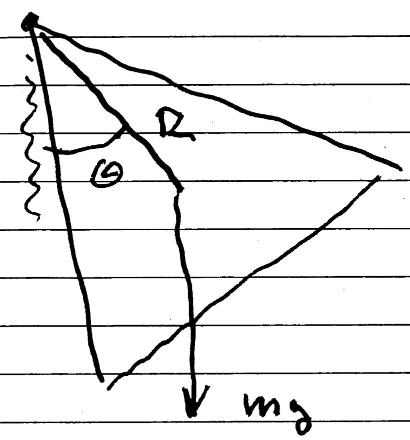
Y

Problem Set 12 Last Problem Set

7-5



$N = -mgR\sin\theta$

$L = I\dot{\theta}$

small ∢

so $N = \frac{dL}{dt} \Rightarrow I\ddot{\theta} = -mgR\theta$

$\ddot{\theta} = -\frac{mgR}{I}\theta$

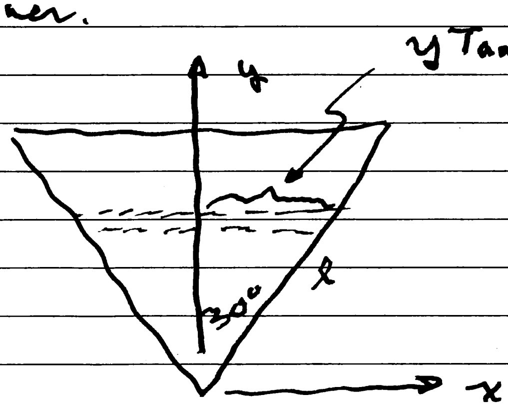
$\omega^2 = \frac{mgR}{I}$

I is prop. to m and dep. entirely on the length. Also it is prop. to length^2

so ω^2 is indep. of mass & g &

has dim. $1/\text{time}^2$ $\Rightarrow \omega^2 \propto \frac{g}{R}$

To find the freq we need R (distance to the center of mass) and I about a corner.



center of mass — clearly it is at $x = 0$ but what y ?

$$R = \frac{\int dm\, y}{\int dm} \qquad \int dm = M$$

$$dm = \rho\, dy\, (2y \tan 30^\circ)$$

density

Total mass

$$\rho \approx \frac{M}{\frac{1}{2}\, l\, y_0}$$

$y_0 =$ max y

$$\cos 30^\circ = \frac{y_0}{l} \Rightarrow y_0 = l \cos 30$$

Recall $\sin 30^\circ = \frac{1}{2}$ $\quad \cos 30^\circ = \frac{\sqrt{3}}{2}$ $\quad \tan 30^\circ = \frac{\sqrt{3}}{3}$

So

$$y_0 = \ell \frac{\sqrt{3}}{2}$$

$$\rho = \frac{M}{\frac{1}{2}\ell^2 \frac{\sqrt{3}}{2}} = \frac{4M}{\sqrt{3}\,\ell^2}$$

$$dm = \frac{4M}{\sqrt{3}\,\ell^2}\, dy \left(2y \frac{\sqrt{3}}{3}\right)$$

$$= \frac{8M}{3\ell^2}\, y\, dy$$

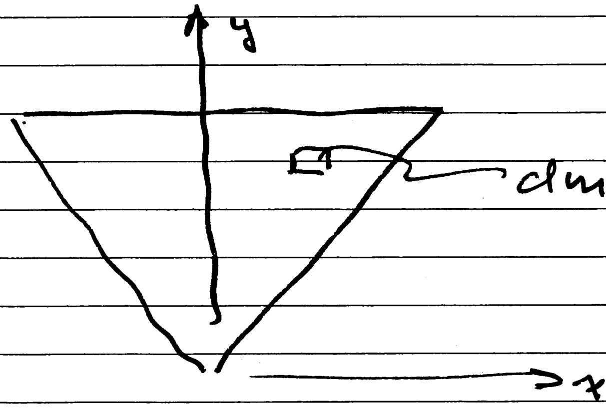
$$R = \frac{8}{3\ell^2}\int_0^{\ell\sqrt{3}/2} y^2\, dy = \frac{8}{3\ell^2}\left(\frac{y^3}{3}\right|_0^{\ell\sqrt{3}/2}$$

$$= \cancel{\frac{8}{3\ell^2}\,\frac{1}{2}\left(\ell\frac{\sqrt{3}}{2}\right)^3}$$

$$= \cancel{\frac{4}{3\ell^2}\,\ell^3\,\frac{\sqrt{3}\,3}{8}} = \cancel{6\sqrt{3}\,\ell}$$

$$R = \frac{\cancel{8}}{\cancel{3}\ell^2}\,\ell^3\,\frac{\frac{\cancel{3}\sqrt{3}}{\cancel{8}}}{3} = \frac{1}{\sqrt{3}}\,\ell$$

I



$$I = \int dm\,(x^2+y^2)$$

$$dm = \rho\, dx\, dy$$

$$I = \rho \iint dx\, dy\, (x^2+y^2)$$

$$= \rho \int_0^{\ell\sqrt{3}/2} \left\{ \int_{-\frac{1}{\sqrt{3}}y}^{+\frac{1}{\sqrt{3}}y} dx\, (x^2+y^2) \right\} dy$$

$$\int_{-1/\sqrt{3}\,y}^{+\frac{1}{\sqrt{3}}y} dx\,(x^2+y^2) = \left(\frac{x^3}{3} + y^2 x\right)\Bigg|_{-\frac{1}{\sqrt{3}}y}^{+\frac{1}{\sqrt{3}}y}$$

$$= 2y^3 \left\{ \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}} \right\} = 2y^3 \left\{ \frac{1}{9\sqrt{3}} + \frac{1}{\sqrt{3}} \right\}$$

$$= \frac{2y^3}{\sqrt{3}}\left(\frac{1}{9}+1\right) = \frac{20}{9\sqrt{3}}\,y^3$$

$$I = \rho \int_0^{\ell\sqrt{3}/2} \frac{20}{9\sqrt{3}}\,y^3\,dy$$

$$= \rho\,\frac{20}{9\sqrt{3}}\left(\frac{y^4}{4}\right)\Bigg|_0^{\ell\sqrt{3}/2} = \frac{\rho 5}{9\sqrt{3}}\left(\ell^4\,\frac{9}{16}\right)$$

$$= \frac{\rho 5}{16\sqrt{3}}\,\ell^4 = \frac{4M}{\sqrt{3}\,\ell^2}\left(\frac{5\ell^4}{16\sqrt{3}}\right)$$

$$I = \frac{20\,M\ell^2}{3 * 16} = \frac{5}{12}\,M\ell^2$$

so

$$\omega^2 = Mg\left(\frac{1}{\sqrt{3}}\,\ell\right)\left(\frac{12}{5M\ell^2}\right)$$

$$\omega^2 = \frac{12\,g}{5\sqrt{3}\,\ell}$$

$$\omega = \sqrt{\frac{12}{5\sqrt{3}}}\,\sqrt{\frac{g}{\ell}}$$