Preparation for August 30.

In Section 2.3, we will be studying problems in which you have to model a situation with a differential equation before solving it. The text breaks the process of doing these problems into three steps: Construction of the Model, Analysis of the Model, and Comparison with Experiment or Observation. We'll focus most of our attention on the construction of the model. Most differential equations you will need to solve at that point will be either first order linear or separable.

In Calculus, you probably studied related rates problems and optimization problems. There are a few steps that you learned are useful in doing these types of problems. The modelling problems we see now involve many of the same steps:

1. Read the problem carefully. Ask what quantities are unknown and what are given. What are you ultimately trying to find? Typically, time will be one of your variables.

2. Draw a picture (if possible)

3. Introduce notation – that is assign variables that you think are relevant to the problem.

4. If there are several variables, identify relations between the variables (not including time).

5. Try to find a differential equation that describes the system.

As an example, let's look at the first problem in section 2.3, which I copy below:

Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

After reading through the problem, we note that the quantities involved are the amount of dye and the amount of solution in the tank, as well as time. The concentration of dye is simply the quotient of these two quantities.

We are given that the initial quantity of solution is 200 liters. To get the initial quantity of dye, we multiply the concentration by the quantity of solution, and we get 200 grams. We are also given that water is flowing in at 2 L/min and solution is flowing out at 2 L/min.

Our final goal is to find how much time elapses before the concentration is 1% of its original value. (Note here that 1% of the original value would be .01 g/L.) When the concentration is at .01 g/L, the amount of dye will be $.01\text{g/L} * 200\text{L} = 2\text{g}$. So, we want to find when the quantity of dye is 2g.

On to step 2. Draw a picture. (I won't do that here).

Now, we need some notation. It is customary to use t for time. We can use D for the amount of dye. You might let S be the amount of solution in the tank. However, notice that there will always be 200 liters of solution, since the water is flowing in at the same rate as the dye is flowing out. It is not necessary here to assign a letter to a quantity that is constant; since the amount of solution in the tank is always 200, we could just write 200 any time we wanted to refer to the amount of solution in the tank. However, if you have a fairly complicated constant, like $4\frac{\pi}{\sqrt{37}}$, you might want to use a letter for that constant in order to simplify notation.

Other than time, we only have one variable D here, so we skip step 4.

Now for the hard part. We want to think about how D is changing over time; i.e. we want D'. Note at this point that the units on D' should be g/min, since D is measured in grams and time is measured in minutes.

Notice there are two things happening – water is flowing in and the solution is flowing out. The influx of water does not itself add to or take away from the amount of dye in the tank. The solution flowing out is going to cause the amount of dye to decrease. The solution flows out at a rate of 2 L/min, so this number is probably relevant. But it is not measuring the rate at which the amount of *dye* is decreasing. Notice that the units are not what we would have expected. We want to know how many grams of dye are in those 2 liters of water that flow out every minute. The amount of dye is D grams, so the concentration of dye is $D/200$ g/L. Multiply this by 2 L/min to get $2D/200 = D/100$ g/min. This gives the rate at which dye itself is flowing out. Thus, $D' = -D/100$. Notice how useful it can be to use units!

We might go back at this point and tie up some loose ends. We were given that the initial quantity of dye was 200. That tells us that $D(0) = 200$. We want to find t so that $D(t) = 2$.

The differential equation $D' = -D/100$ is separable:

$$\frac{D'}{D} = -\frac{1}{100}$$

Integrate both sides with respect to time to get

$$\ln(D) = -t/100 + C$$

Exponentiate to get

$$D = e^{-t/100+C} = e^C e^{-t/100}$$

Setting $t = 0$, we get $200 = e^C$, so

$$D = 200e^{-t/100}.$$

Now set $D = 2$ and solve for t:

$$e^{-t/100} = 2/200 = .01$$

Thus,

$$t = -100\ln(.01) = 460.517.$$

Of course, the last step is to check your answer in the back of the book, since this is an odd numbered problem. It's right!!!