Preparation for September 25

As promised, a proof of Theorem 2.

Theorem 2 Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, and c and d are constant. Then $\sum_{n=0}^{\infty}(ca_n + db_n)$ converges; in fact,

$$\sum_{n=0}^{\infty}(ca_n + db_n) = c\sum_{n=0}^{\infty} a_n + d\sum_{n=0}^{\infty} b_n$$

Proof For each m, let $A_m = \sum_{n=0}^{m} a_n$ and let $B_m = \sum_{n=0}^{m} b_n$.

By definition of series convergence, the sequences of partial sums (A_m) and (B_m) converge. Let A and B be the numbers to which these sequences converge.

By Axiom 1, $(cA_m + dB_m)$ converges to $cA + dB$.

Notice that for all m,

$$\begin{aligned} cA_m + db_m &= c\sum_{n=0}^{m} a_n + d\sum_{n=0}^{m} b_n \\ &= \sum_{n=0}^{m} (ca_n + db_n) \end{aligned}$$

Thus, by definition of series convergence, $\sum_{n=0}^{\infty} ca_n + db_n$ converges to $cA + dB$. QED

You might notice that in the above argument, we said that

$$c\sum_{n=0}^{m} a_n + d\sum_{n=0}^{m} b_n = \sum_{n=0}^{m} (ca_n + db_n)$$

You might think this looks just like the statement of Theorem 2, so it might feel a little circular. The difference is that the above statement only involves sums of finitely many terms, so we are just using usual algebraic properties of distributivity, associativity, and commutativity. The theorem allows us to extend those properties to series.