Solutions to Written Assignment 7

Problem 1: No, this operation does *not* always preserve the solution set. Consider the following system:

$$\begin{array}{ccccc} x & - & y & = & 1 \\ x & + & y & = & 5. \end{array}$$

Suppose that we take R_2 and replace it by $R_2 + (-1) \cdot R_2$. We then obtain the following linear system:

$$\begin{array}{ccccc} x & - & y & = & 1 \\ & & 0 & = & 0. \end{array}$$

Notice that $(2,1)$ is a solution to this latter linear system because $2 - 1 = 1$ and $0 = 0$ trivially. On the other hand, $(2,1)$ is not a solution to the first linear system because $2 + 1 = 3$ and $3 \neq 5$ (so $(2,1)$ does not satisfy the second equation in the original system). Therefore, the solution sets of the two systems are different, and we have produced a specific counterexample to the statement.

Although not part of the problem, let me elaborate on one point. If we replace row i by the sum of itself and c times row i, then all that we have done is replace row i by $(c+1)$ times row i, i.e. we've multiplied row i by the constant $c+1$. Now *if* $c+1$ is nonzero, then this corresponds to performing an elementary row operation, so Corollary 4.2.5 applies to tell us that the solution set will be preserved. However, if $c + 1 = 0$, that is if $c = -1$, then we can not apply Corollary 4.2.5. Furthermore as the above counterexample demonstrates, then the solution set may not be preserved when $c = -1$.

Problem 2: Let $a, b, c \in \mathbb{R}$ be arbitrary. We can view each of our matrices as augmented matrices of certain linear systems. Let "System 1" be

$$\begin{array}{ccccc} 4x & + & 2y & = & 1 \\ ax & - & y & = & 0 \\ bx & + & cy & = & 3, \end{array}$$

and let "System 2" be

$$\begin{array}{ccccc} x & + & y & = & 2 \\ -2x & & & = & -1 \\ x & + & 3y & = & 5. \end{array}$$

and notice our first matrix is the augmented matrix of System 1, while our second matrix is the augmented matrix of System 2. Using Corollary 4.2.5, we know that if there is a sequence of elementary row operations turning our first matrix into our second, then System 1 and System 2 must have the same solution set. Thus, to show that that no such sequence exists, it suffices to show that System 1 and System 2 have different solution sets.

We now determine the solution set of System 2 using Gaussian Elimination. Applying elementary row operations to System 2, we obtain

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix} && \begin{matrix} \\ (2R_1 + R_2) \\ (-R_1 + R_3) \end{matrix} \\ &\to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} && \begin{matrix} \\ \\ (-R_2 + R_3). \end{matrix} \end{aligned}$$

The second line tells us that $y = \frac{3}{2}$, and using back-substitution into the first line, we see that $x + \frac{3}{2} = 2$, so $x = \frac{1}{2}$. Thus, $(\frac{1}{2}, \frac{3}{2})$ is the unique solution to System 2.

Let's examine whether $(\frac{1}{2}, \frac{3}{2})$ is a solution to System 1. We have

$$4 \cdot \frac{1}{2} + 2 \cdot \frac{3}{2} = 2 + 3 = 5.$$

Since $5 \neq 1$, we see that $(\frac{1}{2}, \frac{3}{2})$ is not a solution to System 1 because it does not satisfy the first equation (we don't even have to check the second and third equations because if it fails one equation then it is not a solution to the system). Since the two linear systems have different solution sets, there can be no sequence of elementary row operations turning the first matrix into the second.

Problem 3: Suppose that U and W are subspaces of V. To show that $U + W$ is a subspace of V, we need to verify the three properties.

- $\vec{0} \in U + W$: Notice that $\vec{0} \in U$ because U is a subspace of V and also $\vec{0} \in W$ because W is a subspace of V. Now $\vec{0} = \vec{0} + \vec{0}$, so since we have written $\vec{0}$ as the sum of an element of U with an element of W, we conclude that $\vec{0} \in U + W$.

- $U + W$ is closed under addition: Let $\vec{v}_1, \vec{v}_2 \in U + W$ be arbitrary. Since $\vec{v}_1 \in U + W$, we may fix $\vec{u}_1 \in U$ and $\vec{w}_1 \in W$ with $\vec{v}_1 = \vec{u}_1 + \vec{w}_1$. Similarly, since $\vec{v}_2 \in U + W$, we may fix $\vec{u}_2 \in U$ and $\vec{w}_2 \in W$ with $\vec{v}_2 = \vec{u}_2 + \vec{w}_2$. Using the commutative and associative laws of addition (which hold in our vector space V), we have

  $$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= (\vec{u}_1 + \vec{w}_1) + (\vec{u}_2 + \vec{w}_2) \\ &= (\vec{u}_1 + \vec{u}_2) + (\vec{w}_1 + \vec{w}_2). \end{aligned}$$

  Now we know that $\vec{u}_1 \in U$ and $\vec{u}_2 \in U$, so $\vec{u}_1 + \vec{u}_2 \in U$ because U is a subspace of V. Similarly, we know that $\vec{w}_1 \in W$ and $\vec{w}_2 \in W$, so $\vec{w}_1 + \vec{w}_2 \in W$ because W is a subspace of V. It follows that the above equation shows that $\vec{v}_1 + \vec{v}_2$ can be written as a sum of a vector in U and a vector in W. We conclude that $\vec{v}_1 + \vec{v}_2 \in U + W$.

- $U + W$ is closed under scalar multiplication: Let $\vec{v} \in U + W$ and $c \in \mathbb{R}$ be arbitrary. Since $\vec{v} \in U + W$, we can fix $\vec{u} \in U$ and $\vec{w} \in W$ with $\vec{v} = \vec{u} + \vec{w}$. Using Property 7 in our vector space V, we then have

  $$\begin{aligned} c \cdot \vec{v} &= c \cdot (\vec{u} + \vec{w}) \\ &= c \cdot \vec{u} + c \cdot \vec{w}. \end{aligned}$$

  Now we know that $\vec{u} \in U$, so $c \cdot \vec{u} \in U$ because U is a subspace of V. Similarly we know that $\vec{w} \in W$, so $c \cdot \vec{w} \in W$ because W is a subspace of V. It follows that the above equation shows that $\vec{v}$ can be written as a sum of a vector in U and a vector in W. We conclude that $c \cdot \vec{v} \in U + W$.

Therefore, $U + W$ is a subspace of V.