Solutions to Problem Set 4

Problem 1: Notice that by definition of the absolute value function, we have

$$\{x \in \mathbb{R} : |x| < 5\} = \{x \in \mathbb{R} : -5 < x < 5\},$$

so we want to describe the set

$$\{x \in \mathbb{R} : -5 < x < 5\} \cup \{x \in \mathbb{R} : x \geq 3\}.$$

We claim that this set equals

$$\{x \in \mathbb{R} : x > -5\}.$$

To see this, first notice that if x satisfies $-5 < x < 5$, then we trivially have $x > -5$, and if x satisfies $x \geq 3$, then we certainly have $x \geq -5$, so

$$\{x \in \mathbb{R} : -5 < x < 5\} \cup \{x \in \mathbb{R} : x \geq 3\} \subseteq \{x \in \mathbb{R} : x > -5\}.$$

For the other containment, suppose that x satisfies $x > -5$. In this case, either $x \geq 3$, or we have $x < 3$, in which case $-5 < x < 5$. Thus

$$\{x \in \mathbb{R} : x > -5\} \subseteq \{x \in \mathbb{R} : -5 < x < 5\} \cup \{x \in \mathbb{R} : x \geq 3\}.$$

We have argued that both containments are true, so

$$\{x \in \mathbb{R} : |x| < 5\} \cup \{x \in \mathbb{R} : x \geq 3\} = \{x \in \mathbb{R} : x > -5\}.$$

Problem 2a: Notice that

$$\{10n : n \in \mathbb{N}\} = \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, \dots\}$$

consists of the nonnegative multiples of 10, which are those nonnegative numbers that end in a 0. Thus, to determine the smallest 3 elements of

$$\{6n : n \in \mathbb{N}\} \cap \{10n : n \in \mathbb{N}\},$$

we just need to determine the first 3 elements of $\{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, \dots\}$ that are multiples of 6. Checking through these in order, we see that the first 3 elements are 0, 30, and 60 (the fourth element is 90). Thus, we have

$$\{6n : n \in \mathbb{N}\} \cap \{10n : n \in \mathbb{N}\} = \{0, 30, 60, 90, \dots\}.$$

Problem 2b: Looking at the pattern of the first four elements, it appears that $\{6n : n \in \mathbb{N}\} \cap \{10n : n \in \mathbb{N}\}$ consists of the nonnegative multiples of 30, so we guess that

$$\{6n : n \in \mathbb{N}\} \cap \{10n : n \in \mathbb{N}\} = \{30n : n \in \mathbb{N}\}.$$

Problem 3a: We have $\{1, 3, 8, 9\} \triangle \{2, 3, 4, 7, 8\} = \{1, 2, 4, 7, 9\}$.

Problem 3b: We begin by listing out the first few elements of each of the two sets:

- $\{2n : n \in \mathbb{N}\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, \dots\}$
- $\{3n : n \in \mathbb{N}\} = \{3, 6, 9, 12, 15, 18, 21, \dots\}$.

Looking at these two sets, we see that the smallest 9 elements of the set $\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\}$ are:

$$2, 3, 4, 8, 9, 10, 14, 15, 16.$$

Thus, our set starts out as

$$\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\} = \{2, 3, 4, 8, 9, 10, 14, 15, 16, \dots, \}.$$

Problem 3c: Examining the above list, we see that it starts with 3 consecutive elements $(2, 3, 4)$, then skips the next 3, then includes the next 3, and this pattern seems to repeat. Since we appear to cycle around after 6 numbers, we use this value of 6 in our sets. In other words, we pull apart the following strands:

- $2, 8, 14, \dots$
- $3, 9, 15, \dots$
- $4, 10, 16, \dots$

Since the set of elements in each of these lists increase by 6 from each element to the next, we define the following sets:

- $A_1 = \{6n + 2 : n \in \mathbb{N}\}$.
- $A_2 = \{6n + 3 : n \in \mathbb{N}\}$.
- $A_3 = \{6n + 4 : n \in \mathbb{N}\}$.Since the numbers in each of these sets leave different remainders upon division by 6 compared to the numbers in the other two sets, these three sets appear to be pairwise disjoint and, if the pattern continues, then we would have

These three sets appear to be pairwise disjoint and, if the pattern continues, then it looks like we have

$$\{2n : n \in \mathbb{N}\} \triangle \{3n : n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3.$$

Problem 4a: We work through the values of f on each of the 12 inputs. We have the following:

- The set of positive divisors of 1 is $\{1\}$, so $f(1) = 1$.
- The set of positive divisors of 2 is $\{1, 2\}$, so $f(2) = 2$.
- The set of positive divisors of 3 is $\{1, 3\}$, so $f(3) = 2$.
- The set of positive divisors of 4 is $\{1, 2, 4\}$, so $f(4) = 3$.
- The set of positive divisors of 5 is $\{1, 5\}$, so $f(5) = 2$.
- The set of positive divisors of 6 is $\{1, 2, 3, 6\}$, so $f(6) = 4$.
- The set of positive divisors of 7 is $\{1, 7\}$, so $f(7) = 2$.
- The set of positive divisors of 8 is $\{1, 2, 4, 8\}$, so $f(8) = 4$.

- The set of positive divisors of 9 is $\{1, 3, 9\}$, so $f(9) = 3$.
- The set of positive divisors of 10 is $\{1, 2, 5, 10\}$, so $f(10) = 4$.
- The set of positive divisors of 11 is $\{1, 11\}$, so $f(11) = 2$.
- The set of positive divisors of 12 is $\{1, 2, 3, 4, 6, 12\}$, so $f(12) = 6$.

Since a function is formally defined as the set of input-output ordered pairs (see Definition 1.21), we have

$$f = \{(1,1), (2,2), (3,2), (4,3), (5,2), (6,4), (7,2), (8,4), (9,3), (10,4), (11,2), (12,6)\}.$$

Problem 4b: Recall that if $f \colon A \to B$ is a function, then

$$\text{range}(f) = \{b \in B : \text{There exists } a \in A \text{ with } f(a) = b\}.$$

In other words, range(f) is the set of elements of B that are actually outputs of f. In our example of f, we have

$$\text{range}(f) = \{1, 2, 3, 4, 6\}.$$

Problem 5: We claim that range$(f) \neq \mathbb{N}$. To see this, it suffices to give one example of an element of $\mathbb{N}$ that is not in range(f). We claim that 3 is such an example. We clearly have that $3 \in \mathbb{N}$.

We now show that $3 \notin \text{range}(f)$. To show this, we need to argue that $f((a,b)) \neq 3$ whenever $a, b \in \mathbb{N}$. To that end, let $a, b \in \mathbb{N}$ be arbitrary. We consider three cases:

- *Case 1:* Suppose that $a \geq 2$. In this case, we have

$$\begin{aligned} f((a,b)) &= a^2 + b^2 \\ &\geq 2^2 + 0 \\ &= 4, \end{aligned}$$

  so $f((a,b)) \neq 3$.

- *Case 2:* Suppose that $b \geq 2$. In this case, we have

$$\begin{aligned} f((a,b)) &= a^2 + b^2 \\ &\geq 0 + 2^2 \\ &= 4, \end{aligned}$$

  so $f((a,b)) \neq 3$.

- *Case 3:* Suppose that both $a \leq 1$ and $b \leq 1$. Since both $a, b \in \mathbb{N}$, we have that $a \in \{0, 1\}$ and $b \in \{0, 1\}$. Thus, we have

$$\begin{aligned} f((a,b)) &= a^2 + b^2 \\ &\leq 1^2 + 1^2 \\ &= 2, \end{aligned}$$

  so $f((a,b)) \neq 3$.

These cases exhaust all possibilities, so $f((a,b)) \neq 3$ unconditionally. Since $a, b \in \mathbb{N}$ were arbitrary, we can conclude that $3 \notin \text{range}(f)$, and this completes the argument.

Problem 6a: We have

$$f(2) = 5 \cdot 2 - 3 = 7,$$

so we can fill in the blank with $2 \in \mathbb{Q}$.

Problem 6b: We have

$$f(-10) = 5 \cdot (-10) - 3 = -53,$$

so we can fill in the blank with $-10 \in \mathbb{Q}$.

Problem 6c: We have

$$f\left(\frac{4}{5}\right) = 5 \cdot \frac{4}{5} - 3 = 1,$$

so we can fill in the blank with $\frac{4}{5} \in \mathbb{Q}$.

Problem 6d: We argue that $\mathbb{Q} \subseteq \text{range}(f)$. Let $b \in \mathbb{Q}$ be arbitrary. We then have that $b + 3 \in \mathbb{Q}$ (because the sum of rational numbers is a rational number), hence $\frac{b+3}{5} \in \mathbb{Q}$ (because the quotient of a rational number by a nonzero rational number is a rational number). Now

$$\begin{aligned} f\left(\frac{b+3}{5}\right) &= 5 \cdot \left(\frac{b+3}{5}\right) - 3 \\ &= (b+3) - 3 \\ &= b. \end{aligned}$$

Thus, we can fill in the blank with $\frac{b+3}{5} \in \mathbb{Q}$.