Preparation for September 18

Maybe you have been wondering why we care about the second-order differential equations we have been looking at for the last five days. The reason is that these come up a lot in Physics. Let me explain how.

Imagine that you have a mass attached to a spring, where the spring is hung from a fixed ceiling. The mass is constrained to a vertical line below the spring. If we pull the spring, it will pull the mass back. If we compress the spring, it will push the mass back. We might expect the mass to oscillate back and forth if we pull it a little and let it go.

Since we have attached the mass to the spring, the spring stretches to a certain length. Let u denote the distance that the spring has been extended beyond this length. A negative u will mean the spring has been compressed. The force exerted on the mass by the spring is then negative if u is positive and positive if u is negative. Hooke's Law tells us that there is a positive spring constant k so that the force of the spring is given by

$$F_{\text{spring}} = -ku.$$

At the same time, there may be some damping effect caused by friction. This damping force is negatively proportional to the velocity (like air resistance in some of the examples we encountered earlier in the course). Thus, there is some constant γ so that the damping force is given by

$$F_{\text{damping}} = -\gamma u'.$$

Finally, we might have some time-dependent extra force $F(t)$ that is driving the mass.

Newton's second law tells us $F = mu''$. Thus, adding the forces from the spring and the damping, we get

$$mu'' = -\gamma u' - ku + F(t).$$

This becomes

$$mu'' + \gamma u' + ku = F(t).$$

This is the kind of second-order differential equation we have been studying!

Suppose a mass weighs 4 pounds. When attached to a spring, the mass pulls the spring downward 8 inches. Suppose the mass is then pulled down another 4 inches and released.

To find u as a function of t, first we need to find k. It's best to convert everything to feet – there are 12 inches in a foot, so 8 inches is 2/3 feet Since 4 pounds extends the spring 8 inches, we get $k * (2/3) = 4$, so $k = 6$ (the units on k are lbs/ft.

Since there is no damping and no external force, $\gamma = 0$ and $F(t) = 0$.

Now, since pounds are a unit of force, we need to convert to mass by dividing by $g = 32$; 4 pounds has a mass of $4/32 = 1/8$.

So, our differential equation is

$$(1/8)u'' + 6u = 0 \implies u'' + 48u = 0.$$

The characteristic polynomial is $r^2 + 48 = 0$, which has roots $r = \pm i\sqrt{48}$ Thus, our general solution is

$$u = c_1 \cos\left(\sqrt{48}t\right) + c_2 \sin\left(\sqrt{48}t\right)$$

This gives

$$u' = -\sqrt{48}c_1 \sin\left(\sqrt{48}t\right) + \sqrt{48}c_2 \cos\left(\sqrt{48}t\right)$$

We are given that $u(0) = 1/3$ (remember 4 inches is (1/3) feet), and $u'(0) = 0$. Thus,

$$(1/3) = c_1 * 1 + c_2 * 0 \quad 0 = -\sqrt{48}c_1 * 0 + \sqrt{48} * c_2 * 1$$

So, $c_1 = 1/3$ and $c_2 = 0$. So, $u = (1/3)\cos\left(\sqrt{48}t\right)$.