Preparation for September 29

We learned last time the ratio test: given a series $\sum_{n=0}^{\infty} c_n$, suppose $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = R$. If $R < 1$, the series converges, while if $R > 1$, the series diverges. (If $R = 1$, the ratio test does not tell us what happens.)

Now, we consider a *power series*. Given a number x_0, and letting x be a variable, a power series at x_0 is a series of the from

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots .$$

Thus, the n^{th} term of the power series is $a_n(x - x_0)^n$.

Let $\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. Notice that ρ looks a little different from the limit we used to define R – we have a_n on top and a_{n+1} on the bottom.

Claim. The power series converges when $|x - x_0| < \rho$ and diverges when $|x - x_0| > \rho$. That is, the series converges when the distance between x and x_0 is less than ρ, and diverges when that distance is greater than ρ. (We call ρ the *radius of convergence* of the power series.)

Here is why the claim holds. To find where the series converges, we use the ratio test. Since the n^{th} term of the series is $a_n(x - x_0)^n$, we must study

$$R = \lim_{n\to\infty} \left|\frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n}\right|$$

Simplifying the fraction, we get

$$R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x - x_0)\right|$$

Since $|x - x_0|$ does not depend on n, we can pull it out of the limit:

$$R = \left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right) |x - x_0|$$

The series converges when $R < 1$, i.e. when

$$|x - x_0| < \frac{1}{\left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right)} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \rho$$

The series diverges when $R > 1$, i.e. when

$$|x - x_0| > \rho$$

Example

Consider the power series $\sum_{n=0}^{\infty} \frac{nx^n}{3^n}$. Here $a_n = \frac{n}{3^n}$ and $x_0 = 0$. Thus,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{n/3^n}{(n+1)/3^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{3^{n+1}}{3^n} \right| \\
&= \lim_{n\to\infty} \frac{n}{n+1} \cdot 3 = 3
\end{aligned}$$

Thus, the radius of convergence of the power series is 3, so the series converges whenever $|x - 0| < 3$, i.e. on the interval $(-3, 3)$. We also know that when $|x - 0| > 3$, the series diverges.

The only values left that you might care about are $x = 3$ and $x = -3$. If you plug in $x = 3$, the series becomes $\sum_{n=0}^{\infty} n$. If you plug in $x = -3$, the series becomes $\sum_{n=0}^{\infty} (-1)^n * n$. Both of these diverge by the divergence test.