1 Classical Game Theory

1.1 Introduction

A *game* consists of a meeting of two players, each of whom must pick one from a collection of strategies. For each pair of strategies, there is an associated payoff to each player, and each player desires to maximize their payoff.

For example, in the game *rock-paper-scissors*, each player can choose one of three strategies. If the first player chooses *rock* and the second player chooses *scissors*, then the first player wins, so we could say the payoff for the first player is 1 and the payoff for the second player is -1. If both players choose *paper*, then the game results in a tie, so each player receives a payoff of 0. We can illustrate the payoffs each player receives using a table:

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

To find the payoffs, select the row corresponding to the strategy selected by the first player and the column corresponding to the strategy selected by the second player. In the corresponding square, the first number is the payoff for the first player, and the second number is the payoff for the second player.

Of course, in this game, there is no one strategy that is better than any other. If you do not know what the other player's strategy is going to be, then any strategy you choose has the same odds of resulting in a win, a lose, or a tie.

The game above is called a *zero-sum game*, since the sum of the payoffs for the two players is 0 regardless of the strategies chosen. As an example of a game which is not a zero-sum game, consider the game: *odd or even.* Each player selects a number. If the sum of the two numbers is even, both players win. If the sum is odd, both players lose. All that matters about each number is whether it is odd or even, so the strategies are Odd and Even. Here is the payoff table:

	O	E
O	1,1	-1,-1
E	-1,-1	1,1

Here again, there is no obviously good strategy. If the players could communicate ahead of time what they are planning to do, they could coordinate their actions. However, if such coordination is not allowed, then each player has a 50% chance of selecting the same strategy as the other (winning) and a 50% chance of selecting the opposite strategy (losing).

Probably the most famous game in classical game theory is *prisoner's dilemma*. In this game, two individuals have been arrested by the police on suspicion of some serious crime. The police do not have enough evidence to convict either of these prisoners unless one of them confesses. So the police separate the two prisoners and tell each, "If you confess to the crime and your accomplice does not confess, then we can convict your accomplice, who will serve a ten-year sentence in jail, but we'll let you go free. If your accomplice confesses and you do not, then you will serve a ten year sentence, while your accomplice will go free. If you both confess, we will put you both in jail for 8 years. If neither of you confesses, we have enough evidence to put you both in prison for a year on a minor charge."

Each player has two strategies: Confess or Silence. The payoff table is below

	C	S
C	-8,-8	0,-10
S	-10,0	-1,-1

Note that in this game, the best strategy for each prisoner is to confess. For example, the first prisoner might reason as follows: "If the other prisoner confesses, then confessing puts me in prison for only 8 years, rather than 10. If the other prisoner stays silent, then confessing sets me free, whereas staying silent puts me in prison for a year. Either way, confessing is a better strategy."

Another game is called *chicken*. Two players are driving their cars toward one another. If one player turns, that player is the "chicken" and loses, and must pay $1000 to the other player. If neither player turns, then the players crash into each other, which results in \$10,000 of damage to each car. If both players turn at the same time, then no one wins, but at least a crash is avoided. Let's say the two strategies are "Turn" and "Straight". One might use the following payoff table:

	T	S
T	0,0	-1,1
S	1,-1	-10,-10

Here again, there is no optimal strategy, though turning might seem wise, since at least one avoids the danger of a \$10,000 repair charge. But if the other player thinks this too, I might be able to get that \$1000 by not turning! But what if the other player thinks *that* and is therefore not going to turn?

1.2 Nash equilibria

A *Nash equilibrium* is a pair of strategies such that each strategy is a best response to the other strategy. That is, if player 1 chooses α and player 2 chooses β, then we say that (α, β) is a Nash equilibrium if, given player 2's choice of β, there is no better choice for player 1 than α, and given player 1's choice of α, there is no better choice for player 2 than β.

In the game *odd or even*, there are two Nash equilibria. One of these is (O, O). To see why, notice that given player 2's choice of O, player 1's best response is O. Similarly, given player 1's choice of O, player 2's best response is O. By a similar argument, (E, E) is also a Nash equilibrium.

We can be more formal about the definition of a Nash equilibrium by introducing some notation. Let P_1 be the function that takes a given pair of strategies and outputs the payoff for player 1. For example, in *prisoner's dilemma*, $P_1(C, C) = -8$, $P_1(C, S) = 0$, $P_1(S, C) = -10$ and $P_1(S, S) = -1$. Similarly, we define P_2 to be the function that outputs the payoffs for player 2.

Then a strategy (α, β) is a Nash equilibirum if

- $P_1(\alpha', \beta) \leq P_1(\alpha, \beta)$ for each strategy α' available to player 1.
- $P_2(\alpha, \beta') \leq P_2(\alpha, \beta)$ for each strategy β' available to player 2.

In the game *rock-paper-scissors*, there is no Nash equilibrium. For example, consider the pair (R, R), where both players choose Rock. We have $P_1(P, R) > P_1(R, R)$, since if player 1 chooses Paper, player 1 beats player 2. So, (R, R) is not a Nash equilibrium.

Consider the pair (R, P). We have $P_1(S, P) > P_1(R, P)$. So, (R, P) is not a Nash equilibrium.

Finally, consider the pair (R, S). It is true that $P_1(R, S) > P_1(S, S)$ and $P_1(R, S) > P_1(P, S)$. But, $P_2(R, P) > P_2(R, S)$. So, (R, S) is not a Nash equilibrium.

In a similar way, we could check that each of the other 6 pairs of strategies is not a Nash equilibrium.

Suppose we changed the game by modifying the payoffs. Let *modified odd or even* be the game where the strategies are the same as in *odd or even*, but the payoff is as follows:

	O	E
O	2,2	-1,-1
E	-1,-1	1,1

One might think that (E, E) would be eliminated as a Nash equilibrium, but in fact, this modified game has the same Nash equilibria as the original. But, notice that $P_1(E, E) \geq P_1(O, E)$ and $P_2(E, E) \geq P_2(E, O)$, so (E, E) is still a Nash equilibrium.

In *prisoner's dilemma*, the only Nash equilibrium is (C, C), even though (S, S) would have been better for both players. In *chicken*, (S, T) and (T, S) are the two Nash equilibria. If I know the other player is going to turn, my best response is to go straight. If I know the other player is going to go straight, my best response is to turn. So the best response for each player would be to do the opposite of what the other player does.

1.3 Mixed Nash equilibria

Suppose that we get to play a particular game many times. In *odd or even*, player 1 might decide to choose an odd number one third of the time and an even number two thirds of the time. This is called a *mixed strategy*. A *pure strategy* is just a mixed strategy in which the player chooses the same strategy with probability 100%. We have already seen that there are two pure strategy Nash equilibria in *odd or even*. Let's see if there are any mixed strategy Nash equilibia.

Say player 1 chooses nonnegative numbers a and b. Here, a stands for the probability of choosing strategy O, and b for the probability of choosing E. (Thus, we must have $a + b = 1$). Similarly, player 2 chooses probabilities x and y for the two strategies, with $x + y = 1$.

In order for this pair to be a mixed Nash equilibrium, it must be the case that if player 1 knows that player 2 is going to choose O with probability x and E with probability y, then player 1 has no incentive to change to a different mixed strategy. That is, the two strategies O and E must yield the same payoff for player 1, since if one of these strategies yielded a higher payoff, choosing that strategy all the time would be a better response to player 2's mixed strategy than for player 1 to use a mixed strategy. We conclude as follows:

For a mixed pair of strategies to be a mixed Nash equilibrium, the payoff for each player would not change if they alone changed their strategy.

Note that when player 1 chooses strategy O, the expected payoff is

$$x * 1 + y * (-1),$$

since player 1 gets payoff $+1$ in those cases when player 2 chooses O and gets payoff -1 in those cases when player 2 chooses E. When player 1 chooses strategy E, the expected payoff is

$$x * (-1) + y * 1.$$

Since these expected payoffs must be equal, we need

$$x - y = -x + y, \quad x + y = 1.$$

The only solution is $x = y = 1/2$. A similar analysis shows that $a = b = 1/2$.

Let's think about what this means. To say $x = y = 1/2$ means that player 2 chooses an odd number with the same probability as an even number. If player 1 knows this is going to be the case, then there is no reason for player 1 to favor O or E. The same holds with the two players reversed. This pair of mixed strategies is a Nash equilibrium, since neither player can improve, even with advance knowledge of the strategy of the other player.

We can also compare the payoffs of the various Nash equilibria. For the pure strategy Nash equilibria in the game *odd or even*, the payoffs to each player are both 1. In the mixed strategy Nash equilibrium, notice that there is a $1/4$ chance that both players choose O, and this yields a payoff of $+1$ for each player. Similarly, there is a $1/4$ chance that player 1 chooses O and player 2 chooses E, and this yields a payoff of -1 for each player. Continuing in this way, we find the total expected payoff for each player to be

$$(1/4)(1) + (1/4)(-1) + (1/4)(-1) + (1/4)(1) = 0.$$

Clearly, in this game, it would benefit both players if they could coordinate beforehand!

With *rock-paper-scissors*, to get a mixed strategy, player one might select three numbers, a, b, c, where a is the probability of choosing Rock, b of choosing Paper, and c of choosing Scissors. Similarly, player 2 might choose three numbers x, y, z. Again, we must have $a + b + c = 1$ and $x + y + z = 1$, and these numbers must be nonnegative.

There are a few cases to consider here, some of which are left for the exercises. We have already seen that there is no pure strategy Nash equilibrium (i.e. none of the numbers a, b, c, x, y, or z can be 1.) But we may have a mixed strategy Nash equilibrium where $c = 0$ and $a + b = 1$. (You will explain in the exercises why this cannot, in fact, happen – think about what player 2 would certainly do if they knew $c = 0$, and then think about whether player 1 really had the best response to this.)

Now, let us suppose that none of the probabilities are 0. Then, player 2 chooses R with probability $x > 0$, P with probability $y > 0$, and S with probability $z > 0$. Then the payoff for player 1 when choosing R, P, and S is (respectively)

$$x * 0 + y * -1 + z * 1 = -y + z$$

$$x * 1 + y * 0 + z * (-1) = x - z$$

$$x * (-1) + y * (1) + z * (0) = -x + y$$

If any of these strategies yielded a larger payoff than any other, then the mixed strategy is not a Nash equilibrium. So, we must have

$$-y + z = x - z = -x + y, \quad x + y + z = 1$$

Solving these equations with a little algebra yields $x = y = z = 1/3$.

So, the only Nash equilibrium in this game is the mixed strategy where each player chooses from each strategy with equal probability. If you compute the expected payoff to each player as we did in the previous example, you should find it to be 0.

1.4 Exercises

(See last page for selected answers; of course, you need to show your work!)

1. Explain why there are no mixed Nash equilibria in the *prisoner's dilemma* game.

2. What are the mixed Nash equilibria in *modified odd or even*? What are the expected payoffs for the two players under this mixed equilibrium? How does this compare with the expected payoffs for the pure strategies?

3. Find a mixed Nash equilibrium in *chicken*? What are the expected payoffs to the two players for this mixed Nash equilibrium? How does this compare to the expected payoffs for the pure strategy Nash equilibria?

4. Explain why any mixed Nash equilibria in the game *rock-paper-scissors* cannot have any of the probabilities equal to 0. (By the symmetry of the game, it is enough to explain why we cannot have $c = 0$.)

5. Consider a game with three strategies α, β, γ and the following payoff table:

	α	β	γ
α	-1,1	-3,3	-2,2
β	4,-4	3,-3	-1,1
γ	2,-2	-2,2	2,-2

   - Explain why neither player would ever select α.
   - If there are any pure strategy Nash equilibria, find them.
   - If there are any mixed strategy Nash equilibria, find them.

2 Evolutionarily Stable Strategies

Acknowledgment: I learned much of the material in this and the following section from the book *Game Theory Evolving* by Herbert Gintis. Thanks to Professor William Ferguson for recommending this book to me.

Suppose that instead of having players with chosen strategies (pure or mixed), one has a single population of players, each of which has its own pure strategy. For example, one might have a population of animals, and each animal might behave according to their genetic makeup. The game is played many times, with players of different types paired off at each iteration. Each strategy then has an average payoff, but that payoff will depend on how popular the various strategies are in the population.

For example, if a population of players is playing the game *chicken*, and 99.9% of the population uses the strategy Turn, then the 0.1% of the population that has adopted the strategy Straight is going to do fairly well. But if 50% of the population uses the strategy Turn, then the 50% that always go straight are going to do poorly, since they will crash whenever they are paired up against each other.

We will restrict our attention to "symmetric" games. That is, we consider a game with a collection of strategies, denoted $1, 2, \ldots, n$. If two players (one using strategy i and one using strategy j) meet, then the payoff to the player using strategy i is A_{ij}; the payoff to a player using stragegy j against a player using stragegy i is then A_{ji}. The matrix A is then called the *payoff matrix*. We no longer distinguish between a "first player" and a "second player". Both players select their strategy from the same list of options: $\{1, 2, \ldots, n\}$, and the payoff only depends on the pair of options, not on who goes first and who goes second.

For example, the game *rock-paper-scissors* is symmetric; the payoff matrix is

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The game *odd or even* has payoff matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The game *prisoner's dilemma* has payoff matrix

$$\begin{pmatrix} -8 & 0 \\ -10 & -1 \end{pmatrix}.$$

The game *chicken* has payoff matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & -10 \end{pmatrix}.$$

On the other hand, the game described in Exercise 5 of the previous section cannot be described with a payoff matrix, since in that case, if player 1 uses strategy β and player 2 uses strategy α, then player 1 gets a payoff of 4, while if player 1 uses strategy α and player 2 uses strategy β, then player 2 gets a payoff of 3.

Now, suppose that if we select a player at random from our population, then the probability of that player using strategy i is w_i. The sum of these probabilities must be 1, i.e.

$$\sum_{i=1}^{n} w_i = w_1 + w_2 + \cdots + w_n = 1.$$

Now if I play the game using strategy i against a randomly selected player, then my expected payoff will be the weighted average

$$A_{i1}w_1 + A_{i2}w_2 + \cdots + A_{in}w_n = \sum_{j=1}^{n} A_{ij}w_j.$$

If we let $\mathbf{w}$ denote the column vector with i^{th} entry equal to w_i, then the above sum is just the i^{th} entry of $A\mathbf{w}$. We will call $\mathbf{w}$ the *strategy profile* of the population.

Now, suppose we have two groups of players, with strategy profiles $\mathbf{v}$ (for the first population) and $\mathbf{w}$ (for the second population). If a player from the first population plays one from the second population, then the expected payoff to the player from the first population is the weighted average of the expected payoffs for each strategy used against the second population:

$$v_1(Aw)_1 + v_2(Aw)_2 + \cdots + v_n(Aw)_n = \sum_{i=1}^{n} v_i(Aw)_i.$$

Letting $\mathbf{v}^T$ denote the row vector obtained by transposing the column vector $\mathbf{v}$, the above sum is just the single number in the 1×1 matrix

$$\mathbf{v}^T A\mathbf{w}.$$

In particular, if a randomly selected player from the first population plays another randomly selected player from the first population, the expected payoff to each player is given by

$$\mathbf{w}^T A\mathbf{w}.$$

Suppose we have a population W with strategy profile $\mathbf{w}$. A small fraction μ of the population mutates, adopting strategy profile $\mathbf{v}$. The presence of a mutant subpopulation will change the strategy profile of the population as a whole: the new strategy profile will be the weighted average

$$\bar{\mathbf{w}} = \mu\mathbf{v} + (1-\mu)\mathbf{w}.$$

Then, the average payoff to the mutants will be

$$\mathbf{v}^T A\bar{\mathbf{w}} = (\mu)\mathbf{v}^T A\mathbf{v} + (1-\mu)\mathbf{v}^T A\mathbf{w}. \tag{1}$$

The average payoff for the population of nonmutants is

$$\mathbf{w}^T A\bar{\mathbf{w}} = \mathbf{w}^T A\left(\mu\mathbf{v} + (1-\mu)\mathbf{w}\right) = (\mu)\left(\mathbf{w}^T A\mathbf{v}\right) + (1-\mu)\left(\mathbf{w}^T A\mathbf{w}\right). \tag{2}$$

Suppose that there is a positive number M such that whenever a fraction $\mu < M$ of the population mutates, the mutant population does at least as well as the nonmutant population. Then the mutation may well persist. We say that the mutants can *invade* the population. A strategy profile $\mathbf{w}$ represents an *evolutionarily stable strategy* if such an invasion is not possible.

That is, a strategy profile $\mathbf{w}$ is *not* evolutionarily stable if for *some* strategy profile $\mathbf{v}$, and for sufficiently small μ, we have

$$\mu(\mathbf{w}^T A\mathbf{v}) + (1-\mu)(\mathbf{w}^T A\mathbf{w}) \leq \mu(\mathbf{v}^T A\mathbf{v}) + (1-\mu)(\mathbf{v}^T A\mathbf{w})$$

2.1 Examples

Consider the game *odd or even*. Recall that this game has three Nash equilibria: the two players both play O, the two players both play E, or both players randomize their selection with probability $1/2$ for both O and E.

Suppose we had a population with strategy profile $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, corresponding to the mixed Nash equilibrium. Now, suppose that a fraction μ of the population mutates and the mutants always choose Odd. Thus, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

By Equation (1), the average payoff for the mutants is

$$\mu\left(\mathbf{v}^T A\mathbf{v}\right) + (1-\mu)\left(\mathbf{v}^T A\mathbf{w}\right) =$$

$$\mu\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-\mu)\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$= (\mu)(1) + (1-\mu)(0) = \mu.$$

By Equation (2), the average payoff for the nonmutants is

$$\mu\left(\mathbf{w}^T A\mathbf{v}\right) + (1-\mu)\left(\mathbf{w}^T A\mathbf{w}\right) =$$

$$(\mu)\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-\mu)\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$= (\mu)(0) + (1-\mu)(0) = 0.$$

Thus, in this game, the mutants (whose strategy is always to choose Odd), end up doing better than the nonmutants, so the Nash equilibrium for the nonmutants is not an evolutionarily stable strategy. Since the mutants are doing better than everyone else, the mutants will no doubt take over!

Now, let's suppose instead that $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, corresponding to the Nash equilibrium in which all players always choose Odd. Suppose that a fraction μ of the population mutates, where the mutants have strategy profile $\mathbf{v} = \begin{pmatrix} 1-x \\ x \end{pmatrix}$.

By Equation 1, the average payoff for the mutants is

$$\mu\left(v^T Av\right) + (1-\mu)\left(v^T Aw\right) =$$

$$\mu\begin{pmatrix} 1-x & x \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1-x \\ x \end{pmatrix} + (1-\mu)\begin{pmatrix} 1-x & x \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \mu(1-2x)^2 + (1-\mu)(1-2x) = (1-2x)(1-2\mu x).$$

By Equation 2, the average payoff for the nonmutants is

$$\begin{aligned}
&\mu\left(w^T A v\right) + (1-\mu)\left(w^T A w\right) = \\
&\mu\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1-x\\ x\end{pmatrix} + (1-\mu)\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} \\
&\quad = (\mu)(1-2x) + (1-\mu)(1) = (1-2\mu x).
\end{aligned}$$

As long as $x > 0$ and $0 < \mu < \frac{1}{2x}$, the nonmutants do better than the mutants. So, this strategy is evolutionarily stable.

Note that if we took x to be 1, and allowed μ to be larger than $1/2$, then the mutant population could invade, but this would involve more than half of the population suddenly mutating. Since mutations within a population are typically assumed to be slow and gradual, we do not consider the case in which a large portion of the population suddenly mutates.

2.2 Evolutionarily stable strategies and Nash equilibria

Given a strategy profile $\mathbf{w}$ for a population W, we get a corresponding mixed strategy that each of two players could use when playing the other, using the same payoff table. If this mixed strategy were not a Nash equilibrium, that would mean that if the second player adopts this mixed strategy, there would be some pure strategy i that the first player could adopt that would be a better response for the first player. Returning to the population W, we could imagine a very small population of mutants who always use strategy i. We then know that these mutants are going to do better against the nonmutants than the nonmutants do against each other.

Notice, however, that the mutants also play against each other, and the pure strategy i might yield a terrible payoff when played against itself. However, as long as the mutant population is sufficiently small, such interactions between two mutants will be rare, so that the benefit of playing the pure strategy i against the general population of nonmutants will outweigh the rare loss when two mutants play each other. Thus, such a strategy would not be evolutionarily stable.

In short, a non-Nash equilibrium cannot be evolutionarily stable. Another way of saying this is that a strategy that is evolutionarily stable must represent a Nash equilibrium. This makes it much easier to find evolutionarily stable strategies – we only need to investigate each of the Nash equilibria to see which are susceptible to invasion. Moreover, since an evolutionarily stable strategy is described by a single strategy profile vector, we can ignore those Nash equilibria for which the two players use different strategies, such as the pure Nash equilibria in the game *chicken*.

2.3 Exercises

(See last page for selected answers; of course, you need to show your work!)

1. Is the pure Nash equilibrium of *prisoner's dilemma* an evolutionary stable strategy?

2. Is the mixed Nash equilibrium of *chicken* an evolutionarily stable strategy?

3. Is the mixed Nash equilibrium of *rock-paper-scissors* an evolutionarily stable strategy? (Hint: consider what happens if mutants who always choose Rock invade, and recall that an invasion is considered successful if the mutants do *at least* as well as the nonmutants.)

4. Consider a game with the following payoff matrix:

$$\begin{pmatrix} 0 & 7 & 3 \\ -3 & 7 & 7 \\ 10 & 4 & -2 \end{pmatrix}$$

   - Are there any pure strategy Nash equilibria?
   - Find the unique Nash equilibrium involving all three strategies; i.e. where all entries of the strategy profile are positive.
   - Show that this strategy profile is not evolutionarily stable, as it is subject to invasion by mutants using the second strategy.

3 Dynamical Game Theory

Suppose a population has strategy profile $\mathbf{w}$. Recall that $\mathbf{w}$ has components $w_1, w_2, \ldots, w_n$. The components represent the probability that a randomly selected agent of the population will use the corresponding strategy.

Now, suppose various agents of the population play our game. If the payoff for a given strategy j is frequently higher than the payoff for a given strategy i, then whenever a player who uses strategy i encounters a player using strategy j, they may decide they should change strategies. If the difference is payoffs is very large, then the player using strategy j might be all the more inclined to change. Thus, we might expect that the j-component of the strategy profile for the population will increase if the payoff for using strategy j against randomly selected members of the population is greater than the average payoff for the population as a whole; the rate of such an increase might be proportional to the difference between these payoffs, since players are more likely to change if they perceive a greater benefit. The rate will also be proportional to w_j itself. If w_j is very small, then strategy j is hardly ever used, so even though it may be a wonderful strategy, players of the population may only find out about it on rare occasions. In particular, if no agents in the population are using strategy j, then strategy j has, in a sense, become extinct. A more advanced model might take into account the possibility that a strategy that has gone extinct could become rediscovered, but we will keep things more simple here.

The above suggests some differential equations to model the change over time of the strategy profile. We will let p_i denote the average payoff of using strategy i; as we saw in the previous section, $p_i = (A\mathbf{w})_i$. We will let p denote the average payoff to the population of playing the game; as we saw in the previous section, $p = \mathbf{w}^T A\mathbf{w}$. The preceding paragraph suggests the following differential equations:

$$w_i' = w_i(p_i - p), \quad 1 \leq i \leq n. \tag{3}$$

That is, if the payoff of strategy i is better than average, w_i will increase, at a rate proportional to the extent of the benefit, and also proportional to how much the strategy is used in the population as a whole.

We will call the differential equations above the *replicator* equations. (One might imagine that we have a population of organisms; succesful genotypes will get replicated as the organisms evolve.)

One might be concerned that the differential equations above will yield nonsensible results, like negative probabilities, or probabilities that sum up to more than 1.

Note that as long as we start with nonnegative probabilities, we will not get negative probabilities, because as p_i gets smaller, the rate at which p_i decreases itself gets smaller. Indeed, setting $p_i = 0$ makes $p_i' = 0$, so the value of p_i will never cross through 0.

Also, if we assume that at time $t = 0$, we have $\sum_{i=1}^{n} p_i = 1$, then we get

$$\begin{aligned}
\sum_{i=1}^{n} w_i' &= \sum_{i=1}^{n} w_i p_i - \sum_{i=1}^{n} w_i p \\
&= \sum_{i=1}^{n} w_i p_i - p \sum_{i=1}^{n} w_i \quad \text{(factor out } p\text{)} \\
&= \sum_{i=1}^{n} w_i p_i - p \quad \text{(since } \sum_{i=1}^{n} w_i = 1\text{)} \\
&= \sum_{i=1}^{n} w_i (Aw)_i - p \quad \text{(since } p_i = (Aw)_i\text{)} \\
&= \mathbf{w}^T A\mathbf{w} - p \quad \text{(rewriting the sum as a product of a matrices)} \\
&= 0 \quad \text{(cancelling)}
\end{aligned}$$

This shows that the derivative of $(\sum_{i=1}^{n} w_i)$ is 0, so this sum will stay constant, namely equal to 1.

Since our model seems reasonable and will not lead to nonsensible results, let's see how it behaves with some of our examples!

3.1 Examples

Let's begin with the simplest game in our collection: *odd or even.* In this case, there are two strategies, so the strategy profile of a population is just a column vector $w = \begin{pmatrix} w_1 \\ w_w \end{pmatrix}$. But we know $w_2 = 1 - w_1$, so in the end, knowing what happens to w_1 will tell us what happens to w_2. To simplify notation, we will let $x = w_1$, so $1 - x = w_2$

In this case,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Thus,

$$Aw = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1-x \end{pmatrix} = \begin{pmatrix} 2x-1 \\ -2x+1 \end{pmatrix}$$

Thus, the expected payoffs for the two strategies are $p_1 = 2x - 1$ and $p_2 = -2x + 1$.

Also, the expected payoff to a randomly selected agent in the population is

$$\begin{aligned} p &= w^T A w \\ &= \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 2x-1 \\ -2x+1 \end{pmatrix} \\ &= x(2x-1) + (1-x)(-2x+1) \\ &= 4x^2 - 4x + 1 \end{aligned}$$

There is one replicator equation:

$$x' = x(p_1 - p) = x((2x-1) - (4x^2 - 4x + 1)) = -4x^3 + 6x^2 - 2x.$$

This is an autonomous differential equation, the kind we studied in Section 2.5, in the second week of the course. We study these by first finding where $x' = 0$. Factoring lets us find such points:

$$x' = 2x(1-x)(2x-1)$$

Thus, we find three equilibria $x = 0$, $x = 1$, and $x = 1/2$. Note that these correspond precisely to the Nash equilibria for this game.

We can classify these equilibria as stable or unstable by looking at the phase line. In this case, when $x > 1/2$, x' is positive, and when $x < 1/2$, x' is negative. This means that if $x > 1/2$, the x value is increasing, while if $x < 1/2$, the x value is decreasing. Thus, $x = 0$ and $x = 1$ are stable equilibria, while $x = 1/2$ is an unstable equilibrium.

It is interesting that the strategy profile with $x = 1/2$ was also not an evolutionarily stable strategy, since a mutant population of individuals who always choose Odd can invade (or a mutant population of individuals who always choose Even). We see something similar here – if even a slight majority of agents in the population choose Odd, then that majority will grow, since these agents will get a greater payoff than those who use even.

Consider a game with three strategies, with payoff matrix

$$A = \begin{pmatrix} 8 & 9 & -2 \\ 9 & 10 & -20 \\ 2 & 20 & -10 \end{pmatrix}$$

Then letting $x = w_1$ and $y = w_2$, so $w_3 = 1 - x - y$, we have

$$A\mathbf{w} = \begin{pmatrix} 8 & 9 & -2 \\ 9 & 10 & -20 \\ 2 & 20 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 - x - y \end{pmatrix} = \begin{pmatrix} -2 + 10x + 11y \\ -20 + 29x + 30y \\ -10 + 12x + 30y \end{pmatrix}$$

Thus, the expected payoffs for the three strategies are

$$p_1 = -2 + 10x + 11y,$$

$$p_2 = -20 + 29x + 30y,$$
$$p_3 = -10 + 12x + 30y.$$

The expected payoff to a randomly selected agent in the population is

$$\begin{aligned} p &= \mathbf{w}^T A\mathbf{w} \\ &= \begin{pmatrix} x & y & 1 - x - y \end{pmatrix} \begin{pmatrix} -2 + 10x + 11y \\ -20 + 29x + 30y \\ -10 + 12x + 30y \end{pmatrix} \\ &= -2x^2 - 2xy + 20x + 20y - 10 \end{aligned}$$

The replicator equations say

$$\begin{aligned} x' &= x(p_1 - p) \\ &= x((-2 + 10x + 11y) - (-2x^2 - 2xy + 20x + 20y - 10)) \\ &= x(2x^2 + 2xy - 9y - 10x + 8), \\ y' &= y(p_2 - p) \\ &= y((-20 + 29x + 30y) - (-2x^2 - 2xy + 20x + 20y - 10)) \\ &= y(2x^2 + 2xy + 10y + 9x - 10). \end{aligned}$$

At this point, we have a nonlinear system, so we search for critical points. I will ignore critical points where $x = 0$ or $y = 0$ (in these cases one of the strategies is not being used at all). Then we have

$$2x^2 + 2xy - 9y - 10x + 8 = 0,$$
$$2x^2 + 2xy + 10y + 9x - 10 = 0$$

Subtracting the top from the bottom equation yields

$$19y + 19x - 18 = 0, \text{ or } x + y = \frac{18}{19}.$$

Setting $y = \frac{18}{19} - x$ in $2x^2 + 2xy - 9y - 10x + 8 = 0$ and simplifying yields

$$\frac{17x - 10}{19} = 0.$$

Thus, $x = \frac{10}{17}$ and $y = \frac{116}{323}$. (This leaves a fraction of $\frac{1}{19}$ using the third strategy).

Now, we have

$$x' = x(2x^2 + 2xy - 9y - 10x + 8)$$

and

$$y' = y(2x^2 + 2xy + 10y + 9x - 10).$$

We can find the Jacobian:

$$J(x, y) = \begin{pmatrix} 8 - 20x + 6x^2 - 9y + 4xy & -9x + 2x^2 \\ 9y + 4xy + 2y^2 & -10 + 9x + 2x^2 + 20y + 4xy \end{pmatrix}$$

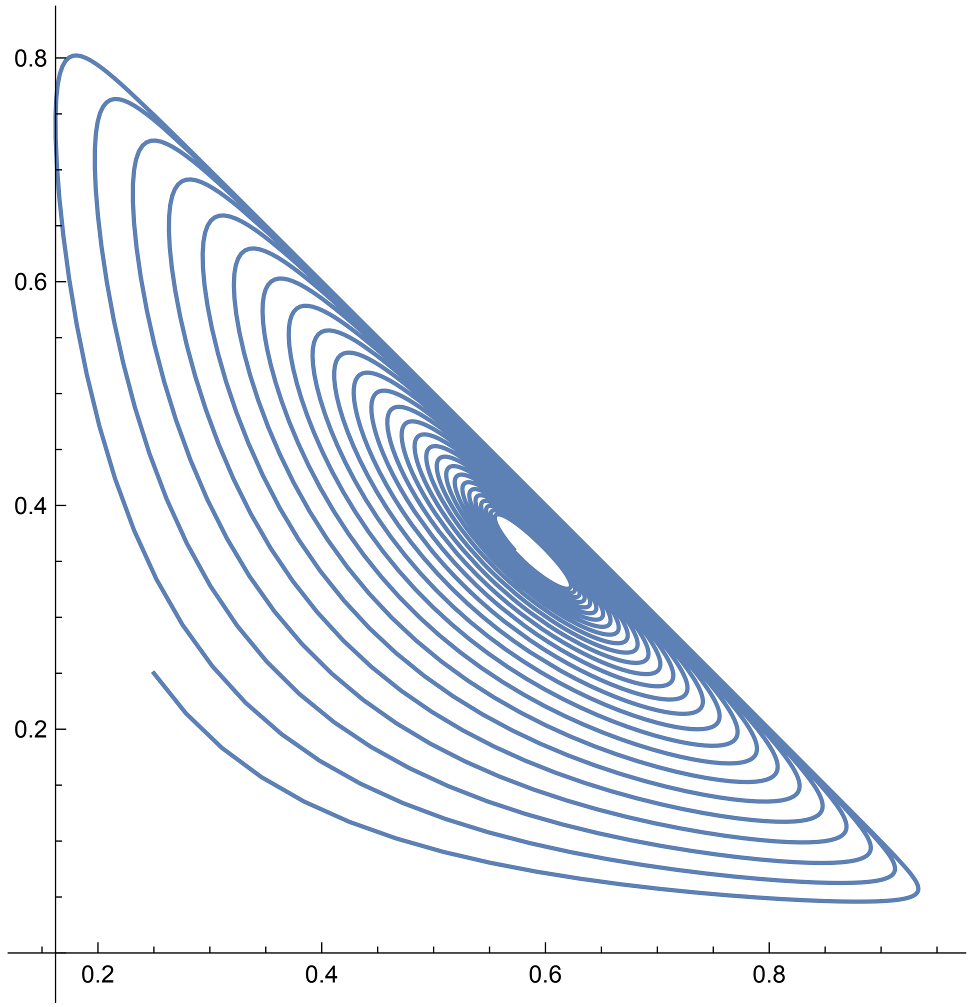
In particular,

$$J(10/17, 116/323) = \begin{pmatrix} -4.07576 & -4.60208 \\ 4.33517 & 4.01384 \end{pmatrix}$$

This has eigenvalues

$$-0.0309598 \pm 1.89483i$$

Since the real part of these complex eigenvalues is negative, the critical point would be an asymptotically spiral. Below is a plot of a solution trajectory, beginning at $x = .25, y = .25$.



Initially, a quarter of the population is using the first strategy, a quarter is using the second strategy, and the other half is using the third strategy (which is not plotted on the picture). At this point, the expected payoffs are

$$p_1 = -2 + 10x + 11y = 3.25,$$

$$p_2 = -20 + 29x + 30y = -5.25,$$

$$p_3 = -10 + 12x + 30y = .5.$$

Thus, initially, anyone using strategy 2 is not doing well, so there will be a tendency for those using strategy 2 to switch to strategy 1.

After a unit of time, it turns out that 93.3% of the agents are using strategy 1, while only 5.6% are using strategy 2, and only 1.1% are using strategy 3. (I used NDSolve in Mathematica to get these numbers). At this point, the payoffs are

$$p_1 = -2 + 10x + 11y = 7.95,$$
$$p_2 = -20 + 29x + 30y = 8.74,$$
$$p_3 = -10 + 12x + 30y = 2.88.$$

That is, the abundance of individuals using strategy 1 makes strategy 2 more profitable, as can be seen in the payoff table, where $A_{2,1} = 9$. So, agents begin to switch from strategy 1 to strategy 2.

At time $t = 5.45$, strategy 1 is used 17.5% of the time, while strategy 2 is used 80.1% of the time, and strategy 3 only 2.4% of the time. With so many in the population now using strategy 2, strategy 3 becomes more profitable. The payoffs are now

$$p_1 = -2 + 10x + 11y = 8.56,$$
$$p_2 = -20 + 29x + 30y = 9.11,$$
$$p_3 = -10 + 12x + 30y = 16.13.$$

So, agents in the population start switching from strategy 1 and 2 toward strategy 3.

This process of switching strategies continues, and the strategy profile asymptotically approaches the equilibrium, where $x = 10/17$ and $y = 116/323$. This is, as you can check, a Nash equilibrium. The payoffs at this point are

$$p_1 = -2 + 10x + 11y = 7.833,$$
$$p_2 = -20 + 29x + 30y = 7.833,$$
$$p_3 = -10 + 12x + 30y = 7.833.$$

3.2 Exercises

(See last page for selected answers; of course, you need to show your work!)

1. Find the replicator equations for the game *modified odd or even* described in the first section. (Say the strategy profile is $\begin{pmatrix} x \\ 1-x \end{pmatrix}$, and find an autonomous first-order differential equation for x.) What are the equilibria of this differential equation. Which are stable, and which are unstable?

2. Determine whether the mixed Nash equilibrium in *chicken* is a stable or unstable equilibrium of the replicator equations.

3. Consider the game *rock-paper-scissors*. Suppose a population has strategy profile $\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}$.

   - Find the expected payoffs for the three strategies in terms of x and y.
   - Find the expected payoff to a randomly selected agent in the population.
   - Find and simplify the replicator equations for x' and y'.
   - Find any equilibria for these replicator equations with $x > 0, y > 0, 1-x-y > 0$. Are these also Nash equilibria?
   - Find the Jacobian at any equilibria, and find the eigenvalues of this Jacobian. What do these tell you about the behavior of trajectories close to the equilibrium?
   - Plot a solution trajectory that starts at $x = .25$ and $y = .25$. What do you observe? Explain what is happening (as I did in the notes above!)

4. Suppose that in the game *rock-paper-scissors*, each player gets a payoff a in the event of a tie. Repeat the first five steps in the above problem. You should find that when $a < 0$, the Nash equilibrium with all probablities equal to $1/3$ is asymptotically stable, while if $a > 0$, this Nash equilibrium is unstable. (You may want to use the Solve command in Mathematica when computing the equilibria).

5. Repeat the six steps in problem 3, but this time use the game whose payoff matrix is that of Exercise 4 in section 2. You should find that there is one Nash equilibrium with all probabilities positive, and the corresponding point is an asymptotically stable equilibrium of the replicator equations. Recall that you found that the strategy profile was not evolutionarily stable. Thus, an asymptotically stable equilibrium of the replicator equations does not need to be an evolutionarily stable strategy. It has been proved, on the other hand, that an evolutionarily

stable strategy is an asymptotically stable equilibrium of the replicator equations. See Taylor and Jonker, "Evolutionarily Stable Strategies and Game Dynamics." *Mathematical Biosciences* 40: 145-156.

Answers to Selected Exercises (You can use these answers to check your work, but be sure to show your work!)

Section 1.

2. The mixed Nash equilibrium is $(2/5, 3/5)$, with expected payoff $1/5$. The pure Nash equilibria do better.

3. The mixed Nash equilibrium is $(9/10, 1/10)$, with expected payoff $-1/10$. The expected payoffs for the pure equilibria are 1 and -1.

5b) None.

5c) First player: $(1/2, 1/2)$. Second player: $(3/8, 5/8)$.

Section 2.

1. Yes
2. Yes
3. No

4a) Both players choose β

4b) $(12/46, 25/46, 9/46)$

Section 3.

1. $x' = x(x-1)(5x-2)$. Stable equilibria at $x = 0$, and $x = 1$; unstable equilibrium at $x = 2/5$.

2. Stable

3a) $p_1 = 1 - x - 2y, p_2 = -1 + 2x + y, p_3 = -x + y$

3b) 0

3c) $x' = x - x^2 - 2xy, y' = -y + 2xy + y^2$

3d) $(1/3, 1/3)$ is a Nash equilibrium.

3e) $J(1/3, 1/3) = \begin{pmatrix} -1/3 & -2/3 \\ 2/3 & 1/3 \end{pmatrix}$.

Eigenvalues are $\pm i\sqrt{1/3}$. We cannot determine stability from the Jacobian.

4a) $p_1 = 1 + (a-1)x - 2y$, $p_2 = -1 + 2x + (a+1)y$, $p_3 = -(a+1)x + (1-a)y + a$

4b) $p = a - 2ax + 2ax^2 - 2ay + 2axy + 2ay^2$

4c) $x' = x - ax - x^2 + 3ax^2 - 2ax^3 - 2xy + 2axy - 2ax^2y - 2axy^2$

$y' = -y - ay + 2xy + 2axy - 2ax^2y + y^2 + 3ay^2 - 2axy^2 - 2ay^3$

4d) $(1/3, 1/3)$ is the only Nash equilibrium.

4e) $J(1/3, 1/3) = \begin{pmatrix} (a-1)/3 & -2/3 \\ 2/3 & (a+1)/3 \end{pmatrix}$.

The eigenvalues are $\frac{a \pm i\sqrt{3}}{3}$. When $a > 0$, the equilibrium is an unstable spiral. When $a < 0$, the equilibrium is a stable spiral.

5a) $p_1 = 3 - 3x + 4y$, $p_2 = 7 - 10x$, $p_3 = -2 + 12x + 6y$

5b) $p = -2 + 17x - 15x^2 + 15y - 24xy - 6y^2$

5c) $x' = 5x - 20x^2 + 15x^3 - 11xy + 24x^2y + 6xy^2$

$y' = 9y - 27xy + 15x^2y - 15y^2 + 24xy^2 + 6y^3$

5d) $(6/23, 25/46)$. (Thus, $z = 9/46$. Note this was the Nash equilibrium we found in the last problem in section 2.)

5e) $J(6/23, 25/46) = \begin{pmatrix} 120/529 & 246/529 & -3525/1058 & -1275/1058 \end{pmatrix}$

The eigenvalues are $\frac{15}{92}\left(-3 \pm i\sqrt{39}\right)$. The equilibrium is therefore an asymptotically stable spiral.