Preparation for October 23

We have now learned how to find a function $u(x,t)$ satsifying the following four conditions:

1. $u_t = \alpha^2 u_{xx}$
2. $u(0,t) = 0$ for $t > 0$
3. $u(L,t) = 0$ for $t > 0$

The general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-n\pi\alpha t}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

If we also want the initial condition $u(x,0) = f(x)$ for $0 < x < L$, then our coefficients c_n are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right).$$

The boundary values $u(0,t) = 0$ and $u(L,t) = 0$ are like homogeneous conditions. Suppose we had nonhomogeneous conditions $u(0,t) = T_1$ and $u(L,t) = T_2$. For example, the endpoints of our rod might be held at constant temperatures T_1 and T_2.

The technique we will use should remind you of how we found the general solution of a nonhomogeneous solution by finding a particular solution and then adding the general solution of the homogeneous equation. The terms T_1 and T_2 above are analogous to nonhomogeneous terms.

We will first look for a *steady-state* solution $v(x,t)$ (this will be our "particular" solution). A steady-state solution is a solution of the heat equation that does not change with respect to time (so we could write $v(x)$ instead of $v(x,t)$), but has the required values at the endpoints 0 and L. Since v does not change with respect to time, we get

$$0 = v_t = \alpha^2 v_{xx} \implies v_{xx} = 0$$

Since v is really just a function of x, saying $v_{xx} = 0$ is the same as saying the the graph of v is just a line. Since we want $v(0) = T_1$ and $v(1) = T_2$, it is straightforward to find a formula for v:

$$v(x) = T_1 + \frac{T_2 - T_1}{L} x.$$

Now, suppose $u(x,t)$ is any solution of the “homogeneous” equation:

1. $u_t = \alpha^2 u_{xx}$
2. $u(0,t) = 0$ for $t > 0$
3. $u(L,t) = 0$ for $t > 0$

Then, letting $w = v + u$, we get

1. $w_t = (v+u)_t = v_t + u_t = \alpha^2 u_{xx} + \alpha^2 v_{xx} = \alpha^2 w_{xx}$
2. $w(0,t) = v(0,t) + u(0,t) = T_1 + 0 = T_1$ for $t > 0$
3. $w(L,t) = v(L,t) + u(L,t) = T_2 + 0 = T_2$ for $t > 0$.

So, $w(x,t) = v(x) + u(x,t)$ is now also a solution of the nonhomogeneous equation.

Finally, if we want $w(x,0) = f(x)$, then we need

$$u(x,0) = w(x,0) - v(x,0) = f(x) - v(x).$$

So,

$$\begin{aligned} w(x,t) &= v(x) + u(x,t) \\ &= T_1 + \frac{T_2 - T_1}{L} x + \sum_{n=1}^{\infty} c_n e^{\frac{-n\pi\alpha t}{L}} \sin\left(\frac{n\pi x}{L}\right), \end{aligned}$$

where

$$c_n = \frac{2}{L} \int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi x}{L}\right).$$