Solutions to Written Assignment 8

Problem 1: Since $(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ is linearly independent, we can fix $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{0},$$

and such that at least one of the c_i is nonzero. Notice then that since

$$0\vec{w}_1 + 0\vec{w}_2 + \cdots + 0\vec{w}_m = \vec{0},$$

we have that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n + 0\vec{w}_1 + 0\vec{w}_2 + \cdots + 0\vec{w}_m = \vec{0}.$$

Now since at least one of the c_i is nonzero, it follows that at least one of the coefficients of the $n+m$ many vectors is nonzero, and hence $(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_m)$ is linearly dependent.

Problem 2: We need to show that $(\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w})$ is linearly independent. Let $c_1, c_2, c_3 \in \mathbb{R}$ be arbitrary with

$$c_1(\vec{u}+\vec{v}) + c_2(\vec{u}+\vec{w}) + c_3(\vec{v}+\vec{w}) = \vec{0}.$$

Using the vector space axioms, we then have

$$c_1\vec{u} + c_1\vec{v} + c_2\vec{u} + c_2\vec{w} + c_3\vec{v} + c_3\vec{w} = \vec{0},$$

and hence

$$(c_1+c_2)\vec{u} + (c_1+c_3)\vec{v} + (c_2+c_3)\vec{w} = \vec{0}.$$

We now have a linear combination of $(\vec{u}, \vec{v}, \vec{w})$ giving the vector $\vec{0}$, so since $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent, we conclude that $c_1 + c_2 = 0$, that $c_1 + c_3 = 0$, and that $c_2 + c_3 = 0$. In other words, c_1, c_2, c_3 satisfy the following three equations:

$$\begin{array}{ccccccc} c_1 & + & c_2 & & & = & 0 \\ c_1 & & & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0. \end{array}$$

We find all possible solutions to this linear system by applying elementary row operations to the corresponding augmented matrix:

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} && (-R_1 + R_2) \\ &\to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} && (R_2 + R_3). \end{aligned}$$

Since there is a leading entry in each non-augmented column, there is only one solution to this system, namely the trivial one when all three variables are 0. Since (c_1, c_2, c_3) is a solution, we must have that $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Thus, we have shown that whenever $c_1, c_2, c_3 \in \mathbb{R}$ satisfy

$$c_1(\vec{u}+\vec{v}) + c_2(\vec{u}+\vec{w}) + c_3(\vec{v}+\vec{w}) = \vec{0},$$

then we must have $c_1 = c_2 = c_3 = 0$. Therefore, $(\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w})$ is linearly independent.

Problem 3a: Let $V = \mathbb{R}^2$. Consider the example where $n = 2$ and $m = 2$, and we have

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \vec{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

Notice that $(\vec{u}_1, \vec{u}_2)$ is linearly independent because if $c_1, c_2 \in \mathbb{R}$ are such that $c_1\vec{u}_1 + c_2\vec{u}_2 = 0$, then we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and hence $c_1 = c_2 = 0$. For $(\vec{w}_1, \vec{w}_2)$, applying one elementary row operation to the corresponding matrix, we see that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \qquad (-R_1 + R_2).$$

Since this latter matrix in echelon form has a leading entry in each column, we may use Proposition 4.3.3 to conclude that $(\vec{w}_1, \vec{w}_2)$ is linearly independent.

Finally, notice that $(\vec{u}_1, \vec{u}_2, \vec{w}_1, \vec{w}_2)$ is linearly dependent by Corollary 4.3.5 because $4 > 2$.

Problem 3b: We need to show that the sequence $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ is linearly independent. Let $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_m \in \mathbb{R}$ be arbitrary with

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n + d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_m\vec{w}_m = \vec{0}.$$

Subtracting the latter m terms on the left from both sides of the equation, and then using the vector space axioms, we conclude that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = (-d_1)\vec{w}_1 + (-d_2)\vec{w}_2 + \cdots + (-d_m)\vec{w}_m$$

Now the left-hand side is a linear combination of the $\vec{u}_i$, so is an element of $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$. Similarly, the right-hand side is a linear combination of the $\vec{w}_i$, so is an element of $\text{Span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$. Since the two sides equal the same vector, it follows that both sides are elements of the set

$$\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) \cap \text{Span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m).$$

Since we are assuming that

$$\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) \cap \text{Span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m) = \{\vec{0}\}.$$

we conclude that both sides equal $\vec{0}$, i.e. we have both of the following:

$$\begin{aligned} c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n &= \vec{0} \\ (-d_1)\vec{w}_1 + (-d_2)\vec{w}_2 + \cdots + (-d_m)\vec{w}_m &= \vec{0}. \end{aligned}$$

Since $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ is linearly independent, we may use the first equation to conclude that $c_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. Since $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ is linearly independent, we may use the second equation to conclude that $-d_i = 0$ for all $i \in \{1, 2, \ldots, m\}$, and hence that $d_i = 0$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, all of the coefficients in our original equation must be 0. It follows that $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ is linearly independent.