Preparation for November 13

Last time, we considered the following system:

$$x' = xy + x$$
$$y' = 2xy - 2x - 3y + 3$$

We found two equilibria: $(0, 1)$ and $(3/2, -1)$. By zooming in on the graphs, we guessed that $(0, 1)$ is an unstable saddle, and $(3/2, -1)$ is a stable center.

Rather than zooming in with a computer, it would be nice to have a way to classify these critical points by hand.

Recall from Calculus 1 that you can approximate a function $f(x)$ near a point x_0 using the equation of the tangent line:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

In Calculus 2, you learned that you can approximate a function $f(x, y)$ near a point (x_0, y_0) using the equation of the tangent plane:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Now, we will approximate vector fields. Say we have a system of differential equations:

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned}$$

Also, suppose that (x_0, y_0) is an equilibrium solution. That means $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$. Thus, near (x_0, y_0), we have

$$\begin{aligned} P(x, y) &\approx P(x_0, y_0) + P_x(x_0, y_0)(x - x_0) + P_y(x_0, y_0)(y - y_0) \\ &= P_x(x_0, y_0)(x - x_0) + P_y(x_0, y_0)(y - y_0) \\ Q(x, y) &\approx Q(x_0, y_0) + Q_x(x_0, y_0)(x - x_0) + Q_y(x_0, y_0)(y - y_0) \\ &= Q_x(x_0, y_0)(x - x_0) + Q_y(x_0, y_0)(y - y_0) \end{aligned}$$

We can write the two equations above in matrix form:

$$\begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix} \approx \begin{pmatrix} P_x(x_0, y_0) & P_y(x_0, y_0) \\ Q_x(x_0, y_0) & Q_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

Near the equilibrium solution (x_0, y_0), the system above approximates the system we began with, just as a tangent plane approximates a surface near a point of tangency. The matrix in the above equation is called the *Jacobian* of the vector field $\begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$. For a specified critical point (x_0, y_0), this matrix consists of constants, so we can use the methods of chapter 7 to guess at the classification of the type of the critical point (e.g. stable node or unstable spiral).

Returning to our example above, $P(x,y) = xy + x$ and $Q(x,y) = 2xy - 2x - 3y + 3$. Thus,

$$P_x = y + 1 \quad P_y = x$$

$$Q_x = 2y - 2 \quad Q_y = 2x - 3$$

So, the Jacobian is

$$\begin{pmatrix} y+1 & x \\ 2y-2 & 2x-3 \end{pmatrix}$$

One of the equilibrium points we were interested in was $(0,1)$. At this point, the Jacobian is

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

The characteristic polynomial of this matrix is $(2-\lambda)(-3-\lambda)$, so the eigenvalues of this matrix are 2 and -3. Thus, in the linear approximation, $(0,1)$ is an unstable saddle point.

The other equilibrium point was $(3/2, -1)$. At this point, the Jacobian is

$$\begin{pmatrix} 0 & 3/2 \\ -4 & 0 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 + 6$, which has purely imaginary roots. Thus, in the linear approximation, $(3/2, -1)$ is a stable center.

Note that the solution curves of the nonlinear system will involve terms like $\cos\left(\sqrt{6}t\right)$ and $\sin\left(\sqrt{6}t\right)$. Thus, we can approximate the period of the orbit as $2\pi/\sqrt{6}$.