MAT215 Exam 2

Olek Yardas

TOTAL POINTS

18.6 / 24

QUESTION 1

Definitions 6 pts

1.1 Definition 1 **2 / 2**

✓ **- 0 pts Correct**

- 2 pts Incorrect or not precise enough

- 2 pts Stated Proposition 3.4.7. instead of the definition.

1.2 Definition 2 **2 / 2**

✓ **- 0 pts Correct**

- 2 pts Incorrect or not precise enough

1.3 Definition 3 **2 / 2**

✓ **- 0 pts Correct**

- 2 pts Incorrect or not precise enough

QUESTION 2

Short Answers 6 pts

2.1 Short Answer 1 **2.5 / 3**

✓ **+ 2.25 pts Stated that matrix multiplication is defined to represent the composition of the associated linear transformations.**

+ 0.5 pts Illustrated the the connection between matrix multiplication and composition of linear transformations using appropriate notation correctly. (e.g. $$[T\circ S]=[T]\cdot [S]$$)

✓ **+ 0.25 pts Description written in complete sentences.**

+ 0.75 pts Included a precise and complete construction of the definition of matrix multiplication, from scratch, by composing arbitrary linear transformations.

+ 0.5 pts Described the connection between linear transformations and the matrix vector product instead of the product of matrices.

+ 0 pts Description does not illustrate an understanding of the connection between linear transformations and matrix multiplication.

+ 0.5 pts Illustrated with a specific example.

2.2 Short Answer 2 **1.6 / 3**

✓ **+ 1.4 pts Correctly identified $$1$$ as an eigenvalue associated to an eigenvector belonging to $$\text{Span}(\vec{w})$$, with justification.**

+ 1.4 pts Correctly identified $$-1$$ as an eigenvalue associated to an eigenvector belonging to $$\text{Span}(\vec{v})$$, where $$\vec{v}$$ is any non-zero vector perpendicular to $$\vec{w}$$, with justification.

✓ **+ 0.2 pts Response written in complete sentences.**

+ 0.1 pts Provided an appropriate diagram to support claims.

+ 0.1 pts Provided a specific example to support claims.

+ 0.5 pts Correctly set up the computation to find the eigenvalues of a general reflection using the characteristic polynomial, but did not identify $$-1$$ or $$1$$ as eigenvalues or find the associated eigenvectors.

+ 1 pts Correctly identified potential eigenvectors, with supporting reasoning, but did not correctly identify eigenvalues.

+ 0 pts Omitted, or reasoning and claims are not relevant to the question, are not specific enough, or are incorrect.

+ 1 pts Correctly identified eigenvalues and eigenvectors for a specific example, but did not sufficiently generalize these observations to all reflections.

QUESTION 3

Proofs 12 pts

3.1 Proof 1 3 / 6

+ **3 pts** Showed the "only if" direction of the biconditional, completely and correctly.

✓ + **3 pts** **Showed the "if" direction of the biconditional, completely and correctly.**

+ **1.25 pts** Proof of the "only if" direction relies too heavily on supporting results that sidestep the details of the proof from definitions. Formal proofs of those results are omitted. (See comments)

+ **1.25 pts** Proof of the "if" direction relies too heavily on supporting results that sidestep the details of the proof from definitions. Formal proofs of those results are omitted. (See comments)

+ **0 pts** Proof of the "if" direction relies heavily on faulty reasoning, large gaps, or on results that follow from the one to be proved here. (See comments)

✓ + **0 pts** **Proof of the "only if" direction relies heavily on faulty reasoning, large gaps, or on results that follow from the one to be proved here. (See comments)**

+ **0 pts** Neither direction of the proof is sufficiently complete. (See comments)

In the "only if" direction, you cannot pick elements whose image is the zero vector. This assumes too much, and the reasoning that follows does not allow you to make a general conclusion.

3.2 Proof 2 5.5 / 6

✓ + **1.5 pts** **Correctly identified both eigenvalues, with supporting computations.**

✓ + **2.5 pts** **Correctly identified an eigenvector associated to each eigenvalue, with supporting computations.**

+ **2 pts** Used the eigenvectors from Part (b) as a basis to explicitly demonstrate that $$A$$ is diagonalizable in the coordinates determined by that basis.

✓ + **1.5 pts** **Correctly justified the claim that $$A$$ is diagonalizable, but did not demonstrate explicitly enough.**

+ **0 pts** Did not correctly identify the eigenvalues

+ **0 pts** Did not correctly identify eigenvectors.

+ **0 pts** Did not demonstrate that $$A$$ is diagonalizable.

- **0.3 pts** Computation error.

Exam 2

MAT215 - Spring 2018

- All work must be your own.
- Do not start until instructed to do so.
- Scrap paper is available at the front of the room.
- Only use one side, of any page, for work you wish to have scored for credit. If you use scrap paper, write your name on the back of each page and turn it in with your exam.
- You are expected to remain in the exam room until you have completed your exam.
- Notes, or other references, are NOT permitted for use during the exam.
- You are expected to refrain from glancing at other students' exams, and to be reasonably careful that your responses are not visible to others. (This includes work on scrap pages.)
- Everything except for the writing utensils you will be using to complete the exam must be secured in your bag, and kept at the front of the room.
- Please, be sure that your phone is silenced completely (no vibrate mode), and no alarms are scheduled to go off during the exam.
- Phones, smart watches, or any device that might be used to communicate with others must be disabled, or completely inaccessible, during the exam.

Name: Orea Yardees

1 Definitions - 6 points

For each of the following, state the definition as precisely as possible. No partial credit will be given. These statements should be identical, or nearly identical, to the statements given in the textbook. Proposition or theorem statements that provide conditions which are equivalent to the formal definition will not earn credit.

1. Let α be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Define *matrix of T relative to α.*

Let $\alpha = (\vec{u}_1, \vec{u}_2)$, and Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Fix $a, b, c, d \in \mathbb{R}$ with $Coord_\alpha(T(\vec{u}_1)) = [T(\vec{u}_1)]_\alpha = \begin{pmatrix} a \\ c \end{pmatrix}$ and $Coord_\alpha(T(\vec{u}_2)) = [T(\vec{u}_2)]_\alpha = \begin{pmatrix} b \\ d \end{pmatrix}$. We define the matrix of T relative to α as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote this matrix by $[T]_\alpha$. In other words, we let the first column of $[T]_\alpha$ be the coordinates of $T(\vec{u}_1)$ relative to α and we let the second column of $[T]_\alpha$ be the coordinates of $T(\vec{u}_2)$ relative to α.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Define *Null*(T).

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. We define

$$Null(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}$$

We call $Null(T)$ the null space of T (or the kernel of T).

3. Define *characteristic polynomial.*

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We define the Characteristic Polynomial of A to be the following polynomial in variable λ:

$$(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$$

2 Short Answers - 6 points

1. Describe the connection between linear transformations and multiplication of matrices.

~~A~~ Lined transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ has a standard matrix $[T]$ that defines it.

We can view ~~the~~ composition of two linear transformations T and S both from $\mathbb{R}^2 \to \mathbb{R}^2$ as the multiplication of ~~their~~ their standard matrices.

So matrix multiplication ~~is~~ - the composition of linear transformations

Ran out of time here.

2. Let $\vec{w} \in \mathbb{R}^2$ be non-zero, and let $W = \text{Span}(\vec{w})$. In the homework we showed that the function $F_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by letting $F_{\vec{w}}(\vec{v})$ to be the result of reflecting $\vec{v}$ across the line W, is a linear transformation. What are the eigenvalues and eigenvectors of $F_{\vec{w}}$? You do not need to give a formal proof, but you should explain your reasoning.

By definition, the eigenvalue of $F_{\vec{w}}$ is a nonzero $\vec{v} \in \mathbb{R}^2$ such that there exists a $\lambda \in \mathbb{R}$ with $F_{\vec{w}}(\vec{v}) = \lambda \vec{v}$.

By def. An eigenvalue of $F_{\vec{w}}$ is a scalar $\lambda \in \mathbb{R}$ such that there exists a nonzero $\vec{v} \in \mathbb{R}^2$ with $F_{\vec{w}}(\vec{v}) = \lambda \vec{v}$.

Whenever $\vec{v} \in \mathbb{R}^2$ is nonzero and $\lambda \in \mathbb{R}$ are such that $F_{\vec{w}}(\vec{v}) = \lambda \vec{v}$, we say that $\vec{v}$ is an eigenvector of $F_{\vec{w}}$ corresponding to the eigenvalue λ. Geometrically, this means that for some eigenvector $\vec{v}_1$ of $F_{\vec{w}}$, $F_{\vec{w}}(\vec{v}_1)$ has the same effect as scaling $\vec{v}_1$ by its eigenvalue λ_1. The reflection of a vector across a line that that vector lies on is simply that vector. So in this case, the eigenvalues are 1 and 1 and the eigenvectors are all elements of $\text{Span}(\vec{w})$.

3 Proofs - 12 points

Prove the following statements. You must formally justify each of your claims by stating the definition or result from which it follows. Do not refer to results by their number in the text, state them completely.

3.1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Show that $\text{Null}(T) = \{\vec{0}\}$ if and only if T is injective. We first prove that If T is injective

Suppose that T is injective. By definition, whenever $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ satisfy $T(\vec{v_1}) = T(\vec{v_2})$, we have that $\vec{v_1} = \vec{v_2}$. ~~[illegible]~~

Let $\vec{U} \in Null(T)$ be arbitrary. By definition, $T(\vec{U}) = \vec{0}$

Notice that $T(\vec{0}) = \vec{0}$ (by the Proposition that states $T(\vec{0}) = \vec{0}$).

So $T(\vec{U}) = T(\vec{0})$! Because T is injective, it follows that $\vec{U} = \vec{0}$

Because $\vec{U} \in Null(T)$ was arbitrary the result follows.

We now prove that If $Null(T) = \{\vec{0}\}$ then T is injective.

Let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ be arbitrary. Suppose $\boxed{T(\vec{v_1}) = T(\vec{v_2}) = \vec{0}.}$

So $\vec{v_1}, \vec{v_2} \in Null(T)$ by definition. But $Null(T) = \{\vec{0}\}$

So it must be that $\vec{v_1} = \vec{v_2} = \vec{0}$. So T is injective.

~~It follows~~ Because $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ were arbitrary the result follows.

3.2

Consider the matrix

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

(a) Find all real eigenvalues of A.

(b) Find an eigenvector associated to each eigenvalue.

(c) Demonstrate that A is diagonalizable.

a) We can find the eigenvalues by finding the roots of the characteristic polynomial.

$$(5-\lambda)(5-\lambda) - 3\cdot 3 = 0.$$

$$\Rightarrow (5-\lambda)^2 - 9 = 0 \Rightarrow (5-\lambda)^2 = 9 \Rightarrow 5-\lambda = \pm 3 \Rightarrow \lambda = 5 \mp 3$$

So we have two real eigenvalues of A, $\boxed{\lambda_1 = 5+3,\ \lambda_2 = 5-3}$

b) Eigenvectors $\vec{v}$ satisfy $(A - \lambda I)\vec{v} = \vec{0}$, where λ is the eigenvalue corresponding to $\vec{v}$.

$\underline{\lambda = \lambda_1}$ $\vec{v} = \vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$\left[\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 5+3 & 0 \\ 0 & 5+3 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3x_1 + 3y_1 \\ 3x_1 - 3y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is easy to see that it must be the case that $x_1 = -y_1$, so $\vec{v}_1 = \begin{pmatrix} -y_1 \\ y_1 \end{pmatrix}$. Any $y_1 \in \mathbb{R}$ will do, so $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\underline{\lambda = \lambda_2}$, $\vec{v} = \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\left[\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 5-3 & 3 \\ 3 & 5-3 \end{pmatrix}\right]\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3x_2 + 3y_2 \\ 3x_2 + 3y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is easy to see that it must be the case that $x_2 = x_2$, so $\vec{v}_2 = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$

Any $x_2 \in \mathbb{R}$ will do, so we say that $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

PAGE 1 OF 2 For Problem 3.2

Run low on time here.

So the $\begin{pmatrix}-1\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}5&3\\3&5\end{pmatrix}$ with eigenvalue $5+3=8$ and $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}5&3\\3&5\end{pmatrix}$ with eigenvalue $5-3=2$

c) There is a proposition that states that for a linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ with standard matrix $[T]$, and letting $\alpha=(\vec{v_1},\vec{v_2})$ be a basis of $\mathbb{R}^2$, the following are equivalent:

1) $[T]_\alpha$ is diagonal

2) $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of T.

(2)

(1) By definition, a linear transformation is diagonizable if there exists a basis $\alpha=(\vec{v_1},\vec{v_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha$ is a diagonal matrix

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation defined by letting $[T]=\begin{pmatrix}5&3\\3&5\end{pmatrix}$.

Thus, the eigen vectors of T are $\vec{v_1}=\begin{pmatrix}-1\\1\end{pmatrix}$ and $\vec{v_2}=\begin{pmatrix}1\\1\end{pmatrix}$.

Note that $-1\cdot1-1\cdot1=-1-1=-2\neq0$, so $\operatorname{Span}(\vec{v_1},\vec{v_2})=\mathbb{R}^2$ by Theorem 2.3.10, By definition of basis, $(\vec{v_1},\vec{v_2})$ is a basis of $\mathbb{R}^2$.

By (2), $[T]_\alpha$ is diagonal. By (1), T is diagonizable

PAGE 2 of 2 for Problem 3.2