Preparation for November 15

Last time, we saw that if (x_0, y_0) is an equilibrium solution for

$$x' = P(x, y)$$

$$y' = Q(x, y)$$

then we can approximate the direction field near (x_0, y_0) using the Jacobian matrix to find a corresponding linear system:

$$\begin{pmatrix} P_x(x_0, y_0) & P_y(x_0, y_0) \\ Q_x(x_0, y_0) & Q_y(x_0, y_0) \end{pmatrix}$$

It is usually the case that the type and the stability of an equilibrium solution in a nonlinear system coincides with that of the corresponding linear system. For example, if we find that the corresponding linear system has two complex eigenvalues with negative real part, then the same will hold for the original nonlinear system.

But there are two notable exceptions, and both of these happen when a small change in the eigenvalues would have caused us to classify the equilibrium solution differently. Be sure to know the following:

1. If the two eigenvalues of the linear system are purely imaginary (i.e. their real part is 0), then the type of critical point of the linear system is a stable center. Note that a small change in the entries of the Jacobian could push the real part of the eigenvalues into positive or negative territory. Since the linear system only approximates the nonlinear system, we cannot tell from the linear system whether the equilibrium point will be a stable spiral, an unstable spiral, or a center.

2. If the linear system has repeated eigenvalues, then the type of the critical point of the linear system is an improper or a proper node.[1] But a small change in the Jacobian could cause the repeated eigenvalue to separate – either we could get two real eigenvalues or two complex eigenvalues, very close to the repeated eigenvalue. If the repeated eigenvalue is positive, we can say that the equilibrium point will be unstable,

[1] When the geometric multiplicity is less than the algebraic multiplicity, we have an improper node. When the geometric multiplicity is equal to the algebraic multiplicity, we have a proper node.

and if the repeated eigenvalue is negative, we can say that it will be asymptotically stable. But just as above, we cannot tell from the linear system whether the equilibrium point will be a node or a spiral point.

We will study examples of this in class.