Assignment: Problem Set 11

Name: Oleksandr Yardas

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Consider the unique linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix}.$$

Let $\alpha = (\vec{u_1}, \vec{u_2})$, where

$$\vec{u_1} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \vec{u_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

In this problem, we compute $[T]_\alpha$ directly from the definition.

a. Show that $\alpha = (\vec{u_1}, \vec{u_2})$ is a basis of $\mathbb{R}^2$.

Solution: Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be arbitrary. By definition, the ordered pair $(\vec{v}, \vec{u})$ is a basis for $\mathbb{R}^2$ if $\text{Span}(\vec{v}, \vec{u}) = \mathbb{R}^2$. Consider the case where $\vec{v} = \vec{u_1}, \vec{u} = \vec{u_2}$, and denote the ordered pair $(\vec{u_1}, \vec{u_2})$ by α, that is, let $\alpha = \left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$. Notice that $5 \cdot 1 - 2 \cdot 3 = 5 - 6 = -1 \neq 0$. Applying Theorem 3.4.1, it follows that $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$. Therfore, $\alpha = (\vec{u_1}, \vec{u_2})$ satisfies the definition of a basis of $\mathbb{R}^2$.

b. Determine $T(\vec{u_1})$ and then use this to compute $[T(\vec{u_1})]_\alpha$.

Solution: $[T(\vec{u_1})]_\alpha$ is shorthand for $Coord_\alpha(T(\vec{u_1}))$, where $Coord_\alpha$ is the linear transformation with standard matrix $[Coord_\alpha] = \frac{1}{-1} \cdot \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$ (by Definition 2.3.12). $Coord_\alpha(T(\vec{u_1})) = (Coord_\alpha \circ T)(\vec{u_1})$ by definition of function composition. T and $Coord_\alpha$ are both linear transformations, so by Proposition 2.4.8, $Coord_\alpha \circ T$ is also a linear transformation. By Proposition 3.1.4, $(Coord_\alpha \circ T)(\vec{u_1}) = [Coord_\alpha \circ T]\vec{u_1}$. By Proposition 3.2.2, $[Coord_\alpha \circ T] = [Coord_\alpha] \cdot [T]$, and it follows that $[Coord_\alpha \circ T]\vec{u_1} = [Coord_\alpha] \cdot [T] \cdot \vec{u_1}$. We conclude that $[T(\vec{u_1})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{u_1}$ and we compute:

$$\begin{aligned}
[T(\vec{u_1})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{u_1} &= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -1 \cdot 6 + 2 \cdot 4 & -1 \cdot -7 + 2 \cdot -5 \\ 3 \cdot 6 + -5 \cdot 4 & 3 \cdot -7 + -5 \cdot -5 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -6 + 8 & 7 + -10 \\ 18 + -20 & -21 + 25 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 5 + -3 \cdot 3 \\ -2 \cdot 5 + 4 \cdot 3 \end{pmatrix} \\
&= \begin{pmatrix} 10 - 9 \\ -10 + 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{aligned}$$

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Problem 2: With the same setup as Problem 1, compute $[T]_\alpha$ using Proposition 3.4.7.

Solution: Proposition 3.4.7 states that, given a basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ and a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, and fixing $a, b, c, d \in \mathbb{R}$ with $\vec{u_1} = \begin{pmatrix} a \\ c \end{pmatrix}, \vec{u_2} = \begin{pmatrix} b \\ d \end{pmatrix}$ so as to let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have that P is invertible and $[T]_\alpha = P^{-1}[T]P$. In this case, we have that $[T] = \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix}$ and $\alpha = \left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$, so $P = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$, and By Proposition 3.3.16, $P^{-1} = \frac{1}{5 \cdot 1 - 2 \cdot 3} \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} = \frac{1}{5-6} \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} = -1 \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$. Applying Proposition 3.4.7, we have:

$$\begin{aligned}
[T]_\alpha = P^{-1}[T]P &= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6 \cdot 5 + -7 \cdot 3 & 6 \cdot 2 + -7 \cdot 1 \\ 4 \cdot 5 + -5 \cdot 3 & 4 \cdot 2 + -5 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 30 - 21 & 12 - 7 \\ 20 - 15 & 8 - 5 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} \\
&= \begin{pmatrix} -1 \cdot 9 + 2 \cdot 5 & -1 \cdot 5 + 2 \cdot 3 \\ 3 \cdot 9 + -5 \cdot 5 & 3 \cdot 5 + -5 \cdot 3 \end{pmatrix} \\
&= \begin{pmatrix} -9 + 10 & -5 + 6 \\ 27 + -25 & 15 - 15 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}
\end{aligned}$$

This result agrees with our result from Problem 1.

Problem 3: Again, use the same setup as in Problem 1. Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

In this problem, we compute $[T(\vec{v})]_\alpha$ in two different ways.

a. First determine $T(\vec{v})$, and then use this to compute $[T(\vec{v})]_\alpha$.

Solution: Recall that in Problem 1, we reasoned that $[T(\vec{u_1})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{u_1}$. By similar reasoning, we have that $[T(\vec{v})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{v}$. Recall also that we found $[Coord_\alpha] \cdot [T] = \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix}$. So $[T(\vec{v})]_\alpha = \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + -3 \cdot 2 \\ -2 \cdot 1 + 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 - 6 \\ -2 + 8 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$.

b. First determine $[\vec{v}]_\alpha$, and then multiply by the result by your matrix $[T]_\alpha$ to compute$[T(\vec{v})]_\alpha$.

Solution: $[\vec{v}]_\alpha$ is shorthand for $Coord_\alpha(\vec{v})$. Recall from Problem 1 we reasoned that $Coord_\alpha(T(\vec{u_1})) = [Coord_\alpha] \cdot [T] \cdot \vec{u_1}$. By similar reasoning, we have that $Coord_\alpha(\vec{v}) = [Coord_\alpha] \cdot \vec{v}$. Recall also that we found $[Coord_\alpha] = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$. So we have that $[\vec{v}]_\alpha = [Coord_\alpha] \cdot \vec{v} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot 1 + -5 \cdot 2 \end{pmatrix} = \begin{pmatrix} -1 + 4 \\ 3 - 10 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$. Taking the matrix product with $[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, we get $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 1 \cdot -7 \\ 2 \cdot 3 + 0 \cdot -7 \end{pmatrix} = \begin{pmatrix} 3 - 7 \\ 6 + 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = [T(\vec{v})]_\alpha$.

Problem 4: Consider the unique linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}.$$

Let $\alpha = (\vec{u_1}, \vec{u_2})$, where

$$\vec{u_1} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \vec{u_2} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}.$$

Compute $[T]_\alpha$ using any method.

Solution: Let $P = \begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix}$, and so by Proposition 3.3.16, $P^{-1} = \frac{1}{-4 \cdot 4 - 9 \cdot -2} \begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} = \frac{1}{-16+18} \begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix}$. Applying Proposition 3.4.7, we have that

$$\begin{aligned}
[T]_\alpha = P^{-1}[T]P &= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 3 \cdot -4 + 2 \cdot -2 & 3 \cdot 9 + 2 \cdot 4 \\ 4 \cdot -4 + -1 \cdot -2 & 4 \cdot 9 + -1 \cdot 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -12 + -4 & 27 + 8 \\ -16 + 2 & 36 - 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -16 & 35 \\ -14 & 32 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot -16 + -\frac{9}{2} \cdot -14 & 2 \cdot 35 + -\frac{9}{2} \cdot 32 \\ 1 \cdot -16 + -2 \cdot -14 & 1 \cdot 35 + -2 \cdot 32 \end{pmatrix} \\
&= \begin{pmatrix} -32 + 63 & 70 - 144 \\ -16 + 28 & 35 - 64 \end{pmatrix} = \begin{pmatrix} 31 & -74 \\ 12 & -29 \end{pmatrix}
\end{aligned}$$

Therefore $[T]_\alpha = \begin{pmatrix} 31 & -74 \\ 12 & -29 \end{pmatrix}$.

Problem 5: Let A and B be 2×2 matrices. Assume that $A\vec{v} = B\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. Show that $A = B$.

Solution: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $S : \mathbb{R}^2 \to \mathbb{R}^2$ be arbitrary linear transformations, and let $[T] = A, [S] = B$. We assume that for all $\vec{v} \in \mathbb{R}^2, A\vec{v} = B\vec{v}$, so by definition of A, B, $[T]\vec{v} = [S]\vec{v}$ for for all $\vec{v} \in \mathbb{R}^2$. By Proposition 3.1.4, it follows that $T(\vec{v}) = S(\vec{v})$, and so by Proposition 3.1.6, $[T] = [S]$. By definition of $[T], [S]$, we conclude that $A = B$.

So we have that $[T(\vec{u_1})]_\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Note that $[Coord_\alpha] \cdot [T] = \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix}$. We will use this in the rest of the problem.

c. Determine $T(\vec{u_2})$ and then use this to compute $[T(\vec{u_2})]_\alpha$.

Solution: By the similar reasoning as in part b, we have that $[T(\vec{u_2})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{u_2}$. We compute:

$$\begin{aligned}
[T(\vec{u_2})]_\alpha = [Coord_\alpha] \cdot [T] \cdot \vec{u_2} &= \begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 2 + -3 \cdot 1 \\ -2 \cdot 2 + 4 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 - 3 \\ -4 + 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

So we have that $[T(\vec{u_2})]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

d. Using parts b and c, determine $[T]_\alpha$.

Solution: We have that $[T(\vec{u_1})]_\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, [T(\vec{u_2})]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So $[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ (by Definition 3.4.2).