

Problem Set 21: Due Wednesday, May 2

Problem 1: Define $T: \mathcal{P}_2 \rightarrow \mathbb{R}^2$ by letting

$$T(f) = \begin{pmatrix} f(0) \\ f(2) \end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for \mathcal{P}_2 .

a. Let

$$\varepsilon_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

be the standard basis of \mathbb{R}^2 . What is $[T]_{\alpha}^{\varepsilon_2}$?

b. Let

$$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),$$

which is a basis of \mathbb{R}^2 . What is $[T]_{\alpha}^{\beta}$?

Problem 2: Let V be the vector space of all 2×2 matrices. Define $T: V \rightarrow V$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Notice that the function T takes an input matrix and outputs the result of switching the rows and columns (which is called the *transpose* of the original matrix). It turns out that T is a linear transformation. Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

and recall that α is a basis for V . What is $[T]_{\alpha}^{\alpha}$? Explain briefly.

Problem 3: Working in \mathbb{R}^4 , let

$$W = \text{Span} \left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ -7 \\ 0 \\ -3 \end{pmatrix} \right).$$

Find, with explanation, a basis for W and also $\dim(W)$.

Problem 4: Consider the unique linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ with

$$[T] = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix}.$$

- Find bases for each of $\text{range}(T)$ and $\text{Null}(T)$.
- Calculate $\text{rank}(T)$ and $\text{nullity}(T)$.

Problem 5: Define $T: \mathcal{P}_5 \rightarrow \mathcal{P}_5$ by letting $T(f) = f''$, i.e. $T(f)$ is the second derivative of f . Determine, with explanation, both $\text{rank}(T)$ and $\text{nullity}(T)$.