Evaluate

$$\int \frac{x^4+5\,x^3+7\,x^2+14\,x+6}{x^3+x^2+3\,x}dx. \tag{1}$$

First, we notice that the degree of the numerator is greater than that of the denominator. So, we do polynomial long division to express (1) as

$$\int \left(x+4+\frac{2x+6}{x^3+x^2+3x}\right)dx. \tag{2}$$

The last summand in the integrand,

$$\frac{2x+6}{x^3+x^2+3x} \tag{3}$$

is now a rational function with the degree of its denominator larger than the degree of its numerator. So, we factor the denominator to get,

$$x^3+x^2+3x=(x)(x^2+x+3). \tag{4}$$

We have one linear factor, and one irreducible quadratic factor. So, we consider the following expression,

$$\frac{A}{x}+\frac{Bx+C}{x^2+x+3}. \tag{5}$$

If we express (5) as a single fraction, we get

$$\frac{(A+B)x^2+(A+C)x+A3}{(x)(x^2+x+3)}. \tag{6}$$

We picked the denominators in the terms of (5) so that (6) and (3) have the same denominators. So, if we can choose values for A, B, and C that make the numerators equal, we can use (5) to “take apart” (3).

Thus, we want to find a solution to the following:

$$(A+B)x^2+(A+C)x+A3=2x+6.$$

We solve the resulting system of equations

$$\begin{aligned} A+B&=0,\\ A+C&=2,\\ 3A&=6, \end{aligned}$$

to get that

$$\begin{aligned} A&=2,\\ B&=-2,\\ C&=0. \end{aligned}$$

Plugging these back into (5), we deduce that

$$\frac{2x+6}{x^3+x^2+3x}=\frac{2}{x}-\frac{2x}{x^2+x+3}.$$

So, we can re-write (1) as

$$\int \left(x+4+\frac{2}{x}-\frac{2x}{x^2+x+3}\right)dx.$$

We immediately see that

$$\int \left(x + 4 + \frac{2}{x} - \frac{2x}{x^2 + x + 3}\right) dx = \frac{x^2}{2} + 4x + 2\ln(x) - \int \left(\frac{2x}{x^2 + x + 3}\right) dx, \tag{7}$$

so we may focus our efforts on evaluating

$$\int \left(\frac{2x}{x^2 + x + 3}\right) dx. \tag{8}$$

After looking for obvious substitutions, or possible component functions to try our standard techniques, we resort to trigonometric substitution. This proceeds as follows:

$$\begin{aligned}
\int \left(\frac{2x}{x^2 + x + 3}\right) dx &= 2\int \left(\frac{x}{(x + \frac{1}{2})^2 + \frac{11}{4}}\right) dx && \text{(completing the square)} \\
&= \sqrt{11}\int \left(\frac{\frac{\sqrt{11}}{2}\tan(\theta) - \frac{1}{2}}{\frac{11}{4}\sec^2(\theta)}\right)\sec^2(\theta)d\theta && \text{(substitution: } x = \frac{\sqrt{11}}{2}\tan(\theta) - \frac{1}{2}\text{)} \quad (9) \\
&= \frac{2\sqrt{11}}{11}\left(\sqrt{11}\int \tan(\theta)d\theta - \int d\theta\right) \\
&= 2\left(\ln|\sec(\theta)|\right) - \frac{2\sqrt{11}}{11}\theta + C. && (10)
\end{aligned}$$

Finally, we "undo" the substitution we made in (9). We do this by, first, solving for θ,

$$\theta = \arctan\left(\frac{2x + 1}{\sqrt{11}}\right),$$

and plugging this back in to (10) to get

$$\int \left(\frac{2x}{x^2 + x + 3}\right) dx = 2\left(\ln\left|\sec(\arctan\left(\frac{2x + 1}{\sqrt{11}}\right))\right|\right) - \frac{2\sqrt{11}}{11}\arctan\left(\frac{2x + 1}{\sqrt{11}}\right) + C.$$

Last, but not least, we tie this together with (7) to get that

$$\begin{aligned}
\int \frac{x^4 + 5x^3 + 7x^2 + 14x + 6}{x^3 + x^2 + 3x}dx = \frac{x^2}{2} &+ 4x + 2\ln(x) \\
&- 2\left(\ln\left|\sec(\arctan\left(\frac{2x + 1}{\sqrt{11}}\right))\right|\right) \\
&+ \frac{2\sqrt{11}}{11}\arctan\left(\frac{2x + 1}{\sqrt{11}}\right) + C,
\end{aligned}$$

and we are done.