Preparation for November 8

We have looked at solutions of the differential equation

$$\mathbf{x}' = A\mathbf{x}$$

in the cases in which the eigenvalues of A are real and distinct, or complex. Now, we consider the following example, in which we will see the eigenvalues are real but repeated

$$\mathbf{x}' = \begin{pmatrix} 11 & 27 \\ -3 & -7 \end{pmatrix} \mathbf{x}$$

The characteristic polynomial is

$$(11 - \lambda)(-7 - \lambda) - (-3)(27) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

We see that 2 is a root of the characteristic polynomial. Since the power of $\lambda - 2$ in the factorization of the characteristic polynomial is 2, we say the *algebraic multiplicity* of the eigenvalue is 2.

To find the eigenspace of $\lambda = 2$, we look at the nullspace of

$$A - \lambda I = \begin{pmatrix} 9 & 27 \\ -3 & -9 \end{pmatrix}$$

Every vector in that nullspace is a scalar multiple of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. That is the nullspace could be thought of as a line containing a single vector. Since a line is one-dimensional, we say that the eigenvector has *geometric multiplicity* 1.

At this point, we only know that functions of the form

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{2t}$$

are solutions of the original system. That is, we only have one degree of freedom – c_1. If we want to be able to solve the system given an initial condition vector $\mathbf{x}(0) = \begin{pmatrix} a \\ b \end{pmatrix}$, we will need two degrees of freedom. This is analogous to the situation we saw in Section 3.4 when we encountered repeated eigenvalues of the characteristic polynomial.

Our strategy will be to find a solution of the form

$$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{v}t + \mathbf{w}\right).$$

Inserting this into $\mathbf{x}' = A\mathbf{x}$, and using the product rule, we get

$$\lambda e^{\lambda t}\left(\mathbf{v}t + \mathbf{w}\right) + e^{\lambda t}\left(\mathbf{v}\right) = A\left(e^{\lambda t}\mathbf{v}t + \mathbf{w}\right)$$

Pulling out $e^{\lambda t}$ and cancelling it on both sides yields

$$\lambda\mathbf{v}t + \lambda\mathbf{w} + \mathbf{v} = A\mathbf{v}t + A\mathbf{w}$$

For this to hold for all t, we need

$$A\mathbf{v} = \lambda\mathbf{v} \quad A\mathbf{w} = \lambda\mathbf{w} + \mathbf{v}$$

In other words, we need

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \quad (A - \lambda I)\mathbf{w} = \mathbf{v}$$

We already have a good example for $\mathbf{v}$, namely $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. So, we need a vector $\mathbf{w}$ such that

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

i.e. such that

$$\begin{pmatrix} 9 & 27 \\ -3 & -9 \end{pmatrix}\mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Note that the first column of $A - \lambda I$, multiplied by $1/3$, gives $\mathbf{w}$. Thus, we can set $\mathbf{w} = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$.

Now that we have a second solution, we get a general solution:

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{2t}\left(\begin{pmatrix} 3 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}\right)$$