Solutions to Problem Set 16

Problem 1: The augmented matrix of our linear system is:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 3 & 1 & 0 & 1 \\ -1 & 1 & 1 & 4 \end{pmatrix}.$$

Applying elementary row operations, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & -1 & 0 \\ 3 & 1 & 0 & 1 \\ -1 & 1 & 1 & 4 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 4 \end{pmatrix} && \begin{matrix} (-3R_1 + R_2) \\ (R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & 3 \end{pmatrix} && (-R_2 + R_3) \\
&\to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} && (-\tfrac{1}{3} \cdot R_3).
\end{aligned}$$

Therefore, our original system has the same solution set as the following system:

$$\begin{array}{ccccccc} x & & & - & z & = & 0 \\ & & y & + & 3z & = & 1 \\ & & & & z & = & -1. \end{array}$$

To satisfy the last equation, we must have $z = -1$. Plugging this into the second equation gives $y - 3 = 1$, so $y = 4$. Plugging these into the first equation yields $x + 1 = 0$, so $x = -1$. Thus, the unique solution is $(-1, 4, 1)$, and the solution set is $\{(-1, 4, -1)\}$.

Problem 2: We want to find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ pass through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

- We want $f(1) = 2$, so we need the coefficients to satisfy $a + b + c = 2$.
- We want $f(-1) = 6$, so we need the coefficients to satisfy $a - b + c = 6$.
- We want $f(2) = 3$, so we need the coefficients to satisfy $4a + 2b + c = 3$.

Thus, we want to solve the following system of equations.

$$\begin{array}{ccccccc} a & + & b & + & c & = & 2 \\ a & - & b & + & c & = & 6 \\ 4a & + & 2b & + & c & = & 3. \end{array}$$

The augmented matrix of our linear system is:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{pmatrix}.$$

Applying elementary row operations, we obtain

$$
\begin{aligned}
\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -3 & -5 \end{pmatrix} && \begin{matrix} \\ (-R_1 + R_2) \\ (-4R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{pmatrix} && \begin{matrix} \\ \\ (-R_2 + R_3). \end{matrix}
\end{aligned}
$$

Therefore, our original system has the same solution set as the following system:

$$
\begin{array}{ccccccc}
a & + & b & + & c & = & 2 \\
 & & -2b & & & = & 4 \\
 & & & & -3c & = & -9.
\end{array}
$$

The second and third equations tell us that $b = -2$ and that $c = 3$. Back-substituting these into the first equation gives $a - 2 + 3 = 2$, so $a = 1$. Therefore, the unique coefficients that work are $a = 1$, $b = -2$, and $c = 3$, and hence that unique such function is $f(x) = x^2 - 2x + 3$.

Problem 3: We want to know if there exists $c_1, c_2, c_3 \in \mathbb{R}$ with

$$
c_1 \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 5 \\ 10 \end{pmatrix}.
$$

Thus, we want to know if there exists $c_1, c_2, c_3 \in \mathbb{R}$ with

$$
\begin{pmatrix} 4c_2 + c_3 \\ 2c_1 - 2c_2 + c_3 \\ c_1 + c_3 \\ c_1 + c_2 - c_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 5 \\ 10 \end{pmatrix}.
$$

Thus, we want to know if the linear system

$$
\begin{array}{ccccccc}
 & & 4c_2 & + & c_3 & = & 20 \\
2c_1 & - & 2c_2 & + & c_3 & = & 0 \\
c_1 & & & + & c_3 & = & 5 \\
c_1 & + & c_2 & - & c_3 & = & 10
\end{array}
$$

has a solution. The augmented matrix of our linear system is:

$$
\begin{pmatrix} 0 & 4 & 1 & 20 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & -1 & 10 \end{pmatrix}.
$$

Applying elementary row operations, we obtain

$$
\begin{aligned}
\begin{pmatrix} 0 & 4 & 1 & 20 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & -1 & 10 \end{pmatrix}
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 2 & -2 & 1 & 0 \\ 0 & 4 & 1 & 20 \\ 1 & 1 & -1 & 10 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_3) \\ \\ (R_1 \leftrightarrow R_3) \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & -2 & -1 & -10 \\ 0 & 4 & 1 & 20 \\ 0 & 1 & -2 & 5 \end{pmatrix} && \begin{matrix} (-2R_1 + R_2) \\ \\ (-R_1 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 5 \\ 0 & 4 & 1 & 20 \\ 0 & -2 & -1 & -10 \end{pmatrix} && \begin{matrix} (R_2 \leftrightarrow R_4) \\ \\ (R_2 \leftrightarrow R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & -5 & 0 \end{pmatrix} && \begin{matrix} (-4R_2 + R_3) \\ (2R_2 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{pmatrix} && (\tfrac{1}{9} \cdot R_3) \\
&\to \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} && (5R_3 + R_4).
\end{aligned}
$$

Therefore, our original system has the same solution set as the following system:

$$
\begin{alignedat}{4}
c_1 & & &+{} & c_3 &= 5 \\
& & c_2 &-{} & 2c_3 &= 5 \\
& & & & c_3 &= 0.
\end{alignedat}
$$

Solving this system, we see that we must have $c_3 = 0$, hence $c_2 = 5$ and $c_2 = 5$. It follows that our system has a unique solution $(5, 5, 0)$, so the solution set is $\{(5, 5, 0)\}$. Therefore, this is indeed a solution, and hence

$$
\begin{pmatrix} 20 \\ 0 \\ 5 \\ 10 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right).
$$

Problem 4: Applying elementary row operations to the augmented matrix gives

$$
\begin{aligned}
\begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 1 & 0 & 1 & 4 \\ 1 & -1 & 1 & 1 & 1 \end{pmatrix}
&\to \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & 1 & -2 \\ 0 & -3 & 2 & 1 & -2 \end{pmatrix} && \begin{matrix} (-2R_1 + R_2) \\ (-R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -3 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} && (-R_2 + R_3).
\end{aligned}
$$

Notice that the third and fourth columns do not have leading entries. We therefore introduce parameters for the variables z and w, say $z = s$ and $w = t$. We can now solve the second equation $-3y + 2z + w = -2$ for y in terms of s and t to get $-3y = -2 - 2z - w = -2 - 2s - t$ and hence

$$y = \frac{2}{3} + \frac{2}{3} \cdot s + \frac{1}{3} \cdot t.$$

We next solve the first equation $x + 2y - z = 3$ for x in terms of s and t to get

$$\begin{aligned} x &= 3 - 2y + z \\ &= 3 - 2 \cdot \left(\frac{2}{3} + \frac{2}{3} \cdot s + \frac{1}{3} \cdot t\right) + s \\ &= \frac{5}{3} - \frac{1}{3} \cdot s - \frac{2}{3} \cdot t. \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{pmatrix} 5/3 - (1/3)s - (2/3)t \\ 2/3 + (2/3)s + (1/3)t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\},$$

which we can also write as follows:

$$\left\{ \begin{pmatrix} 5/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Problem 5: For any $h, k \in \mathbb{R}$, the augmented matrix of our linear system is

$$\begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix}.$$

Applying one elementary row operation, we obtain

$$\begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix} \to \begin{pmatrix} 1 & h & 2 \\ 0 & 8 - 4h & k - 8 \end{pmatrix} \qquad (-4R_1 + R_2).$$

The interesting cases to consider now are when $8 - 4h = 0$, i.e. when $h = 2$, and when $k - 8 = 0$, i.e. when $k = 8$. We handle the possibilities now.

- Suppose that $h \neq 2$, and $k \in \mathbb{R}$ is arbitrary. Since $h \neq 2$, we have that $8 - 4h \neq 0$. Therefore, the second row has a leading entry in the second column, and hence the last column does not have a leading entry. Using Proposition 4.2.12, we conclude that there is a unique solution to the system in this case.
- Suppose that $h = 2$ and $k \neq 8$. We then have that $8 - 4h = 0$ and also that $k - 8 \neq 0$. In this case, the leading entry in the second row lies in the third column, so the third column does have a leading entry. Using Proposition 4.2.12, we conclude that there are no solutions in this case.
- Suppose finally that $h = 2$ and $k = 8$. In this case, the second row is all zeros, and hence has no leading entry. Thus, neither the second nor third columns have a leading entry. Using Proposition 4.2.12, we conclude that there are infinitely many solutions in this case.

Problem 6: Let $a, b, c, d \in \mathbb{R}$ be arbitrary. The augmented matrix of our linear system is

$$\begin{pmatrix} 1 & -3 & a \\ 3 & 1 & b \\ 1 & 7 & c \\ 2 & 4 & d \end{pmatrix}.$$

Applying elementary row operations, we obtain

$$\begin{aligned} \begin{pmatrix} 1 & -3 & a \\ 3 & 1 & b \\ 1 & 7 & c \\ 2 & 4 & d \end{pmatrix} &\to \begin{pmatrix} 1 & -3 & a \\ 0 & 10 & -3a+b \\ 0 & 10 & -a+c \\ 0 & 10 & -2a+d \end{pmatrix} && \begin{matrix} \\ (-3R_1 + R_2) \\ (-R_1 + R_3) \\ (-2R_1 + R_4) \end{matrix} \\ &\to \begin{pmatrix} 1 & -3 & a \\ 0 & 10 & -3a+b \\ 0 & 0 & 2a-b+c \\ 0 & 0 & a-b+d \end{pmatrix} && \begin{matrix} \\ \\ (-R_2 + R_3) \\ (-R_2 + R_4). \end{matrix} \end{aligned}$$

Now if either $2a - b + c \neq 0$ or $a - b + d \neq 0$ (or both), then the last column has a leading entry, so the system has no solution by Proposition 4.2.12. Conversely, if both $2a - b + c = 0$ and $a - b + d = 0$, then the last column has no leading entry, so the system has a solution by Proposition 4.2.12. Therefore, the system has a solution if and only if both $2a - b + c = 0$ and $a - b + d = 0$.

Although not part of the problem, we can explicitly solve the system in this case. We can then solve the second equation in uniquely for y to obtain

$$y = \frac{-3a + b}{10}.$$

With can plug this value into the first equation to solve uniquely for x. Concretely, this gives

$$x - \frac{-9a + 3b}{10} = a,$$

so

$$x = a + \frac{-9a + 3b}{10} = \frac{a + 3b}{10}.$$

Hence, if both $2a - b + c \neq 0$ and $a - b + d \neq 0$, then

$$\left(\frac{a + 3b}{10}, \frac{-3a + b}{10} \right)$$

is the unique solution to the original system.