Name:____________________________________

Worksheet 04

1 Partial Fractions Review and Examples

The goal is the reverse of finding a common denominator, we want to take a single fraction with a complicated denominator and turn it into a sum of fractions with simpler denominators. Let's consider a simple example with integers, rather than polynomials.

Example Write

$$\frac{3}{10}$$

as the sum of two fractions (each smaller, in absolute value, than 1).

Now, lets consider a simple polynomial.

Example Write

$$\frac{3}{x^2 + 3x + 2}$$

as the sum of two rational functions.

Example Write

$$\frac{x^2}{x^3 + x^2 + x + 1}$$

as the sum of two fractions.

Example Write

$$\frac{x^2 + x + 4}{x^4 + 2x^2 + 1}$$

as the sum of two fractions.

Notice, the reasons we pick the numerators the way that we do are:

- We've already done polynomial long division, so we know the numerator is of larger degree than the denominator.
- We don't know, ahead of time, how the numerators will combine with the factors in the denominator so we assign a general polynomial of one degree lower.

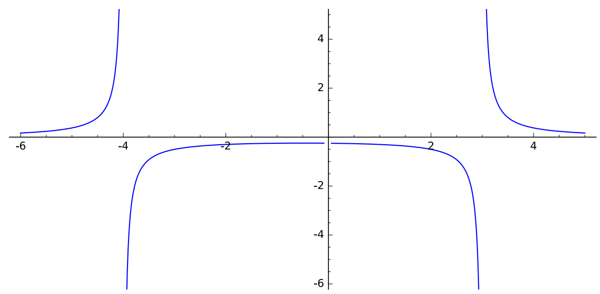
2 Improper Integrals

Improper integrals are, essentially, definite integrals whose bounds of integration are not in the domain of the integrand. They are most commonly encountered when the integrand has vertical asymptotes, or asymptotically approaches the x-axis.

Example The graph of

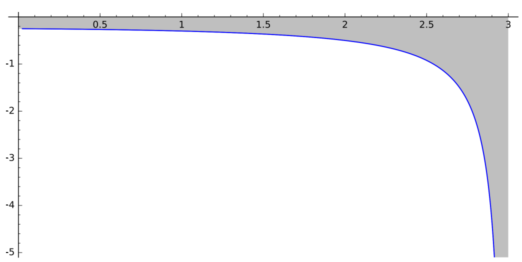
$$f(x) = \frac{3}{x^2 + x - 12}$$

looks like this

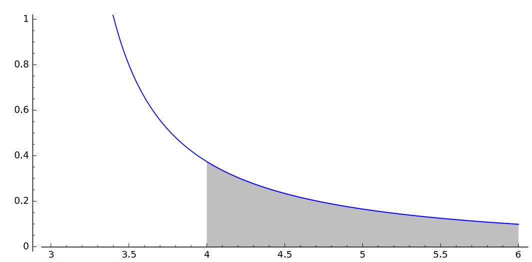


We might be interested in computing the definite integral near, or away from, the singularities. That is, we might be interested evaluating any of the following:

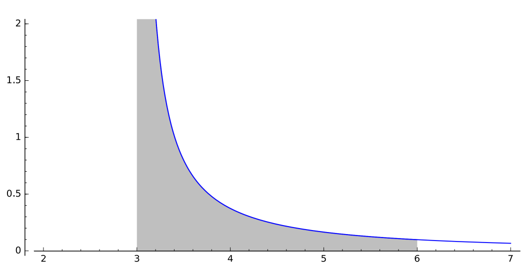
$$\int_0^3 \left(\frac{3}{x^2 + x - 12}\right) dx$$



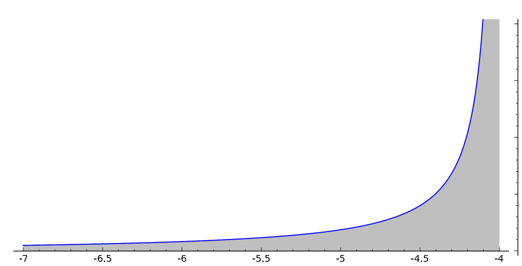
$$\int_4^\infty \left(\frac{3}{x^2 + x - 12}\right) dx$$



$$\int_3^6 \left(\frac{3}{x^2 + x - 12}\right) dx$$



$$\int_{-\infty}^{-4} \left(\frac{3}{x^2 + x - 12}\right) dx$$



A key point to remember is that an improper integral is a definite integral where *exactly one* of the bounds of integration does not lie in the domain of the integrand, which is a continuous function on the rest of the interval defined by these bounds. To formally define such an integral, we replace the problematic bound with a variable, use the properties of limits to make our computations.

Exercise 1. *Write*

$$\frac{4}{21}$$

as the sum of two fractions (each smaller, in absolute value, than 1).

Exercise 2. *Write*

$$\frac{5}{12}$$

as the sum of three fractions (each with a distinct denominator and smaller, in absolute value, than 1). How many different ways can you write it as the sum of two fractions (with the same restrictions)?

Exercise 3. *Write*

$$\frac{3}{x^2 + x - 12}$$

as the sum of two rational functions.

Exercise 4. *Write*

$$\frac{x^2 - x + 4}{x^4 + 6x^2 + 9}$$

as the sum of two rational functions.

Exercise 5. *Compute the integrals in the example at the beginning of Section 2. (Hint: Use the partial fraction decomposition you came up with in Exercise 3 to get started.)*

Exercise 6. *Compute*

$$\int_0^\infty \frac{x^2 - x + 4}{x^4 + 6x^2 + 9}$$

(Hint: Use the partial fraction decomposition you came up with in Exercise 4 to get started.)

3 Optional Practice

Exercise 7. *How might you apply your observations about the number of ways you can write*

$$\frac{5}{12}$$

from, Exercise 2 to find a partial fraction decomposition of

$$\frac{x}{(x+1)(x-3)^2}.$$

What if one of the factors were repeated three times?

Exercise 8. *Consider integrals of the following forms:*

i) $\displaystyle\int_0^1 \left(\frac{1}{x^p}\right) dx$

ii) $\displaystyle\int_1^\infty \left(\frac{1}{x^p}\right) dx$

For what values of p does each converge? For what values of p do they diverge?

Exercise 9. *Partial fraction decompositions allow us to compute the integrals of many rational functions. The principal being, if we can break our initial integral into a sum of parts with less complicated denominators, then we can use tools we have to evaluate the integral. The following are general forms of the types of integrands we need to evaluate after completing a partial fraction decomposition. If you are able to find general rules for these forms, then you are able to evaluate entire families of rational functions.*

Find the general rule for each of the following integrals. Pay careful attention to any assumptions you make about the coefficients, and break your solutions into cases if needed.

i) $\displaystyle\int \left(\frac{A}{Bx+E}\right) dx$

ii) $\displaystyle\int \left(\frac{Ax+B}{(Cx+D)^2}\right) dx$

iii) $\displaystyle\int \left(\frac{Ax^2+Bx+C}{(Dx+E)^3}\right) dx$

iv) $\displaystyle\int \left(\frac{a_1x+a_0}{b_2x^2+b_1x+b_0}\right) dx$

v) $\displaystyle\int \left(\frac{a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{(b_2x^2+b_1x+b_0)^2}\right) dx$