Worksheet 11 - The Range of a Linear Transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

1 Recap

1.1 Vectors

- Our study of linear algebra began by thinking of *vectors in* $\mathbb{R}^2$ as the fundamental building blocks for giving directions in two-dimensional space.
- These directions come in the form of *linear combinations* of vectors, via vector addition and scalar multiplication.
- This led to notion of the *span* of a vector, or pair of vectors, gave us conditions under which we can give directions to anywhere in $\mathbb{R}^2$.
- Being able to use different pairs of vectors to give directions to anywhere in $\mathbb{R}^2$ led to the notion of different *coordinate* systems.

1.2 Linear Transformations

- *Linear transformations* are "nice" functions between "different" $\mathbb{R}^2$'s.
- They are "nice" because they act predictably with linear combinations of vectors, i.e.

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$$

- This predictability and preservation of structure results in the property that: *every linear transformation is completely determined by where it sends a pair of basis vectors in its domain.* This is established by Theorem 2.4.5 and Theorem 2.4.6.

1.3 Matrices

The most important thing to remember when working matrices is that matrices are *not an entirely new structure*, they provide shorthand that encodes all of the information we need to define and apply linear transformations.

2 The Range of a Linear Transformation

Now that we have "nice" functions that will take us from "one $\mathbb{R}^2$ to another $\mathbb{R}^2$", we want to know how the choices we make when defining these functions influences the output we can expect.

Exercise 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$[T] = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

Prove that $\text{range}(T) = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$

Exercise 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Prove that $\text{range}(T) = \text{Span}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$

Exercise 3. Given the result you proved in Exercise 2, the range of any liner transformation can be completely described by where it sends the standard basis. Give an example of a linear transformation whose range is:

(a) $\text{Span}\left(\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right)$

(b) $\mathbb{R}^2$

(c) $\{\vec{0}\}$

STOP

Before moving on, check in with the other team of the same color, and compare answers. If your group is ahead, help them catch up.

3 Functions and Inverses

Before proceeding, it will be helpful to establish a few facts about functions, in general.

Definition. Let $f : A \to B$ be a function.

- A *left inverse* for f is a function $g : B \to A$ such that $g \circ f = id_A$.
- A *right inverse* for f is a function $g : B \to A$ such that $f \circ b = id_B$.
- An *inverse* for f is a function $g : B \to A$ that is both a left and right inverse.

Exercise 4. Let $f : A \to B$ be a function. Prove that f is injective if and only if there exists a left inverse for f.

Exercise 5. Let $f : A \to B$ be a function. Prove that f is surjective if and only if there exists a right inverse for f.

Exercise 6. Let $f : A \to B$ be a function. Prove that f is a bijection if and only if there exists an inverse for f.(Hint: Explain why your proofs for Exercises 4 and 5 guarantee that this is true.)

The following establishes that, if an inverse exists, then it is unique.

Exercise 7. Let $f : A \to B$ be a function. Prove that, if g is a left inverse for f, and h is a right inverse for f, then $g = h$. (Hint: Consider Proposition 1.6.5.)

STOP

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4 Applications and Practice

Exercise 8. Let $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and consider the linear transformation $P_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$. Which vectors in the domain are sent to $\vec{0}$ by $P_{\vec{w}}$? Which vectors are sent to a scalar multiple of themselves by $P_{\vec{w}}$?

Exercise 9. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $\text{range}(T) = \text{Span}(\vec{v})$ for some non-zero $\vec{v} \in \mathbb{R}^2$. Is it the case that T has the geometric effect of projection onto some vector? If so, prove it. If not, give a counterexample.

Exercise 10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$[T] = \begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix}$$

What is the area of the quadrilateral with vertices $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, and $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$?

Exercise 11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$[T] = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

What is the area of the quadrilateral with vertices $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, and $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$?

Exercise 12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with

$$[T] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Make a conjecture about the area of the quadrilateral with vertices $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, and $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, based on your observations from Exercises 10 and 11.

For next time

- Complete this worksheet
- Finish reading Section 3.2
- Read Section 3.3, through the examples leading up to Theorem 3.3.3.

5 Optional Challenge

Exercise 13. Can you define a function with a left inverse, but no right inverse? How about a right inverse, but no left inverse?

Exercise 14. How would you generalize our definitions of the following to $\mathbb{R}^3$?:

- linear combination
- span
- linear transformation
- basis

Exercise 15. How might you generalize the following results to $\mathbb{R}^3$?:

(a) Proposition 2.3.6

(b) Proposition 2.3.8

(c) Theorem 2.3.10 (Hint: for the fourth statement, you probably have to complete Exercise 16)

(d) Theorem 2.4.5

(e) Theorem 2.4.6

Exercise 16. How might we modify Proposition 2.1.1. so that it applies to $\mathbb{R}^3$? (This seems tedious and terrible, but sometimes that can't be avoided.)