Worksheet 03 - Conditional Statements and Set Basics

1 Warm-Up

Exercise 1. Complete the following truth table:

A	B	(Not A) or B	A and (Not B)
T	T		
T	F		
F	T		
F	F		

STOP

2 Conditionals

One of our most used logical connectives is the "If. . . then. . . ", or *conditional*, connective. We use the shorthand "$A \Rightarrow B$" to denote "If A then B." We call A the *hypothesis*, and B the *conclusion*, of the conditional. The truth table for this connective is:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that a conditional is always true if the hypothesis is false. This may seem strange, but the reasoning comes from the interpretation of this argument structure as a sort of "promise". If the first part of the promise is not kept, then the second part cannot be considered broken.

Exercise 2. Complete the following truth table:

A	B	$A \Rightarrow B$	Not ($A \Rightarrow B$)	$A \Leftarrow B$	($A \Rightarrow B$) and ($A \Leftarrow B$)
T	T	T			
T	F	F			
F	T	T			
F	F	T			

Compare this table to the one in Exercise 1. What can you conclude?

The last column of the truth table in Exercise 2 is equivalent to the *bi-conditional* connective. You might see this a "A if and only if B", or "A iff B", or "$A \Leftrightarrow B$".

Exercise 3. Complete the following truth table:

A	B	$A \Rightarrow B$	(Not A)$\Rightarrow$ (Not B)	$B\Rightarrow$ (Not A)
T	T	T		
T	F	F		
F	T	T		
F	F	T		

Compare this table to the one in Exercise 2. What can you conclude?

STOP

We have names for most of these related statements. Given the conditional: "If A then B", we define:

- the *converse*: "If B then A"
- the *inverse*: "If not A then not B"
- the *contrapositive*: "If not B then not A"

Exercise 4. Given the conditional: "If A then B", use names and/or symbols to complete the following. (Note: there may be more than one correct answer for each.)

- the converse is logically equivalent to:
- the inverse is logically equivalent to:
- the contrapositive is logically equivalent to:
- the negation is logically equivalent to:

STOP

3 Sets and Beginning Proof Strategies

3.1 Sets

You can read about sets in more depth in Section 1.5 of the text, but the basics are as follows:

- A set is a collection of elements. Order doesn't matter. Elements are not repeated. A set without elements is the empty set.
- Sets are equal if they contain exactly the same elements.
- The notation $a \in X$ means: "a is an element of X" (and $a \notin X$ means: "a is not an element of X")

There are two primary ways that we construct or describe sets.

> To *carve out a subset of a larger set* we can say "S is the set of all elements of X with a given property", denoted:
>
> $$S = \{a \in X : x \text{ has Property P}\}$$
>
> or, possibly,
>
> $$S = \{a \in X : P(x)\}$$
>
> To *give a parametric description of a set*, we can say "S is the set of all outputs of the function, f, given any elements from a given subset of its domain", denoted:
>
> $$S = \{f(a) : a \in X\}$$
>
> I.e. S is all of the outputs of f that result from using each element of X as n input.

3.2 Beginning Proof Strategy

3.2.1 Identifying the hypotheses and conclusions

We want to clearly separate our hypotheses and conclusions.

- Hypotheses are the things we are assuming are true. They describe the world we are living in at the beginning of our proof.
- The conclusion is the consequence that we hope to show follows from our living in this world.

Example. Consider the proposition:

> If $a \in \mathbb{Z}$ is even, then a^2 is even.

The hypothesis is that we live in a world where we have an arbitrary even integer in our hands, a. The conclusion we hope to make is that a^2 in an even integer.

3.2.2 Applying definitions or results to make connections

We want to apply any results or definitions that might help us understand the connections between our hypothesis and conclusion.

Example. Continuing our example, we have

- hypothesis: we have an arbitrary even integer in our hands, a.
- conclusion: we want to show that a^2 is an even integer.

The only thing we can really work with, is our definition of *even*. So, we apply it:

- We have an arbitrary even integer, a, which means that $a = 2k$ for some integer k.
- We want to show that a^2 is an even integer, so we want to show that $a^2 = 2t$ for some integer t.

Note: It is important to keep what we "have" and what we "want" separate.

3.2.3 Following your nose

Once we have identified some properties from what we started with in our hypothesis, we can see where they get us.

Example. To finish our proof, we use what we have learned to connect our hypothesis to our desired conclusion.

- We have that $a = 2k$ for some integer k, so $a^2 = 4k^2$.
- We wanted that $a^2 = 2t$ for some integer t, so we let $t = 2k^2$, and we have reached our conclusion.

To clean up this draft, we might write:

> Let a be any even integer. By the definition of even, $a = 2k$ for some integer k. Now, $a^2 = 4k^2$ and, since $2k^2$ is an integer, we have that $a^2 = 2(2k^2)$, which is even by definition.

4 Practice

Exercise 5. If n and m are even integers, then nm is even.

Exercise 6. Write the contrapositive of:

> If n^2 is an odd integer, then n is an odd integer.

Which statement seems more straightforward to prove?(Prove it.)

Exercise 7. Suppose a and b are integers. Write the contrapositive of

> If $a + b$ is even then a and b have the same parity.

Which statement seems more straightforward to prove?(Prove it.)

Preparation for 1/31

- Complete this worksheet.
- Read Section 1.5