Solutions to Problem Set 9

Problem 1: Let

$$\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \vec{w} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Notice that we can parameterize the equation $y = 3x$ by $x = t$ and $y = 3t$, so the solution set to $y = 3x$ equals $\text{Span}(\vec{u})$. Similarly, we can parameterize the equation $y = 4x$ by $x = t$ and $y = 4t$, so the solution set to $y = 4x$ equals $\text{Span}(\vec{w})$. Using the notation of Proposition 3.1.11, we then have $T = P_{\vec{w}} \circ P_{\vec{u}}$ because in compositions we apply the function on the right first. Since $P_{\vec{w}}$ and $P_{\vec{u}}$ are both linear transformations by Proposition 3.1.11, we know that T is linear transformation by Proposition 2.4.8. Furthermore, we know that $[T] = [P_{\vec{w}}] \cdot [P_{\vec{u}}]$ by Proposition 3.2.2. Using the formulas from Proposition 3.1.11, it follows that

$$\begin{aligned}
[T] &= [P_{\vec{w}}] \cdot [P_{\vec{u}}] \\
&= \begin{pmatrix} \frac{1}{17} & \frac{4}{17} \\ \frac{4}{17} & \frac{16}{17} \end{pmatrix} \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{170} & \frac{39}{170} \\ \frac{52}{170} & \frac{156}{170} \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{170} & \frac{39}{170} \\ \frac{26}{85} & \frac{78}{85} \end{pmatrix}.
\end{aligned}$$

Problem 2a: Letting

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix},$$

we have

$$\begin{aligned}
A \cdot A &= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{25} & \frac{10}{25} \\ \frac{10}{25} & \frac{20}{25} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \\
&= A.
\end{aligned}$$

Problem 2b: Notice that if we let

$$\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

then

$$[P_{\vec{w}}] = A$$

(see p. 86 of the notes).

Problem 2c: By hitting a vector by A, we are projecting that vector onto the line $y = 2x$. Now notice that $P_{\vec{w}} \circ P_{\vec{w}} = P_{\vec{w}}$ because if we project a point onto $y = 2x$, and then project the result onto the line

$y = 2x$, then the second projection does not move the point (since it is already on $y = 2x$). Therefore, we have $A \cdot A = A$ by Proposition 3.2.2.

Problem 3: No, T is not a linear transformation. Notice that $(0,0)$ is not a point on the line $y = x + 1$, so we must have

$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, T is not a linear transformation by Proposition 2.4.2.

In fact, one can show that

$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

although this is not necessary for the argument.

Problem 4: Let $\vec{v} \in \mathbb{R}^2$ be arbitrary. Notice the vector with tail at the head of $\vec{v}$ and tip at the point $P_{\vec{w}}(\vec{v})$ can be written as $P_{\vec{w}}(\vec{v}) - \vec{v}$. If we add this vector to $\vec{v}$, then of course we land at $P_{\vec{w}}(v)$, which is $\vec{w}$ itself. Now if we want to reflect *across* W, then we want to add this vector again. In other words, we want to add 2 times $P_{\vec{w}}(\vec{v}) - \vec{v}$ to $\vec{v}$. Therefore, we have

$$F_{\vec{w}}(\vec{v}) = 2 \cdot (P_{\vec{w}}(\vec{v}) - \vec{v}) + \vec{v}$$

for all $\vec{v} \in \mathbb{R}^2$, so

$$F_{\vec{w}}(\vec{v}) = 2 \cdot P_{\vec{w}}(\vec{v}) - \vec{v}$$

for all $\vec{v} \in \mathbb{R}^2$.

Fix $a, b \in \mathbb{R}$ with

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now given arbitrary $x, y \in \mathbb{R}$, we can use Proposition 3.1.11 to compute

$$\begin{aligned}
F_{\vec{w}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= 2 \cdot P_{\vec{w}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 2 \cdot \begin{pmatrix} \frac{a^2}{a^2+b^2} \cdot x + \frac{ab}{a^2+b^2} \cdot y \\ \frac{ab}{a^2+b^2} \cdot x + \frac{b^2}{a^2+b^2} \cdot y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} (\frac{2a^2}{a^2+b^2} - 1) \cdot x + \frac{2ab}{a^2+b^2} \cdot y \\ \frac{2ab}{a^2+b^2} \cdot x + (\frac{2b^2}{a^2+b^2} - 1) \cdot y \end{pmatrix} \\
&= \begin{pmatrix} \frac{a^2-b^2}{a^2+b^2} \cdot x + \frac{2ab}{a^2+b^2} \cdot y \\ \frac{2ab}{a^2+b^2} \cdot x + \frac{b^2-a^2}{a^2+b^2} \cdot y \end{pmatrix}.
\end{aligned}$$

Therefore, using Proposition 2.4.3, we conclude that $F_{\vec{w}}$ is a linear transformation and that

$$[F_{\vec{w}}] = \begin{pmatrix} \frac{a^2-b^2}{a^2+b^2} & \frac{2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{pmatrix}.$$

Problem 5a: Let $T_x \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects across the x-axis, and let $T_y \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects across the y-axis. We can use Problem 4 applied to the vector

$(1,0)$ to conclude that

$$[T_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Without using Problem 4, we can derive this by simply calculating that

$$T_x(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad T_x(\vec{e}_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

or by noting that

$$T_x\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ -y \end{pmatrix}$$

for all $x, y \in \mathbb{R}$, and then using Proposition 2.4.3. Similarly, we can use any of these three methods to conclude that

$$[T_y] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now notice that $T = T_y \circ T_x$ (again recall that in a composition, we perform the function on the right first). Since $[T] = [T_y] \cdot [T_x]$ by Proposition 3.2.2, it follows that

$$\begin{aligned} [T] &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Problem 5b: Using Proposition 3.1.10, notice that

$$\begin{aligned} [R_\pi] &= \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= [T]. \end{aligned}$$

Therefore, we have $T = R_\pi$ by Proposition 3.1.6. It follows the action of T is the same as rotating a point by π radians (i.e. 180^o) counterclockwise around the origin.

Problem 6: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $r \in \mathbb{R}$. Suppose that

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In order to determine $[r \cdot T]$, we need to determine both $(r \cdot T)(\vec{e}_1)$ and $(r \cdot T)(\vec{e}_2)$. Looking at the first column of $[T]$, we know that

$$T(\vec{e}_1) = \begin{pmatrix} a \\ c \end{pmatrix},$$

so we have

$$\begin{aligned} (r \cdot T)(\vec{e}_1) &= r \cdot T(\vec{e}_1) && \text{(by definition)} \\ &= r \cdot \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} ra \\ rc \end{pmatrix}. \end{aligned}$$

Similarly, looking at the second column of $[T]$, we know that

$$T(\vec{e}_2) = \begin{pmatrix} a \\ c \end{pmatrix},$$

so we have

$$\begin{aligned} (r \cdot T)(\vec{e}_2) &= r \cdot T(\vec{e}_2) && \text{(by definition)} \\ &= r \cdot \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} rb \\ rd \end{pmatrix}. \end{aligned}$$

Therefore, by definition of $[r \cdot T]$, we have

$$[r \cdot T] = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$