Solutions

PHY 232: Modern Physics, Fall 2016
Unit T Practice Exam (50 points)

You may use a calculator and a single 8.5 "x 11" page of notes written in your own handwriting on one side only (which you will turn in with the exam). You may also ask me questions of clarification. You are prohibited from using any other resources, including but not limited to internet resources, electronic devices, worked solutions to any problems, books, notes, and other people. If you have any doubt as to whether a specific resource is allowed, please ask me.

When I ask you to show your work, I want to see an explanation of your physical reasoning, as well as all your mathematical work. The "right answer" by itself is worth only a small fraction of the possible points. Generous partial credit will be given for correct reasoning and mathematical work, though you will lose points for explanations that are wrong.

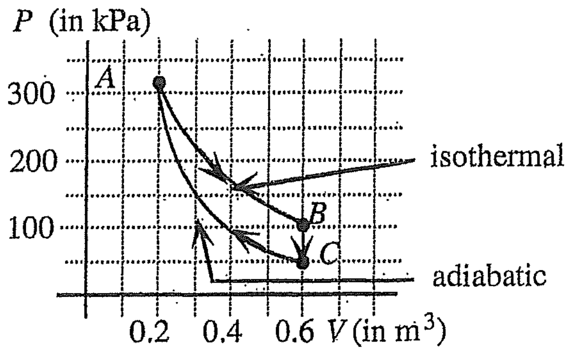
Be sure to include appropriate units for all numerical quantities.

Please staple your equation sheet to the exam.

SHORT ANSWER PROBLEMS: 30 points

1. An ideal gas is confined in a cylinder by a movable piston. The gas is then constrained to follow the cyclic process $A \to B \to C \to A$ shown in the graph below and to the left.

   a. (5 pts) Fill in each blank on the chart below and to the right, indicating the sign of each quantity in the vertical column during the process corresponding to each row.



	Q	W	ΔU
$A \to B$	−	+	0
$B \to C$	−	0	−
$C \to A$	0	+	+

$A \to B$ Isothermal $\Rightarrow \Delta U = 0 \Rightarrow Q = -W$. Expansion so $W < 0 \Rightarrow Q > 0$

$B \to C$ Isochoric $\Rightarrow W = 0 \Rightarrow \Delta U = Q$. Moving to lower isotherm $\Rightarrow \Delta U < 0$

$C \to A$ Adiabatic $\Rightarrow Q = 0 \Rightarrow \Delta U = W$. heating up $\Rightarrow \Delta U > 0 \Rightarrow W > 0$

   b. (1 pt) Is this gas ☒ monatomic or ☐ diatomic?
   (2 pts) Please explain your reasoning (briefly). (*Hint*: Consider the adiabatic expansion phase: monatomic and diatomic gases behave differently during such an expansion.)

$$PV^{\gamma} = \text{const} \Rightarrow P \sim \begin{cases} V^{-5/3} & \text{monatomic} \\ V^{-7/5} & \text{diatomic} \end{cases} \qquad \gamma = \frac{f+2}{f} = 1 + \frac{2}{f}$$

$$\left.\begin{array}{l} \gamma \stackrel{?}{=} \frac{5}{3} \quad (50)(0.6)^{5/3} \stackrel{?}{=} (320)(0.2)^{5/3} \\ \qquad\qquad 21.3 \stackrel{?}{=} 21.9 \end{array}\right\} \text{Monatomic}$$

$$\gamma \stackrel{?}{=} \frac{7}{5} \quad (50)(0.6)^{7/5} \stackrel{?}{=} (320)(0.2)^{7/5}$$

$$24.4 \stackrel{?}{=} 33.6 \qquad \text{NO}$$

   c. (3 pts) If the temperature of the gas when it is at point B is 600 K, roughly how many molecules are confined in this cylinder?

$N =$ 7.2×10^{24}

Ideal gas law

$$N = \frac{PV}{k_B T} = \frac{(10^5\,\text{Pa})(0.6\,\text{m}^3)}{(1.38 \times 10^{-23}\,\text{J/K})(600\,\text{K})}$$

SHORT ANSWER PROBLEMS (continued):

2. Imagine that one has two bottles of gas, one containing a certain amount of a monatomic gas, and the other the same number of molecules of a diatomic gas with more massive molecules. Imagine that the gases in both bottles are initially at room temperature and atmospheric pressure, but then we add enough heat to each to raise their temperature by the same amount ΔT.

   a. (2 pts) To which gas did we have to add more energy?

   ☐ the monatomic gas ☒ the diatomic gas ☐ we added the same amount of heat to both

   Equipartition $U = \frac{3}{2}Nk_BT \Rightarrow C_v = \frac{3}{2}Nk_B$ monatomic

   $U = \frac{5}{2}Nk_BT \Rightarrow C_v = \frac{5}{2}Nk_B$ diatomic

   $Q = C_v\Delta T \Rightarrow$ diatomic needs more energy (heat)

   b. (2 pts) Which will have the higher pressure at the new temperature?

   ☐ the monatomic gas ☐ the diatomic gas ☒ both gases have the same pressure

   $P = \frac{Nk_BT}{V}$ Same for both gases

3. (3 pts) Imagine that we place two Einstein solids in thermal contact. Each has 10,000 atoms, and initially, solid *B* has 999 units of energy and solid *A* has 1 unit of energy. A listing of a few lines from the macropartition table *StatMech* generates for this situation appears below:

U(A)	U(B)	g(A)	g(B)	g(AB)	% of states
0	1000	1	4.663e+1916	4.663e+1916	(tiny)
1	999	30,000	1.504e+1915	4.512e+1919	(tiny)
2	998	450,015,000	4.847e+1913	2.181e+1922	(tiny)

Imagine now that random energy exchanges between the atoms during a certain period of time change the total energy in each solid by one unit (so that U_A is either 0 units or 2 units). How many times more likely is it that we end up with $U_A = 2$ units rather than zero units? **Be sure to state what you are assuming to get your answer.**

$$\frac{Pr(2:998)}{Pr(0:1000)} = \frac{\Omega(2:998)}{\Omega(0:1000)}$$ by ergodic hypothesis (all accessible states equally probable)

$$= \frac{2.181\times10^{1922}}{4.663\times10^{1916}} = \underline{0.4677\times10^{6}} = 4.677\times10^{5}$$

subtract powers of ten

SHORT ANSWER PROBLEMS (continued):

4. a. (9 pts) In many climates, it is both energy-efficient and economical (in the long run) to heat a home using a *heat pump*, a device like a refrigerator that uses work to pump heat from a cold reservoir at temperature T_C (often the groundwater near the house) to a hot reservoir at temperature T_H (the interior of the house itself). The advantage of using a heat pump as opposed to just heating the house directly with electricity is that the heat energy $|Q_H|$ that the house ends up receiving can be many times the electrical work W that one puts into the heat pump (whereas if you put electrical work W into an ordinary electric heater you get the same amount of heat out). We can define the energy productivity EP of a heat pump to be the heat benefit $|Q_H|$ we get out divided by the work energy W we have to put in. What limit does the second law of thermodynamics impose on EP? The equations below outline a derivation of this limit. **To the right of each equation, explain what principle or assumption it expresses or how it follows from the previous equations.** You can be pretty brief in your explanations. (I have already filled in a few for you.)

Equation:	**Explanation:**
(1) $EP \equiv \dfrac{\lvert Q_H\rvert}{W}$	Definition of the energy productivity
(2) $\lvert Q_H\rvert = W + \lvert Q_C\rvert$	Energy conservation
(3) $EP = \dfrac{\lvert Q_H\rvert}{\lvert Q_H\rvert - \lvert Q_C\rvert}$	Solve (2) for W and plug into (1)
(4) $EP = \dfrac{1}{1-\left\lvert\dfrac{Q_C}{Q_H}\right\rvert}$	Divide num & denom by $\lvert Q_H \rvert$
(5) $\Delta S_H + \Delta S_C + \Delta S_{hp} \geq 0$	Second law of thermodynamics
(6) $\Delta S_{hp} = 0$	Once the heat pump reaches steady-state operating conditions, it will return to the same macrostate after a cycle, so its entropy must be the same after a cycle. (If the heat pump is noncyclic, then it just remains in a fixed macrostate, and its change in entropy is also zero.)
(7) $\Delta S_H = \dfrac{\lvert Q_H\rvert}{T_H},\ \Delta S_C = -\dfrac{\lvert Q_C\rvert}{T_C}$	For isothermal processes $dS = \frac{dQ}{T} \rightarrow \Delta S = \frac{Q}{T}$
(8) $\dfrac{\lvert Q_H\rvert}{T_H} \geq \dfrac{\lvert Q_C\rvert}{T_C}$	Substitute (7) & (6) into (5) & rearrange
(9) $\dfrac{T_C}{T_H} \geq \left\lvert\dfrac{Q_C}{Q_H}\right\rvert$	multiply both sides of (8) by $\dfrac{T_C}{\lvert Q_H\rvert}$
(10) $1 - \left\lvert\dfrac{Q_C}{Q_H}\right\rvert \geq 1-\dfrac{T_C}{T_H}$	because $\dfrac{T_C}{T_H} \geq \left\lvert\dfrac{Q_C}{Q_H}\right\rvert$, $1-\left\lvert\dfrac{Q_C}{Q_H}\right\rvert$ is a bigger number than $1-\dfrac{T_C}{T_H}$
(11) $EP \leq \dfrac{1}{1-\dfrac{T_C}{T_H}}$	Substitute (4) into (10) and solve for EP

SHORT ANSWER PROBLEMS (continued):

4. b. (3 pts) In the heat pump situation discussed on the previous page, if the house temperature is 22° C and the groundwater temperature is 12° C, what is the maximum heat energy that a heat pump might put into a house for every 100 J of electrical energy it uses?

Here we have $T_H = 22°C = 295K$, $T_C = 12C = 285K$,

$W = 100J$

$$Q_H = W\,EP \leq W \frac{1}{1-\frac{T_C}{T_H}} = \frac{100J}{1-\frac{285}{295}} = \frac{100J}{10} \times 295$$

$$\underline{\underline{Q_H^{max} = 2950\ J}}$$

ESSAY PROBLEM:

1. (10 pts) The multiplicity of an ideal gas is approximately given by $\Omega = CV^N U^{5N/2}$, where V is the gas' volume, N is the number of molecules, U is the gas' internal energy, and C is a constant that does not depend on V or U. Use the definitions of entropy S and temperature T to determine how the internal energy U depends on N, T, and maybe V.

$$S = k_B \ln \Omega = k_B\left[\ln[C] + N \ln V + \tfrac{5}{2} N \ln U\right]$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{5}{2} N k_B \frac{1}{U} \Rightarrow U = \frac{5}{2} N k_B T$$

Using the definition of entropy, we

ESSAY PROBLEM:

2. (10 pts) The ground ($n = 1$) state of a hydrogen atom is actually two different quantum states with slightly different energies: because of a magnetic interaction between the proton and electron, the quantum state where the proton and electron spins are anti-aligned has an energy a few microelectron-volts above the energy of the state where the spins are aligned. If the hydrogen atom undergoes a transition between these states, it emits a photon with a characteristic wavelength of 21 cm. (Astronomers can detect such photons with a radio telescope and thus determine the distribution of hydrogen gas in the sky.) This means that the energy difference between the states is

$$\Delta E = \frac{hc}{\lambda} = \frac{1240\ \text{eV}\cdot\text{nm}}{21\ \text{cm}}\left(\frac{100\ \text{cm}}{1\ \text{m}}\right)\left(\frac{10^{-9}\ \text{m}}{1\ \text{nm}}\right) = 5.9\times10^{-6}\ \text{eV}$$

If a hydrogen cloud in intergalactic space has a temperature of about 2.74 K (the temperature of the cosmic background radiation), about what fraction of hydrogen atoms in the $n = 1$ level will be in the aligned state (and thus be capable of emitting the 21-cm photons)?

We use the Boltzmann factor.

$5.9\,\mu\text{eV}$ — A $|\uparrow\uparrow\rangle$, AA $|\uparrow\downarrow\rangle$

$$\frac{P_A}{P_{AA}} = e^{-\Delta E/k_B T}$$

$$= \exp\left[-\frac{5.9\times10^{-6}\ \text{eV}}{(8.617\times10^{-5}\ \text{eV}\,\text{K}^{-1})(2.74\ \text{K})}\right]$$

$= 0.9753$ This is the ratio of probabilities

The fraction aligned is $f_A = \dfrac{N_A}{N_A + N_{AA}} = \dfrac{\frac{N_A}{N_{AA}}}{\frac{N_A}{N_{AA}} + 1}$

$$f_A = \frac{\frac{P_A}{P_{AA}}}{1 + \frac{P_A}{P_{AA}}} = \frac{0.9753}{1 + 0.9753} = \underline{\underline{0.4938}}$$

So 49.4% of the atoms can emit.