.3.4

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{b}$ by 6.3.1.1

$= 2(\vec{a} \times \vec{b})$ ✓

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.4

Are the planes defined by

$$x + 2y + 2z = 1 \tag{1}$$
$$2x - y + 2z = 1 \tag{2}$$

parallel, perpendicular, or neither? If they are neither, what is the angle between them?

$\vec{n}_1 = \langle 1, 2, 2 \rangle$ is normal to the first

$\vec{n}_2 = \langle 2, -1, 2 \rangle$ is normal to the second

$\vec{n}_1 \times \vec{n}_2 = \left\langle \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}, -, - \right\rangle$

$= \langle 4, -, - \rangle \neq 0$ so the planes are not parallel by 6.3.4.

$\vec{n}_1 \bullet \vec{n}_2 = 2 - 2 + 4 = 4 \neq 0$ so the planes are not orthogonal, by 6.2.3

$\vec{n}_1 \bullet \vec{n}_2 = \|\vec{n}_1\| \cdot \|\vec{n}_2\| \cdot \cos(\theta)$ where θ is the angle between $\vec{n}_1$ and $\vec{n}_2$, by 6.2.2.

Now,

$\|\vec{n}_1\| = \sqrt{9} = 3$

$\|\vec{n}_2\| = \sqrt{9} = 3$

so, $\theta = \arccos\left(\frac{4}{9}\right)$.

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

2.5

Find a vector equation for the line tangent to the curve defined by

$$x = \ln(t),$$
$$y = 2\sqrt{t},$$
$$z = t^2,$$

for $t > 0$, at the point $(0, 2, 1)$.

$\vec{r}\,'(t) = \langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \rangle$

$\vec{r}(1) = \langle 0, 2, 1 \rangle$

$\vec{r}\,'(1) = \langle 1, 1, 2 \rangle$

$\ell(s) = \langle 0, 2, 1 \rangle + s \cdot \langle 1, 1, 2 \rangle, \quad -\infty < s < \infty$

$= \langle s, 2+s, 1+2s \rangle, \quad -\infty < s < \infty$

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

3 Intermediate Skills (8 points)

Your responses in this section should be precise and detailed, with justification given for all steps that rely on results from calculus. Clearly indicated your final answer to each exercise. Correct answers with insufficient work or justification may not receive full credit.

3.1

Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

This plane contains the points

$P(a,0,0)$

$Q(0,b,0)$

$R(0,0,c)$

So, it contains $\vec{v} = \langle a, -b, 0\rangle = \overrightarrow{QP}$

$\vec{u} = \langle 0, b, -c\rangle = \overrightarrow{RQ}$

Thus, it has normal vector

$$\vec{n} = \langle a, -b, 0\rangle \times \langle 0, b, -c\rangle$$

$$= \left\langle \begin{vmatrix} -b & 0 \\ b & -c \end{vmatrix}, -\begin{vmatrix} a & 0 \\ 0 & -c \end{vmatrix}, \begin{vmatrix} a & -b \\ 0 & b \end{vmatrix} \right\rangle$$

$$= \langle bc, ac, ab\rangle$$

$$\vec{n} \bullet (\langle x,y,z\rangle - \langle a,0,0\rangle) = 0$$

$\Rightarrow bcx + acy + abz - abc = 0$ is an equation for the plane.

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

4 Advanced Skills (6 points)

Your response in this section should be precise and detailed, with justification given for all steps that rely on results from calculus. Clearly indicated your final answer. Correct answers with insufficient work or justification may not receive full credit.

4.1

Prove: If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \bullet \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta).$$

Hint: Recall the Law of Cosines: In a triangle with sides of length a, b and c, and θ the angle opposite the side of length c, we have $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

See the proof of Theorem 3 in section 13.3 of the text.

☐ Put an "X" in this box if work to be scored, for this problem, is located on a **scratch page**.

5 True/False (10 points)

Determine the truth value of the following statements.
Clearly mark them as **TRUE** or **FALSE** in the space provided. Be aware that a statement is only true if it is *always* true. That is to say, if there is *even one example* that makes a statement false, then statement is false. You do not need to provide a proof, or counterexample, to justify your answers. No partial credit will be awarded.

1. If $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$, then $\vec{b} = \vec{c}$.

$\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c} \Leftrightarrow \vec{a} \bullet (\vec{b} - \vec{c}) = 0$ .... TRUE FALSE

$\Leftrightarrow \vec{b} - \vec{c}$ is orthogonal to $\vec{a}$

this does not force $\vec{b} = \vec{c}$.

2. Any three distinct points in $\mathbb{R}^3$ uniquely determine a plane.

only if they are not colinear .... TRUE FALSE

3. Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.

Consider $\vec{r}(t)$ and $\vec{r}(2t)$ .... TRUE FALSE

4. If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors of the same dimension, then $(\underbrace{\vec{u} \bullet \vec{v}}_{\text{scalar}}) \bullet \underbrace{\vec{w}}_{\text{vector}} = \vec{u} \bullet (\vec{v} \bullet \vec{w})$.

the equation is meaningless. .... TRUE FALSE

5. Any level set of a given function, f, is a subset of the domain of f.

.... TRUE FALSE

1 Trigonometric Identities

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\sec^2(x) - \tan^2(x) = 1$
3. $2\sin^2(x) = 1 - \cos(2x)$
4. $2\cos^2(x) = 1 + \cos(2x)$
5. $2[\sin(A)\cos(B)] = \sin(A - B) + \sin(A + B)$
6. $2[\sin(A)\sin(B)] = \cos(A - B) - \cos(A + B)$
7. $2[\cos(A)\cos(B)] = \cos(A - B) + \cos(A + B)$

3 Derivatives

1. $\frac{d}{dx}(a^x) = a^x \ln(a)$
2. $\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}$
3. $\frac{d}{dx}(\sin(x)) = \cos(x)$
4. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
5. $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
6. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
7. $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
8. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
9. $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$
11. $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$
12. $\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}}$
13. $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$
14. $\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}$

5 Integrals

For functions u and v satisfying the appropriate hypotheses, we have

1. $\int (a^u)du = a^u/\ln(u) + C$
2. $\int (1/u)du = \ln|u| + C$
3. $\int \sin(u)du = -\cos(u) + C$
4. $\int \cos(u)du = \sin(u) + C$
5. $\int \sec^2(u)du = \tan(u) + C$
6. $\int \csc(u)\cot(u)du = -\csc(u) + C$
7. $\int \sec(u)\tan(u)du = \sec(u) + C$
8. $\int \csc^2(u)du = -\cot(u) + C$
9. $\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}(u/a) + C$
10. $\int \frac{1}{a^2+u^2}du = (1/a)\tan^{-1}(u/a) + C$
11. $\int \frac{1}{u\sqrt{u^2-a^2}}du = (1/a)\sec^{-1}(u/a) + C$
12. $\int \tan(u)du = \ln|\sec(u)| + C$
13. $\int \cot(u)du = \ln|\sin(u)| + C$
14. $\int \sec(u)du = \ln|\sec(u) + \tan(u)| + C$
15. $\int \csc(u)du = \ln|\csc(u) - \cot(u)| + C$

6 Vectors

6.1 Properties of Vectors

If (V_n, F) is a vector space, with $\vec{a}, \vec{b}$ and $\vec{c}$ in V_n, and c and d in F, then

Theorem 6.1.1.

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a}=a\vec{a}+d\vec{a}$
7. $(cd)\vec{a}=c(d\vec{a})$
8. $1\vec{a}=\vec{a}$

6.2 Properties of the Dot Product

In the vector space $(\mathbb{R}^n, \mathbb{R})$, with $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbb{R}^n$, and c is in $\mathbb{R}$, the *dot product* has the following properties:

Theorem 6.2.1.

1. $\vec{a}\bullet\vec{a}=||\vec{a}||^2$
2. $\vec{a}\bullet\vec{b}=\vec{b}\bullet\vec{a}$
3. $\vec{a}\bullet(\vec{b}+\vec{c})=\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$
4. $(c\vec{a})\bullet\vec{b}=c(\vec{a}\bullet\vec{b})=\vec{a}\bullet(c\vec{b})$
5. $\vec{0}\bullet\vec{a}=0$

Theorem 6.2.2. *If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, then*

$$\vec{a} \bullet \vec{b}=||\vec{a}||\cdot||\vec{b}||\cdot\cos(\theta).$$

Theorem 6.2.3. *Vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if*

$$\vec{a} \bullet \vec{b}=0.$$

6.3 Properties of the Cross Product

In the vector space $(\mathbb{R}^3, \mathbb{R})$, with $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$, and c is in $\mathbb{R}$, the *cross product* has the following properties:

Theorem 6.3.1.

1. $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$
2. $(c\vec{a})\times\vec{b}=c(\vec{a}\times\vec{b})=\vec{a}\times(c\vec{b})$
3. $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$
4. $(\vec{a}+\vec{b})\times\vec{c}=(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})$
5. $\vec{a}\bullet(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\bullet\vec{c}$
6. $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\bullet\vec{c})\vec{b}-(\vec{a}\bullet\vec{c})\vec{c}$

Theorem 6.3.2. *The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.*

Theorem 6.3.3. *If θ is the angle between $\vec{a}$ and $\vec{b}$, with $0\le\theta\le\pi$, then*

$$||\vec{a}\times\vec{b}||=||\vec{a}||\cdot||\vec{b}||\cdot\sin(\theta)$$

Theorem 6.3.4. *Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if*

$$\vec{a}\times\vec{b}=\vec{0}.$$

6.4 Calculus and Vector Valued Functions

Theorem 6.4.1. *In the vector space $(\mathbb{R}^n, \mathbb{R})$, given vector-valued functions $\vec{u}(t)$ and $\vec{v}(t)$, real-valued function $f(t)$, and constant c in $\mathbb{R}$, when differentiating with respect to t, we have*

1. $[\vec{u}(t)+\vec{v}(t)]'=\vec{u}\,'(t)+\vec{v}\,'(t)$
2. $[c\vec{u}(t)]'=c(\vec{u\,'}(t))$
3. $[f(t)\vec{u}(t)]'=f'(t)\vec{u}(t)+f(t)\vec{u}\,'(t)$
4. $[\vec{u}(t)\bullet\vec{v}(t)]'=(\vec{u}\,'(t)\bullet\vec{v}(t))+(\vec{u}(t)\bullet\vec{v}\,'(t))$
5. $[\vec{u}(t)\times\vec{v}(t)]'=(\vec{u}\,'(t)\times\vec{v}(t))+(\vec{u}(t)\times\vec{v}\,'(t))$
6. $[\vec{u}(f(t))]'=f'(t)\vec{u}\,'(f(t))$