**Preparation for October 25**

We have now spent three days getting a feel for the general idea of a Fourier series, and two days looking at how Fourier series can be used to find solutions of the heat equation. We will turn now to another famous partial differential equation, the *wave equation*. The first equation below is the wave equation. The next three are homogeneous boundary conditions.

1. $u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, t > 0$
2. $u(0,t) = 0, \quad t > 0$
3. $u(L,t) = 0, \quad t > 0$
4. $u_t(x,0) = 0$

As with the heat equation, the principle of superposition will apply just as well. (If you can't remember why, you might want to skim over the notes from October 13).

As with the heat equation, we will begin by seeking solutions of the form

$$u(x,t) = T(t)X(x)$$

Since $u(0,t) = 0$ and $u(L,t) = 0$, we get $X(0) = 0$ and $X(L) = 0$. Since $u_t(x,0) = 0$, we need $T'(0) = 0$.

Since $u_{tt} = \alpha^2 u_{xx}$, we need

$$T''(t)X(x) = \alpha^2 T(t)X''(x)T$$

or

$$\frac{1}{\alpha^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of $t$ and the right side is a function of $x$, both sides will be a constant, which we will denote $-\lambda$.

So,

$$\frac{X''(x)}{X(x)} = -\lambda$$

or

$$X''(x) + \lambda X(x) = 0.$$

If $\lambda > 0$,

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Now, $X(0) = 0$, so $A = 0$.

Also $X(L) = 0$. If $B = 0$, then $X$ is a trivial solution, so we want $B \neq 0$. Thus $\sin\left(\sqrt{\lambda}L\right) = 0$. This can only happen if $\sqrt{\lambda}L$ is a multiple of $\pi$. That is, for some positive integer $n$,

$$\sqrt{\lambda} = \frac{n\pi}{L}.$$

We will set $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$. Then, every solution of the equations

1. $X''(x) = \lambda X(x)$
2. $X(0) = 0 = X(L)$

is a constant multiple of $X_n(x)$.

Turning to $T$, when $\sqrt{\lambda} = \frac{n\pi}{L}$, we have

$$\frac{1}{\alpha^2}\frac{T''(t)}{T(t)} = -\lambda = -\frac{n^2\pi^2}{L^2}$$

or

$$T''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 T(t) = 0.$$

This has general solution

$$T(t) = A\cos\left(\frac{n\pi\alpha t}{L}\right) + B\sin\left(\frac{n\pi\alpha t}{L}\right).$$

Since we want $T'(0) = 0$, we need $B = 0$. So, we set

$$T_n(t) = \cos\left(\frac{n\pi\alpha t}{L}\right).$$

Then since $u(x,t) = T(t)X(x)$, we have

$$u_n(x,t) = \cos\left(\frac{n\pi\alpha t}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Now, we get a general solution by taking superpositions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi\alpha t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$