1 Background

In this Differential Equations course, there will be times when we need to use some of the results from Linear Algebra, though we will not generally need to know the proofs of such results. This document is meant to provide the basics, and comes with exercises to give you practice with the kinds of computations you will need to do.

2 Matrices and vectors

A matrix is an array of numbers. (In what follows, we will be considering only real numbers, but one could also consider a matrix with complex numbers, and this is more useful than you might initially think!) Here are some matrices:

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & -5 \end{pmatrix}$$

Notice that the first matrix has two rows and two columns; the second has two rows and three columns, and the third has three rows and three columns. We say that the first matrix is a 2×2 (read "two by two") matrix, the second is a 2×3 matrix, and the third is a 3×3 matrix. A matrix which has the same number of rows as columns (as in the first and third examples above) is called a square matrix.

Each of the numbers in a matrix is called an entry. If you want to refer to the entry in the second row and third column of the matrix A, you would write A_{23}. For example, in the third matrix listed above, A_{23} is -1, while A_{32} is 1.

If A is an $m \times n$ matrix, you can get an $n \times m$ matrix by simply interchanging rows and columns – that is, list the first row of A as the first column of a new matrix, the second row as the second column, and so on. For example, the matrix

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

would become

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 3 & 2 \end{pmatrix}.$$

As another example, the matrix

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & -5 \end{pmatrix}$$

would become

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & -5 \end{pmatrix}.$$

The new matrix is called the *transpose* of the old matrix. If A is an $m \times n$ matrix, its transpose is denoted A^T, and this is an $n \times m$ matrix.

In Calculus 2, you may have seen a vector $\mathbf{v}$ denoted like $\langle 1, 3, -2 \rangle$. This vector points 1 unit in the positive x-direction, 3 units in the positive y-direction, and 2 units in the negative z-direction. In this class, we will typically think of a vector as a matrix with a single column. In this case, we get a 3×1 matrix:

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

Such a matrix (with one column) is called a column vector. The tranpose of a 3×1 matrix is a 1×3 matrix, and this is typically called a row vector. The tranpose of the column vector above is

$$\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}$$

Notice that you could think of a 3×3-matrix as consisting of three column vectors listed from left to right, or three row vectors listed from top to bottom.

We can multiply a 3×3 matrix by a column vector of length 3. To do this, we take the dot product of each of the row vectors of the 3×3 matrix with the column vector, and list the result as a vector. Here is an example. Suppose we wish to compute the product

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

The first row vector of the matrix is $\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$. The dot product of this vector with the column vector $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ is $3 * 1 + 1 * 3 + 2 * (-2) = 2$, so the first entry of the product is 2.

Here is the full computation:

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 * 1 + 1 * 3 + 2 * (-2) \\ 1 * 1 + 0 * 3 + -1 * (-2) \\ 2 * 1 + 1 * 3 + (-5)(-2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 15 \end{pmatrix}.$$

2.1 Exercise

Compute the vector

$$\begin{pmatrix} 4 & -4 & 1 \\ 1 & 0 & 4 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

3 Matrices and systems of equations

Suppose we wish to find a column vector $\mathbf{v}$ such that

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

If we write $\begin{pmatrix} x \\ y \end{pmatrix}$ for $\mathbf{v}$, then the above equation becomes

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

Computing the left side makes this equation become

$$\begin{pmatrix} 2x + 3y \\ 4x - 7y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

So, finding a solution $\mathbf{v}$ to the original equation amounts to finding a solution of a system of equations

$$2x + 3y = 5$$

$$4x - 7y = 7$$

One could argue in the other direction – trying to find a solution of a system of equations can be phrased as trying to solve an equation involving matrices and vectors.

3.1 Exercise

Find rewrite the equation

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & 0 & -4 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

as a system of three equations in three unknowns. Hint: let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Solve the system to determine a vector $\mathbf{v}$ that solves the equation.

4 Addition and multiplication of matrices

If A is an $m \times n$ matrix and B is also an $m \times n$ matrix, we can add A and B by adding the corresponding entries. For example,

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1+3 & (-1)+2 & 3+4 \\ 0+1 & 1+1 & 2+(-2) \end{pmatrix} = \begin{pmatrix} 4 & 1 & 7 \\ 1 & 2 & 0 \end{pmatrix}.$$

Note that the number of rows in A and B must be the same, and the number of columns in A and B must be the same.

As with vectors, matrices can be multiplied by scalars. If c is a number and A is an $m \times n$ matrix, then cA is just the $m \times n$ matrix obtained by multiplying each entry of A by the constant c. For example,

$$3\begin{pmatrix}1 & -1 & 3\\ 0 & 1 & 2\end{pmatrix} = \begin{pmatrix}3 & -3 & 9\\ 0 & 3 & 6\end{pmatrix}$$

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, we can compute a matrix product AB. Note that in this case, the number of columns of A must be the same as the number of rows of B. The product AB will then be an $m \times p$ matrix. To compute AB, we take the dot product of each of the row vectors of A with each of the column vectors of B. More specifically, the i, j entry of AB is obtained by taking the dot product of the i^{th} row of A and the j^{th} column of B.

For example, we can compute the product

$$\begin{pmatrix}1 & -1 & 3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}3 & 1 & 2\\ 1 & 0 & -1\\ 2 & 1 & -5\end{pmatrix}$$

The $1, 1$ entry is the dot product of the row vector $\begin{pmatrix}1 & -1 & 3\end{pmatrix}$ and the column vector $\begin{pmatrix}3 & 1 & 2\end{pmatrix}$. This is 8. The full product of these two matrices is

$$\begin{pmatrix}8 & 4 & -12\\ 5 & 2 & -11\end{pmatrix}.$$

You might notice that multiplying a 3×3 matrix by a column vector of length 3 can be viewed as a special case of multiplying a $3\times$ matrix by a 3×1 matrix, since a column vector of length 3 is just a 3×1 matrix.

It can be useful to have a succinct formula for the i, j entry of the product of two matrices. As mentioned above, the i, j entry of the product AB is the dot product of the i^{th} row of A and the j^{th} column of B. The i^{th} row of A is $\begin{pmatrix}A_{i,1} & A_{i,2} & \cdots & A_{i,n}\end{pmatrix}$. The j^{th} column of B is

$$\begin{pmatrix}B_{1,j}\\ B_{2,j}\\ \vdots\\ B_{n,j}\end{pmatrix}$$

Therefore,

$$(AB)_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j}.$$

More succinctly,

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}.$$

An equation like the one above allows us to demonstrate one interesting property of matrices. Suppose A and B are both $n \times n$ matrices. While it is not often the case that $AB = BA$, we can show that $(AB)^T = B^T A^T$. To see why, consider the i, j entry of $(AB)^T$, where i and j are arbitrary. By definition of transpose,

$$(AB)^T_{i,j} = (AB)_{j,i}$$

By the formula above,

$$(AB)_{j,i} = \sum_{k=1}^{n} A_{j,k} B_{k,i}$$

Now, consider the i, j entry $B^T A^T$. By the formula above,

$$(B^T A^T)_{i,j} = \sum_{k=1}^{n} B^T_{i,k} A^T_{k,j}$$

By definition of transpose,

$$\sum_{k=1}^{n} B^T_{i,k} A^T_{k,j} = \sum_{k=1}^{n} B_{k,i} A_{j,k}$$

But since $B_{k,i}$ and $A_{j,k}$ are ordinary numbers, we have

$$\sum_{k=1}^{n} B_{k,i} A_{j,k} = \sum_{k=1}^{n} A_{j,k} B_{k,i}.$$

Thus, each entry of $(AB)^T$ is equal to the corresponding entry of $B^T A^T$.

For each n, there is a particularly important $n \times n$ matrix, called the identity matrix, and denoted I. This is defined by the formula

$$I_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We must sometimes use context to determine the size n of I.

For example, the 2×2 identity matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The 3×3 identity matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For now, suppose I is the $n \times n$ identity matrix. If A is a matrix with n columns, then $AI = A$. If B is a matrix with n rows, then $IB = B$. In particular, if $\mathbf{v}$ is a column vector of length n, then $I\mathbf{v} = \mathbf{v}$, since such a column vector is really just a $n \times 1$ matrix.

4.1 Exercises

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -3 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 5 & 3 \\ -4 & 4 & 4 \\ -4 & -1 & -4 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & -1 & -2 \\ -5 & -4 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

Compute the following, or say why the question does not make sense.

1. $A + B$
2. $C + D$
3. AB
4. BA

5. BC
6. $(AB)C$
7. $(BC)D$

5 Properties of matrix operations

Matrices under addition, scalar mutliplication, and matrix multiplication satisfy many algebraic properties. You do not need to memorize this list, but you should be familiar with these terms, in case I use them in class, and if you are skeptical about any of these, you should ask me about them.

1. (Associativity) Suppose A, B, and C all have m rows and n columns. Then
$$(A+B)+C = A+(B+C)$$
On the other hand, if A is $m \times n$, B is $n \times p$ and C is $p \times q$, then
$$(AB)C = A(BC)$$

2. (Additive commutativity) If A and B both have m rows and both have n columns, then
$$A+B = B+A.$$

3. (Additive and multiplicative identity) We have already mentioned the $n \times n$ identity matrix I; this is sometimes called the *multiplicative identity*. If A is $m \times n$ and B is $n \times p$, then
$$AI = A$$
$$IB = B$$
There is also an $m \times n$ matrix that acts like the number 0, and this is just the matrix with all entries equal to 0. We typically denote this matrix 0, and call it the *additive identity*. For any other $m \times n$ matrix A, we have
$$A + 0 = A.$$

4. (Additive inverse) If we multiply all the entries of a matrix A by -1, then the resulting matrix, denoted $-A$, is an *additive inverse* of A, in the sense that
$$A + (-A) = 0.$$

5. (Compatibility between scalar and matrix multiplication) If c and d are numbers and A is a matrix, then
$$(cd)A = c(dA)$$
Also, if c is a number, and A and B are matrices where the number of columns in A is equal to the number of rows in B, then
$$(cA)B = c(AB) = A(cB)$$
Finally, multiplication by the scalar 1 does nothing. That is, $1A = A$.

6. (Distributivity) If c and d are numbers, and A and B are $m \times n$ matrices, then we can say
$$c(A + B) = cA + cB$$
and
$$(c + d)A = cA + dA.$$
If A and B and $m \times n$ matrices and C and D are $n \times p$ matrices, then
$$(A + B)C = AC + BC$$
$$A(C + D) = AC + AD$$

There are some properties that do not hold. Students often forget that matrix multiplication need not be commutative – if A and B are both $n \times n$ matrices, it is often true that
$$AB \neq BA.$$

Also, students forget that you can't divide by matrices, and there is no such thing as cancellation. So, if you know $AB = AC$, you cannot conclude that $B = C$. Or if you know that $BC = AC$, you cannot conclude that $A = B$.

5.1 Exercises

1. Choose examples of two 2×2 matrices A and B, and check if $AB = BA$. If they do, look for another pair of matrices until you find two such that $AB \neq BA$.

2. Let $A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ Find matrices B and C such that $B \neq C$, but $AB = AC$.

6 Inverses and determinants

It sometimes is the case that square matrices have multiplicative inverses. The inverse of an $n \times n$ matrix A is an $n \times n$ matrix B such that $AB = I$ and $BA = I$. (For square matrices, it turns out that if $AB = I$ then $BA = I$ and vice versa, so you only have to check one.)

For example, let $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$, and let $B = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$. Then

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1*7+3*(-2) & 1*(-3)+3(1) \\ 2*7+7*(-2) & 2*(-3)+7*1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 7*1+(-3)*2 & 7*3+(-3)*7 \\ (-2)*1+1*2 & -2*3+1*7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

We typically write A^{-1} for the inverse of A, if A has an inverse. But some matrices, like $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, do not have an inverse. Also, notice that matrices that are not square never have inverses!

There is a useful way to tell whether or not a matrix has an inverse, called the *determinant*. For 2×2 matrices, the determinant is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For 3×3 matrices, the calculation is a little more involved. The determinant of

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg.$$

There are ways of computing determinants for larger (square) matrices, but you won't need to know any of them in this class.

What you will need to know is that a matrix has an inverse if and only if the determinant of that matrix is not 0.

For example, the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$ has determinant $2 * 4 - 5 * 1 = 3$, so this matrix has an inverse.

In fact, for 2×2 matrices, there is a simple formula for the inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

You can see that this formula only makes sense when $ad - bc \neq 0$.

6.1 Exercises

1. Compute the determinant of

$$\begin{pmatrix} -1 & -2 & 1 \\ -1 & 1 & -1 \\ 5 & -2 & 2 \end{pmatrix}$$

2. Compute the inverse of

$$\begin{pmatrix} -5 & 4 \\ -4 & 2 \end{pmatrix}$$

7 The nullspace of a matrix

Suppose A is an $n \times n$ matrix, and suppose we want to find all solutions of the equation

$$A\mathbf{v} = \mathbf{0}.$$

Since $A\mathbf{0} = \mathbf{0}$, we have one solution already: $\mathbf{v} = \mathbf{0}$. But are there others?

If A has an inverse, and $\mathbf{v}$ is any solution, then

$$A^{-1}(A\mathbf{v}) = A^{-1}\mathbf{0}.$$

Since $A^{-1}\mathbf{0} = \mathbf{0}$, we have

$$A^{-1}(A\mathbf{v}) = \mathbf{0}$$

Since matrix multiplication is associative, this gives

$$(A^{-1}A)\mathbf{v} = \mathbf{0}$$

Since $A^{-1}A$ is the identity matrix I, we have

$$I\mathbf{v} = \mathbf{0}$$

Since $I\mathbf{v} = \mathbf{v}$, we must have $\mathbf{v} = \mathbf{0}$.

To summarize, if the matrix A has an inverse, then the only solution of $A\mathbf{v} = \mathbf{0}$ is $\mathbf{0}$ itself. This is sometimes called the trivial solution.

Important fact: If A does not have an inverse, then the equation $A\mathbf{v} = \mathbf{0}$ will always have solutions other than the trivial solution.

As an example, suppose $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. Then the determinant of A is $1*6-3*2 = 0$, so we know A does not have an inverse. If we set $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, we find that

$$A\mathbf{v} = \begin{pmatrix} 1*3+3*(-1) \\ 2*3+6*(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As another example, suppose

$$A = \begin{pmatrix} 0 & -5 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{pmatrix}$$

Then the determinant of A is

$$0*1*(-4)+(-5)*5*(-1)+3*2*2-0*5*2-(-5)*2*(-4)-3*1*(-1) = 0$$

To find a vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$, we could write $\mathbf{v}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$A\mathbf{v} = \begin{pmatrix} 0x - 5y + 3z \\ 2x + y + 5z \\ x - 2y + 4z \end{pmatrix}$$

So, we just need to solve the system of three equations in three unknowns:

$$0x - 5y + 3z = 0$$

$$2x + y + 5z = 0$$

$$x - 2y + 4z = 0$$

The first equation tells us that $z = 5y/3$. Substituting into the second and third equation gives

$$2x + (28/3)y = 0$$

$$x + (14/3)y = 0$$

This tells us that $y = -3x/14$. Now, we can set x equal to any number, e.g. 14. Then $x = 14, y = -3, z = -5$, so

$$\mathbf{v} = \begin{pmatrix} 14 \\ -3 \\ -5 \end{pmatrix}.$$

Notice that any scalar multiple of this vector would also work here.

7.1 Exercises

Let $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. Let $B = \begin{pmatrix} -5 & 5 & 3 \\ -4 & 4 & 5 \\ 2 & -2 & -5 \end{pmatrix}$. Let $C = \begin{pmatrix} 5 & 2 & -3 \\ 2 & 5 & 3 \\ 4 & 4 & 0 \end{pmatrix}$. Find nonzero vectors in the nullspaces of A, B, and C.

8 Eigenvalues and eigenvectors

Given a matrix A, we will want to find vectors $\mathbf{v}$ such that $A\mathbf{v}$ is just a scalar multiple of $\mathbf{v}$. That is, $A\mathbf{v} = \lambda\mathbf{v}$ for some number λ. Of course, setting $\mathbf{v} = \mathbf{0}$ will work for any λ, since $A\mathbf{0} = \mathbf{0}$ and $\lambda\mathbf{0} = \mathbf{0}$. Thus, we will ignore this possibility.

If λ is a number and $\mathbf{v}$ is a nonzero vector such that $A\mathbf{v} = \lambda\mathbf{v}$, then we say that λ is an *eigenvalue* of A and $\mathbf{v}$ is an *eigenvector* of A (with eigenvalue λ).

In order to find such eigenvectors, we first look for eigenvalues. The equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

can be written

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

Now, in order to use some of the properties described above, we note that $\mathbf{v} = I\mathbf{v}$, where I is the identity matrix, so the above equation can be written

$$A\mathbf{v} - \lambda(I\mathbf{v}) = \mathbf{0}$$

If you haven't seen this before, this step of rewriting $\mathbf{v}$ as $I\mathbf{v}$ probably seems rather odd – why put in more symbols when we had enough already? The answer will become clear momentarily! First, remember that multiplication of matrices is associative, so we can rewrite the equation above as

$$A\mathbf{v} - (\lambda I)\mathbf{v} = \mathbf{0}$$

Now, using the distributive property, this becomes

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

This step explains why we needed to rewrite $\mathbf{v}$ as $I\mathbf{v}$ – it wouldn't make sense to write $A - \lambda$ since you can't subtract a number from a matrix. But, $A - \lambda I$ makes sense, since A and λI are both matrices.

Remember that we are looking for a nonzero vector $\mathbf{v}$. Thus, in order for such a vector to exist, the matrix $A - \lambda I$ must not have an inverse. That is, $A - \lambda I$ must have determinant equal to 0.

Let's take an example:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

Then

$$\begin{aligned}
A - \lambda I &= \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix}
\end{aligned}$$

The determinant of this matrix is

$$\begin{aligned}
(1-\lambda)(4-\lambda) - (-2)(1) = 4 - \lambda - 4\lambda + \lambda^2 + 2 &= \lambda^2 - 5\lambda + 6 \\
&= (\lambda - 2)(\lambda - 3)
\end{aligned}$$

Since we want values of λ that make the determinant equal to 0, we must have $\lambda = 2$ or $\lambda = 3$. Thus, these are the two eigenvalues of this matrix.

To find the eigenvectors, we consider each eigenvalue separately.

First, consider $\lambda = 2$. Then

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

We need a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, we want $-1 * x - 2 * y = 0$ and $x + 2y = 0$. Setting $y = 1$ gives $x = -2$. So, one eigenvector would be $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Notice that any scalar multiple of this would also be an eigenvector with eigenvalue 2.

Now, we consider $\lambda = 3$. Then

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}$$

We need a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, we want $-2*x-2*y=0$ and $x+y=0$. Setting $y=1$ gives $x=-1$, so, one eigenvector would be $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Notice that any scalar multiple of this would also be an eigenvector with eigenvalue 3.

8.1 Exercises

Find the eigenvalues and corresponding eigenvectors for each of the following matrices:

1. $\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$

4. $\begin{pmatrix} 4 & 1 & -1 \\ 5 & -1 & 3 \\ 5 & 3 & -1 \end{pmatrix}$