Preparation for September 4.

A differential equation of the form $y' = f(y)$ is called *autonomous.* E.g.

- $y' = y(y-1)(y+1)$
- $y' = \sin(y)$
- $y' = e^y(y-3)$

Autonomous differential equations are always separable. For example, we could rewrite the first equation above as

$$\frac{y'}{y(y-1)(y+1)} = 1$$

But who wants to find $\int \frac{dy}{y(y-1)(y+1)}$? Instead, we could try to get a qualitative feel for such equations by first finding constant solutions. In order for $y = c$ to be a solution to $y' = f(y)$, we need

$$\frac{d}{dt}(c) = f(c)$$

Since the derivative of a constant is 0, we need $f(c) = 0$.

In the three examples above, the constant solutions are therefore

- $y = 0, y = 1, y = -1$
- $y = n\pi$ (where n is any integer)
- $y = 3$

These constant solutions are called *equilibria.* If a system is described by an autonomous differential equation, then an equilibrium is a state of the system that would not change – all forces that are trying to increase y are perfectly balanced by forces trying to decrease y.

Now, we could try to figure out what happens when the value of y is a little different from such a constant. Consider again the first case

$$y' = y(y-1)(y+1)$$

- If $y > 1$, then we can deduce that each of the terms y, $y-1$ and $y+1$ are positive. Thus, their product is positive. So $y' > 0$. In words, if y is bigger than 1, then y is increasing. (In fact, y is increasing faster and faster as y gets bigger and bigger).

- If $0 < y < 1$, then y and $y+1$ are positive, but $y-1$ is negative. Thus, the product is negative, so $y' < 0$. In words, if y is between 0 and 1, then y is decreasing. (When y is very close to 1, $y-1$ is very small, so the magnitude of y' won't be so big. Similarly, when y is close to 0, y is small, so the magnitude of y' won't be so big then either. Thus, if y starts out at .99, then y will decrease very slowly, then faster when y is about .5, and then slower again as y gets closer and closer to 0.

- If $-1 < y < 0$, then $y+1$ is positive, but $y-1$ and y are negative. Thus, $y' > 0$. So, if y is between -1 and 0, y will be increasing.

- If $y < -1$, then all three quantities y, $y-1$ and $y+1$ are negative, so the product is negative. Thus, $y' < 0$. So, if $y < -1$, then y will be decreasing. (As y gets more and more negative, it will decrease faster and faster.

Notice how we never had to solve a differential equation in order to get insight into what the solutions might look like.

Suppose $y = c$ is an equilibrium solution to an autonomous differential equation. Suppose that when y is a little bigger than c, we find $y' < 0$, while if y is a little less than c, we find $y' > 0$. That means that no matter which side of c we start, the value of y will move toward c (down if we're above, up if we're below). Such an equilibrium is called a *stable* equilibrium. In the above example, $y = 0$ is a stable equilibrium.

If, on the other hand, we find that when y is a little bigger than c, $y' > 0$, while if y is a little less than c, $y' < 0$, then no matter which side of c we start, the value of y will move away from c. Such an equilibrium is called an *unstable* equilibrium. In the example above, $y = -1$ and $y = 1$ are unstable equilibria.

On the next page, we sketch some solutions: