Preparation for October 9

Suppose $f(x)$ is a periodic function with period 2, and

$$f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1. \end{cases}$$

Let's compute the Fourier coefficients for $f(x)$. Since the period is 2, and L is half the period, $L = 1$.

First,

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\ dx = \int_{-1}^{1} f(x)\ dx.$$

Recall from calculus that when $f(x)$ is an odd function, any integral $\int_{-a}^{a} f(x)\ dx$ is 0. So, $a_0 = 0$.

Next,

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right)\ dx = \int_{-1}^{1} f(x) \cos(m\pi x)\ dx.$$

Recall that the product of an odd function and an even function is odd (e.g. $x^3 * x^4 = x^7$). So, $a_m = 0$ as well.

The above comments hold in general. If $f(x)$ is an odd function, then all the a_m coefficients are going to be 0. Similarly, if $f(x)$ were an even function, then all the b_m coefficients would be 0. (This is because $\sin\left(\frac{m\pi x}{L}\right)$ is an odd function, and the product of an even function and an odd function is an odd function.)

Now, let's compute b_m for $f(x)$.

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right)\ dx = \int_{-1}^{1} f(x) \sin(m\pi x)\ dx.$$

Recall that the product of an odd function and an odd function is an even function (e.g. $x^3 * x^5 = x^8$.) So, $f(x)\sin(m\pi x)$ is an even function. Recall from calculus that if $g(x)$ is an even function, then $\int_{-a}^{a} g(x)\ dx = 2\int_0^a g(x)\ dx$. So,

$$\int_{-1}^{1} f(x) \sin(m\pi x)\ dx = 2\int_0^1 f(x) \sin(m\pi x)\ dx.$$

This is rather convenient, since $f(x) = 1$ for $0 < x < 1$. So, we get

$$\begin{aligned} 2\int_0^1 f(x)\sin(m\pi x)\ dx &= 2\int_0^1 \sin(m\pi x)\ dx \\ &= \frac{-2}{m\pi}\cos(m\pi x)\big|_0^1 \\ &= \frac{-2}{m\pi}\left(\cos(m\pi) - \cos(0)\right) \\ &= \frac{-2}{m\pi}\left((-1)^m - 1\right). \end{aligned}$$

In the last equation above, we used the fact that $\cos(m\pi)$ is 1 if m is even and -1 if m is odd.

So, we find

$$b_m = \frac{-2}{m\pi}\left((-1)^m - 1\right) = \begin{cases} 4/m\pi & m \text{ odd} \\ 0 & m \text{ even.} \end{cases}$$

Thus, the Fourier series for $f(x)$ is

$$\sum_{m=1}^{\infty} \frac{-2}{m\pi}\left((-1)^m - 1\right)\sin(m\pi x).$$

Here is a graph of the 10^{th} partial sum of that series: