Preparation for October 2

Consider the following differential equation:

$$y'' - (x+1)y' - y = 0$$

Since the coefficient of y' is not a constant, none of the techniques we learned in Chapter 3 are going to work here. Instead, we will search for a power series solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

It turns out that you can take the derivative term by term even with infinite series. (This is not obvious, but should be plausible).

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Now, we are interested in $(x+1)y'$. This is $xy' + y'$. When we multiply y' by x, we can distribute the x across the series (this is by Axiom 1).

$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$

Substituting into the left side of the original differential equation gives

$$\begin{aligned} & y'' - (x+1)y' - y \\ & \qquad = y'' - xy' - y' - y \\ & \qquad = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

We cannot combine these series, since the powers of x are different in the different series. To adjust this, we change the index. By replacing n with $n+2$, we get

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n+2=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

The $n = -2$ and $n = -1$ terms of this series are both 0 because of the $(n+2)(n+1)$ term in the coefficient of x^n. So, this becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Similarly, we change n to $n+1$ in the third term:

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n+1=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=-1}^{\infty}(n+1)a_{n+1}x^n$$

Again, the $n = -1$ term is 0 because of the $n+1$ term. So, this becomes

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

Making these substitutions, we now have

$$\begin{aligned} &y'' - (x+1)y' - y \\ &\quad = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Now, we can combine these series to get

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - n a_n - (n+1)a_{n+1} - a_n\right)x^n.$$

Simplifying the coefficient of x^n, we get

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n\right)x^n.$$

Since we want this series to be 0 for all x, we need all the coefficients to be 0. That is, we need

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n = 0 \qquad \text{for all } n \in \mathbb{N}$$

Equivalently, after a bit of algebra, we get

$$a_{n+2} = \frac{a_{n+1} + a_n}{n+2} \qquad \text{for all } n \in \mathbb{N}$$

This is called a *recurrence relation.* If you know a_0 and a_1, you can compute a_2. Then, knowing a_1 and a_2 would allow you to find a_3. Continuing thus, you could find all the terms in the series.