Preparation for September 13

We have been looking at second-order equations of the form

$$ay'' + by' + cy = 0. \tag{1}$$

Such equations are called *homogeneous*. If we replace the 0 on the right by a function of t, we get a *nonhomogeneous* equation:

$$ay'' + by' + cy = f(t). \tag{2}$$

Suppose that y_p is a solution of the nonhomogeneous equation, and y_1 is a solution of the homogeneous equation. Then

$$\begin{aligned} a(y_1 + y_p)'' + b(y_1 + y_p)' + &c(y_1 + y_p) \\ &= (ay_1'' + by_1' + cy_1) + (ay_p'' + by_p' + cy_p) \\ &= 0 + f(t) = f(t). \end{aligned} \tag{3}$$

(We know $ay_1'' + by_1' + cy_1 = 0$ since y_1 is a solution of Equation (1). We know $ay_p'' + by_p' + cy_p = f(t)$ since y_p is a solution of Equation (2).

So, once we find one solution of Equation 2, we can add on the general solution of Equation (1) and get other solutions of Equation (2). (In fact, this process will give us all such solutions.)

We will spend some time looking at what to do in various cases for finding a particular solution y_p.

First, let's suppose $f(t)$ is an exponential function. For example,

$$y'' - y' - 6y = e^{2t}$$

In such a case, it is reasonable to try $y_p = Ae^{2t}$.
Then

$$\begin{aligned} y_p &= Ae^{2t} \\ y_p' &= 2Ae^{2t} \\ y_p'' &= 4Ae^{2t} \end{aligned}$$

Thus,

$$y_p'' - y_p' - 6y_p = 4Ae^{2t} - 2Ae^{2t} - 6Ae^{2t} = -4Ae^{2t}.$$

Since we want this to be e^{2t}, we set $A = -1/4$. That is, $y_p = (-1/4)e^{2t}$ is a particular solution of the given differential equation.

The characteristic equation is $r^2 - r - 6 = 0$, which holds for $r = 3$ and $r = -2$. So, the general solution is

$$y = c_1 e^{3t} + c_2 e^{-2t} + (-1/4)e^{2t}.$$

In general, if you are trying to solve $ay'' + by' + cy = De^{\alpha t}$, a good starting guess is $y_p = Ae^{\alpha t}$, since then we get

$$\begin{aligned} ay_p'' + by_p' + cy_p &= aA\alpha^2 e^{\alpha t} + bA\alpha e^{\alpha t} + cAe^{\alpha t} \\ &= A(a\alpha^2 + b\alpha + c)e^{\alpha t} \end{aligned}$$

So, we could set $A = \frac{D}{a\alpha^2 + b\alpha + c}$.

Of course, that won't work if $a\alpha^2 + b\alpha + c = 0$, since we can't divide by 0. In other words, if α is a root of the characteristic polynomial, we will need to try something else.

We will look at this more in class.