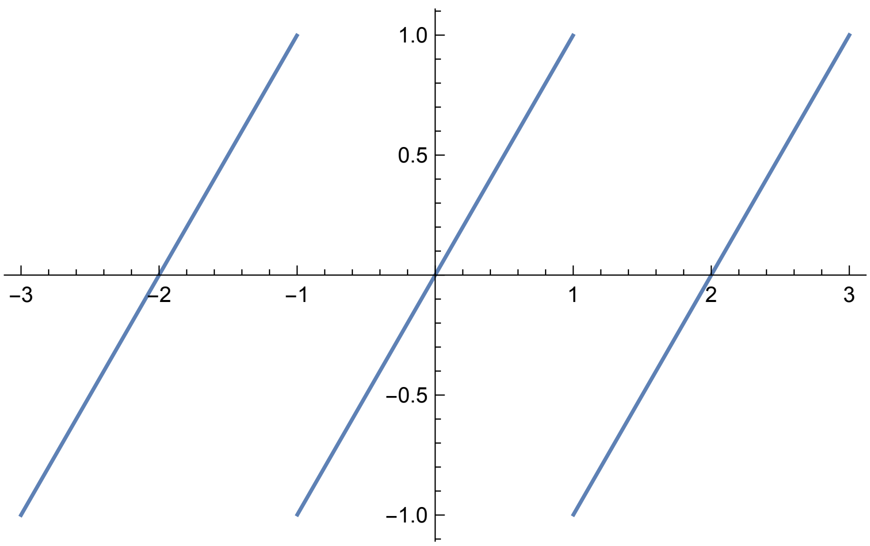
Preparation for October 6

A function $f(t)$ is *periodic* with period L if for all t, we have $f(t+L) = f(t)$. For example, the following is a graph of a periodic function with period 2 (though I have only included the part of the graph where t is between -3 and 3).



Note that if you shift the graph to the left by 2, you get the same graph. That's what it means to be periodic with period 2. But you should also observe that if you shift the graph to the left by 4 or 6, you get the same graph – this graph could be said to have period 4 or 6 as well.

The most familiar examples of periodic functions are the sin and cos functions. If n is a positive integer, then the function $\sin\left(\frac{n\pi t}{L}\right)$ is periodic with period $2L$, since for any t,

$$\sin\left(\frac{n\pi(t+2L)}{L}\right) = \sin\left(\frac{n\pi t}{L} + 2n\pi\right) = \sin\left(\frac{n\pi t}{L}\right)$$

Similarly, $\cos\left(\frac{n\pi t}{L}\right)$ is periodic with period $2L$.

We will attempt to write functions that are periodic with period L in the form of a series, called a *Fourier series*:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi t}{L}\right) + b_m \sin\left(\frac{m\pi t}{L}\right)\right).$$

These differ from power series in that rather than having powers of t, we have sines and cosines in the series. The Fourier coefficients of the series are the numbers a_0, a_1, a_2, a_3, $\ldots$ and b_1, b_2, b_3, $\ldots$.

To find the power series coefficients in chapter 5, we looked at higher derivatives of f at x_0. To find the Fourier coefficients, we will need to use some integrals.

From the angle addition/subtraction formulas, we have

$$\begin{aligned}
&\cos(A+B)+\cos(A-B) = \\
&\quad (\cos(A)\cos(B)-\sin(A)\sin(B)) + (\cos(A)\cos(B)+\sin(A)\sin(B)) \\
&\quad = 2\cos(A)\cos(B)
\end{aligned}$$

We therefore have

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right).$$

Now, we can compute an important integral. Let m and n be distinct positive integers. Then by the equation above, we have

$$\begin{aligned}
&\int_{-L}^{L} \cos\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)\,dt \\
&\quad = \int_{-L}^{L} \frac{1}{2}\left(\cos\left(\frac{(m+n)\pi t}{L}\right) + \cos\left(\frac{(m-n)\pi}{L}\right)\right)\,dt \\
&\quad = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi t}{L}\right)\,dt + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m-n)\pi t}{L}\right)\,dt \\
&\quad = \frac{L}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi t}{L}\right)\Big|_{-L}^{L} + \frac{L}{2(m-n)\pi}\sin\left(\frac{(m-n)\pi t}{L}\right)\Big|_{-L}^{L} \\
&\quad = 0
\end{aligned}$$

The last term is 0 because when we evaluate at L or $-L$, we are getting the sine of an integral multiple of π.

Using similar methods, we can prove the following formulas:

$$\int_{-L}^{L} \cos\left(\frac{m\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)\,dt = \begin{cases} 0 & m \neq n \\ L & m = n. \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)\,dt = 0$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi t}{L}\right) \sin\left(\frac{n\pi t}{L}\right) \, dt = \begin{cases} 0 & m \neq n \\ L & m = n. \end{cases}.$$

We will use these in class to figure out what the Fourier coefficients have to be.