Solutions to Problem Set 3

Problem 1a: For the statement

$$\text{“If } a \in \mathbb{Z} \text{ and } a \geq 2, \text{ then } 4a > 7\text{”,}$$

the converse is

$$\text{“If } a \in \mathbb{Z} \text{ and } 4a > 7, \text{ then } a \geq 2\text{”,}$$

while the contrapositive is

$$\text{“If } a \in \mathbb{Z} \text{ and } \textbf{Not}(4a > 7), \text{ then } \textbf{Not}(a \geq 2)\text{”,}$$

which is the same as

$$\text{“If } a \in \mathbb{Z} \text{ and } 4a \leq 7, \text{ then } a < 2\text{”.}$$

Problem 1b: For the statement

$$\text{“If } x, y \in \mathbb{R} \text{ and } x^4 + y^4 = 1, \text{ then } x^2 + y^2 \leq 2\text{”,}$$

the converse is

$$\text{“If } x, y \in \mathbb{R} \text{ and } x^2 + y^2 \leq 2, \text{ then } x^4 + y^4 = 1\text{”,}$$

while the contrapositive is

$$\text{“If } x, y \in \mathbb{R} \text{ and } \textbf{Not}(x^2 + y^2 \leq 2), \text{ then } \textbf{Not}(x^4 + y^4 = 1)\text{”,}$$

which is the same as

$$\text{“If } x, y \in \mathbb{R} \text{ and } x^2 + y^2 > 2, \text{ then } x^4 + y^4 \neq 1\text{”.}$$

Problem 1c: For the statement

$$\text{“If } a \in \mathbb{Z} \text{ and there exists } m \in \mathbb{Z} \text{ with } a = 10m, \text{ then there exists } m \in \mathbb{Z} \text{ with } a = 5m\text{”,}$$

the converse is

$$\text{“If } a \in \mathbb{Z} \text{ and there exists } m \in \mathbb{Z} \text{ with } a = 5m, \text{ then there exists } m \in \mathbb{Z} \text{ with } a = 10m\text{”,}$$

while the contrapositive is

$$\text{“If } a \in \mathbb{Z} \text{ and } \textbf{Not}(\text{there exists } m \in \mathbb{Z} \text{ with } a = 5m), \text{ then } \textbf{Not}(\text{there exists } m \in \mathbb{Z} \text{ with } a = 10m)\text{”,}$$

which is the same as

$$\text{“If } a \in \mathbb{Z} \text{ and for all } m \in \mathbb{Z} \text{ we have } \textbf{Not}(a = 5m), \text{ then for all } m \in \mathbb{Z} \text{ we have } \textbf{Not}(a = 10m)\text{”,}$$

which is the same as

$$\text{“If } a \in \mathbb{Z} \text{ and for all } m \in \mathbb{Z} \text{ we have } a \neq 5m, \text{ then for all } m \in \mathbb{Z} \text{ we have } a \neq 10m\text{”.}$$

Problem 2a: The contrapositive of

"If $a \in \mathbb{Z}$ and $3a + 5$ is even, then a is odd"

is

"If $a \in \mathbb{Z}$ and a is not odd, then $3a + 5$ is not even".

Problem 2b: We prove the contrapositive statement. Let $a \in \mathbb{Z}$ be arbitrary such that a is not odd. Using Fact 1.4.6, it follows that a is even. Thus, we can fix $n \in \mathbb{Z}$ with $a = 2n$. We then have

$$\begin{aligned} 3a + 5 &= 3 \cdot (2n) + 5 \\ &= 6n + 5 \\ &= 6n + 4 + 1 \\ &= 2 \cdot (3n + 2) + 1. \end{aligned}$$

Notice that $3n + 2 \in \mathbb{Z}$ because $n \in \mathbb{Z}$, so we can conclude that $3a + 5$ is odd. Using Proposition 1.4.5, it follows that $3a + 5$ is not even. Since $a \in \mathbb{Z}$ was an arbitrary odd number, we have proven the contrapositive of the original statement. Therefore, the original statement is true as well.

Problem 3a: We have

$$A = \{y \in \mathbb{R} : \text{There exists } x \in \mathbb{R} \text{ with } y = e^x\}.$$

Problem 3b: We claim that $A = \{y \in \mathbb{R} : y > 0\}$. To see this, we give a double containment proof. Notice that if $a \in A$ is arbitrary, then we can fix $x \in \mathbb{R}$ with $a = e^x$, and hence $a > 0$ (recall that the output of the exponential function is always positive). For the reverse direction, if $y \in \mathbb{R}$ is arbitrary with $y > 0$, then we can consider $\ln y$ (recall that the domain of ln is the set of positive numbers), and we have $e^{\ln y} = y$, so $y \in A$. Thus, $A = \{y \in \mathbb{R} : y > 0\}$.

Problem 4a: Recall that the statement $B \subseteq A$ says that

"For all $b \in B$, we have $b \in A$".

Thus, the statement $B \not\subseteq A$ can be written as

"There exists $b \in B$ with $b \notin A$".

To prove this, we need only give an example, with justification, of such a b.

- Notice that $1 \in B$ because $1 = 4 \cdot 0 + 1$.
- We prove that $1 \notin A$ by contradiction. Suppose instead that $1 \in A$. We can then fix $n \in \mathbb{Z}$ with $1 = 12n - 7$. We then have $12n = 8$, so $n = \frac{12}{8} = \frac{4}{3}$, and hence $n \notin \mathbb{Z}$. This is a contradiction, so $1 \in A$.

We have shown that $1 \in B$ and $1 \notin A$. Therefore, $B \not\subseteq A$.

Problem 4b: Let $a \in A$ be arbitrary. By definition of A, we can fix $n \in \mathbb{Z}$ with $a = 12n - 7$. Now notice that

$$\begin{aligned} a &= 12n - 7 \\ &= 12n - 8 + 1 \\ &= 4 \cdot (3n - 2) + 1. \end{aligned}$$

Since $3n - 2 \in \mathbb{Z}$, we conclude that $a \in B$. Since $a \in A$ was arbitrary, the result follows.

Problem 5a: Let $a \in A$ be arbitrary. By definition of A, we can fix $x \in \mathbb{R}$ with $a = x^2 + 5$. We have $x^2 \geq 0$, so

$$\begin{aligned} a &= x^2 + 5 \\ &\geq 0 + 5 \\ &= 5. \end{aligned}$$

Thus, $a \geq 5$, and hence $a \in B$. Since $a \in A$ was arbitrary, the result follows.

Problem 5b: Let $y \in B$ be arbitrary. By definition of B, we know that $y \geq 5$. Now notice that $y - 5 \geq 0$ and that

$$\begin{aligned} (\sqrt{y-5})^2 + 5 &= (y - 5) + 5 \\ &= y, \end{aligned}$$

so $y \in A$. Since $y \in B$ was arbitrary, the result follows.