Preparation for August 28.

Recall from Calculus that you can define y implicitly as a function of x by an equation

$$f(x, y) = C.$$

You can find y' using implicit differentiation:

$$f_x + f_y y' = 0.$$

If we were given the second equation, then the solution of this differential equation is given implicitly using the first equation.

For example, suppose we are given

$$(2x + y) + (x + 2y)y' = 0.$$

Then we want this to be $f_x + f_y y' = 0$, so we need a function f that satisfies

$$f_x = 2x + y, \quad f_y = x + 2y.$$

Since we first want $f_x = 2x + y$, we can look for all functions satisfying this equation. We need the partial anti-derivative of $2x + y$ with respect to x. That is, we take the anti-derivative treating y like a constant, much like when you took partial derivatives of functions with respect to x, you treated y like a constant. The partial anti-derivative of $2x$ with respect to x is x^2, and the partial anti-derivative of y with respect to x is xy (since y is treated like a constant.) Remember that when you take antiderivatives, you must not forget to write "$+C$" at the end. Here, any function of y will be a constant, so we will write

$$f(x, y) = x^2 + xy + C(y)$$

(The partial derivative of any such function with respect to x is $2x + y$.)

Now, we also need $f_y = x + 2y$. So let's compute f_y assuming $f(x, y) = x^2 + xy + C(y)$:

$$f_y = x + C'(y)$$

We want this to be $x + 2y$. So, we need

$$C'(y) = 2y$$

That is, $C(y) = y^2$ (plus any constant.) Thus, we can set

$$f(x, y) = x^2 + xy + y^2.$$

You could add any constant to this that you like, but the final answer will be just to set the function equal to a constant, so why not just bring all constants to the right side:

$$x^2 + xy + y^2 = C.$$

And that's the answer!

Maybe you'd like to try an example on your own:

$$(\cos x)y + (\sin x - 2y\sin(y^2))y' = 0$$

The answer is in the footnote.[1]

Will this technique work in general? Suppose we are given

$$M(x, y) + N(x, y)y' = 0.$$

Will we always be able to find f so that $f_x = M(x, y)$ and $f_y = N(x, y)$? In fact, we will not, and there is a fairly easy way to see why. Suppose

$$f_x = M(x, y) \quad f_y = N(x, y).$$

Then take the partial with respect to y of the first equation and the partial with respect to x of the second equation to get

$$f_{xy} = M_y \quad f_{yx} = N_x.$$

By Clairaut's Theorem, we must have $f_{xy} = f_{yx}$, so if there is to be any hope of finding f, we better have $M_y = N_x$.

For example, if you were asked to solve

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0,$$

you would be wise first to compute

$$M_y = 3x^2 + 2x + 3y^2 \quad N_x = 2x.$$

Since these are not the same, there is no way you will be able to find f with $f_x = M$ and $f_y = N$. We will learn a different method for dealing with this kind of situation in class.

[1] $(\sin x)y + \cos(y^2) = C$