Assignment: Problem Set 6

Name: Oleksandr Yardas

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List Your Collaborators:

Note :

In Problems 1 and 4, I have written in a different color (blue) the parts that go into the blanks.

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the following statement: If $\vec{u}, \vec{w} \in \mathbb{R}$ and $\vec{w} \in \text{Span}(\vec{u})$, then $\text{Span}(\vec{w}) \in \text{Span}(\vec{u})$.

Note :

I have written in a different color (blue) the parts that go into the blanks.

Solution: Let $\vec{v} \in \text{Span}(\vec{w})$ be arbitrary. Since $\vec{w} \in \text{Span}(\vec{u})$, we can fix a $c \in \mathbb{R}$ with $\vec{w} = c \cdot \vec{u}$ (by definition of $\text{Span}(\vec{u})$). Since $\vec{v} \in \text{Span}(\vec{w})$, we can fix a $d \in \mathbb{R}$ with $\vec{v} = d \cdot \vec{w}$ (by definition of $\text{Span}(\vec{w})$). Now notice that $\vec{v} = d \cdot \vec{w} = d \cdot (c \cdot \vec{u}) = (cd) \cdot \vec{u}$ (by Proposition 2.2.1.9). Since $cd \in \mathbb{R}$, we conclude that $\vec{v} \in \text{Span}(\vec{u})$. Since $\vec{v} \in \text{Span}(\vec{u})$ was arbitrary, the result follows.

Problem 2: Given $\vec{u} \in \mathbb{R}^2$, is the set $\text{Span}(\vec{u})$ always closed under componentwise multiplication? In other words, if

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \text{Span}(\vec{u}) \qquad \text{and} \qquad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \text{Span}(\vec{u}),$$

must it be the case that

$$\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} \in \text{Span}(\vec{u})?$$

Either argue that this is always true, or provide a specific counterexample (with justification).

Solution: Let $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \vec{v_1}$, $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \vec{v_2}$. Consider the case in which $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v_1} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$. Note that

$$3 \cdot \vec{u} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \\ 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \vec{v_1}$$

and

$$4 \cdot \vec{u} = 4 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 \\ 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \vec{v_2}$$

So $\vec{v_1}, \vec{v_2} \in \text{Span}(\vec{u})$ by definition of $\text{Span}(\vec{u})$. If we take the componentwise product of $\vec{v_1}, \vec{v_2}$, we get:

$$\begin{aligned} \begin{pmatrix} 3 \cdot 4 \\ 6 \cdot 8 \end{pmatrix} &= \begin{pmatrix} 12 \\ 48 \end{pmatrix} \\ &= \begin{pmatrix} 12 \cdot (1) \\ 24 \cdot (2) \end{pmatrix} \end{aligned}$$

This vector cannot be represented by the product of a scalar and $\vec{u}$. We have found a $\vec{v_1}, \vec{v_2} \in \text{Span}(\vec{u})$ such that their componentwise product is not an element of $\text{Span}(\vec{u})$, therefore it is not the case that the set $\text{Span}(\vec{u})$ always closed under componentwise multiplication.

Problem 3: Let $\vec{u_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, let $\vec{u_2} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, and let $\alpha = (\vec{u_1}, \vec{u_2})$. In each part, briefly explain how you carried out your computation.

a. Show that $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$, so $\alpha = (\vec{u_1}, \vec{u_2})$ is a basis for $\mathbb{R}^2$.

Solution: Let $\vec{v} \in \mathbb{R}^2$ be arbitrary. We define $\text{Span}(\vec{u_1}, \vec{u_2})$ as

$$\text{Span}(\vec{u_1}, \vec{u_2}) = \{c_1\vec{u_1} + c_2\vec{u_2} : c_1, c_2 \in \mathbb{R}\},$$

which we can rewrite as

$$\text{Span}(\vec{u_1}, \vec{u_2}) = \{\vec{v} \in \mathbb{R}^2 : \text{There exist } c_1, c_2 \in \mathbb{R} \text{ with } \vec{v} = c_1\vec{u_1} + c_2\vec{u_2}\}.$$

Now, notice that

$$((-1) \cdot 1) - (5 \cdot 2) = -1 - 10 = -11 \neq 0$$

By Theorem 2.3.10, it follows that for every $\vec{v} \in \mathbb{R}^2$, there does indeed exist $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u_1} + c_2\vec{u_2}$ for our particular values of $\vec{u_1}, \vec{u_2}$. This means that the rule by which $\vec{v}$ is carved out of $\mathbb{R}^2$ to construct $\text{Span}(\vec{u_1}, \vec{u_2})$ is true for all $\vec{v} \in \mathbb{R}^2$, and it follows that

$$\text{Span}(\vec{u_1}, \vec{u_2}) = \{\vec{v} : \vec{v} \in \mathbb{R}^2\},$$

which is just the set of all vectors $\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. But the set of all vectors $\vec{v} \in \mathbb{R}^2$ is just the set $\mathbb{R}^2$, therefore, $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$. By definition 2.3.11 that $\alpha = (\vec{u_1}, \vec{u_2})$ is a basis for $\mathbb{R}^2$.

b. Find the coordinates of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ relative to α. In other words, calculate $Coord_\alpha\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$.

Solution: By the Proposition 2.3.13, we have

$$Coord_\alpha\left(\begin{pmatrix} j \\ k \end{pmatrix}\right) = \frac{1}{(-11)} \cdot \begin{pmatrix} (1) \cdot j - (5) \cdot k \\ (-1) \cdot k - (2) \cdot j \end{pmatrix}$$

So we have

$$\begin{aligned} Coord_\alpha\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) &= \frac{1}{(-11)} \cdot \begin{pmatrix} (1) \cdot (5) - (5) \cdot (1) \\ (-1) \cdot (1) - (2) \cdot (5) \end{pmatrix} \\ &= \frac{1}{(-11)} \cdot \begin{pmatrix} 5 - 5 \\ -1 - 10 \end{pmatrix} \\ &= \frac{1}{(-11)} \cdot \begin{pmatrix} 0 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{(-11)} \cdot 0 \\ \frac{1}{(-11)} \cdot (-11) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

So the coordinates of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ relative to α are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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Problem 4: In this problem we work through the proof of Proposition 2.3.8 in the notes, which says the following: Let $\vec{u_1}, \vec{u_2} \in \mathbb{R}^2$. The following are equivalent.

1. $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$.
2. $\vec{u_2} \in \text{Span}(\vec{u_1})$.

Fill in the blanks below with appropriate phrases so that the result is a correct proof:

Note :

I have written in a different color (blue) the parts that go into the blanks.

Solution: We first show that 1 implies 2. Assume that 1 is true, so assume that $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$. Notice that $\vec{u_2} = 0 \cdot \vec{u_1} + c_2\vec{u_2} = \vec{0} + c_2\vec{u_2} = c_2\vec{u_2}$. Since $0, c_2 \in \mathbb{R}$, it follows that $\vec{u_2} \in \text{Span}(\vec{u_1}, \vec{u_2})$. Since $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$, we conclude that $\vec{u_2} \in \text{Span}(\vec{u_1})$.

We now show that 2 implies 1. Assume then that 2 is true, so assume that $\vec{u_2} \in \text{Span}(\vec{u_1})$. By definition, we can fix a $c \in \mathbb{R}$ with $\vec{u_2} = c \cdot \vec{u_1}$. To show that $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$, we give a double containment proof.

- Using Proposition 2.3.7, we know immediately that $\text{Span}(\vec{u_1}) \subseteq \text{Span}(\vec{u_1}, \vec{u_2})$.
- We now show that $\text{Span}(\vec{u_1}, \vec{u_2}) \subseteq \text{Span}(\vec{u_1})$. Let $\vec{v} \in \text{Span}(\vec{u_1}, \vec{u_2})$ be arbitrary. By definition we can fix a $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u_1} + c_2\vec{u_2}$. Notice that $\vec{v} = c_1\vec{u_1} + c_2(c \cdot \vec{u_1}) = c_1\vec{u_1} + (c_2c) \cdot \vec{u_1} = (c_1c_2c) \cdot \vec{u_1}$. Since $c_1c_2c \in \mathbb{R}$, it follows that $\vec{v} \in \text{Span}(\vec{u_1})$. Since $\vec{v} \in \text{Span}(\vec{u_1}, \vec{u_2})$ was arbitrary, we conclude that $\text{Span}(\vec{u_1}, \vec{u_2}) \subseteq \text{Span}(\vec{u_1})$.

Since we have shown both $\text{Span}(\vec{u_1}) \subseteq \text{Span}(\vec{u_1}, \vec{u_2})$ and $\text{Span}(\vec{u_1}, \vec{u_2}) \subseteq \text{Span}(\vec{u_1})$, we conclude that $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$.

c. Find the coordinates of $\begin{pmatrix} 8 \\ 17 \end{pmatrix}$ relative to α. In other words, calculate $Coord_\alpha\left(\begin{pmatrix} 8 \\ 17 \end{pmatrix}\right)$.

Solution: By the Proposition 2.3.13, we have

$$Coord_\alpha\left(\begin{pmatrix} j \\ k \end{pmatrix}\right) = \frac{1}{(-11)} \cdot \begin{pmatrix} (1) \cdot j - (5) \cdot k \\ (-1) \cdot k - (2) \cdot j \end{pmatrix}$$

So we have

$$\begin{aligned} Coord_\alpha\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) &= \frac{1}{(-11)} \cdot \begin{pmatrix} (1) \cdot (8) - (5) \cdot (17) \\ (-1) \cdot (17) - (2) \cdot (8) \end{pmatrix} \\ &= \frac{1}{(-11)} \cdot \begin{pmatrix} 8 - 85 \\ -17 - 16 \end{pmatrix} \\ &= \frac{1}{(-11)} \cdot \begin{pmatrix} -93 \\ -33 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{(-11)} \cdot (-93) \\ \frac{1}{(-11)} \cdot (-33) \end{pmatrix} \\ &= \begin{pmatrix} \frac{93}{11} \\ 3 \end{pmatrix} \end{aligned}$$

So the coordinates of $\begin{pmatrix} 8 \\ 17 \end{pmatrix}$ relative to α are $\begin{pmatrix} \frac{93}{11} \\ 3 \end{pmatrix}$.

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