Solutions to Problem Set 10

Problem 1: We follow the proof of Theorem 3.3.3 to find $\vec{u}$ and $\vec{w}$. Notice that $2 \cdot 15 - (-5) \cdot (-6) = 0$, both columns of $[T]$ are nonzero, so we are in Case 3 of the Theorem. Following the proof, we do indeed have an $r \in \mathbb{R}$ with

$$\begin{pmatrix} -5 \\ 15 \end{pmatrix} = r \cdot \begin{pmatrix} 2 \\ -6 \end{pmatrix},$$

namely $r = -\frac{5}{2}$. Therefore, we have

$$\text{Null}(A) = \text{Span}\left(\begin{pmatrix} \frac{5}{2} \\ 1 \end{pmatrix} \right)$$

by the proof in Case 3. Finally, reading the last few lines of the proof of the Case 3, we conclude that

$$\text{range}(T) = \text{Span}\left(\begin{pmatrix} 2 \\ -6 \end{pmatrix} \right).$$

We can see this directly by noticing that

$$\text{range}(T) = \text{Span}\left(\begin{pmatrix} 2 \\ -6 \end{pmatrix}, \begin{pmatrix} -5 \\ 15 \end{pmatrix} \right)$$

by Proposition 3.3.1, so since

$$\begin{pmatrix} -5 \\ 15 \end{pmatrix} = -\frac{5}{2} \cdot \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

from above, we can apply Proposition 2.3.8 to conclude that

$$\text{range}(T) = \text{Span}\left(\begin{pmatrix} 2 \\ -6 \end{pmatrix} \right).$$

Problem 2a: Let $\vec{v}_1, \vec{v}_2 \in \text{Null}(T)$ be arbitrary. By definition of $\text{Null}(T)$, we have that $T(\vec{v}_1) = \vec{0}$ and $T(\vec{v}_2) = \vec{0}$. Since T is a linear transformation, we have

$$\begin{aligned} T(\vec{v}_1 + \vec{v}_2) &= T(\vec{v}_1) + T(\vec{v}_2) \\ &= \vec{0} + \vec{0} \\ &= \vec{0}. \end{aligned}$$

Therefore $\vec{v}_1 + \vec{v}_2 \in \text{Null}(T)$.

Problem 2b: Let $\vec{v} \in \text{Null}(T)$ and $c \in \mathbb{R}$ be arbitrary. By definition of $\text{Null}(T)$, we have that $T(\vec{v}) = \vec{0}$. Since T is a linear transformation, we have

$$\begin{aligned} T(c \cdot \vec{v}) &= c \cdot T(\vec{v}) \\ &= c \cdot \vec{0} \\ &= \vec{0}. \end{aligned}$$

Therefore $c \cdot \vec{v} \in \text{Null}(T)$.

Problem 3: Notice that $7 \cdot 4 - (-9) \cdot (-3) = 1$, which is nonzero, so T is bijective by Corollary 3.3.5. Using Proposition 3.3.8, we conclude that T has an inverse, i.e. that T is invertible. Applying Proposition 3.3.14, we know that

$$\begin{aligned}[T^{-1}] &= \frac{1}{7 \cdot 4 - (-9) \cdot (-3)} \cdot \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) &= \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ 22 \end{pmatrix}.\end{aligned}$$

Problem 4a: Let

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}. \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

For any $x, y \in \mathbb{R}$, we have that

$$\begin{array}{ccccc} x & + & 4y & = & -3 \\ 2x & + & 5y & = & 8 \end{array}$$

if and only if

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}.$$

In other words, we want to solve $A\vec{v} = \vec{b}$.

Problem 4b: Notice that $1 \cdot 5 - 4 \cdot 2 = -3$, which is nonzero. Therefore, by Proposition 3.3.16, we have that A is invertible and that

$$\begin{aligned}A^{-1} &= \frac{1}{1 \cdot 5 - 4 \cdot 2} \begin{pmatrix} 5 & -4 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{-3} \begin{pmatrix} 5 & -4 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -5 & 4 \\ 2 & -1 \end{pmatrix}.\end{aligned}$$

Problem 4c: As discussed on the top of page 112, the unique solution to $A\vec{v} = \vec{b}$ is given by $\vec{v} = A^{-1}\vec{b}$. In other words, the unique solution is

$$\begin{aligned}\vec{v} &= \frac{1}{3} \begin{pmatrix} -5 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 8 \end{pmatrix} \\ &= \frac{1}{3} \cdot \begin{pmatrix} 47 \\ -14 \end{pmatrix} \\ &= \begin{pmatrix} \frac{47}{3} \\ -\frac{14}{3} \end{pmatrix}.\end{aligned}$$

Problem 5a: Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Notice that A is a nonzero 2×2 matrix and that

$$\begin{aligned} A \cdot A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Problem 5b: Suppose that A is an invertible 2×2 matrix and that $A \cdot A = 0$. Multiplying both sides on the left by A^{-1}, we have that $A^{-1} \cdot (A \cdot A) = A^{-1} \cdot 0$. Now

$$\begin{aligned} A^{-1} \cdot (A \cdot A) &= (A^{-1} \cdot A) \cdot A && \text{(by Proposition 3.2.6)} \\ &= I \cdot A \\ &= A && \text{(by Proposition 3.2.8),} \end{aligned}$$

and $A^{-1} \cdot 0 = 0$ by Proposition 3.2.8. Therefore, we must have $A = 0$.

Problem 6: We first derive forward to solve for a potential X. Suppose that we have an X with $A(X+B)C = I$. Multiplying both sides on the left by A^{-1} (since A is invertible) and applying both Proposition 3.2.6 and Proposition 3.2.8, we conclude that $(X+B)C = A^{-1}$. Now multiplying both sides on the right by C^{-1} (since C is invertible) and applying Proposition 3.2.6, we conclude that $X + B = A^{-1}C^{-1}$. Finally, subtracting B from both sides, it follows that $X = A^{-1}C^{-1} - B$. Thus, the only possible solution is $X = A^{-1}C^{-1} - B$.

Now that we have a candidate value for X, we can just plug it in to see if it works. Notice that

$$\begin{aligned} A((A^{-1}C^{-1} - B) + B)C &= A(A^{-1}C^{-1})C \\ &= (AA^{-1})(C^{-1}C) && \text{(by Proposition 3.2.6)} \\ &= II \\ &= I && \text{(by Proposition 3.2.8).} \end{aligned}$$

Therefore, we can indeed take $X = A^{-1}C^{-1} - B$.