Assignment: Problem Set 10

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: None

Problem 1: Consider the unique linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}$$

Find, with explanation, vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $\text{Null}(T) = \text{Span}(\vec{u})$ and $\text{range}(T) = \text{Span}(\vec{w})$.

Solution: Applying Theorem 3.3.3 to our given linear transformation, all of a, b, c, d are nonzero and $ad - bc = 2 \cdot 15 - -5 \cdot -6 = 30 - 30 = 0$ so it follows that there do indeed exist vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $\text{Null}(T) = \text{Span}(\vec{u})$ and $\text{range}(T) = \text{Span}(\vec{w})$. We start by finding a $\vec{u} \in \mathbb{R}^2$ with $\text{Null}(T) = \text{Span}(\vec{u})$. Let $\vec{v} \in \text{Null}(T)$ be arbitrary. Because $\text{Null}(T) = \text{Span}(\vec{u})$, $\vec{v} \in \text{Span}(\vec{u})$. By the definition of $\text{Span}(\vec{u})$, we can fix $a \in \mathbb{R}$ such that $\vec{v} = a \cdot \vec{u}$. By the definition of $\text{Null}(T)$, $T(\vec{v}) = \vec{0}$, so $T(a \cdot \vec{u}) = \vec{0} = a \cdot T(\vec{u})$ (by definition of linear transformation), and so $T(\vec{u}) = \vec{0}$ by definition of scalar multiplication of a vector. So we want to find a $\vec{u} \in \mathbb{R}^2$ such that $T(\vec{u}) = \vec{0}$. Let's try $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Applying Proposition 3.1.4, we get:

$$\begin{aligned} T(\vec{u}) = [T]\vec{u} &= \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} && \text{(By definition of } [T], \vec{u}) \\ &= \begin{pmatrix} 2 \cdot 5 - 5 \cdot 2 \\ -6 \cdot 5 + 15 \cdot 2 \end{pmatrix} \\ &= \begin{pmatrix} 30 - 30 \\ -30 + 30 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ satisfies $\text{Null}(T) = \text{Span}(\vec{u})$.

Now we find a $\vec{w} \in \mathbb{R}^2$ with $\text{range}(T) = \text{Span}(\vec{w})$. Let $\vec{p} \in \text{range}(T)$ be arbitrary. By definition of range, there exists a $\vec{d} \in \mathbb{R}^2$ with $T(\vec{d}) = \vec{p}$. Because $\text{range}(T) = \text{Span}(\vec{w})$, $\vec{p} \in \text{Span}(\vec{w})$ so $T(\vec{d}) \in \text{Span}(\vec{w})$. By definition of Span, we can fix $b \in \mathbb{R}$ with $b \cdot \vec{w} = T(\vec{d})$. Let $\vec{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By definition 3.1.1, $T(\vec{d}) = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. If we pick $b = 2$, we have

$$\begin{aligned} 2\vec{w} &= \begin{pmatrix} 2 \\ -6 \end{pmatrix} \\ 2\vec{w} &= 2 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ \vec{w} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \end{aligned}$$

So $\vec{w} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ satisfies $\text{range}(T) = \text{Span}(\vec{w})$.

Problem 2: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$\text{Null}(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}$$

a. Show that if $\vec{v_1}, \vec{v_2} \in \text{Null}(T)$, then $\vec{v_1} + \vec{v_2} \in \text{Null}(T)$.

Solution: Let $\vec{v_1}, \vec{v_2} \in \text{Null}(T)$ be arbitrary. By definition of $\text{Null}(T)$, we have that $T(\vec{v_1}) = \vec{0}$ and $T(\vec{v_2}) = \vec{0}$. Notice that $T(\vec{v_1}+\vec{v_2}) = T(\vec{v_1})+T(\vec{v_2})$ (by the definition of linear transformation) $= \vec{0} + \vec{0} = \vec{0}$. So we have $T(\vec{v_1} + \vec{v_2}) = \vec{0}$. Because $\vec{v_1} + \vec{v_2} \in \mathbb{R}^2$, $\vec{v_1} + \vec{v_2} \in \text{Null}(T)$. Because $\vec{v_1}, \vec{v_2}$ were arbitrary, the result follows.

b. Show that if $\vec{v} \in \text{Null}(T)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \text{Null}(T)$.

Solution: Let $c \in \mathbb{R}, \vec{v} \in \text{Null}(T)$ be arbitrary. By definition of $\text{Null}(T)$, we have that $T(\vec{v}) = \vec{0}$. Notice that $T(c \cdot \vec{v}) = c \cdot T(\vec{v})$ (by the definition of linear transformation) $= c \cdot \vec{0} = \vec{0}$. So we have $T(c \cdot \vec{v}) = \vec{0}$. Because $c \cdot \vec{v} \in \mathbb{R}^2$, $c \cdot \vec{v} \in \text{Null}(T)$. Because $c, \vec{v}$ were arbitrary, the result follows.

Problem 3: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}.$$

Explain why T has an inverse and calculate

$$T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right).$$

Solution: We have $ad - bc = 7 \cdot 4 - -9 \cdot -3 = 28 - 27 \neq 0$. By Corollary 3.3.5, it follows that T is bijective. By Proposition 3.3.8, it follows that there exists an inverse for T. By Proposition 3.3.14, it follows that the inverse of T, denoted T^{-1}, has the standard matrix $[T^{-1}] = \frac{1}{28-27}\begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}$. By Proposition 3.1.4, $T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 & 9 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \cdot 5 + 9 \cdot 1 \\ 3 \cdot 5 + 7 \cdot 1 \end{pmatrix} = \begin{pmatrix} 20 + 9 \\ 15 + 7 \end{pmatrix} = \begin{pmatrix} 29 \\ 22 \end{pmatrix}$. We check our answer by computing $T\left(\begin{pmatrix} 29 \\ 22 \end{pmatrix}\right) = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 29 \\ 22 \end{pmatrix} = \begin{pmatrix} 7 \cdot 29 + -9 \cdot 22 \\ -3 \cdot 29 + 4 \cdot 22 \end{pmatrix} = \begin{pmatrix} 203 - 198 \\ -87 + 88 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. This is what we expect. We conclude that we have correctly computed $T^{-1}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$ to be $\begin{pmatrix} 29 \\ 22 \end{pmatrix}$.

Problem 4: Consider the following system of equations:

$$\begin{aligned} x + 4y =& -3 \\ 2x + 5y =& 8 \end{aligned}$$

a. Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix A and vector $\vec{b}$.

Solution: Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}, \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{b} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$. Let $A\vec{v} = \vec{b}$. Notice that $A\vec{v} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 4y \\ 2x + 5y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$. This is simply the system of equations we have above, and we can rewrite this system of equations as $\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$.

b. Explain why A is invertible and calculate A^{-1}.

Solution: Notice that $1 \cdot 5 - 4 \cdot 2 = 5 - 8 \neq 0$. By Proposition 3.3.16, A is invertible, and its unique inverse is $\frac{1}{5-8} \begin{pmatrix} 5 & -4 \\ -2 & 1 \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} 5 & -4 \\ -2 & 1 \end{pmatrix}$. This is denoted by A^{-1} by definition.

c. Use A^{-1} to solve the system.

Solution: We have $A\vec{v} = \vec{b}$. Taking the matrix product on both sides, we get $A^{-1}(A\vec{v}) = A^{-1}\vec{b} = (A^{-1}A)\vec{v}$(By Proposition 3.2.5). Because A is invertible, $A^{-1}A = I$, where I is the identity matrix. So we have $A^{-1}\vec{b} = I\vec{v} = \vec{v}$. We compute:

$A^{-1}\vec{b} = \frac{1}{-3} \begin{pmatrix} 5 & -4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 8 \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} 5 \cdot -3 + -4 \cdot 8 \\ -2 \cdot -3 + 1 \cdot 8 \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} -15 + -32 \\ 6 + 8 \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} -47 \\ 14 \end{pmatrix} = \begin{pmatrix} \frac{47}{3} \\ -\frac{14}{3} \end{pmatrix} = \vec{v}$. So $x = \frac{47}{3}, y = -\frac{14}{3}$.

Problem 5: In this problem, let 0 denote the 2×2 zero matrix, i.e the 2×2 where all four entries are 0.

a. Give an example of a nonzero 2×2 matrix A with $A \cdot A$ =0.

Solution: Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Notice that A is nonzero. We then have that $A \cdot A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We have found a nonzero 2×2 matrix A, namely $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, for which $A \cdot A$ =0.

b. Show that if A is invertible and $A \cdot A$ =0, then A =0.
Hint: Since 0 is not invertible, it follows from part b that there is no invertible matrix A with $A \cdot A$ =0.

Solution: We assume that $A \cdot A = 0$ and that A is invertible, so there exists a 2×2 matrix B with $AB = I$ and $BA = I$, where I is the identity matrix. We take the matrix product of $A \cdot A = 0$ with B and get

$$\begin{aligned} B \cdot (A \cdot A) &= B \cdot 0 \\ (B \cdot A) \cdot A &= 0 && \text{(By Propositions 3.2.6 and 3.2.8)} \\ I \cdot A &= 0 && \text{(By definition of } B\text{)} \\ A &= 0 && \text{(By Proposition 3.2.7)} \end{aligned}$$

We conclude that $A = 0$.

Problem 6: Let A, B, C all be invertible 2×2 matrices. Must there exist a 2×2 matrix X with

$$A(X+B)C = I?$$

Either justify carefully of give a counterexample.

Solution: Let A, B, C be arbitrary invertible 2×2 matrices. By definition of invertible, there exist 2×2 matrices A^{-1}, B^{-1}, C^{-1} with $A \cdot A^{-1} = I$, $A^{-1} \cdot A = I$, $B \cdot B^{-1} = I$,$B^{-1} \cdot B = I$, $C \cdot C^{-1} = I$ and $C^{-1} \cdot C = I$, where I is the identity matrix. Applying Proposition 3.2.6, we do the following: We start with our expression:

$$\begin{aligned}
A(X+B)C =& I && \text{and then take the matrix product with } A^{-1}\text{:}\\
A^{-1}A(X+B)C =& A^{-1}I\\
I(X+B)C =& A^{-1} && \text{(By definition of } A^{-1}\text{). Now we take the matrix product with } C^{-1}\text{:}\\
(X+B)CC^{-1} =& A^{-1}C^{-1}\\
(X+B)I =& A^{-1}C^{-1} && \text{(By definition of } C^{-1}\text{)}\\
X+B =& A^{-1}C^{-1} && \text{(By Proposition 3.2.7)}\\
X =& A^{-1}C^{-1} - B
\end{aligned}$$

Notice that B need not be invertible in order for this equation to be true, however in this case it is. Notice that the existence of X is dependent on A, C being invertible 2×2 matrices, otherwise the matrices we have defined above as A^{-1}, C^{-1} would not exist (by Proposition 3.3.16), and so X would be undefined for arbitrary 2×2 matrices A, B, C. In this case, A, B, C are all arbitrary invertible 2×2 matrices, so we conclude that there must exist such a 2×2 matrix X with

$$A(X+B)C = I$$

which is given by $X = A^{-1}C^{-1} - B$.