Assignment: Problem Set 21

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Define $T : \mathcal{P}_2 \to \mathbb{R}^2$ bt letting

$$T(f) = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}.$$

It turns out that T is a linear transformation. Let $\alpha = (x^2, x, 1)$, which is a basis for $\mathcal{P}_2$.

a. Let

$$\epsilon_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^{\epsilon_2}$?

Solution: By Definition 5.1.7, $[T]_\alpha^{\epsilon_2}$ is the matrix where the i^{th} column is $[T(\vec{u_i})]_{\epsilon_2}$, where $\vec{u_i}$ is the i^{th} element in α. We have that $T(x^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $T(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Notice that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so by Definition 4.4.3, $[T(x^2)]_{\epsilon_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $[T(x)]_{\epsilon_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $[T(1)]_{\epsilon_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, $[T]_\alpha^{\epsilon_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

b. Let

$$\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).$$

be the standard basis of $\mathbb{R}^2$. What is $[T]_\alpha^\beta$?

Solution: By Definition 5.1.7, $[T]_\alpha^\beta$ is the matrix where the i^{th} column is $[T(\vec{u_i})]_\beta$, where $\vec{u_i}$ is the i^{th} element in α. We have the same values of $T(x^2)$, $T(x)$, and $T(1)$ as before. Notice that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 0.5 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so by Definition 4.4.3, $[T(x^2)]_\beta = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$, $[T(x)]_\beta = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$, and $[T(1)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, $[T]_\alpha^\beta = \begin{pmatrix} 0.5 & 0.5 & 1 \\ -0.5 & -0.5 & 0 \end{pmatrix}$.

Problem 2: Let V be the vector space of all 2×2 matrices. Define $T : V \to V$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Note that the function T takes an input matrix and outputs the result of switching the rows and columns (which is called the *transpose* of the original matrix). It turns out that T is a linear transformation. Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and recall that α is a basis for V. What is $[T]_\alpha^\alpha$? Explain briefly.

Solution: By Definition 5.1.7, $[T]_\alpha^\alpha$ is the matrix where the i^{th} column is $[T(\vec{u_i})]_\alpha$, where $\vec{u_i}$ is the i^{th} element in α. Let $\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\alpha_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We have that $T(\alpha_1) = \alpha_1$, $T(\alpha_2) = \alpha_3$, $T(\alpha_3) = \alpha_2$, and $T(\alpha_4) = \alpha_4$. Notice that

$$\begin{aligned}
\alpha_1 &= 1 \cdot \alpha_1 + 0 \cdot \alpha_2 + 0 \cdot \alpha_3 + 0 \cdot \alpha_4 \\
\alpha_2 &= 0 \cdot \alpha_1 + 1 \cdot \alpha_2 + 0 \cdot \alpha_3 + 0 \cdot \alpha_4 \\
\alpha_3 &= 0 \cdot \alpha_1 + 0 \cdot \alpha_2 + 1 \cdot \alpha_3 + 0 \cdot \alpha_4 \\
\alpha_4 &= 0 \cdot \alpha_1 + 0 \cdot \alpha_2 + 0 \cdot \alpha_3 + 1 \cdot \alpha_4
\end{aligned}$$

so by Definition 4.4.3, $[T(\alpha_1)]_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $[T(\alpha_2)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $[T(\alpha_3)]_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and $[T(\alpha_4)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, $[T]_\alpha^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Problem 3: Working in $\mathbb{R}^4$, let

$$W = \text{Span}\left(\begin{pmatrix}1\\3\\0\\2\end{pmatrix}, \begin{pmatrix}2\\6\\1\\-1\end{pmatrix}, \begin{pmatrix}3\\9\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-1\\7\end{pmatrix}, \begin{pmatrix}-4\\-7\\0\\3\end{pmatrix}\right).$$

Find, with explanation, a basis for W and also $\dim(W)$.

Solution: Let $\alpha_1 = \begin{pmatrix}1\\3\\0\\2\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}2\\6\\1\\-1\end{pmatrix}$, $\alpha_3 = \begin{pmatrix}3\\9\\1\\1\end{pmatrix}$, $\alpha_4 = \begin{pmatrix}1\\3\\-1\\7\end{pmatrix}$, and $\alpha_5 = \begin{pmatrix}-4\\-7\\0\\3\end{pmatrix}$, so $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\text{Span}(\alpha) = W$. Notice immediately that $\alpha_3 = 1\cdot\alpha_1 + 1\cdot\alpha_2 + 0\cdot\alpha_4 + 0\cdot\alpha_5$, so by definition we have that $\alpha_3 \in \text{Span}(\alpha_1, \alpha_2, \alpha_4, \alpha_5)$. It follows from Proposition 4.4.4 that $\text{Span}(\alpha) = \text{Span}(\alpha_1, \alpha_2, \alpha_4, \alpha_5)$, so $\text{Span}(\alpha_1, \alpha_2, \alpha_4, \alpha_5) = W$. Notice also that $\alpha_4 = 3\cdot\alpha_1 - 1\cdot\alpha_2 + 0\cdot\alpha_5$, so by definition we have that $\alpha_4 \in \text{Span}(\alpha_1, \alpha_2, \alpha_5)$. It follows from Proposition 4.4.4 that $\text{Span}(\alpha_1, \alpha_2, \alpha_4, \alpha_5) = \text{Span}(\alpha_1, \alpha_2, \alpha_5)$, so $\text{Span}(\alpha_1, \alpha_2, \alpha_5) = W$. Let A_0 be the 4×3 matrix with α_1, α_2, and α_5 as the first, second, and third columns respectively. We perform Gaussian Elimination on A_0 to get:

$$\begin{aligned}
\begin{pmatrix}1 & 2 & -4\\3 & 6 & -7\\0 & 1 & 0\\2 & -1 & 3\end{pmatrix} &\rightarrow \begin{pmatrix}1 & 2 & -4\\0 & 0 & 5\\0 & 1 & 0\\0 & -5 & 11\end{pmatrix} \begin{matrix}\\-3R_1 + R_2\\ \\-2R_1 + R_4\end{matrix}\\
&\rightarrow \begin{pmatrix}1 & 0 & -4\\0 & 0 & 5\\0 & 1 & 0\\0 & 0 & 11\end{pmatrix} \begin{matrix}-2R_3 + R_1\\ \\ \\5R_3 + R_4\end{matrix}\\
&\rightarrow \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 5\\0 & 0 & 0\end{pmatrix} \begin{matrix}\frac{4}{5}R_2 + R_1\\R_3 \leftrightarrow R_2\\R_2 \leftrightarrow R_3\\-\frac{11}{5}R_2 + R_4\end{matrix}
\end{aligned}$$

Notice that this matrix is indeed in echelon form and has a leading entry in every column, so by Proposition 4.3.3 $(\alpha_1, \alpha_2, \alpha_5)$ is linearly independent. Because $\text{Span}(\alpha_1, \alpha_2, \alpha_5) = W$ and $(\alpha_1, \alpha_2, \alpha_5)$ is linearly independent, it follows from Definition 4.4.1 that $(\alpha_1, \alpha_2, \alpha_5)$ is a basis for W. Notice that there are three elements in this basis, so by Definition 4.4.9, $\dim(W) = 3$.

Problem 4: Consider the unique linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with

$$[T] = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -1 \end{pmatrix}.$$

a. Find bases for each of range(T) and Null(T).

Solution: We first work with range(T). Let $R = \text{range}(T)$. By definition of range, we have that $R = \{T(\vec{v}) : \vec{v} \in \mathbb{R}^4\}$. Let $\vec{a} \in \mathbb{R}^4$ be arbitrary and fix $w, x, y, x \in \mathbb{R}$ with $\vec{a} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$.

Notice that

$$\begin{aligned} T(\vec{a}) &= \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 1w + 0x + 2y - 1z \\ 3w + 1x + 9y - 5z \\ -1w + 2x + 4y - 1z \end{pmatrix} \\ &= w \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + x \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + y \cdot \begin{pmatrix} 2 \\ 9 \\ 4 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ -5 \\ -1 \end{pmatrix} \end{aligned}$$

Letting $\alpha_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 9 \\ 4 \end{pmatrix}$, and $\alpha_4 = \begin{pmatrix} -1 \\ -5 \\ -1 \end{pmatrix}$, we have that range($T$) = Span($\alpha_1, \alpha_2, \alpha_3, \alpha_4$). Notice that these are just the columns of $[T]$. Performing Gaussian Elimination on $[T]$, we get:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -1 \end{pmatrix} &\to \begin{pmatrix} 0 & 2 & 6 & -2 \\ 0 & 7 & 21 & -8 \\ -1 & 2 & 4 & -1 \end{pmatrix} \begin{matrix} R_3 + R_1 \\ 3R_3 + R_2 \\ \end{matrix} \\ &\to \begin{pmatrix} 0 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 3 & -\frac{8}{7} \\ -1 & 0 & \frac{22}{7} & \frac{9}{7} \end{pmatrix} \begin{matrix} -\frac{2}{7}R_2 + R_1 \\ \frac{1}{7}R_2 \\ -\frac{2}{7}R_2 + R_3 \end{matrix} \\ &\to \begin{pmatrix} 0 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 3 & 0 \\ -1 & 0 & \frac{22}{7} & 0 \end{pmatrix} \begin{matrix} \\ 4R_1 + R_2 \\ -\frac{9}{2}R_1 + R_3 \end{matrix} \\ &\to \begin{pmatrix} -1 & 0 & \frac{22}{7} & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & \frac{2}{7} \end{pmatrix} \begin{matrix} R_3 \leftrightarrow R_1 \\ \\ R_1 \leftrightarrow R_3 \end{matrix} \end{aligned}$$

Problem 5: Define $T : \mathcal{P}_5 \to \mathcal{P}_5$ by letting $T(f) = f''$, i.e $T(f)$ is the second derivative of f. Determine, with explanation, both rank(T) and nullity(T).

Solution: Letting $N = \text{Null}(T)$, by definition we have that $N = \{\vec{v} \in \mathcal{P}_5 : T(\vec{v}) = \vec{0}_{\mathcal{P}_5}\}$. $\vec{0}_{\mathcal{P}_5}$ is just the function $f_0(x) = 0x^5+0x^4+0x^3+0x^2+0x+0 = 0x^3+0x^2+0x+0$. Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary polynomial in Null(T), and fix $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{R}$ with $g(x) = a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$. By definition of T, we have that $T(g) = 20a_5x^3 + 12a_2x^2 + 6a_3x + 2a_4$. Because $g \in \text{Null}(T)$, we have that $20a_1x^3 + 12a_2x^2 + 6a_3x + 2a_4 = 0x^3 + 0x^2 + 0x + 0$ for all $x \in \mathbb{R}$. Because polynomials functions are equal exactly when their coefficients are equal, we have that $a_1 = a_2 = a_3 = a_4 = 0$. So $g(x) = a_5x + a_6$. Because $g \in \text{Null}(T)$ was arbitrary, we have that $\text{Null}(T) = \{a_5x + a_6 : a_5, a_6 \in \mathbb{R}\} = \text{Span}(x, 1) = \mathcal{P}_1$. We know that $(x, 1)$ is linearly independent, so by definition we have that $(x, 1)$ is a basis for Null(T). Our basis has two elements, so by definition we have that nullity$(T) = 2$.

By definition, we have that range$(T) = \{T(f) : f \in \mathcal{P}_5\}$. Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary polynomial in $\mathcal{P}_5$, and fix $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{R}$ with $h(x) = b_1x^5+b_2x^4+b_3x^3+b_4x^2+b_5x+b_6$. By definition of T, we have that $T(h) = 20b_1x^3 + 12b_2x^2 + 6b_3x + 2b_4$. Because $h \in \mathcal{P}_5$ was arbitrary, it follows that $\text{range}(T) = \{20b_1x^3 + 12b_2x^2 + 6b_3x + 2b_4 : b_1, b_2, b_3, b_4 \in \mathbb{R}\}$. $20b_1, 12b_2, 6b_3, 2b_4 \in \mathbb{R}$, so by definition we have that $\text{range}(T) = \text{Span}(x^3, x^2, x, 1) = \mathcal{P}_3$. We know that $(x^3, x^2, x, 1)$ is linearly independent, so by definition we have that $(x^3, x^2, x, 1)$ is a basis for range(T). Our basis has four elements, so by definition we have that rank$(T) = 4$.

Notice that if we take the matrix with α_1, α_2, and α_4 as its columns, and append α_3 on as the last column, we then have an augmented matrix that encodes the linear system given by $w\alpha_1 + x\alpha_2 + z\alpha_4 = \alpha_3$. Performing the same sequence of elementary row operations as above, we get

$$\begin{pmatrix} -1 & 0 & 0 & \frac{22}{7} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & \frac{2}{7} & 0 \end{pmatrix}.$$

Notice that there is a leading entry in every column but the last, so by Proposition 4.2.12 the system is consistent and has a solution, so $\alpha_3 \in \mathrm{Span}(\alpha_1, \alpha_2, \alpha_4)$. By Proposition 4.4.4, we have that $\mathrm{Span}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathrm{Span}(\alpha_1, \alpha_2, \alpha_4)$, and it follows that $\mathrm{range}(T) = \mathrm{Span}(\alpha_1, \alpha_2, \alpha_4)$. Notice that if we take the matrix with α_1, α_2, and α_4 as its columns, and perform the same sequence of elementary row operations as above, we get

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix}.$$

Notice that there is a leading entry in every column, so by Proposition 4.3.3 we have that $(\alpha_1, \alpha_2, \alpha_4)$ is linearly independent. Because $\mathrm{range}(T) = \mathrm{Span}(\alpha_1, \alpha_2, \alpha_4)$ and $(\alpha_1, \alpha_2, \alpha_4)$ is linearly independent, it follows from Definition 4.4.1 that $(\alpha_1, \alpha_2, \alpha_4)$ is a basis for $\mathrm{range}(T)$.

Now we work with $\mathrm{Null}(T)$. Let $N = \mathrm{Null}(T)$. By Definition of $\mathrm{Null}(T)$, we have that $N = \{\vec{v} \in \mathbb{R}^4 : T(\vec{v}) = \vec{0}_{\mathbb{R}^3}\}$. We found the echelon form of $[T]$ when finding a basis for $\mathrm{range}(T)$. Appending $\vec{0}_{\mathbb{R}^3}$ to the end of $[T]$, we get the augmented matrix that encodes the system of linear equations given by $w\alpha_1 + x\alpha_2 + y\alpha_3 + z\alpha_4 = \vec{0}_{\mathbb{R}^3}$. Performing the same sequence of elementary row operations as above, we get

$$\begin{pmatrix} -1 & 0 & \frac{22}{7} & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 \end{pmatrix}.$$

Notice that this matrix is in echelon form, and that there is no leading entry in the last and third columns, so by Proposition 4.2.12 the system is consistent and has infinitely many solutions. Changing y to the parameter t, we solve to determine $z = 0$, $y = t$, $x = -3t$, and $w = \frac{22}{7}t$. Therefore, $\mathrm{Null}(T) = \left\{ \begin{pmatrix} \frac{22}{7}t \\ -3t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} \frac{22}{7} \\ -3 \\ 1 \\ 0 \end{pmatrix} \right)$. Let $\beta = \begin{pmatrix} \frac{22}{7} \\ -3 \\ 1 \\ 0 \end{pmatrix}$, and notice that (β) is linearly independent, so by Definition 4.4.1 (β) is a basis for $\mathrm{Null}(T)$.

b. Calculate $rank(T)$ and $nullity(T)$.

Solution: By Definition 5.2.7, $rank(T) = \dim(\text{range}(T))$ and $nullity(T) = \dim(\text{Null}(T))$. In part a, we found a basis for range(T) that had three elements, and we found a basis for Null(T) that had one element, so by Definition 4.4.9 $\dim(\text{range}(T)) = 3$ and $\dim(\text{Null}(T)) = 1$, so $rank(T) = 3$ and $nullity(T) = 1$. Notice that $\dim(\mathbb{R}^4) = 4$ and that $rank(T) + nullity(T) = 4$, so $rank(T) + nullity(T) = \dim(\mathbb{R}^4)$ which is what we expect (given Theorem 5.2.8) because T is a linear transformation with domain $\mathbb{R}^4$.