Preparation for November 10

In the previous four sections, we have been looking at material from Chapter 7, considering differential equations of the form

$$\mathbf{x}' = A\mathbf{x}$$

This was shorthand notation for a system of equations. If we write $\mathbf{x}$ as $\begin{pmatrix} x \\ y \end{pmatrix}$, and A as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the above vector-valued differential equation can be written as the system of two first-order differential equations:

$$x' = ax + by$$

$$y' = cx + dy$$

We now move to chapter 9. Here, we will look at equations of the form

$$x' = F(x, y)$$

$$y' = G(x, y)$$

That is, we do not assume that $F(x, y)$ and $G(x, y)$ are so simple. We could have something like

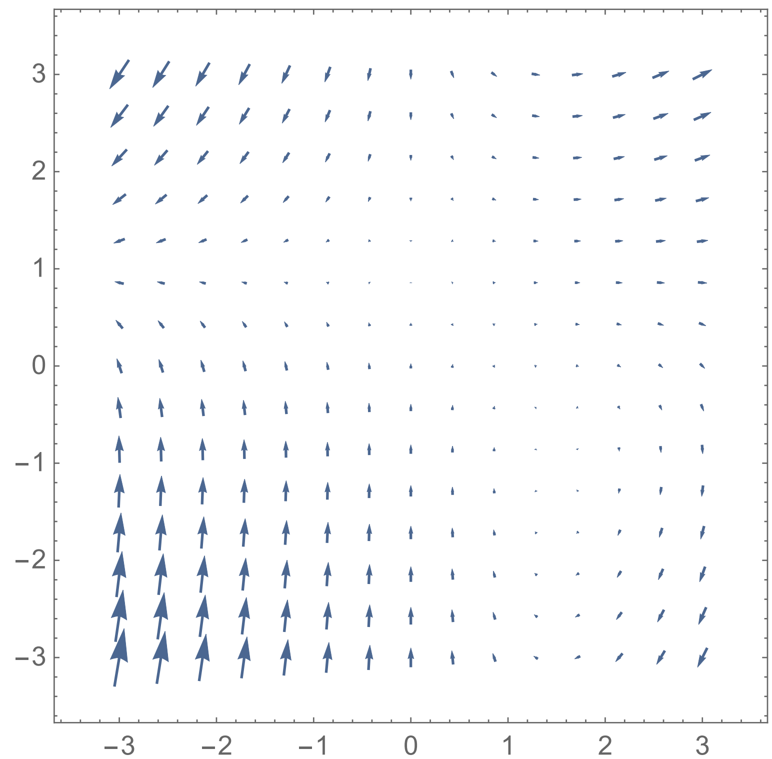
$$x' = xy + x$$

$$y' = 2xy - 2x - 3y + 3$$

We could look at a direction field using the VectorPlot command in Mathematica.

```
VectorPlot[{x*y + x, 2 x*y - 2 x - 3 y + 3},
 {x, -3, 3}, {y, -3, 3}]
```

The result is



But if we want to get a qualitative sense of the behavior of solutions of the differential equation without resorting to an interpolating function, we could begin by looking for equilibria solutions, much as we did in Chapter 2.5. Recall that an equilibrium solution is a constant solution. For a constant solution, $x' = 0$ and $y' = 0$. So, we want to solve

$$0 = xy + x$$

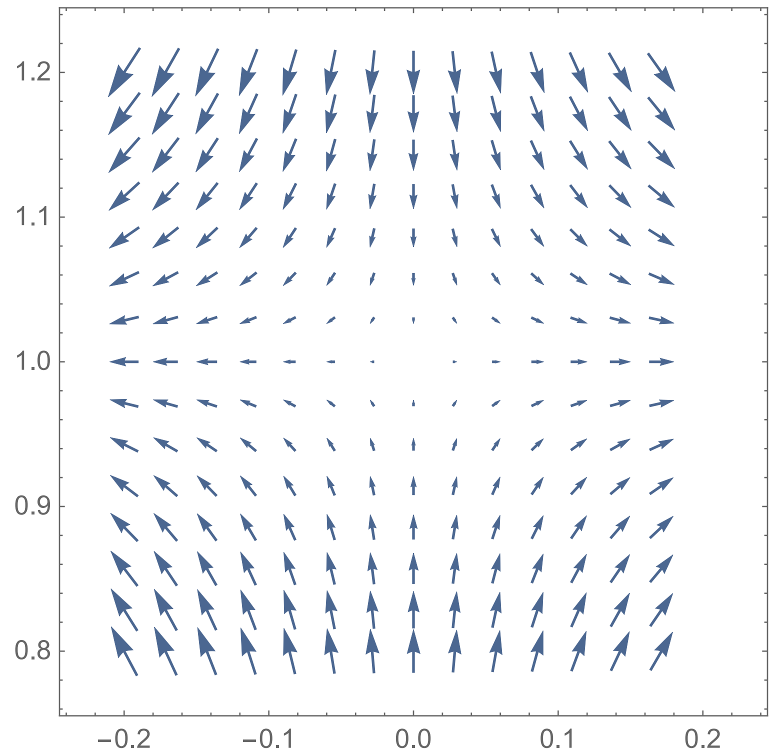
$$0 = 2xy - 2x - 3y + 3$$

In order to get $xy + x = 0$, we need $x(y+1) = 0$, so either $x = 0$ or $y = -1$. If $x = 0$, then the second equation tells us $-3y + 3 = 0$, so $y = 1$. If $y = -1$, then the second equation tells us $-2x - 2x + 3 + 3 = 0$, so $x = 3/2$. Thus, we have two equilibrium solutions: $(0, 1)$ and $(1.5, -1)$.

We could study these equilibrium solutions by zooming in on the direction field around $(0, 1)$. We will use an x-window going from $-.2$ to $.2$, and a y-window going from $.8$ to 1.2. Entering

```
VectorPlot[{x*y + x, 2 x*y - 2 x - 3 y + 3},
 {x, -.2, .2}, {y, .8, 1.2}]
```

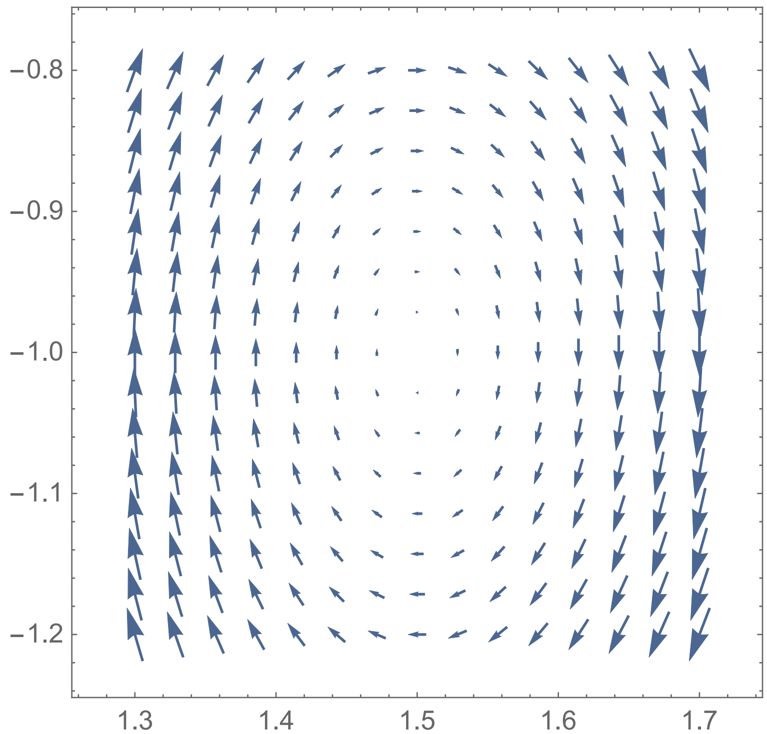
we get



You might guess from this that $(0, 1)$ is a saddle point, hence unstable. Similarly, we can zoom in on the equilibrium point $(1.5, -1)$. Entering

```
VectorPlot[{x*y + x, 2 x*y - 2 x - 3 y + 3},
 {x, 1.3, 1.7}, {y, -1.2, -.8}]
```

we get



You might guess from this that $(1.5, -1)$ is a spiral point, or perhaps a center. It is difficult to tell from the picture whether or not the equilibrium is an asymptotically stable spiral, a stable center, or an unstable spiral.

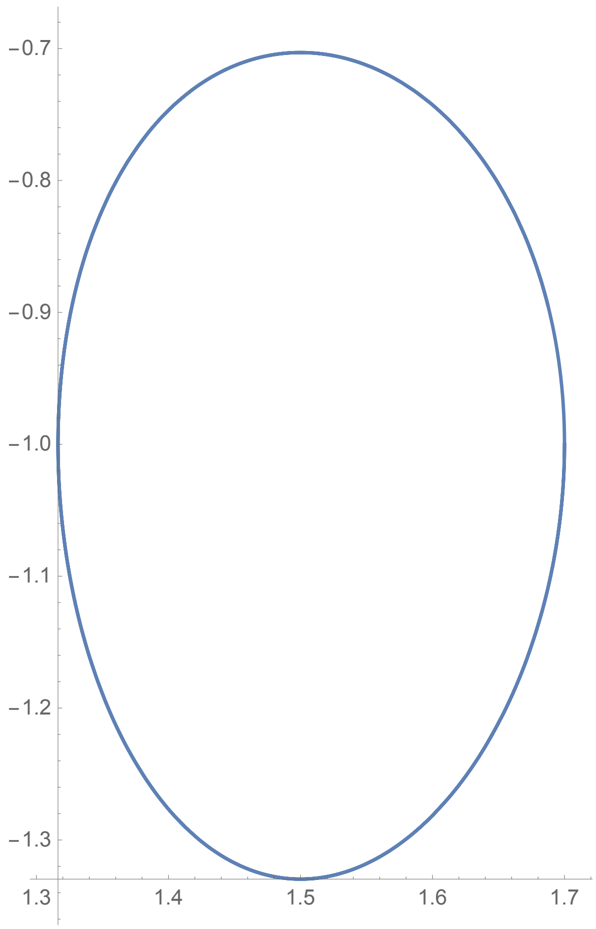
We might choose inital values for x and y, say $x(0) = 1.7$ and $y(0) = -1$. We could use the NDSolve command in Mathematica to find a solution with these initial values. We could enter the following

```
sol = NDSolve[{x'[t] == x[t] y[t] + x[t],
    y'[t] == 2 x[t] y[t] - 2 x[t] -
      3 y[t] + 3, x[0] == 1.7, y[0] == -1},
   {x, y}, {t, 0, 5}];
```

This outputs an interpolating function for the solution. To graph it, we could type

```
ParametricPlot[{x[t], y[t]} /. sol,
  {t, 0, 5}]
```

The output appears as follows:



The picture suggests that $(1.5, -1)$ is actually a stable center, though it is difficult to use a picture to be sure.