Worksheet 16 - Vector spaces

1 Definitions

Here is a definition, equivalent to the one in your text. It is slightly more general, as it does not assign any notation ahead of time for the zero vector.

Definition (Vector Space)**.** A vector space is a non-empty set, V, of objects, called vectors, equipped with operations of addition and scalar multiplication, such that the following properties hold:

1. For all $\vec{v}, \vec{w} \in V$, we have that $\vec{v} + \vec{w} \in V$.
2. For all $\vec{v} \in V$ and all $c \in \mathbb{R}$, we have that $c \cdot \vec{v} \in V$.
3. For all $\vec{v}, \vec{w} \in V$, we have that $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
4. For all $\vec{u}, \vec{v}, \vec{w} \in V$, we have that $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
5. There exists a $\vec{z} \in V$ such that, for all $\vec{v} \in V$, such that that $\vec{v} + \vec{z} = \vec{v}$. (There is a zero vector, or an additive identity, under vector addition.)
6. For all $\vec{v} \in V$, there exists a $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{z}$.
7. For all $\vec{v}, \vec{w} \in V$ and all $c \in \mathbb{R}$, we have that $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{w} + c \cdot \vec{v}$.
8. For all $\vec{v}$ and all $c, d \in \mathbb{R}$, we have that $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$.
9. For all $\vec{v} \in V$ and all $c, d \in \mathbb{R}$, we have that $c \cdot (d \cdot \vec{v}) = cd \cdot \vec{v}$.
10. For all $\vec{v} \in V$, we have that $1 \cdot \vec{v} = \vec{v}$.

Definition (Subspace)**.** Let V be a vector space. A subspace of V is a subset $W \subseteq V$ with the following properties:

- The additive identity vector is in W.
- For all $\vec{w_1}, \vec{w_2} \in W$, we have $\vec{w_1} + \vec{w_2} \in W$.
- For all $\vec{w} \in W$ and all $c \in \mathbb{R}$, we have that $c \cdot \vec{w} \in W$.

You can see that $\mathbb{R}^2$ is a vectors space with scalars from $\mathbb{R}$ and scalar multiplication and vector addition defined in the usual way, and all of the subsets we gave names to are, in fact, subspaces.

So, a vector space is a set of vectors with vector addition defined, and a set of scalars with scalar multiplication with vectors defined. (We also happen to be assuming some things about our set of scalars, you can look up the mathematical definition “field” to see what we’re including by using $\mathbb{R}$.)

2 Examples of Vector Spaces and Non-vector Spaces

In your groups, determine if the sets and operations I've defined for you make vector spaces. If they do not form a vector space, state which properties do they satisfy and which do they fail to exhibit.

For these examples, the notation $\oplus$ is used for vector addition and $\odot$ is used for scalar multiplication, to emphasize that these operations need not be the same as the ones we are used to. The set of scalars in all examples is simply $\mathbb{R}$ with its usual operations.

Exercise 1. Determine if the following set and operations form a real vector space:

- $V = \mathbb{P}_n$, where $\mathbb{P}_n$ is the set of all polynomials of degree less than or equal to n.
- $\oplus$ is defined as standard polynomial addition.
- $\odot$ is defined as standard real number multiplication.

Exercise 2. Determine if the following set and operations form a real vector space:

- $V = \{f : X \to \mathbb{R} : f \text{ is a function}\}$ for some fixed, non-empty, set X.
- $\oplus$ is defined as function addition, i.e. $(f \oplus g)(x) = f(x) + g(x)$.
- $\odot$ is defined as standard multiplication in $\mathbb{R}$, i.e. $c \odot f(x) = c \cdot (f(x))$.

Exercise 3. Determine if the following set and operations form a real vector space:

- $V = \mathbb{R}^2$.
- $\oplus$ is defined by $(a, b) \oplus (c, d) = (a + c, bd)$.
- $\odot$ is defined by $c \odot (a, b) = (ca, b^c)$.

Exercise 4. Determine if the following set and operations form a real vector space:

- $V = \mathbb{R}$.
- $\oplus$ is defined by $x \oplus y = x + y + 7$.
- $\odot$ is defined by $c \odot x = cx + 7(c - 1)$.

Exercise 5. Determine if the following set and operations form a real vector space:

- $V = \mathbb{R}^2$.
- $\oplus$ is defined by $(a, b) \oplus (c, d) = (a + c, 0)$.
- $\odot$ is defined by $c \odot (a, b) = (ca, 0)$.

For this next example, the set of scalars is $\mathbb{F}_2 = \{0, 1\}$ with the operations within this set defined by,

$$\begin{array}{cccc} 0+0=0 & 0+1=1 & 1+0=1 & 1+1=0 \\ 0\times 0=0 & 0\times 1=0 & 1\times 0=0 & 1\times 1=1 \end{array}$$

Note: With these operations, $1 = -1$.

Exercise 6. Let G be a finite set of points, called nodes, long with a finite set, E, of edges such that for any two distinct nodes in G, there is at most one edge associated to that pair. If a pair of nodes in G are associated to an edge in E, we say those nodes are adjacent. Such a set of nodes and edges is called a *finite simple graph.* A *spanning subgraph* of a finite simple graph, is a finite simple graph that shares the same set of nodes, G, along with any subset of the edges, $E' \subseteq E$.

Let G be a finite simple graph. Determine if the following set and operations form a vector space:

- $V = H : H$ is a spanning subgraph of G.
- $\oplus$ is defined by $H \oplus H' =$ the subgraph of G with edges that are in exactly one of H or H'.
- $\odot$ is defined by $1 \odot H = H$ and $0 \odot H =$ the set of nodes, G, and no edges (the set of edges is the empty set).