(Name)

Exam I

Mathematics 220

November 1, 2017

Please complete five out of six questions. (There are a few blank pages at the end if you need scrap paper.)

1. (Flashback) A spring is stretched 0.1 m by a force of 5 N. A mass of 2 kg is attached to the spring. For what damping constant γ is the system critically damped. What is the general solution to the differential equation $mx'' + \gamma x' + kx = 0$ for this γ?

2. Find the radius of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n n^3 (x-1)^n}{2^n}$$

3. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 3 \end{cases}$$

It is not necessary to simplify trigonometric expressions such as $\cos(n\pi)$ or $\sin(n\pi/2)$.

4. Consider the differential equation

$$(x^2 - 1)y'' + xy' + 2y = 0$$

We seek a power series solution of the differential equation at $x_0 = 0$.

- Find a recurrence relation for the coefficients of such a solution.
- Let $a_0 = 1, a_1 = 2$. Find a_2 and a_3.

5. Consider the heat equation: $u_t = \alpha^2 u_{xx}$, together with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. The derivation of the general solution is begun below. Give short answers at the five prompts. (The problem is continued on the next page).

   - We seek solutions of the form $u(x,t) = T(t)X(x)$. Then

     $$u_t = \alpha^2 u_{xx}$$

     becomes

     $$T'(t)X(x) = \alpha^2 T(t)X''(x).$$

     We rewrite this as

     $$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

     Now, we set both sides equal to $-\lambda$. Why can we do that?

   - We now get $X''(x) + \lambda X(x) = 0$. Assuming $\lambda > 0$, we can solve this differential equation using techniques we saw earlier in the course. What is the general solution we get for $X(x)$?

- Since $u(0,t) = 0$ and $u(L,t) = 0$, we get $X(0) = 0$ and $X(L) = 0$. What do these facts allow us to conclude about the coefficients of the general solution we obtained in the previous step, and about the possible values of λ?

- After we have found possible values for λ, we can substitute these into the equation $T'(t) + \lambda\alpha^2 T(t) = 0$. What solution do we get for $T(t)$?

- After multiplying together the solutions for $T(t)$ and $X(x)$, we got a sequence of solutions $u_n(x,t)$ for the heat equation. What did we do to these to get the general solution?

6. The proof of Theorem 3 is outlined below, but with justifications for the claims removed. Justify the steps – when your reason involves a definition, axiom, or theorem, precisely identify which one applies. (The definitions, axioms and theorems are on the next page).

   We suppose $0 \leq a_n \leq b_n$ for all n and $\sum_{n=0}^{\infty} b_n$ converges.

   - We let $A_m = \sum_{n=0}^{m} a_n$ and $B_m = \sum_{n=0}^{m} b_n$. Then we know that the sequence (B_m) converges to some number B.

   - We know $B_m \leq B_{m+1}$ and $A_m \leq A_{m+1}$ for all m.

   - We know $B_m \leq B$ for all m.

   - From the above steps, we have $A_m \leq A_{m+1}$ for all m, and $A_m \leq B$ for all m. It follows that the sequence (A_m) converges.

   - Thus, $\sum_{n=0}^{\infty} a_n$ converges.

Definition of sequential convergence. A sequence (a_n) *converges* to a number L if for any number $\epsilon > 0$, there is a number N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. We then write $(a_n) \to L$ or $\lim_{n\to\infty} a_n = L$.
Definition of partial sum. Given a series $\sum_{n=0}^{\infty} a_n$, the corresponding sequence of partial sums is the sequence whose m^{th} term is $\sum_{n=0}^{m} a_n$. That is, the sequence of partial sums begins

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \cdots$$

Definition of series convergence. We say a series $\sum_{n=0}^{\infty} a_n$ *converges* if the corresponding sequence of partial sums converges. We use the notation $\sum_{n=0}^{\infty} a_n$ to refer to the number to which the series converges.

Axiom 1. If $(a_n) \to L$ and $(b_n) \to M$, then $(a_n + b_n) \to L + M$. Also, if c and d are constants, then $(ca_n + db_n) \to cL + dM$.

Axiom 2. If c is a constant, then the sequence (c) (which is a sequence of all c's) converges to c.

Axiom 3. If $|x| < 1$, then $(x^n) \to 0$.

Axiom 4. The sequence $(1/n)$ converges to 0.

Axiom 5. If $a_n \leq a_{n+1}$ for all n and $(a_n) \to L$, then $a_n \leq L$ for all n.

Axiom 6. If $a_n \leq a_{n+1}$ for all n, and there exists a bound M such that $a_n \leq M$ for all n, then (a_n) converges to some number L.

Axiom 7. If $(a_n) \to L$, and we replace a finite number of the terms at the beginning of the sequence with another finite number of terms, then the new sequence also converges to L.

Theorem 1. If $|x| < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Theorem 2. Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B, and c and d are constants. Then $\sum_{n=0}^{\infty} ca_n + db_n$ converges to $cA + dB$.

Theorem 3. (Comparison Test) If $0 \leq a_n \leq b_n$ for all n and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Theorem 4. (Absolute Convergence Test) If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Theorem 5. (Divergence Test) If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to infty} a_n = 0$.

Theorem 6. (Ratio Test) Let $R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $R < 1$, then $\sum_{n=0}^{\infty} a_n$ converges. If $R > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.