Preparation for September 11

With second-order equations, another trick that we can use is called *reduction of order*. This requires that we have one solution of an equation, and makes it easier to find another.

I'll illustrate this with an example, which is problem 25.

$$t^2y'' + 3ty' + y = 0, \quad t > 0$$

Suppose that by some luck, we discover that $y_1 = \frac{1}{t}$ is a solution. Then, we let $v = \frac{y}{y_1}$, so $y = vy_1 = v/t$. Now, we compute y' and y'':

$$y' = v'y_1 + vy_1' = v'\left(\frac{1}{t}\right) + v\left(\frac{-1}{t^2}\right)$$
$$\begin{aligned} y'' =& v''\left(\frac{1}{t}\right) + v'\left(\frac{-1}{t^2}\right) + v'\left(\frac{-1}{t^2}\right) + v\left(\frac{2}{t^3}\right) \\ =& v''\left(\frac{1}{t}\right) + v'\left(\frac{-2}{t^2}\right) + v\left(\frac{2}{t^3}\right) \end{aligned}$$

Plugging all this into the original equation and simplifying, we get

$$\begin{aligned} 0 = t^2y'' + 3ty' + y &= (tv'' - 2v' + 2v/t) + (3v' - 3v/t) + (v/t) \\ &= tv'' + v' \end{aligned}$$

This is a second-order differential equation in which v does not appear, so we use the trick from last time, letting $w = v'$. So, we get

$$tw' + w = 0.$$

This is first-order linear. We can solve it to get

$$w = C/t$$

Thus

$$v' = C/t$$

Integrating,

$$v = C\ln|t| + D$$

Now, recall $y = v/t$, so

$$y = C\ln|t|/t + D/t.$$

Notice there are two degrees of freedom.