Preparation for September 1.

The following is similar (at least at the beginning) to example 4 in section 2.3, so you can cross-reference with that. The goal is to determine the initial velocity v_0 at which you would have to fire a projectile in order for it o keep travelling away from the surface of the earth without ever falling back. We show this is $\sqrt{2gR}$, where R is the radius of the earth in meters, and $g = 9.8m/s^2$. Notice that the units on $2gR$ would be m^2/s^2, so the square root of this quantity would give a velocity.

If you fire a projectile into space, the force of gravity diminishes as it gets further away from the center of the earth. You may have learned that the force is inversely proportional to the square of the distance between the projectile and the center of the earth.

If the distance of the projectile from the surface of the earth is x, then $R+x$ is the distance between the projectile and the center of the earth. Since the force F is *inversely* proportional to the square of this distance, there is a constant k so that

$$F = \frac{k}{(R+x)^2}.$$

To find k, recall that when $x = 0$, $F = -mg$, where m is the mass of the projectile. (The negative sign is because the force points in the negative direction.) So, $-mg = \frac{k}{R^2}$, so $k = -mgR^2$. Thus,

$$F = \frac{-mgR^2}{(R+x)^2}.$$

You probably know $F = ma$, where a is acceleration, or the derivative of velocity; that is $a = v'$. Thus, we get

$$m\frac{dv}{dt} = ma = \frac{-mgR^2}{(R+x)^2}$$

We could write $\frac{dv}{dt}$ as x'', but we have not yet studied second order differential equations. Instead, notice that since the projectile is always going up, we could theoretically compute the velocity v given the position x. That is, v is a function of x.

Then, we can use the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}.$$

Since $\frac{dx}{dt} = v$, we get

$$\frac{dv}{dt} = v\frac{dv}{dx}.$$

So, the above equation

$$m\frac{dv}{dt} = \frac{-mgR^2}{(R+x)^2}$$

becomes

$$mv\frac{dv}{dx} = \frac{-mgR^2}{(R+x)^2}$$

which is a separable differential equation for v in terms of x.

After cancelling m from both sides, we get

$$vv' = \frac{-gR^2}{(R+x)^2}$$

Integrating both sides with respect to x gives

$$\int vv'\ dx = \int \frac{-gR^2}{(R+x)^2}\ dx$$

Rewrite $v'\ dx$ as dv, and pull constants out from the integral:

$$\int v\ dv = -gR^2 \int (R+x)^{-2}.$$

So,

$$\frac{v^2}{2} = gR^2(R+x)^{-1} + C = \frac{gR^2}{R+x} + C.$$

So far, the above is essentially what the book does. I have a shortcut for the last part.

When $x = 0$ (at the surface of the earth), the velocity is v_0. Thus,

$$\frac{v_0^2}{2} = \frac{gR^2}{R} + C.$$

Solving for C and plugging that back into the previous formula gives

$$\frac{v^2}{2} = \frac{gR^2}{R+x} - \frac{gR^2}{R} + \frac{v_0^2}{2}.$$

That is,

$$v_0^2 - v^2 = 2gR - \frac{2gR^2}{R+x}$$

Now, think about what it would mean for the projectile to fall back to the earth. In order for that to happen, the velocity of the projectile would have to be 0 at some x value, when it reaches its maximum height. Setting $v = 0$, we would have

$$v_0^2 = 2gR^2 \left(\frac{1}{R} - \frac{1}{R+x} \right)$$

If you solve for x and simplify, you get

$$x = \frac{Rv_0^2}{2gR - v_0^2}$$

As long as $v_0^2 < 2gR$, this fraction is a positive number, so there will be some x value where the velocity where the projectile falls back. But when $v_0 = \sqrt{2gR}$, then we get a 0 in the denominator, so the projectile never falls back.