Preparation for September 6

In chapter 2, we studied first-order differential equations; these equations involved y and y' (and sometimes the independent variable x). Remember that the solution to such equations often had a "c" in it, since after you integrate, you must not forget to write "$+c$". That is, the general solution of a first-order differential equation typically has one degree of freedom.

In chapter 3, we will consider second-order differential equations; these equations can include y, y', and y'' (and sometimes x). Typically, the general solution of such an equation has two degrees of freedom. (You can perhaps guess how many degrees of freedom to expect for a third order differential equation, but we won't go there.)

Much of our attention will focus on equations of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constant.

A good guess for solving such equations is to try $y = e^{rt}$, where r is a constant that needs to be determined. For example, consider

$$y'' - 3y' + 2y = 0 \tag{1}$$

If we set $y = e^{rt}$, then by the chain rule, $y' = re^{rt}$; by the chain rule again, $y'' = r^2e^{rt}$. Plugging these into the differential equation yields

$$r^2e^{rt} - 3re^{rt} + 2e^{rt} = 0$$

Dividing through by e^{rt} (which is never 0) yields

$$r^2 - 3r + 2 = 0.$$

This happens when $r = 1$ or $r = 2$, so $y = e^t$ and $y = e^{2t}$ are solutions.

Notice that we started with the differential equation $y'' - 3y' + 2y = 0$, and we found that $y = e^{rt}$ is a solution if $r^2 - 3r + 2 = 0$. In general, and by the same reasoning, $y = e^{rt}$ is a solution of $ay'' + by' + cy = 0$ when $ar^2 + br + c = 0$. We call $ar^2 + br + c = 0$ the *characteristic equation* of the differential equation $ay'' + by' + cy = 0$. We call $ar^2 + br + c$ the *characteristic polynomial*.

There is an important principle, called the Principle of Superposition, that works for this particular type of differential equation. It says that any constant multiple of a solution is also a solution, and any sum of two solutions is also a solution. (In class, I will show you why that works for these differential equations; if you are so inclined, you might try thinking a bit about this a bit first. If you enjoyed Math 215, you might think about how to phrase this in Linear Algebra terms.)

For example, since $y = e^t$ is a solution of Equation (1), so is $y = c_1 e^t$ for any constant c_1. And since $y = e^{2t}$ is another solution, so is $y = c_2 e^{2t}$ for any constant c_2. Adding these, we get a general solution

$$y = c_1 e^t + c_2 e^{2t}.$$

Notice that c_1 and c_2 provide the two expected degrees of freedom in describing the general solution.