Written Assignment 5: Due Friday, March 9

Problem 1: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Is it always possible to find a basis $\alpha = (\vec{u}_1, \vec{u}_2)$ of $\mathbb{R}^2$ such that $[T]_\alpha \neq [T]$? Either prove this is true, or give a counterexample (with justification).

Problem 2: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $\alpha = (\vec{u}_1, \vec{u}_2)$ and $\beta = (\vec{w}_1, \vec{w}_2)$ be bases of $\mathbb{R}^2$. Show that there exists an invertible 2×2 matrix R with $[T]_\beta = R^{-1} \cdot [T]_\alpha \cdot R$, and explicitly describe how to calculate R.

Problem 3: Given two 2×2 matrices A and B, write $A \sim B$ to mean that there exists a 2×2 invertible matrix P with $B = P^{-1}AP$.
a. Show that $A \sim A$ for all 2×2 matrices A.
b. Show that if A and B are 2×2 matrices with $A \sim B$, then $B \sim A$.
c. Show that if A, B and C are 2×2 with both $A \sim B$ and $B \sim C$, then $A \sim C$.
Cultural Aside: Using Problem 2 along with our work in class, it follows that $A \sim B$ if and only if A and B are both representations of a common linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, but with respect to possibly different coordinates. In this problem, you are proving that $\sim$ is something called an *equivalence relation*, a concept that you will see repeatedly throughout your mathematical journey.