Preparation for September 20

In the examples in the previous section, we had only a mass and a spring with some damping, but no external driving force. Let's look at a problem with external force, which is number 6 in section 3.8.

A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

From last time, we know the differential equation will take the form

$$mu'' + \gamma u' + ku = F(t).$$

Since the mass is set in motion from its equilibrium position, the initial position can be taken to be 0, so $u(0) = 0$. Since the initial velocity is .03 m/s, we have $u'(0) = .03$.

We are given $m = 5$.

We are given that $\gamma * .04 = 2$. (I converted 4 centimeters to .04 meters.) So, $\gamma = 2/.04 = 50$.

We are given that a force of $5 * 9.8$ N stretches the spring .1 meters, so $k * (.1) = 5 * 9.8$, or $k = 490$.

We are given the external force is $10\sin(t/2)$ (measured in Newtons), so $F(t) = 10\sin(t/2)$.

So, our initial value problem is

$$5u'' + 50u' + 490u = 10\sin(t/2), \quad u(0) = 0, u'(0) = .03.$$

Or we could divide the differential equation by 5 to get

$$u'' + 10u' + 98u = 2\sin(t/2), \quad u(0) = 0, u'(0) = .03.$$

In problem 8a, we are asked to solve for u. The characteristic equation is

$$r^2 + 10r + 98 = 0.$$

This gives $r = -5 \pm i\sqrt{73}$. Thus, the general solution of the homogeneous equation is

$$u = c_1 e^{-5t} \cos\left(\sqrt{73}t\right) + c_2 e^{-5t} \sin\left(\sqrt{73}t\right).$$

To get a particular solution of the nonhomogeneous equation, we try

$$u_p = A\sin(t/2) + B\cos(t/2)$$

Then after taking some derivatives and simplifying, we get

$$u_p'' + 10u_p' + 98u_p = (5A + (391/4)B)\cos(t/2) + ((391/4)A - 5B)\sin(t/2)$$

We want this to be $2\sin(t/2)$, so we need

$$5A + (391/4)B = 0 \quad (391/4)A - 5B = 2$$

Solving this gives $A = 3128/153281, B = -160/153281$. So, we get a general solution for u:

$$c_1 e^{-5t} \cos\left(\sqrt{73}t\right) + c_2 e^{-5t} \sin\left(\sqrt{73}t\right) + \\ (3128/153281)\sin(t/2) - (160/153281)\cos(t/2)$$

Setting $u(0) = 0$ and $u'(0) = .006$, we get

$$c_1 = 160/153281, c_2 = \frac{383443}{15328100\sqrt{73}}$$

Notice that as t increases, the e^{-5t} dies off, so the dominant term is

$$(3128/153281)\sin(t/2) - (160/153281)\cos(t/2).$$

This is called the *steady state solution*, while the part that dies off is called the *transient solution*. The steady state solution does not depend on the initial conditions, and continues to have significance as t goes to infinity. The transient solution dies off.