Worksheet 20 - Linear Independence

Definition. Let V be a vector space and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in V$. we say that $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u})$ is a *linearly independent sequence* if and only if the following statement is true:

For all $c_1, c_2, \ldots, c_n \in \mathbb{R}$, if $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u} = \vec{0}$ then $c_1 = c_2 = \cdots = c_n = 0$.

Otherwise, we say that the sequence is *linearly dependent*.

1 Warm-up

Exercise 1. Find vectors, $\vec{u}$ and $\vec{v}$, in $\mathbb{R}^2$ such that

(a) $(\vec{u}, \vec{v})$ is linearly dependent.

(b) $(\vec{u}, \vec{v})$ is linearly independent.

Exercise 2. Find vectors, $\vec{u}, \vec{v}$ and $\vec{w}$, in $\mathbb{R}^2$ such that

(a) $(\vec{u}, \vec{v}, \vec{w})$ is linearly dependent.

(b) $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent.

Exercise 3. Find vectors, $\vec{u}$ and $\vec{v}$, in $\mathbb{R}^3$ such that

(a) $(\vec{u}, \vec{v})$ is linearly dependent.

(b) $(\vec{u}, \vec{v})$ is linearly independent.

Exercise 4. Find vectors, $\vec{u}, \vec{v}$ and $\vec{w}$, in $\mathbb{R}^3$ such that

(a) $(\vec{u}, \vec{v}, \vec{w})$ is linearly dependent.

(b) $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent.

Exercise 5. Find vectors, $\vec{u}, \vec{v}, \vec{w}$ and $\vec{z}$, in $\mathbb{R}^3$ such that

(a) $(\vec{u}, \vec{v}, \vec{w}, \vec{z})$ is linearly dependent.

(b) $(\vec{u}, \vec{v}, \vec{w}, \vec{z})$ is linearly independent.

2 Applications and Generalizations

Exercise 6. Consider the following set of vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

Determine whether these are linearly independent. Describe a step-by-step, general, approach for determining whether three vectors in $\mathbb{R}^3$ are linearly independent.

Exercise 7. Consider the following set of vectors:

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

Demonstrate explicitly that these are linearly dependent. Describe a step-by-step, general, approach for demonstrating that four vectors in $\mathbb{R}^3$ are linearly dependent.

Exercise 8. Let V be the vector space made up of all real valued functions. Which of the following sets of functions are linearly independent in V?:

(a)	$f_1(t) = 3t;$	$f_2(t) = t + 5;$	$f_3(t) = 2t^2;$	$f_4(t) = (t+1)^2$
(b)	$f_1(t) = (t+1)^2;$	$f_2(t) = t^2 - 1;$	$f_3(t) = 2t^2 + 2t - 3$	
(c)	$f_1(t) = 1;$	$f_2(t) = e^t;$	$f_3(t) = e^{-t}$	
(d)	$f_1(t) = t^2;$	$f_2(t) = t;$	$f_3(t) = 1$	
(e)	$f_1(t) = 1 - t;$	$f_2(t) = t(1-t);$	$f_3(t) = 1 - t^2$	