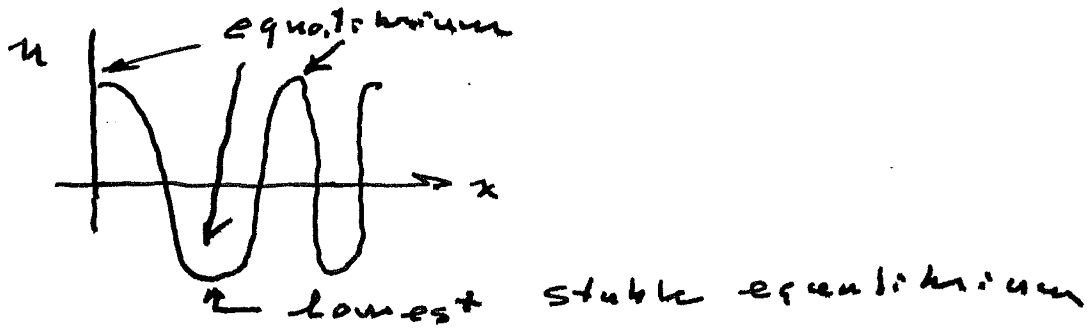
1. Consider a particle of mass m moving along the x axis in a potential given by:

$$U = A\,\mathrm{Cos}\left(\frac{\pi x^2}{a^2}\right)$$

Where A and a are real, positive constants.

a) Find the location of the equilibrium points. Which points are stable?
b) Suppose the particle is placed near the first stable equilibrium point x (lowest positive value of x). Find the frequency for small oscillations.



a) $$F = -\frac{\partial U}{\partial x} = +A\,\mathrm{Sin}\left(\frac{\pi x^2}{a^2}\right)\left(\frac{2\pi x}{a^2}\right)$$

This $=0$ when $\frac{\pi x^2}{a^2} = n\pi \qquad n = 0, 1, 2 \ldots$

$$x = \sqrt{n}\,a$$

b) $$\frac{\pi x^2}{a^2} = \pi \Rightarrow x = +a$$

Expand U about $x = a$

$$U = U(a) + \underbrace{\left.\frac{dU}{dx}\right|_{x=a}}_{=0}(x-a) + \frac{1}{2}\underbrace{\left.\frac{d^2U}{dx^2}\right|_{x=a}}_{\text{This gives k for small osc.}}(x-a)^2$$

$$\frac{d^{*}U}{dx} = -A\,\mathrm{Sin}\left(\frac{\pi x^2}{a^2}\right)\left(\frac{2\pi x}{a^2}\right)$$

$$\frac{d^2u}{dx^2} = -A\left(\frac{2\pi}{a^2}\right)\sin\left(\frac{\pi x^2}{a^2}\right) +$$

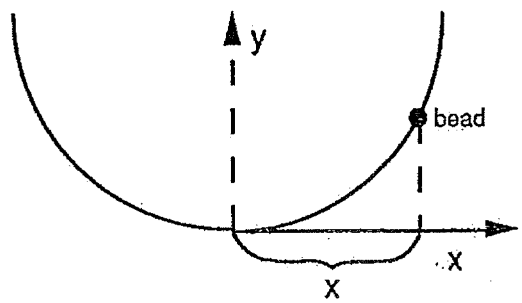
$$- A\cos\left(\frac{\pi x^2}{a^2}\right)\left(\frac{2\pi x}{a^2}\right)^2$$

$$\left.\frac{d^2u}{dx^2}\right|_{x=a} = -A(-1)\left(\frac{2\pi a}{a^2}\right)^2 = A\left(\frac{2\pi}{a}\right)^2$$

$$\omega^2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{A}{m}\left(\frac{2\pi}{a}\right)^2}$$

$$\omega^2 = \left(\frac{2\pi}{a}\right)\sqrt{\frac{A}{m}}$$

2. A fixed wire in the shape of $y=ax^2$. A bead of mass m slides on the wire in a frictionless fashion. The system is shown in the figure below:



Using the variable x to describe the beads position, find the Lagrangian and the differential equation for the motion of the bead. You need not solve the equation.

$$L = \frac{1}{2} m v^2 - U$$

$$v = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2ax\frac{dx}{dt}$$

$$U = mgy = mgax^2$$

So

$$L = \frac{1}{2}m\left(1 + (2ax)^2\right)\dot{x}^2 - mgax^2$$

$$\frac{\partial L}{\partial x} = -2mgax + \frac{1}{2}m\dot{x}^2(4a^2x)$$

$$= m\left[-2ga + 2\dot{x}^2a^2\right]x = 2m(\dot{x}^2a^2 - ga)x$$

$$\frac{\partial L}{\partial \dot{x}} = m(1 + 4a^2x^2)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m(1+4a^2x^2)\ddot{x} + m(+ 8a^2 x\dot{x})\dot{x}$$

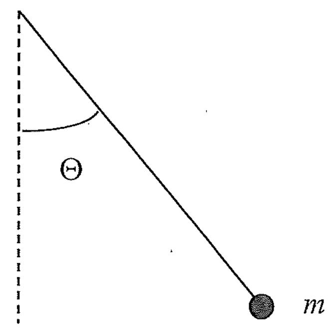
so eq. of motion is

$$2m(\dot{x}^2a^2 - ag)x = m(1+4a^2x^2)\ddot{x} + m(+ 8a^2x\dot{x})\dot{x}$$

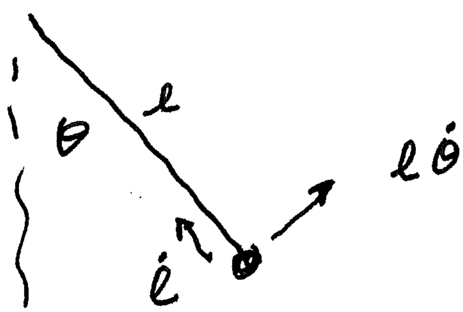
$$2a^2\dot{x}^2x - 2agx = (1+4a^2x^2)\ddot{x} + 8a^2x\dot{x}^2$$

$$\boxed{-6a^2x\dot{x}^2 - 2agx = (1+4a^2x^2)\ddot{x}}$$

3. The figure below shows a pendulum consisting of a ball of mass m attached to a mass-less string.



The string however is quite magical in that its length varies with time according to $l = l_0 + b\sin(\omega t)$ where l_0, b and ω are real, positive constants. Obtain a differential equation for Θ. You need not solve the differential equation.



$\dot{l} = b\omega\cos(\omega t)$

$T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \underbrace{b^2\omega^2\cos^2(\omega t)}_{\dot{l}^2}\right)$

$U = -mg\left(\underbrace{l_0 + b\sin(\omega t)}_{l}\right)\cos\theta$

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{l}^2\right) + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) = 2ml\dot{l}\dot{\theta} + ml^2\ddot{\theta}$$

So

$$-mgl\sin\theta = 2ml\dot{l}\dot{\theta} + ml^2\ddot{\theta}$$

$$-g\sin\theta = 2\dot{l}\dot{\theta} + l\ddot{\theta}$$

$$l = l_0 + b\sin(\omega t)$$

$$\dot{l} = b\omega\cos(\omega t)$$

$$\rightarrow -g\sin\theta = 2b\omega\cos(\omega t)\dot{\theta} + (l_0 + b\sin(\omega t))\ddot{\theta}$$