Preparation for October 13

We now turn our attention to using Fourier series to find solutions of two famous *partial differential equations*. (A partial differential equation involves functions of more than one variable and partial derivatives of those functions.) One of these equations is the heat equation, and one is the wave equation.

Suppose given a rod of metal of length L. Let x represent the distance from one end of the rod, so for points on the rod, $0 \leq x \leq L$. Let $u(x,t)$ be the temperature at position x and time t. Then the heat equation tells us

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L.$$

The constant α is called a thermal diffusivity constant; it measures how easily heat can travel through the material. For metals like copper and silver, α is large. For materials like granite and brick, which insulate against heat, α is small.

We will look for functions $u(x,t)$ satisfying the following three conditions:

1. $u_t = \alpha^2 u_{xx}$
2. $u(0,t) = 0$ for $t > 0$
3. $u(L,t) = 0$ for $t > 0$

That is, we suppose that the endpoints of the rod (where $x = 0$ and $x = L$) are held at temperature 0.

We will want to use superposition. Suppose u_1 and u_2 are both functions that satisfy the three conditions, and c_1 and c_2 are constants, then

$$\begin{aligned}(c_1u_1 + c_2u_2)_t &= c_1(u_1)_t + c_2(u_2)_t \\ &= c_1\left(\alpha^2(u_1)_{xx}\right) + c_2\left(\alpha^2(u_2)_{xx}\right) \\ &= \alpha^2\left(c_1(u_1)_{xx} + c_2(u_2)_{xx}\right) \\ &= \alpha^2\left(c_1u_1 + c_2u_2\right)_{xx}\end{aligned}$$

So, $c_1u_1 + c_2u_2$ also satisfies the first condition. Also, since $u_1(0,t) = 0$ and $u_2(0,t) = 0$, we have

$$\left(c_1u_1 + c_2u_2\right)(0,t) = 0.$$

So, $c_1u_1 + c_2u_2$ satisfies the second condition. Similarly, $c_1u_1 + c_2u_2$ satisfies the third condition.

Remember that when we were trying to solve a problem like

$$y'' - 5y' + 6y = 0$$

we started out by finding two solutions of $y'' - 5y' + 6y = 0$, namely $y = e^{2t}$ and $y = e^{3t}$. Then, we used the principle of superposition to say that any function

$$y = c_1 e^{2t} + c_2 e^{3t}$$

satsifies $y'' - 5y' + 6y = 0$. Finally, if we were given initial conditions for y and y', we used them to find the constants c_1 and c_2.

Our general strategy here is similar:

- First, seek functions $u(x,t)$ which satisfy the first three conditions above.
- Look at all possible superpositions of the answers obtained.
- If given $u(x,0)$, use this to find constants.

When we were looking for solutions of $y'' - 5y' + 6y = 0$, we looked for solutions of the form $y = e^{rt}$, where r was a constant.

Here, we will look for functions of the form $u(x,t) = X(x)T(t)$, where X and T are functions of one variable.

We'll do this in class!