Solutions to Problem Set 23

Problem 1: We do a cofactor expansion down the second column. Notice that since three of these values are 0, we have

$$\begin{aligned}
\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix} &= (-1)^{3+2} \cdot 2 \cdot \begin{vmatrix} 3 & -2 & 1 \\ 4 & 2 & 0 \\ 3 & 3 & -1 \end{vmatrix} \\
&= (-2) \cdot \begin{vmatrix} 3 & -2 & 1 \\ 4 & 2 & 0 \\ 3 & 3 & -1 \end{vmatrix}.
\end{aligned}$$

We now continue by performing a cofactor expansion along the second row of this 3×3 matrix (making use of the 0):

$$\begin{aligned}
\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix} &= (-2) \cdot \begin{vmatrix} 3 & -2 & 1 \\ 4 & 2 & 0 \\ 3 & 3 & -1 \end{vmatrix} \\
&= (-2) \cdot \left((-1)^{2+1} \cdot 4 \cdot \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} \right) \\
&= (-2) \cdot [(-4) \cdot (2-3) + 2 \cdot (-3-3)] \\
&= (-2) \cdot (4 - 12) \\
&= 16.
\end{aligned}$$

Problem 2: We perform elementary row operations while keeping track of how these affect the determinant. We have

$$\begin{aligned}
\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix} &= (-1) \cdot \begin{vmatrix} a & b & c \\ 2g+3a & 2h+3b & 2i+3c \\ -d & -e & -f \end{vmatrix} && \begin{matrix} (R_1 \leftrightarrow R_3) \\ \\ (R_1 \leftrightarrow R_3) \end{matrix} \\
&= (-1)^2 \cdot \begin{vmatrix} a & b & c \\ -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \end{vmatrix} && \begin{matrix} \\ (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3) \end{matrix} \\
&= (-1)^3 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ 2g+3a & 2h+3b & 2i+3c \end{vmatrix} \\
&= (-1)^3 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ 2g & 2h & 2i \end{vmatrix} && (-3R_1 + R_3) \\
&= (-1)^3 \cdot 2 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \\
&= (-1)^3 \cdot 2 \cdot 5 \\
&= -10.
\end{aligned}$$

Problem 3: We perform elementary row operations while keeping track of how these affect the determinant. We have

$$\begin{aligned}
\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} &= \begin{vmatrix} a & b & c \\ x & x & x \\ y & y & y \end{vmatrix} && \begin{matrix} \\ (-R_1+R_2) \\ (-R_1+R_3) \end{matrix} \\
&= x \cdot \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ y & y & y \end{vmatrix} \\
&= x \cdot y \cdot \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\
&= x \cdot y \cdot \det\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \\
&= x \cdot y \cdot 0 \\
&= 0,
\end{aligned}$$

where the second to last line follows from the fact that two of the vectors are equal.

Problem 4a: Performing a cofactor expansion along the first row, we see that

$$\begin{aligned}
\det(A_c) &= 1 \cdot \begin{vmatrix} 9 & c \\ c & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & c \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 9 \\ 1 & c \end{vmatrix} \\
&= 1 \cdot (27 - c^2) - 1 \cdot (3 - c) + 1 \cdot (c - 9) \\
&= (27 - c^2) + (-3 + c) + (c - 9) \\
&= -c^2 + 2c + 15.
\end{aligned}$$

Problem 4b: We know from Corollary 5.3.11 that A_c is invertible if and only if $\det(A_c) \neq 0$. Therefore, A_c is invertible if and only if $-c^2 + 2c + 15 \neq 0$. Now

$$\begin{aligned}
-c^2 + 2c + 15 &= -(c^2 - 2c - 15) \\
&= -(c-5)(c+3).
\end{aligned}$$

Therefore, A_c is invertible exactly when $c \notin \{-3, 5\}$.

Problem 5: Let

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 6 & 6 & 2 \\ -3 & -4 & -3 \end{pmatrix}.$$

We then have that

$$A - (-2)I = \begin{pmatrix} 3 & 4 & 1 \\ 6 & 8 & 2 \\ -3 & -4 & -1 \end{pmatrix}.$$

To find the eigenspace corresponding to -2, we want to find $\text{Null}(A - (-2)I)$, and so we want to solve the corresponding homogeneous linear system. Applying elementary row operations to this matrix, we obtain

$$\begin{pmatrix} 3 & 4 & 1 \\ 6 & 8 & 2 \\ -3 & -4 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} (-2R_1 + R_2) \\ (R_1 + R_3). \end{matrix}$$

If we use the three variables x, y, z, then we can set y and z equal to parameters because the corresponding columns do not have leading entries. Let $y = s$ and $z = t$. The first line tells us that $3x + 4y + z = 0$, so $3x = -4s - t$, hence $x = (-4/3) \cdot s + (-1/3) \cdot t$. It follows that the solution set to the corresponding homogeneous linear system, and hence the eigenspace corresponding to -2, is

$$\left\{ \begin{pmatrix} (-4/3) \cdot s + (-1/3) \cdot t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Therefore, the eigenspace corresponding to 2 equals

$$\text{Span}\left(\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right).$$

Now the sequence

$$\left(\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right)$$

is linearly independent because if $c_1, c_2 \in \mathbb{R}$ are such that

$$c_1 \cdot \begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$\begin{pmatrix} (-4/3) \cdot c_1 + (-1/3) \cdot c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and hence both $c_1 = 0$ and $c_2 = 0$. Therefore,

$$\left(\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis for the eigenspace corresponding to $\lambda = -2$.