Assignment: Problem Set 5

Name: Oleksandr Yardas

Due Date: 02/12/2018

List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 8x$. Show that f is not injective

Solution: By the definition of injective, for all $a, b \in \mathbb{R}$, if $f(a) = f(b)$, then $a = b$. The contrapositive of this statement: For all $a, b \in \mathbb{R}$, if $a \neq b$, then $f(a) \neq f(b)$. We assume f is injective, so the original statement is true, and it follows that the contrapositive is true. Let $a = 0, b = 8^{\frac{1}{2}}$. So $a \neq b$, and we have

$$\begin{aligned} a^3 - 8a &= (0)^3 - 8(0) & \qquad b^3 - 8b &= (8^{\frac{1}{2}})^3 - 8(8^{\frac{1}{2}}) \\ &= 0 - 0 & &= (8^{\frac{3}{2}} - 8^{\frac{3}{2}}) \\ &= 0 & &= 0 \end{aligned}$$

We conclude that $f(0) = f((8^{\frac{1}{2}})$. We have found an $a, b \in \mathbb{R}$ with $a \neq b$ and $f(a) = f(b)$. However, it must be the case that $f(a) \neq f(b)$ (by the contrapositive). Our assumption has lead to a logical contradiction, so it must be the case that the contrapositive is false, and it follows that the original statement is false. Therefore, f is not injective.

Problem 2: Determine if the three lines $2x + y = 5$, $7x - 2y = 1$, and $-5x + 3y = 4$ intersect. Explain your reasoning using a few sentences.

Solution: If the three lines intersect, then their solution sets will be the same. This is because the solution set of two distinct lines is a point, so if the solution sets of each pair of lines is the same, they all intersect at the point that is the singe element in their solution sets. By proposition 2.1.1, the solution set S of the linear equations of the form

$$ax + by = j$$

$$cx + dy = k$$

is

$$S = \left\{ \left(\frac{dj - bk}{ad - bc}, \frac{ak - cj}{ad - bc} \right) \right\}$$

as long as $ad - bc \neq 0$. We compute each solution set:

$$\begin{aligned} 2x + y &= 5 \\ 7x - 2y &= 1 \end{aligned} \qquad \begin{aligned} 7x - 2y &= 1 \\ -5x + 3y &= 4 \end{aligned} \qquad \begin{aligned} -5x + 3y &= 4 \\ 2x + y &= 5 \end{aligned}$$

Let their solution set be S_1 Let their solution set be S_2 Let their solution set be S_3

We get:

$$S_1 = \left\{ \left(\frac{-2 \cdot 5 - 1 \cdot 1}{(2 \cdot (-2) - 1 \cdot 7}, \frac{2 \cdot 1 - 7 \cdot 5}{2 \cdot (-2) - 1 \cdot 7} \right) \right\}$$

$$S_2 = \left\{ \left(\frac{3 \cdot 1 - (-2) \cdot 4}{7 \cdot 3 - (-2) \cdot (-5)}, \frac{7 \cdot 4 - (-5) \cdot 1}{7 \cdot 3 - (-2) \cdot (-5)} \right) \right\}$$

$$S_3 = \left\{ \left(\frac{1 \cdot 4 - 3 \cdot 5}{-5 \cdot 1 - 3 \cdot 2}, \frac{-5 \cdot 5 - 2 \cdot 4}{-5 \cdot 1 - 3 \cdot 2} \right) \right\}$$

which become

$$\begin{aligned} S_1 &= \left\{ \left(\frac{-10 - 1}{-4 - 7}, \frac{2 - 35}{-4 - 7} \right) \right\} \\ &= \left\{ \left(\frac{-11}{-11}, \frac{-33}{-11} \right) \right\} \\ &= \left\{ (1, 3) \right\} \end{aligned} \quad \begin{aligned} S_2 &= \left\{ \left(\frac{3 + 8}{21 - 10}, \frac{28 + 5}{21 - 10} \right) \right\} \\ &= \left\{ \left(\frac{11}{11}, \frac{33}{11} \right) \right\} \\ &= \left\{ (1, 3) \right\} \end{aligned} \quad \begin{aligned} S_3 &= \left\{ \left(\frac{4 - 15}{-5 - 6}, \frac{-25 - 8}{-5 - 6} \right) \right\} \\ &= \left\{ \left(\frac{-11}{-11}, \frac{-33}{-11} \right) \right\} \\ &= \left\{ (1, 3) \right\} \end{aligned}$$

All the solution sets are the same, and it follows that all three lines intersect at the point $(1, 3)$.

Problem 3: For each part, explain your reasoning using a sentence or two.

a. Find an example of a choice for $\vec{v}, \vec{u} \in \mathbb{R}^2$ such that the solution set to $-x + 9y = -6$ is $\{\vec{v} + t\vec{u} : t \in \mathbb{R}\}$.

Solution: We can parameterize $-x+9y = -6$ as $\vec{q}(t) = \vec{b} + t\vec{a}$ with $\vec{a} = \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}$. Let $\vec{r}(t) = \vec{v} + t\vec{u}$ with $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v_1, v_2, w_1, w_2 \in \mathbb{R}$. By definition, the solution set of $\vec{q}(t) = \vec{r}(t)$ is

$$\{t \in \mathbb{R} : \vec{b} + t\vec{a} = \vec{v} + t\vec{u}\}$$

Looking at the components, we see

$$\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

So it must be the case that

$$\begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So we have found $\vec{v}, \vec{u} \in \mathbb{R}$, namely $\vec{u} = \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}$, for which the solution set of $-x + 9y = -6$ is $\{\vec{v} + t\vec{u} : t \in \mathbb{R}\}$.

b. Find an example of $\vec{u} \in \mathbb{R}^2$ such that the solution set of $5x + 3y = 0$ is $\text{Span}(\vec{u})$.

Solution: We can parameterize $5x + 3y = 0$ as $\vec{q}(t) = \vec{a}t$ with $\vec{a} = \begin{pmatrix} -\frac{5}{3} \\ 1 \end{pmatrix}$. Let $\vec{r}(t) = \vec{u}t$ with $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $u_1, u_2 \in \mathbb{R}$. By definition, the solution set of $\vec{q}(t) = \vec{r}(t)$ is

$$\{t \in \mathbb{R} : \vec{a}t = \vec{u}t\}$$

Looking at the components, we see

$$\begin{pmatrix} -\frac{5}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

So it must be the case that $\vec{a} = \vec{u}$, and our solution set becomes

$$\{\vec{a}t : t \in \mathbb{R}\} = \{\vec{u}t : t \in \mathbb{R}\} = \text{Span}(\vec{u})$$

by definition.

Problem 4: Let

$$A = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R} \right\} \text{ and } B = \left\{ \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} : c \in \mathbb{R} \right\}$$

In this problem, we will prove that $A = B$ by giving a double containment proof.

a. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement that $A \subseteq B$.

Let $\vec{u} \in A$ be arbitrary. By definition of A, we can __________. Now notice that __________$=$ $\vec{u}$. Since __________$\in \mathbb{R}$, we conclude that $\vec{u} \in B$. Since $\vec{u} \in A$ was arbitrary, the result follows.

Solution: Let $\vec{u} \in A$ be arbitrary. By definition of A, we can fix a $c \in \mathbb{R}$ such that $\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Now notice that

$$\begin{aligned} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} + (c-2) \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \vec{u} \end{aligned}$$

Since $(c-2) \in \mathbb{R}$, we conclude that $\vec{u} \in B$. Since $\vec{u} \in A$ was arbitrary, the result follows.

b. Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement that $B \subseteq A$.

Let $\vec{w} \in B$ be arbitrary. By definition of B, we can __________. Now notice that __________$=$ $\vec{w}$. Since __________$\in \mathbb{R}$, we conclude that $\vec{w} \in A$. Since $\vec{w} \in B$ was arbitrary, the result follows.

Solution: Let $\vec{w} \in B$ be arbitrary. By definition of B, we can fix a $c \in \mathbb{R}$ such that $\vec{w} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Now notice that

$$\begin{aligned} \begin{pmatrix} 5 \\ 7 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (c+2) \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \vec{w} \end{aligned}$$

Since $(c+2) \in \mathbb{R}$, we conclude that $\vec{w} \in A$. Since $\vec{w} \in B$ was arbitrary, the result follows.

Problem 5: Given $a, b \in \mathbb{R}$, define two functions $f_a : \mathbb{R} \to \mathbb{R}$ and $g_b : \mathbb{R} \to \mathbb{R}$ by letting $f_a(x) = ax$ and letting $g_b(x) = x + b$. Determine, with explanation, all possible values of $a, b \in \mathbb{R}$ so that $f_a \circ g_b = g_b \circ f_a$.
Hint: Recall that to prove that two functions are equal, you need to argue that they give the same output for every input. To prove that two functions are not equal, you just need to give one example of an input that produces different outputs.

Solution: Let $f_a : \mathbb{R} \to \mathbb{R}$ and $g_b : \mathbb{R} \to \mathbb{R}$ be functions defined by letting $f_a(x) = ax$ and letting $g_b(x) = x + b$. By definition of function composition,

$$(f_a \circ g_b)(x) = f_a(g_b(x)) = a(x + b) \text{ and } (g_b \circ f_a)(x) = g_b(f_a(x)) = (ax) + b$$

Let $(f_a \circ g_b)(x) = (g_b \circ f_a)(x)$. We have

$$a(x + b) = (ax) + b$$

$$ax + ab = ax + b$$

So it must be the case that $ab = b$, that is that, $a = 1, b \in \mathbb{R}$.

c. Find an example of a choice for $a, b, c \in \mathbb{R}$ such that the solution set of $ax + by = c$ is $\text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right)$.

Solution: We parameterize the equation $ax + by = c$ into the form $\vec{r}(t) = \vec{s} + t \cdot \vec{n}$, with $\vec{n} = \begin{pmatrix} -\frac{a}{b} \\ 1 \end{pmatrix}$, $\vec{s} = \begin{pmatrix} \frac{c}{b} \\ 0 \end{pmatrix}$. If $\vec{r}(t) \in \text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right)$ then the solution set of $\vec{r}(t)$ is equal to $\text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right)$ because for every $h \in \mathbb{R}$, there exists a $d \in \mathbb{R}$ with $d \cdot \vec{r}(t) = h \cdot \begin{pmatrix} 2 \\ -7 \end{pmatrix}$. So we have

$$\begin{aligned} d \cdot \vec{r}(t) &= d \cdot t \begin{pmatrix} -\frac{a}{b} \\ 1 \end{pmatrix} + d \cdot \begin{pmatrix} \frac{c}{b} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{d \cdot t \cdot a}{b} \\ d \cdot t \end{pmatrix} + \begin{pmatrix} \frac{d \cdot c}{b} \\ d \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{d \cdot t \cdot a - d \cdot c}{b} \\ d \cdot t \end{pmatrix} \end{aligned}$$

Let $c = 0, d \cdot t = -7, a = 2, b = 7$. We get

$$\begin{aligned} &\begin{pmatrix} -\frac{-7 \cdot 2 - d \cdot 0}{7} \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} \frac{7 \cdot 2}{7} \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix} \end{aligned}$$

We have found values for $a, b, c \in \mathbb{R}$, namely $a = 2, b = 7, c = 0$, such that the solution set of $ax + by = c$ is $\text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right)$

.