Solutions to Problem Set 12

Problem 1: Let
$$A = \begin{pmatrix} 5 & -1 \\ -7 & 3 \end{pmatrix}.$$
For any $\lambda \in \mathbb{R}$, we have
$$A - \lambda I = \begin{pmatrix} 5-\lambda & -1 \\ -7 & 3-\lambda \end{pmatrix},$$
so the characteristic polynomial of A is
$$\begin{aligned}(5-\lambda)(3-\lambda) - (-1)(-7) &= 15 - 8\lambda + \lambda^2 - 7 \\ &= \lambda^2 - 8\lambda + 8.\end{aligned}$$
Using the quadratic formula, we determine that the roots of this polynomial are
$$\frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 8}}{2} = \frac{8 \pm \sqrt{32}}{2}.$$
Since $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$, the roots of the characteristic polynomial are $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$. Hence, the eigenvalues of A are $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$.

Problem 2: Let
$$A = \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}.$$
For any $\lambda \in \mathbb{R}$, we have
$$A - \lambda I = \begin{pmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{pmatrix},$$
so the characteristic polynomial of A is
$$\begin{aligned}(1-\lambda)(1-\lambda) - 8 \cdot 2 &= 1 - 2\lambda + \lambda^2 - 16 \\ &= \lambda^2 - 2\lambda - 15 \\ &= (\lambda - 5)(\lambda + 3).\end{aligned}$$
Thus, the eigenvalues of A are 5 and -3. To find an eigenvector of A corresponding to 5, we want to find an element of $\text{Null}(A - 5I)$. We have
$$A - 5I = \begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix}.$$
Therefore, an eigenvector corresponding to 5 is
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
To find an eigenvector of A corresponding to -3, we want to find an element of $\text{Null}(A + 3I)$. We have
$$A + 3I = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}.$$

Therefore, an eigenvector corresponding to -3 is

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Problem 3: Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}.$$

For any $\lambda \in \mathbb{R}$, we have

$$A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{pmatrix},$$

so the characteristic polynomial of A is

$$\begin{aligned} (2-\lambda)(4-\lambda) - (-1)\cdot 1 &= 8 - 6\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 6\lambda + 9 \\ &= (\lambda-3)^2. \end{aligned}$$

Therefore, 3 is the only eigenvalue of A. To find an eigenvector of A corresponding to 3, we want to find an element of $\text{Null}(A - 3I)$. We have

$$A - 3I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, an eigenvector corresponding to 3 is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

In fact, by following the proof of Theorem 3.3.3, it is straightforward to show that the null space of $A - 3I$ is

$$\text{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right),$$

so every eigenvector of A corresponding to 3 is a multiple of this one.

Problem 4: Let $c, d \in \mathbb{R}$, and let

$$A = \begin{pmatrix} 3 & 1 \\ c & d \end{pmatrix}.$$

We then have

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ c & d-\lambda \end{pmatrix},$$

hence the characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(d-\lambda) - 1\cdot c \\ &= 3d - 3\lambda - d\lambda + \lambda^2 - c \\ &= \lambda^2 - (d+3)\lambda + (3d - c). \end{aligned}$$

Now A has both 4 and 7 as eigenvalues if and only if both 4 and 7 are roots of this polynomial, which is if and only if

$$\lambda^2 - (d+3)\lambda + (3d-c) = (\lambda-4)(\lambda-7).$$

Multiplying out the the polynomial on the right gives

$$\lambda^2 - (d+3)\lambda + (3d - c) = \lambda^2 - 11\lambda + 28.$$

Two polynomials are equal exactly when the corresponding coefficients are equal, so this is true exactly when both

$$d + 3 = 11 \qquad \text{and} \qquad 3d - c = 28.$$

To determine which c and d work we just need to solve this system. The first equation gives $d = 8$. Plugging this into the second gives $24 - c = 28$, so $c = -4$. Furthermore, we can plug these in to check that this is indeed a solution. It follows that

$$A = \begin{pmatrix} 3 & 1 \\ -4 & 8 \end{pmatrix}$$

is the unique such matrix having both 4 and 7 as eigenvalues.

Problem 5: Let

$$A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}.$$

We then have

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{pmatrix},$$

so the characteristic polynomial of A is

$$\begin{aligned} (1-\lambda)(-1-\lambda) - 0 \cdot 6 &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1). \end{aligned}$$

Therefore, the eigenvalues of A are 1 and -1. Since T has two distinct eigenvalues, we know that T is diagonalizable by Corollary 3.5.20.

We now determine specific eigenvectors. We first examine the case when $\lambda = 1$. We have

$$A - 1I = \begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}.$$

To find an eigenvector corresponding to 1, we just need to find a nonzero element of $\text{Null}(A - 1I)$. One such example is

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

We next examine the case when $\lambda = -1$. We have

$$A - (-1)I = \begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}.$$

To find an eigenvector corresponding to -1, we just need to find a nonzero element of $\text{Null}(A - (-1)I)$. One such example is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, let $\alpha = (\vec{u}_1, \vec{u}_2)$ where

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notice that α is a basis of $\mathbb{R}^2$ because $1 \cdot 1 - 3 \cdot 0 = 1$ is nonzero (or by using Proposition 3.5.19). Finally, by Proposition 3.5.13, we have that

$$[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 6: Let

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}.$$

We then have

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{pmatrix},$$

so the characteristic polynomial of A is

$$\begin{aligned} (3-\lambda)(5-\lambda) - (-1) \cdot 1 &= 15 - 8\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 8\lambda + 16 \\ &= (\lambda - 4)^2. \end{aligned}$$

Therefore, the only eigenvalues of A is 4. We have

$$A - 4I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

To find the eigenvectors of A corresponding to 4, we want to find the nonzero elements of $\text{Null}(A - 4I)$. We claim that

$$\text{Null}(A - 4I) = \text{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right).$$

One way to prove this is by a double containment argument, but we also get it immediately from the proof of Theorem 3.3.3. Therefore, any two eigenvectors are nonzero multiples of

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and so are nonzero multiples of each other. Hence, for any eigenvectors $\vec{u}_1$ and $\vec{u}_2$ of T, we have that $\text{Span}(\vec{u}_1, \vec{u}_2) \neq \mathbb{R}^2$, so $(\vec{u}_1, \vec{u}_2)$ is not a basis of $\mathbb{R}^2$. Using Corollary 3.5.14, it follows that T is not diagonalizable.