Review

Ch 15

Concept Check

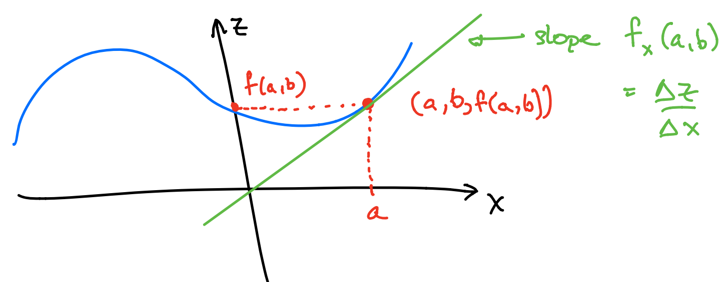
#5

a) $f_x(a,b) = \lim_{h \to 0} \dfrac{f(a+h, b) - f(a,b)}{h}$

only the x-coordinate is moving

b) $f_x(a,b)$ is the rate of change in the z-direction for every unit change in the x-direction.

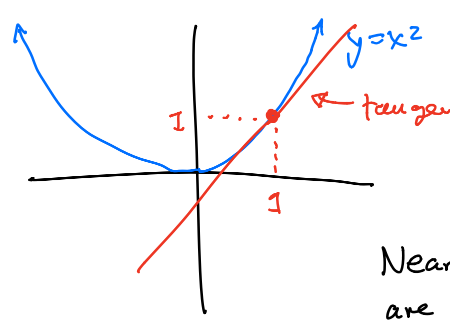
In the $y=b$ plane we see



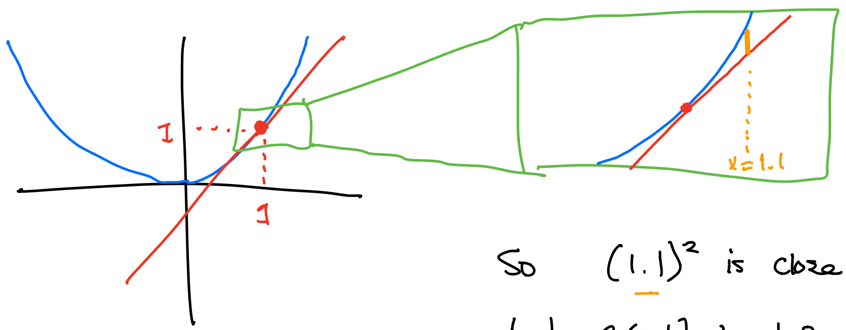
c) To compute f_x, treat all other independent variables as constants and differentiate as always.

#8

In two dimensions, we used tangent lines to approximate function values:



Near $x=1$, $y=x^2$ and $y=2x-1$ are pretty close to one another



So $(1.1)^2$ is close to $2(1.1)-1$

but $2(1.1)-1=1.2$ is easier to compute.

The same works w/ tangent planes.

If $z=T(x,y)$ is tangent to $z=f(x,y)$ at (a,b) then $T(a_0,b_0)\approx f(a_0,b_0)$ for (a_0,b_0) near (a,b).

#9) a) $f(x,y)$ is differentiable at (a,b) if

$$\Delta z \approx f_x\,\Delta x + f_y\,\Delta y$$

i.e. the "increment in z" can be roughly expressed in terms of the increment in x and y.

b) (Thm 15.4.8)

If f_x and f_y are continuous at (a,b) then f is differentiable at (a,b)

#10) (pp 932) (examples 15.4.4, 15.4.5, 15.4.6)

dx and dy and dz are "differentials"

they represent rates of change or "increments"

If $z = f(x,y)$, then the "total differential" of z is

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial x}\,dy$$

So dz is a function of 4 variables:

$$x,\ y,\ dx,\ dy$$

standard x, y values

"increment" in x and y

#11) Section 15.5 red boxes 2, 3, and 4

#12) (p 942)

<u>ex</u>

If $F(x,y) = 0$ defines y as a function of x, then $F(x,y) = F(x, f(x))$ for some f.

then

Chain rule

$$\frac{d}{dx}\Big(F(x, f(x))\Big) = F_x \overbrace{\frac{dx}{dx}}^{1} + F_y \overbrace{\frac{df}{dx}}^{\frac{dz}{dx}} = 0$$

$$\Rightarrow \frac{df}{dx} = \frac{-F_x}{F_y}$$

$$\Rightarrow \frac{dz}{dx} = -\frac{F_x}{F_y}$$

#13) 15.6 red boxes 2 and 3

interpret as: "the rate of change in z for each unit change in the $\vec{u}$ direction"

#14) a,b) 15.6 red boxes 8 and 9

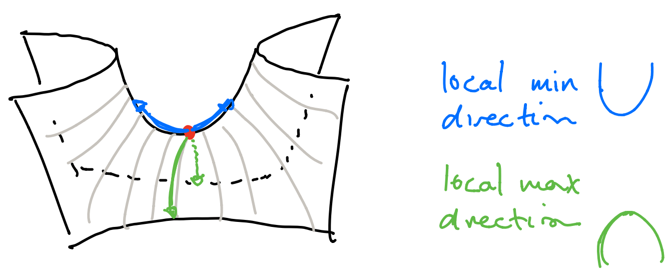
c) on a graph of $f(x,y)=z$, $\nabla f(a,b)$ points in the direction of steepest ascent from the point $(a,b,f(a,b))$.

#15) a) $f(a,b) \geq f(a_0,b_0)$ for all (a_0,b_0) nearby (a,b)
(local to)

b) $f(a,b) \geq f(x,y)$ for all (x,y) in the domain

c) change $\geq$ to $\leq$ in a)
d) change $\geq$ to $\leq$ in d)

e) $f(a,b)$ is a local min in one direction and a local max in another



#16a) (Thm 15.7.2)

$$f_x(a,b)=f_y(a,b)=0$$

#18c) 15.7 red box 9

#19) pp.974 "Two Constraints"

Ch 15 T/F

#4) T $D_{\vec{k}} f(x,y,z) = \nabla f(x,y,z) \cdot \langle 0,0,1 \rangle = f_z(x,y,z)$

#7) T Thm 15.7.2 and def of ∇f

#9) F $\nabla f(x,y) = \langle 0, \frac{1}{y} \rangle$

Ch 15 Exercises

#11) a) $T_x(6,4) = \lim_{h \to 0} \dfrac{T(6+h,4) - T(6,4)}{h}$

we can estimate by using $h=-2$ and $h=2$ and averaging the results. (we choose these as they are the closest values in our table to $(6,4)$ in the x-coordinate)

$T_y(6,4)$ can be estimated similarly

b) Use the estimates from a) and the formula $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

c) too involved for the exam

#18) $C(T,S,D)$ = speed of sound in water w/
temp T
salinity S
pressure P

$C_T(T_0, S_0, D_0)$ is the rate of change in the speed that sound travels in water w/
temp T_0
salinity S_0
ressure P_0
for each unit change in temp (T).

#19) $f(x,y) = 4x^3 - xy^2$

$f_x(x,y) = 12x^2 - y^2$ $\quad f_{xx} = 24x$ $\quad f_{xy} = -2y = f_{yx}$

$f_y(x,y) = -2xy$ $\quad f_{yy} = -2x$

#23) $z = xy + xe^{y/x}$

$$\frac{\partial z}{\partial x} = y + \left(e^{y/x} - xe^{y/x} \cdot \frac{y}{x^2}\right)$$

$$= y + e^{y/x} - ye^{y/x}$$

$$\frac{\partial z}{\partial y} = x + e^{y/x}$$

the rest is arithmetic ...

#29 a)

Tangent plane to

$\sin(xyz) = x + 2y + 3z$ @ $(2,-1,0)$

To use 15.6 red box 19 we set

$F(x,y,z) = \sin(xyz) - x - 2y - 3z = 0$

$\nabla F = \langle yz\cos(xyz) - 1, \; xz\cos(xyz) - 2, \; xy\cos(xyz) - 3 \rangle$

$\nabla F(2,-1,0) = \langle -1, -2, -5 \rangle$

$\langle -1,-2,-5\rangle \cdot \langle x-2, y+1, z-0\rangle = 0$

$-x + 2 - 2y - 2 - 5z = 0 \Rightarrow -x - 2y - 5z = 0$ is the tangent plane.

#47) $f(x,y) = x^2y + \sqrt{y}$ find max rate of change @ $(2,1)$

$\nabla f = \langle 2xy, \; x^2 + \frac{1}{2}y^{-1/2} \rangle$

$\nabla f(2,1) = \langle 4, 9/2 \rangle$

$\|\langle 4, \frac{9}{2}\rangle\| = \sqrt{4^2 + (\frac{9}{2})^2} = \frac{\sqrt{145}}{2}$ is the max rate of change

and it occurs in the direction $\langle 4, 9/2 \rangle$.

#56) find absolute max and min of

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2) \quad \text{on} \quad D = \{(x,y) \mid x^2+y^2 \le 4\}$$

1) find crit pts.

$$f_x = (-2x)(e^{-x^2-y^2})(x^2+2y^2) + e^{-x^2-y^2}(2x)$$

$$= 0 \text{ if } \quad x=0 \text{ or } x^2+2y^2 = 1$$

$$f_y = (-2y)(e^{-x^2-y^2})(x^2+2y^2) + (e^{-x^2-y^2})(4y)$$

$$= -2y(e^{-x^2-y^2})(x^2+2y^2-2)$$

$$= 0 \text{ if } \quad y=0 \text{ or } \underbrace{2y^2-2=0}_{} \quad (\text{ie. } y = \pm 1)$$

if $x^2+2y^2=1$ then $x^2+2y^2-2 \neq 0$, so f_x and f_y are only simultaneously zero when $x=0$.
So, if $x=0$ then $x^2+2y^2-2 = 2y^2-2$ and this is zero if $y=\pm 1$

Similarly, we have $y=0$ and $x^2+2y=1$ is satisfied when $x=\pm 1$, so...

Our crit points are

$x=0$ and $y=0 \to (0,0)$
$x=0$ and $y=\pm 1 \to (0,1), (0,-1)$
and $y=0$ and $x=\pm 1 \to (1,0), (-1,0)$

2) plug in crit pts.

$f(0,0)=0 \qquad f(0,\pm 1) = 2e^{-1} \qquad f(\pm 1, 0) = e^{-1}$

3) Check the boundary for crit pts.

On the boundary of $D = \{(x,y) \mid x^2+y^2 \le 4\}$ we have $x^2+y^2=4$

So $f(x,y) = e^{-x^2-y^2}(x^2+2y^2) = e^{-4}(4+y^2)$ on the boundary.

and $0 \le y^2 \le 4$ so this is largest when $y^2=4$ and smallest when $y^2=0$

4) Compare min/max on boundary to points inside D.

Then $f(\pm 2, 0) = 4e^{-4}$ and $f(0,\pm 2) = 8e^{-4}$ so the max on D is at $f(0,\pm 1)$ and the min is at $f(0,0)$.

#63 find pts on $xy^2z^3=2$ closest to $(0,0,0)$

Goal:
minimize distance from $(0,0,0)$ subject to the constraint $xy^2z^3=2$

1) Set up function to minimize

The distance from (x_0,y_0,z_0) to $(0,0,0)$ is given by

$$\sqrt{(x_0-0)^2+(y_0-0)^2+(z_0-0)^2}$$

we can see that minimizing

$$x_0^2+y_0^2+z_0^2$$

will minimize this value so we set

$$f(x,y,z)=x^2+y^2+z^2$$

and proceed to minimize $f(x,y,z)$ subject to the constraint $xy^2z^3=2$. (Let $g(x,y,z)=xy^2z^3$ and restrict to $g(x,y,z)=2$)

2) Compute gradients and set up constraints

To use the method of Lagrange Multipliers we find

$$\nabla f=\langle 2x,2y,2z\rangle$$

$$\lambda\nabla g=\langle \lambda y^2z^3,\ \lambda 2xyz^3,\ \lambda 3xy^2z^2\rangle$$

and we know $x,y,z\neq 0$ as $xy^2z^3=2$ and we set

$$2x=\lambda y^2z^3$$
$$2y=\lambda 2xyz^3$$
$$2z=\lambda 3xy^2z^2$$

Solving each for λ gives us

① $\lambda=\dfrac{2x}{y^2z^3}$

② $\lambda=\dfrac{1}{xz^3}$

③ $\lambda=\dfrac{2}{3xy^2z}$

then

② = ③ $\Rightarrow$ $y^2=\frac{2}{3}z^2\Rightarrow y=\pm\sqrt{\frac{2}{3}}\cdot z$ ④

① = ③ $\Rightarrow$ $3x^2=z^2\Rightarrow x=\pm\sqrt{\frac{1}{3}}\cdot z$ ⑤

and $xy^2z^3=2$ so x and z have the same sign, so $x=\sqrt{\frac{1}{3}}\,z$

3) Check the values that satisfy the constraints by plugging in to f.

So $f(x,y,z)$ has a minima or maxima whenever

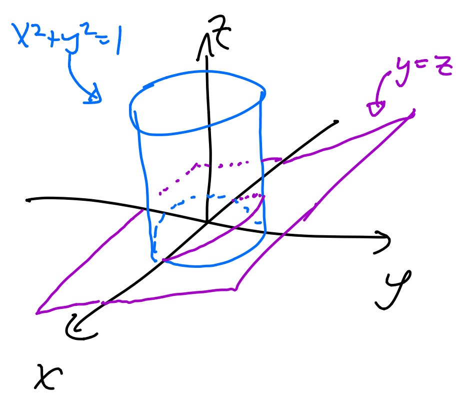
$$\sqrt{\tfrac{1}{3}}z\left(\sqrt{\tfrac{2}{3}}\,z\right)^2\cdot z^3=2 \quad \text{or} \quad \sqrt{\tfrac{1}{3}}z\cdot\left(-\sqrt{\tfrac{2}{\sqrt{3}}}\cdot z\right)^2\cdot z^3=2 \quad \text{or, when } z=\pm\sqrt[4]{3}$$

and x and y satisfy ④ and ⑤

Both options, when plugged into f, give $2\sqrt{3}$. If we check any other point on $g(x,y,z)=2$ ($(2,1,1)$, for example) $f(2,1,1)=6>2\sqrt{3}$ so $2\sqrt{3}$ is a minimum and there is no maximum.

16.3 # 27

Find the volume bounded by $x^2+y^2=1$ and $y=z$ and $x=0$ and $z=0$ in the first octant.



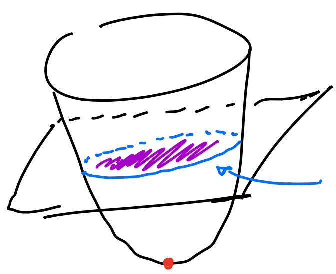
$$V=\iiint_E 1\,dV$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y 1\,dz\,dy\,dx$$

16.4

#24 volume bounded by

$z = 1 + 2x^2 + 2y^2$ and $z = 7$ in first octant



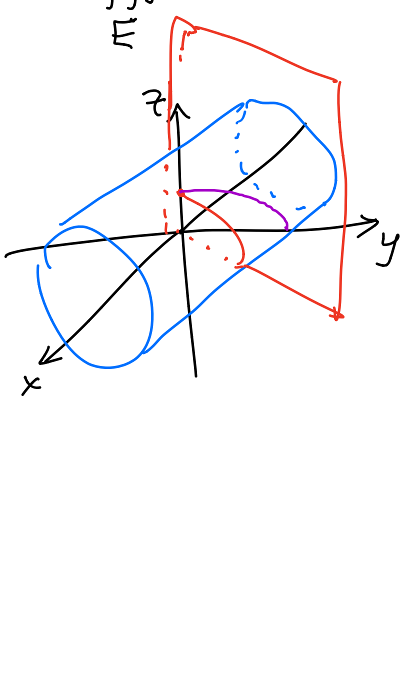
$1 + 2x^2 + 2y^2 = 7 \Rightarrow x^2 + y^2 = 3$

$x^2 + y^2 \le 3$

$$\int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{1+2r^2}^{7} ((1)r)\, dz\, dr\, d\theta$$

16.6 #18

$\iiint_E z\, dV$ E bounded by $y^2 + z^2 = 9$, $x = 0$, $y = 3x$, $z = 0$ in the first octant



$$\int_0^1 \int_{3x}^{3} \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx$$

16.8

#17

Sketch the solid w/ volume

$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$