Numerical simulation of a Coupled Double Pendulum Oscillator

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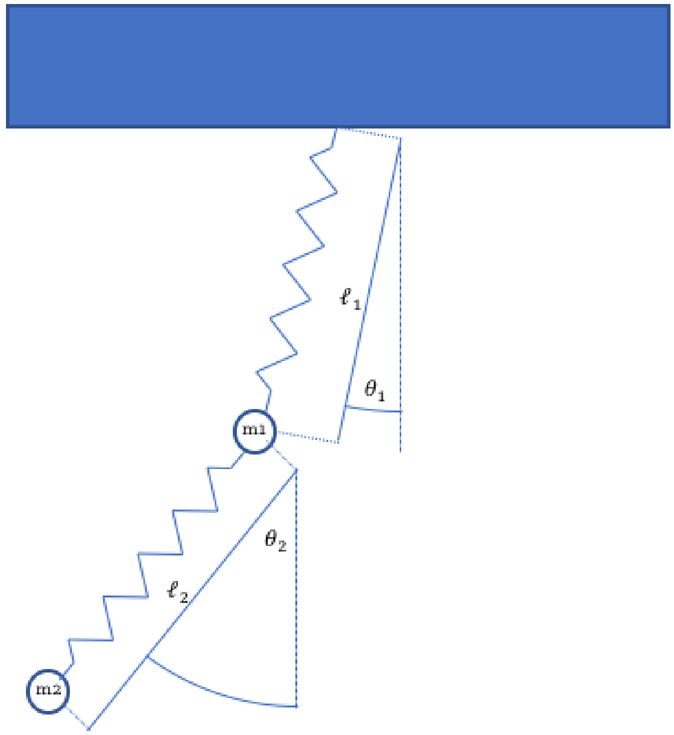


Figure 1: Coupled oscillator double pendulum.

Problem 1: We set out to gain a better understanding of a double spring pendulum system, that is, a system where a mass is connected to a fixed point by a spring, and then a second mass is connected to that mass by a second spring, as shown above.

We began by constructing a Lagrangian for this system:

$$
\begin{aligned}
\mathcal{L} =& \mathcal{T} - \mathcal{V} \\
\mathcal{T} =& \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
\mathcal{V} =& m_1 g y_1 + m_2 g y_2 + \frac{1}{2}k_1(l_1 - l_{0_1})^2 + \frac{1}{2}k_2(l_2 - l_{0_2})^2 \\
l_1 =& l_{1_0} + \Delta l_1(t) \\
l_2 =& l_{2_0} + \Delta l_2(t) \\
y_1 =& -l_1\cos(\theta_1) \\
y_2 =& -l_1\cos(\theta_1) - l_2\cos(\theta_2) \\
v_1^2 =& \dot{l}_1^2 + l_1^2\dot{\theta}_1^2 \\
v_2^2 =& \dot{l}_1^2 + l_1^2\dot{\theta}_1^2 + \dot{l}_2^2 + l_2^2\dot{\theta}_2^2 + 2\sin(\theta_1 - \theta_2)\cdot(\dot{l}_1 l_2\dot{\theta}_2 - \dot{l}_2 l_1\dot{\theta}_1) + 2\cos(\theta_1 - \theta_2)\cdot(\dot{l}_1\dot{l}_2 + l_1\dot{\theta}_1 l_2\dot{\theta}_2)
\end{aligned}
$$

So we get

$$
\begin{aligned}
\mathcal{L}(t, l_1, \dot{l}_1, \theta_1, \dot{\theta}_1, l_2, \dot{l}_2, \theta_2, \dot{\theta}_2) =& \frac{1}{2}m_1(\dot{l}_1^2 + l_1^2\dot{\theta}_1^2) + \frac{1}{2}m_2(\dot{l}_1^2 + l_1^2\dot{\theta}_1^2 + \dot{l}_2^2 + l_2^2\dot{\theta}_2^2 \\
&+ 2\sin(\theta_1 - \theta_2) \cdot (\dot{l}_1 l_2 \dot{\theta}_2 - \dot{l}_2 l_2 \dot{\theta}_1) + 2\cos(\theta_1 - \theta_2) \cdot (\dot{l}_1 \dot{l}_2 + l_1 \dot{\theta}_1 l_2 \dot{\theta}_2)) \\
&+ m_1 g l_1 \cos(\theta_1) + m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_2)) - \frac{1}{2}k_1(l_1 - l_{0_1})^2 - \frac{1}{2}k_2(l_2 - l_{0_2})^2
\end{aligned}
$$

We then solved the Lagrangian for each variable, obtaining four coupled equations:

$$
\begin{aligned}
\ddot{\theta}_1 =& \frac{(m_1 + m_2)(g\sin(\theta_1) + 2\dot{l}_1\dot{\theta}_1) + m_2((2\dot{l}_2\dot{\theta}_2 + l_2\ddot{\theta}_2)\cos(\theta_1 - \theta_2) + (l_2\dot{\theta}_2^2 - \ddot{l}_2)\sin(\theta_1 - \theta_2))}{-(m_1 + m_2)l_1} \\
\ddot{l}_1 =& \frac{(-m_1 - m_2)(g\cos(\theta_1) + l_1\dot{\theta}_1^2) + k_1(l_1 - l_{0_1}) + m_2((2\dot{l}_2\dot{\theta}_2 + l_2\ddot{\theta}_2)\sin(\theta_1 - \theta_2) + (\ddot{l}_2 - l_2\dot{\theta}_2^2)\cos(\theta_1 - \theta_2))}{-(m_1 + m_2)} \\
\ddot{\theta}_2 =& \frac{g\sin(\theta_2) + 2\dot{l}_2\dot{\theta}_2 + (2\dot{l}_1\dot{\theta}_1 + l_1\ddot{\theta}_1)\cos(\theta_1 - \theta_2) + (\ddot{l}_1 - l_1\dot{\theta}_1^2)\sin(\theta_1 - \theta_2)}{-l_2} \\
\ddot{l}_2 =& g\cos(\theta_2) + l_2\dot{\theta}_2^2 - \frac{k_2}{m_2}(l_2 - l_{0_2}) + ((2\dot{l}_1\dot{\theta}_1 + l_1\ddot{\theta}_1)\sin(\theta_1 - \theta_2) + (l_1\dot{\theta}_1^2 - \ddot{l}_1)\cos(\theta_1 - \theta_2))
\end{aligned}
$$

This system can be solved (with the assistance of Mathematica) to give unique solutions for $\ddot{\theta}_1, \ddot{l}_1, \ddot{\theta}_2, \ddot{l}_2$:

$$
\begin{aligned}
\ddot{\theta}_1 =& \frac{-gm_1\sin(\theta_1) - k_2(l_2 - l_{0_2})\sin(\theta_1 - \theta_2) - 2m_1\dot{l}_1\dot{\theta}_1}{m_1 l_1} \\
\ddot{l}_1 =& \frac{gm_1\cos(\theta_1) - k_1(l_1 - l_{0_1}) + k_2(l_2 - l_{0_2})\cos(\theta_1 - \theta_2) + l_1 m_1 \dot{\theta}_1^2}{m_1} \\
\ddot{\theta}_2 =& \frac{k_1(l_1 - l_{0_1})\sin(\theta_1 - \theta_2) - 2m_1\dot{l}_2\dot{\theta}_2}{m_1 l_2} \\
\ddot{l}_2 =& \frac{k_1 m_2(l_1 - l_{0_1})\cos(\theta_1 - \theta_2) - k_2(l_2 - l_{0_2})(m_1 + m_2) + l_2 m_1 m_2 \dot{\theta}_2^2}{m_1 m_2}
\end{aligned}
$$

Unfortunately, given the complexity of this system, it isn't really possible to analyze the full breadth of this system using some the usual methods, like looking for equilibrium points (there are so many variables in each equation that the solutions are essentially meaningless) and Poincare sections (given the 4 different coupled oscillators in this, there isn't a good way to find the natural frequency of the system). However, we can begin to understand this system by starting with the simplest case and then gradually increasing the total energy of the system in order find where it begins to get chaotic.

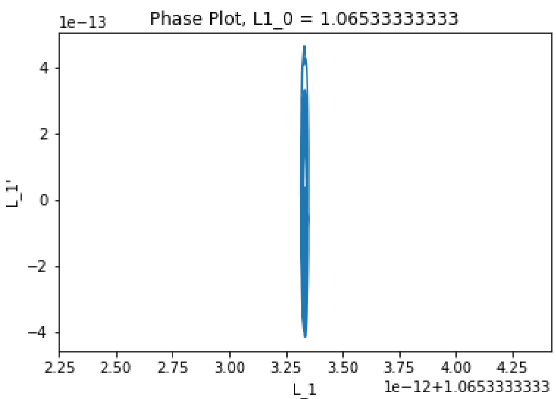
Minimum Energy System

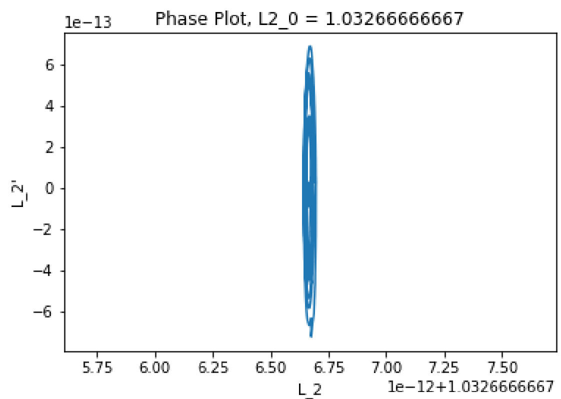
Before starting on this, it is useful to look at a system with no motion. To do so, we need to find the lengths of each spring in the hanging pendulum when the system is at rest. We can accomplish this using Newton 2:

$$F_{g2} = F_{s_2}$$
$$m_2 g = k(l_2 - l_{0_2})$$
$$l_2 = l_{0_2} + \frac{m_2 g}{k}$$

$$F_{s_1} = F_{s_2} + F_{g_1}$$
$$k(l_1 - l_{0_1}) = m_1 g + m_2 g$$
$$l_1 = l_{0_1} + \frac{g(m_1 + m_2)}{k}$$

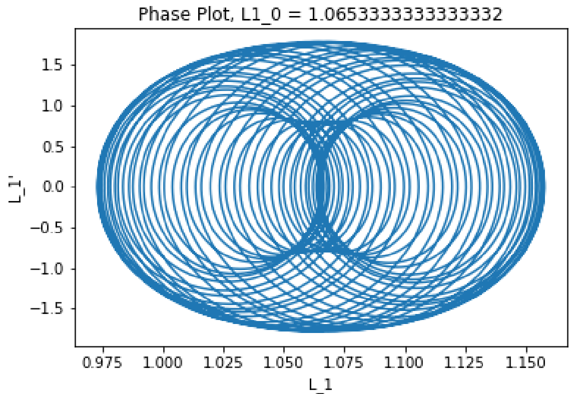
When we input this into our simulation, it allows us to check that everything is working as expected, and to look at how extreme the inaccuracy introduced by our numerical integration (rtol and atol were set to 1e-13). The phase portraits for θ in this were empty, but those for l_1 and l_2 actually did have some small correction:

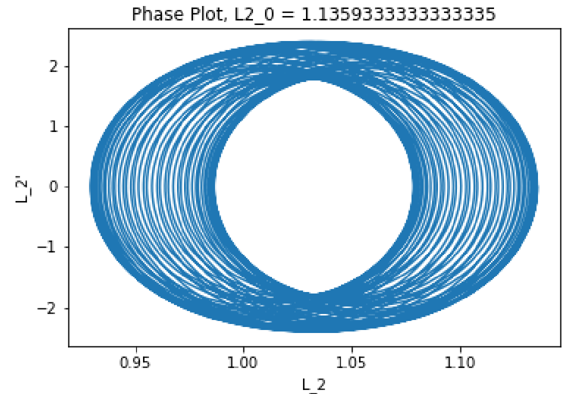




Systems with $\theta_1 = \theta_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$

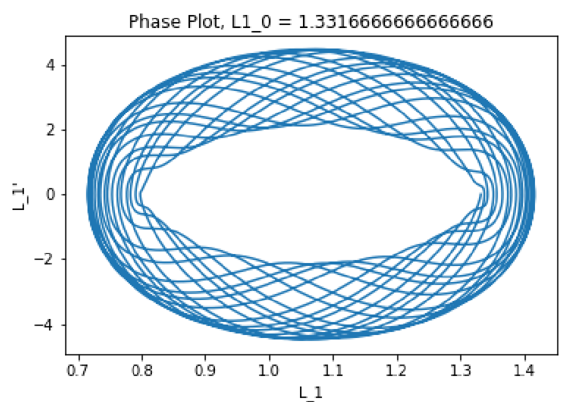
The most basic version of this system is one that starts with $\theta_1 = \theta_2 = 0$, and some small displacement from the rest length adjusted for gravity for m_2, and no initial velocities. For these starting conditions, we get the following phase plots:

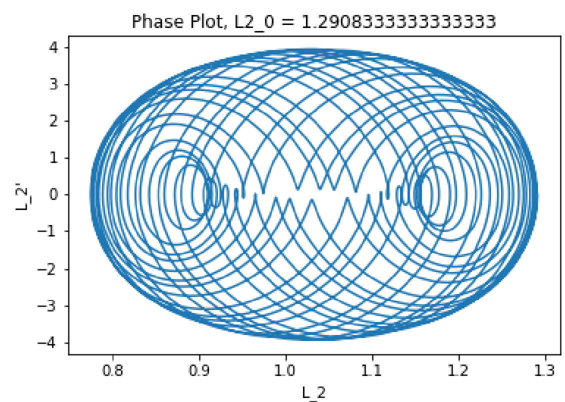




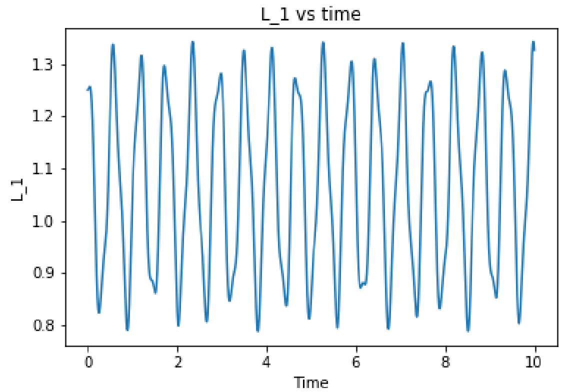
We can see from these that this looks similar to a pair of coupled springs. l_2 shows regular periodic motion with displacement back and forth over a small area of the length because of the first spring going up and down. l_1 is a little more complex looking because of the force from l_2 pushing it up and down in a way that causes some periods to be smaller and some to be larger.

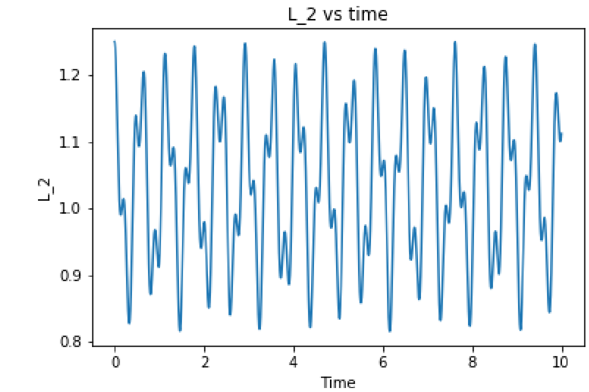
If we adjust both initial lengths to be equal to each other, we can get some interesting periodic behavior, as seen below:





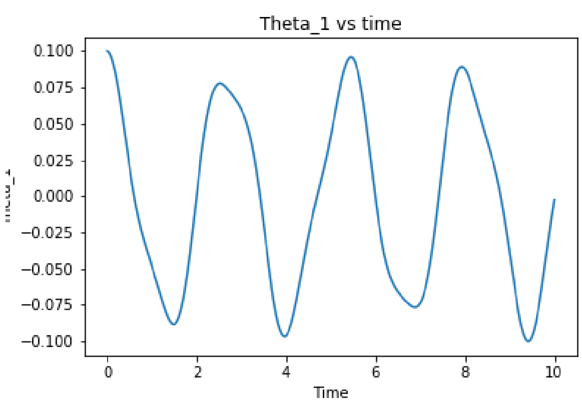
One might expect that these orbits would be perfect periodic since the springs have equal stretch, however, due to a combination of the force of gravity on the masses and the fact that the second spring pushes up on the first string, this causes a beating pattern between the two springs, which can more be more easily seen in the time plots:

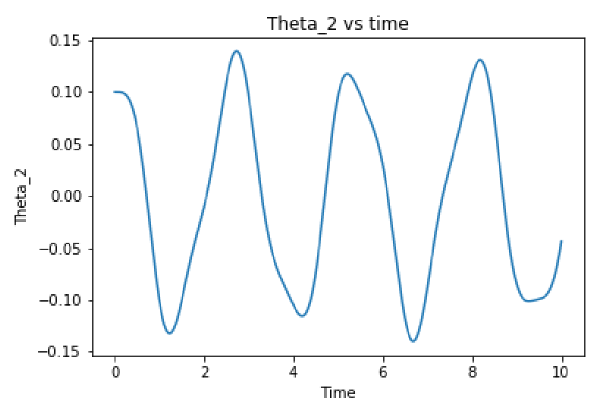


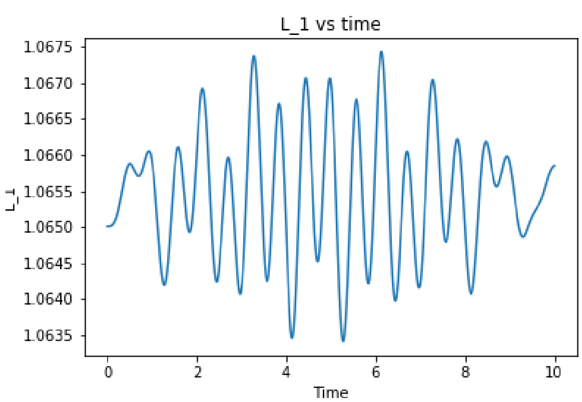


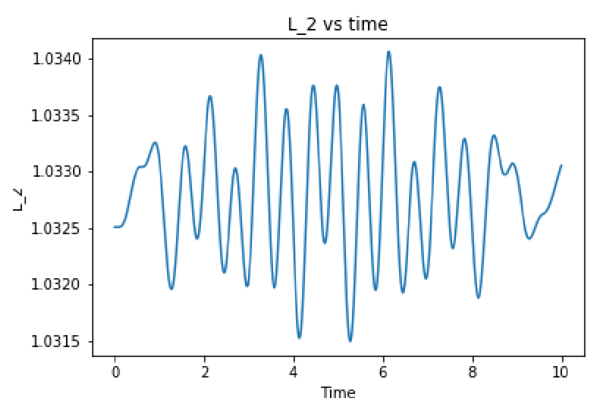
Introducing Displacements in θ

To start with, we can work with small displacements in θ to see how those complicate the system. Unfortunately, there is no simple way to get isolate the motion in the θ-dimension from that of the spring because as soon as you introduce the swinging motion it essentially creates a centrifugal force on the mass (in its frame) which causes a change in spring length. To combat this, one could adjust the effective spring length to that it would be at its gravitational rest length when the pendulum is at its fastest. However, that math becomes very complex for this system, so we will look at less specialized systems. First, we will look at $\theta_1 = \theta_2 = .1$. For this, we get the following plots for θ_1, θ_1, l_1, and l_2 respectively:



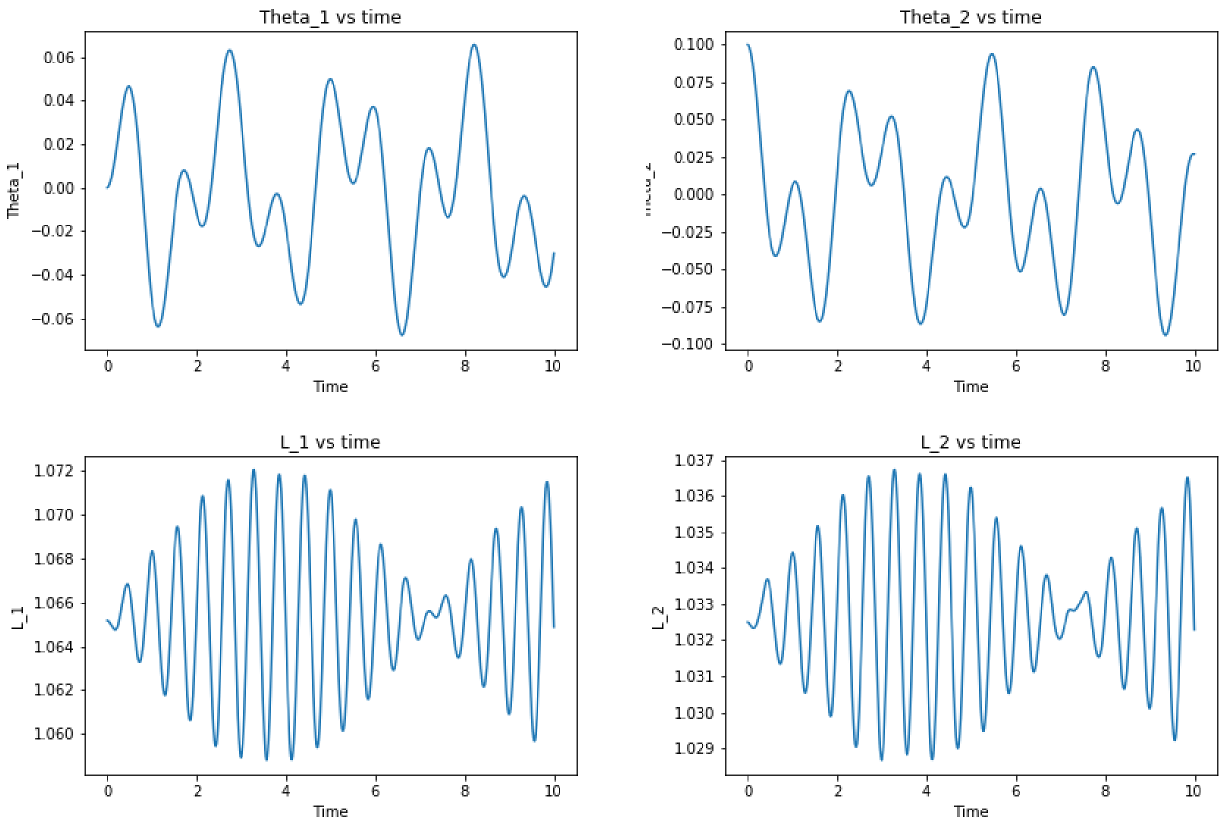






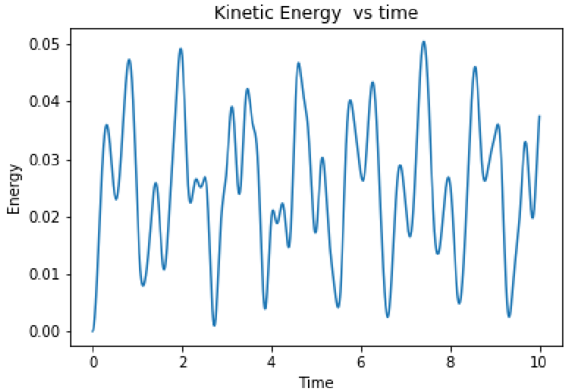
From these, we can see while this isn't chaotic motion, it is certainly more complex that the motion from before, as it has all 4 oscillators in play. The motion of each mass in the θ-dimension is relatively steady, while each of the springs are barely moving (the range of motion is about .004) with a very short period.

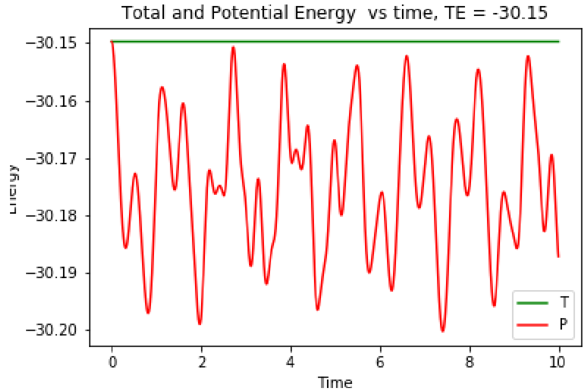
Another interesting system is where $\theta_1 = 0$ and $\theta_2 = 0.1$ (this behavior is also very similar for $\theta_1 = 0.1$ and $\theta_2 = 0$. In this system, we can see a beating pattern in the springs.



Finding the Threshold Energy for Chaos

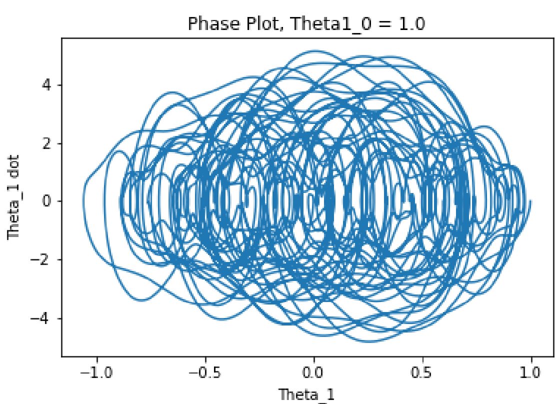
Now that we have a basic understanding of how this system acts when there is no angular displacement but there is spring displacement and when there are small angular displacements but not spring displacements, we can move on to trying to find a threshold energy for chaotic motion. For reference, the energy vs. time graphs for the last system mentioned above ($\theta_1 = 0$, $\theta_2 = 0.1$ look like the following:

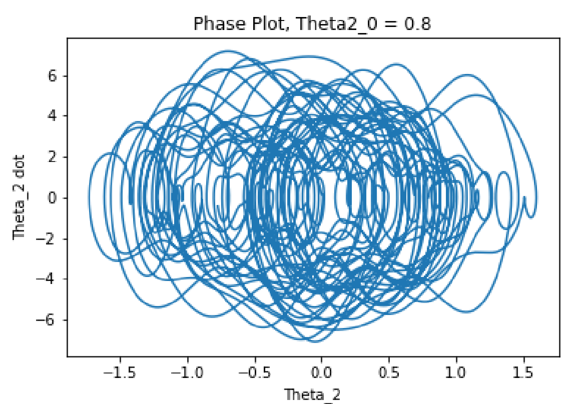


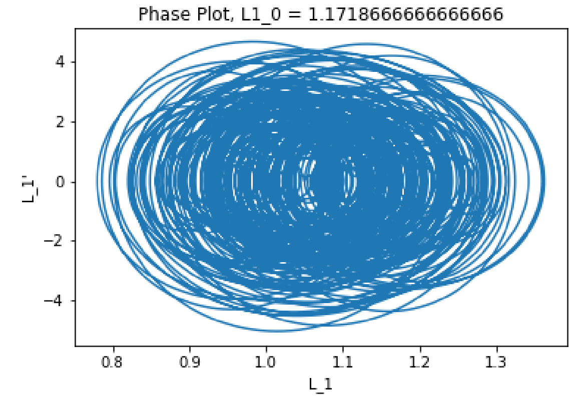


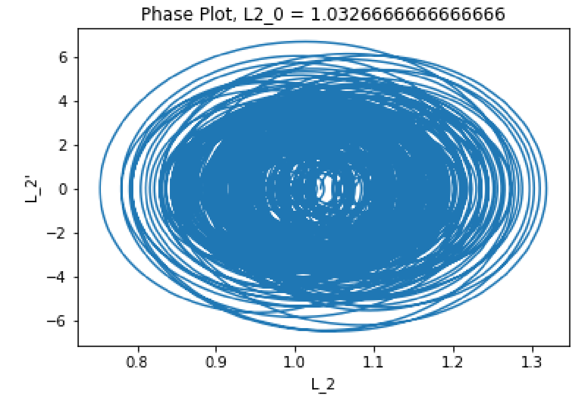
Once we increase the energy to get to the chaotic regime, we should expect to see a strong sensitivity to initial conditions. To find this, we can slowly increase the total energy by essentially lifting the pendulum up, until the system begins to show signs of chaos. As this is already a highly complex system, one would expect this threshold to be fairly low. While this method isn't entirely analytical, it does provide us with some interesting insights into the system.

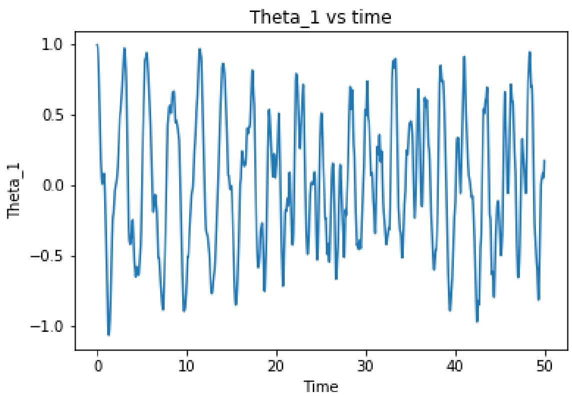
To find this threshold energy, we increased the timescale from 10 seconds to 50 seconds, and gradually adjusted the angles of the pendulum and the length of the springs upward until chaotic motion started to appear, and then adjusted back downward in different ways until the chaotic motion ceased. Given the complexity of the system, this may not actually be the lowest energy state where chaotic motion appears (it seems that different combinations of spring lengths and angles can lead to non-chaotic motion, even at higher energies). The lowest energy state where chaotic motion appeared that we found was at TE=-14.87, with initial conditions $\theta_{1_0} = 1, \theta_{2_0} = .8, l_{1_0} = 1.1 * L_{rest,gravity}, l_{2_0} = L_{rest,gravity}$. Below are a variety of graphs to help understand the motion of this. It may also be helpful to look at the simulation of this using the python notebook.

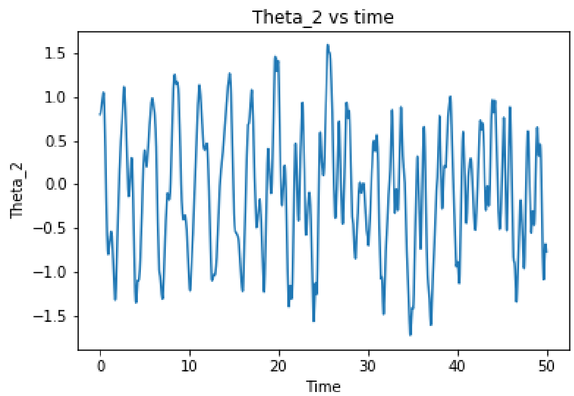


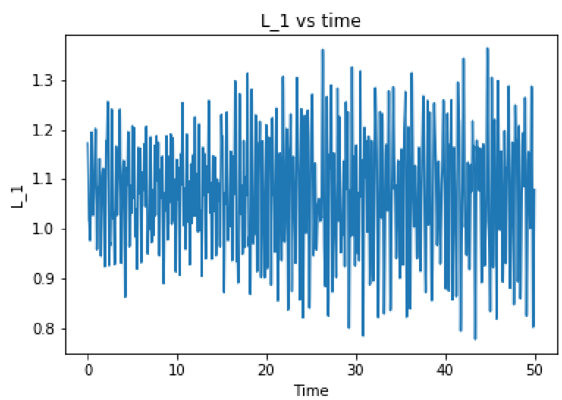


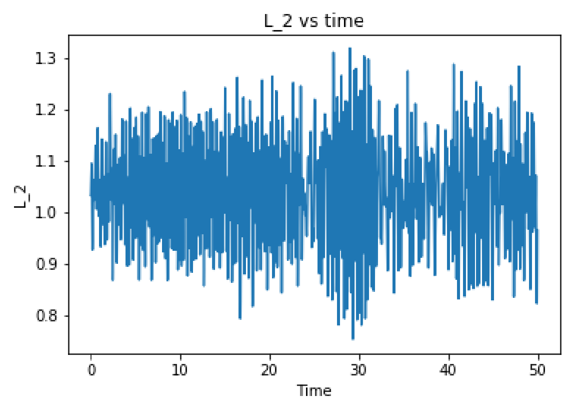

Phase Plot, L1_0 = 1.1718666666666666
L_1'
L_1

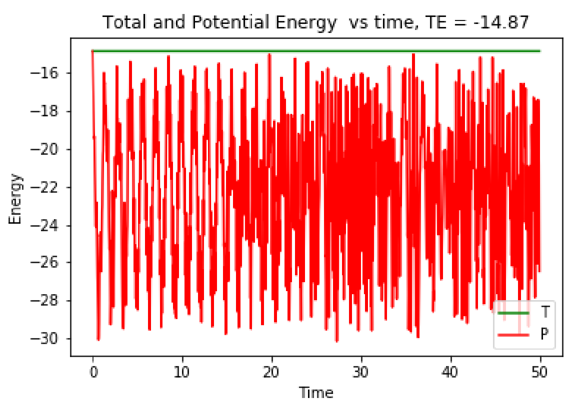

Phase Plot, L2_0 = 1.0326666666666666
L_2'
L_2

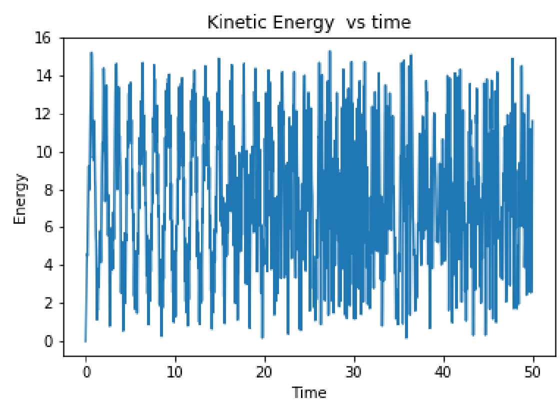

Theta_1 vs time
Theta_1
Time


Theta_2 vs time
Theta_2
Time


L_1 vs time
L_1
Time


L_2 vs time
L_2
Time


Total and Potential Energy vs time, TE = -14.87
Energy
Time
T
P


Kinetic Energy vs time
Energy
Time