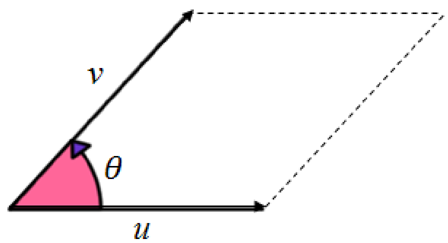
The Cross Product

Definition

The cross product is an operation on two three-dimensional vectors which results in a third vector orthogonal to the first two. The length of the cross-product is equivalent to the area of the parallelogram formed by the two vectors.

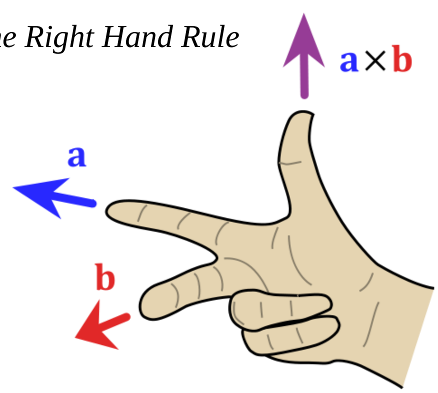


Here, the cross product $u \times v$ would point out of the page, and would have magnitude equal to the area of the parallelogram shown.

The cross product of two vectors $\boldsymbol{u} = u_1 i + u_2 j + u_3 k$ and $\boldsymbol{v} = v_1 i + v_2 j + v_3 k$ is defined as $\boldsymbol{u} \times \boldsymbol{v} = |\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot sin\theta$, where θ is the angle formed by the two vectors.

Determining Direction

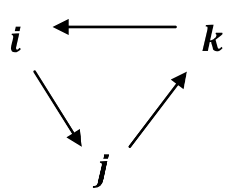
The Right Hand Rule



To find the direction of $\boldsymbol{u} \times \boldsymbol{v}$, begin with the fingers of your **right hand** pointing in the direction of $\boldsymbol{u}$. Curl your fingers toward $\boldsymbol{v}$. Your thumb is now pointing in the direction of the cross product. Notice that $\boldsymbol{v} \times \boldsymbol{u}$ and $\boldsymbol{u} \times \boldsymbol{v}$ are not the same.

Determining Sign

By convention, we follow the arrows in the following diagram to determine the cross product of two basis vectors. Moving with the arrows gives a positive result, and against the arrows gives a negative result. Crossing two vectors that point in the same direction gives zero.



$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, because it goes with the arrows.

$\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$, because we move against the arrows.

$\boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{0}$, because they are in the same direction.

Calculating the Cross Product

The cross product of two vectors $\boldsymbol{u} = u_1 i + u_2 j + u_3 k$ and $\boldsymbol{v} = v_1 i + v_2 j + v_3 k$ can be calculated by either of the following methods.

Method 1: Diagonals

First set up the following matrix:

$$\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Rewrite the first two columns next to the matrix, and multiply along the following diagonals.

$$\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{matrix} i & j \\ u_1 & u_2 \\ v_1 & v_2 \end{matrix}$$

Sum the products to obtain $(u_2 v_3)i + (u_3 v_1)j + (u_1 v_2)k$. Then multiply again along the following diagonals:

$$\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{matrix} i & j \\ u_1 & u_2 \\ v_1 & v_2 \end{matrix}$$

Subtract these terms to obtain the cross product:

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2 v_3)\boldsymbol{i} + (u_3 v_1)\boldsymbol{j} + (u_1 v_2)\boldsymbol{k} - (v_2 u_3)\boldsymbol{i} - (v_3 u_1)\boldsymbol{j} - (v_1 u_2)\boldsymbol{k}$$

$$= (u_2 v_3 - v_2 u_3)\boldsymbol{i} + (u_3 v_1 - v_3 u_1)\boldsymbol{j} + (u_1 v_2 - v_1 u_2)\boldsymbol{k}$$

Method 2: Determinants

First set up the following matrix:

$$\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

For each basis vector, $i, j,$ and k, cross out the row and column containing that vector, as shown:

$$\begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Find the determinant of each remaining 2×2 matrix. Use them as coefficients in the following formula for the cross product. (Don't forget the negative sign in front of $\boldsymbol{j}$.)

$$\boldsymbol{u} \times \boldsymbol{v} = \ det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} \boldsymbol{i} - det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} \boldsymbol{j} + det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \boldsymbol{k}$$

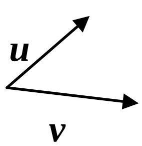
$$= (u_2 v_3 - v_2 u_3)\boldsymbol{i} - (v_3 u_1 - u_3 v_1)\boldsymbol{j} + (u_1 v_2 - v_1 u_2)\boldsymbol{k}$$

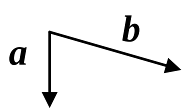
Notice that both methods yield the same formula for the cross product.

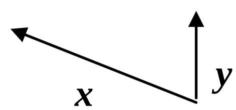
Practice Problems

1. Use the right hand rule to determine the direction of each cross product.

a. b. c.







$\boldsymbol{u} \times \boldsymbol{v}$ $\qquad$ $\boldsymbol{b} \times \boldsymbol{a}$ $\qquad$ $\boldsymbol{x} \times \boldsymbol{y}$

2. Determine the following cross products using the correct sign convention:

a. $\boldsymbol{j} \times \boldsymbol{i} =$ $\qquad$ b. $\boldsymbol{k} \times \boldsymbol{i} =$ $\qquad$ c. $\boldsymbol{j} \times \boldsymbol{k} =$ $\qquad$ d. $\boldsymbol{i} \times \boldsymbol{i} =$

3. Use the "diagonals" method to find the cross product $\boldsymbol{u} \times \boldsymbol{v}$ of the vectors:

a. $u = 3i + 2j - k$ $\qquad$ $v = i + j + 2k$

b. $u = -i + 7j + 2k$ $\qquad$ $v = 3i - j + k$

4. Use the "determinants" method to find the cross product $\boldsymbol{u} \times \boldsymbol{v}$ of the vectors:

a. $u = 4i + j + k$ $\qquad v = i + 2j + 3k$

b. $u = -i + 5j - 2k$ $\qquad v = 2i + j + 3k$

Solutions to Practice Problems:

1.
a. into the page b. out of the page c. into the page

2.
a. $-\boldsymbol{k}$ b. $\boldsymbol{j}$ c. $\boldsymbol{i}$ d. 0

3.
a. $5i - 7j + 2k$
b. $9i + 7j - 20k$

4.
a. $\boldsymbol{i} - 11j + 7k$
b. $17i - j - 11k$