Chapter 1

Introduction

1.1 What is Linear Algebra?

1.1.1 Generalizing Lines and Planes

1.1.2 Transformations of Space

1.2 Mathematical Statements and Mathematical Truth

1.3 Quantifiers and Proofs

Proposition 1.3.1. *For all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $m < n$.*

Proposition 1.3.2. *For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $m \geq n$.*

1.4 Evens and Odds

Definition 1.4.1. *Let $a \in \mathbb{Z}$.*

- *We say that a is* even *if there exists $m \in \mathbb{Z}$ with $a = 2m$.*
- *We say that a is* odd *if there exists $m \in \mathbb{Z}$ with $a = 2m + 1$.*

Proposition 1.4.2. *If $a \in \mathbb{Z}$ is even, then a^2 is even.*

Proposition 1.4.3. *If $a \in \mathbb{Z}$ is even and $b \in \mathbb{Z}$ is odd, then $a + b$ is odd.*

Proposition 1.4.4. *The integer 3 is not even.*

Proposition 1.4.5. *No integer is both even and odd.*

Fact 1.4.6. *Every integer is either even or odd.*

Proposition 1.4.7. *If $a \in \mathbb{Z}$ and a^2 is even, then a is even.*

Theorem 1.4.8. *There does not exist $q \in \mathbb{Q}$ with $q^2 = 2$. In other words, $\sqrt{2}$ is irrational.*

Proposition 1.4.9. *If $a \in \mathbb{Z}$ is odd, then there exist $b, c \in \mathbb{Z}$ with $a = b^2 - c^2$. In other words, every odd integer is the difference of two perfect squares.*

1.5 Sets, Set Construction, and Subsets

1.5.1 Sets and Set Construction

Definition 1.5.1. *A* set *is a collection of elements without regard for repetition or order.*

Definition 1.5.2. *Given two sets A and B, we say that $A = B$ if A and B have exactly the same elements.*

Definition 1.5.3. *Given an object x and a set A, we write $x \in A$ to mean that x* is an element of *A, and we write $x \notin A$ to mean that x* is not an element of *A*

1.5.2 Subsets and Set Equality

Definition 1.5.4. *Given two sets A and B, we write $A \subseteq B$ to mean that every element of A is an element of B. More formally, $A \subseteq B$ means that for all x, if $x \in A$, then $x \in B$.*

Proposition 1.5.5. *Let $A = \{6n : n \in \mathbb{Z}\}$ and $B = \{2n : n \in \mathbb{Z}\}$. We have $A \subseteq B$.*

Proposition 1.5.6. *Let $A = \{7n - 3 : n \in \mathbb{Z}\}$ and $B = \{7n + 11 : n \in \mathbb{Z}\}$. We have $A = B$.*

Proposition 1.5.7. *We have $\{9m + 15n : m, n \in \mathbb{Z}\} = \{3m : m \in \mathbb{Z}\}$.*

1.5.3 Ordered Pairs and Sequences

Definition 1.5.8. *An* ordered pair *is a collection of two (not necessarily distinct) objects, where order and repetition do matter.*

1.5.4 Operations on Sets

Definition 1.5.9. *Given two sets A and B, we define $A \cup B$ to be the set consisting of those elements that are in A or B (or both). In other words, we have*

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

We call this set the union *of A and B.*

Definition 1.5.10. *Given two sets A and B, we define $A \cap B$ to be the set consisting of those elements that are in both of A and B. In other words, we have*

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

We call this set the intersection *of A and B.*

Definition 1.5.11. *We say that two sets A and B are* disjoint *if $A \cap B = \emptyset$.*

Definition 1.5.12. *Given two sets A and B, we define*

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$$

and call this set the (relative) complement *of B (in A).*

Definition 1.5.13. *Given two sets A and B, we let $A \times B$ be the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, and we call this set the* Cartesian product *of A and B.*

1.6 Functions

Definition 1.6.1. *Let A and B be sets. A* function *from A to B is a subset f of $A \times B$ with the property that for all $a \in A$, there exists a unique $b \in B$ with $(a, b) \in f$. Also, instead of writing "f is a function from A to B", we typically use the shorthand notation "$f\colon A \to B$".*

Notation 1.6.2. *Let A and B be sets. If $f\colon A \to B$ and $a \in A$, we write $f(a)$ to mean the unique $b \in B$ such that $(a, b) \in f$.*

Definition 1.6.3. *Let $f\colon A \to B$ be a function.*

- *We call A the* domain *of f.*
- *We call B the* codomain *of f.*
- *We define $range(f) = \{b \in B : \text{There exists } a \in A \text{ with } f(a) = b\}$.*

Definition 1.6.4. *Suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. The* composition *of g and f, denoted $g \circ f$, is the function $g \circ f\colon A \to C$ defined by letting $(g \circ f)(a) = g(f(a))$ for all $a \in A$.*

Proposition 1.6.5. *Let A, B, C, D be sets. Suppose that $f\colon A \to B$, that $g\colon B \to C$, and that $h\colon C \to D$ are functions. We then have that $(h \circ g) \circ f = h \circ (g \circ f)$. Stated more simply, function composition is associative whenever it is defined.*

Definition 1.6.6. *Let A be a set. The function $id_A\colon A \to A$ defined by $id_A(a) = a$ for all $a \in A$ is called the* identity function *on A.*

Proposition 1.6.7. *For any function $f\colon A \to B$, we have $f \circ id_A = f$ and $id_B \circ f = f$.*

Definition 1.6.8. *Let $f\colon A \to B$ be a function.*

- *We say that f is* injective *(or* one-to-one*) if whenever $a_1, a_2 \in A$ satisfy $f(a_1) = f(a_2)$, we have $a_1 = a_2$.*
- *We say that f is* surjective *(or* onto*) if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.*
- *We say that f is* bijective *if f is both injective and surjective.*

Proposition 1.6.9. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is bijective.*

1.7 Solving Equations

Definition 1.7.1. *Suppose that we have two functions $f\colon A \to B$ and $g\colon A \to B$. We define the* solution set *of the equation $f(x) = g(x)$ to be $\{a \in A : f(a) = g(a)\}$.*

Chapter 2

Linear Algebra of $\mathbb{R}^2$

2.1 Intersections of Lines in $\mathbb{R}^2$

Proposition 2.1.1. *Suppose that $a, b, c, d, j, k \in \mathbb{R}$ and consider the following system of two equations in two unknowns:*

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

If $ad - bc \neq 0$ then the solution set, S, of this system has a unique element. In particular,

$$S = \left\{ \left(\frac{dj - bk}{ad - bc}, \frac{ak - cj}{ad - bc} \right) \right\}$$

2.2 Vectors in $\mathbb{R}^2$

Proposition 2.2.1. *See full text.*

2.3 Spans and Coordinates

Definition 2.3.1. *Let $\vec{u} \in \mathbb{R}^2$. We define a subset of $\mathbb{R}^2$ as follows:*

$$Span(\vec{u}) = \{c \cdot \vec{u} : c \in \mathbb{R}\}$$

We call this set the span *of the vector $\vec{u}$.*

Proposition 2.3.2. *Let $\vec{u} \in \mathbb{R}^2$ be arbitrary, and let $S = Span(\vec{u})$. We have the following.*

1. *$\vec{0} \in S$.*
2. *For all $\vec{v}_1, \vec{v}_2 \in S$, we have $\vec{v}_1 + \vec{v}_2 \in S$ (i.e. S is* closed *under addition).*
3. *For all $\vec{v} \in S$ and all $d \in \mathbb{R}$, we have $d\vec{v} \in S$ (i.e. S is* closed *under scalar multiplication).*

Proposition 2.3.3. *For all $\vec{u} \in \mathbb{R}^2$, we have that $Span(\vec{u}) \neq \mathbb{R}^2$.*

Definition 2.3.4. *Let $\vec{u}_1, \vec{u}_2, \vec{v} \in \mathbb{R}^2$. We say that $\vec{v}$ is a* linear combination *of $\vec{u}_1$ and $\vec{u}_2$ if there exists $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$.*

Definition 2.3.5. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. We define a subset of $\mathbb{R}^2$ as follows:*

$$Span(\vec{u}_1, \vec{u}_2) = \{c_1\vec{u}_1 + c_2\vec{u}_2 : c_1, c_2 \in \mathbb{R}\}.$$

In other words, $Span(\vec{u}_1, \vec{u}_2)$ is the set of all linear combinations of $\vec{u}_1$ and $\vec{u}_2$. We call this set the span *of the vectors $\vec{u}_1$ and $\vec{u}_2$.*

Proposition 2.3.6. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be arbitrary, and let $S = Span(\vec{u}_1, \vec{u}_2)$. We have the following.*

1. *$\vec{0} \in S$.*
2. *For all $\vec{v}_1, \vec{v}_2 \in S$, we have $\vec{v}_1 + \vec{v}_2 \in S$ (i.e. S is* closed *under addition).*
3. *For all $\vec{v} \in S$ and all $d \in \mathbb{R}$, we have $d\vec{v} \in S$ (i.e. S is* closed *under scalar multiplication).*

Proposition 2.3.7. *For all $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, we have*

$$Span(\vec{u}_1) \subseteq Span(\vec{u}_1, \vec{u}_2).$$

Proposition 2.3.8. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. The following are equivalent, i.e. if one is true, then so is the other (and hence if one is false, then so is the other).*

1. *$Span(\vec{u}_1, \vec{u}_2) = Span(\vec{u}_1)$.*
2. *$\vec{u}_2 \in Span(\vec{u}_1)$.*

Proposition 2.3.9. *For all $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, we have $Span(\vec{u}_1, \vec{u}_2) = Span(\vec{u}_2, \vec{u}_1)$.*

Theorem 2.3.10. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, say*

$$\vec{u}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

The following are equivalent, i.e. if any one of the statements is true, then so are all of the others (and hence if any one of the statements is false, then so are all of the others).

1. *For all $\vec{v} \in \mathbb{R}^2$, there exist $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$, i.e. $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$.*
2. *Both $\vec{u}_1$ and $\vec{u}_2$ are nonzero, and there does not exist $c \in \mathbb{R}$ with $\vec{u}_2 = c\vec{u}_1$.*
3. *Both $\vec{u}_1$ and $\vec{u}_2$ are nonzero, and there does not exist $c \in \mathbb{R}$ with $\vec{u}_1 = c\vec{u}_2$.*
4. *$a_1b_2 - a_2b_1 \neq 0$.*
5. *For all $\vec{v} \in \mathbb{R}^2$, there exist a unique pair of numbers $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$.*

Definition 2.3.11. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. We say that the ordered pair $(\vec{u}_1, \vec{u}_2)$ is a* basis *for $\mathbb{R}^2$ if $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$.*

Definition 2.3.12. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. We define a function $Coord_\alpha \colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows. Given $\vec{v} \in \mathbb{R}^2$, let $c_1, c_2 \in \mathbb{R}$ be the unique values such that $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$, and define*

$$Coord_\alpha(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Proposition 2.3.13. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. Fix $a, b, c, d \in \mathbb{R}$ with*

$$\vec{u}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} b \\ d \end{pmatrix}.$$

For any $j, k \in \mathbb{R}$, we have

$$\begin{aligned} Coord_\alpha\left(\begin{pmatrix} j \\ k \end{pmatrix}\right) &= \begin{pmatrix} \frac{dj-bk}{ad-bc} \\ \frac{ak-cj}{ad-bc} \end{pmatrix} \\ &= \frac{1}{ad-bc} \cdot \begin{pmatrix} dj-bk \\ ak-cj \end{pmatrix} \end{aligned}$$

2.4 Linear Transformations of $\mathbb{R}^2$

Definition 2.4.1. *A* linear transformation *from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ with the following two properties:*

1. *$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ (i.e. T* preserves addition*).*
2. *$T(c \cdot \vec{v}) = c \cdot T(\vec{v})$ for all $\vec{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ (i.e. T* preserves scalar multiplication*).*

Proposition 2.4.2. *Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. We have the following:*

1. *$T(\vec{0}) = \vec{0}$.*
2. *$T(-\vec{v}) = -T(\vec{v})$ for all $\vec{v} \in \mathbb{R}^2$.*
3. *$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 \cdot T(\vec{v}_1) + c_2 \cdot T(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ and all $c_1, c_2 \in \mathbb{R}$.*

Proposition 2.4.3. *Let $a, b, c, d \in \mathbb{R}$. Define a function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$\begin{aligned} T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \\ &= x \cdot \begin{pmatrix} a \\ c \end{pmatrix} + y \cdot \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

We then have that T is a linear transformation.

Proposition 2.4.4. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. The function $Coord_\alpha \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.*

Theorem 2.4.5. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis for $\mathbb{R}^2$. Suppose that $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S \colon \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations with the property that $T(\vec{u}_1) = S(\vec{u}_1)$ and $T(\vec{u}_2) = S(\vec{u}_2)$. We then have that $T = S$, i.e. $T(\vec{v}) = S(\vec{v})$ for all $\vec{v} \in \mathbb{R}^2$.*

Theorem 2.4.6. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ and assume that $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$. Let $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^2$. There exists a linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\vec{u}_1) = \vec{w}_1$ and $T(\vec{u}_2) = \vec{w}_2$.*

Definition 2.4.7. *We define the following.*

- *Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S \colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations. We define a new function $T+S \colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $(T+S)(\vec{v}) = T(\vec{v}) + S(\vec{v})$ for all $\vec{v} \in \mathbb{R}^2$.*
- *Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformations and let $r \in \mathbb{R}$. We define a new function $r \cdot T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $(r \cdot T)(\vec{v}) = r \cdot T(\vec{v})$ for all $\vec{v} \in \mathbb{R}^2$.*

Proposition 2.4.8. *Let $T, S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations.*

1. *The function $T + S$ is a linear transformation.*
2. *For all $r \in \mathbb{R}$, then function $r \cdot T$ is a linear transformation.*
3. *The function $T \circ S$ is a linear transformation.*

Chapter 3

Matrices and Linear Transformations in Two Dimensions

3.1 The Standard Matrix of a Linear Transformation

Definition 3.1.1. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Fix $a, b, c, d \in \mathbb{R}$ with*

$$T(\vec{e}_1) = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \textit{and} \qquad T(\vec{e}_2) = \begin{pmatrix} b \\ d \end{pmatrix}$$

We define the standard matrix *of T to be the following 2×2 matrix:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In other words, the standard matrix of T has the entries of $T(\vec{e}_1)$ in the first column, and the entries of $T(\vec{e}_2)$ in the second column.

Notation 3.1.2. *If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then we use the notation $[T]$ to denote the standard matrix of T.*

Definition 3.1.3. *Let $a, b, c, d, x, y \in \mathbb{R}$. Let A be the 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and let $\vec{v} \in \mathbb{R}^2$ be

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We define the matrix-vector product, written as $A\vec{v}$, to be the vector

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

In other words, we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Proposition 3.1.4. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If A is the standard matrix of T, then $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. In other words, $T(\vec{v}) = [T]\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.*

Definition 3.1.5. *Given two 2×2 matrices, say*

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

we define $A = B$ to mean that $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$.

Proposition 3.1.6. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ both be linear transformations. We then have that $T = S$ if and only if $[T] = [S]$.*

Proposition 3.1.7. *Let $id\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $id(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. In other words, id is the identity function. We then have that id is a linear transformation and*

$$[id] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proposition 3.1.8. *Let $a, b, c, d \in \mathbb{R}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

We then have that T is a linear transformation and

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 3.1.9. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. If*

$$\vec{u}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \textit{and} \qquad \vec{u}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

then

$$[Coord_\alpha] = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.$$

Proposition 3.1.10. *Let $\theta \in \mathbb{R}$. Define a function $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $R_\theta(\vec{v})$ be the result of rotating $\vec{v}$ by θ radians counterclockwise around the origin. We then have that R_θ is a linear transformation and*

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Proposition 3.1.11. *Let $\vec{w} \in \mathbb{R}^2$ be a nonzero vector and let $W = Span(\vec{w})$. Fix $a, b \in \mathbb{R}$ with*

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Define $P_{\vec{w}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $P_{\vec{w}}(\vec{v})$ be the vector in W that is closest to $\vec{v}$. We then have that $P_{\vec{w}}$ is a linear transformation and

$$[P_{\vec{w}}] = \begin{pmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{pmatrix}.$$

3.2 Matrix Algebra

Definition 3.2.1. *Given two matrices*

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

we define

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Proposition 3.2.2. *Let $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations. We have the following.*

1. $[T_1 + T_2] = [T_1] + [T_2]$.
2. $[T_1 \circ T_2] = [T_1] \cdot [T_2]$.

Definition 3.2.3. *Given a matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and an $r \in \mathbb{R}$, we define

$$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

Proposition 3.2.4. *Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $r \in \mathbb{R}$. We have $[r \cdot T] = r \cdot [T]$.*

Proposition 3.2.5. *Let A and B be 2×2 matrices*

1. *For $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, we have $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$.*
2. *For all $\vec{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have $A(c\vec{v}) = c \cdot A\vec{v}$.*
3. *For all $\vec{v} \in \mathbb{R}^2$, we have $(A + B)\vec{v} = A\vec{v} + B\vec{v}$.*
4. *For all $\vec{v} \in \mathbb{R}^2$, we have $A(B\vec{v}) = (AB)\vec{v}$.*

Proposition 3.2.6. *Let A, B, and C be 2×2 matrices. We have the following*

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. $(AB)C = A(BC)$.
4. $A(B + C) = AB + AC$.
5. $(A + B)C = AC + BC$.
6. *For all $r \in \mathbb{R}$, we have $A(r \cdot B) = r \cdot (AB) = (r \cdot A)B$.*

Definition 3.2.7. *We define two special matrices.*

- *We let* 0 *denote the* 2×2 *of all zeros, i.e.*
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
We have to distinguish between the number 0 *and the matrix* 0 *from context.*
- *We let* $I = [id]$ *where* $id \colon \mathbb{R}^2 \to \mathbb{R}^2$ *is the function given by* $id(\vec{v}) = \vec{v}$ *for all* $\vec{v} \in \mathbb{R}^2$*. Thus,*
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
by Proposition 3.1.7.

Proposition 3.2.8. *Let* A *be a* 2×2 *matrix.*

1. $A + 0 = A = 0 + A$.
2. $A + (-1) \cdot A = 0 = (-1) \cdot A + A$.
3. $A \cdot 0 = 0 = 0 \cdot A$.
4. $A \cdot I = A = I \cdot A$.

3.3 Range, Null Space, and Inverses

Proposition 3.3.1. *Let* $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation. Let* $\vec{u}_1$ *be the first column of* $[T]$*, and let* $\vec{u}_2$ *be the second column of* $[T]$*. We then have that* $range(T) = Span(\vec{u}_1, \vec{u}_2)$*.*

Definition 3.3.2. *Let* $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation. We define*
$$Null(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}.$$
We call $Null(T)$ *the* null space *of* T *(or the* kernel *of* T*).*

Theorem 3.3.3. *Let* $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation, and fix* $a, b, c, d \in \mathbb{R}$ *with*
$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
We have the following cases.

1. *If* $ad - bc \neq 0$*, then* $Null(T) = \{\vec{0}\}$ *and* $range(T) = \mathbb{R}^2$*.*
2. *If all of* a, b, c, d *equal* 0*, then* $Null(T) = \mathbb{R}^2$ *and* $range(T) = \{\vec{0}\}$*.*
3. *If* $ad - bc = 0$ *and at least one of* a, b, c, d *is nonzero, then there exist nonzero* $\vec{u}, \vec{w} \in \mathbb{R}^2$ *with* $Null(T) = Span(\vec{u})$ *and* $range(T) = Span(\vec{w})$*.*

Proposition 3.3.4. *Let* $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation. We then have that* T *is injective if and only if* $Null(T) = \{\vec{0}\}$*.*

Corollary 3.3.5. *Let* $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation, and fix* $a, b, c, d \in \mathbb{R}$ *with*
$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
The following are equivalent:

1. *$ad - bc \neq 0$.*
2. *T is bijective.*
3. *T is injective.*
4. *T is surjective.*

Definition 3.3.6. *Let $f\colon A \to B$ be a function. An* inverse function *for f is a function $g\colon B \to A$ such that both $g \circ f = id_A$ and $f \circ g = id_B$.*

Definition 3.3.7. *Let $f\colon A \to B$ be a function.*

- *A* left inverse *for f is a function $g\colon B \to A$ such that $g \circ f = id_A$.*
- *A* right inverse *for f is a function $g\colon B \to A$ such that $f \circ g = id_B$.*

Proposition 3.3.8. *Let $f\colon A \to B$ be a function.*

1. *f is injective if and only if there exists a left inverse for f.*
2. *f is surjective if and only if there exists a right inverse for f.*
3. *f is bijective if and only if there exists an inverse for f.*

Proposition 3.3.9. *Let $f\colon A \to B$ be a function. If $g\colon B \to A$ is a left inverse of f and $h\colon B \to A$ is a right inverse of f, then $g = h$.*

Corollary 3.3.10. *If $f\colon A \to B$ is a function, then there exists at most one function $g\colon B \to A$ that is an inverse of f.*

Corollary 3.3.11. *If $f\colon A \to B$ is a bijective function, then there exists a unique inverse for f.*

Notation 3.3.12. *Suppose that $f\colon A \to B$ is bijective. We let $f^{-1}\colon B \to A$ be the the unique inverse for f. More concretely, f^{-1} is defined as follows. Given $b \in B$, we define $f^{-1}(b)$ to equal the unique $a \in A$ with $f(a) = b$.*

Proposition 3.3.13. *Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a bijective linear transformation. We then have that the function $T^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.*

Proposition 3.3.14. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective linear transformation, and fix $a, b, c, d \in \mathbb{R}$ with*

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then have that $ad - bc \neq 0$, and

$$\begin{aligned}[T^{-1}] &= \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \\ &= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.\end{aligned}$$

Definition 3.3.15. *Let A be a 2×2 matrix. We say that A is* invertible *if there exists a 2×2 matrix B with both $AB = I$ and $BA = I$.*

Proposition 3.3.16. *Let A be a 2×2 matrix, say*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We then have that A is invertible if and only if $ad - bc \neq 0$. In this case, A has a unique inverse, given by

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Definition 3.3.17. *If A is an invertible 2×2 matrix, we denote its unique inverse by A^{-1}.*

Proposition 3.3.18. *We have the following.*

1. *If A is an invertible 2×2 matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *If A and B are both invertible 2×2 matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

3.4 Matrices with Respect to Other Coordinates

Notation 3.4.1. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. Given $\vec{v} \in \mathbb{R}^2$, we use the notation $[\vec{v}]_\alpha$ as shorthand for $Coord_\alpha(\vec{v})$.*

Definition 3.4.2. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$, and let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Fix $a, b, c, d \in \mathbb{R}$ with*

$$[T(\vec{u}_1)]_\alpha = \begin{pmatrix} a \\ c \end{pmatrix}$$

and

$$[T(\vec{u}_2)]_\alpha = \begin{pmatrix} b \\ d \end{pmatrix}.$$

We define the matrix of T relative to α *to be*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote this matrix by $[T]_\alpha$. In other words, we let the first column of $[T]_\alpha$ be the coordinates of $T(\vec{u}_1)$ relative to α, and we let the second column of $[T]_\alpha$ be the coordinates of $T(\vec{u}_2)$ relative to α.

Proposition 3.4.3. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$, and let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. We have*

$$[T(\vec{v})]_\alpha = [T]_\alpha \cdot [\vec{v}]_\alpha$$

for all $\vec{v} \in \mathbb{R}^2$.

Proposition 3.4.4. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$ and let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let $A = [T]$, let $B = [T]_\alpha$, and let $C = [Coord_\alpha]$. We then have that $(CA)\vec{v} = (BC)\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.*

Proposition 3.4.5. *Let A and B be 2×2 matrices. If $A\vec{v} = B\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$, then $A = B$.*

Corollary 3.4.6. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$ and let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let $A = [T]$, let $B = [T]_\alpha$, and let $C = [Coord_\alpha]$. We then have that $CA = BC$.*

Proposition 3.4.7. *Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Fix $a, b, c, d \in \mathbb{R}$ with*

$$\vec{u}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

and notice that $ad - bc \neq 0$ (because α is a basis of $\mathbb{R}^2$). Let

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then have that P is invertible and

$$[T]_\alpha = P^{-1}[T]P.$$

3.5 Eigenvalues and Eigenvectors

Definition 3.5.1. *Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.*

- *An* eigenvector *of T is a nonzero vector $\vec{v} \in \mathbb{R}^2$ such that there exists $\lambda \in \mathbb{R}$ with $T(\vec{v}) = \lambda\vec{v}$.*
- *An* eigenvalue *of T is a scalar $\lambda \in \mathbb{R}$ such that there exists a* nonzero *$\vec{v} \in \mathbb{R}^2$ with $T(\vec{v}) = \lambda\vec{v}$.*

When $\vec{v} \in \mathbb{R}^2$ is nonzero and $\lambda \in \mathbb{R}$ are such that $T(\vec{v}) = \lambda\vec{v}$, we say that $\vec{v}$ is an eigenvector of T corresponding to the eigenvalue λ.

Proposition 3.5.2. *Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $\vec{v}$ is an eigenvector of T, then there exists a unique $\lambda \in \mathbb{R}$ such that $\vec{v}$ is an eigenvector of T corresponding to λ. In other words, if $\vec{v}$ is an eigenvector of T corresponding to both $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, then $\lambda = \mu$.*

Definition 3.5.3. *Let A be a 2×2 matrix.*

- *An* eigenvector *of A is a nonzero vector $\vec{v} \in \mathbb{R}^2$ such that there exists $\lambda \in \mathbb{R}$ with $A\vec{v} = \lambda\vec{v}$.*
- *An* eigenvalue *of A is a scalar $\lambda \in \mathbb{R}$ such that there exists a* nonzero *$\vec{v} \in \mathbb{R}^2$ with $A\vec{v} = \lambda\vec{v}$.*

When $\vec{v} \in \mathbb{R}^2$ is nonzero and $\lambda \in \mathbb{R}$ are such that $A\vec{v} = \lambda\vec{v}$, we say that $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

Proposition 3.5.4. *Let A be a 2×2 matrix, let $\vec{v} \in \mathbb{R}^2$, and let $\lambda \in \mathbb{R}$. We have that $A\vec{v} = \lambda\vec{v}$ if and only if $\vec{v} \in Null(A - \lambda I)$. Therefore, $\vec{v}$ is an eigenvector of A corresponding to λ if and only if $\vec{v} \neq \vec{0}$ and $\vec{v} \in Null(A - \lambda I)$.*

Corollary 3.5.5. *Let A be a 2×2 matrix and let $\lambda \in \mathbb{R}$. We have that λ is an eigenvalue of A if and only if $Null(A - \lambda I) \neq \{\vec{0}\}$.*

Definition 3.5.6. *Given a 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define the characteristic polynomial *of A to be the following polynomial in variable λ:*

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Definition 3.5.11. *A* diagonal *2×2 matrix is a 2×2 matrix D such that there exists $a, d \in \mathbb{R}$ with*

$$D = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

Definition 3.5.12. *A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is* diagonalizable *if there exists $\alpha = (\vec{u}_1, \vec{u}_2)$ with $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$ such that $[T]_\alpha$ is a diagonal matrix.*

Proposition 3.5.13. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $\alpha = (\vec{u}_1, \vec{u}_2)$ be a basis of $\mathbb{R}^2$. The following are equivalent.*

1. *$[T]_\alpha$ is a diagonal matrix.*
2. *$\vec{u}_1$ and $\vec{u}_2$ are eigenvectors of T.*

Furthermore, in this case, the diagonal entries of $[T]_\alpha$ are the eigenvalues corresponding to $\vec{u}_1$ and $\vec{u}_2$, i.e. the upper left entry of $[T]_\alpha$ is the eigenvalue of $[T]$ corresponding to $\vec{u}_1$, and the lower right entry of $[T]_\alpha$ is the eigenvalue of T corresponding to $\vec{u}_2$.

Corollary 3.5.14. *A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is diagonalizable if and only if there exists eigenvectors $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ of T such that $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$.*

Proposition 3.5.19. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Suppose that $\vec{u}_1, \vec{u}_2$ are eigenvectors of T corresponding to* distinct *eigenvalues λ_1 and λ_2 respectively. We then have that $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, and so $\alpha = (\vec{u}_1, \vec{u}_2)$ is a basis of $\mathbb{R}^2$.*

Corollary 3.5.20. *If a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ has 2 distinct eigenvalues, then it is diagonalizable.*

3.6 Determinants

Consider a function, $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, with the following properties:

1. $f(\vec{e_1}, \vec{e_2}) = 1$.
2. $f(\vec{v}, \vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^2$.
3. $f(c \cdot \vec{v}, \vec{w}) = c \cdot f(\vec{v}, \vec{w}) = f(\vec{v}, c \cdot \vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^2$ and all $c \in \mathbb{R}$.
4. $f(\vec{v}, \vec{u} + \vec{w}) = f(\vec{v}, \vec{u}) + f(\vec{v}, \vec{w})$ and $f(\vec{v} + \vec{w}, \vec{u}) = f(\vec{v}, \vec{u}) + f(\vec{w}, \vec{u})$ for all $\vec{v}, \vec{u}, \vec{w} \in \mathbb{R}^2$.

Proposition 3.6.1. *If $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a function with the above 4 properties, then $f(\vec{w}, \vec{v}) = -f(\vec{v}, \vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^2$.*

Proposition 3.6.2. *If $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a function with the above 4 properties, then*

$$f\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) = ad - bc$$

for all $a, b, c, d \in \mathbb{R}$.

Proposition 3.6.3. *If we define $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$f\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) = ad - bc,$$

then f satisfies the above 4 properties.

Definition 3.6.4. *Give a matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define $\det(A) = ad - bc$ *and call this number the* determinant *of A. We also write*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

for the determinant of A.