

Problem Set 10: Due Monday, March 10

Problem 1: Consider the unique linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}.$$

Find, with explanation, vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $\text{Null}(T) = \text{Span}(\vec{u})$ and $\text{range}(T) = \text{Span}(\vec{w})$.

Problem 2: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation. Recall that

$$\text{Null}(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}.$$

- a. Show that if $\vec{v}_1, \vec{v}_2 \in \text{Null}(T)$, then $\vec{v}_1 + \vec{v}_2 \in \text{Null}(T)$.
- b. Show that if $\vec{v} \in \text{Null}(T)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \text{Null}(T)$.

Problem 3: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}.$$

Explain why T has an inverse and calculate

$$T^{-1} \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} \right).$$

Problem 4: Consider the following system of equations:

$$\begin{array}{rclcl} x & + & 4y & = & -3 \\ 2x & + & 5y & = & 8. \end{array}$$

- a. Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix A and vector \vec{b} .
- b. Explain why A is invertible and calculate A^{-1} .
- c. Use A^{-1} to solve the system.

Problem 5: In this problem, let 0 denote the 2×2 zero matrix, i.e. the 2×2 matrix where all four entries are 0 .

- a. Give an example of a nonzero 2×2 matrix A with $A \cdot A = 0$.
- b. Show that if A is invertible and $A \cdot A = 0$, then $A = 0$.

Note: Since 0 is not invertible, it follows from part b that there is no invertible matrix A with $A \cdot A = 0$.

Problem 6: Let A, B, C all be invertible 2×2 matrices. Must there exist a 2×2 matrix X with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.