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Phys 232-0
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HW for 10/24/17
QAM.6, QIM.7

1

QAM.6
Consider an electron in a magnetic field that points in the +z direction, and let its energies when its spin is aligned and anti-aligned with the field be E_+ and E_- respectively. Define $\omega = [E_+ - E_-]/\hbar$. Suppose this electron has an initial state $|\psi(0)\rangle = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$

a) Calculate the expectation values $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ in this situation

The initial state of the quanton is $|\psi(0)\rangle = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = \frac{4}{5}|+z\rangle - \frac{3}{5}|-z\rangle$
By the TE rule, the quanton's state as a function of time is
$|\psi(t)\rangle = \frac{4}{5}e^{-iE_+t/\hbar}|+z\rangle - \frac{3}{5}e^{-iE_-t/\hbar}|-z\rangle = \begin{bmatrix} \frac{4}{5}e^{-iE_+t/\hbar} \\ -\frac{3}{5}e^{-iE_-t/\hbar} \end{bmatrix}$

$\langle S_x \rangle = +\frac{\hbar}{2}|\langle +x|\psi\rangle|^2 - \frac{\hbar}{2}|\langle -x|\psi\rangle|^2$ $\quad |+x\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}, \quad |-x\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{bmatrix}$

$\langle S_y \rangle = +\frac{\hbar}{2}|\langle +y|\psi\rangle|^2 - \frac{\hbar}{2}|\langle -y|\psi\rangle|^2$ $\quad |+y\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ i\sqrt{\frac{1}{2}} \end{bmatrix}, \quad |-y\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -i\sqrt{\frac{1}{2}} \end{bmatrix}$

$\langle S_z \rangle = +\frac{\hbar}{2}|\langle +z|\psi\rangle|^2 - \frac{\hbar}{2}|\langle -z|\psi\rangle|^2$ $\quad |+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\langle S_x \rangle = +\frac{\hbar}{2}\left|\begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{4}{5}e^{-iE_+t/\hbar} \\ -\frac{3}{5}e^{-iE_-t/\hbar} \end{bmatrix}\right|^2 - \frac{\hbar}{2}\left|\begin{bmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{4}{5}e^{-iE_+t/\hbar} \\ -\frac{3}{5}e^{-iE_-t/\hbar} \end{bmatrix}\right|^2$$

$$\langle S_x \rangle = +\frac{\hbar}{2}\left|\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right|^2 - \frac{\hbar}{2}\left|\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right|^2$$

$$\langle S_x \rangle = +\frac{\hbar}{2}\left(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right)\left(\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{iE_-t/\hbar}\right) - \frac{\hbar}{2}\left(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right)\left(\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{iE_-t/\hbar}\right)$$

$$\langle S_x \rangle = +\frac{\hbar}{2}\left(\frac{16}{50} - \frac{12}{50}e^{-i(E_+-E_-)t/\hbar} - \frac{12}{50}e^{i(E_+-E_-)t/\hbar} + \frac{9}{50}\right) - \frac{\hbar}{2}\left(\frac{16}{50} + \frac{12}{50}e^{-i(E_+-E_-)t/\hbar} + \frac{12}{50}e^{i(E_+-E_-)t/\hbar} + \frac{9}{50}\right)$$

$$\langle S_x \rangle = +\frac{\hbar}{2}\left(\frac{25}{50} - \frac{12}{50}\left(e^{-i\omega t} + e^{i\omega t}\right)\right) - \frac{\hbar}{2}\left(\frac{25}{50} + \frac{12}{50}\left(e^{-i\omega t} + e^{i\omega t}\right)\right)$$

$$\langle S_x \rangle = -\frac{2\hbar}{2}\frac{12}{50}\left(e^{-i\omega t} + e^{i\omega t}\right) = -\hbar\frac{12}{50}\left(2\cos(\omega t)\right) = \boxed{\frac{-12\hbar}{25}\cos(\omega t) = \langle S_x \rangle}$$

$$\langle S_y \rangle = +\frac{\hbar}{2}\left|\begin{bmatrix} \sqrt{\frac{1}{2}}, & -i\sqrt{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{4}{5}e^{-iE_+t/\hbar} \\ -\frac{3}{5}e^{-iE_-t/\hbar} \end{bmatrix}\right|^2 - \frac{\hbar}{2}\left|\begin{bmatrix} \sqrt{\frac{1}{2}}, & i\sqrt{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{4}{5}e^{-iE_+t/\hbar} \\ -\frac{3}{5}e^{-iE_-t/\hbar} \end{bmatrix}\right|^2$$

$$\langle S_y \rangle = +\frac{\hbar}{2}\left|\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + i\frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right|^2 - \frac{\hbar}{2}\left|\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - i\frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right|^2$$

$$\langle S_y \rangle = +\frac{\hbar}{2}\left(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + i\frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right)\left(\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} - i\frac{3}{5\sqrt{2}}e^{iE_-t/\hbar}\right) - \frac{\hbar}{2}\left(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - i\frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}\right)\left(\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} + i\frac{3}{5\sqrt{2}}e^{iE_-t/\hbar}\right)$$

$$\langle S_y \rangle = +\frac{\hbar}{2}\left(\frac{16}{50} - i\frac{12}{50}e^{-i(E_+-E_-)t/\hbar} + i\frac{12}{50}e^{i(E_+-E_-)t/\hbar} + \frac{9}{50}\right) - \frac{\hbar}{2}\left(\frac{16}{50} + i\frac{12}{50}e^{-i(E_+-E_-)t/\hbar} - i\frac{12}{50}e^{i(E_+-E_-)t/\hbar} + \frac{9}{50}\right)$$

$$\langle S_y \rangle = +\frac{\hbar}{2}\left(\frac{25}{50} + \frac{12i}{50}\left(e^{i\omega t} - e^{-i\omega t}\right)\right) - \frac{\hbar}{2}\left(\frac{25}{50} - i\frac{12}{50}\left(e^{i\omega t} - e^{-i\omega t}\right)\right)$$

$$\langle S_y \rangle = +\frac{2\hbar}{2}\frac{12i}{50}\left(e^{i\omega t} - e^{-i\omega t}\right) = +\hbar\frac{12i}{50}\left(2i\sin(\omega t)\right) = \boxed{\frac{-12\hbar}{25}\sin(\omega t) = \langle S_y \rangle}$$

$$\frac{\hbar}{2}\cdot\frac{12}{50} + \frac{\hbar}{2}\cdot\frac{12}{50} = \frac{12\hbar}{100}\cdot 2 =$$

$$\langle S_z\rangle = +\frac{\hbar}{2}\left|[1\ 0]\begin{bmatrix}\frac{4}{5}e^{-iE_+t/\hbar}\\ -\frac{3}{5}e^{-iE_-t/\hbar}\end{bmatrix}\right|^2 - \frac{\hbar}{2}\left|[0\ 1]\begin{bmatrix}\frac{4}{5}e^{-iE_+t/\hbar}\\ -\frac{3}{5}e^{-iE_-t/\hbar}\end{bmatrix}\right|^2$$

$$\langle S_z\rangle = +\frac{\hbar}{2}\left|\frac{4}{5}e^{-iE_+t/\hbar}\right|^2 - \frac{\hbar}{2}\left|-\frac{3}{5}e^{-iE_-t/\hbar}\right|^2$$

$$\langle S_z\rangle = +\frac{\hbar}{2}\left(\frac{4}{5}e^{-iE_+t/\hbar}\right)\left(\frac{4}{5}e^{iE_+t/\hbar}\right) - \frac{\hbar}{2}\left(-\frac{3}{5}e^{-iE_-t/\hbar}\right)\left(-\frac{3}{5}e^{iE_-t/\hbar}\right)$$

$$\langle S_z\rangle = +\frac{\hbar}{2}\left(\frac{16}{25}\right) - \frac{\hbar}{2}\left(\frac{9}{25}\right) = +\frac{\hbar}{2}\left(\frac{16-9}{25}\right) = \boxed{+\frac{7\hbar}{50} = \langle S_z\rangle}$$

b) Considering Ehrenfest's theorem, do you results make sense?

Ehrenfest's Theorem ~~states~~ th the expectation values of the spin observables behave like we would expect their classical counterparts to behave.

This means that the square root of the sum of the squares of the expectation values should give us the total spin of our quwoon.

$$\sqrt{\langle S_x\rangle^2+\langle S_y\rangle^2+\langle S_z\rangle^2} = \sqrt{\left(\frac{-12\hbar}{25}\cos(\omega t)\right)^2+\left(\frac{-12\hbar}{25}\sin(\omega t)\right)^2+\left(\frac{7\hbar}{50}\right)^2}$$

$$\Rightarrow \sqrt{\frac{144\hbar^2}{625}\cos^2\omega t + \frac{144\hbar^2}{625}\sin^2\omega t + \frac{49\hbar^2}{2500}}$$

$$\Rightarrow \sqrt{\frac{144\hbar^2}{625}\left(\cos^2(\omega t)+\sin^2(\omega t)\right) + \frac{49\hbar^2}{2500}} \Rightarrow \sqrt{\frac{144\hbar^2}{625}+\frac{49\hbar^2}{2500}} \Rightarrow \sqrt{\frac{576\hbar^2+49\hbar^2}{2500}}$$

$$\Rightarrow \sqrt{\frac{625\hbar^2}{2500}} = \frac{25\hbar}{50} = \boxed{\frac{\hbar}{2}}$$

This is the spin of an electron so our expectation values make sense and satisfy Ehrenfest's theorem

Q11M.7

Imagine that the only visible photons an electron in a box of unknown length emits have wavelength of 620nm and 443 nm. Identify the transitions and find the box length.

$\lambda_1 = 620nm, \quad \lambda_2 = 443 nm$

$E_{Ph} = \frac{hc}{\lambda} \rightarrow E_{Ph1} = \frac{hc}{620nm} = \frac{1240eV\cdot nm}{620nm} = 2eV$

$E_{Ph2} = \frac{hc}{443nm} = \frac{1240eV\cdot nm}{443nm} = 2.8eV$

The energy of a quantum in a box is $E_n = \frac{h^2n^2}{8mL^2} = \frac{(hc)^2 n^2}{8mc^2 L^2}$, where $hc = 1240eV\cdot nm$, $mc^2 = 511,000eV$ for an electron.

The smaller λ the higher transition. So E_2 was a higher transition than E_1. Energies are quantized and because E_1 and E_2 are the only energies that are in the visible spectrum they are consecutive.

So $E_2/E_1 = \frac{E_0((n_1+1)^2-n_0^2)}{E_0(n_1^2-n_0^2)} = \frac{n_1^2+2n_1+1-n_0^2}{n_1^2-n_0^2} = 1 + \frac{2n_1+1}{n_1^2-n_0^2}$

$E_{P2}/E_{P1} = 1.4 \Rightarrow 1.4 = 1 + \frac{2n_1+1}{n_1^2-n_0^2} \Rightarrow 0.4 = \frac{2n_1+1}{n_1^2-n_0^2} \Rightarrow 0.4 = \frac{2n_1+1}{(n_1-n_0)(n_1+n_0)}$

$\Rightarrow 5n_1 + 2.5 = n_1^2 - n_0^2 \Rightarrow n_1^2 - 5n_1 - n^2 - 2.5 = 0 \Rightarrow$

I wrote a code in FORtran to solve for n_1 numerically. The best fit pair was $(n_1, n_0) = (11, 8)$

So the transitions were most certainly $12 \rightarrow 8$ and $11 \rightarrow 8$
E_{P2} E_{P1}

So solving for E_0 we check:

$2.8eV = E_0 \cdot (12^2 - 8^2) \Rightarrow \frac{2.8eV}{80} = E_0 = 0.035eV$

$2eV = E_0 \cdot (11^2 - 8^2) \Rightarrow \frac{2eV}{57} = E_0 = 0.035eV$ checks out

solving for L: $0.035eV = \frac{(hc)^2}{8mc^2 L^2} \Rightarrow L = \frac{hc}{\sqrt{8mc^2 \cdot 0.035eV}}$

$L = \frac{1240eV\cdot nm}{\sqrt{8 \cdot 511000eV \cdot 0.035eV}} = 3.28nm$

Eval 0 answers

QAM.6) Reasonable magnitude, right units

QAM.7) Reasonable magnitudes, right units