Preparation for September 22

In Chapter 5, we will be considering how one can solve differential equations using power series. Here is an example of a power series for e^x, which we briefly mentioned earlier in the course:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sometimes, this is written in an expanded form

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The "dot dot dot" at the end of the expression above means that we need to take a limit. For a fixed positive number x, as we add terms on the right side, the sum keeps increasing, so we get an increasing sequence of numbers; this sequence will approach e^x in the limit.

We will try to make this picture more precise in the next two classes. In what follows, we will give a precise definition of convergence, but we will then state without proof some simple facts about convergence. (For full proofs, you would need to take MAT-316, Foundations of Analysis). We will then use these simple facts to give careful arguments for a number of theorems that we will be able to use to test for convergence of power series.

Before we talk more about series, we really should discuss sequences. I will write (a_n) for a sequence of numbers a_0, a_1, a_2, a_3, ... Thus, for each natural number n, we have a real number a_n.

Intuitively, we say that a sequence (a_n) converges to a number L if the numbers a_n can be made arbitrarily close to L by making n sufficiently large. Let's try making this more precise. Notice that we can measure the distance between two real numbers a and b by taking the absolute value of their difference. For example, the distance between 3 and 7 is $|3 - 7| = 4$. So, we want to say that $|a_n - L|$ can be made arbitrarily small by making n large enough. We will use ϵ to quantify our tolerance for how small is small enough, and we will use N to quantify how large n must actually be. We are now ready for a precise definition.

Definition. A sequence (a_n) *converges* to a number L if for any number $\epsilon > 0$, there is a number N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. We then write $(a_n) \to L$ or $\lim_{n\to\infty} a_n = L$.

Here is a simple example, the sequence $(1/n)$ converges to 0. If you say $\epsilon = 1/10$, then for $n > 10$, I know $|1/n - 0| = 1/n < \epsilon$. If you say $\epsilon = 1/100$, then for $n > 100$, I know that $|1/n - 0| = 1/n < \epsilon$. You should be able to see that no matter what tolerance ϵ you give me, I can choose N to be an integer a little bigger than $1/\epsilon$. Then, provided $n > N$, the distance between $1/n$ and 0 will be smaller than your tolerance ϵ.

Now, for some assumptions about convergence, which should seem plausible.

Axiom 1. If $(a_n) \to L$ and $(b_n) \to M$, then $(a_n + b_n) \to L + M$. Also, if c and d are constants, then $(ca_n + db_n) \to cL + dM$.

Axiom 2. If c is a constant, then the sequence (c) (which is a sequence of all c's) converges to c.

Axiom 3. If $|x| < 1$, then $(x^n) \to 0$.

Axiom 4. The sequence $(1/n)$ converges to 0.

Axiom 5. If $a_n \leq a_{n+1}$ for all n and $(a_n) \to L$, then $a_n \leq L$ for all n.

Axiom 6. If $a_n \leq a_{n+1}$ for all n, and there exists a bound M such that $a_n \leq M$ for all n, then (a_n) converges to some number L.

Axiom 7. If $(a_n) \to L$, and we replace a finite number of the terms at the begining of the sequence with another finite number of terms, then the new sequence also converges to L.

Now that we have discussed sequences, we can consider series.

Definition. Given a series $\sum_{n=0}^{\infty} a_n$, the corresponding sequence of partial sums is the sequence whose m^{th} term is $\sum_{n=0}^{m} a_n$. That is, the sequence of partial sums begins

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \cdots$$

For example, for the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$, the sequence of partial sums is

$$1, 1 + 1/2 = 1.5, 1 + 1/2 + 1/4 = 1.75, 1 + 1/2 + 1/4 + 1/8 = 1.875, \cdots$$

Definition. We say a series $\sum_{n=0}^{\infty} a_n$ *converges* if the corresponding sequence of partial sums converges. We use the notation $\sum_{n=0}^{\infty} a_n$ to refer to the number to which the series converges.

In the example above, the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges to 2.