Solutions to Written Assignment 1

Problem 1: Let $a \in \mathbb{Z}$ be arbitrary. Since a is an integer, we know that it is of either type 0, or type 1, or type 2, or type 3. We consider each of these four cases in turn.

- *Case 1:* Suppose that a has type 0. By definition, we can fix $n \in \mathbb{Z}$ with $a = 4n$. We then have

$$\begin{aligned} a^2 &= (4n)^2 \\ &= 16n^2 \\ &= 4 \cdot (4n^2). \end{aligned}$$

  Since $4n^2 \in \mathbb{Z}$, it follows that a^2 has type 0 in this case.

- *Case 2:* Suppose that a has type 1. By definition, we can fix $n \in \mathbb{Z}$ with $a = 4n + 1$. We then have

$$\begin{aligned} a^2 &= (4n+1)^2 \\ &= 16n^2 + 8n + 1 \\ &= 4 \cdot (4n^2 + 2n) + 1. \end{aligned}$$

  Since $4n^2 + 2n \in \mathbb{Z}$, it follows that a^2 has type 1 in this case.

- *Case 3:* Suppose that a has type 2. By definition, we can fix $n \in \mathbb{Z}$ with $a = 4n + 2$. We then have

$$\begin{aligned} a^2 &= (4n+2)^2 \\ &= 16n^2 + 16n + 4 \\ &= 4 \cdot (4n^2 + 4n + 1). \end{aligned}$$

  Since $4n^2 + 4n + 1 \in \mathbb{Z}$, it follows that a^2 has type 0 in this case.

- *Case 4:* Suppose that a has type 3. By definition, we can fix $n \in \mathbb{Z}$ with $a = 4n + 3$. We then have

$$\begin{aligned} a^2 &= (4n+3)^2 \\ &= 16n^2 + 24n + 9 \\ &= 16n^2 + 24n + 8 + 1 \\ &= 4 \cdot (4n^2 + 6n + 2) + 1. \end{aligned}$$

  Since $4n^2 + 6n + 2 \in \mathbb{Z}$, it follows that a^2 has type 1 in this case.

Since a must fall into one of the above cases (because every integer is of one of the four types), we have shown that a^2 either has type 0 or has type 1 unconditionally.

Problem 2a: We examine the following type 0 integers as examples:

$$\begin{array}{ccccccc} 4 \cdot 1 & = & 4 & = & 2^2 & - & 0^2 \\ 4 \cdot 2 & = & 8 & = & 3^2 & - & 1^2 \\ 4 \cdot 3 & = & 12 & = & 4^2 & - & 2^2 \\ 4 \cdot 4 & = & 16 & = & 5^2 & - & 3^2 \\ 4 \cdot 5 & = & 20 & = & 6^2 & - & 4^2 \end{array}$$

Now we are not claiming that these are the only ways to write the given type 0 integer as the difference of two squares. For example, we can also write $16 = 4^2 - 0^2$. However, our examples suggest a general pattern that might continue.

Problem 2b: Looking at our above examples, it appears that if we have a type 0 integer a, and we fix $n \in \mathbb{Z}$ with $a = 4n$, then a natural guess is that we can let b be one more than n, and let c be one less than n, i.e. that we can let $b = n + 1$ and $c = n - 1$.

Problem 2c: Let $a \in \mathbb{Z}$ be an arbitrary type 0 integer. By definition, we can fix $n \in \mathbb{Z}$ with $a = 4n$. Notice that $n + 1 \in \mathbb{Z}$ and $n - 1 \in \mathbb{Z}$, and that

$$\begin{aligned}(n+1)^2 - (n-1)^2 &= n^2 + 2n + 1 - (n^2 - 2n + 1) \\ &= n^2 + 2n + 1 - n^2 + 2n - 1 \\ &= 4n \\ &= a.\end{aligned}$$

Therefore, we have shown the existence of integers b and c (namely $b = n + 1$ and $c = n - 1$) for which $a = b^2 - c^2$. Since $a \in \mathbb{Z}$ was an arbitrary type 0 integer, the result follows.