If $\mathrm{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$

then $\vec{u}_1$ and $\vec{u}_2$ are not zero and $\vec{u}_2 \notin \mathrm{Span}(\vec{u}_1)$

If $\vec{u}_1 = \vec{0}$ or $\vec{u}_2 = 0$ or $\vec{u}_2 \in \mathrm{Span}(u_1)$ then

$\mathrm{span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2$

Case 1: $\vec{u}_1 = 0$

or

Case 2: $\vec{u}_2 = 0$

or

Case 3: $\vec{u}_2 \in \mathrm{Span}(u_1)$

Goal

$\mathrm{span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2.$

The benefit of the contrapositive in this instance is that we can treat each "or" case separately.

Case 1

If $\vec{u}_1 = \vec{0}$ then $\text{Span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2$

By def of $\text{Span}(\vec{u}_1, \vec{u}_2)$ we have $\{c_1\vec{u} + c_2\vec{u} : c_1, c_2 \in \mathbb{R}\}$

$= \{c_1 \cdot \vec{0} + c_2\vec{u} : c_1, c_2 \in \mathbb{R}\}$

by prop 1.5 we get

$= \{c_2\vec{u}_2 : c_2 \in \mathbb{R}\}$

$= \text{Span}(\vec{u}_2)$

by prop 3 $\text{Span}(\vec{u}_2) \neq \mathbb{R}^2$

Case 2

If $\vec{u}_2 = \vec{0}$ then $\mathrm{Span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2$

By 2.1 $\vec{0} \in \mathrm{Span}(\vec{u}_1, \cancel{\vec{u}_2})$

$\vec{u}_2 = \vec{0} \Rightarrow \vec{u}_2 \in \mathrm{Span}(\vec{u}_1, \cancel{\vec{u}_2})$

$\vec{u}_2 = \vec{0} \Rightarrow \vec{u}_2 \in \mathrm{Span}(\vec{u}_1)$

Case 3

If $\vec{u}_2 \in \text{Span}(\vec{u}_1)$ then $\text{Span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2$

By prop 6. $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$

and prop 3 says $\text{Span}(\vec{u}_1) \neq \mathbb{R}^2$