Assignment: Problem Set 22

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}.$$

We know from Proposition 3.3.16 that A is invertible, and we also know a formula for the inverse. Now compute A^{-1} using our new method by applying elementary row operations to the matrix

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}.$$

Solution: We use the algorithm given at the end of section 5.2. Notice that we already have appended $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to A, so we proceed to perform elementary row operations to until the 2 $\times$ 2 matrix on the left is in echelon form:

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} -\tfrac{5}{3}R_1 + R_2$$

Notice that the 2 $\times$ 2 matrix on has a leading entry in every row, so we continue to step

$$\to \begin{pmatrix} 3 & 0 & -9 & 6 \\ 0 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix} 6R_2 + R_1$$

$$\to \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{pmatrix} \begin{matrix} \frac{1}{3}R_1 \\ -3R_2 \end{matrix}$$

We have the 2 $\times$ 2 identity matrix on the left, and so by step 5 the right 2 columns contains A^{-1}. So $A^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$. Checking our calculation for A^{-1} using our formula from Chapter 3, we get $A^{-1} = \frac{1}{3\cdot3-5\cdot2}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \frac{1}{9-10}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = -1\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$. So A^{-1} does indeed equal $\begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$.

Problem 2: Let V be the vector space of all 2×2 matrices. Explain why there is no injective linear transformation $T : \mathcal{P}_4 \to V$.

Solution: Let $T : \mathcal{P}_4 \to V$ be an arbitrary linear transformation. We know that $(1, x, x^2, x^3, x^4)$ is a basis for $\mathcal{P}_4$ and has 5 elements, so by definition of dimension we have that $\dim(\mathcal{P}_4) = 5$. We know that $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is a basis for V and has 4 elements, so by definition of dimension we have that $\dim(V) = 4$. $5 > 4$, so by Corollary 5.2.15 we have that T is not injective. Because $T : \mathcal{P}_4 \to V$ was arbitrary, the result follows.

Problem 3: Determine whether each of the following matrices is invertible, and if so, find the inverse.

a. $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$. Notice that A is a 3×3 matrix, so we can use the algorithm given at the end of section 5.2. We first form the 3×6 matrix obtained by augmenting A with the 3×3 identity matrix:

$$\begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We then perform elementary row operations until the 3×3 matrix on the left is in echelon form:

$$\begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ R_1 + R_3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{matrix} \\ \\ -R_2 + R_3 \end{matrix}$$

Notice that there is a leading entry in every row and column of the 3×3 matrix on the left, so applying Proposition 4.2.14 and Proposition 4.3.3 we have that the columns of this matrix span $\mathbb{R}^3$ and that the sequence of the columns of this matrix are linearly independent, so by definition the columns of this matrix form a basis of $\mathbb{R}^3$. Treating A as the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, by Proposition 5.2.19 we have that T is bijective, so by Proposition 3.3.8 T has an inverse, and it follows that A has an inverse, so by definition A is invertible. We continue performing elementary row operations to eliminate nonzero entries above the diagonal and to make the diagonal entries equal to 1:

$$\to \begin{pmatrix} 1 & 1 & 0 & 4 & -3 & 3 \\ 0 & 2 & 0 & 4 & -3 & 4 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{matrix} 3R_3 + R_1 \\ 4R_3 + R_2 \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 2 & -\frac{3}{2} & 1 \\ 0 & 1 & 0 & 2 & -\frac{3}{2} & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} \begin{matrix} -\frac{1}{2}R_2 + R_1 \\ \frac{1}{2}R_2 \\ -R_3 \end{matrix}$$

We have the 3×3 identity matrix on the left, and the rightmost 3 columns contain the matrix A^{-1}. So $A^{-1} = \begin{pmatrix} 2 & -\frac{3}{2} & 1 \\ 2 & -\frac{3}{2} & 2 \\ -1 & 1 & -1 \end{pmatrix}$.

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Problem 4: Either prove or find a counterexample: If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible.

Solution: Consider the two 2×2 matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that $1 \cdot 1 - 0 \cdot 0 = 1 - 0 = 1 \neq 0$, so by Proposition 3.3.16 we have that A is invertible. Notice also that $0 \cdot 0 - 1 \cdot 1 = 0 - 1 = -1 \neq 0$, so by Proposition 3.3.16 we have that B is invertible. Now consider the 2×2 matrix $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Notice that $1 \cdot 1 - 1 \cdot 1 = 1 - 1 = 0$, so by Proposition 3.3.16 we have that $A + B$ is not invertible. Therefore, if A and B are invertible $n \times n$ matrices, it need not be the case that $A + B$ is invertible.

Problem 5: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

a. Explain why A has no left inverse.

Solution: Notice that A is a 2×3 matrix. Notice that $3 > 2$, so by Corollary 5.2.18 A has no left inverse.

b. Show that A has infinitely many right inverses.

Solution: Let $a, b \in \mathbb{R}$ are arbitrary. Let B be the 3×2 matrix defined by letting $B = \begin{pmatrix} a & b \\ 0 & 1 \\ 1-a & -b \end{pmatrix}$. Notice that

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \\ 1-a & -b \end{pmatrix} \\ &= \begin{pmatrix} 1(a) + 0(0) + 1(1-a) & 1(b) + 0(1) + 1(-b) \\ 0(a) + 1(0) + 0(1-a) & 0(b) + 1(1) + 0(-b) \end{pmatrix} \\ &= \begin{pmatrix} a + 1 - a & b - b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \end{aligned}$$

So $AB = I_2$. We then have that B is a right inverse of A by Definition 5.1.16. Because $a, b \in \mathbb{R}$ were arbitrary, and there are infinitely many elements in $\mathbb{R}$, it follows that there are infinitely many B with $AB = I_2$. Therefore, there are infinitely many right inverses of A.

b. $\begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 3 & -4 & 2 \end{pmatrix}$

Solution: Let $B = \begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 3 & -4 & 2 \end{pmatrix}$. We then perform elementary row operations until the 3×3 matrix on the left is in echelon form:

$$\begin{aligned} \begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 3 & -4 & 2 \end{pmatrix} &\to \begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{matrix} \\ \\ -3R_2 + R_3 \end{matrix} \\ &\to \begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ -\frac{1}{2}R_2 + R_3 \end{matrix} \\ &\to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_2 \leftrightarrow R_1 \\ R_1 \leftrightarrow R_2 \\ \\ \end{matrix} \end{aligned}$$

Notice that there is not a leafing entry in every row of the 3×3 matrix on the left, so by Proposition 4.2.14 the columns of B do not span $\mathbb{R}^3$, so the columns of B cannot be a basis for $\mathbb{R}^3$ by definition. It follows from Proposition 5.2.19 that the linear transformation with B as its augmented matrix is not bijective, so by Proposition 3.3.8 the linear transformation does not have an inverse, and it follows that B does not have an inverse, so by definition B is not invertible.

c. $\begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$

Solution: Let $C = \begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$. Notice that C is a 3×3 matrix, so we can use the algorithm given at the end of section 5.2. We first form the 3×6 matrix obtained by augmenting A with the 3×3 identity matrix:

$$\begin{pmatrix} 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix}$$

We then perform elementary row operations until the 3×3 matrix on the left is in echelon form:

$$\begin{pmatrix} 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 7 & 1 & \frac{1}{2} & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \frac{1}{2}R_2 + R_1 \\ \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 7 & 1 & \frac{1}{2} & 0 \end{pmatrix} \begin{matrix} R_3 \leftrightarrow R_1 \\ \\ R_1 \leftrightarrow R_3 \end{matrix}$$

Notice that there is a leading entry in every row and column of the 3×3 matrix on the left, so applying Proposition 4.2.14 and Proposition 4.3.3 we have that the columns of this matrix span $\mathbb{R}^3$ and that the sequence of the columns of this matrix are linearly independent, so by definition the columns of this matrix form a basis of $\mathbb{R}^3$. Treating A as the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, by Proposition 5.2.19 we have that T is bijective, so by Proposition 3.3.8 T has an inverse, and it follows that C has an inverse, so by definition C is invertible. We continue performing elementary row operations to eliminate nonzero entries above the diagonal and to make the diagonal entries equal to 1:

$$\to \begin{pmatrix} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & 0 \end{pmatrix} \begin{matrix} \\ -\frac{1}{2}R_2 \\ \frac{1}{7}R_3 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 3 & 0 & \frac{2}{7} & \frac{1}{7} & 1 \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{5}{14} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & 0 \end{pmatrix} \begin{matrix} 2R_3 + R_1 \\ 2R_3 + R_2 \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 2 & 0 & 0 & -\frac{4}{7} & \frac{17}{14} & 1 \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{5}{14} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & 0 \end{pmatrix} \begin{matrix} -3R_2 + R_1 \\ \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{7} & \frac{17}{28} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{5}{14} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{14} & 0 \end{pmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \\ \\ \end{matrix}$$

We have the 3×3 identity matrix on the left, and the rightmost 3 columns contain the matrix C^{-1}. So $C^{-1} = \begin{pmatrix} -\frac{2}{7} & \frac{17}{28} & \frac{1}{2} \\ \frac{2}{7} & -\frac{5}{14} & 0 \\ \frac{1}{7} & \frac{1}{14} & 0 \end{pmatrix}$.

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