MAT215 Exam 1

Olek Yardas

TOTAL POINTS

24 / 24

QUESTION 1

Definitions 6 pts

1.1 Definition 1 **2 / 2**

✓ **- 0 pts** **Correct**

- 2 pts Incorrect or not precise enough

1.2 Definition 2 **2 / 2**

✓ **- 0 pts** **Correct**

- 2 pts Incorrect or not precise enough

1.3 Definition 3 **2 / 2**

✓ **- 0 pts** **Correct**

- 2 pts Incorrect or not precise enough

QUESTION 2

Short Answers 6 pts

2.1 Short Answer 1 **2 / 2**

✓ **+ 1 pts** **Negation correct**

✓ **+ 1 pts** **Contrapositive correct**

+ 0 pts Negation incorrect

+ 0 pts Contrapositive incorrect

✓ **+ 0.25 pts** **Correctly stated the general form for the negation of a conditional statement.**

✓ **+ 0.25 pts** **Correctly stated the general form for the contrapositive of a conditional statement.**

2.2 Short Answer 2 **4 / 4**

✓ **+ 4 pts** **Completely correct**

+ 3.5 pts Nearly correct, minor detail or technical issue.

+ 3 pts Nearly correct, significant technical issue.

QUESTION 3

Proofs 12 pts

3.1 Proof 1 **6 / 6**

✓ **+ 2 pts** **Part a) Correct**

✓ **+ 2 pts** **Part b) Correct**

✓ **+ 2 pts** **Part c) Correct**

+ 1.25 pts Part a) mostly correct, but omits formal justifications, or includes technical errors.

+ 1.25 pts Part b) mostly correct, but omits formal justifications, or includes technical errors.

+ 1.25 pts Part c) mostly correct, but omits formal justifications, or includes technical errors.

+ 0.75 pts Part a) Evidence of progress derailed by major errors.

+ 0.75 pts Part b) Evidence of progress derailed by major errors.

+ 0.75 pts Part c) Evidence of progress derailed by major errors.

+ 0 pts a) Not enough correct progress to score.

+ 0 pts b) Not enough correct progress to score.

+ 0 pts c) Not enough correct progress to score.

+ 0.25 pts Included the complete statement of a minor supporting result.

+ 0.25 pts Included the complete statement of a minor supporting result. (2)

3.2 Proof 2 **6 / 6**

✓ **+ 3 pts** **Complete, correct, and detailed proof that the composition of linear transformations preserve addition.**

✓ **+ 3 pts** **Complete, correct, and detailed proof that the composition of linear transformations preserves scalar multiplication.**

+ 2.75 pts Correct proof that the composition of linear transformations preserve addition, aside from a minor omission or missed detail.

+ 2.75 pts Correct proof that the composition of linear transformations preserves scalar multiplication, aside from a minor omission or missed detail.

+ 2.25 pts Correct, or mostly correct, sketch of a proof for the preservation of addition. Missing formal justifications and some technical detail.

+ **2.25 pts** Correct, or mostly correct, sketch of a proof for the preservation of scalar multiplication. Missing formal justifications and some technical detail.

+ **3 pts** There is some good intuition here. However, there is not enough detail or progress to make up a complete proof of either property.

+ **0 pts** Omitted or not enough correct progress to score.

Exam 1
MAT215 - Spring 2018

- All work must be your own.
- Do not start until instructed to do so.
- Scrap paper is available at the front of the room.
- A proof attempt outline template is available at the front of the room. You may use this to help you organize your ideas, or as a reference for doing so, but you must transfer any work that you want scored for credit onto your main exam pages, or plain scrap pages that you turn in with your exam.
- Only use one side, of any page, for work you wish to have scored for credit. If you use scrap paper, write your name on the back of each page and turn it in with your exam.
- You are expected to remain in the exam room until you have completed your exam.
- No notes, or other references, are permitted for use during the exam.
- You are expected to refrain from glancing at other students' exams, and to be reasonably careful that your responses are not visible to others. (This includes work on scrap pages.)
- Everything except for the writing utensils you will be using to complete the exam must be secured in your bag, and kept at the front of the room.
- Please, be sure that your phone is silenced completely (no vibrate mode), and no alarms are scheduled to go off during the exam.
- Phones, smart watches, or any device that might be used to communicate with others must be disabled, or completely inaccessible, during the exam.

Name: Olac Yardes

1 Definitions - 6 points

For each of the following, state the definition as precisely as possible. No partial credit will be given. These statements should be identical, or nearly identical, to the statements given in the textbook. Proposition or theorem statements that provide conditions which are equivalent to the formal definition will not earn credit.

1. Define *compostion of functions.*

Let $f: A \to B$ and $g: B \to C$ be functions,
The composition of g and f, denoted by $g \circ f$, is
the function $(g \circ f): A \to C$ defined by letting $(g \circ f)(a) = g(f(a))$
for all $a \in A$

2. Define *surjective function.*

Let $f: A \to B$ be a function
We say that f is surjective (or onto) if, for all $b \in B$,
there exists an $a \in A$ with $f(a) = b$.
In other words, a function is surjective when the
codomain (B) is equal to the range $(\text{Range}(f) = \{b \in B: \text{there exists } a \in A \text{ with } f(a) = b\})$
of the function

3. Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. Define $Span(\vec{u}_1, \vec{u}_2)$.

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. We define a subset of $\mathbb{R}^2$ as follows:

$$Span(\vec{u}_1, \vec{u}_2) = \{c_1 \vec{u}_1 + c_2 \vec{u}_2 : c_1, c_2 \in \mathbb{R}\}$$

In other words, $Span(\vec{u}_1, \vec{u}_2)$ is the set of all linear combinations
of $\vec{u}_1$ and $\vec{u}_2$. We call this set the span of the vectors $\vec{u}_1$ and $\vec{u}_2$

2 Short Answers - 6 points

1. Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, and say

$$\vec{u}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

Consider the statement: If $\underbrace{\text{for all } \vec{v} \in \mathbb{R}^2, \text{ there exist } c_1, c_2 \in \mathbb{R} \text{ with } \vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2}_{A}$, then $\underbrace{a_1b_2 - a_2b_1 \neq 0}_{B}$.

a) Give the negation of the statement:

For all $\vec{v} \in \mathbb{R}^2$, there exist $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$ and $a_1b_2 - a_2b_1 = 0$

The negation of a statement of the form "If A, then B", is "A and NOT(B)."

b) Give the contrapositive of the statement:

If $a_1b_2 - a_2b_1 = 0$, then there exists $\vec{v} \in \mathbb{R}^2$ for all $c_1, c_2 \in \mathbb{R}$ with $\vec{v} \neq c_1\vec{u}_1 + c_2\vec{u}_2$

The contrapositive of a statement of the form "If A, then B" is "If NOT(A), then NOT(A)."

2. Replace **(A)**,**(B)**,**(C)**, and **(D)** below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a + b$ is even":

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can **(A)**. Since b is odd, we can **(B)**. Now notice that $a + b =$ **(C)**. Since **(D)** $\in \mathbb{Z}$, we conclude that $a + b$ is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

(A) fix an $m \in \mathbb{Z}$ given that $a = 2m+1$

(B) Fix an $n \in \mathbb{Z}$ such that $b = 2n+1$

(C) $2m+1+2n+1 = 2m+2n+2 = 2(m+n+1)$

(D) $m+n+1$

3 Proofs - 12 points

Prove the following statements. You must formally justify each of your claims by stating the definition or result from which it follows. Do not refer to results by their number in the text, state them completely.

3.1

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be arbitraty and let $S = Span(\vec{u}_1, \vec{u}_2)$. Prove that

a) $\vec{0} \in S$

b) For all $\vec{v}_1, \vec{v}_2 \in S$, we have that $\vec{v}_1 + \vec{v}_2 \in S$

c) For all $\vec{v} \in S$ and all $d \in \mathbb{R}$, we have $d \cdot \vec{v} \in S$

a) By the definition of $Span(\vec{u}_1, \vec{u}_2)$, we have $Span(\vec{u}_1, \vec{u}_2) = \{c_1\vec{u}_1 + c_2\vec{u}_2 : c_1, c_2 \in \mathbb{R}\}$
Consider the case in which $c_1 = c_2 = 0$. We have

$$\begin{aligned} c_1\vec{u}_1 + c_2\vec{u}_2 &= 0\cdot\vec{u}_1 + 0\cdot\vec{u}_2 \\ &= \vec{0} + \vec{0} && (\text{For all } \vec{u} \in \mathbb{R}^2,\ 0\cdot\vec{u} = \vec{0}) \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (\text{by the definition of zero vector}) \\ &= \begin{pmatrix} 0+0 \\ 0+0 \end{pmatrix} && (\text{by the definition of vector addition}) \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} && (\text{by the definition of zero vector}). \end{aligned}$$

We have found a $c_1, c_2 \in \mathbb{R}$ for which $c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0}$,
Therefore, $\vec{0} \in S$ by definition of span. Because $\vec{u}_1, \vec{u}_2$ were arbitrary, the result follows.

$\square$

b) Let $\vec{v}_1, \vec{v}_2 \in S$ be arbitrary. By the definition of $Span(\vec{u}_1, \vec{u}_2)$, we can find $a, b, c, d \in \mathbb{R}$ with $\vec{v}_1 = a\vec{u}_1 + b\vec{u}_2$ and $\vec{v}_2 = c\vec{u}_1 + d\vec{u}_2$
Notice that

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_1 + d\vec{u}_2 \\ &= (a+c)\vec{u}_1 + (b+d)\vec{u}_2 && (\text{distributive property}) \end{aligned}$$

Because $(a+c), (b+d) \in \mathbb{R}$, $(a+c)\vec{u}_1 + (b+d)\vec{u}_2 \in S$, so $\vec{v}_1 + \vec{v}_2 \in S$.
Because $\vec{v}_1, \vec{v}_2$ were arbitrary, the result follows.

$\square$

Definition. A function $T : \mathbb{R}^2 \to \mathbb{R}^2$ where, for all $c_1, c_2 \in \mathbb{R}$ and all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, we have that

1. $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
2. $T(c_1 \cdot \vec{v}_1) = c_1 \cdot T(\vec{v}_1)$

is called a *linear transformation.*

3.2

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear transformations. Show that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. (*Hint:* You do not need to use any properties of linear transformations aside from the definition.)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $S: \mathbb{R}^2 \to \mathbb{R}^2$ be arbitrary linear transformations.

Let $\vec{V}_1, \vec{V}_2, \vec{W} \in \mathbb{R}^2$ be arbitrary vectors

Let $C \in \mathbb{R}$ be an arbitrary scalar.

Condition 1:

By the definition of function composition $(T \circ S)(\vec{v}_1 + \vec{v}_2) = T(S(\vec{v}_1 + \vec{v}_2))$
By definition of linear transformation, $S(\vec{v}_1 + \vec{v}_2) = S(\vec{v}_1) + S(\vec{v}_2)$
So $T(S(\vec{v}_1 + \vec{v}_2)) = T(S(\vec{v}_1) + S(\vec{v}_2))$. By definition of linear transformation
$T(S(\vec{v}_1) + S(\vec{v}_2)) = T(S(\vec{v}_1)) + T(S(\vec{v}_2))$. By definition of function composition,
$T(S(\vec{v}_1)) = (T \circ S)(\vec{v}_1)$, $T(S(\vec{v}_2)) = (T \circ S)(\vec{v}_2)$, so $T(S(\vec{v}_1)) + T(S(\vec{v}_2)) = (T \circ S)(\vec{v}_1) + (T \circ S)(\vec{v}_2)$
So $(T \circ S)(\vec{v}_1 + \vec{v}_2) = (T \circ S)(\vec{v}_1) + (T \circ S)(\vec{v}_2)$. The first condition is satisfied.

Condition 2:

By definition of function composition, $(T \circ S)(C \cdot \vec{W}) = T(S(C \cdot \vec{W}))$
By definition of linear transformation, $S(C \cdot \vec{W}) = C \cdot S(\vec{W})$, so $T(S(C \cdot \vec{W})) = T(C \cdot S(\vec{W}))$
Because $S: \mathbb{R}^2 \to \mathbb{R}^2$, its range is a set of vectors, that is to say, for a given vector $\vec{W}$, $S(\vec{W})$ is also a vector. By definition of linear transformation,
$T(C \cdot S(\vec{W})) = C \cdot T(S(\vec{W}))$. By definition of function composition, $T(S(\vec{W})) = (T \circ S)(\vec{W})$, so
$C \cdot T(S(\vec{W})) = C \cdot (T \circ S)(\vec{W})$. So $(T \circ S)(C\vec{W}) = C \cdot (T \circ S)(\vec{W})$, the second condition is satisfied. Because $C, \vec{V}_1, \vec{V}_2, \vec{W}$ were arbitrary, the result follows.

So $(S \circ T): \mathbb{R}^2 \to \mathbb{R}^2$ is also a linear transformation.

PROBLEM 3.1 Part C

c) For all $\vec{v} \in S$ and all $d \in \mathbb{R}$, we have $d \cdot \vec{v} \in S$.

Let $\vec{v} \in S$ be arbitrary. Let $d \in \mathbb{R}$ be arbitrary.
By the definition of $\text{Span}(\vec{u}_1, \vec{u}_2)$, we can fix a $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$. Note that

$$\begin{aligned} d \cdot \vec{v} &= d \cdot (c_1\vec{u}_1 + c_2\vec{u}_2) \\ &= d \cdot c_1 \cdot \vec{u}_1 + d \cdot c_2 \cdot \vec{u}_2 \end{aligned}$$

Because $d \cdot c_1, d \cdot c_2 \in \mathbb{R}$, $d \cdot c_1 \vec{u}_1 + d \cdot c_2 \cdot \vec{u}_2 \in S$, so $d \cdot \vec{v} \in S$. Because d, $\vec{v}$ were arbitrary, the result follows.

$\square$