Solutions to Problem Set 5

Problem 1: To show that f is not injective, we need to show that the statement

$$\text{“For all } x_1, x_2 \in \mathbb{R}, \text{ if } f(x_1) = f(x_2), \text{ then } x_1 = x_2\text{”}$$

is false, which is the same as showing that its negation

$$\text{“There exists } x_1, x_2 \in \mathbb{R} \text{ with } f(x_1) = f(x_1) \text{ and } x_1 \neq x_2\text{”}$$

is true. In order to verify this, we just need to provide examples of such x_1 and x_2 (with justification). Notice that

$$\begin{aligned} f(0) &= 0^3 - 8 \cdot 0 \\ &= 0 - 0 \\ &= 0, \end{aligned}$$

and that

$$\begin{aligned} f(\sqrt{8}) &= (\sqrt{8})^3 - 8\sqrt{8} \\ &= (\sqrt{8})^2 \cdot \sqrt{8} - 8\sqrt{8} \\ &= 8\sqrt{8} - 8\sqrt{8} \\ &= 0. \end{aligned}$$

Therefore, we have that $f(\sqrt{8}) = 0 = f(0)$. Since $\sqrt{8} \neq 0$ and $\sqrt{8}, 0 \in \mathbb{R}$, we have shown that f is not injective.

Problem 2: To determine if the three lines intersect, we want to determine if the system

$$\begin{array}{rcrcr} 2x & + & y & = & 5 \\ 7x & - & 2y & = & 1 \\ -5x & + & 3y & = & 4 \end{array}$$

has a solution. Suppose that (x, y) is a solution to this system. Adding twice the first equation to the second equation, we conclude that (x, y) must also satisfy $11x = 11$, and hence we must have $x = 1$. Plugging this into the first equation, we see that (x, y) must also satisfy $2 + y = 5$, and hence we must have $y = 3$. Therefore, the only possible solution is $(1, 3)$. We now verify that $(1, 3)$ is indeed a solution by checking that it satisfies all three equations.

- We have $2 \cdot 1 + 3 = 5$, so $(1, 3)$ satisfies the first equation.
- We have $7 \cdot 1 - 2 \cdot 3 = 1$, so $(1, 3)$ satisfies the second equation.
- We have $(-5) \cdot 1 + 3 \cdot 3 = 4$, so $(1, 3)$ satisfies the third equation.

Therefore, $(1, 3)$ is a solution to the above system. It follows that the three lines intersect at $(1, 3)$.

Problem 3a: We can rewrite our given equation as $x = 6 + 9y$. From here, we can give a parametric equation that traces out the solution set to our equation by making y our parameter:

$$\begin{aligned} x &= 6 + 9t \\ y &= t \end{aligned}$$

Thus, the solution set to our equation can be written as

$$\left\{\begin{pmatrix} 6+9t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}.$$

We can rewrite this set as

$$\left\{\begin{pmatrix} 6+9t \\ 0+1t \end{pmatrix} : t \in \mathbb{R}\right\},$$

which is the same as

$$\left\{\begin{pmatrix} 6 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 9 \\ 1 \end{pmatrix} : t \in \mathbb{R}\right\}.$$

Thus, we can take

$$\vec{v} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{u} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$$

Problem 3b: We can rewrite our given equation as $y = -\frac{5}{3}x$. From here, we can give a parametric equation that traces out the solution set to our equation by making x our parameter:

$$\begin{aligned} x &= t \\ y &= (-5/3) \cdot \cdot t \end{aligned}$$

Thus, the solution set to our equation can be written as

$$\left\{\begin{pmatrix} t \\ (-5/3) \cdot t \end{pmatrix} : t \in \mathbb{R}\right\}.$$

We can rewrite this set as

$$\left\{t \cdot \begin{pmatrix} 1 \\ -5/3 \end{pmatrix} : t \in \mathbb{R}\right\},$$

and this set is the definition of

$$\text{Span}\left(\begin{pmatrix} 1 \\ -5/3 \end{pmatrix}\right).$$

Therefore, we can take

$$\vec{u} = \begin{pmatrix} 1 \\ -5/3 \end{pmatrix}.$$

Problem 3c: Since we want the solution set of $ax + by = c$ to equal

$$\text{Span}\left(\begin{pmatrix} 2 \\ -7 \end{pmatrix}\right),$$

which is a line through the origin, we should take $c = 0$. Now the direction of our line is $\langle 2, -7 \rangle$, so our line should have slope $\frac{-7}{2}$. Since the slope of $ax + by = c$ equals $-\frac{a}{b}$ (assuming that $b \neq 0$), we can guess that we should take $a = 7$ and $b = 2$.

To see that this works, consider the equation $7x + 2y = 0$. If we parametrize as in part b, we arrive at

$$\begin{aligned} x &= t \\ y &= (-7/2) \cdot t. \end{aligned}$$

Thus, the solution set to our equation can be written as

$$\left\{ \begin{pmatrix} t \\ (-7/2)\cdot t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \cdot \begin{pmatrix} 1 \\ -7/2 \end{pmatrix} : t \in \mathbb{R} \right\},$$

which equals

$$\text{Span}\left(\begin{pmatrix} 1 \\ -7/2 \end{pmatrix} \right).$$

Since the vectors

$$\begin{pmatrix} 2 \\ -7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -7/2 \end{pmatrix}$$

are multiples of each other, they have the same span. Therefore, our equation $7x + 2y = 0$ has the required solution set.

Problem 4 Preamble: Before jumping into parts a and b of this problem, let's first play around to get a feel for what is happening. We'll pick examples of vectors in one set, and show why those vectors are in the other set.

- Notice that $\begin{pmatrix} 3 \\ -1 \end{pmatrix} \in A$ using $c = 0$, and that $\begin{pmatrix} 3 \\ -1 \end{pmatrix} \in B$ using $c = -2$.
- Notice that $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \in A$ using $c = 1$, and that $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \in B$ using $c = -1$.
- Notice that $\begin{pmatrix} 5 \\ 7 \end{pmatrix} \in A$ using $c = 2$, and that $\begin{pmatrix} 5 \\ 7 \end{pmatrix} \in B$ using $c = 0$.

From these small examples, it looks likely that if a vector is in A because of a certain value of c, then the number $c - 2$ will witness the fact that the given vector is B. In the other direction, it looks likely that if a vector is in B because of a certain value of c, then the number $c + 2$ will witness the fact that the given vector is A.

Problem 4a: Let $\vec{u} \in A$ be arbitrary. By definition of A, we can fix $d \in \mathbb{R}$ with

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Now notice that

$$\begin{aligned} \begin{pmatrix} 5 \\ 7 \end{pmatrix} + (d-2)\cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + (-2)\cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -8 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \vec{u} \end{aligned}$$

Since $d - 2 \in \mathbb{R}$, we conclude that $\vec{u} \in B$. Since $\vec{u} \in A$ was arbitrary, the result follows.

Problem 4b: Let $\vec{w} \in B$ be arbitrary. By definition of B, we can fix $d \in \mathbb{R}$ with

$$\vec{w} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Now notice that

$$\begin{aligned} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (d+2) \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} + d \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \vec{w} \end{aligned}$$

Since $d + 2 \in \mathbb{R}$, we conclude that $\vec{w} \in A$. Since $\vec{w} \in B$ was arbitrary, the result follows.

Problem 5: Notice that for any $x \in \mathbb{R}$, we have

$$\begin{aligned} (f_a \circ g_b)(x) &= f_a(g_b(x)) \\ &= f_a(x+b) \\ &= a(x+b) \\ &= ax + ab. \end{aligned}$$

Also for any $x \in \mathbb{R}$, we have

$$\begin{aligned} (g_b \circ f_a)(x) &= g_b(f_a(x)) \\ &= g_b(ax) \\ &= ax + b. \end{aligned}$$

Looking at these two results, it seems likely that we want $ab = b$ for these two functions to be equal, which is the same as saying that either $b = 0$ or $a = 1$. We now prove this. Let $a, b \in \mathbb{R}$ be arbitrary.

- *Case 1:* Suppose that $b = 0$. In this case, we have

  $$\begin{aligned} (f_a \circ g_b)(x) &= ax + ab && \text{(from above)} \\ &= ax + a \cdot 0 \\ &= ax \end{aligned}$$

  for all $x \in \mathbb{R}$. Also, we have

  $$\begin{aligned} (g_b \circ f_a)(x) &= ax + b && \text{(from above)} \\ &= ax + 0 \\ &= ax \end{aligned}$$

  for all $x \in \mathbb{R}$. Therefore, we conclude that $(f_a \circ g_b)(x) = (g_b \circ f_a)(x)$ for all $x \in \mathbb{R}$, so $f_a \circ g_b = g_b \circ f_a$.

- *Case 2:* Suppose that $a = 1$. In this case, we have

  $$\begin{aligned} (f_a \circ g_b)(x) &= ax + ab && \text{(from above)} \\ &= 1 \cdot x + 1 \cdot b \\ &= x + b \end{aligned}$$

for all $x \in \mathbb{R}$. Also, we have

$$\begin{aligned}
(g_b \circ f_a)(x) &= ax + b && \text{(from above)} \\
&= 1 \cdot x + b \\
&= x + b
\end{aligned}$$

for all $x \in \mathbb{R}$. Therefore, we conclude that $(f_a \circ g_b)(x) = (g_b \circ f_a)(x)$ for all $x \in \mathbb{R}$, so $f_a \circ g_b = g_b \circ f_a$.

- *Case 3:* Suppose that $b \neq 0$ and that $a \neq 1$. Notice that

$$\begin{aligned}
(f_a \circ g_b)(0) &= a \cdot 0 + ab && \text{(from above)} \\
&= ab
\end{aligned}$$

and that

$$\begin{aligned}
(g_b \circ f_a)(0) &= a \cdot 0 + b && \text{(from above)} \\
&= b.
\end{aligned}$$

Now if $ab = b$, then since $b \neq 0$ we can divide both sides by it to conclude that $a = 1$, which is a contradiction. Therefore, we have $ab \neq b$, and hence $(f_a \circ g_b)(0) \neq (g_b \circ f_a)(0)$. Since we have found a value where the two functions disagree, we conclude that $f_a \circ g_b \neq g_b \circ f_a$.