Preparation for September 8

Some 2nd order differential equations can be solved by reducing them to first order equations. For example, suppose we have a second-order equation in which t, y' and y'' appear, but y does not. Here is an example:

$$ty'' + y' = 1, t > 0$$

Then we could introduce a new variable $v = y'$. Then $v' = y''$. Now, the equation above becomes

$$tv' + v = 1, t > 0,$$

or

$$v' + \left(\frac{1}{t}\right) v = \frac{1}{t}$$

which is just a first-order linear equation. Using the techniques we learned the first day, we get

$$v = 1 + \frac{C}{t}$$

Then substituting back, we get

$$y' = 1 + \frac{C}{t}$$

Integrating this gives

$$y = t + C \ln |t| + D$$

Here's another trick. Suppose we have a second-order equation in which y, y' and y'' appear, but t does not. For example,

$$y'' + y(y')^3 = 0.$$

The trick is actually the same – let $v = y'$. Then $v' = y''$. Substituting, we get

$$v' + yv^3 = 0$$

On the other hand, we can view v as a function of y, and by the chain rule,

$$v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v.$$

Substituting again, we get

$$\frac{dv}{dy}v + yv^3 = 0.$$

This is a first-order separable equation (with v as a function of y), and simplifies to

$$\frac{1}{v^2}\frac{dv}{dy} = -y.$$

Integrating with respect to y,

$$\frac{-1}{v} = -y^2/2 + C$$

This simplifies to

$$v = \frac{1}{y^2/2 - C} = \frac{2}{y^2 - 2C}$$

Now, replace v by y' again to get

$$y' = \frac{2}{y^2 - 2C}$$

This is separable:

$$(y^2 - 2C)y' = 2$$

Integrating both sides with respect to t gives

$$\frac{y^3}{3} - 2Cy = 2t + D.$$

Notice we have two degrees of freedom.