Solutions to Problem Set 8

Problem 1: We first try to find $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In other words, we want to find $c_1, c_2 \in \mathbb{R}$ satisfying both of the following equations:

$$\begin{array}{rcrcr} 9c_1 & + & 2c_2 & = & 6 \\ 4c_1 & + & c_2 & = & 2. \end{array}$$

Since $9 \cdot 1 - 2 \cdot 4 = 1$ is nonzero, we may use Proposition 2.1.1 to conclude that there is a unique such pair of numbers $c_1, c_2 \in \mathbb{R}$, and it is when

$$c_1 = \frac{1 \cdot 6 - 2 \cdot 2}{9 \cdot 1 - 2 \cdot 4} = 2,$$

and

$$c_2 = \frac{9 \cdot 2 - 6 \cdot 4}{9 \cdot 1 - 2 \cdot 4} = -6.$$

Thus, we have

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix} + (-6) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since T is a linear transformation, we have

$$\begin{aligned} T\left(\begin{pmatrix} 6 \\ 2 \end{pmatrix}\right) &= T\left(2 \cdot \begin{pmatrix} 9 \\ 4 \end{pmatrix} + (-6) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \\ &= 2 \cdot T\left(\begin{pmatrix} 9 \\ 4 \end{pmatrix}\right) + (-6) \cdot T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \\ &= 2 \cdot \begin{pmatrix} 1 \\ -5 \end{pmatrix} + (-6) \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -28 \end{pmatrix}. \end{aligned}$$

Problem 2: We have

$$\begin{aligned} \begin{pmatrix} 4 & 3 \\ -7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} &= \begin{pmatrix} 4 \cdot 1 + 3 \cdot 5 \\ (-7) \cdot 1 + 1 \cdot 5 \end{pmatrix} \\ &= \begin{pmatrix} 19 \\ -2 \end{pmatrix}. \end{aligned}$$

In terms of linear transformations,this computation says that if $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the unique linear transformation with

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

then we have

$$T\left(\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} 19 \\ -2 \end{pmatrix}.$$

Problem 3: We can view a clockwise rotation around the origin by an angle of θ as the same thing as a counterclockwise rotation around the origin by an angle of $-\theta$. Thus, if $\theta \in \mathbb{R}$, then we have $C_\theta = R_{-\theta}$. Using Proposition 3.1.10, it follows that

$$[C_\theta] = [R_{-\theta}] = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}.$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, it follows that

$$[C_\theta] = [R_{-\theta}] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Alternatively, you can use some geometry and trigonometry to determine $C_\theta(\vec{e}_1)$ and $C_\theta(\vec{e}_2)$, and then put these into the columns of the matrix (simplifying as necessary).

Problem 4a: Notice that for any $x, y \in \mathbb{R}$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Thus, this transformation takes an arbitrary point in the plane, and projects it vertically up/down onto the x-axis.

We can also verify this through our work on projections. Notice that if we let

$$\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then using Proposition 3.1.11, we have

$$[P_{\vec{w}}] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so indeed our matrix gives a projection onto

$$\text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right),$$

which is the x-axis.

Problem 4b: Let $k \in \mathbb{R}$ be arbitrary with $k > 0$. Notice that for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} kx \\ ky \end{pmatrix} \\ &= k \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Thus, this transformation takes an arbitrary point in the plane, and scales it by a factor of k. In other words, we have the following cases.

- If $k > 1$, then every point is moved outward from the origin by factor of k.
- If $0 < k < 1$, then every point is contracted inward toward the origin by a factor of k
- If $k = 1$, then every point is left alone.

Problem 4c: Let $k \in \mathbb{R}$ be arbitrary with $k > 0$. Notice that for any $x, y \in \mathbb{R}$, we have

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ y \end{pmatrix}.$$

Thus, this transformation takes an arbitrary point in the plane and scales the x-component, but leaves it on the same horizontal line. In other words, we have the following cases.

- If $k > 1$, then every point is expanded horizontally away from the y-axis by factor of k.
- If $0 < k < 1$, then every point is contracted horizontally inward toward the y-axis by a factor of k.
- If $k = 1$, then every point is left alone.

Alternatively, we can describe the action as a horizontal stretch if $k > 1$, and a horizontal contraction if $0 < k < 1$.

Problem 5a: Let A be an arbitrary 2×2 matrix, and let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ be arbitrary. Fix $a, b, c, d, x_1, y_1, x_2, y_2 \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad \text{and} \qquad \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

We have

$$\begin{aligned}
A(\vec{v}_1 + \vec{v}_2) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} a(x_1 + x_2) + b(y_1 + y_2) \\ c(x_1 + x_2) + d(y_1 + y_2) \end{pmatrix} \\
&= \begin{pmatrix} ax_1 + ax_2 + by_1 + by_2 \\ cx_1 + cx_2 + dy_1 + dy_2 \end{pmatrix} \\
&= \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix} + \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= A\vec{v}_1 + A\vec{v}_2.
\end{aligned}$$

Problem 5b: We show that $A(r \cdot \vec{v}) = r \cdot A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$ and all $r \in \mathbb{R}$ (we changed the letter so that we can use c in our matrix). Let A be an arbitrary 2×2 matrix, and let $\vec{v} \in \mathbb{R}^2$ and $r \in \mathbb{R}$ be arbitrary. Fix $a, b, c, d, x, y \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We have

$$\begin{aligned}
A(r\cdot\vec{v}) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(r \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} rx \\ ry \end{pmatrix} \\
&= \begin{pmatrix} arx + bry \\ crx + dry \end{pmatrix} \\
&= \begin{pmatrix} r(ax+by) \\ r(cx+dy) \end{pmatrix} \\
&= r \cdot \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \\
&= r \cdot A\vec{v}.
\end{aligned}$$

Problem 6: To find $[T]$, we first determine $T(\vec{e}_1)$ and $T(\vec{e}_2)$. To do this, we first try to find $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

In other words, we want to find $c_1, c_2 \in \mathbb{R}$ satisfying both of the following equations:

$$\begin{array}{rcrcl} c_1 & + & -2c_2 & = & 1 \\ -c_1 & + & 3c_2 & = & 0. \end{array}$$

Since $1 \cdot 3 - (-2) \cdot (-1) = 1$ is nonzero, we may use Proposition 2.1.1 to conclude that there is a unique such pair of numbers $c_1, c_2 \in \mathbb{R}$. Adding the equations tells us that $c_2 = 1$, and plugging this into the first gives $c_1 - 2 = 1$, so $c_1 = 3$. Since $(3, 1)$ is the only possible solution, and we know that there is a solution, it follows that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Since T is a linear transformation, we have

$$\begin{aligned}
T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= T\left(3 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) \\
&= 3 \cdot T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) + 1 \cdot T\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) \\
&= 3 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 19 \end{pmatrix}.
\end{aligned}$$

We now determine $T(\vec{e}_2)$. As above, we first try to find $c_1, c_2 \in \mathbb{R}$ with

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

In other words, we want to find $c_1, c_2 \in \mathbb{R}$ satisfying both of the following equations:

$$\begin{array}{rcrcl} c_1 & + & -2c_2 & = & 0 \\ -c_1 & + & 3c_2 & = & 1. \end{array}$$

Since $1 \cdot 3 - (-2) \cdot (-1) = 1$ is nonzero, we may use Proposition 2.1.1 again to conclude that there is a unique such pair of numbers $c_1, c_2 \in \mathbb{R}$. Adding the equations tells us that $c_2 = 1$, and plugging this into the first gives $c_1 - 2 = 0$, so $c_1 = 2$. Since $(2, 1)$ is the only possible solution, and we know that there is a solution, it follows that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Since T is a linear transformation, we have

$$\begin{aligned} T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= T\left(2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) \\ &= 2 \cdot T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) + 1 \cdot T\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) \\ &= 2 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 15 \end{pmatrix}. \end{aligned}$$

It follows that

$$[T] = \begin{pmatrix} 5 & 4 \\ 19 & 15 \end{pmatrix}.$$