Preparation for November 6

Consider problem 2 in section 7.6:

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

Just as we did last time, we look for solutions of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$. So we need eigenvalues and eigenvectors of the matirx.

The characteristic polynomial is

$$(-1-\lambda)(-1-\lambda) - (1)(-4) = \lambda^2 + 2\lambda + 5$$

The roots are $-1 \pm 2i$.

For $\lambda_1 = -1 + 2i$, we get

$$\begin{pmatrix} -1-\lambda_1 & -4 \\ 1 & -1-\lambda_1 \end{pmatrix} = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}$$

One vector in the nullspace of this matrix is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$.

For $\lambda_2 = -1 - 2i$, we get

$$\begin{pmatrix} -1-\lambda_1 2 & -4 \\ 1 & -1-\lambda_2 \end{pmatrix} = \begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}$$

One vector in the nullspace of this matrix is $\mathbf{v}_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$.

Note that λ_1 and λ_2 are complex conjugates. The corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are also complex conjugates. This sort of thing will always happen: provided the entries of a matrix are all real, if λ is a complex eigenvalue with eigenvector $\mathbf{v}$, then the complex conjugate of λ will also be a complex eigenvalue with eigenvector the complex conjugate of $\mathbf{v}$.

From λ_1 and $\mathbf{v}_1$, we get one solution, and we can use Euler's formula to simplify it:

$$\begin{aligned}
\mathbf{x}(t) = e^{(-1+2i)t}\begin{pmatrix} 2 \\ -i \end{pmatrix} &= \left(e^{-t}\cos(2t) + ie^{-t}\sin(2t)\right)\begin{pmatrix} 2 \\ -i \end{pmatrix} \\
&= \begin{pmatrix} 2(e^{-t}\cos(2t) + ie^{-t}\sin(2t) \\ -i(e^{-t}\cos(2t) + ie^{-t}\sin(2t)) \end{pmatrix} \\
&= \begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix} + i\begin{pmatrix} 2e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) \end{pmatrix}
\end{aligned}$$

From λ_2 and $\mathbf{v}_2$, we get another solution. This will be the complex conjugate of the solution above:

$$\mathbf{x}(t) = \begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix} - i\begin{pmatrix} 2e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) \end{pmatrix}$$

In chapter 3, when we had two complex-valued solutions to a second order equation, we found two real-valued solutions. One was obtained by adding the two complex-valued solutions and then dividing by 2. The other was obtained by subtracting the two complex-valued solutions and then dividing by $2i$. We can do the same thing here to get two real-valued solutions:

$$\begin{aligned}
\mathbf{x}_1(t) &= \begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix} \\
\mathbf{x}_2(t) &= \begin{pmatrix} 2e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) \end{pmatrix}
\end{aligned}$$

We can take any superposition of these solutions to get a general solution:

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix} + c_2\begin{pmatrix} 2e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) \end{pmatrix}$$

In class, we will look at what these look like in the plane. But notice at this point that the solutions must all asymptotically approach the origin, because of the e^{-t} terms. The sine and cosine terms will cause some kind of oscillation. If you're seeing spirals, that's good!