Assignment: Written Assignment 9

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: Not Applicable

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Let U and W be subspaces of $\mathbb{R}^6$ with $\dim(U) = 4$ and $\dim(W) = 3$. Show that $U \cap W \neq \{\vec{0}\}$.
Hint: Do a proof by contradiction. Start by fixing bases of U and W. What would happen if $U \cap W = \{\vec{0}\}$? A previous homework problem will be helpful.

Solution: In Problem 3 of Written Assignment 8, we showed that, given a vector space V, and letting $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_m} \in V$ such that $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n})$ and $(\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m})$ are both linearly independent sequences, if

$$\mathrm{Span}(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}) \cap \mathrm{Span}(\vec{w_1}, \vec{w_2}, \ldots, \vec{w_m}) = \{\vec{0}\},$$

then $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_m})$ is linearly independent.

We know that $\dim(\mathbb{R}^6) = 6$, so by definition we can fix six linearly independent vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}, \vec{v_6} \in \mathbb{R}^6$ such that $(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}, \vec{v_6})$ is a basis of $\mathbb{R}^6$. It follows from the definition of basis that $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}, \vec{v_6}) = \mathbb{R}^6$, so by Theorem 4.4.6, we have that for all $\vec{q_1}, \vec{q_2}, \ldots, \vec{q_m} \in \mathbb{R}^6$, if $m > 6$ then $(\vec{q_1}, \vec{q_2}, \ldots, \vec{q_m})$ is linearly dependent. For the sake of obtaining a contradiction, we assume that given subspaces U and W of $\mathbb{R}^6$ with $\dim(U) = 4$ and $\dim(W) = 3$, we have that $U \cap W = \{\vec{0}\}$. Fix $\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}, \vec{w_1}, \vec{w_2}, \vec{w_3} \in \mathbb{R}^6$ such that $\alpha = (\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4})$ and $\beta = (\vec{w_1}, \vec{w_2}, \vec{w_3})$ are linearly independent sequences. Let $\mathrm{Span}(\alpha) = M$ and let $\mathrm{Span}(\beta) = N$. By Proposition 4.1.16 we have that M and N are subspaces of $\mathbb{R}^6$. Because α and β are linearly independent, by definition we have that α is a basis for M and β is a basis for N. Notice that there are 4 elements in α and 3 elements in β, so by Definition 4.4.9 we have that $\dim(M) = 4$ and $\dim(N) = 3$. By assumption we have that $M \cap N = \{\vec{0}\}$, so it follows from our result in Problem 3 of WA 8 that $(\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}, \vec{w_1}, \vec{w_2}, \vec{w_3})$ is linearly independent. But we showed earlier that any sequence of $m > 6$ vectors in $\mathbb{R}^6$ is linearly *dependent*, and so we have reached a contradiction. We know that Theorem 4.4.6 is true, so it must be the case that our assumption that, for all subspaces U and W of $\mathbb{R}^6$ with $\dim(U) = 4$ and $\dim(W) = 3$, we have that $U \cap W = \{\vec{0}\}$, is false. Therefore, it must be the case that, given subspaces U and W of $\mathbb{R}^6$ with $\dim(U) = 4$ and $\dim(W) = 3$, we have that $U \cap W \neq \{\vec{0}\}$

Problem 2: Let V and W be vector spaces. Suppose that $T : V \to W$ is an injective linear transformation and that $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ is a linearly independent sequence in V. Show that $(T(\vec{u_1}), T(\vec{u_2}), \dots, T(\vec{u_n}))$ is a linearly independent sequence in W.

Solution: Let $T : V \to W$ be an injective linear transformation. Let $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ be a linearly independent sequence in V. Let $a_1, a_2, \dots, a_n \in \mathbb{R}$ be arbitrary and suppose that $a_1 \cdot T(\vec{u_1}) + a_2 \cdot T(\vec{u_2}) + \cdots + a_n \cdot T(\vec{u_n}) = \vec{0}_W$. Notice that

$$\begin{aligned} a_1 \cdot T(\vec{u_1}) + a_2 \cdot T(\vec{u_2}) + \cdots + a_n \cdot T(\vec{u_n}) &= T(a_1 \cdot \vec{u_1}) + T(a_2 \cdot \vec{u_2}) + \cdots + T(a_n \cdot \vec{u_n}) && \text{(By Definition 5.1.1)} \\ &= T(a_1 \cdot \vec{u_1} + a_2 \cdot \vec{u_2} + \cdots + a_n \cdot \vec{u_n}) && \text{(By Defintion 5.1.1)} \end{aligned}$$

So $T(a_1 \cdot \vec{u_1} + a_2 \cdot \vec{u_2} + \cdots + a_n \cdot \vec{u_n}) = \vec{0}_W$. Because T is a linear transformation, by Proposition 5.1.4 we have that $T(\vec{0}_V) = \vec{0}_W$. We then have that $T(\vec{0}_V) = T(a_1 \cdot \vec{u_1} + a_2 \cdot \vec{u_2} + \cdots + a_n \cdot \vec{u_n})$. Because T is injective, by definition we have that $a_1 \cdot \vec{u_1} + a_2 \cdot \vec{u_2} + \cdots + a_n \cdot \vec{u_n} = \vec{0}_V$. Because $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ is linearly independent, by definition we have that $a_1 = a_2 = \cdots = a_n = 0$. Because $a_1, a_2, \dots, a_n \in \mathbb{R}$ were arbitrary, it follows that for all $a_1, a_2, \dots, a_n \in \mathbb{R}$, if $a_1 \cdot T(\vec{u_1}) + a_2 \cdot T(\vec{u_2}) + \cdots + a_n \cdot T(\vec{u_n}) = \vec{0}_W$, then $a_1 = a_2 = \cdots = a_n = 0$. It follows from Definition 4.3.1 that $(T(\vec{u_1}), T(\vec{u_2}), \dots, T(\vec{u_n}))$ is a linearly independent sequence in W.