Problem Set 8: Due Monday, February 26

Problem 1: Consider the unique linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix} 9 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Determine, with explanation, the value of

$$T\left(\begin{pmatrix} 6 \\ 2 \end{pmatrix}\right).$$

Problem 2: Compute

$$\begin{pmatrix} 4 & 3 \\ -7 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

What does your answer mean in terms of linear transformations? Explain.

Problem 3: Let $\theta \in \mathbb{R}$. Define $C_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $C_\theta(\vec{v})$ be the result of rotating $\vec{v}$ *clockwise* around the origin by an angle of θ. It can be shown geometrically that C_θ is a linear transformation (no need to do this). What is $[C_\theta]$? Explain your reasoning, and simplify your answer as much as possible.

Problem 4: For each of following, consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that has the given matrix as its standard matrix. Describe the action of T geometrically. It may help to plug in a few points and/or make some case distinctions.

a. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for a fixed $k \in \mathbb{R}$ with $k > 0$.

c. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ for a fixed $k \in \mathbb{R}$ with $k > 0$.

Problem 5: Let A be a 2×2 matrix. Verify each of the following using the formula for the matrix-vector product.

a. $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$.

b. $A(c \cdot \vec{v}) = c \cdot A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$ and all $c \in \mathbb{R}$.

Note: Since matrices encode linear transformations, you should expect these to be true. In fact, we can argue that they are true by interpreting the matrix as being the standard matrix of a certain linear transformation, and then just appealing to the fact that linear transformation preserve addition and scalar multiplication. However, in this problem, you should just work through the computations directly.

Problem 6: Consider the unique linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

What is $[T]$? Explain.