Preparation for September 15

Last time, we considered differential equations of the form

$$ay'' + by' + cy = g(t)$$

where $g(t)$ was either an exponential function, a polynomial function, or a product of an exponential and a polynomial function. Let's consider trigonometric functions. As an example, suppose we have

$$y'' - y' - 6y = \sin t$$

We might guess that we should try $y_p = A\sin t$, and solve for A. Look what happens:

$$\begin{aligned} y_p &= A\sin t \\ y_p' &= A\cos t \\ y_p'' &= -A\sin t \end{aligned}$$

If we plug these into the differential equation, we get

$$\begin{aligned} y_p'' - y_p' - 6y_p &= -A\sin t - A\cos t - 6A\sin t \\ &= -7A\sin t - A\cos t \end{aligned}$$

There is no way to get this equal to $\sin t$ – if we set $A = -1/7$, then we get $\sin t + (1/7)\cos t$. The problem is that we do not have enough degrees of freedom in order to solve the equation

$$-7A\sin t - A\cos t = \sin t$$

In order to get more degrees of freedom, we could try $y_p = A\sin t + B\cos t$. Then, we have

$$\begin{aligned} y_p &= A\sin t + B\cos t \\ y_p' &= A\cos t - B\sin t \\ y_p'' &= -A\sin t - B\cos t \end{aligned}$$

Thus,

$$\begin{aligned} &y_p'' - y_p' - 6y_p \\ &\qquad = (-A\sin t - B\cos t) - (A\cos t - B\sin t) - 6(A\sin t + B\cos t) \\ &\qquad = (-7A + B)\sin t + (-A - 7B)\cos t \end{aligned}$$

Since we want the above expression to be $\sin t$, we need to solve the system of equations

$$(-7A + B) = 1 \qquad -A - 7B = 0.$$

A little algebra gives $A = -7/50, B = 1/50$, so a particular solution is

$$y_p = (-7/50)\sin t + (1/50)\cos t$$

The general solution is

$$y = c_1 e^{3t} + c_2 e^{-2t} + (-7/50)\sin t + (1/50)\cos t$$

In general, if the nonhomogeneous term is of the form $a\sin t + b\cos t$, we try setting $y_p = A\sin t + B\cos t$, and then solving for A and B.

It should not be too surprising that if the nonhomogeneous term is of the from $e^{\alpha t}(a\sin t + b\cos t)$, we try $y_p = e^{\alpha t}(A\sin t + B\cos t)$.

There is an exception, which can be illustrated by the following example. Suppose we want

$$y'' + y = \sin t$$

Then, if we set $y_p = A\sin t + B\cos t$, we're going to get $y_p'' + y_p = 0$. There is no way that will work. The problem is that the solution of the homogeneous equation is $c_1 \sin t + c_2 \cos t$, so we can't have y_p take that form.

As before, in such a case, we would instead try $y_p = t(A\sin t + B\cos t)$. We will discuss this more in class!