Assignment: Problem Set 3

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Problem 1

Write both the converse and contrapositive of each of the following statements (No need to argue whether any of them are true or false). In each case, get rid of all occurrences of NOT in the final statement.

The contrapositive of the statement "If A, then B" is "If NOT(B), then NOT(A)."

a) If $a \in \mathbb{Z}$ and $a \geq 2$, then $4a > 7$

We can rewrite this as "For all $a \in \mathbb{Z}$, If $\underbrace{a \geq 2}_{A}$, Then $\underbrace{4a > 7}_{B}$"

Applying the definition of contrapositive, we get:
"For all $a \in \mathbb{Z}$, If NOT$(4a > 7)$, Then NOT$(a \geq 2)$."
Which becomes, "For all $a \in \mathbb{Z}$, If $4a \leq 7$, Then $a < 2$."
This last statement is the contrapositive of the statement given.

b) If $x, y \in \mathbb{R}$ and $x^4 + y^4 = 1$, Then $x^2 + y^2 \leq 2$

We can rewrite this as "For all $x, y \in \mathbb{R}$, If $\underbrace{x^4+y^4=1}_{A}$, Then $\underbrace{x^2+y^2 \leq 2}_{B}$."

Applying the definition of contrapositive, We get:
"For all $x, y \in \mathbb{R}$, If NOT$(x^2+y^2 \leq 2)$, then NOT$(x^4+y^4=1)$."
Which becomes:
"For all $x, y \in \mathbb{R}$, If $x^2+y^2 > 2$, then $x^4+y^4 \neq 1$."
This last statement is the contrapositive of the statement given

Problem 2

Consider the following statement:

If $a \in \mathbb{Z}$ and $3a+5$ is even then a is odd

a) Write down the contrapositive of the given statement

Applying the definition of contrapositive (see Problem 1), we get:
"If $a \in \mathbb{Z}$ and NOT(a is odd), then NOT($3a+5$ is even)"
Which becomes:
"If $a \in \mathbb{Z}$ and a is even, then $3a+5$ is odd." (By Fact 1A.6).
This last statement is the contrapositive of the statement given.

b) Show that the original statement is true by proving that the contrapositive is true.

Proof: We prove the contrapositive. Let $a \in \mathbb{Z}$ be an arbitrary even integer. So we can fix a $m \in \mathbb{Z}$ with $a = 2m$. So we have:

$$\begin{aligned} 3a+5 &= 3(2m)+5 \\ &= 6m+5 \\ &= 6m+4+1 \\ &= 2(3m+2)+1 \end{aligned}$$

Because $m \in \mathbb{Z}$, $3m+2 \in \mathbb{Z}$ so $2\cdot(3m+2)+1$ is odd by definition. Because a was arbitrary, the result follows. Because we have proved the contrapositive, the original statement must be true.

Problem 3.

Let $A = \{e^x : x \in \mathbb{R}\}$

a) Write a description of A by carving it out of a set using a property with a "there exists" quantifier.

Let $A = \{y \in \mathbb{R}, \text{ There exists } x \in \mathbb{R} \text{ with } y = e^x\}$

b) Find another way to describe A by carving it out of a set using a property without any quantifiers. Briefly explain why your set is equal.

$A = \{y \in \mathbb{R} : y > 0\}$

This set is equivalent to the set given because for every $y \in \mathbb{R}$ with $y > 0$, there exists an $x \in \mathbb{R}$ such that $y = e^x$.

Problem 4

Let $A=\{12n-7 : n \in \mathbb{Z}\}$ and Let $B=\{4n+1, n \in \mathbb{Z}\}$

a) Show that $B \not\subseteq A$

By definition, $B \subseteq A$ means that every element of B is also an element of A. So we just need to find One element of B that is not an element of A. We Choose 9 as our example. Notice that $9=4\cdot2+1$, and $2 \in \mathbb{Z}$ so 9 is an element of B. Now Suppose that 9 is an element of A, that is, that there exists a $n \in \mathbb{Z}$ with $9=12n-7$. We manipulate this equation to find n:

$$9+7=12n$$
$$16=12n$$
$$\frac{16}{12}=n=\frac{4}{3}$$

Our assuming that $9 \in A$ led to the contradiction that $n=\frac{4}{3}$, but $n \in \mathbb{Z}$ by definition and $\frac{4}{3} \notin \mathbb{Z}$, so $9 \notin A$. Therefore, it must be the case that $B \not\subseteq A$.

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b) Fill in the blanks below with appropriate Phrases So that the result is a correct proof of the Statement that $A \subseteq B$

Let $a \in A$ be arbitrary. By definition of A, we can fix a $m \in \mathbb{Z}$ with $a=12m-7$, Now notice that

$$a=12m-7$$
$$=12m-8+1$$
$$=4(3m-2)+1$$

Since $3m-2 \in \mathbb{Z}$, we conclude that $a \in B$. Since $a \in A$ was arbitrary, The result follows.

Problem 5

Let $A = \{x^2+5 : x \in \mathbb{R}\}$ and let $B = \{x \in \mathbb{R} : x \geq 5\}$. In this problem, we show that $A = B$ by doing a double containment proof.

a) Prove that $A \subseteq B$.

Let $a \in A$ be arbitrary. By definition of A, we can fix a $y \in \mathbb{R}$ with $a = y^2+5$. By definition, for all $a \in \mathbb{R}$, $a^2 \geq 0$. So $a \geq 5$ for every $y \in \mathbb{R}$. Because $y \in \mathbb{R}$, $a \in \mathbb{R}$, so $a \in B$ by definition. Because a was arbitrary, the result follows.

b) Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement that $B \subseteq A$.

Let $y \in B$ be arbitrary. By definition of B, we know that for all $y \in \mathbb{R}$, $y \geq 5$. Now notice that for every $x \in \mathbb{R}$, $x^2 \geq 0$, so $x^2+5 \in \mathbb{R}$, and that $x^2+5 = y$, so $y \in A$. Since $y \in B$ was arbitrary, the result follows.

c) If $a \in \mathbb{Z}$ and there exists $m \in \mathbb{Z}$ with $a = 10m$, then there exists $m \in \mathbb{Z}$ with $a = 5m$.

We can rewrite this statement as:
"$\underbrace{\text{For all } a \in \mathbb{Z}, \text{ If there exists } m \in \mathbb{Z} \text{ with } a = 10m}_{A}$, $\underbrace{\text{Then there exists } m \in \mathbb{Z} \text{ with } a = 5m}_{B}$."

Applying the definition of contrapositive, we get:
"For all $a \in \mathbb{Z}$, if NOT(There exists $m \in \mathbb{Z}$ with $a = 5m$), Then NOT(There exists $m \in \mathbb{Z}$ with $a = 10m$)."
which becomes:
"For all $a \in \mathbb{Z}$, if for all $m \in \mathbb{Z}$ we have that $a \neq 5m$, Then for all $m \in \mathbb{Z}$ we have that $a \neq 10m$."
This last statement is the contrapositive of the statement given.

In my original scan I realized that I had forgotten to write the converse statements. They are listed here (The converse is simply the negation of the contrapositive.

a) For all $a \in \mathbb{Z}$, if $4a > 7$, then $a \geq 2$

b) For all $x, y \in \mathbb{R}$, if $x^2 + y^2 \leq 2$, then $x^4 + y^4 = 1$

c) For all $a \in \mathbb{Z}$, if there exists $m \in \mathbb{Z}$ with $a = 5m$, Then there exists $m \in \mathbb{Z}$ with $a = 10m$