Preparation for November 3

Consider problem 2 in section 7.5:

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$$

We begin by searching for solutions of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$. Then we need

$$\lambda e^{\lambda t}\mathbf{v} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} e^{\lambda t}\mathbf{v}$$

That is, we need

$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{v} = \lambda \mathbf{v}$$

So, we need eigenvalues and eigenvectors of the matrix.

The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} = (1-\lambda)(-4-\lambda) - (-2)(3) = \lambda^2 + 3\lambda + 2$$

The roots of this polynomial are -1 and -2.

Let $\lambda_1 = -1$. Then

$$\begin{pmatrix} 1-\lambda_1 & -2 \\ 3 & -4-\lambda_1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}$$

One vector in the nullspace of this matrix is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $\lambda_2 = -2$. Then

$$\begin{pmatrix} 1-\lambda_2 & -2 \\ 3 & -4-\lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$$

One vector in the nullspace of this matrix is $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Thus, we get two solutions:

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

By superposition, the general solution is

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

As $t \to \infty$, all solutions tend to the origin, since the exponential terms all go to the zero vector.

In class, we will look at what the solutions look like as curves in the plane.