Worksheet 19 - Spans

Exercise 1. Let V be a vector space, with $\vec{u}, \vec{v} \in V$. Show that if $\vec{w} \in \text{Span}(\vec{u}, \vec{v})$ then $\text{Span}(\vec{u}, \vec{v}, \vec{w}) = \text{Span}(\vec{u}, \vec{v})$.

Exercise 2. Let V be a vector space, with $\vec{u}, \vec{v} \in V$. Show that if $\vec{w} = c_1\vec{u} + c_2\vec{v}$ for non-zero $c_1, c_2 \in \mathbb{R}$, then $\text{Span}(\vec{u}, \vec{v}) = \text{Span}(\vec{u}, \vec{w}) = \text{Span}(\vec{w}, \vec{v})$.

Exercise 3. Let V be a vector space, with $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in V$. Show that if $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$ for non-zero $c_1, c_2, c_3 \in \mathbb{R}$, then $\text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{v})$.

Exercise 4. In Exercise 2 is it necessary that *both* c_1 and c_2 are not zero? Explain.

Exercise 5. In Exercise 3 is it necessary that c_1, c_2 and c_3 are *all* not zero? Explain.

Exercise 6. Let V be a vector space, with $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{w} \in V$. If $\text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) \neq \text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{w})$, then what can you say about $\vec{w}$?

Exercise 7. Let V be a vector space with $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n, \vec{w} \in V$. Under what conditions can we say that $\text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n) = \text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n, \vec{w})$? Prove that your answer is correct.

Exercise 8. Let V be a vector space with $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in V$, and suppose

$$\vec{w} = d_1\vec{u}_1 + d_2\vec{u}_2 + \cdots + d_n\vec{u}_n$$

for $d_i \in \mathbb{R}$. State conditions under which we can replace $\vec{u}_k$ with $\vec{w}$ in $\text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ and conclude that

$$\text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{k-1}, \vec{u}_k, \vec{u}_{k+1}, \dots, \vec{u}_n) = \text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{k-1}, \vec{w}, \vec{u}_{k+1}, \dots, \vec{u}_n)$$

Prove that your answer is correct.