Homework Assignment 2
PHYSICS 314 - THERMODYNAMICS & STATISTICAL PHYSICS (Spring 2018)
Due Friday, February 9th, by noon, Noyce 1135

I cannot award full credit for work that I am unable to read or follow. For my benefit and for yours, please:
- o   *Write neatly*
- o   *Show and EXPLAIN all steps*
- o   *Make diagrams large and clearly-labeled*

You are welcome to collaborate with others on this assignment. However, the work you turn in should be your own. Please cite collaborators and outside sources. See the syllabus for details.

Regardless of the number of parts, all homework problems are weighted equally. Regardless of the number of questions, all homework assignments are weighted equally.

1) A cylindrical container with cross-sectional area *S* and closed off at its top by a piston of mass *m*, contains n moles of an ideal gas. The walls of the cylinder are *diathermal*, which means they conduct heat readily. It is placed in a room where the pressure is P_{ext}, and the temperature is T_0. In its initial state, the temperature of the gas is also T_0, its volume is V_0, and its pressure is P_0. The piston is attached to the ceiling with a string. You cut the string and wait for the gas to attain its final state.

Outside
T_0, V>>V_0, P_{ext}
Inside
N, T_0, V_0, P_0
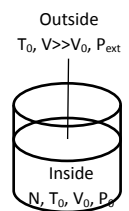
a) Determine the final state of the gas by finding:
   i) P_f, (*Hint: When does the piston stop moving? Think Physics 131.*)

   This piston will stop moving when the net force on it equal zero. The forces downward are from the external pressure and from gravity. The force upward comes from the internal pressure P_f. Multiplying the pressures by the cross-sectional area *S* gives the forces from the pressures.

   $$P_f S = P_{ext}S + mg$$

   $$P_f = P_{ext} + \frac{mg}{S}$$

   ii) T_f, and

   The walls readily conduct heat, and this energy transfer will keep the internal temperature the same as the external temperature. (If the internal temperature were different then the external temperature, they would quickly equalize. Because there is much more gas outside of the cylinder, the equilibrium temperature would essentially be the same as the original external temperature). Thus, there is no temperature change.
   $$T_f = T_0$$

   iii) V_f.

   With P_f and T_f known, use the ideal gas law to find V_f.

   $$V_f = \frac{nRT_f}{P_f}$$

$$V_f = \frac{nRT_0}{P_{ext} + \frac{mg}{S}}$$

b) Calculate the variation in the total energy of the gas (ΔU).

Use the equipartition theorem.

$$\Delta U = \frac{f}{2} nR\Delta T$$

As discussed in ii), the initial and final temperatures are the same.

$$\Delta U = \frac{f}{2} nR(0)$$

$$\Delta U = 0$$

c) The amount of heat that flows into the room in the process is measured to be Q_0. What is the work done on the gas by the piston? (*Be careful about the sign!*)

Use the First Law and your answer from b).

$$\Delta U = Q + W = 0$$

$$W = -Q$$

This is the work done on the gas by the piston. However, note that this is the amount of heat flowing into the gas; the negative sign means Q flows out of the gas and into the room. Thus, -Q is Q_0.

$$W = Q_0$$

This makes sense from a conservation of energy standpoint. We put energy W into the gas through work. The energy of the gas does not change, however, so that same energy W must be released into the room through Q_0.

2) A cylinder with adiabatic (non-thermally conducting) walls is divided into two compartments, A and B. Each one contains one mole of a monatomic ideal gas. The separating wall between A and B is a non-permeable piston that is free to move. Initially, in compartment A, a small heater (of negligible heat capacity) slowly heats the gas until a *long time* later, t_f, when a thermometer is placed in compartment B, indicating its final temperature, $T_B = 321$ K.
Assume that $C_V = 3/2$ R for both compartments, and assume that initially for both compartments $T_0 = 273$ K and $P_0 = 105$ Pa. (*Note: This is not a typo: 105 Pa, NOT 10^5 Pa.*) Calculate the initial and final temperatures, pressures, and volumes in each of the two compartments, as well as the energy provided by the heater in the case of:
a) a diathermal piston.

The initial temperatures and pressures are given.

$$T_{Ai} = T_{Bi} \equiv T_i = 273\ K$$

$$P_{Ai} = P_{Bi} \equiv P_i = 105\ Pa$$

The initial volumes can be calculated using the ideal gas law and n = 1 mole for both sides. Because the initial temperatures, pressures, and number of mole are the same for both sides, the initial volumes will be the same, V_i.

$$V_i = \frac{nRT_0}{P_0}$$

$$V_i = \frac{1\ mol\ \left(8.314\ ^J/_{mol\ K}\right)273\ K}{105\ Pa}$$

$$V_{Ai} = V_{Bi} \equiv V_i \approx 21.6\ m^3$$

The piston is diathermal and thus conducts heat readily, so the two temperatures must be the same after a long time. In the final state, they are in thermal equilibrium.

$$T_{Af} = T_{Bf} \equiv T_f = 321\ K$$

Similarly, after a long time, they will move the piston to be in mechanical equilibrium.

$$P_{Af} = P_{Bf}$$

Use the ideal gas law. T_f is the same for both, as mentioned above.

$$\frac{nRT_f}{V_{Af}} = \frac{nRT_f}{V_{Bf}}$$

$$V_{Af} = V_{Bf} \equiv V_f$$

The overall volume of the system cannot change, and above it is shown that the volumes of the two sides are equal in both cases. Thus all for volumes must be the same.

$$V_{Ai} = V_{Bi} = V_{Af} = V_{Bf} \approx 21.6\ m^3$$

Plug the volumes and temperatures back in to the ideal gas law to find the common final pressure.

$$P_{Af} = P_{Bf} = \frac{nRT_f}{V_f}$$

$$P_{Af} = P_{Bf} \approx \frac{1\ mol\ \left(8.314\ ^J/_{mol\ K}\right)321\ K}{21.6\ m^3}$$

$$P_{Af} = P_{Bf} \approx 124 \; Pa$$

$T_{Ai} = 273 \; K$	$T_{Bi} = 273 \; K$	$T_{Af} = 321 \; K$	$T_{Bf} = 321 \; K$
$P_{Ai} = 105 \; Pa$	$P_{Bi} = 105 \; Pa$	$P_{Af} \approx 124 \; Pa$	$P_{Bf} \approx 124 \; Pa$
$V_{Ai} \approx 21.6 \; m^3$	$V_{Bi} \approx 21.6 \; m^3$	$V_{Af} \approx 21.6 \; m^3$	$V_{Bf} \approx 21.6 \; m^3$

Now consider the energy provided by the heater. There is no change in volume, and so there is no work done. Use the first law for the overall system (that is, the two compartments considered together).

$$\Delta U = Q + W = Q$$

Note that this heat comes from the heater. Use the Equipartition Theorem. The number of moles, n, does change from its initial value of two.

$$Q = \Delta U = \frac{f}{2} nR\Delta T$$

The gasses are monatomic, so f =3. Plug in the temperature change values.

$$Q = \frac{3}{2} \times 2 \; mol \left(8.314 \; ^J\!/_{mol \; K}\right)(321 - 273) \; K$$

$$Q_{heater} \approx 1{,}197 \; J$$

b) an adiabatic piston.
In both cases, present your results in a table.
Hint: For the energy from the heater in b), consider the First Law for each compartment individually.

The initial temperatures, pressures, and volumes are the same as in a). From the problem, the final temperature for B is also the same.

The process is adiabatic, so PV^γ is constant. Since we know more about temperature than volume, use the ideal gas law to eliminate volume.

$$PV^\gamma = P\left(\frac{nRT}{P}\right)^\gamma = P^{1-\gamma}T^\gamma(nR)^\gamma = constant$$

The term in parentheses is just a constant, too, so we conclude that $P^{\gamma-1}T^\gamma$ is constant. (This was a result from HW1, too.) Use this to figure out the final pressure in B.

$$P_{Bf}{}^{1-\gamma}T_{Bf}{}^\gamma = P_{Bi}{}^{1-\gamma}T_{Bi}{}^\gamma$$

$$P_{Bf}{}^{1-\gamma} = \frac{P_{Bi}{}^{1-\gamma}T_{Bi}{}^{\gamma}}{T_{Bf}{}^{\gamma}}$$

$$P_{Bf} = \left(\frac{P_{Bi}{}^{1-\gamma}T_{Bi}{}^{\gamma}}{T_{Bf}{}^{\gamma}}\right)^{\frac{1}{1-\gamma}}$$

$$P_{Bf} = \frac{P_{Bi}T_{Bi}{}^{\frac{\gamma}{1-\gamma}}}{T_{Bf}{}^{\frac{\gamma}{1-\gamma}}}$$

The gases are monatomic, so f = 3.

$$\gamma = \frac{f+2}{f} = \frac{3+2}{3} = \frac{5}{3}$$

Plug in known values to find the final pressure in B.

$$P_{Bf} = \frac{105 \; Pa \; (273 \; K)^{\frac{5/3}{1-5/3}}}{(321 \; K)^{\frac{5/3}{1-5/3}}}$$

$$P_{Bf} \approx 157 \; Pa$$

Now use the ideal gas law to find the final volume of B.

$$V_{Bf} = \frac{nRT_{Bf}}{P_{Bf}}$$

$$V_{Bf} = \frac{1 \; mol \; \left(8.314 \; {}^{J}\!/_{mol \; K}\right) 321 \; K}{157 \; Pa}$$

$$V_{Bf} \approx 17.0 \; m^3$$

Now to compartment A. After a long time, the piston moves to be in mechanical equilibrium.

$$P_{Af} = P_{Bf}$$

$$P_{Af} = P_{Bf} \approx 157 \; Pa$$

The overall volume does not change. Use this fact to find the final value of A.

$$V_{Ai} + V_{Bi} = V_{Af} + V_{Bf}$$

$$V_{Ai} + V_{Bi} - V_{Bf} = V_{Af}$$

$$21.6 \ m^3 + 21.6 \ m^3 - 17 \ m^3 \approx V_{Af}$$

$$26.2 \ m^3 \approx V_{Af}$$

Use the ideal gas law to find the final temperature of A.

$$T_{Af} = \frac{V_{Af} P_{Af}}{nR}$$

$$T_{Af} \approx \frac{26.2 \ m^3 (157 \ Pa)}{1 \ mol \left(8.314 \ ^J/_{mol \ K}\right)}$$

$$T_{Af} \approx 495 \ K$$

$T_{Ai} = 273 \ K$	$T_{Bi} = 273 \ K$	$T_{Af} \approx 495 \ K$	$T_{Bf} = 321 \ K$
$P_{Ai} = 105 \ Pa$	$P_{Bi} = 105 \ Pa$	$P_{Af} \approx 157 \ Pa$	$P_{Bf} \approx 157 \ Pa$
$V_{Ai} \approx 21.6 \ m^3$	$V_{Bi} \approx 21.6 \ m^3$	$V_{Af} \approx 26.2 \ m^3$	$V_{Bf} \approx 17 \ m^3$

The container is adiabatic, so no heat is exchanged. Thus, by the First Law, this is also the work done on B.

$$\Delta U_B = W_{A \to B}$$

This work is done by A. Thus, it is the opposite of the work done on A by B.

$$W_{A \to B} = -W_{B \to A}$$

There is heat added to A, from the heater. Use the First Law on A.

$$\Delta U_A = W_{B \to A} + Q_{heater}$$

$$\Delta U_A = -W_{A \to B} + Q_{heater}$$

Now look at the change in energy for the entire system.

$$\Delta U = \Delta U_A + \Delta U_B = -W_{A \to B} + Q_{heater} + W_{A \to B}$$

$$\Delta U = Q_{heater}$$

Use the Equipartition Theorem on the whole system.

$$Q_{heater} = \Delta U = \Delta U_A + \Delta U_B = \frac{f}{2} nR\Delta T_A + \frac{f}{2} nR\Delta T_B$$

$$Q_{heater} = \frac{f}{2}nR(\Delta T_A + \Delta T_B)$$

$$Q_{heater} \approx \frac{3}{2}1\ mol\left(8.314\ ^J/_{mol\ K}\right)\left((495\ K - 273\ K) + (321\ K - 273\ K)\right)$$

$$Q_{heater} \approx 3{,}367\ J$$

3) You wish to keep your dorm room at a constant temperature of T_1 = 22° C while the outside temperature in Grinnell is 0° C. To evaluate the thermal losses, you turn off the heat. After two hours, the temperature in your dorm room has dropped to 17° C. We assume that during an interval of time, dt, the amount of heat lost is given by:

$$Q = -a(T - T_{ext})dt$$

a) Derive a relationship T(t) between the temperature T (in K) and time t (in seconds), in terms of a and the heat capacity of the dorm room, C.

Use the definition of heat capacity to replace heat in the given equation.

$$C = \frac{Q}{dT}$$

$$Q = -a(T - T_{ext})dt$$

$$CdT = -a(T - T_{ext})dt$$

Rearrange to get all temperature dependence on one side and all time dependence on the other.

$$\frac{dT}{(T - T_{ext})} = -\frac{adt}{C}$$

Integrate both sides. The integrals are indefinite, so there will be constants of integration. Combine the constants into a single constant X_1.

$$\int \frac{dT}{(T - T_{ext})} = -\int \frac{adt}{C}$$

$$\ln(T - T_{ext}) = -\frac{at}{C} + X_1$$

(There should be an absolute value in the natural log, but T > T_{ext}, so it is not necessary.) Raise e to both sides to eliminate the natural log.

$$T - T_{ext} = e^{-\frac{at}{C}+X_1} = e^{-\frac{at}{C}}e^{X_1} = X_2 e^{-\frac{at}{C}}$$

X_2 is another constant based on X_1. Solve for T as a function of time.

$$T(t) = X_2 e^{-\frac{at}{C}} + T_{ext}$$

To determine the constant, plug in the given initial values for t=0. Recall $e^0 = 1$.

$$T_{ext} = 0°C = 273\ K$$

$$T(0\ s) = 22°C = 295\ K$$

$$T(0\ s) = X_2 + 273\ K = 295\ K$$

$$X_2 = 22\ K$$

Plug this back in.

$$T(t) = \left(22\ e^{-\frac{at}{C}} + 273\right) K$$

b) Calculate the numerical value for a, given that C = 10^7 J K^{-1}.

Solve the expression from a) for a.

$$T(t) = \left(22\ e^{-\frac{at}{C}} + 273\right) K$$

$$T(t) - 273\ K = 22\ e^{-\frac{at}{C}}\ K$$

$$\frac{T(t) - 273\ K}{22\ K} = e^{-\frac{at}{C}}$$

Take the natural log of both sides.

$$\ln\left(\frac{T(t) - 273\ K}{22\ K}\right) = -\frac{at}{C}$$

$$-\frac{C}{t}\ln\left(\frac{T(t) - 273\ K}{22\ K}\right) = a$$

Plug known values into the expression from a) and C.

$$T(t = 2\ \text{hours}) = T(t = 7{,}200s) = 17°C = 290\ K$$

$$-\frac{10^7 \, ^J/_K}{2700 \, s} \ln\left(\frac{290 \, K - 273 \, K}{22 \, K}\right) = a$$

$$a \approx 358 \frac{J}{K \, s}$$

c) Calculate the power that must be exerted by your heater to maintain a constant temperature of 22° C in your dorm room when the temperature outside is 0° C.

The heating rate must be equal and opposite the heat loss rate to maintain a constant temperature. The heat loss rate is given by dividing Q by the change in time in the initial formula.

$$Power_{heater} = -\frac{Q}{dt} = a(T - T_{ext})$$

$$Power_{heater} \approx 358 \frac{J}{K \, s}(295 \, K - 273 \, K)$$

$$Power_{heater} \approx 7880 \, ^J/_S \approx 7880 \, W$$

4) Mount Ogden in Weber County, Utah, is 9,579 feet above sea level. (For contrast, the highest point in Iowa is Hawkeye Point in the northwest part of the state, which is 1,670 feet above sea level.) A 60-kg hiker wishes to climb to the summit of Mt. Ogden, an ascent of 5,000 vertical feet ($\approx$ 1,500 m) above the surrounding terrain.

a) Assuming that she is 25% efficient at converting chemical energy from food into mechanical work, and that essentially all of the mechanical work is used to climb vertically, roughly how many bowls of corn flakes (standard serving size 1 oz., 100 kcal) should the hiker eat before setting out? (1 kcal = 4,184 J)

The hiker must perform mechanical work equal to her increase in gravitational potential energy.

$$W = mgh = (60 \, kg)\left(9.8 \, ^m/_{s^2}\right)(1500 \, m) = 880 \, kJ$$

Convert to kilocalories.

$$W = 880 \, kJ \times \frac{1 \, kcal}{4.184 \, kJ} \approx 210 \, kcal$$

If she is 25% efficient, she must consume four times this much.

$$210 \, kcal \times 4 \times \frac{1 \, bowl}{100 \, kcal} \approx 8.5 \, bowls$$

b) As the hiker climbs the mountain, 75% of the energy from the corn flakes is converted to thermal energy. If there were no way to dissipate this energy, and this energy were equivalent to the heat that flows into her body, by how many degrees would her body temperature increase? (The specific heat capacity of the human body is roughly 3,500 J K^{-1} kg^{-1}.)

Find the amount of total energy converted to thermal energy, that is, the heat that flows into her.

$$Q = 840 \; kcal \times 75\% = 630 \; kcal \times \frac{4.184 \; kJ}{1 \; kcal} = 2635.92 \; kJ$$

Use the definition of specific heat capacity and heat capacity. (1.41 and 1.42 in the text.)

$$c = \frac{C}{m} = \frac{Q}{m\Delta T}$$

$$\Delta T = \frac{Q}{mc}$$

Insert the known values.

$$\Delta T = \frac{2635.92 \; kJ}{60 \; kg \times 3.5 \; kJ/_{K \cdot kg}}$$

$$\Delta T \approx 12.6 \; K$$

c) Thankfully, the extra energy does not warm the hiker's body significantly. (The temperature increase you found in part b) would probably be fatal.) Instead, it goes (mostly) into evaporating water from her skin. If the temperature is 25° C, how many liters of water should she drink during the hike to replace fluids lost due to the thermal energy gained? (At 25° C, the latent heat of vaporization of water is 580 cal / g. Recall that 1 L of water has a mass of about 1 kg.)

In b), we found that the thermal energy gained was 630 kcal. Using the latent heat given, calculate the amount of water that can be vaporized at 25° C with this energy.

$$630 \; kcal \times \frac{1 \; kg}{580 \; kcal} \approx 1.1 \; kg \approx 1.1 \; L$$

5) Consider the combustion of one mole of methane (CH_4) gas. Assume the system is at standard temperature (298 K) and pressure (10^5 Pa = 1 bar) both before and after the reaction.
$$CH_4 \; (gas) + 2 \; O_2 \; (gas) \rightarrow CO_2 \; (gas) + 2 \; H_2O \; (gas)$$
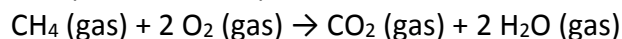
a) First imagine the process of converting a mole of methane into its elemental constituents (graphite and hydrogen gas). Use the reference data at the back of Schroeder to find ΔH for this process. *Careful: You cannot quite just read the value from the chart on page 404.*

From the table on page 404 of Schroeder, ΔH for forming methane into its constituent parts is -74.81 kJ. Thus, ΔH for breaking down methane into its constituent parts, the reverse reaction, is the opposite: **ΔH = +74.81 kJ.**

b) Now imagine forming a mole of CO_2 and two moles of water vapor from their elemental constituents. Determine ΔH for each of these processes, and find the total ΔH.

Use the same table. To form one mole of CO_2, the ΔH is **-393.51 kJ**. To form TWO moles of H_2O gas, the ΔH is $2 \times -241.82 \text{ kJ} = -483.64 \text{ kJ}$. The total ΔH is the sum.

$$-393.51 \text{ kJ} + -483.64 \text{ kJ} = -877.15 \text{ kJ}$$

c) What is the ΔH for the actual reaction in which methane and oxygen form carbon dioxide and water vapor directly? You probably will want to use your answers from a) and b). However, the processes described in part a) and b) are not actually how methane forms. Explain why it is still ok to use a) and b).

Enthalpy is a state function, so it is path independent. As long as we have the same initial and final states, we have the same change in enthalpy. Thus, since the reactions described in parts a) and b) have the same initial and final states as the actual reaction, the ΔH in the two processes must be the same. We can sum our answers from a) and b) to get the total change in enthalpy.
$$\Delta H = +74.81 \text{ kJ} - 393.51 \text{ kJ} + -483.64 \text{ kJ} = -802.34 \text{ kJ}$$

d) How much heat is <u>given off</u> during this reaction, assuming that no "other" (non-compression-expansion) forms of work are done?

From Schroeder (1.55), when pressure is constant and when no "other" forms of work are done, the heat INTO the system is equal to the change in enthalphy.

$$\Delta H = Q$$

From c), this means that -802.34 kJ of heat flows INTO the system. In other words, 802.34 kJ is given off.

e) What is the change in the system's energy during this reaction?

We have a constant pressure, so we can use the result for change in enthalpy given in Schroeder (1.53) and plug to find the change in energy.

$$\Delta H = \Delta U + P\Delta V$$

$$\Delta U = \Delta H - P\Delta V$$

Pressure and temperature are constant. You can check that the reaction equation is balanced, and so the number of moles is constant as well. With all three of these quantities constant, by the ideal gas law, volume must be constant, so ΔV=0.
Plug in the change in enthalpy from c).

$$\Delta U = \Delta H = -802.34 \text{ kJ}$$

f) What would the change in enthalpy be if the H_2O ended up as liquid water instead of vapor?

Use the method and table as a) and b). ΔH for <u>breaking down</u> methane into its constituent parts is +74.81 kJ, just as in a). To form one mole of CO_2, the ΔH is -393.51 kJ, as in b). However, to form two moles of H_2O <u>LIQUID</u>, the ΔH is 2×-285.83 kJ $= -571.66$ kJ. The total ΔH is the sum of all of the processes.

$$74.81 \text{ kJ} - 393.51 \text{ kJ} + -571.66 \text{ kJ} = -890.36 \text{ kJ}$$

g) What would the change in energy be if the H_2O ended up as liquid water instead of vapor? Assume the volume of the liquid is negligible. *Hint: In the Ideal Gas Law, n = moles of <u>gas</u>.*

Use the same constant pressure formula as in e).

$$\Delta U = \Delta H - P\Delta V$$

However, the change in volume is no longer zero. Use the ideal gas law to calculate ΔV with constant P and T.

$$\Delta V = \frac{RT}{P}\Delta n$$

Examine the reaction equation. If the water is a liquid, we have three moles of gas on the left side and only one on the right side. Thus $\Delta n = -2$.

$$\Delta V = \frac{-2RT}{P}$$

Plug this in to find the change in energy.

$$\Delta U = \Delta H - P\Delta V$$

$$\Delta U = \Delta H + 2RT$$

$$\Delta U = -890.36 \text{ kJ} + 2 \times 8.315 \times 10^{-3} \,{}^{kJ}\!/_K \times 298 \, K = -885.40 \, kJ$$

h) The Sun has a mass of 2×10^{30} kg and gives off energy at a rate of 3.9×10^{26} watts. Suppose the Sun were made of methane and oxygen (in the proper proportions to react all of each). Suppose further that the source of the Sun's energy were ordinary combustion this chemical fuel. About how long would the Sun last?

First find the number of moles of methane and oxygen in proper portions in 2×10^{30} kg. A mole of methane has a mass of 16 g. (C + 4 x H = 12 + 4 x 1) Two moles of oxygen gas has a mass of 64 g. (2 x 2 x O = 4 x 16) The mass of the fuel in the reaction of each mole of methane is then 80 g = .080 kg. Calculate the number of moles of methane that would be contained in the Sun.

$$\frac{2 \times 10^{30} \text{ kg}}{.080 \text{ kg}} = 2.5 \times 10^{31} \, mol$$

From d) 802.34 kJ is given off per mole of methane. Calculate the total amount of energy given off.

$$2.5 \times 10^{31} \, mol \times \frac{802.34 \text{ kJ}}{mol} \approx 2 \times 10^{34} \, kJ = 2 \times 10^{37} \, J$$

Calculate the time it would take for this amount of energy to be radiated away at 3.9×10^{26} watts.

$$\frac{2 \times 10^{37} \, J}{3.9 \times 10^{26} \, W} \approx 5 \times 10^{10} \, s \approx 1630 \, uears$$

The sun is much older than this! Chemical fuels could only power the Sun for a few thousand years at most.

6) I flip a two-sided coin five times. My results, in order, were T T H H H. (H = heads and T = tails)
   a) If you flip your own coin five times, what is the probability that you get this same sequence?

   Each flip has two possible outcomes. The outcome of each flip is independent of the other flips. Thus the number of possible outcomes for five flips is the product of the possible outcomes for each flip.

   $$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

   The particular sequence shown is one out of thirty-two equally likely outcomes. Thus, the probability of repeating it is 1/32.

   b) Try it with your own coin. What was your sequence? Did you match mine?

   I will never know if you lie.

   c) There are 15 people in this class. Assume they all do part b). What is the probability that not a single person out of the 15 matched my sequence? *Hint: What is the probability for any particular person to NOT match?*

   The probability of any particular person to NOT match is $1 -$ (the probability to match) = 31/32. The outcome of each person's match is independent of the others' outcomes. Thus, the probability of every single person not matching is then the product of the probability of each person not matching.

   $$\frac{31}{32} \times \frac{31}{32} \times \frac{31}{32} \times \dots = \left(\frac{31}{32}\right)^{15} \approx 62\%$$

   The probability of someone matching is what is left.

   $$100\% - 62\% = 38\%$$

7)

a) When rolling three six-sided dice, what is the probability that their sum is six or lower?

The probability of rolling a six or lower is the sum of the individual multiplicities (Ω) for each number six or lower divided by the total multiplicity.

$$P_{\leq 6} = \frac{\Omega(1) + \Omega(2) + \Omega(3) + \Omega(4) + \Omega(5) + \Omega(6)}{\Omega(all)}$$

The lowest number on each die is one, so the lowest sum is three.

$$\Omega(1) = \Omega(2) = 0$$

Find the other multiplicities by finding the combinations that give you the other sums.
$$\Omega(3) = 1$$
$$1, 1, 1$$

$$\Omega(4) = 3$$
$$1, 1, 2 \ ; \ 1, 2, 1 \ ; \ 2, 1, 1$$

$$\Omega(5) = 6$$
$$1, 1, 3 \ ; \ 1, 3, 1 \ ; \ 3, 1, 1 \ ; \ 1, 2, 2 \ ; \ 2, 1, 2 \ ; \ 2, 2, 1$$

$$\Omega(6) = 10$$
$$2, 2, 2 \ ; \ 1, 1, 4 \ ; \ 1, 4, 1 \ ; \ 4, 1, 1 \ ; \ 1, 2, 3 \ ; \ 3, 1, 2 \ ; \ 2, 3, 1 \ ; \ 2, 1, 3 \ ; \ 1, 3, 2 \ ; \ 3, 2, 1$$

For the total number of multiplicities, there are six choices for each of the three independent dice.

$$\Omega(all) = 6 \times 6 \times 6 = 216$$

Plug in the multiplicities.

$$P_{\leq 6} = \frac{0 + 0 + 1 + 3 + 6 + 10}{216}$$

$$P_{\leq 6} = \frac{5}{54} \approx 9.26\%$$

b) Compare the results of a) to the Einstein solids results from Schroeder (2.8) and (2.9). Explain qualitatively how the two situations are analogous. What is the difference that explains why $\Omega(3) = 1$ for dice and $\Omega(3) = 10$ for oscillators?

The number of oscillators is analogous to the number of dice. Just as the number of energy units are spread amongst the oscillators, we the sum of the roll is spread amongst the individual dice. The difference in multiplicity stems from the fact that the oscillators minimum energy is zero, while the minimum sum on the die is three.

You can use (2.9) to calculate the same multiplicities as in a) if you let N equal the number of dice and let q equal the sum of the roll minus three.

8) When rolling a single six-sided die (with numbers 1-6) N times:

a) What is the probability that not a single one of the rolls yields a one?

The probability of NOT rolling a one in each roll is $^5/_6$.

The probability of two independent events both happening is the product of the individual probabilities. The probability of NOT rolling a one in one roll, and then NOT rolling a one in the second roll is

$$^5/_6 \times {}^5/_6 = \left(^5/_6\right)^2.$$

The probability of NOT rolling a one in one roll, and then NOT rolling a one in the second roll, and then NOT rolling one in the Nth roll is $^5/_6 \times {}^5/_6 \times \dots = \left(^5/_6\right)^N$.

b) What is the probability of not rolling a single one in the first (N-1) rolls, and then rolling a one on the Nth try?

From a), the probability of NOT rolling a one in (N-1) rolls is $\left(^5/_6\right)^{N-1}$.

The probability of rolling a one on the Nth (or any) roll is $^1/_6$.

As above, the probability of the two independent events is the product of the probabilities.

$$\left(^5/_6\right)^{N-1} \times {}^1/_6 = \frac{5^{N-1}}{6^N}$$

c) What is the mean number of rolls it takes to roll a one?

Hint: For $|x| < 1$, $\sum_{n=1}^{\infty} nx^n = \frac{x}{(x-1)^2}$.

From b), the probability of first getting a one on your N^{th} roll is $\frac{5^{N-1}}{6^N}$. It's going to take one roll 1/6th of the time, two rolls 5/36th of the time... To find the mean, then, use these probabilities as weights for a weighted average of the number of rolls.

$$N_{mean} = \frac{1}{6} + 2\frac{5^{2-1}}{6^2} + \dots + N\frac{5^{N-1}}{6^N} + \dots$$

$$N_{mean} = \sum_{n=1}^{\infty} n\frac{5^{n-1}}{6^n}$$

Rearrange to match the form of the hint. Multiply and divide by 5.

$$N_{mean} = \frac{1}{5} \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^n$$

Evaluate sum using hint.

$$N_{mean} = \frac{1}{5} \left(\frac{5/6}{(5/6 - 1)^2}\right)$$

$$N_{mean} = \frac{1}{5}(30) = 6$$

The die is six-sided, so this seems plausible.

9) List _three_ main ideas from this homework assignment.   For example, you could write a few-sentence explanation of a concept, or list an equation and explain the variables and in what circumstances the equation applies.
The goal is for you to review and to reflect on the big picture.   Think about what you might want to remember when you look back at this homework before the test.   I hope that this will be useful for your studying.   I am not looking for anything specific here; you will be graded on effort and completion.