Assignment: Problem Set 15

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: None

Problem 1: Recall that $\mathcal{P}$ is the vector space of all polynomial functions $f : \mathbb{R} \to \mathbb{R}$. Let W be the subset of $\mathcal{P}$ consisting of those polynomials that have a nonnegative constant term (i.e. the constant terms is greater than or equal to 0). Is W a subspace of $\mathcal{P}$? Either prove or give a counterexample.

Solution: If W is a subspace of $\mathcal{P}$, then W has the following the properties as laid out in Definition 4.1.12:

1. There exists a $\vec{w_0} \in W$ such that $\vec{w} + \vec{w_0} = \vec{w}$ for all $\vec{w} \in W$.
2. For all $\vec{w_1}, \vec{w_2} \in W$, we have $\vec{w_1} + \vec{w_2} \in W$
3. For all $\vec{w} \in W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{w} \in W$

Consider the third property:

Define a polynomial $w : \mathbb{R} \to \mathbb{R}$ by fixing $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$ such that $a_0 > 0$ and $w(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for all $x \in \mathbb{R}$. Notice that a_0 is a nonnegative constant term, so $w \in W$ by definition. Consider $(-1) \in \mathbb{R}$. Notice that

$$\begin{aligned}
(-1) \cdot w(x) &= -1 \cdot (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\
&= (-1)a_n x^n + (-1)a_{n-1}x^{n-1} + \cdots + (-1)a_1 x + (-1)a_0 \\
&= -a_n x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0
\end{aligned}$$

So $(-1) \cdot w(x) = -a_n x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$ for all $x \in \mathbb{R}$. Notice that the constant term is negative, so $(-1) \cdot w(x) \notin W$. Because $(-1) \in \mathbb{R}$, it follows that Property 3 is not true for W, and so W is not a subspace of $\mathcal{P}$.

Problem 2: Let $V = \mathbb{R}^4$. Write down a system of four equations in three unknowns such that

$$\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} \in \text{ Span}\left(\begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right)$$

if and only if the system has a solution.

Solution: By definition of Span, $\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} \in \text{ Span}\left(\begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right)$ if and only if there exist $a, b, c \in \mathbb{R}$ with $\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} = a \cdot \begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix} + b \cdot \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}$. We can express this sum of vectors as the following linear system of 4 equations in the variables a, b, and c:

$$\begin{aligned} 2a + 6b + 0c &= 1 \\ -5a + 1b + 3c &= 7 \\ 1a - 8b + 3c &= 0 \\ 4a + 2b + 3c &= 6 \end{aligned}$$

If this system has a solution (a_0, b_0, c_0), then the result follows.

Problem 3: Let V be the vector space of all 2×2 matrices. Show that
$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \text{ Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}\right).$$

Solution: Let $A \in \text{ Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}\right)$ be arbitrary. By definition of Span, we can fix $x, y \in \mathbb{R}$ with $A = x \cdot \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} + y \cdot \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}$. Notice that
$$\begin{aligned} A = x \cdot \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} + y \cdot \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix} &= \begin{pmatrix} 1x & 1x \\ 2x & -3x \end{pmatrix} + \begin{pmatrix} 3y & 0y \\ 5y & -4y \end{pmatrix} \\ &= \begin{pmatrix} 1x + 3y & 1x + 0y \\ 2x + 5y & -3x - 4y \end{pmatrix} \end{aligned}$$

So $A = \begin{pmatrix} 1x + 3y & 1x + 0y \\ 2x + 5y & -3x - 4y \end{pmatrix}$. If $\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \text{ Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}\right)$, then there is a solution to the following linear system of four equations in the variables (x, y):
$$1x + 3y = -2 \tag{1}$$
$$1x + 0y = 7 \tag{2}$$
$$2x + 5y = -1 \tag{3}$$
$$-3x - 4y = -9 \tag{4}$$

Notice that (2) implies $x = 7$. Using this value for x, our system becomes
$$7 + 3y = -2 \tag{5}$$
$$x = 7 \tag{6}$$
$$14 + 5y = -1 \tag{7}$$
$$-21 - 4y = -9 \tag{8}$$

Solving simultaneously for y, we get
$$y = \frac{-2 - 7}{3} = \frac{-9}{3} = -3 \tag{9}$$
$$x = 7 \tag{10}$$
$$y = \frac{-1 - 14}{5} = \frac{-15}{5} = -3 \tag{11}$$
$$y = \frac{-9 + 21}{-4} = \frac{12}{-4} = -3 \tag{12}$$

So the solution to first system of equations is $(x, y) = (7, -3)$. Checking our answer, we compute: $A(7, -3) = \begin{pmatrix} 1(7) + 3(-3) & 1(7) + 0(-3) \\ 2(7) + 5(-3) & -3(7) - 4(-3) \end{pmatrix} = \begin{pmatrix} 7 - 9 & 7 \\ 14 - 15 & -21 + 12 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix}$.

Therefore, $\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \text{ Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}\right)$.

Problem 4: Let $\mathcal{D}$ be the vector space of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be the function $f_1(x) = \sin^2(x)$ and let $f_2 : \mathbb{R} \to \mathbb{R}$ be the function $f_2(x) = \cos^2(x)$. Finally, let $W = \text{Span}(f_1, f_2)$, and notice that W is a subspace of $\mathcal{D}$. Determine, with explanation, whether the following functions are elements of W.

a. The function $g_1 : \mathbb{R} \to \mathbb{R}$ given $g_1(x) = 3$.

Solution: If $g_1 \in W$, then by definition of Span, there exist $a, b \in \mathbb{R}$ with $g_1(x) = a \cdot f_1(x) + b \cdot f_2(x)$ for all $x \in \mathbb{R}$, that is, there exist $a, b \in \mathbb{R}$ such that $3 = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. Notice that $3 = 3(1) = 3(\sin^2(x) + \cos^2(x)) = 3\sin^2(x) + 3\cos^2(x)$ for all $x \in \mathbb{R}$. Because $3, 3 \in \mathbb{R}$, it follows that $g_1 \in W$.

b. The function $g_2 : \mathbb{R} \to \mathbb{R}$ given $g_2(x) = x^2$.

Solution: If $g_2 \in W$, then by definition of Span, there exist $a, b \in \mathbb{R}$ with $g_2(x) = a \cdot f_1(x) + b \cdot f_2(x)$ for all $x \in \mathbb{R}$, that is, there exist $a, b \in \mathbb{R}$ such that $x^2 = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. We assume that there exists $a_2, b_2 \in \mathbb{R}$ such that $x^2 = a_2\sin^2(x) + b_2\cos^2(x)$ for all $x \in \mathbb{R}$. We then have that $\frac{d}{dx}(x^2) = \frac{d}{dx}(a_2\sin^2(x) + b_2\cos^2(x)$ for all $x \in \mathbb{R}$. Computing the derivatives, we get $2x = 2a_2\sin(x)\cos(x) - 2b_2\sin(x)\cos(x) = (a_2 - b_2)2\sin(x)\cos(x) = (a_2 - b_2)\sin(2x)$ for all $x \in \mathbb{R}$. So we have that $2x = (a_2 - b_2)\sin(2x)$ for all $x \in \mathbb{R}$. Consider $x = \pi$. We then have $2\pi = (a_2 - b_2)\sin(2\pi) = (a_2 - b_2)0 = 0$. So $2\pi = 0$. Our assumption has led us to a contradiction, so it must be the case that there exist no $a, b \in \mathbb{R}$ such that $x^2 = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. Therefore, $g_2 \notin W$.

c. The function $g_3 : \mathbb{R} \to \mathbb{R}$ given $g_3(x) = \sin x$.

Solution: If $g_3 \in W$, then by definition of Span, there exist $a, b \in \mathbb{R}$ with $g_3(x) = a \cdot f_1(x) + b \cdot f_2(x)$ for all $x \in \mathbb{R}$, that is, there exist $a, b \in \mathbb{R}$ such that $\sin(x) = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. We assume that there exists $a_3, b_3 \in \mathbb{R}$ such that $\sin(x) = a_3\sin^2(x) + b_3\cos^2(x)$ for all $x \in \mathbb{R}$. We then have that $\frac{d}{dx}(\sin(x)) = \frac{d}{dx}(a_3\sin^2(x) + b_3\cos^2(x))$ for all $x \in \mathbb{R}$. Computing the derivatives, we get $\cos(x) = 2a_3\sin(x)\cos(x) - 2b_3\sin(x)\cos(x) = (a_3 - b_3)2\sin(x)\cos(x) = (a_3 - b_3)\sin(2x)$ for all $x \in \mathbb{R}$ So we have $\sin(x) = (a_3 - b_3)\sin(2x)$ for all $x \in \mathbb{R}$. Consider $x = \frac{\pi}{2}$. We then have that $\sin(\frac{\pi}{2}) = (a_3 - b_3)\sin(2\frac{\pi}{2})$. This reduces to $1 = (a_3 - b_3)\sin(\pi) = 0$. So $1 = 0$. Our assumption has let us to a contradiction, so it must be the case that there exist no $a, b \in \mathbb{R}$ such that $\sin(x) = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. Therefore, $g_3 \notin W$.

d. The function $g_4 : \mathbb{R} \to \mathbb{R}$ given $g_4(x) = \cos 2x$.

Solution: If $g_4 \in W$, then by definition of Span, there exist $a, b \in \mathbb{R}$ with $g_4(x) = a \cdot f_1(x) + b \cdot f_2(x)$ for all $x \in \mathbb{R}$, that is, there exist $a, b \in \mathbb{R}$ such that $\cos 2x = a\sin^2(x) + b\cos^2(x)$ for all $x \in \mathbb{R}$. Notice that $\cos 2x = \cos^2(x) - \sin^2(x) = 1 \cdot \cos^2(x) + (-1) \cdot \sin^2(x)$ for all $x \in \mathbb{R}$. Because $1, -1 \in \mathbb{R}$, it follows that $g_4 \in W$.

Problem 5: Let V be a vector space, and let W be a subspace of V. Recall that

$$V \setminus W = \{\vec{v} \in V : \vec{v} \notin W\}$$

i.e. $V \setminus W$ is the set of elements of V that are *not* in W. Is $V \setminus W$ always a subspace V? Sometimes a subspace of V? Never a subspace of V? Explain.

Solution: Let V be an arbitrary vector space. By Definition 4.1.1, $\vec{0} \in V$. Let W be an arbitrary subspace of V. By Defintion 4.1.12, $\vec{0} \in W$. By By Definition 1.5.2, $V \setminus W = \{\vec{v} : \vec{v} \in V \text{ and } \vec{v} \notin W\}$. Because $\vec{0} \in V$ and $\vec{0} \in W$, it follows that $\vec{0} \notin V \setminus W$, and so by Definition 4.1.12, $V \setminus W$ is not a subspace of V. Because vector space V and subspace W were arbitrary, it follows that $V \setminus W$ is *never* a subspace of V.

Problem 6: Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Let W be the set of all even functions, i.e

$$W = \{f \in \mathcal{F} : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}.$$

Is W a subspace of $\mathcal{F}$? Either prove or give a counterexample.

Solution: If W is a subspace of $\mathcal{F}$, then W has the following properties as laid out in Definition 4.1.12:

1. There exists a $\vec{u_0} \in W$ such that $\vec{u} + \vec{u_0} = \vec{u}$ for all $\vec{u} \in W$.
2. For all $\vec{u_1}, \vec{u_2} \in W$, we have $\vec{u_1} + \vec{u_2} \in W$
3. For all $\vec{u} \in W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{u} \in W$

We check all three properties

1. We define a function $u_0 : \mathbb{R} \to \mathbb{R}$ by letting $u_0(x) = 0$ for all $x \in \mathbb{R}$. Notice that $u_0(x) = 0 = u_0(-x)$, so by definition u_0 is an even function, and it follows that $u_0 \in W$. Now let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary even function, so $u \in W$ by definition. Notice that $u(x) + u_0(x) = u(x) + 0 = u(x)$. Because $u \in W$ was arbitrary, it follows that $u(x) + u_0(x) = u(x)$ for all $u \in W$. So u_0 satisfies the definition of $\vec{u_0}$, and thus the first property is satisfied.

2. Let $u_1 : \mathbb{R} \to \mathbb{R}$, $u_2 : \mathbb{R} \to \mathbb{R}$ be arbitrary even functions. So $u_1, u_2 \in W$ by definition. Because u_1 is even, we have that $u_1(x) = u_1(-x)$ for all $x \in \mathbb{R}$. Similarly, u_2 is even, so $u_2(x) = u_2(-x)$ for all $x \in \mathbb{R}$. Notice that $(u_1+u_2)(x) = u_1(x)+u_2(x) = u_1(-x)+u_2(-x) = (u_1 + u_2)(-x)$ for all $x \in \mathbb{R}$. So $(u_1 + u_2)(x) = (u_1 + u_2)(-x)$ for all $x \in \mathbb{R}$, so by definition $u_1 + u_2$ is an even function, and it follows that $u_1 + u_2 \in W$. Since $u_1, u_2 \in U$ were arbitrary, we have that $u_1 + u_2 \in W$ for all $u_1, u_2 \in W$, and thus the second property is satisfied.

3. Let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary even function, so $u \in W$ by definition. Because u is even, we have that $u(x) = u(-x)$ for all $x \in \mathbb{R}$. Now let $r \in \mathbb{R}$ be arbitrary. Notice that $(r \cdot u)(x) = r \cdot u(x) = r \cdot u(-x) = (r \cdot u)(-x)$ for all $x \in \mathbb{R}$. So $(r \cdot u)(x) = (r \cdot u)(-x)$ for all $x \in \mathbb{R}$, so by definition, $r \cdot u$ is an even function, and it follows that $r \cdot u \in W$. Since $u \in W$ and $r \in \mathbb{R}$ were arbitrary, we have that $r \cdot u \in W$ for all $u \in W$ and all $r \in \mathbb{R}$, and thus the third property is satisfied.

We have shown that W has all three properties of a subspace of $\mathcal{F}$, therefore, W is indeed a subspace of $\mathcal{F}$.