Homework Assignment 11

PHYSICS 314 - THERMODYNAMICS & STATISTICAL PHYSICS (Spring 2018)

Due Friday, May 11ᵗʰ, by noon, Noyce 1135

I cannot award full credit for work that I am unable to read or follow. For my benefit and for yours, please:

- o *Write neatly*
- o *Show and EXPLAIN all steps*
- o *Make diagrams large and clearly-labeled*

You are welcome to collaborate with others on this assignment. However, the work you turn in should be your own. Please cite collaborators and outside sources. See the syllabus for details.

Regardless of the number of parts, all homework problems are weighted equally. Regardless of the number of questions, all homework assignments are weighted equally.

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1) Gaussian integrals and v_{rms}
    a) Use the result of the first integral to evaluate the second integral. *Hint: Use an integration trick similar to one used on HW10.*

$$\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^4 e^{-ax^2}\, dx = ?$$

Based on SCHROEDER B.2

Use the "differentiation with respect to a" trick from HW10 #6 part d). Differentiate both sides of the first equation.

$$\frac{\partial}{\partial a}\left(\int_0^\infty e^{-ax^2}\, dx\right) = \frac{\partial}{\partial a}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)$$

Move the derivative inside the integral on the left side. Pull out the constants on the right side.

$$\int_0^\infty \frac{\partial}{\partial a} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\pi}\,\frac{\partial}{\partial a}\left(a^{-\frac{1}{2}}\right)$$

Differentiate.

$$\int_0^\infty (-x^2) e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\pi}\left(-\frac{1}{2} a^{-\frac{3}{2}}\right)$$

$$-\int_0^\infty x^2 e^{-ax^2}\, dx = -\frac{1}{4}\sqrt{\pi}\, a^{-\frac{3}{2}}$$

$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\pi}\, a^{-\frac{3}{2}}$$

Use the trick again. Proceed similarly.

$$\frac{\partial}{\partial a}\left(\int_0^\infty x^2 e^{-ax^2}\, dx\right) = \frac{\partial}{\partial a}\left(\frac{1}{4}\sqrt{\pi}\, a^{-\frac{3}{2}}\right)$$

$$\int_0^\infty \frac{\partial}{\partial a} x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\pi}\, \frac{\partial}{\partial a}\left(a^{-\frac{3}{2}}\right)$$

$$\int_0^\infty x^2(-x^2) e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\pi}\left(-\frac{3}{2} a^{-\frac{5}{2}}\right)$$

$$-\int_0^\infty x^4 e^{-ax^2}\, dx = -\frac{3}{8}\sqrt{\pi}\, a^{-\frac{5}{2}}$$

$$\int_0^\infty x^4 e^{-ax^2}\, dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

b) Beginning with the Maxwell speed distribution, find the root-mean-square speed, v_{rms}. *Hint: There is a reason that this question appears directly after part a).*

Based on SCHROEDER 6.37

Use the weighted probability approach to calculating the average speed squared (similar to Schroeder (6.51)).

$$\overline{v^2} = \sum_{all\ v} v^2\, \mathcal{D}(v)\, dv$$

$$\overline{v^2} = \int_0^\infty v^2 \left(\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}}\right) dv$$

$$\overline{v^2} = \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^4 e^{-\frac{mv^2}{2kT}}\, dv$$

Pull out the constants.

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \int_0^\infty v^4 e^{-\frac{mv^2}{2kT}}\, dv$$

Note that this integral matches that from part a) with the following constant.

$$a = \frac{m}{2kT}$$

Use the result from part a) to evaluate the integral.

$$\int_0^\infty x^4 e^{-ax^2} \, dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \int_0^\infty v^4 e^{-\frac{mv^2}{2kT}} \, dv$$

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \left(\frac{3}{8}\sqrt{\frac{\pi}{\left(\frac{m}{2kT}\right)^5}}\right)$$

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \left(\frac{3}{8}\left(\frac{2kT}{m}\right)^{\frac{5}{2}}\sqrt{\pi}\right)$$

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left(\frac{2kT}{m}\right)^{\frac{5}{2}} \pi\sqrt{\pi}\left(\frac{3}{8} \times 4\right)$$

$$\overline{v^2} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left(\frac{2kT}{m}\right)^{\frac{5}{2}} \pi^{\frac{3}{2}}\left(\frac{3}{2}\right)$$

$$\overline{v^2} = \left(\frac{m}{2kT}\right)^{\frac{3}{2}} \left(\frac{2kT}{m}\right)^{\frac{5}{2}} \left(\frac{3}{2}\right)$$

$$\overline{v^2} = \frac{2kT}{m}\left(\frac{3}{2}\right)$$

$$\overline{v^2} = \frac{3kT}{m}$$

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}}$$

2) More Maxwell speed distributions
   a) Beginning with the Maxwell speed distribution, find the most probable speed, v_{max}.

   Based on SCHROEDER 6.35

   The most probable speed is at the peak of the distribution function. Find this point by differentiating the distribution function and setting it equal to zero.

$$\frac{d}{dv}\mathcal{D}(v) = 0$$

$$\frac{d}{dv}\left[\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}}\right] = 0$$

Pull out the constants.

$$\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \times \frac{d}{dv}\left[v^2 e^{-\frac{mv^2}{2kT}}\right] = 0$$

The constants will not change where the function is zero.

$$\frac{d}{dv}\left[v^2 e^{-\frac{mv^2}{2kT}}\right] = 0$$

Differentiate using the product rule and chain rule.

$$v^2\left(-\frac{mv}{kT} e^{-\frac{mv^2}{2kT}}\right) + e^{-\frac{mv^2}{2kT}}(2v) = 0$$

$$\left[v^2\left(-\frac{mv}{kT}\right) + 2v\right] e^{-\frac{mv^2}{2kT}} = 0$$

$$\left[-\frac{mv^3}{kT} + 2v\right] e^{-\frac{mv^2}{2kT}} = 0$$

The exponential is only zero at $v = \infty$, and that is a minimum, not a maximum. Examine when the term in the brackets is zero.

$$\left[-\frac{mv^3}{kT} + 2v\right] = 0$$

$$\left[-\frac{mv^2}{kT} + 2\right] v = 0$$

$v = 0$ is also an uninteresting solution and a minimum, not a maximum. Examine when the term in the brackets is zero. This yields the most probable speed, v_{max}.

$$-\frac{mv_{max}^2}{kT} + 2 = 0$$

Solve for v_{max}.

$$2 = \frac{mv_{max}^2}{kT}$$

$$\frac{2kT}{m} = v_{max}^2$$

$$v_{max} = \sqrt{\frac{2kT}{m}}$$

b) Beginning with the Maxwell speed distribution, find the average speed, $\bar{v}$.
   Hint: Use an integral result from HW10.

Based on SCHROEDER 6.36

Use the weighted probability approach to calculating the average speed (Schroeder (6.51)).

$$\bar{v} = \sum_{all\ v} v\, \mathcal{D}(v)\, dv$$

As in class, approximate this sum as an integral over all speeds.

$$\bar{v} = \int_0^\infty v\, \mathcal{D}(v)\, dv$$

$$\bar{v} = \int_0^\infty v\left(\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}}\right) dv$$

$$\bar{v} = \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^3 e^{-\frac{mv^2}{2kT}}\, dv$$

Pull out the constants.

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \int_0^\infty v^3 e^{-\frac{mv^2}{2kT}}\, dv$$

Note that this integral matches that from HW10 #6 part d) with the following constant.

$$a = \frac{m}{2kT}$$

Use the result from HW10 to evaluate the integral.

$$\int_0^\infty x^3 e^{-ax^2}\, dx = \frac{1}{2a^2}$$

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \int_0^\infty v^3 e^{-\frac{mv^2}{2kT}}\, dv$$

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \left(\frac{1}{2\left(\frac{m}{2kT}\right)^2}\right)$$

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \left(\frac{1}{\frac{1}{2}\left(\frac{m}{kT}\right)^2}\right)$$

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi \left(2\left(\frac{kT}{m}\right)^2\right)$$

$$\bar{v} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 8\pi \left(\frac{kT}{m}\right)^2$$

$$\bar{v} = \left(\frac{1}{2}\right)^{\frac{3}{2}} 2^3 \sqrt{\frac{kT}{\pi m}}$$

$$\bar{v} = 2^{3-3/2} \sqrt{\frac{kT}{\pi m}}$$

$$\bar{v} = 2^{3/2} \sqrt{\frac{kT}{\pi m}}$$

$$\bar{v} = \sqrt{\frac{2^3 kT}{\pi m}}$$

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

c) Calculate v_{rms}, v_{max} and $\bar{v}$ for oxygen molecules at room temperature.

Based on SCHROEDER 6.33

From the first problem and from parts a) and b), use expressions for the speed.

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$v_{max} = \sqrt{\frac{2kT}{m}}$$

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

Note that they are all the same within a numerical factor. Calculate the common term for oxygen at room temperature.

$$m_{O_2} = 2 \times 16 \; amu \approx 32 \times 1.66 \times 10^{-27} \; kg \approx 5.31 \times 10^{-26} \; kg$$

$$\sqrt{\frac{kT}{m}} \approx \sqrt{\frac{(1.38 \times 10^{-23} \; J \, / \, K)(300 \; K)}{5.31 \times 10^{-26} \; kg}}$$

$$\sqrt{\frac{kT}{m}} \approx 279 \frac{m}{s}$$

Use this to calculate the various speeds.

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{3}\sqrt{\frac{kT}{m}}$$

$$v_{rms} \approx \sqrt{3} \times 279 \frac{m}{s}$$

$$v_{rms} \approx 484 \frac{m}{s}$$

$$v_{max} = \sqrt{\frac{2kT}{m}} = \sqrt{2}\sqrt{\frac{kT}{m}}$$

$$v_{max} \approx \sqrt{2} \times 279 \frac{m}{s}$$

$$v_{max} \approx 395 \frac{m}{s}$$

$$v_{max} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8}{\pi}}\sqrt{\frac{kT}{m}}$$

$$v_{max} \approx \sqrt{\frac{8}{\pi}} \times 279 \frac{m}{s}$$

$$v_{max} \approx 446 \frac{m}{s}$$

3) Maxwell speed distribution plots
   a) Use a computer program to plot the Maxwell speed distributions for nitrogen molecules at $T = 150\ K$, $T = 300\ K$, and $T = 600\ K$. Plot all three distributions on the same set of axes. Include axis labels with units.
   Briefly comment on how temperature affects the distribution.

   Based on SCHROEDER 6.34

   I used Mathematica, but many programs would work equally well. Because I did not want to type in the same thing several times, I defined a function to calculate the Maxwell distribution. It is a function of mass (in atomic mass units – the conversion to kilograms is in the function itself), temperature (in Kelvin), and velocity (in meters per second).

   Here is the code to define the function.

   Maxwell[m_, T_, v_]:
   $$= (m * 1.66 * 10^\wedge - 27/(2 * \text{Pi} * k * T))^\wedge(3/2) * 4 * \text{Pi} * v^\wedge 2 * \text{Exp}[-m * 1.66 * 10^\wedge - 27 * v^\wedge 2/(2 * k * T)]$$

   Here is the code for the plots. Note that the mass of a nitrogen atom is $14\ amu$, so the mass of a nitrogen molecule is twice that.

   Plot[{Maxwell[214,150, v], Maxwell[214,300, v], Maxwell[214,600, v]}, {v, 0,1600}, PlotLegends$-$
   $>$ {"150 K", "300 K", "600 K"}, PlotStyle
   $\rightarrow$ {{Blue, Dashed}, Black, {Red, DotDashed}}, AxesLabel
   $\rightarrow$ {"Velocity (m/s)", "$\mathcal{D}$(v) (s/m)"}, ImageSize $\rightarrow$ Large, LabelStyle $\rightarrow$ Larger]

   The plot is below.
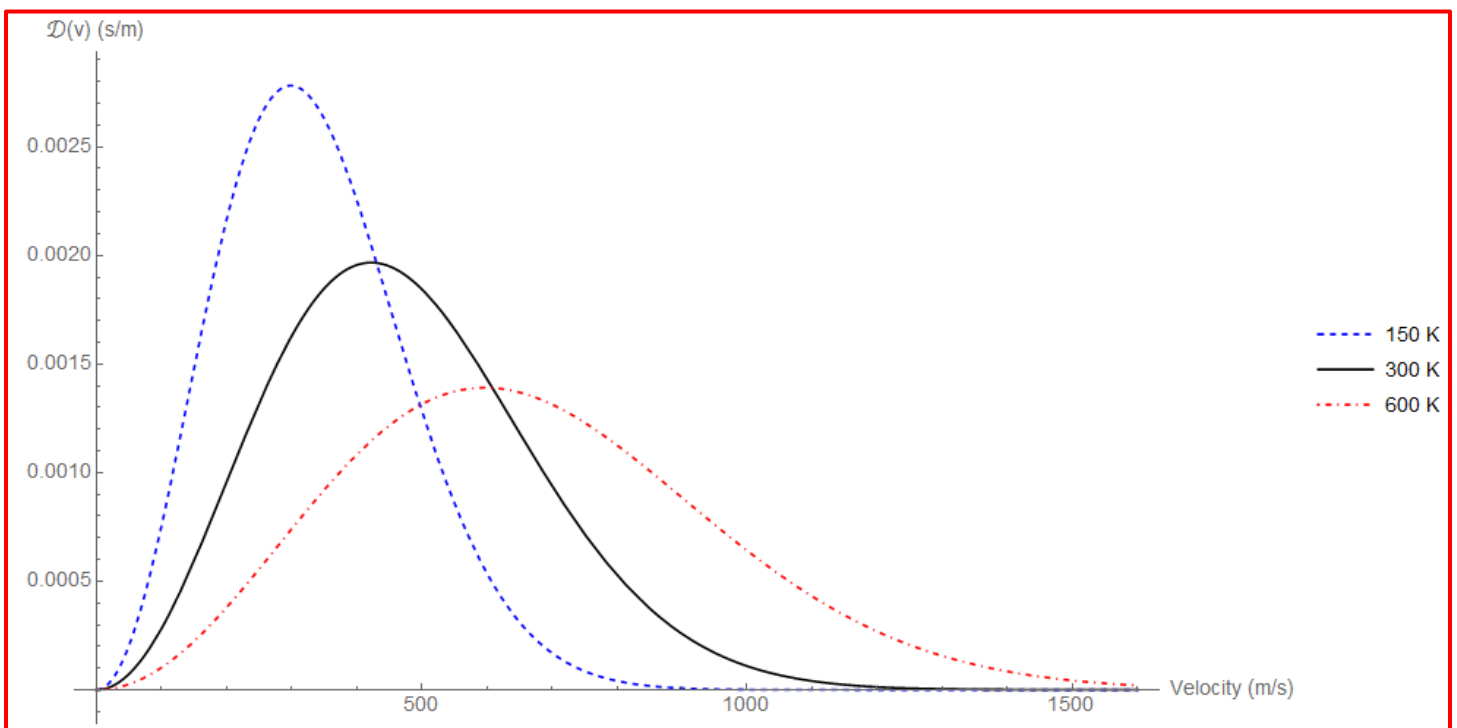
b) Use a computer program to plot the Maxwell speed distribution at $T = 300\ K$ for hydrogen molecules, nitrogen molecules, and oxygen molecules. Plot all three on the same set of axes. Include axis labels with units.
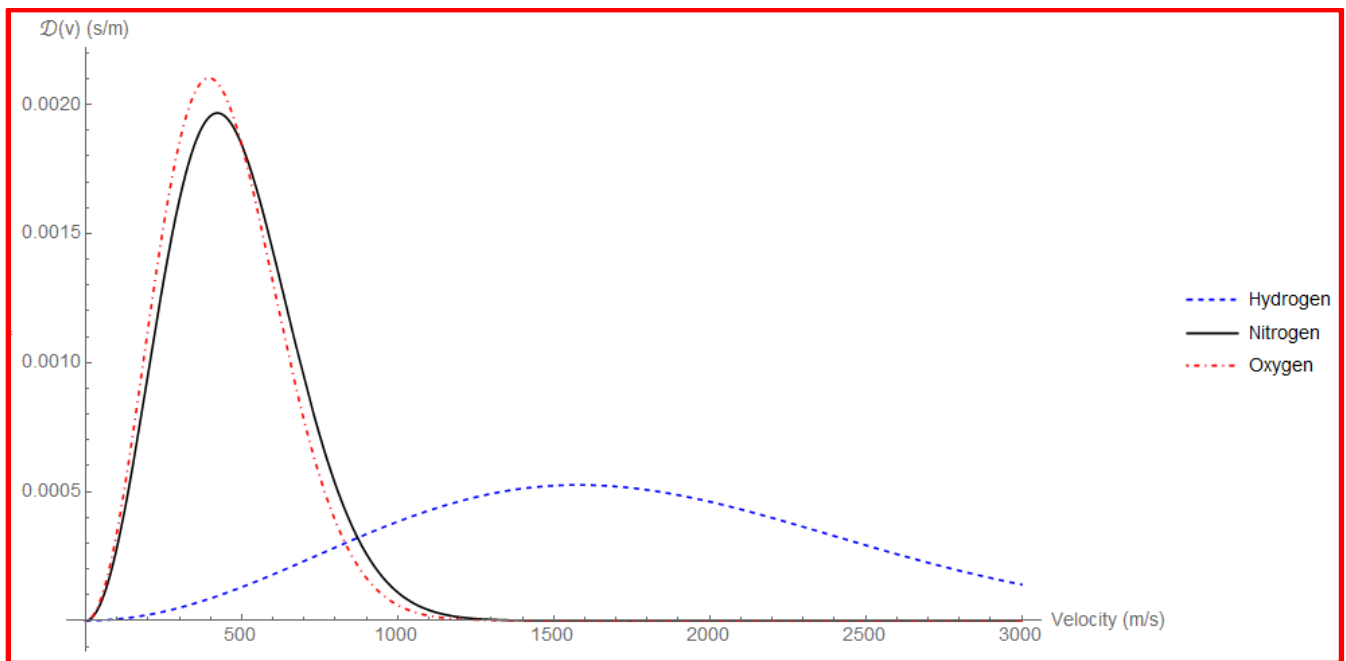Briefly comment on how particle mass affects the distribution.

$$m_{H_2} = 2\ amu$$

$$m_{N_2} = 28\ amu$$

$$m_{O_2} = 32\ amu$$

Plot[{Maxwell[21,300, v], Maxwell[214,300, v], Maxwell[216,300, v]}, {v, 0,3000}, PlotLegends$-$
$>$ {"Hydrogen", "Nitrogen", "Oxygen"}, PlotStyle
$\rightarrow$ {{Blue, Dashed}, Black, {Red, DotDashed}}, AxesLabel
$\rightarrow$ {"Velocity (m/s)", "$\mathcal{D}(v)$ (s/m)"}, ImageSize $\rightarrow$ Large, LabelStyle
$\rightarrow$ Larger, PlotRange $\rightarrow$ All]

4) The *escape speed* of a planet (or other celestial body) is defined as the minimum speed at which an object must be moving in order to escape the gravitational attraction of the planet. The escape speed is thus the speed for which the kinetic energy of the object is just enough to overcome the gravitational potential energy.

a) Derive an expression for the escape speed in terms of the mass and radius of the planet. Use this expression to calculate the escape speed on Earth and the escape speed on Jupiter. You will probably need to look up properties of the planets.

Set the kinetic energy equal to the gravitational potential energy.

$$E_{kinetic} = E_{gravitational}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

M is the mass of the planet, and m is the mass of the particle. v is the escape speed of the particle. G is the gravitational constant, and r is the radius of the planet – the distance between an object on the surface and the center of the planet.

Solve for speed.

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

$$\frac{1}{2}v^2 = \frac{GM}{r}$$

$$v^2 = \frac{2GM}{r}$$

$$v = \sqrt{\frac{2GM}{r}}$$

Look up the masses of the planets from NASA's planet fact sheet.
https://nssdc.gsfc.nasa.gov/planetary/factsheet/

$$m_{Earth} \approx 5.97 \times 10^{24} \, kg$$

$$m_{Jupiter} \approx 1898 \times 10^{24} \, kg$$

$$r_{Earth} \approx 6{,}378 \, km$$

$$r_{Jupiter} \approx 71{,}492 \, km$$

Plug in values, remembering to convert to meters.

$$v_{Earth} = \sqrt{\frac{2GM_{Earth}}{r_{Earth}}}$$

$$v_{Earth} \approx \sqrt{\frac{2\left(6.67 \times 10^{-11} \frac{m^3}{kg\ s^2}\right)(5.97 \times 10^{24}\ kg)}{6{,}378 \times 10^3\ m}}$$

$$v_{Earth} \approx 11{,}200\ m/s$$

$$v_{Jupiter} = \sqrt{\frac{2GM_{Jupiter}}{r_{Jupiter}}}$$

$$v_{Jupiter} \approx \sqrt{\frac{2\left(6.67 \times 10^{-11} \frac{m^3}{kg\ s^2}\right)(1{,}898 \times 10^{24}\ kg)}{71{,}492 \times 10^3\ m}}$$

$$v_{Jupiter} \approx 59{,}500\ m/s$$

b) Gases in the upper atmosphere with high enough speeds can escape into space. The region of the atmosphere from which gases escape is known as the *exobase*, and it can be very hot. Make the reasonable assumption that both for Earth and for Jupiter, the temperature of the exobase is $1000\ K$. Use a computer program to calculate the probability that hydrogen molecules in the exobase have speeds higher than the escape velocity of Earth. Repeat the calculation for the escape velocity of Jupiter. *Hint: The values will be small, but do not just round them to zero.*

To find the probability of a certain range of speeds, integrate the distribution function with the ends of the range as the bounds of integration.

$$\mathcal{P}(v_1, v_2) = \int_{v_1}^{v_2} \mathcal{D}(v)\ dv$$

In this case, the lower bound is the escape speed, and the upper bound is infinity.

$$\mathcal{P}(v > v_{escape}) \equiv \mathcal{P}_{escape} = \int_{v_{escape}}^{\infty} \mathcal{D}(v)\ dv$$

Plug in the Maxwell speed distribution function.

$$\mathcal{P}_{escape} = \int_{v_{escape}}^{\infty} \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}}\ dv$$

To evaluate the integral, I will use my Maxwell function in Mathematica from problem 3. The mass of H_2 is $2\ amu$. The temperature is given to be $1000\ K$. The escape speeds are from part a).

Here is the code for the Earth numerical integration.

$$\text{NIntegrate}[\text{Maxwell}[21, 1000, v], \{v, 11200, \infty\}]$$

$$\mathcal{P}_{escape\ Earth} \approx 1.27 \times 10^{-6}$$

Repeat the calculation with the escape speed for Jupiter.

$$\text{NIntegrate}[\text{Maxwell}[21,1000,v],\{v,59500,\infty\}]$$

$$\mathcal{P}_{escape\ Jupiter} \approx 2.63 \times 10^{-184}$$

c) Based on your result in part b), what can you infer about the hydrogen content in the atmosphere of the Earth compared to the atmosphere of Jupiter?

In both cases, the probability of hydrogen escaping is small, so little hydrogen will escape in either case. However, the probability of hydrogen escaping from Earth (which is itself very unlikely), is still over 150 orders of magnitude more likely than hydrogen escaping Jupiter. Compared to on Earth, the probability of hydrogen escaping on Jupiter is vanishingly small. Small amounts of hydrogen are very slowly lost from Earth, but over time, this small loss adds up. The losses are much, much slower on Jupiter. Thus, we expect the atmosphere of Jupiter to contain more hydrogen (per volume) than the atmosphere of Earth. This is in fact the case.

5) Suppose that there is a system that consists of ten particle states. To make the calculations easier, assume that all the states have the same energy, $E \equiv 0$.

Based on SCHROEDER 7.8

a) What is the partition function of the system, Z, if the system contains only one particle?

Because there is only particle, the partition function for the system is just the single-particle partition function.

$$Z = Z_1$$

The single particle partition function is the sum of the Boltzmann factors.

$$Z = Z_1 = \sum_{all\ 10\ particle\ states} e^{-\frac{E(s)}{kT}}$$

The energy of each state is given as zero.

$$Z = Z_1 = \sum_{all\ 10\ particle\ states} e^0$$

$$Z = Z_1 = \sum_{all\ 10\ particle\ states} 1$$

$$Z = Z_1 = 10$$

The partition function is just equal to the number of possible system states, the multiplicity.

b) What is the partition function of the system, Z, if the system contains only two distinguishable particles?

With two distinguishable particles, each particle can independently be in any one of the ten states. Thus, the partition function is the product of the two independent (identical) single-particle partition functions. The single-particle partition function was found in part a).

$$Z = Z_1 Z_2 = Z_1^2$$

$$Z = 10^2$$

$$Z = 100$$

c) What is the partition function of the system, Z, if the system contains only two identical bosons? *Hint: The answer is not 50.*

The particles are no longer distinguishable, so the partition function is no longer just the product of the single particle states. (The state of the system is the same regardless of which of the two identical particles is in which of two positions.)

Because the number of available states is so low, it is NOT a good assumption to use the approximate indistinguishable particle partition function.

$$Z \neq \frac{1}{2} N^2 = 50$$

With this low number of states, double occupancy of a single state is reasonably likely, and thus the "double counting error" is significant. Instead explicitly consider the possibilities.

First consider system states in which the particles are both in the same particle state. There are ten particle states, and thus ten double-occupancy system states.

$$10 \ double - occupancy \ states$$

Now for system states in which the particles are in different states. There are ten choices for the state of the first particle and then the remaining nine choices for the state of the second particle. Thus there are ninety such combinations. However, since the particles are indistinguishable, this number must be divided by two to avoid double counting of the same system states.

$$\frac{10 \times 9}{2} = 45 \ single - occupancy \ states$$

The total number of system states is the double-occupancy system states plus the number of single-occupancy states.

$$10 + 45 = 55 \ system \ states$$

By the same logic as part a), the number of possible system states is equal to the partition function because the energy is zero.

$$Z = 55$$

Note the double counting error is about 10%.

d) What is the partition function of the system, Z, if the system contains only two identical fermions?

The analysis is similar to the boson case with one exception. Double-occupancy states are not allowed by the Pauli Exclusion Principle. Thus, only the single-occupancy states from part c) contribute to the number of system states and therefore to the partition function.

$$Z = 45$$

e) For each of the parts a)-d), what is the probability of finding the system in a double-occupancy state (that is, with two particles in the same state)?

By the fundamental assumption of statistical mechanics, all states are equally likely. Thus, the probability of a double-occupancy state is just the number of double-occupancy states divided by the total number of states.

For a), there is only one particles, so there are no double-occupancy states.

$$\mathcal{P}_{double\ occupancy} = \frac{0}{10} = 0\%$$

For b), there are 10 states, and thus 10 chances for double occupancy.

$$\mathcal{P}_{double\ occupancy} = \frac{10}{100} = 10\%$$

For c), there are 10 double-occupancy states.

$$\mathcal{P}_{double\ occupancy} = \frac{10}{55} \approx 18\%$$

For d), double-occupancy is not allowed.

$$\mathcal{P}_{double\ occupancy} = \frac{0}{45} \approx 0\%$$

6) Fermions and Bosons

a) For a system of fermions at $T = 300\ K$, calculate the probabilities that single-particle states with the following energies, ϵ, are occupied.

Based on SCHROEDER 7.11

The probability of a state being occupied by a fermion (the occupancy of a state) is given by the Fermi-Dirac Distribution (Schroeder (7.23)).

$$\bar{n}_{FD} = \frac{1}{e^{\frac{\epsilon - \mu}{kT}} + 1}$$

For ease of calculations, calculate the Boltzmann constant times the temperature in electron volts.

$$kT \approx 8.62 \times 10^{-5} \frac{eV}{K} \times 300\ K$$

$$kT \approx 0.026\ eV$$

Note that because the energies are given relative to the chemical potential ($\epsilon = \mu + x$), the difference between the energies and the chemical potentials are just the "difference term," x.

$$\epsilon - \mu = (\mu + x) - \mu = x$$

Thus, the Fermi Dirac Distribution can be rewritten.

$$\bar{n}_{FD} = \frac{1}{e^{\frac{\epsilon - \mu}{kT}} + 1}$$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{x}{0.026\ eV}} + 1}$$

Evaluate each case. Note that states with energy significantly greater than the chemical potential are unlikely to be occupied, while states with energy significantly less than the chemical potential are likely to be occupied.

i) $\epsilon = \mu + 1\ eV$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{1\ eV}{0.026\ eV}} + 1}$$

$$\bar{n}_{FD} \approx 2 \times 10^{-17}$$

ii) $\epsilon = \mu + 0.01\ eV$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{0.01\ eV}{0.026\ eV}} + 1}$$

$$\bar{n}_{FD} \approx 0.41$$

iii) $\epsilon = \mu$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{0\ eV}{0.026\ eV}} + 1}$$

$$\bar{n}_{FD} \approx 0.5$$

iv) $\epsilon = \mu - 0.01 \; eV$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{-0.01 \; eV}{0.026 \; eV}} + 1}$$

$$\bar{n}_{FD} \approx 0.59$$

v) $\epsilon = \mu - 1 \; eV$

$$\bar{n}_{FD} \approx \frac{1}{e^{\frac{-1 \; eV}{0.026 \; eV}} + 1}$$

$$\bar{n}_{FD} \approx 1$$

b) Suppose there is a system of bosons at $T = 300 \; K$. For states with each of the following energies, ϵ, find the probabilities that the state is unoccupied. Then find the probability that the state contains one particle. Then find the probability that the state contains two particles. Then find the probability that the state contains three particles. Finally, find the average occupancy of the state.

Based on SCHROEDER 7.13

The average occupancy of a state for a boson is given by the Bose-Einstein Distribution (Schroeder (7.28)).

$$\bar{n}_{BE} = \frac{1}{e^{\frac{\epsilon - \mu}{kT}} - 1}$$

As in part a), it is useful to have a value for the Boltzmann constant times the temperature in electron volts.

$$kT \approx 8.62 \times 10^{-5} \frac{eV}{K} \times 300 \; K$$

$$kT \approx 0.026 \; eV$$

Note that because the energies are given relative to the chemical potential ($\epsilon = \mu + x$), the difference between the energies and the chemical potentials are just the "difference term," x.

$$\epsilon - \mu = (\mu + x) - \mu = x$$

Thus, the Bose-Einstein Distribution can be rewritten.

$$\bar{n}_{BE} = \frac{1}{e^{\frac{\epsilon - \mu}{kT}} - 1}$$

$$\bar{n}_{BE} \approx \frac{1}{e^{\frac{x}{0.026\ eV}} - 1}$$

The probability of a state being occupied by n bosons is given by the Gibbs factor divided by the partition function (Schroeder (7.20)).

$$\mathcal{P}(n) = \frac{1}{\mathcal{Z}} e^{-\frac{n(\epsilon-\mu)}{kT}}$$

Insert the partition function for bosons (Schroeder (7.24)).

$$\mathcal{P}(n) = \left(\frac{1}{\frac{1}{1 - e^{-\frac{(\epsilon-\mu)}{kT}}}} \right) e^{-\frac{n(\epsilon-\mu)}{kT}}$$

$$\mathcal{P}(n) = \left(1 - e^{-\frac{(\epsilon-\mu)}{kT}}\right) e^{-\frac{n(\epsilon-\mu)}{kT}}$$

For ease of calculations, factor out the n in the exponent so that the same exponential appears twice.

$$\mathcal{P}(n) = \left(1 - e^{-\frac{(\epsilon-\mu)}{kT}}\right) \left(e^{-\frac{(\epsilon-\mu)}{kT}}\right)^{n}$$

As above, we can rewrite this exponent because of how the energies are given in terms of the chemical potential. (Note that there is a sign difference from the exponential in the Bose-Einstein Distribution.)

$$\mathcal{P}(n) \approx \left(1 - e^{-\frac{x}{0.026\ eV}}\right) \left(e^{-\frac{x}{0.026\ eV}}\right)^{n}$$

Calculate each case. Note that the average occupancy is higher for states with energy closer to the chemical potential.

i) $\epsilon = \mu + 1\ eV$

$$\mathcal{P}(0) \approx \left(1 - e^{-\frac{1\ eV}{0.026\ eV}}\right) \left(e^{-\frac{1\ eV}{0.026\ eV}}\right)^{0}$$

$$\mathcal{P}(0) \approx 1.0$$

$$\mathcal{P}(1) \approx \left(1 - e^{-\frac{1\ eV}{0.026\ eV}}\right) \left(e^{-\frac{1\ eV}{0.026\ eV}}\right)^{1}$$

$$\mathcal{P}(1) \approx 2.0 \times 10^{-17}$$

$$\mathcal{P}(2) \approx \left(1 - e^{-\frac{1\ eV}{0.026\ eV}}\right) \left(e^{-\frac{1\ eV}{0.026\ eV}}\right)^{2}$$

$$\mathcal{P}(2) \approx 3.9 \times 10^{-34}$$

$$\mathcal{P}(3) \approx \left(1 - e^{-\frac{1\,eV}{0.026\,eV}}\right)\left(e^{-\frac{1\,eV}{0.026\,eV}}\right)^{3}$$

$$\mathcal{P}(3) \approx 7.7 \times 10^{-51}$$

$$\bar{n}_{BE} \approx \frac{1}{e^{\frac{1\,eV}{0.026\,eV}} - 1}$$

$$\bar{n}_{BE} \approx 2.0 \times 10^{-17}$$

Note all of these are essentially equal to zero. This state will most likely be unoccupied.

ii) $\epsilon = \mu + 0.1\,eV$

$$\mathcal{P}(0) \approx \left(1 - e^{-\frac{1\,eV}{0.026\,eV}}\right)\left(e^{-\frac{1\,eV}{0.026\,eV}}\right)^{0}$$

$$\mathcal{P}(0) \approx 0.98$$

$$\mathcal{P}(1) \approx \left(1 - e^{-\frac{0.1\,eV}{0.026\,eV}}\right)\left(e^{-\frac{0.1\,eV}{0.026\,eV}}\right)^{1}$$

$$\mathcal{P}(1) \approx 0.02$$

$$\mathcal{P}(1) \approx \left(1 - e^{-\frac{0.1\,eV}{0.026\,eV}}\right)\left(e^{-\frac{0.1\,eV}{0.026\,eV}}\right)^{2}$$

$$\mathcal{P}(2) \approx 4.5 \times 10^{-4}$$

$$\mathcal{P}(3) \approx \left(1 - e^{-\frac{0.1\,eV}{0.026\,eV}}\right)\left(e^{-\frac{0.1\,eV}{0.026\,eV}}\right)^{3}$$

$$\mathcal{P}(3) \approx 9.5 \times 10^{-6}$$

$$\bar{n}_{BE} \approx \frac{1}{e^{\frac{0.1\,eV}{0.026\,eV}} - 1}$$

$$\bar{n}_{BE} \approx .022$$

iii) $\epsilon = \mu + 0.01\,eV$

$$\mathcal{P}(0) \approx \left(1 - e^{-\frac{0.01\,eV}{0.026\,eV}}\right)\left(e^{-\frac{0.01\,eV}{0.026\,eV}}\right)^{0}$$

$$\mathcal{P}(0) \approx 0.32$$

$$\mathcal{P}(1) \approx \left(1 - e^{-\frac{0.01\ eV}{0.026\ eV}}\right)\left(e^{-\frac{0.01\ eV}{0.026\ eV}}\right)^{1}$$

$$\mathcal{P}(1) \approx 0.22$$

$$\mathcal{P}(1) \approx \left(1 - e^{-\frac{0.01\ eV}{0.026\ eV}}\right)\left(e^{-\frac{0.01\ eV}{0.026\ eV}}\right)^{2}$$

$$\mathcal{P}(2) \approx 0.15$$

$$\mathcal{P}(3) \approx \left(1 - e^{-\frac{0.01\ eV}{0.026\ eV}}\right)\left(e^{-\frac{0.01\ eV}{0.026\ eV}}\right)^{3}$$

$$\mathcal{P}(3) \approx 0.10$$

$$\bar{n}_{BE} \approx \frac{1}{e^{\frac{0.01\ eV}{0.026\ eV}} - 1}$$

$$\bar{n}_{BE} \approx 2.1$$

NOTICE!!! THIS PROBLEM IS TIME SENSITIVE!

7) You have two choices for the reflection problem this week. You may choose to do either Option I or Option II. *I strongly encourage you to do Option I if you are able,* but both options are worth the same number of points.

Option I

Attend the final Physics seminar of the year on Tuesday, May 8th, at noon.
The speaker is Professor Steve Kawaler from Iowa State University. The presentation is entitled, "Sounding stars while hunting for planets with the Kepler, K2, and TESS spacecraft". This talk may not be as directly related to thermodynamics as others, but it is still important to broaden your physics horizons.

After attending the lecture, write a half-page reflection on the presentation.
- Summarize the main points of the presentation.
- Discuss the connections between the science discussed in the presentation and the material covered in this course.
- Fulfilling the above requirements will earn you a $3.25/4$. The rest of the points will be awarded based on the depth and quality of your reflection.

Option II

Read the following material related to recent topics from class.

Satyendra Nath Bose obituary
    https://physicstoday.scitation.org/doi/10.1063/1.3128568
New state of matter revealed: Bose-Einstein Condensate
    https://www.nobelprize.org/nobel_prizes/physics/laureates/2001/popular.html

Write a short response to the two articles. One paragraph per article is sufficient.
- Summarize the main points of each article in a few sentences.

- For each article, discuss in a few sentences the connections between the science discussed in the article and the material covered in this course.
- Which article did you feel better conveyed the relevant science? Why? (Keep in mind the intended audience of each piece.)
- Fulfilling the above requirements will earn you a $^{3.25}/_4$. The rest of the points will be awarded based on the depth and quality of your explanations of the connections. (For example, a response that discusses the energy flow of a negative temperature system will earn more points than a response that simply states that negative temperature is something we saw in class.)

8) Peer Problem
   Complete the problem created by your classmate. I will distribute problems in class on Tuesday.