Solutions to Written Assignment 3

Problem 1: Yes, T is injective. To prove this, we need to show that the statement

$$\text{For all } \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2, \text{ if } T(\vec{v}_1) = T(\vec{v}_2), \text{ then } \vec{v}_1 = \vec{v}_2$$

is true. Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ be arbitrary with $T(\vec{v}_1) = T(\vec{v}_2)$. Fix $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with

$$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Since $T(\vec{v}_1) = T(\vec{v}_2)$, we know that

$$\begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \end{pmatrix} = \begin{pmatrix} x_2 - y_2 \\ x_2 + y_2 \end{pmatrix},$$

from which we can conclude that $x_1 - y_1 = x_2 - y_2$ and that $x_1 + y_1 = x_2 + y_2$. Adding these two equalities together, we conclude that $2x_1 = 2x_2$. Dividing both sides of this by 2, it follows that $x_1 = x_2$. Plugging this into the first equation gives $x_1 + y_1 = x_1 + y_2$, and by subtracting x_1 from both sides we can conclude that $y_1 = y_2$. We have shown that both $x_1 = x_2$ and $y_1 = y_2$, so we conclude that

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

and hence $\vec{v}_1 = \vec{v}_2$. Since $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ were arbitrary with $T(\vec{v}_1) = T(\vec{v}_2)$, it follows that T is injective.

Problem 2a: Let $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$ be arbitrary. By definition of $\text{range}(T)$, we can fix $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$. We then have

$$\begin{aligned} \vec{w}_1 + \vec{w}_2 &= T(\vec{v}_1) + T(\vec{v}_2) \\ &= T(\vec{v}_1 + \vec{v}_2) && \text{(since } T \text{ is a linear transformation).} \end{aligned}$$

Since $\vec{v}_1 + \vec{v}_2 \in \mathbb{R}^2$, it follows that $\vec{w}_1 + \vec{w}_2 \in \text{range}(T)$. Since $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$ were arbitrary, the result follows.

Problem 2b: Let $\vec{w} \in \text{range}(T)$ and $c \in \mathbb{R}$ be arbitrary. By definition of $\text{range}(T)$, we can fix $\vec{v} \in \mathbb{R}^2$ with $\vec{w} = T(\vec{v})$. We then have

$$\begin{aligned} c\vec{w} &= c \cdot T(\vec{v}) \\ &= T(c\vec{v}) && \text{(since } T \text{ is a linear transformation).} \end{aligned}$$

Since $c\vec{v} \in \mathbb{R}^2$, it follows that $c\vec{w} \in \text{range}(T)$. Since $\vec{w} \in \text{range}(T)$ and $c \in \mathbb{R}$ were arbitrary, the result follows.

Problem 3a: Given arbitrary $x, y \in \mathbb{R}$, we have

$$\begin{aligned} g_r(x + y) &= r(x + y) \\ &= rx + ry \\ &= g_r(x) + g_r(y), \end{aligned}$$

so g_r satisfies the first condition. Also given arbitrary $x, c \in \mathbb{R}$, we have

$$\begin{aligned} g_r(cx) &= r \cdot cx \\ &= c \cdot rx \\ &= c \cdot g_r(x), \end{aligned}$$

so g_r satisfies the second condition. Combining both of these, it follows that g_r is a linear transformation.

Problem 3b: Let $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ be arbitrary linear transformations with $f(1) = g(1)$. To show that $f = g$, we need to show that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ is arbitrary. We have

$$\begin{aligned} f(x) &= f(x \cdot 1) \\ &= x \cdot f(1) && \text{(since } f \text{ is a linear transformation)} \\ &= x \cdot g(1) && \text{(since } f(1) = g(1)\text{)} \\ &= g(x \cdot 1) && \text{(since } g \text{ is a linear transformation)} \\ &= g(x). \end{aligned}$$

Since $x \in \mathbb{R}$ was arbitrary, it follows that $f = g$.