Worksheet 22 - Test Your Understanding

Exercise 1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 5 & 4 \\ 0 & -2 & 1 & -3 & 0 \\ 0 & 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}$$

Which of the following statements are true? (Use precise terminology to explain your reasoning):

(a) If we regard A as an augmented matrix, encoding a system of linear equations, then that system is consistent.

(b) The columns of A are linearly independent.

Exercise 2. Consider the matrix

$$B = \begin{pmatrix} 2 & -3 & 1 & 0 & 6 \\ 0 & 6 & -1 & 7 & 4 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & -9 & 12 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Which of the following statements are true? (Use precise terminology to explain your reasoning):

(a) If we regard B as an augmented matrix, encoding a system of linear equations, then that system is consistent.

(b) The columns of B are linearly independent.

Exercise 3. Does there exist a matrix, C, such that both of the following statements are true? (Use precise terminology to explain your reasoning):

(a) If we regard C as an augmented matrix, encoding a system of linear equations, then that system is consistent.

(b) The columns of C are linearly independent.

Exercise 4. Does there exist a matrix, D, such that both of the following statements are false? (Use precise terminology to explain your reasoning):

(a) If we regard D as an augmented matrix, encoding a system of linear equations, then that system is consistent.

(b) The columns of D are linearly independent.

Exercise 5. How many distinct subspaces does $\mathbb{R}^3$ have? Explain.

Exercise 6. Suppose V is vector space, and $\vec{u}, \vec{v}, \vec{w} \in V$ such that the following subspaces are all distinct:

- $\mathrm{Span}(\vec{u}, \vec{v})$
- $\mathrm{Span}(\vec{u}, \vec{w})$
- $\mathrm{Span}(\vec{v}, \vec{w})$
- $\mathrm{Span}(\vec{u}, \vec{v}, \vec{w})$

What can you conclude about $\vec{u}, \vec{v}$, and $\vec{w}$? Justify your reasoning.

Exercise 7. Give a specific example of a system of four linear equations such that, in any echelon form of the associated augmented matrix, the last column has no leading entry and exactly one row has no leading entry. Then, find the solution set.

Exercise 8. Does there exists a sequence of six polynomials in $\mathcal{P}_4$ that is linearly independent? Prove that your answer is correct.

Exercise 9. Does there exists a sequence of four polynomials in $\mathcal{P}_4$ that span $\mathcal{P}_4$? Prove that your answer is correct.

Exercise 10. Suppose $p_0, p_1, p_2, \ldots, p_m$ are polynomials in $\mathcal{P}_m$ (note that the list starts at p_0). If $p_j(2) = 0$ for all $j \in \{0, 1, 2, \ldots, m\}$, then is $(p_0, p_1, p_2, \ldots, p_m)$ linearly independent? Prove that your answer is correct.