Assignment: Problem Set 12

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: None

Problem 1: Find the eigenvalues of the matrix

$$\begin{pmatrix} 5 & -1 \\ -7 & 3 \end{pmatrix}.$$

Solution: We can find the eigenvalues λ by solving the characteristic polynomial of $\begin{pmatrix} 5 & -1 \\ -7 & 3 \end{pmatrix}$, that is, finding values of λ such that the equation $(5-\lambda)(3-\lambda)-(-7)(-1)=0$ is true. We assume that the characteristic polynomial is true. We then have

$$\begin{aligned} 0 = (5-\lambda)(3-\lambda)-(-7)(-1) =& (5-\lambda)(3-\lambda)-7 \\ =& 15-8\lambda+\lambda^2-7 \\ =& \lambda^2-8\lambda+8 \end{aligned}$$

We complete the square, adding 8 to both sides:

$$\begin{aligned} 8 =& \lambda^2-8\lambda+16 && (1) \\ 8 =& (\lambda-4)^2 && (2) \\ \pm 2\sqrt{2} =& \lambda-4 && (3) \\ 4 \pm 2\sqrt{2} =& \lambda && (4) \end{aligned}$$

We have two eigenvalues, $\lambda_1 = 4+2\sqrt{2}, \lambda_2 = 4-2\sqrt{2}$.

Problem 2: Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix},$$

and then find (at least) one eigenvector for each eigenvalue.

Solution: We can find the eigenvalues λ by solving the characteristic polynomial of $\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$, that is, finding values of λ such that the equation $(1-\lambda)^2 - (2)(8) = 0$ is true. We assume that the characteristic polynomial is true. We then have

$$\begin{aligned} 0 = (1-\lambda)^2 - (2)(8) =& (1-\lambda)^2 - 16 \qquad (5) \\ 16 =& (\lambda - 1)^2 \qquad (6) \\ \pm 4 =& \lambda - 1 \qquad (7) \\ 1 \pm 4 =& \lambda \qquad (8) \end{aligned}$$

We have two eigenvalues, $\lambda_1 = 1 + 4 = 5, \lambda_2 = 1 - 4 = -3$. We now find eigenvectors of $\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$ corresponding to eigenvalues 5 and -3, that is, we find the value of the vectors $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ that satisfy

$$\left(\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} - 5I \right) \vec{v_1} = \vec{0} \qquad \text{and} \qquad \left(\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} + 3I \right) \vec{v_2} = \vec{0}$$

Letting, $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we get,

$$\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

which become

$$\begin{pmatrix} -4x_1 + 8y_1 \\ 2x_1 - 4y_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 4x_2 + 8y_2 \\ 2x_2 + 4y_2 \end{pmatrix}$$

Letting $x_1 = 2, y_1 = 1, x_2 = -2, y_2 = 1$, we get

$$\begin{pmatrix} -4 \cdot 2 + 8 \cdot 1 \\ 2 \cdot 2 - 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} -8 + 8 \\ 4 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \text{and} \qquad \begin{pmatrix} 4 \cdot -2 + 8 \cdot 1 \\ 2 \cdot -2 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} -8 + 8 \\ -4 + 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$$

So $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$ corresponding to eigenvalue -5, and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$ corresponding to eigenvalue 3.

Problem 3: Find the eigenvalues of the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix},$$

and then find (at least) one eigenvector for each eigenvalue.

Solution: We can find the eigenvalues λ by solving the characteristic polynomial of $\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$, that is, finding values of λ such that the equation $(2-\lambda)(4-\lambda)-(1)(-1)=0$ is true. We assume that the characteristic polynomial is true. We then have

$$\begin{aligned} 0 = (2-\lambda)(4-\lambda)-(1)(-1) =& (2-\lambda)(4-\lambda)+1 \\ =& 8-6\lambda+\lambda^2+1 \\ =& \lambda^2-6\lambda+9 \\ =& (\lambda-3)^2 \end{aligned}$$

We have a repeated eigenvalue, $\lambda = 3$. We now find an eigenvector of $\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$ corresponding to eigenvalue 3, that is, we find the value of the vectors $\vec{v} \in \mathbb{R}^2$ that satisfies

$$\left(\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} - 3I \right) \vec{v} = \vec{0}$$

Letting, $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, we get,

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which becomes

$$\begin{pmatrix} -x-y \\ x+y \end{pmatrix}$$

Letting $x = 1, y = -1$, we get

$$\begin{pmatrix} -1-(-1) \\ 1+(-1) \end{pmatrix} = \begin{pmatrix} -1+1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$$

So $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$ corresponding to eigenvalue 3.

Problem 4: Find values for c and d such that the matrix

$$\begin{pmatrix} 3 & 1 \\ c & d \end{pmatrix}$$

has both 4 and 7 as eigenvalues. You should show the derivation for how you arrived at your choice.

Solution: We want to find values of c and d such that the solutions of the characteristic polynomial are 4 and 7, that is, we want to find c and d such that the equation $0 = (3-\lambda)(d-\lambda) - (c)(1)$ is true for $\lambda = 4$ and $\lambda = 7$. In other words, we need to find c, d that solve the system of equations

$$0 = 4 - d - c \qquad \text{and} \qquad 0 = 28 - 4d - c$$

These equations are derived by plugging in 4 and 7 for λ, yielding

$$0 = (3-4)(d-4) - c \qquad \text{and} \qquad 0 = (3-7)(d-7) - c$$

which become

$$0 = -1(d-4) - c \qquad \text{and} \qquad 0 = -4(d-7) - c,$$

and finally

$$0 = 4 - d - c \qquad \text{and} \qquad 0 = 28 - 4d - c.$$

We solve for d:

$$\begin{aligned} 28 - 4d - c &= 4 - d - c && (9) \\ 28 - 4d &= 4 - d && (10) \\ 28 - 4 &= 4d - d && (11) \\ 24 &= 3d && (12) \\ 8 &= d && (13) \end{aligned}$$

So $d = 8$. We now solve for c:

$$\begin{aligned} 0 = 28 - 32 - c &= 4 - 8 - c = 0 && (14) \\ 0 = -4 - c &= -4 - c = 0 && (15) \end{aligned}$$

We know that both the left and the right hand side have to be equal to zero, so it must be the case that $c = -4$. Thus, the values of c and d that give the matrix $\begin{pmatrix} 3 & 1 \\ c & d \end{pmatrix}$ eigenvalues of 4 and 7 are $c = -4, d = 8$.

Problem 5: Consider the unique linear transfomation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}.$$

Determine if T is diagonalizable. If so, find an example of the basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha$ is a diagonal matrix, and determine $[T]_\alpha$ in this case.

Solution: By definition, $[T]$ is diagonalizable if there exists a basis $\alpha = (\vec{u_1}, \vec{u_2})$ such that $[T]_\alpha$ is a diagonal matrix. It follows from Proposition 3.5.13, that T is diagonalizable if and only if $\vec{u_1}$ and $\vec{u_2}$ are eigenvectors of T. So we need to find eigenvectors of T $\vec{u_1}$ and $\vec{u_2}$ form a basis of $\mathbb{R}^2$. As we did before, we first find eigenvalues by solving the characteristic polynomial:

$$\begin{aligned} 0 =& (1-\lambda)(-1-\lambda) - (6)(0) \\ =& -(1-\lambda)(1+\lambda) \\ =& -(1-\lambda^2) = \lambda^2 - 1 \end{aligned}$$

We get $1 = \lambda^2$, which gives us two eigenvalues, $\lambda_1 = 1, \lambda_2 = -1$. We now find the eigenvectors $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ corresponding to λ_1, λ_2 respectively:

$$\left(\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} - 1I \right) \vec{v_1} = \vec{0} \quad \text{and} \quad \left(\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} + 1I \right) \vec{v_2} = \vec{0} \tag{16}$$

$$\begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \vec{0} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \vec{0} \tag{17}$$

$$\begin{pmatrix} 0 \\ 6x_1 - 2y_1 \end{pmatrix} = \vec{0} \quad \text{and} \quad \begin{pmatrix} 2x_2 \\ 6x_2 \end{pmatrix} = \vec{0} \tag{18}$$

It is easy to see that setting $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfies the above equations. Note that $(1)(1) - (3)(0) = 1 \neq 0$, so by Theorem 2.3.10, $\text{Span}(\vec{v_1}, \vec{v_2}) = \mathbb{R}^2$. By definition of basis, $\beta = (\vec{v_1}, \vec{v_2})$ is a basis of $\mathbb{R}^2$. By definition of $[T]_\beta$, the entries in the first row are the coordinates of $T(\vec{v_1})$ with respect to β and the entries in the second row are the coordinates of $T(\vec{v_2})$ with respect to β. We find these as follows:

$$\begin{aligned} T(\vec{v_1}) &= \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1\vec{v_1} + 0\vec{v_2} \\ T(\vec{v_2}) &= \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0\vec{v_1} - 1\vec{v_2} \end{aligned}$$

So $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and it is clear to see that it is indeed diagonal.

Problem 6: Consider the unique linear transfomation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}.$$

Determine if T is diagonalizable. If so, find an example of the basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha$ is a diagonal matrix, and determine $[T]_\alpha$ in this case.

Solution: By definition, $[T]$ is diagonalizable if there exists a basis $\alpha = (\vec{u_1}, \vec{u_2})$ such that $[T]_\alpha$ is a diagonal matrix. It follows from Proposition 3.5.13, that T is diagonalizable if and only if $\vec{u_1}$ and $\vec{u_2}$ are eigenvectors of T. So we need to find eigenvectors of T $\vec{u_1}$ and $\vec{u_2}$ form a basis of $\mathbb{R}^2$. As we did before, we first find eigenvalues by solving the characteristic polynomial:

$$\begin{aligned} 0 =& (3-\lambda)(5-\lambda) - (1)(-1) \\ =& 15 - 8\lambda + \lambda^2 + 1 \\ =& \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 \end{aligned}$$

We get a repeated eigenvalue, $\lambda = 4$, so $[T]$ only has one eigenvector, so there exists no basis $\alpha = (\vec{u_1}, \vec{u_2})$ for which $\vec{u_1}$ and $\vec{u_2}$ are eigenvectors of T. We conclude that $[T]$ is not diagonalizable.