Solutions to Written Assignment 6

Problem 1a: Suppose that $c \in \mathbb{R}$ and $\vec{v} \in V$ are such that $c \cdot \vec{v} = \vec{0}$. We need to show that either $c = 0$ or $\vec{v} = \vec{0}$. To do this, we assume that $c \neq 0$ and prove that $\vec{v} = \vec{0}$ (because if $c = 0$ then we are done). Suppose then that $c \neq 0$. Starting with the equation

$$c \cdot \vec{v} = \vec{0},$$

we multiply both sides by $\frac{1}{c}$ (which makes sense because $c \neq 0$) to conclude that

$$\frac{1}{c} \cdot (c \cdot \vec{v}) = \frac{1}{c} \cdot \vec{0}.$$

Now working with the left-hand side of this equation we have

$$\begin{aligned} \frac{1}{c} \cdot (c \cdot \vec{v}) &= \left(\frac{1}{c} \cdot c\right) \cdot \vec{v} && \text{(by Property 9)} \\ &= 1 \cdot \vec{v} \\ &= \vec{v} && \text{(by Property 10).} \end{aligned}$$

For the right-hand side of the above equation we know that

$$\frac{1}{c} \cdot \vec{0} = \vec{0}$$

by Proposition 4.1.11. Plugging these into the two sides of our above equation, we conclude that $\vec{v} = \vec{0}$.

Problem 1b: Suppose that $\vec{v} \in V$ and $c, d \in \mathbb{R}$ are such that $c \cdot \vec{v} = d \cdot \vec{v}$. Suppose also that $\vec{v} \neq \vec{0}$. Adding $-(d \cdot \vec{v})$ to both sides of $c \cdot \vec{v} = d \cdot \vec{v}$, we conclude that $c \cdot \vec{v} + -(d \cdot \vec{v}) = \vec{0}$. By definition of $-$, this implies that $c \cdot \vec{v} - d\vec{v} = \vec{0}$. Using Problem 4b on Problem set 14, we conclude that $(c - d) \cdot v = \vec{0}$. Now we are assuming that $\vec{v} \neq \vec{0}$, so by part a, we can conclude that $c - d = 0$. Adding d to both sides, it follows that $c = d$.

Problem 1c: Let V be a vector space with more than 1 element. Since V has at least two elements, we can fix $\vec{v} \in V$ with $\vec{v} \neq \vec{0}$. Now if $c, d \in \mathbb{R}$ are arbitrary with $c \neq d$, then we know that $c \cdot \vec{v} \neq d \cdot \vec{v}$ by part b. Since $c \cdot \vec{v} \in V$ for all $c \in \mathbb{R}$, and these distinct scalar multiples of $\vec{v}$ are distinct, we can use the fact that there are infinitely many real numbers to conclude that V has infinitely many elements.

Problem 2a: Suppose that U and W are subspaces of V. To show that $U \cap W$ is a subspace of V, we need to verify the three properties mentioned above.

- $\vec{0} \in U \cap W$: Notice that $\vec{0} \in U$ because U is a subspace of V and also $\vec{0} \in W$ because W is a subspace of V. Therefore, by definition of $U \cap W$, we conclude that $\vec{0} \in U \cap W$.
- $U \cap W$ is closed under addition: Let $\vec{v}_1, \vec{v}_2 \in U \cap W$ be arbitrary. By definition of $U \cap W$, we know that both $\vec{v}_1 \in U$ and $\vec{v}_2 \in U$, so using the fact that U is a subspace of V we conclude that $\vec{v}_1 + \vec{v}_2 \in U$. Similarly, by definition of $U \cap W$, we know that both $\vec{v}_1 \in W$ and $\vec{v}_2 \in W$, so using the fact that W is a subspace of V we conclude that $\vec{v}_1 + \vec{v}_2 \in W$. Since we have shown that both $\vec{v}_1 + \vec{v}_2 \in U$ and $\vec{v}_1 + \vec{v}_2 \in K$, we deduce that $\vec{v}_1 + \vec{v}_2 \in U \cap W$.
- $U \cap W$ is closed under scalar multiplication: Let $\vec{v} \in U \cap W$ and $c \in \mathbb{R}$ be arbitrary. By definition of $U \cap W$, we know that $\vec{v} \in U$, so $c \cdot \vec{v} \in U$ because U is a subspace of V. Similarly, by definition of $U \cap W$, we know that $\vec{v} \in W$, so $c \cdot \vec{v} \in W$ because W is a subspace of V. Since we have shown that both $c \cdot \vec{v} \in U$ and $c \cdot \vec{v} \in W$, we deduce that $c \cdot \vec{v} \in U \cap W$.

Therefore, $U \cap W$ is a subspace of V.

Problem 2b: Consider the vector space $V = \mathbb{R}^2$. Let

- $U = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{R}\right\}$.
- $W = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} 0 \\ a \end{pmatrix} : a \in \mathbb{R}\right\}$.

so U and W are subspaces of V by Proposition 4.1.16. Now we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W,$$

hence we have both

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U \cup W \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in U \cup W.$$

However, notice that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

but

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U \cup W.$$

Therefore, $U \cup W$ is not a subspace of V because it is not closed under addition.