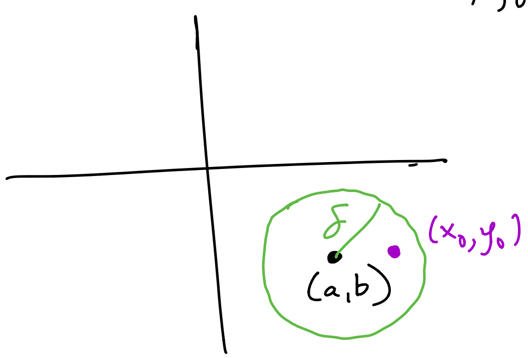
Limits in two dimensions:

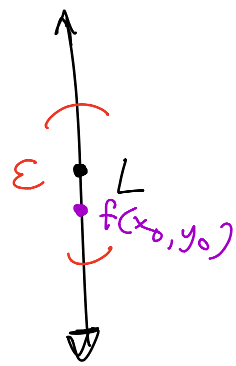
$\lim_{(x,y)\to(a,b)} f(x) = L$ if, for every $\varepsilon > 0$, there is a $\delta > 0$

there is $\delta > 0$ with

(x_0, y_0) in the disk of radius $\delta > 0$ about (a,b)

$\Rightarrow f(x_0, y_0)$ is within $\varepsilon > 0$ of L.





The tricky part in more than one dimension is that there are many, many paths that lead to any given point.

ex's:

$$\lim_{(a,b)\to(0,0)} \left(\frac{x^2+y^2}{x^2-y^2} \right) \quad ①$$

$$\lim_{(a,b)\to(0,0)} \left(\frac{xy}{x^2+y^2} \right) \quad ②$$

$$\lim_{(a,b)\to(0,0)} \left(\frac{xy^2}{x^2+y^4} \right) \quad ③$$

$$\lim_{(a,b)\to(0,0)} \left(\frac{x^2+y^2}{x^2-y^2} \right) \quad ①$$

Ⓐ $\lim_{a\to 0} f(a,0) = \lim_{a\to 0} \frac{a^2}{a^2} = 1$

Ⓑ $\lim_{b\to 0} f(0,b) = \lim_{b\to 0} \frac{b^2}{-b^2} = -1$

$1 \neq -1$ so $\lim_{(a,b)\to(0,0)} f(a,b)$ DNE

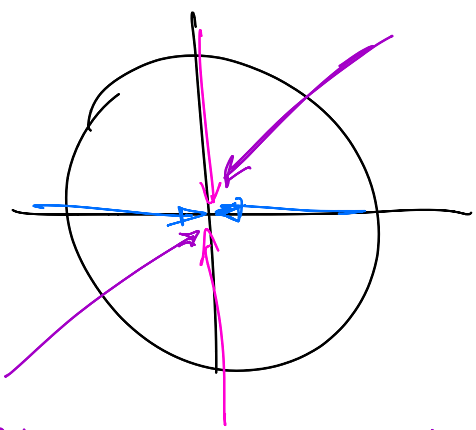
Ⓐ approaches $(0,0)$ along the line $y=0$

Ⓑ approaches zero along the line $x=0$

$$\lim_{(a,b)\to(0,0)} \left(\frac{xy}{x^2+y^2} \right) \quad ②$$

$\lim_{a\to 0} f(a,0) = 0$ Ⓐ

$\lim_{b\to 0} f(0,b) = 0$ Ⓑ



But what about inputs along other lines?

Ⓒ approaches $(0,0)$ along the line $y=x$.

Ⓒ $\lim_{a\to 0} f(a,a) = \lim_{a\to 0} \frac{a^2}{2a^2} = \frac{1}{2}$

$\frac{1}{2} \neq 0$ so $\lim_{(a,b)\to(0,0)} f(a,b)$ DNE

$$\lim_{(a,b)\to(0,0)} \left(\frac{xy^2}{x^2+y^4} \right)$$ (3)

lets do all of the lines at once!

(D) $$\lim_{a\to 0} f(a, ma) = \lim_{a\to 0} \frac{a^3 m^2}{a^2+a^4m^4} = \lim_{a\to 0} \frac{a m^2}{1+a^2 m^4} = 0$$

Does this mean $\lim_{(a,b)\to(0,0)} f(a,b) = 0$? NO

we have to check every, single path to $(0,0)$!!

(E) ex $$\lim_{a\to 0} f(a^2, a) = \lim_{a\to 0} \frac{a^4}{a^4+a^4} = \frac{1}{2} \neq 0!!$$

In general, it is very hard to determine if a limit exists at a discontinuity of a function of several variables.

(D) approaches $(0,0)$ along the line $y=mx$

(E) approaches $(0,0)$ along the parabola $x=y^2$