Solutions to Problem Set 15

Problem 1: No, W is not a subspace of $\mathcal{P}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the polynomial function $f(x) = x^2 + 1$. Notice that $f \in W$ because its constant term is nonnegative. Now $(-1) \cdot f$ is the polynomial function and notice that $((-1) \cdot f)(x) = (-1) \cdot (x^2 + 1) = -x^2 - 1$. Since -1 is not nonnegative, we have that $(-1) \cdot f \notin W$. Therefore, W Is not closed under scalar multiplication, and hence W is not a subspace of $\mathcal{P}$.

Problem 2: We have that

$$\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}\right)$$

if and only if there exists $c_1, c_2, c_3 \in \mathbb{R}$ with

$$c_1 \cdot \begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix},$$

which is the same as saying that there exists $c_1, c_2, c_3 \in \mathbb{R}$ with

$$\begin{pmatrix} 2c_1 + 6c_2 \\ -5c_1 + c_2 + 3c_3 \\ c_1 - 8c_2 + 3c_3 \\ 4c_2 + 2c_2 + 3c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix}.$$

Therefore, we have that

$$\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}\right)$$

if and only if the system

$$\begin{array}{rcrcrcr} 2x & + & 6y & & & = & 1 \\ -5x & + & y & + & 3z & = & 7 \\ x & - & 8y & + & 3z & = & 0 \\ 4x & + & 2y & + & 3z & = & 6 \end{array}$$

has a solution.

Problem 3: We want to know whether there exists $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix},$$

i.e. whether there exists $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 & c_1 \\ 2c_1 + 5c_2 & -3c_1 - 4c_2. \end{pmatrix}$$

Notice that, from the $(1,2)$ entry, the only possibility is $c_1 = 7$. Looking at the $(1,1)$ entry, we must have $c_1 + 3c_2 = -2$, so $7 + 3c_2 = -2$, hence $3c_2 = -9$, and so we must have $c_2 = -3$. We now check

$$\begin{aligned} 7 \cdot \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} + (-3) \cdot \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix} &= \begin{pmatrix} 7 & 7 \\ 14 & -21 \end{pmatrix} + \begin{pmatrix} -9 & 0 \\ -15 & 12 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix} \right).$$

Problem 4a: We claim that $g_1 \in W$. To see this, we need to show that there exists $c_1, c_2 \in \mathbb{R}$ with $g_1 = c_1 f_1 + c_2 f_2$. Now for any $x \in \mathbb{R}$, we have

$$\begin{aligned} (3f_1 + 3f_2)(x) &= 3 \cdot f_1(x) + 3 \cdot f_2(x) && \text{(by definition)} \\ &= 3 \cdot \sin^2 x + 3 \cdot \cos^2 x \\ &= 3 \cdot (\sin^2 x + \cos^2 x) \\ &= 3 \cdot 1 \\ &= 3 \\ &= g_1(x). \end{aligned}$$

Since $g_1(x) = (3f_1 + 3f_2)(x)$ for all $x \in \mathbb{R}$, we conclude that $g = 3f_1 + 3f_2$. Therefore, $g_1 \in W$.

Problem 4b: We claim that $g_2 \in W$. We argue this using a proof by contradiction. Suppose instead that $g_2 \in W$, and fix $c_1, c_2 \in \mathbb{R}$ with $g_2 = c_1 f_1 + c_2 f_2$. Since these two functions are equal, we must have

$$g_2(x) = c_1 f_1(x) + c_2 f_2(x)$$

for all $x \in \mathbb{R}$, and hence

$$x^2 = c_1 \sin^2 x + c_2 \cos^2 x$$

for all $x \in \mathbb{R}$. Since this is true for all $x \in \mathbb{R}$, it must be true whenever we plug in a specific $x \in \mathbb{R}$. Plugging in $x = 0$, we have

$$0^2 = c_1 \cdot 0^2 + c_2 \cdot 1^2,$$

hence $c_2 = 0$. Plugging in $x = \pi$, we have

$$\pi^2 = c_1 \cdot 0^2 + c_2 \cdot (-1)^2,$$

hence $c_2 = \pi^2$. Therefore, we conclude that $0 = \pi^2$, which is a contradiction. It follows that $g_2 \notin W$.

Problem 4c: We claim that $g_3 \notin W$. We argue this using a proof by contradiction. Suppose instead that $g_3 \in W$, and fix $c_1, c_2 \in \mathbb{R}$ with $g_3 = c_1 f_1 + c_2 f_2$. Since these two functions are equal, we must have

$$g_3(x) = c_1 f_1(x) + c_2 f_2(x)$$

for all $x \in \mathbb{R}$, and hence

$$\sin x = c_1 \sin^2 x + c_2 \cos^2 x$$

for all $x \in \mathbb{R}$. Since this is true for all $x \in \mathbb{R}$, it must be true whenever we plug in a specific $x \in \mathbb{R}$. Plugging in $x = \frac{\pi}{2}$, we have

$$1 = c_1 \cdot 1^2 + c_2 \cdot 0^2,$$

hence $c_1 = 1$. Plugging in $x = \frac{3\pi}{2}$, we have

$$-1 = c_1 \cdot (-1)^2 + c_2 \cdot 0^2,$$

hence $c_1 = -1$. Therefore, we conclude that $1 = -1$, which is a contradiction. It follows that $g_3 \notin W$.

Problem 4d: We claim that $g_4 \in W$. To see this, we need to show that there exists $c_1, c_2 \in \mathbb{R}$ with $g_4 = c_1 f_1 + c_2 f_2$. Now for any $x \in \mathbb{R}$, we can use the double-angle identity

$$\cos 2x = \cos^2 x - \sin^2 x$$

to conclude that

$$\begin{aligned}((-1) \cdot f_1 + 1 \cdot f_2)(x) &= (-1) \cdot f_1(x) + 1 \cdot f_2(x) && \text{(by definition)} \\ &= (-1) \cdot \sin^2 x + 1 \cdot \cos^2 x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \\ &= g_4(x).\end{aligned}$$

Since $g_4(x) = ((-1) \cdot f_1 + 1 \cdot f_2)(x)$ for all $x \in \mathbb{R}$, we conclude that $g = (-1) \cdot f_1 + 1 \cdot f_2$. Therefore, $g_4 \in W$.

Problem 5: If W is a subspace of V, then we claim that $V \backslash W$ is never a subspace of V. To see this, let W be an arbitrary subspace of a vector space V. Since W is a subspace of V, we know that $\vec{0} \in W$. It follows that $\vec{0} \notin V \backslash W$ by definition. Since $\vec{0}$ is not an element of $V \backslash W$, we know that $V \backslash W$ is not a subspace of V by definition.

Problem 6: We are working in the vector space $\mathcal{F}$ of all functions $f \colon \mathbb{R} \to \mathbb{R}$ and letting

$$W = \{f \in \mathcal{F} : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$$

be the set of even functions. We claim that W is a subspace of $\mathcal{F}$, and to show this, we need to verify three properties.

- $\vec{0} \in W$: Recall that in $\mathcal{F}$, the vector $\vec{0}$ is the function $z \colon \mathbb{R} \to \mathbb{R}$ given by $z(x) = 0$. Now for all $x \in \mathbb{R}$, we have $z(-x) = 0 = z(x)$. Therefore, $z \in W$, which is to say that $\vec{0} \in W$.
- W is closed under addition: Let $f, g \in W$ be arbitrary. We then have $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and $g(-x) = g(x)$ for all $x \in \mathbb{R}$. Now for any $x \in \mathbb{R}$, we have

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) && \text{(by definition)} \\ &= f(x) + g(x) && \text{(since } f \text{ and } g \text{ are even)} \\ &= (f+g)(x) && \text{(by definition).}\end{aligned}$$

  Thus, $(f+g)(-x) = (f+g)(x)$ for all $x \in \mathbb{R}$, so $f + g \in W$. Therefore, W is closed under addition.

- W is closed under scalar multiplication: Let $f \in W$ and $c \in \mathbb{R}$ be arbitrary. We then have $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Now for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
(c \cdot f)(-x) &= c \cdot f(-x) && \text{(by definition)} \\
&= c \cdot f(x) && \text{(since } f \text{ is even)} \\
&= (c \cdot f)(x).
\end{aligned}$$

  Thus, $(c \cdot f)(-x) = c \cdot f(x)$ for all $x \in \mathbb{R}$, so $c \cdot f \in W$. Therefore, W is closed under scalar multiplication.

It follows that W is a subspace of $\mathcal{F}$.