Name:____________________________________

Worksheet 15

1 Practice

Example Set up an integral, in rectangular coordinates, to compute the volume bounded above by

$$\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{16} = 1$$

and below by

$$z = 2$$

so that we integrate with respect to z, then y, then x.

Exercise 1. *Set up an integral, in rectangular coordinates, to compute the volume bounded below*

$$\frac{x^2}{16} + \frac{y^2}{36} + \frac{z^2}{16} = 1$$

and above

$$\frac{y}{6} + \frac{z}{4} = 1$$

so that we integrate with respect to z*, then* x*, then* y*.*

Example Switching the order of integration can, sometimes, make an impossible integral possible. With this in mind, evaluate the following:

$$\int_0^2 \int_0^1 \int_y^1 \sinh(z^2)\, dz\, dy\, dx$$

Exercise 2. *Switching the order of integration can, sometimes, make an impossible integral possible. With this in mind, evaluate the following:*

$$\int_0^2 \int_0^4 \int_z^2 yze^{x^3}\, dx\, dy\, dz$$

2 Moving Between Coordinate Systems

2.1 Rectangular and Polar

Given x and y, we convert to polar coordinates using

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

Given r and θ, we convert to rectangular coordinates using

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

2.2 Rectangular and Cylindrical

Given x, y and z, we convert to cylindrical coordinates using

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$z = z$$

Given z, r and θ, we convert to rectangular coordinates using

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

2.3 Rectangular and Spherical

Given ρ, θ and ϕ, we can convert to from spherical to rectangular coordinates using:

$$x = \rho\sin(\phi)\cos(\theta)$$
$$y = \rho\sin(\phi)\sin(\theta)$$
$$z = \rho\cos(\phi)$$

To convert from rectangular to spherical, we use on the fact that

$$\rho^2 = x^2 + y^2 + z^2$$

and work from there.

3 Integration in Different Coordinate Systems

The key to setting up integrals in different coordinate systems is to be aware of the basic properties of the divisions of your domain. In one dimension, our divisions had *length*, in two they have *area*, and in three they have *volume*. The following is an informal summary of our tools in these situations.

3.1 One Dimension

When we estimate

$$\int_a^b f(x)dx$$

we break the interval subset of the domain, (a, b), into small lengths, Δx, and use $f(x^*)$ for some x^* in that length to get the height of a rectangle, and compute the area as

$$f(x^*) \cdot \Delta x.$$

3.2 Two Dimensions

When we estimate

$$\iint_R f(x,y)dA$$

we have multiple options for cutting the two dimensional subset of the domain, R, into small pieces.

3.2.1 Rectangular Coordinates

We could cut R into small rectangles with area $\Delta x \cdot \Delta y$, and pick some $(x*, y*)$ in each rectangle to get the height of a rectangular prism with volume

$$f(x^*, y^*) \cdot (\Delta x \cdot \Delta y) \, .$$

Roughly, we can say that, if we choose rectangular coordinates, then $dA = dx\, dy$.

3.2.2 Polar Coordinates

We could cut R into polar rectangles with area $r^*\ \Delta r\ \Delta\theta$, and use the associated θ^* to compute the height of a right solid with volume

$$f(r^*\cos(\theta^*), r^*\sin(\theta^*)) \cdot (r^*\ \Delta r\ \Delta\theta)$$

Roughly, we can say that if we choose polar coordinates, then $dA = r\, dr\, d\theta$.

3.3 Three Dimensions

When we estimate

$$\iiint_E f(x,y,z)dV$$

we have even more options for cutting the three dimensional subset of the domain, E, into small pieces. Note, if $f(x,y,z) = 1$, then this integral computes the volume of E.

3.3.1 Rectangular Coordinates

If we break E into tiny rectangular prisms, each will have volume $\Delta x \Delta y \Delta z$, and if we pick a point in that rectangular prism (x^*, y^*, z^*), then we are adding terms that look like

$$f(x^*, y^*, z^*) \cdot (\Delta x \Delta y \Delta z).$$

Roughly, we can say that if we choose rectangular coordinates, then $dV = dx\, dx\, dz$.

3.3.2 Cylindrical Coordinates

If we break E into pieces that are "thickened" polar rectangles, each will have volume $r^*\ \Delta z \Delta r\ \Delta\theta$ and if we pick a point in each of these pieces, then we are adding terms that look like

$$f(r^*\cos(\theta^*), r^*\sin(\theta^*), z^*) \cdot (r^*\ \Delta z \Delta r\ \Delta\theta).$$

Roughly, we can say that if we choose cylindrical coordinates, then $dV = r\, dz\, dr\, d\theta$.

3.3.3 Spherical Coordinates

If we break E into tiny spherical wedges, each will have a volume $(\rho^*)^2 \sin(\phi^*) \Delta\rho \Delta\theta \Delta\phi$, and if we pick a point in each of these pieces, then we are adding terms that look like

$$f(\rho^*\sin(\phi^*)\cos(\theta^*), \rho^*\sin(\phi^*)\sin(\theta^*), \rho^*\cos(\phi^*)) \cdot ((\rho^*)^2 \sin(\phi^*) \Delta\rho \Delta\theta \Delta\phi).$$

Roughly, we can say that if we choose spherical coordinates, then $dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$.

4 More Practice

Exercise 3. *Complete the following:*

a) Convert $(r, \theta, z) = (\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ *from cylindrical to spherical coordinates,* (ρ, θ, ϕ).

b) Convert $(x, y, z) = (-1, 1, -\sqrt{2})$ *from rectangular to spherical coordinates,* (ρ, θ, ϕ).

Exercise 4. *What surfaces are represented by the following equations in spherical coordinates:*

a) $\rho = 7$

b) $\phi = \frac{\pi}{4}$

c) $\theta = \frac{2\pi}{3}$

d) $\rho \sin \phi = 2$

If you've completed the previous exercises, move on to the exercises in Section 6 until we have a chance to discuss this example as a class.

Example Let $f(x, y, z) = \sqrt{x^2 + y^z}$ and let E be the solid bounded by

$$\begin{aligned} z &= 0 \\ z &= 4 \\ x^2 + y^2 &= 25 \end{aligned}$$

Set up an explicit integral, in cylindrical coordinates, to compute

$$\iiint_E f(x, y, z)dV.$$

Exercise 5. *Let* $f(x,y,z) = \left(x^2+y^2\right)^{3/2}$ *and let* E *be the solid bounded by*

$$\begin{aligned} z &= 0 \\ z &= \frac{1}{2}(x^2+y^2) \\ x^2+y^2 &= 4 \end{aligned}$$

Set up an explicit integral, in cylindrical coordinates, to compute

$$\iiint_E f(x,y,z)dV.$$

Example Let $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$ and let E be the solid in the first octant bounded by

$$\begin{aligned} x^2+y^2+z^2 &= 25 \\ \sqrt{x^2+y^2} &= z \\ 2\sqrt{x^2+y^2} &= z \end{aligned}$$

Set up an explicit integral, in spherical coordinates, to compute

$$\iiint_E f(x,y,z)dV.$$

Exercise 6. *Let* $f(x, y, z) = \sin(\sqrt{x^2 + y^2 + z^2})$ *and let* E *be the solid bounded by*

$$x^2 + y^2 + z^2 = 49$$
$$\sqrt{x^2 + y^2} = z$$

Set up an explicit integral, in spherical coordinates, to compute

$$\iiint_E f(x, y, z)dV.$$

Exercise 7. *Convert the following into cylindrical coordinates:*

a)

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^6 \left(\frac{1}{x^2+y^2}\right) \, dz \, dy \, dx$$

b)

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{x^2+y^2}{2}} \left(\frac{1}{x^2+y^2}\right) \, dz \, dy \, dx$$

Exercise 8. *Convert the following into spherical coordinates:*

a)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \left(\frac{1}{1+x^2+y^2+z^2} \right) \, dz \, dy \, dx$$

b)

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} (xz) \, dz \, dy \, dx$$

5 Even More Practice

Exercise 9. *Evaluate the integrals that were set up, but not evaluated, in the previous examples and exercises.*

6 Challenge

Exercise 10. *The volume of a sphere of radius ρ has volume*

$$V = \frac{4}{3}\pi\rho^3.$$

Without consulting outside references, derive a formula for the volume of the spherical wedge described by

$$W = \{(\rho, \theta, \phi) | \rho_1 \leq \rho_2, \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2\}$$

How is this similar to finding the area of a polar rectangle? How is this related to the dV term in a triple integral in spherical coordinates?

Exercise 11. *Find the average value of the function*

$$f(x) = \int_x^1 \cos(t^2)dt$$

on the interval $[0, 1]$.

Exercise 12. *If f is continuous, show that*

$$\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \frac{1}{2} \int_0^x (x-t)^2 f(t) dt.$$