Solutions to Problem Set 2

Problem 1a: The negation of

$$\text{“For all } x \in \mathbb{R}\text{, we have } e^x \neq 0\text{”}$$

is

$$\text{“}\textbf{Not}(\text{For all } x \in \mathbb{R}\text{, we have } e^x \neq 0)\text{”}$$

which is the same as

$$\text{“There exists } x \in \mathbb{R} \text{ with } \textbf{Not } (e^x \neq 0)\text{”}$$

which is the same as

$$\text{“There exists } x \in \mathbb{R} \text{ with } e^x = 0\text{”.}$$

Problem 1b: The negation of

$$\text{“There exists } m, n \in \mathbb{Z} \text{ with } 4m + 6n = 7\text{”}$$

is

$$\text{“}\textbf{Not}(\text{There exists } m, n \in \mathbb{Z} \text{ with } 4m + 6n = 7)\text{”}$$

which is the same as

$$\text{“For all } x \in \mathbb{R} \text{ we have } \textbf{Not } (4m + 6n = 7)\text{”}$$

which is the same as

$$\text{“For all } m, n \in \mathbb{Z}\text{, we have } 4m + 6n \neq 7\text{”.}$$

Problem 1c: The negation of

$$\text{“There exists } x \in \mathbb{R} \text{ such that for all } y \in \mathbb{R}\text{, we have } x + y^2 \geq 3\text{”}$$

is

$$\text{“}\textbf{Not}(\text{There exists } x \in \mathbb{R} \text{ such that for all } y \in \mathbb{R}\text{, we have } x + y^2 \geq 3)\text{”}$$

which is the same as

$$\text{“For all } x \in \mathbb{R}\text{, we have } \textbf{Not}(\text{For all } y \in \mathbb{R}\text{, we have } x + y^2 \geq 3)\text{”}$$

which is the same as

$$\text{“For all } x \in \mathbb{R}\text{, there exists } y \in \mathbb{R} \text{ with } \textbf{Not}(x + y^2 \geq 3)\text{”}$$

which is the same as

$$\text{“For all } x \in \mathbb{R}\text{, there exists } y \in \mathbb{R} \text{ with } x + y^2 < 3\text{”.}$$

Problem 1d: The negation of

"For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ with both $3 < y - x$ and $x - y < 5$"

is

"**Not**(For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ with both $3 < y - x$ and $x - y < 5$)"

which is the same as

"There exists $y \in \mathbb{R}$ such that **Not**(There exists $x \in \mathbb{R}$ with both $3 < y - x$ and $x - y < 5$)"

which is the same as

"There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have **Not**($3 < y - x$ and $x - y < 5$)"

which is the same as

"There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have either $3 \geq y - x$ or $x - y \geq 5$ (or both)".

Notice that negating an *and* statement turned it into an *or* statement.

Problem 1e: The negation of

"There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$"

is

"**Not**(There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$)"

which is the same as

"For all $y \in \mathbb{R}$ we have **Not**(For all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}^+$ with $x^n + y > 0$)"

which is the same as

"For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that **Not**(There exists $n \in \mathbb{N}^+$ with $x^n + y > 0$)"

which is the same as

"For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}^+$, we have **Not**($x^n + y > 0$)"

which is the same as

"For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}^+$, we have $x^n + y \leq 0$".

Problem 2: Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can fix $n \in \mathbb{Z}$ with $a = 2n + 1$. Since b is odd, we can fix $k \in \mathbb{Z}$ with $b = 2k + 1$. Now notice that

$$\begin{aligned} a + b &= (2n + 1) + (2k + 1) \\ &= 2n + 2k + 2 \\ &= 2 \cdot (n + k + 1). \end{aligned}$$

Since $n + k + 1 \in \mathbb{Z}$, we conclude that $a + b$ is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

Problem 3: Let $a, b \in \mathbb{Z}$ be arbitrary with a even. Since a is even, we can fix $n \in \mathbb{Z}$ with $a = 2n$. Now notice that

$$\begin{aligned} ab &= (2n) \cdot b \\ &= 2 \cdot (nb). \end{aligned}$$

Since $nb \in \mathbb{Z}$, we conclude that ab is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

Problem 4: Let $a \in \mathbb{Z}$ be arbitrary. We have

$$\begin{aligned} 2a^3 + 6a - 3 &= 2a^3 + 6a - 4 - 1 \\ &= 2 \cdot (a^3 + 3a - 2) + 1. \end{aligned}$$

Since $a^3 + 3a - 2 \in \mathbb{Z}$, we conclude that $2a^3 + 6a - 3$ is odd. Since $a \in \mathbb{Z}$ was arbitrary, the result follows.

Problem 5a: The problem is that the number $\frac{8m+5}{2}$ might not actually be an element of $\mathbb{Z}$. In general, if we have an integer, the result of dividing it by 2 might not be an integer. For example, if $m = 1$, then $\frac{8m+5}{2} = \frac{13}{2} \notin \mathbb{Z}$. The argument does show that there exists $n \in \mathbb{Q}$ with $4a + 1 = 2n$, but the assertion that $n \in \mathbb{Z}$ does not follow.

Problem 5b: We claim that the statement in question is false. The statement is

"If $a \in \mathbb{Z}$ is odd, then $4a + 1$ is even"

which is really shorthand for

"For all $a \in \mathbb{Z}$, if a is odd, then $4a + 1$ is even".

Thus, the negation of our statement is

"There exists $a \in \mathbb{Z}$ such that a is odd but $4a + 1$ is not even".

To prove that this negation is true, it suffices to give just one example. Consider the case when $a = 1$ and notice that 1 is odd because $1 = 0 \cdot 1 + 1$. We then have that $4a + 1 = 4 \cdot 1 + 1 = 5$. Now $5 = 2 \cdot 2 + 1$, so 5 is odd, and hence 5 is not even by Proposition 1.4.5. Thus, we have verified that the negation of our statement is true, and hence the original statement is false.