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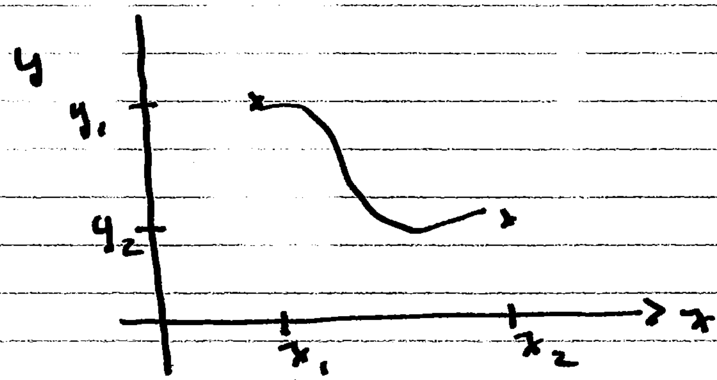
Problem Set #7

4-3 $t = \int \frac{ds}{v}$



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'^2}$$

$$E = \frac{1}{2}mv^2 + mgy$$

$$\shortparallel$$

$$mgy_1$$

$$mg(y_1 - y) = \frac{1}{2}mv^2$$

$$v^2 = 2g(y_1 - y)$$

$$v = \sqrt{2g}\sqrt{y_1 - y}$$

$$t = \int \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2g}\sqrt{y_1 - y}}$$

$$F(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{y_1 - y}} = (1+y'^2)^{1/2}(y_1 - y)^{-1/2}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{\partial F}{\partial y} = -\frac{1}{2}(1+y'^2)^{1/2}(y_1-y)^{-3/2}(-1)$$

$$\frac{\partial F}{\partial y} = \frac{1}{2}(1+y'^2)^{1/2}(y_1-y)^{-3/2}$$

$$\frac{\partial F}{\partial y'} = \frac{1}{2}(1+y'^2)^{-1/2}(2y')(y_1-y)^{-1/2}$$

$$= y'(1+y'^2)^{-1/2}(y_1-y)^{-1/2}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = y''(1+y'^2)^{-1/2}(y_1-y)^{-1/2} +$$

$$-\cancel{\tfrac{1}{2}}(y')(1+y'^2)^{-3/2}(\cancel{2}y'y'')(y_1-y)^{-1/2} +$$

$$-\frac{1}{2}y'(1+y'^2)^{-1/2}(y_1-y)^{-3/2}(-y')$$

$$= \cancel{y''(1+y'^2)^{-3/2}\{(1+y'^2)\}}$$

$$= \cancel{(1+y'^2)^{-1/2}(y_1-y)^{-1/2}}\Big\{\cancel{y''-2y}$$

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$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = y''(1+y'^2)^{-1/2}(y_1-y)^{-1/2} +$$

$$- y'^2 y''(1+y'^2)^{-3/2}(y_1-y)^{-1/2} +$$

$$+ \frac{1}{2} y'^2 (1+y'^2)^{-1/2}(y_1-y)^{-3/2}$$

$$= (1+y'^2)^{-1/2}(y_1-y)^{-1/2}\Big\{ y'' +$$

$$- y'^2 y''(1+y'^2)^{-1} + \frac{1}{2} y'^2 (y_1-y)^{-1}\Big\}$$

So

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\frac{1}{2}(1+y'^2)^{1/2}(y_1-y)^{-3/2} =$$

$$(1+y'^2)^{-1/2}(y_1-y)^{-1/2}\Big\{ y'' +$$

$$- y'^2 y''(1+y'^2)^{-1} + \frac{1}{2} y'^2 (y_1-y)^{-1}\Big\}$$

$$\boxed{\frac{1}{2}(1+y'^2)(y_1-y)^{-1} = y'' - y'^2 y''(1+y'^2)^{-1} + \frac{1}{2} y'^2 (y_1-y)^{-1}}$$

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① $y_1 - y = R(1 - \cos\phi)$ initial y, x $\phi = 0$

② $x - x_1 = R(\phi - \sin\phi)$

① $\Rightarrow$ $-dy = R\sin\phi \, d\phi$

② $\Rightarrow$ $dx = R(1 - \cos\phi)\, d\phi$

$$\underset{\substack{\shortparallel \\ y'}}{\frac{dy}{dx}} = \frac{-R\sin\phi \, d\phi}{R(1-\cos\phi)\, d\phi} = -\frac{\sin\phi}{1-\cos\phi}$$

$$y'^2 = \frac{\sin^2\phi}{(1-\cos\phi)^2}$$

$$1 + y'^2 = \frac{(1-\cos\phi)^2 + \sin^2\phi}{(1-\cos\phi)^2}$$

$$= \frac{1 - 2\cos\phi + \cos^2\phi + \sin^2\phi}{(1-\cos\phi)^2}$$

$$1 + y'^2 = \frac{2(1-\cos\phi)}{(1-\cos\phi)^2} = \frac{2}{1-\cos\phi}$$

$$y' = -\frac{\sin\phi}{1-\cos\phi}$$

$$\frac{dy'}{dx} = \frac{d}{d\phi}\left(-\sin\phi\,(1-\cos\phi)^{-1}\right)\frac{d\phi}{dx}$$

$$x - x_1 = R\,(\phi - \sin\phi)$$

$$\frac{dx}{d\phi} = R\,(1-\cos\phi)$$

$$\frac{d\phi}{dx} = \frac{1}{R(1-\cos\phi)}$$

$$\frac{dy'}{dx} = \left[-\cos\phi\,(1-\cos\phi)^{-1} + \sin\phi\,(1-\cos\phi)^{-2}\sin\phi\right] \times$$

$$\frac{1}{R(1-\cos\phi)}$$

$$= \frac{1}{R(1-\cos\phi)^3}\left[-\cos\phi\,(1-\cos\phi) + \sin^2\phi\right]$$

$$= \frac{1}{R(1-\cos\phi)^3}\left[-\cos\phi + \cos^2\phi + \sin^2\phi\right]$$

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$$y'' = \frac{1 - \cos\phi}{R\,(1-\cos\phi)^3} = \frac{1}{R\,(1-\cos\phi)^2}$$

Sub

$$\frac{1}{2}\;\frac{2}{1-\cos\phi}\;\frac{1}{R\,(1-\cos\phi)} \stackrel{?}{=} \frac{1}{R\,(1-\cos\phi)^2} +$$

$$- \frac{\sin^2\phi}{(1-\cos\phi)^2}\;\frac{1}{R\,(1-\cos\phi)^2}\;\frac{(1-\cos\phi)}{2} +$$

$$+ \frac{1}{2}\;\frac{\sin^2\phi}{(1-\cos\phi)^2}\;\frac{1}{R\,(1-\cos\phi)}$$

$$\frac{1}{R\,(1-\cos\phi)^2} \stackrel{?}{=} \frac{1}{R\,(1-\cos\phi)^2} - \frac{\sin^2\phi}{2R\,(1-\cos\phi)^3} +$$

$$+ \frac{\sin^2\phi}{2R\,(1-\cos\phi)^3}$$

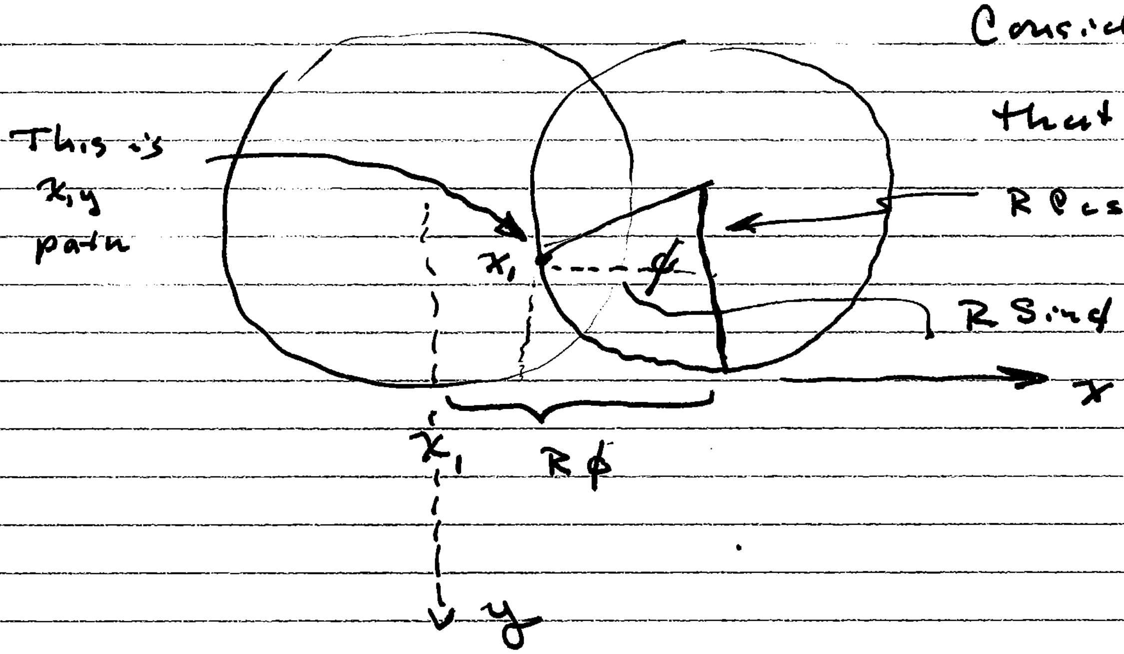
$$\frac{1}{R\,(1-\cos\phi)^2} \stackrel{?}{=} \frac{1}{R\,(1-\cos\phi)^2}$$

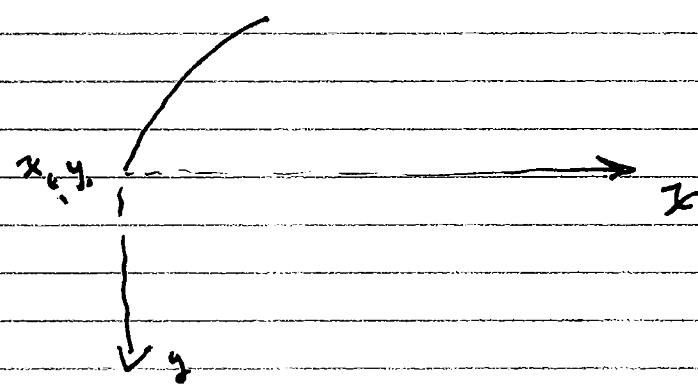
it works

$$y_1 - y = R(1 - \cos\phi)$$

$$x - x_1 = R(\phi - \sin\phi)$$

Consider a disc that rolls.





Jarlan $\quad y' = -\dfrac{\sin\phi}{1-\cos\phi}$

$\phi = 0$ is starting pt.

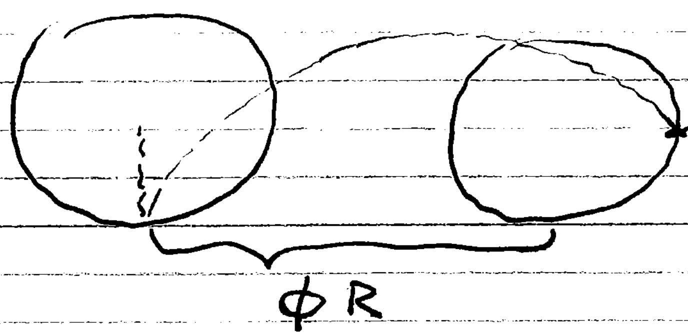
expand. – for small ϕ

$$y' = -\frac{\phi}{1-\left(1-\frac{\phi^2}{2}\right)}$$

$$= \frac{-\phi\ 2}{\phi^2} = -\frac{2}{\phi}$$

So initial slope is ∞.

Drop below ~~initial~~ final y.



$$y_1 - y = R(1-\cos\phi)$$

lowest when $\phi = \pi$

also from $y' = -\frac{\sin\phi}{1-\cos\phi}$ see

$y' = 0$ at $\sin\phi = 0 \Rightarrow \phi = \pi$.

$$\frac{x_2 - x_1}{y_1 - y_2} = \frac{R(\phi - \sin\phi)}{R(1 - \cos\phi)} = \frac{\phi - \sin\phi}{1 - \cos\phi}$$

Sub $\phi = \pi$

$$\frac{x_2 - x_1}{y_1 - y_2} = \frac{\pi}{2}$$

so when $\frac{x_2 - x_1}{y_1 - y_2} > \pi/2$.

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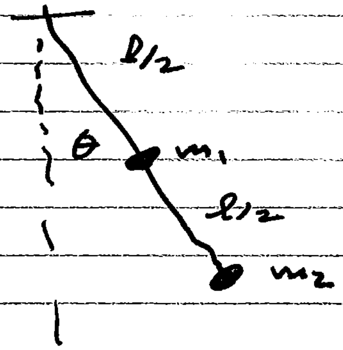

$$\mathcal{L} = T - U$$

$$T = \frac{1}{2}m_1\left(\frac{l}{2}\dot\theta\right)^2 + \frac{1}{2}m_2\left(l\dot\theta\right)^2$$

$$= \frac{1}{8}m_1 l^2\dot\theta^2 + \frac{1}{2}m_2 l^2\dot\theta^2$$

$$= \left(\frac{1}{8}m_1 + \frac{1}{2}m_2\right)l^2\dot\theta^2$$

$$U = -m_1 g\frac{l}{2}\cos\theta - m_2 g l\cos\theta$$

$$= -\left(\frac{m_1}{2} + m_2\right)g l\cos\theta$$

$$\mathcal{L} = \left(\frac{1}{8}m_1 + \frac{1}{2}m_2\right)l^2\dot\theta^2 + \left(\frac{m_1}{2} + m_2\right)g l\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -\left(\frac{m_1}{2} + m_2\right)g l\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = \left(\frac{1}{8}m_1 + \frac{1}{2}m_2\right)2l^2\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \left(\frac{1}{4}m_1 + m_2\right)l^2\ddot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \Rightarrow$$

$$-\left(\frac{m_1}{2} + m_2\right) g\, l \sin\theta - \left(\frac{1}{4} m_1 + m_2\right) l^2 \ddot{\theta} = 0$$

$$-\left(\frac{m_1}{2} + m_2\right) g\, l \sin\theta = \left(\frac{1}{4} m_1 + m_2\right) l^2 \ddot{\theta}$$

$$-\frac{g}{l} \frac{\left(\frac{m_1}{2} + m_2\right)}{\left(\frac{1}{4} m_1 + m_2\right)} \sin\theta = \ddot{\theta}$$

$$-\frac{g}{l} \frac{\left(\frac{m_1}{m_2 2} + 1\right)}{\left(\frac{m_1}{m_2 4} + 1\right)} \sin\theta = \ddot{\theta}$$

for $\frac{m_1}{m_2} \rightarrow 0$ & $\sin\theta \rightarrow \theta$ (small angle)

$$-\frac{g}{l}\theta = \ddot{\theta}$$ usual pendulum result.

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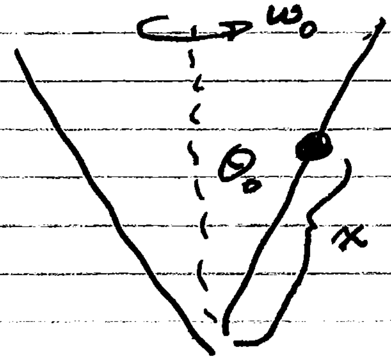

$$T = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \left(\dot{x}^2 + (x \sin\theta_0 \, \omega_0)^2 \right)$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \sin^2\theta_0 \, \omega_0^2 x^2 \right)$$

$$U = mg \, x \cos\theta_0$$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \sin^2\theta_0 \, \omega_0^2 x^2 \right) - mg \cos\theta \, x$$

$$\frac{\partial \mathcal{L}}{\partial x} = m \sin^2\theta_0 \, \omega_0^2 x - mg \cos\theta_0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m \ddot{x}$$

So usin $\quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$

$$m \ddot{x} = m \sin^2\theta_0 \, \omega_0^2 x - mg \cos\theta_0$$

$$\ddot{x} = \sin^2\theta_0 \, \omega_0^2 x - g \cos\theta_0$$

equilibrium pts. $\ddot{x} = 0$

$$\sin^2\theta_0 \omega_0^2 x = g \cos\theta_0$$

$$x = \frac{g\cos\theta_0}{\omega_0^2 \sin^2\theta_0}$$

$$x = \frac{g}{\omega_0^2}\frac{\cos\theta_0}{\sin^2\theta_0} = \frac{g}{\omega_0^2}\frac{\cos\theta_0}{1-\cos^2\theta_0}$$

motion near $x = \frac{g}{\omega_0^2}\frac{\cos\theta_0}{\sin^2\theta_0}$

$$x = \frac{g}{\omega_0^2}\frac{\cos\theta_0}{\sin^2\theta_0} + \varepsilon$$

Sub.

$$\ddot{\varepsilon} = \sin^2\theta_0\,\omega_0^2\left(\frac{g}{\omega_0^2}\frac{\cos\theta_0}{\sin^2\theta_0} + \varepsilon\right) - g\cos\theta_0$$

$$\ddot{\varepsilon} = \underbrace{\sin^2\theta_0\,\omega_0^2}_{\text{pos.}}\,\varepsilon$$

pos. $\Rightarrow$ unstable -

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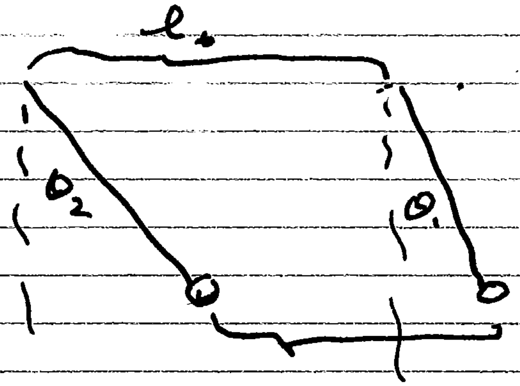

$$l \sin\theta_1 - l \sin\theta_2 + l_0$$

$$T = \tfrac{1}{2} m (l \dot{\theta}_1)^2 + \tfrac{1}{2} m (l \dot{\theta}_2)^2$$

$$= \tfrac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$U = \tfrac{1}{2} k \big(\underbrace{l \sin\theta_1 - l \sin\theta_2 + l_0}_{\text{separation}} - l_0\big)^2 + - mgl\cos\theta_1 - mgl\cos\theta_2$$

$$U = \tfrac{1}{2} k l^2 (\sin\theta_1 - \sin\theta_2)^2 + - mgl(\cos\theta_1 + \cos\theta_2)$$

$$\mathcal{L} = \tfrac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \tfrac{1}{2} k l^2 (\sin\theta_1 - \sin\theta_2)^2 + mgl(\cos\theta_1 + \cos\theta_2)$$

$\underline{\underline{\theta_1}}$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -k\ell^2 (\sin\theta_1 - \sin\theta_2) \cos\theta_1 +$$

$$- mg\ell \sin\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m\ell^2 \dot{\theta}_1 \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = m\ell^2 \ddot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{\partial \mathcal{L}}{\partial \theta_1} \Longrightarrow$$

$$m\ell^2 \ddot{\theta}_1 = -k\ell^2 (\sin\theta_1 - \sin\theta_2) \cos\theta_1 - mg\ell \sin\theta_1$$

$$(1) \quad \boxed{\ddot{\theta}_1 = -\frac{k}{m} (\sin\theta_1 - \sin\theta_2) \cos\theta - \frac{g}{\ell} \sin\theta_1}$$

$\underline{\underline{\theta_2}}$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = -\frac{1}{2} 2k\ell^2 (\sin\theta_1 - \sin\theta_2)(-\cos\theta_2) +$$

$$- mg\ell \sin\theta_2$$

$$= + k\ell^2 (\sin\theta_1 - \sin\theta_2) \cos\theta_2 - mg\ell \sin\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m\ell^2 \dot{\theta}_2 \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m\ell^2 \ddot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2} \Rightarrow$$

$$m l^2 \ddot{\theta}_2 = k l^2 (\sin\theta_1 - \sin\theta_2)\cos\theta_2 - mgl\sin\theta_2$$

(2) $$\boxed{\ddot{\theta}_2 = \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2 - \frac{g}{l}\sin\theta_2}$$

Assume small angles. $\theta_1, \theta_2 \ll 1$

(1) $\rightarrow$

$$\ddot{\theta}_1 = -\frac{k}{m}(\theta_1 - \theta_2) - \frac{g}{l}\theta_1$$

(2) $\rightarrow$

$$\ddot{\theta}_2 = \frac{k}{m}(\theta_1 - \theta_2) - \frac{g}{l}\theta_2$$

add

$$\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{l}(\theta_1 + \theta_2)$$

$$\theta_+ \equiv \theta_1 + \theta_2$$

$$\ddot{\theta}_+ = -\frac{g}{l}\theta_+$$ usual pendulum

$$\omega^2 = \frac{g}{l}$$

Subtract

$$\ddot{\theta}_1 - \ddot{\theta}_2 = -2\frac{k}{m}(\theta_1 - \theta_2) - \frac{g}{l}(\theta_1 - \theta_2)$$

$$\theta_- \equiv \theta_1 - \theta_2$$

$$\ddot{\theta}_- = -\left(\frac{2k}{m} + \frac{g}{l}\right)\theta_-$$

H O with freq. $\omega^2 = 2\frac{k}{m} + \frac{g}{l}$