Solutions to Problem Set 13

Problem 1: Throughout this problem, let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear transformation with $[T] = A$. Also, fix $a, b, c, d \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose first that A is invertible. Since A is invertible, we know from Proposition 3.3.16 that $ad - bc \neq 0$. Using Corollary 3.3.5, it follows that T is injective, and hence $\text{Null}(T) = \{\vec{0}\}$ by Proposition 3.3.4. Therefore, we have $\text{Null}(A) = \{\vec{0}\}$, and since $A - 0I = A$, it follows that $\text{Null}(A - 0I) = \{\vec{0}\}$. Using Corollary 3.5.5, it follows that 0 is not an eigenvalue of A.

Suppose conversely that 0 is not an eigenvalue of A. By Corollary 3.5.5, we then have that $\text{Null}(A - 0I) = \{\vec{0}\}$, so $\text{Null}(A) = \{\vec{0}\}$. Therefore, we have $\text{Null}(T) = \{\vec{0}\}$, so T is injective by Proposition 3.3.4. By Corollary 3.3.5, it follows that $ad - bc \neq 0$. Using Proposition 3.3.16, it follows that A is invertible.

Problem 2a: We have

$$A - \lambda I = \begin{pmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{pmatrix},$$

so the characteristic polynomial of A is

$$\begin{aligned} \left(\frac{1}{2} - \lambda\right)(-\lambda) - \frac{1}{2} &= -\frac{1}{2} \cdot \lambda + \lambda^2 - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2} \cdot \lambda - \frac{1}{2} \\ &= (\lambda - 1)\left(\lambda + \frac{1}{2}\right). \end{aligned}$$

Thus, the eigenvalues of A are 1 and $-\frac{1}{2}$.

We first examine the case when $\lambda = 1$. We have

$$A - I = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}.$$

Therefore, a particular eigenvector of A corresponding to 1 is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We next examine the case when $\lambda = -\frac{1}{2}$. We have

$$A - \left(-\frac{1}{2}\right) I = A + \frac{1}{2} I = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}.$$

Therefore, a particular eigenvector of A corresponding to $-\frac{1}{2}$ is

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Hence, if we let

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

we then have

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and $A = PDP^{-1}$, i.e.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Problem 2b: Let

$$\vec{x}_0 = \begin{pmatrix} g_1 \\ g_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since

$$A\begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} = \begin{pmatrix} g_{n+2} \\ g_{n+1} \end{pmatrix}$$

for all $n \geq 0$ by part a, it follows that

$$A^n \vec{x}_0 = \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix}$$

for all $n \geq 0$. Therefore, we have that g_n is the second coordinate of $A^n \vec{x}_0$. Now

$$\begin{aligned}
A^n \vec{x}_0 &= (PDP^{-1})^n \vec{x}_0 \\
&= PD^nP^{-1}\vec{x}_0 \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2}) \end{pmatrix}^n \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \cdot (-\frac{1}{2})^n \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3} + \frac{1}{3} \cdot (-\frac{1}{2})^n \\ \frac{2}{3} - \frac{2}{3} \cdot (-\frac{1}{2})^n \end{pmatrix}.
\end{aligned}$$

Since g_n is the second coordinate of $A^n \vec{x}_0$, it follows that

$$g_n = \frac{2}{3} - \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^n = \frac{2}{3} \cdot \left(1 - \left(-\frac{1}{2}\right)^n\right).$$

Problem 2c: Since

$$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0,$$

it follows that

$$\lim_{n\to\infty} \left(1 - \left(-\frac{1}{2}\right)^n\right) = 1,$$

and therefore

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{2}{3}\cdot\left(1-\left(-\frac{1}{2}\right)^n\right) = \frac{2}{3}.$$

Problem 3: Fix $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$ with

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

We then have

$$AB = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\det(AB) &= (a_1a_2 + b_1c_2)\cdot(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)\cdot(c_1a_2 + d_1c_2) \\
&= a_1a_2c_1b_2 + a_1a_2d_1d_2 + b_1c_2c_1b_2 + b_1c_2d_1d_2 - (a_1b_2c_1a_2 + a_1b_2d_1c_2 + b_1d_2c_1a_2 + b_1d_2d_1c_2) \\
&= a_1a_2d_1d_2 + b_1c_2c_1b_2 - a_1b_2d_1c_2 - b_1d_2c_1a_2 + (a_1a_2c_1b_2 - a_1b_2c_1a_2 + b_1c_2d_1d_2 - b_1d_2d_1c_2) \\
&= a_1d_1a_2d_2 - a_1d_1b_2c_2 - b_1c_1a_2d_2 + b_1c_1b_2c_2 + 0 \\
&= a_1d_1\cdot(a_2d_2 - b_2c_2) - b_1c_1\cdot(a_2d_2 - b_2c_2) \\
&= (a_1d_1 - b_1c_1)\cdot(a_2d_2 - b_2c_2) \\
&= \det(A)\cdot\det(B).
\end{aligned}$$

Problem 4: Let A be an arbitrary invertible matrix. We know that $AA^{-1} = I$, so

$$\det(AA^{-1}) = \det(I).$$

Now $\det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$ by Problem 4, and

$$\begin{aligned}
\det(I) &= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= 1\cdot 1 - 0\cdot 0 \\
&= 1,
\end{aligned}$$

so

$$\det(A)\cdot\det(A^{-1}) = 1.$$

Now $\det(A)$ and $\det(A^{-1})$ are real numbers whose product is 1, so in particular we must have $\det(A) \neq 0$ (which also follow from the fact that $\det(A) \neq 0$ for any invertible matrix A). Dividing both sides by the nonzero number $\det(A)$, we conclude that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Problem 5: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary linear transformation, and let α an arbitrary basis of $\mathbb{R}^2$. By Proposition 3.4.7, we can fix an invertible matrix P with

$$[T]_\alpha = P^{-1}[T]P.$$

We then have

$$\begin{aligned}
\det([T]_\alpha) &= \det(P^{-1}[T]P) \\
&= \det(P^{-1}) \cdot \det([T]P) && \text{(by Problem 3)} \\
&= \det(P^{-1}) \cdot \det([T]) \cdot \det(P) && \text{(by Problem 3)} \\
&= \frac{1}{\det(P)} \cdot \det([T]) \cdot \det(P) && \text{(by Problem 4)} \\
&= \frac{\det(P)}{\det(P)} \cdot \det([T]) \\
&= \det([T]).
\end{aligned}$$

Problem 6: Let A be a 2×2 matrix and let $r \in \mathbb{R}$. Fix $a, b, c, d \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then have

$$r \cdot A = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix},$$

so

$$\begin{aligned}
\det(r \cdot A) &= (ra)(rd) - (rb)(rc) \\
&= r^2ad - r^2bc \\
&= r^2 \cdot (ad - bc) \\
&= r^2 \cdot \det(A).
\end{aligned}$$

It follows that $\det(r \cdot A) = r^2 \cdot \det(A)$.