Worksheet 10 - Linear Transformations and Matrices

1 Warm-up

In the reading, we introduced a way to encode information about linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ using 2×2 matrices.

Exercise 1. Show that

$$Span\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = \mathbb{R}^2$$

using the *definition of span*.

Exercise 2. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, with

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\3\end{pmatrix}$$
$$T\left(\begin{pmatrix}2\\2\end{pmatrix}\right) = \begin{pmatrix}2\\6\end{pmatrix}$$

then what is $[T]$?

Exercise 3. For the standard basis, $\alpha = (\vec{e}_1, \vec{e}_2)$, what is $[Coord_\alpha]$?

STOP

Before moving on, check in with the other team of the same color, and compare answers. If your group is ahead, help them catch up.

2 Practice and Applications

Exercise 4. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformation with

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}-1\\1\end{pmatrix}$$
$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}$$

and

$$[S] = \begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}$$

then

(a) $T \circ S\left(\begin{pmatrix}1\\0\end{pmatrix}\right) =$

(b) $T \circ S\left(\begin{pmatrix}0\\1\end{pmatrix}\right) =$

Exercise 5. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformation with

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}a\\c\end{pmatrix}$$
$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}b\\d\end{pmatrix}$$

and

$$[S] = \begin{pmatrix}e & f\\g & h\end{pmatrix}$$

then

(a) $[T \circ S] =$

(b) $[S \circ T] =$

STOP

Before moving on, check in with the other team of the same color, and compare answers. If your group is ahead, help them catch up.

3 Check Your Understanding

Exercise 6. Is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which all of the following hold?:

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\-1\end{pmatrix}$$
$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix}$$
$$T\left(\begin{pmatrix}-1\\3\end{pmatrix}\right) = \begin{pmatrix}6\\2\end{pmatrix}$$

Explain your reasoning.

Exercise 7. Suppose a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$, geometrically, rotates points $\pi/2$ radians about the origin.

(a) $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) =$

(b) $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) =$

(c) What is the geometric effect of $T \circ T$? Explain.

(d) What is the geometric effect of $(T \circ T) \circ T$? Explain.

(e) What is $[((T \circ T) \circ T) \circ T]$?

STOP

Before moving on, check in with the other team of the same color, and compare answers. If your group is ahead, help them catch up.

For next time

- Complete this worksheet
- Finish reading Section 3.1
- Read Section 3.2, through the proof of Proposition 3.2.2.