16 Lecture 16: Double pendulum

A good example of normal modes, and of the energy conservation approach we discussed in the previous lecture is that of the double pendulum. A double pendulum consists of two particles, of masses m_1 and m_2, connected by two light rods of lengths a_1 and a_2, respectively, which are connected as shown in figure 14.

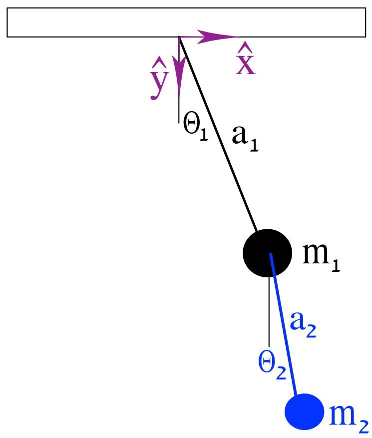

Figure 14: Double pendulum suspended from a fixed point in the ceiling. The axes at the upper end of each rod allow for frictionless rotation in the plane. The upper mass is m_1, the lower is m_2. The lengths of the rods are a_1 (upper) and a_2 (lower). The motion will be described in terms of the angles to the vertical line θ_1 for the upper and θ_2 for the lower.

Potential Energy

We would like to analyse this system using energy conservation. Let us begin by writing down the kinetic and potential energies associated with each of the two masses. To this end it is useful to first define Euclidean coordinates, with the origin located at the axis of the upper pendulum, as indicated in figure 14. The position of m_1 and m_2 are then:

$$\begin{aligned} x_1 &= a_1 \sin(\theta_1) \\ y_1 &= a_1 \cos(\theta_1) \end{aligned} \tag{257}$$

and

$$\begin{aligned} x_2 &= a_1 \sin(\theta_1) + a_2 \sin(\theta_2) \\ y_2 &= a_1 \cos(\theta_1) + a_2 \cos(\theta_2) \end{aligned} \tag{258}$$

Next we write the potential energy due to gravity. Given that the direction $\hat{y}$ is downwards

$$V = -m_1 g\, y_1 - m_2 g\, y_2 \tag{259}$$

So in terms of the angles we get:

$$V(\theta_1, \theta_2) = -m_1 g\, a_1 \cos(\theta_1) - m_2 g \Big(a_1 \cos(\theta_1) + a_2 \cos(\theta_2) \Big) \tag{260}$$

In the small angle approximation the potential is:

$$\begin{aligned} V(\theta_1, \theta_2) &= -m_1 g\, a_1 - m_2 g \Big(a_1 + a_2 \Big) + m_1 g\, a_1 \frac{\theta_1^2}{2} + m_2 g \left(a_1 \frac{\theta_1^2}{2} + a_2 \frac{\theta_2^2}{2} \right) \\ &= V_0 + \frac{1}{2} g \Big[(m_1 + m_2) a_1 \theta_1^2 + m_2 a_2 \theta_2^2 \Big] \\ &= V_0 + \frac{1}{2} g\, \underline{\theta}^T \begin{pmatrix} a_1(m_1 + m_2) & 0 \\ 0 & a_2 m_2 \end{pmatrix} \underline{\theta} \end{aligned} \tag{261}$$

where we introduced the vector notation $\underline{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$.

Kinetic Energy

Turning now to the kinetic terms, differentiating (262) and (263) we get:

$$\begin{aligned} \dot{x}_1 &= a_1 \cos(\theta_1)\,\dot{\theta}_1 \\ \dot{y}_1 &= -a_1 \sin(\theta_1)\,\dot{\theta}_1 \end{aligned} \tag{262}$$

and

$$\begin{aligned} \dot{x}_2 &= a_1 \cos(\theta_1)\,\dot{\theta}_1 + a_2 \cos(\theta_2)\,\dot{\theta}_2 \\ \dot{y}_2 &= -a_1 \sin(\theta_1)\,\dot{\theta}_1 - a_2 \sin(\theta_2)\,\dot{\theta}_2 \end{aligned} \tag{263}$$

So

$$T_1 = \frac{m_1}{2}\left[a_1^2 \cos^2(\theta_1)\,\dot{\theta}_1^2 + a_1^2 \sin^2(\theta_1)\,\dot{\theta}_1^2\right] = \frac{m_1}{2} a_1^2\,\dot{\theta}_1^2 \tag{264}$$

and

$$\begin{aligned} T_2 &= \frac{m_2}{2}\left[\Big(a_1 \cos(\theta_1)\,\dot{\theta}_1 + a_2 \cos(\theta_2)\,\dot{\theta}_2\Big)^2 + \Big(-a_1 \sin(\theta_1)\,\dot{\theta}_1 - a_2 \sin(\theta_2)\,\dot{\theta}_2\Big)^2\right] \\ &= \frac{m_2}{2}\left[a_1^2\dot{\theta}_1^2 + a_2^2\dot{\theta}_2^2 + 2a_1a_2\dot{\theta}_1\dot{\theta}_2\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\Big)\right] \\ &= \frac{m_2}{2}\left[a_1^2\dot{\theta}_1^2 + a_2^2\dot{\theta}_2^2 + 2a_1a_2\dot{\theta}_1\dot{\theta}_2\,\cos(\theta_1 - \theta_2)\right] \end{aligned} \tag{265}$$

In the small angle approximation we get:

$$T_2 = \frac{m_2}{2}\left[a_1^2\dot{\theta}_1^2 + a_2^2\dot{\theta}_2^2 + 2a_1a_2\dot{\theta}_1\dot{\theta}_2\right] \tag{266}$$

so now the kinetic term is quadratic in the angular velocities, and does not depend of the angles themselves. This greatly simplifies the problem. The total kinetic energy in this approximation is then:

$$\begin{aligned} T &= \frac{m_1}{2} a_1^2\,\dot{\theta}_1^2 + \frac{m_2}{2}\left[a_1^2\dot{\theta}_1^2 + a_2^2\dot{\theta}_2^2 + 2a_1a_2\dot{\theta}_1\dot{\theta}_2\right] \\ &= \frac{1}{2}\,\underline{\dot{\theta}}^T \begin{pmatrix} (m_1+m_2)a_1^2 & m_2a_1a_2 \\ m_2a_1a_2 & m_2a_2^2 \end{pmatrix} \underline{\dot{\theta}} \end{aligned} \tag{267}$$

The total energy and the equation of motion

We deduce that the total energy of the double pendulum in the small angle approximation is:

$$E = T + V = \frac{1}{2}\,\underline{\dot{\theta}}^T \underbrace{\begin{pmatrix} (m_1+m_2)a_1^2 & m_2a_1a_2 \\ m_2a_1a_2 & m_2a_2^2 \end{pmatrix}}_{\hat{T}} \underline{\dot{\theta}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\theta}^T \underbrace{\begin{pmatrix} a_1^2(m_1+m_2) & 0 \\ 0 & a_1a_2m_2 \end{pmatrix}}_{\hat{V}} \underline{\theta} \tag{268}$$

where we defied the energy with respect to the equilibrium point (thus V_0 does not appear). We observe that as before we have a quadratic form in the coordinates (here angles) in the potential term, and a quadratic form in the velocities (here angular velocities) in the kinetic term:

$$E = \frac{1}{2}\underline{\dot{\theta}}^T\,\hat{T}\,\underline{\dot{\theta}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\theta}^T\,\hat{V}\,\underline{\theta}$$

What is different in this example compared to the ones we have analysed before, is that the kinetic term is more complicated than the potential one. The former is non-diagonal while the latter is diagonal.

The equations of motion can be obtained as usual by the requirement that the energy is conserved, $\frac{dE}{dt} = 0$. This yields

$$\hat{T}\,\underline{\ddot{\theta}} + \frac{g}{a_1}\hat{V}\,\underline{\theta} = 0$$

This equation can of course be brought to the standard form $\underline{\ddot{\theta}} + G\,\underline{\theta} = 0$ by multiplying by $\hat{T}^{-1}$. One can then diagonalize G and find the normal modes.

Diagonalising the energy

We shall proceed in a different way. Instead of using the equation of motion we can work directly with the expression for the energy and bring it to a diagonal form. Since the potential term is diagonal we can rescale the angles such that

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{a_1^2(m_1+m_2)} & 0 \\ 0 & \sqrt{a_1 a_2 m_2} \end{pmatrix}}_{\hat{V}^{1/2}} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{269}$$

In terms of $\underline{\alpha}$ the total energy is:

$$E = T + V = \frac{1}{2}\,\underline{\dot{\alpha}}^T \underbrace{\begin{pmatrix} 1 & \sqrt{\frac{m_2}{m_1+m_2}}\sqrt{a_2/a_1} \\ \sqrt{\frac{m_2}{m_1+m_2}}\sqrt{a_2/a_1} & a_2/a_1 \end{pmatrix}}_{A} \underline{\dot{\alpha}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\alpha}^T\,\underline{\alpha} \tag{270}$$

We observe that the matrix A depends just on two dimensionless ratios:

$$\mu \equiv \sqrt{\frac{m_2}{m_1+m_2}} \qquad r \equiv \sqrt{a_2/a_1}$$

So we have:

$$E = T + V = \frac{1}{2}\,\underline{\dot{\alpha}}^T A\,\underline{\dot{\alpha}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\alpha}^T\,\underline{\alpha} \tag{271}$$

with

$$A = \begin{pmatrix} 1 & \mu r \\ \mu r & r^2 \end{pmatrix} \tag{272}$$

Since A is a real symmetric matrix, we can diagonalise it by an orthogonal transformation:

$$P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{273}$$

where $P^{-1} = P^T$. This implies that $A = P\Lambda P^T$, so in terms of Λ our expression for the energy takes the form:

$$\begin{aligned} E = T + V &= \frac{1}{2}\,\underline{\dot{\alpha}}^T P\Lambda P^T\,\underline{\dot{\alpha}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\alpha}^T\,\underline{\alpha} \\ &= \frac{1}{2}\,\underline{\dot{\beta}}^T\,\Lambda\,\underline{\dot{\beta}} + \frac{1}{2}\frac{g}{a_1}\,\underline{\beta}^T\,\underline{\beta} \end{aligned} \tag{274}$$

where we defined the normal coordinates by $\underline{\beta} = P^T\underline{\alpha}$ or $\underline{\alpha} = P^T\underline{\beta}$. This illustrates that normal coordinates can be found directly from the expression for the energy.

As usual, the equation of motion for the normal coordinates are independent simple harmonic oscillators. They can be obtained by differentiating (274) with respect to time and requiring $\frac{dE}{dt} = 0$:

$$\Lambda\underline{\ddot{\beta}} + \frac{g}{a_1}\underline{\beta} = 0$$

or

$$\lambda_j\,\ddot{\beta}_j + \frac{g}{a_1}\beta_j = 0$$

whose solution is

$$\beta_j = \gamma_j \cos\left(t\sqrt{\frac{g}{a_1\lambda_j}} - \phi_j\right) \tag{275}$$

where γ_j and ϕ_j are free parameters to be determined by the initial conditions.

Let us then determine the eigenvalues and eigenvectors of Λ. Based on (273)

$$(A - \lambda I)\underline{p} = \underline{0} \tag{276}$$

or

$$\begin{pmatrix} 1-\lambda & \mu r \\ \mu r & r^2-\lambda \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{277}$$

The characteristic equation is:

$$(1-\lambda)(r^2-\lambda) - \mu^2 r^2 = 0 \qquad \Longrightarrow \qquad \lambda^2 - \lambda(1+r^2) + r^2(1-\mu^2) = 0$$

The solutions are:

$$\lambda_{1,2} = \frac{1+r^2 \pm \sqrt{(1+r^2)^2 - 4r^2 + 4r^2\mu^2}}{2} = \frac{1+r^2 \pm \sqrt{(1-r^2)^2 + 4r^2\mu^2}}{2}$$

We see that the eigenvalues simplify in the case where the lengths are the same, $a_2 = a_1 = a$. We then have $r = 1$ so

$$\lambda_{1,2} = 1 \pm \mu$$

Let us then continue and find the eigenvectors in this case: for $\lambda_1 = 1+\mu$ we get:

$$\begin{pmatrix} -\mu & \mu \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \underline{p}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{278}$$

for $\lambda_2 = 1-\mu$ we get:

$$\begin{pmatrix} \mu & \mu \\ \mu & \mu \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \underline{p}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{279}$$

The corresponding matrix P is:

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{280}$$

Note that in these coordinates the eigenvectors are independent of the mass ratio μ (this is special to case $a_1 = a_2$).

The solutions are then:

$$\beta_1 = \gamma_1 \cos\left(t\sqrt{\frac{g}{a(1+\mu)}} - \phi_1\right) \qquad \beta_2 = \gamma_2 \cos\left(t\sqrt{\frac{g}{a(1-\mu)}} - \phi_2\right) \tag{281}$$

Returning to the original coordinates $\underline{\alpha} = P^T \underline{\beta}$ this gives:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta_1 + \beta_2 \\ \beta_1 - \beta_2 \end{pmatrix} \tag{282}$$

and finally using (283) we get the explicit solution in the $\underline{\theta}$ coordinates:

$$\underline{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{a} \begin{pmatrix} \sqrt{1/(m_1+m_2)} & 0 \\ 0 & \sqrt{1/m_2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{283}$$

$$= \frac{1}{\sqrt{2}\, a\sqrt{m_1+m_2}} \begin{pmatrix} \left(\gamma_1 \cos\left(t\sqrt{\frac{g}{a(1+\mu)}} - \phi_1\right) + \gamma_2 \cos\left(t\sqrt{\frac{g}{a(1-\mu)}} - \phi_2\right)\right) \\ \frac{1}{\mu}\left[\gamma_1 \cos\left(t\sqrt{\frac{g}{a(1+\mu)}} - \phi_1\right) - \gamma_2 \cos\left(t\sqrt{\frac{g}{a(1-\mu)}} - \phi_2\right)\right] \end{pmatrix} \tag{284}$$