Assignment: Problem Set 9

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Due Date: 02/28/2018

List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: None

Problem 1: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first projecting $\vec{v}$ onto the line $y = 3x$, and then projecting the result onto the line $y = 4x$. Explain why T is a linear transformation, and then calculate $[T]$.

Solution: Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ be arbitrary, and fix $a, b \in \mathbb{R}$ such that $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Let $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Let $A : \mathbb{R} \to \mathbb{R}^2$ be the linear transformation that projects $\vec{v} \in \mathbb{R}^2$ onto the line $y = 3x$ by letting $A(\vec{v}) = P_{\vec{a}}(\vec{v})$, where $P_{\vec{a}}$ is the projection linear transformation onto $Span(\vec{a})$, as described in Proposition 3.1.11. Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects $\vec{w} \in \mathbb{R}^2$ onto the line $y = 4x$ by letting $B(\vec{v}) = P_{\vec{b}}(\vec{v})$, where $P_{\vec{b}}$ is the projection linear transformation onto $Span(\vec{b})$, as described in Proposition 3.1.11.. Notice that $(B \circ A)(\vec{v}) = B(A(\vec{v}))$ by definition of function composition. Geometrically, this means that $(B \circ A)$ takes the output of P, which is a the projection of a vector $\vec{v}$ onto the line $y = 3x$, and projects it onto the line $y = 4x$. In other words, $(B \circ A)$ takes a vector $\vec{v}$ and projects it onto the line $y = 3x$, and then projects the result onto the line $y = 4x$. But this is exactly how we define T. So $B \circ A = T$. We know that A, B are linear transformations, so By Proposition 2.4.8, $B \circ A$ is a linear transformation, so T is a linear transformation. We want to find $[T]$. We found earlier that $(B \circ A)(\vec{v}) = B(A(\vec{v}))$. By Proposition 3.1.4, $B(A(\vec{v})) = B([A]\vec{v}) = [B] \cdot ([A]\vec{v})$. By Proposition 3.2.5, $[B] \cdot ([A]\vec{v}) = ([B][A])\vec{v}$. So $T(\vec{v}) = ([B][A])\vec{v}$. It follows from Proposition 3.1.4, that $[T] = [B][A]$. We have defined A, B by Proposition 3.1.11, so we have:

Using Definition 3.2.1, we compute:

$$\begin{aligned}
[T] = [B][A] &= \begin{pmatrix} \frac{1}{17} & \frac{4}{17} \\ \frac{4}{17} & \frac{16}{17} \end{pmatrix} \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{17} \cdot \frac{1}{10} + \frac{4}{17} \cdot \frac{3}{10} & \frac{1}{17} \cdot \frac{3}{10} + \frac{4}{17} \cdot \frac{9}{10} \\ \frac{4}{17} \cdot \frac{1}{17} + \frac{16}{17} \cdot \frac{3}{10} & \frac{4}{17} \cdot \frac{3}{10} + \frac{16}{17} \cdot \frac{9}{10} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+12}{170} & \frac{3+36}{170} \\ \frac{4+48}{170} & \frac{12+144}{170} \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{170} & \frac{39}{170} \\ \frac{52}{170} & \frac{156}{170} \end{pmatrix}
\end{aligned}$$

Therfore, $[T] = \begin{pmatrix} \frac{13}{170} & \frac{39}{170} \\ \frac{52}{170} & \frac{156}{170} \end{pmatrix}$.

Problem 2: Let

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

a. Show that $A \cdot A = A$ by simply computing it.

Solution: By Definition 3.2.1, we have $A \cdot A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{2}{5} & \frac{1}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{4}{5} \\ \frac{2}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{2}{5} & \frac{2}{5} \cdot \frac{2}{5} + \frac{4}{5} \cdot \frac{4}{5} \end{pmatrix}$ $= \begin{pmatrix} \frac{1}{25} + \frac{4}{25} & \frac{2}{25} + \frac{8}{25} \\ \frac{2}{25} + \frac{8}{25} & \frac{4}{25} + \frac{16}{25} \end{pmatrix} = \begin{pmatrix} \frac{5}{25} & \frac{10}{25} \\ \frac{10}{25} & \frac{20}{25} \end{pmatrix} = \begin{pmatrix} \frac{5}{5} \cdot \frac{1}{5} & \frac{5}{5} \cdot \frac{2}{5} \\ \frac{5}{5} \cdot \frac{2}{5} & \frac{5}{5} \cdot \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 \cdot \frac{1}{5} & 1 \cdot \frac{2}{5} \\ 1 \cdot \frac{2}{5} & 1 \cdot \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = A$. So $A \cdot A = A$.

b. Find an example of $\vec{w} \in \mathbb{R}^2$ such that $A = [P_{\vec{w}}]$.

Solution: Let $\vec{w} \in \mathbb{R}^2$ be arbitrary, and fix $a, b \in \mathbb{R}$ such that $\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$. By Proposition 3.1.11, $[P_{\vec{w}}] = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$. Notice that $A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Suppose $a = 1, b = 2$. Then we have $[P_{\vec{w}}] = \frac{1}{1^2+2^2} \cdot \begin{pmatrix} 1^2 & 1 \cdot 2 \\ 1 \cdot 2 & 2^2 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = A$. Therefore, $A = [P_{\vec{w}}]$ for $\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

c. By interpreting the action of A geometrically, explain why you should expect that $A \cdot A = A$.

Solution: Let $\vec{v} \in \mathbb{R}^2$ be arbitrary, and fix $\vec{u} \in \mathbb{R}^2$ such that $\vec{u} = A\vec{v}$. We know that $A = [P_{\vec{w}}]$ when $\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. In other words, A is the standard matrix of the linear transformation that takes a vector $\vec{v}$ and gives a vector $\vec{u}$ which is the projection of $\vec{v}$ onto the line $y = 2x$ (by Proposition 3.1.11). The projection of a vector onto the line $y = 2x$ that already lies on the line $y = 2x$ is just that vector. Notice that $\vec{u}$ lies on $y = 2x$ by definition. So we have $A \cdot \vec{u} = \vec{u}$. Substituting $\vec{u} = A\vec{v}$ into this equation, we get $A \cdot (A\vec{v}) = A\vec{v} = (A \cdot A)\vec{v}$ (By Proposition 3.2.5). We know that this equation is true by the way in which we have defined A, therefore it must be the case that $A \cdot A = A$ (and it is!).

Problem 3: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the point on the line $y = x + 1$ that is closest to $\vec{v}$. Is T a linear transformation? Explain.

Solution: We assume that T is a linear transformation. By definition of T, $T(\vec{0}) =$ the point on the line $y = x + 1$ that is closest to $\vec{0}$. The point on the line $y = x + 1$ that is closest to $\vec{0}$ is $(0, 1)$, so $T(\vec{0}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. But by proposition 2.4.2, for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{0}) = \vec{0}$. Our assumption has lead to a contradiction, therefore it must be the case that T is not a linear transformation.

Problem 4: Let $\vec{w} \in \mathbb{R}^2$ be nonzero, and let W =Span($\vec{w}$). Define $F_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $F_{\vec{w}}(\vec{v})$ be the result of reflecting $\vec{v}$ across the line W. Show that $F_{\vec{w}}$ is a linear transformation and determine the standard matrix $[F_{\vec{w}}]$.

Hint: Make use of projections. How can you determine $F_{\vec{w}}(\vec{v})$ using $\vec{v}$ and $P_{\vec{w}}(\vec{v})$?

Solution: Let $\vec{v} \in \mathbb{R}^2$ be arbitrary, and fix $v_1, v_2, u_1, u_2 \in \mathbb{R}, \vec{u} \in \mathbb{R}^2$ such that $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, T_{\vec{w}}(\vec{v}) = \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. So by the definition of $T_{\vec{w}}$, the vector $\vec{u}$ is the reflection of $\vec{v}$ across the line $W = Span(\vec{w})$. Let $P_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection linear transformation (described in Proposition 3.1.11) that takes a vector $\vec{v}$ and gives its projection along the line $W = Span(\vec{u})$. Notice that $P_{\vec{w}}(\vec{u}) = P_{\vec{w}}(\vec{v})$ because the reflection of a vector across a line has the same projection onto that line as the original vector. Because of this, we have that $\vec{u} + \vec{v} = 2 \cdot P_{\vec{w}}(\vec{v})$ (this can be shown graphically by using the "tail to tip" method of adding the reflection of a vector across a line and the original vector, and I would show this but I don't know how to do it in LaTeX.), and it follows that $\vec{u} = 2 \cdot P_{\vec{w}}(\vec{v}) - \vec{v}$, so $T_{\vec{w}}(\vec{v}) = 2 \cdot P_{\vec{w}}(\vec{v}) - \vec{v}$ by defintion of $\vec{u}$. We compute:

$$\begin{aligned}
T_{\vec{w}}(\vec{v}) = 2 \cdot P_{\vec{w}}(\vec{v}) - \vec{v} &= 2 \cdot \begin{pmatrix} \frac{w_1^2}{w_1^2+w_2^2} & \frac{w_1 w_2}{w_1^2+w_2^2} \\ \frac{w_1 w_2}{w_1^2+w_2^2} & \frac{w_2^2}{w_1^2+w_2^2} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} && \text{(By Proposition 3.1.11)} \\
&= \begin{pmatrix} \frac{2v_1 w_1^2}{w_1^2+w_2^2} + \frac{2v_2 w_1 w_2}{w_1^2+w_2^2} \\ \frac{2v_1 w_1 w_2}{w_1^2+w_2^2} + \frac{2v_2 w_2^2}{w_1^2+w_2^2} \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2v_1 w_1^2}{w_1^2+w_2^2} + \frac{2v_2 w_1 w_2}{w_1^2+w_2^2} + \frac{-v_1 w_1^2 - v_1 w_2^2}{w_1^2+w_2^2} \\ \frac{2v_1 w_1 w_2}{w_1^2+w_2^2} + \frac{2v_2 w_2^2}{w_1^2+w_2^2} + \frac{-v_2 w_1^2 - v_2 w_2^2}{w_1^2+w_2^2} \end{pmatrix} \\
&= \frac{1}{w_1^2 + w_2^2} \begin{pmatrix} 2v_1 w_1^2 + 2v_2 w_1 w_2 - v_1 w_1^2 - v_1 w_2^2 \\ 2v_1 w_1 w_2 + 2v_2 w_2^2 - v_2 w_1^2 - v_2 w_2^2 \end{pmatrix} \\
&= \frac{1}{w_1^2 + w_2^2} \begin{pmatrix} v_1(w_1^2 - w_2^2) + v_2(2w_1 w_2) \\ v_1(2w_1 w_2) + v_2(w_2^2 - w_1^2) \end{pmatrix} \\
&= \begin{pmatrix} v_1 \frac{(w_1^2 - w_2^2)}{w_1^2+w_2^2} + v_2 \frac{(2w_1 w_2)}{w_1^2+w_2^2} \\ v_1 \frac{(2w_1 w_2)}{w_1^2+w_2^2} + v_2 \frac{(w_2^2 - w_1^2)}{w_1^2+w_2^2} \end{pmatrix}
\end{aligned}$$

So $T_{\vec{w}}(\vec{v}) = \begin{pmatrix} v_1 \frac{(w_1^2 - w_2^2)}{w_1^2+w_2^2} + v_2 \frac{(2w_1 w_2)}{w_1^2+w_2^2} \\ v_1 \frac{(2w_1 w_2)}{w_1^2+w_2^2} + v_2 \frac{(w_2^2 - w_1^2)}{w_1^2+w_2^2} \end{pmatrix}$, and so it follows (by Proposition 3.1.8) that $T_{\vec{w}}(\vec{v})$ is a linear transformation and $[T_{\vec{w}}] = \begin{pmatrix} \frac{w_1^2 - w_2^2}{w_1^2+w_2^2} & \frac{2w_1 w_2}{w_1^2+w_2^2} \\ \frac{2w_1 w_2}{w_1^2+w_2^2} & \frac{w_2^2 - w_1^2}{w_1^2+w_2^2} \end{pmatrix}$

Problem 5: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\vec{v})$ be the result of first reflecting $\vec{v}$ across the x-axis, and then reflecting the result across the y-axis.

a. Compute $[T]$.

Solution: We define $X(\vec{v}) : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $X(\vec{v})$ be the result of reflecting $\vec{v}$ across the x-axis, and we define $Y(\vec{v}) : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $Y(\vec{v})$ be the result of reflecting $\vec{v}$ across the y-axis. Using our result from Problem 4, we can say that X and Y are both linear transformations, and $[X] = \begin{pmatrix} \frac{1^2-0^2}{1^2+0^2} & \frac{2\cdot1\cdot0}{1^2+0^2} \\ \frac{2\cdot1\cdot0}{1^2+0^2} & \frac{0^2-1^2}{1^2+0^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $[Y] = \begin{pmatrix} \frac{0^2-1^2}{0^2+1^2} & \frac{2\cdot0\cdot1}{0^2+1^2} \\ \frac{2\cdot0\cdot1}{0^2+1^2} & \frac{1^2-0^2}{0^2+1^2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. By the same reasoning used in Problem 1, we come to the conclusion that $[T] = [Y][X] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot -1 + 0\cdot 0 & 1\cdot 0 + 0\cdot -1 \\ 0\cdot -1 + -1\cdot 0 & 0\cdot 0 + -1\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

b. The action of T is the same as a certain rotation. Explain which rotation it is.

Solution: By Proposition 3.1.10, the standard matrix of a rotation of θ degrees counter-clockwise around the origin is $[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Suppose that $\theta = 180°$. So $[R_{180°}] = \begin{pmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = [T]$. So the action of T is the same as a $180°$ rotation about the origin.

Problem 6: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. We know from Proposition 2.4.8 that $r \cdot T$ is a linear transformation. Show that if

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

In other words, if we define the multiplication of a matrix by a scalar as in Definition 3.1.14, then the standard matrix of $r \cdot T$ is obtained by multiplying every element of $[T]$ by r.

Solution: Let $r \in \mathbb{R}, \vec{v} \in \mathbb{R}^2$ be arbitrary. We are given that $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Propositon 3.1.4, $T(\vec{v}) = [T]\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{v}$. Multiplying by r on both sides, we get $r \cdot T(\vec{v}) = r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{v} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \vec{v}$ (By Definition 3.2.3). It follows from Proposition 3.1.4 that $[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$.