Worksheet 13 - Eigenvalues and Eigenvectors

1 The Basic Idea

If we have a linear transformation that describes a phenomena, it may be informative to know which vectors are pointing in the same[1]direction before and after the transformation. That is, for which vectors does our linear transformation act the same as scalar multiplication? In other words:

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then for which $\vec{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ do we have

$$T(\vec{v}) = \lambda\vec{v}$$

Or, using matrix notation, if $[T] = A$, then for which $\vec{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ do we have

$$A\vec{v} = \lambda\vec{v}$$

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the primary method we have for finding eigenvalues of A is by determining which values of λ result in

$$\mathrm{Null}(A - \lambda \cdot I) \neq \{0\}$$

which is equivalent to finding the zeros of the *characteristic polynomial*, i.e. solving

$$(a - \lambda)(d - \lambda) - bc = 0.$$

Finally, if λ_1 is an eigenvalue, then

$$\mathrm{Null}(A - \lambda_1 \cdot I) = \mathrm{Span}(\vec{v}_1)$$

for some vector $\vec{v}_1$, and the vectors in this span are the eigenvectors associated to λ_1.

2 Practice and Applications

Exercise 1. Given

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of M, M^2, M^{-1}, and $M + 4 \cdot I$.

[1] Here we mean "same" to be "parallel". That is, $\vec{v}$ and $-\vec{v}$ point in the same direction.

Exercise 2. Given

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \tag{1}$$

find the eigenvalues and eigenvectors of A.

Exercise 3. If A has eigenvalues λ_1 and λ_2 associated to eigenvectors $\vec{v}_1$ and $\vec{v}_2$, repectively, then what are the eigenvalues and eigenvectors of $A \cdot A$? Prove your claim and illustrate with an example. (Hint: You only need to use the definition of linear transformation.)

Exercise 4. Combining your observations from Exercise 2 and 3, what can you say about the eigenvalues and eigenvectors of the following matrix?:

$$B = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^{99}$$

The matrix in Equation (1) is in the form of a *Markov* matrix. These are used to model probabilistic changes in the state of a phenomena. If a phenomena has states that change probabilistically, we can model that process in the following way:

- Set up a matrix with the entries representing the probabilities of each of the following:

$$\begin{pmatrix} \text{Start in State 1, remain in State 1} & \text{Start in State 2, change to State 1} \\ \text{Start in State 1, change to State 2} & \text{Start in State 2, remain in State 2} \end{pmatrix}$$

- Input a vector of the form

$$\begin{pmatrix} \text{\% of observations currently in State 1} \\ \text{\% of observations currently in State 2} \end{pmatrix} \tag{2}$$

Exercise 5. If the states in the Markov matrix,A, from Equation (1) are

- State 1: A person lives in Los Angeles
- State 2: A person lives in New York

and the probabilities refer to the change in state over the course of a year, then:

(a) What are the observations being referred to in Equation (2) in this context?

(b) How would you interpret the output after applying A to an input vector?

(c) How would you interpret your observations from Exercise 4 in this context? (What do the eigenvalues and eigenvectors mean?)

For Next Time

- Finish this worksheet
- Read through Example 3.5.16