Solutions to Problem Set 18

Problem 1: We apply Proposition 4.3.3. Performing elementary row operations on the matrix having the three vectors as columns, we obtain:

$$\begin{aligned}
\begin{pmatrix} 1 & 2 & 4 \\ -3 & 2 & -4 \\ 5 & 4 & 14 \end{pmatrix} &\to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 8 & 8 \\ 0 & -6 & -6 \end{pmatrix} && \begin{matrix} \\ (3R_1 + R_2) \\ (-5R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} && \begin{matrix} \\ (\frac{1}{8} \cdot R_2) \\ (-\frac{1}{6} \cdot R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \begin{matrix} \\ \\ (-R_2 + R_3). \end{matrix}
\end{aligned}$$

Since the last column does not have a leading entry, Proposition 4.3.3 tells us that the sequence is linearly dependent.

Problem 2: We want to find a choice of $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix} + c_4 \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and where at least one c_i is nonzero. In other words, we want to find $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$\begin{array}{rcrcrcrcr}
 & & 2c_2 & - & 8c_3 & + & 6c_4 & = & 0 \\
c_1 & & & + & 2c_3 & - & c_4 & = & 0 \\
3c_1 & + & 2c_2 & - & 2c_3 & + & 9c_4 & = & 0 \\
-c_1 & - & c_2 & + & 2c_3 & + & 5c_4 & = & 0
\end{array}$$

and where at least one c_i is nonzero. To solve this system, we can consider the corresponding augmented matrix. However, since the last column is all zeros, we omit it in our calculations. Performing elementary row operations on the resulting, we obtain:

$$\begin{pmatrix} 0 & 2 & -8 & 6 \\ 1 & 0 & 2 & -1 \\ 3 & 2 & -2 & 9 \\ -1 & -1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -8 & 6 \\ 3 & 2 & -2 & 9 \\ -1 & -1 & 2 & 5 \end{pmatrix} \qquad \begin{matrix} (R_1 \leftrightarrow R_2) \\ (R_2 \leftrightarrow R_2) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -8 & 12 \\ 0 & -1 & 4 & 4 \end{pmatrix} \qquad \begin{matrix} (\frac{1}{2} \cdot R_2) \\ (-3R_1 + R_2) \\ (R_1 + R_4) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad \begin{matrix} (\frac{1}{2} \cdot R_2) \\ (-2R_1 + R_2) \\ (R_2 + R_4) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad (\tfrac{1}{6} \cdot R_3)$$

$$\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (-7R_3 + R_4).$$

Recall that when we view this matrix as corresponding to a system of equations, we should add a column of zeros on the right. Now the third equation tells us that we must have $c_4 = 0$. Since there is no leading entry in the third column, we can assign any value to c_3, and so we choose to let $c_3 = 1$ (to make the final linear combination nontrivial). From here, the second equation says that

$$c_2 - 4c_3 + 3c_4 = 0,$$

so

$$\begin{aligned} c_2 &= 4c_3 - 3c_4 \\ &= 4 \cdot 1 - 3 \cdot 0 \\ &= 4. \end{aligned}$$

Finally, the first equation says that

$$c_1 + 2c_3 - c_4 = 0,$$

so

$$\begin{aligned} c_1 &= -2c_3 + c_4 \\ &= (-2) \cdot 1 - 0 \\ &= -2. \end{aligned}$$

Thus, one nontrivial solution to our original system is $(-2, 4, 1, 0)$. In other words, we have

$$(-2) \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 3: We claim that (f_1, f_2, f_3) is linearly independent. Let $c_1, c_2, c_3 \in \mathbb{R}$ be arbitrary with $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$. Since functions are equal if and only if they agree on all inputs, we know that for all $x \in \mathbb{R}$, we have

$$c_1 \cdot (9x^2 - x + 3) + c_2 \cdot (3x^2 - 2x + 5) + c_3 \cdot (-5x^2 + x + 1) = 0.$$

We rephrase this as saying that for all $x \in \mathbb{R}$, we have

$$(9c_1 + 3c_2 - 5c_3) \cdot x^2 + (-c_1 - 2c_2 + c_3) \cdot x + (3c_1 + 5c_2 + c_3) = 0x^2 + 0x + 0.$$

Since polynomials give equal values on all inputs precisely when the corresponding coefficients are equal, this implies that

$$\begin{array}{rcrcrcl} 9c_1 & + & 3c_2 & - & 5c_3 & = & 0 \\ -c_1 & - & 2c_2 & + & c_3 & = & 0 \\ 3c_1 & + & 5c_2 & + & c_3 & = & 0. \end{array}$$

Since we know that c_1, c_2, c_3 satisfy this system of equations, we now apply elementary row operations to obtain an system with the same solution set (we omit the augmented column of zeros):

$$\begin{aligned} \begin{pmatrix} 9 & 3 & -5 \\ -1 & -2 & 1 \\ 3 & 5 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} -1 & -2 & 1 \\ 9 & 3 & -5 \\ 3 & 5 & 1 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_2) \\ (R_1 \leftrightarrow R_2) \\ \end{matrix} \\ &\rightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -15 & 4 \\ 0 & -1 & 4 \end{pmatrix} && \begin{matrix} \\ (9R_1 + R_2) \\ (3R_1 + R_3) \end{matrix} \\ &\rightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & 4 \\ 0 & -15 & 4 \end{pmatrix} && \begin{matrix} \\ (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3) \end{matrix} \\ &\rightarrow \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -56 \end{pmatrix} && \begin{matrix} \\ \\ (-15R_2 + R_3). \end{matrix} \end{aligned}$$

Since the system corresponding to the final matrix (with an additional augmented column of zeros) has the same solution as the original system, it follows that c_1, c_2, c_3 satisfy the following system:

$$\begin{array}{rcrcrcl} -c_1 & - & 2c_2 & + & c_3 & = & 0 \\ & & -c_2 & + & 4c_3 & = & 0 \\ & & & & -56c_3 & = & 0. \end{array}$$

From the third equation, we conclude that $c_3 = 0$. Plugging this into the second equation, we conclude that $-c_2 = 0$, hence $c_2 = 0$. Plugging these it into the first allows us to conclude that $-c_1 = 0$, and so $c_1 = 0$. Therefore, we have shown that $c_1 = c_2 = c_3 = 0$. It follows that (f_1, f_2, f_3) is linearly independent.

Problem 4: We claim that (f_1, f_2, f_3) is linearly independent. To see this, let $c_1, c_2, c_3 \in \mathbb{R}$ be arbitrary with $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$. Since functions are equal exactly when they give the same output on every input, we then have that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0,$$

for all $x \in \mathbb{R}$, so

$$c_1 \cdot 2^x + c_2 \cdot x^2 + c_3 \cdot (x - 2) = 0$$

for all $x \in \mathbb{R}$. Since this statement is true for all $x \in \mathbb{R}$, it is true whenever we plug in a specific value for x. Plugging in the values $0, 1, 2$ in turn, we see that the following three equations must be true:

$$\begin{array}{rcrcrcl} c_1 & & & - & 2c_3 & = & 0 \\ 2c_1 & + & c_2 & - & c_3 & = & 0 \\ 4c_1 & + & 4c_2 & & & = & 0. \end{array}$$

Since we know that c_1, c_2, c_3 satisfy this system of equations, we now apply elementary row operations to obtain an system with the same solution set (we omit the augmented column of zeros):

$$\begin{aligned} \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & -1 \\ 4 & 4 & 0 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 4 & 8 \end{pmatrix} && \begin{matrix} (-2R_1 + R_2) \\ (-4R_1 + R_3) \end{matrix} \\ &\to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{pmatrix} && (-4R_2 + R_3). \end{aligned}$$

Since the system corresponding to the final matrix (with an additional augmented column of zeros) has the same solution as the original system, it follows that c_1, c_2, c_3 satisfy the following system:

$$\begin{array}{rcrcrcl} c_1 & & & - & 2c_3 & = & 0 \\ & & c_2 & + & 3c_3 & = & 0 \\ & & & & -4c_3 & = & 0. \end{array}$$

From the third equation, we conclude that $c_3 = 0$. Plugging this into the second equation, we conclude that $c_2 = 0$, and plugging it into the first allows us to conclude that $c_1 = 0$. Therefore, we have shown that $c_1 = c_2 = c_3 = 0$. It follows that (f_1, f_2, f_3) is linearly independent.

Problem 5: Consider the following sequence of vectors in $\mathbb{R}^4$:

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

We claim that whenever we omit one of the 4 vectors, the resulting 3 are linearly independent. We check each of the possibilities.

1. Consider the sequence

   $$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

   To check that this sequence is linearly independent, we use Proposition 4.4.3. The corresponding matrix is:

   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Since there is a leading variable in each column, the sequence of vectors is linearly independent by Proposition 4.4.3.

2. Consider the sequence
$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right).$$
The corresponding matrix is:
$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$
Since there is a leading variable in each column, the sequence of vectors is linearly independent by Proposition 4.4.3.

3. Consider the sequence
$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right).$$
Applying elementary row operations to the corresponding matrix, we have
$$\begin{pmatrix}1 & 1 & 1\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 0 & 1\end{pmatrix}\qquad\qquad(-R_1+R_2).$$
Since there is a leading variable in each column, the sequence of vectors is linearly independent by Proposition 4.4.3.

4. Consider the sequence
$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right).$$
Applying elementary row operations to the corresponding matrix, we have
$$\begin{aligned}\begin{pmatrix}1 & 1 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}&\rightarrow\begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 1 & 0\end{pmatrix}&&(-R_1+R_2)\\&\rightarrow\begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 0 & -1\end{pmatrix}&&(R_2+R_3).\end{aligned}$$
Since there is a leading variable in each column, the sequence of vectors is linearly independent by Proposition 4.4.3.

We have checked all four possibilities, so whenever we omit one of the 4 vectors, the resulting 3 are linearly independent.

Problem 6: Through the argument, let A be the $m \times n$ matrix where the i^{th} column is $\vec{u}_i$, and let B be an echelon form of A. Notice that A and B are $n \times n$ matrices.

Assume first that $\text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n) = \mathbb{R}^n$. By Proposition 4.2.12, we know that every row of B has a leading entry, so B has a total of n leading entries. Now B is in echelon form, so we know that every column of B has at most one leading entry. Since B has n columns, it must be the case that every column of B has a leading entry. Using Proposition 4.4.3, it follows that $(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ is linearly independent.

Assume conversely that $(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ is linearly independent. By Proposition 4.4.3, every column of B has a leading entry. Now B is in echelon form, so we know that every column of B has at most one leading entry. Thus, B has a total of n leading entries. Since every row of B can have at most one leading entry as well (by definition of a leading entry), we conclude that every row of B has a leading entry. Using Proposition 4.2.12, it follows that $\operatorname{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n) = \mathbb{R}^n$.