Worksheet 04 - Functions

1 Definitions

Definition 1.1. Let A and B be sets. A function from A to B is a subset, f, of $A \times B$ with the property that for all $a \in A$, there exists a unique $b \in B$ with $(a, b) \in f$. We denote "f is a function from A to B" by

$$f : A \to B$$

Notation. Let A and B be sets, and let f be a function from A to B. The notation $f(a) = b$ is used to indicate that $(a, b) \in f$.

Definition 1.2. Let $f : A \to B$ be a function.

- A is called the *domain* of f.
- B is called the *codomain* of f.
- We define the *range* of f by $range(f) = \{b \in B : \text{There exists } a \in A \text{ with } f(a) = b\}$

Definition 1.3. Suppose that $f : A \to B$ and $g : B \to C$ are functions. The *composition* of g and f, denoted $g \circ f$, is the function $(g \circ f) : A \to C$ defined by letting $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

Definition 1.4. Let A be a set. The function $\{(a, a) : a \in A\}$ is called the *identity* function on A. We denote it by id_A.

Definition 1.5. Let $f : A \to B$ be a function. We say that f is

- *injective*, or *one-to-one*, whenever $f(a) = f(b)$ implies that $a = b$;
- *surjective*, or *onto*, whenever the $B = range(f)$;
- *bijective*, whenever f is both injective and surjective.

2 Exercises

Exercise 1. Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 7, 9, 11\}$.

a) Use set notation to describe a function, f, from A to B.

b) Use set notation to describe a function, g, from B to A.

Exercise 2. Let $A = \{0, 2, 4, 6\}$, $B = \{3, 5, 7, 9, 11\}$ and $C = \{13, 17, 19, 23, 29\}$.

a) Use set notation to describe a function, f, from A to B.

b) Use set notation to describe a function, g, from B to C.

c) Use set notation to describe $g \circ f$.

Exercise 3. Suppose that $f : A \to B$ and $g : C \to D$ are functions. What conditions might we put on the sets, or the functions, so that

a) $f \circ g$ makes sense;

b) $g \circ f$ makes sense;

c) $f \circ g$ and $g \circ f$ both make sense;

d) neither $f \circ g$ nor $g \circ f$ make sense?

Exercise 4. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 4x + 9$ is injective. Is it surjective?

Exercise 5. Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is

a) bijective

b) surjective but not injective

c) injective but not surjective

d) not injective and not surjective

Exercise 6. Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $\dfrac{5x+1}{x-2}$ is bijective.

Exercise 7. Propose a set theoretic definition for f^{-1}, the inverse of f, that is consistent with what you know about inverses in previous courses.

Exercise 8. Give an example of a function $g : \{a, b, c\} \to \{a, b, c\}$ for which g^{-1} is not a function.

Exercise 9. Let A, B, and C be sets. Prove or disprove the following:

1. If two functions $f : A \to B$ and $g : B \to C$ are each bijective, then $g \circ f : A \to C$ is bijective.

2. Let $f : A \to B$ and $g : B \to C$ be two functions. If g is surjective then $g \circ f : A \to C$ is surjective.

3. There exist functions $f : A \to B$ and $g : B \to C$ such that f is not injective and $g \circ f : A \to C$ is injective.

Preparation for 2/2

- Complete this worksheet.
- Read Section 1.6