Worksheet 02 - Quantifiers and Negations and Implications

1 Warm-Up

Exercise 1. Using set notation, describe the set of all sums of all scalar multiples of the vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. What are the elements of this set?

Exercise 2. What makes *mathematical statements* different from other statements?

2 Quantifiers

Recall that a set is, simply, a collection of elements without regard to repetition or order. Quantifiers allow us to describe which elements in a set we care about. The two most common quantifiers are:

for all
there exists

We might claim that something is true **for all** elements of a set, or we might claim that **there exists** an element of a set with a particular property. There are many English language variations of such claims, so we must be vigilant when reading or writing about properties of sets or their elements.

2.1 For All

A common format for a **for all** statement is

> **For any** element, x, of the set X, we have that x exhibits *Property P*.

Example 1. Some 'for all' statements:

a) For all odd integers, x, we have that x^2 is an odd integer.

b) Given any integer, x, we have that $x^2 > x$.

c) All right triangles, with legs of length a and b, and hypotenuse of length c, we have that $a^2 + b^2 = c^2$.

d) Any pairs of integers with the same parity sums to an even integer.

2.2 There Exists

A common format of a **there exists** statement is

> **There exists** an element, x_0, of the set X with *Property P*.

Example 2. Some 'there exists' statements:

e) There exists an odd integer, x, with $x > 7$.

f) For some natural number, n, it is the case that $n^2 > n!$.

g) We can find a right triangle with all sides having odd integer length.

h) There is a continuous function that is not differentiable.

3 Negations

To *negate* a statement means to reverse its truth value. The simplest way to do this is to simply append the phrase, "It is not the case that…," to the beginning of the statement to be negated. Using this method, the negation of the statement

> For all integers that are divisible by three, we have that their square is a multiple nine.

is the statement

> It is not the case that for all integers that are divisible by three, we have that their square a multiple nine.

However, this method of negation, while logically correct in every case, does not often give us a new perspective on our original statement. Another way to state the negation of this statement is

> There exists an integer that is divisible by three whose square is not a multiple of nine.

Notice the appearance of "there exists" in the negation of a "for all" statement.

Exercise 3. Go back to Examples 1 and 2. Write the negation of each statement, without simply appending "It is not the case that…" How might this exercise help you determine the truth value of a given statement?

Exercise 4. Negate of the following:

a) **For any** element, x, of the set X, we have that x exhibits *Property P*.

b) **There exists** an element, x_0, of the set X where x_0 exhibits *Property P*.

STOP

4 Truth Tables and Compound Statements

In order to make meaningful progress in mathematics, we need to be able to construct, and establish the truth value of more and more intricate statements, or combinations of statements. A truth table helps us organize the truth values for different combinations of statements.

If we use A and B to represent be mathematical statements. All of the possible combinations of truth values for A and B are given by the truth table:

A	B
T	T
T	F
F	T
F	F

This isn't a very interesting truth table. Usually, our truth tables include some logical operator. The truth table for negation is:

A	Not A
T	F
F	T

4.1 Compound Statements

4.1.1 Logical Conjunctions

The words "and" and "or" hold special meaning when used to combine mathematical statements:

A	B	A and B	A or B
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Exercise 5. How is the logical conjunction "or" different from its interpretation in everyday conversation?

Exercise 6. Complete the following truth table:

A	B	Not (A and B)	(Not A) or (Not B)	A and (Not B)
T	T			
T	F			
F	T			
F	F			

STOP

4.1.2 Conditionals

Finally, one of our most used logical connectives is the "If...then...", or *conditional*, connective. We use the shorthand "$A \Rightarrow B$" to denote "If A then B." The truth table for this connective is:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise 7. Compare the truth table for "$A \Rightarrow B$" to the one in Exercise 6. What can you conclude about the negation of "$A \Rightarrow B$"?

Assignment for 1/26

- Complete this worksheet.
- Read Section 1.3