Solutions to Problem Set 6

Problem 1: Let $\vec{v} \in \text{Span}(\vec{w})$ be arbitrary. Since $\vec{w} \in \text{Span}(\vec{u})$, we can fix $c \in \mathbb{R}$ with $\vec{w} = c\vec{u}$. Since $\vec{v} \in \text{Span}(\vec{w})$, we can fix $d \in \mathbb{R}$ with $\vec{v} = d\vec{w}$. Now notice that

$$\begin{aligned} \vec{v} &= d\vec{w} \\ &= d \cdot (c\vec{u}) \\ &= (dc) \cdot \vec{u}. \end{aligned}$$

Since $dc \in \mathbb{R}$, we conclude that $\vec{v} \in \text{Span}(\vec{u})$. Since $\vec{v} \in \text{Span}(\vec{w})$ was arbitrary, the result follows.

Problem 2: We give a counterexample to this statement. Let

$$\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

so that

$$\text{Span}(\vec{u}) = \left\{ c \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} : c \in \mathbb{R} \right\}.$$

Notice that we have the following:

- $\begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \text{Span}(\vec{u})$ because $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
- $\begin{pmatrix} 6 \\ 2 \end{pmatrix} \in \text{Span}(\vec{u})$ because $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Now consider the vector

$$\begin{pmatrix} 18 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 6 \\ 1 \cdot 2 \end{pmatrix}.$$

We claim that

$$\begin{pmatrix} 18 \\ 2 \end{pmatrix} \notin \text{Span}(\vec{u}).$$

We argue this by contradiction. Suppose instead that

$$\begin{pmatrix} 18 \\ 2 \end{pmatrix} \in \text{Span}(\vec{u}).$$

and fix $c \in \mathbb{R}$ with

$$\begin{pmatrix} 18 \\ 2 \end{pmatrix} = c \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

We then have that both $18 = 3c$ and $2 = c$. Since $c = 2$, we can plug this into the first equation to conclude that $18 = 6$, which is a contradiction. It follows that

$$\begin{pmatrix} 18 \\ 2 \end{pmatrix} \notin \text{Span}(\vec{u}).$$

To recap, we have

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \text{Span}(\vec{u}) \quad \text{and} \quad \begin{pmatrix} 6 \\ 2 \end{pmatrix} \in \text{Span}(\vec{u}), \quad \text{but} \quad \begin{pmatrix} 18 \\ 2 \end{pmatrix} \notin \text{Span}(\vec{u}).$$

Problem 3a: We have

$$\begin{aligned}(-1)\cdot 1 - 5\cdot 2 &= -1-10\\ &= -11.\end{aligned}$$

Since this value is nonzero, we can use Theorem 2.3.10 to conclude that $\text{Span}(\vec{u}_1,\vec{u}_2)=\mathbb{R}^2$.

Problem 3b: We want to find the unique pair of numbers $c_1,c_2\in\mathbb{R}$ with

$$\begin{pmatrix}5\\1\end{pmatrix} = c_1\cdot\begin{pmatrix}-1\\2\end{pmatrix} + c_2\cdot\begin{pmatrix}5\\1\end{pmatrix}.$$

Notice that we clearly have

$$\begin{pmatrix}5\\1\end{pmatrix} = 0\cdot\begin{pmatrix}-1\\2\end{pmatrix} + 1\cdot\begin{pmatrix}5\\1\end{pmatrix}.$$

Since we have found a pair of numbers that work, and we know by Theorem 2.3.10 that the pair of numbers is unique, it follows that

$$Coord_{(\vec{u}_1,\vec{u}_2)}\left(\begin{pmatrix}5\\1\end{pmatrix}\right) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Problem 3c: We want to find the unique pair of numbers $c_1,c_2\in\mathbb{R}$ with

$$\begin{pmatrix}8\\17\end{pmatrix} = c_1\cdot\begin{pmatrix}-1\\2\end{pmatrix} + c_2\cdot\begin{pmatrix}5\\1\end{pmatrix}.$$

In other words, we want to find the unique pair of numbers $c_1,c_2\in\mathbb{R}$ with

$$\begin{pmatrix}8\\17\end{pmatrix} = \begin{pmatrix}-c_1+5c_2\\2c_1+c_2\end{pmatrix}.$$

Finding these values amounts to solving the following system of equations:

$$\begin{array}{rcrcr} -x & + & 5y & = & 8\\ 2x & + & y & = & 17\end{array}$$

Adding twice the first equation to the second, we conclude that $11y=33$, and hence $y=3$. Plugging this into the first equation gives $-x+15=8$, so $x=7$. It follows that $(7,3)$ is the only possible solution. Now we can check that

$$\begin{pmatrix}8\\17\end{pmatrix} = 7\cdot\begin{pmatrix}-1\\2\end{pmatrix} + 3\cdot\begin{pmatrix}5\\1\end{pmatrix}.$$

is true, so it follows that

$$Coord_{(\vec{u}_1,\vec{u}_2)}\left(\begin{pmatrix}8\\17\end{pmatrix}\right) = \begin{pmatrix}7\\3\end{pmatrix}.$$

Alternatively, we could have used the formula developed at the end of Section 2.2 instead of solving the system by hand.

Problem 4: We first show that 1 implies 2. Assume then that 1 is true, so assume that $\text{Span}(\vec{u}_1,\vec{u}_2)=\text{Span}(\vec{u}_1)$. Notice that $\vec{u}_2 = 0\cdot\vec{u}_1 + 1\cdot\vec{u}_2$. Since $0,1\in\mathbb{R}$, it follows that $\vec{u}_2\in\text{Span}(\vec{u}_1,\vec{u}_2)$. Since $\text{Span}(\vec{u}_1,\vec{u}_2)=\text{Span}(\vec{u}_1)$, we conclude that $\vec{u_2}\in\text{Span}(\vec{u}_1)$.

We now show that 2 implies 1. Assume then that 2 is true, so assume that $\vec{u}_2\in\text{Span}(\vec{u}_1)$. By definition, we can fix $d\in\mathbb{R}$ with $\vec{u}_2=d\vec{u}_1$. To show that $\text{Span}(\vec{u}_1,\vec{u}_2)=\text{Span}(\vec{u}_1)$, we give a double containment proof.

- Using Proposition 2.3.7, we know immediately that $\text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2)$.
- We now show that $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$. Let $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2)$ be arbitrary. By definition we can fix $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$. Notice that

$$\begin{aligned}
\vec{v} &= c_1\vec{u}_1 + c_2\vec{u}_2 \\
&= c_1\vec{u}_1 + c_2(d\vec{u}_1) \\
&= c_1\vec{u}_1 + (c_2 d)\vec{u}_1 \\
&= (c_1 + c_2 d) \cdot \vec{u}_1.
\end{aligned}$$

  Since $c_1 + c_2 d \in \mathbb{R}$, it follows that $\vec{v} \in \text{Span}(\vec{u}_1)$. Since $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2)$ was arbitrary, we conclude that $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$.

Since we haves shown both $\text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2)$ and $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$, we conclude that $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$.