Worksheet 12 - Changing Coordinates

1 The Basic Idea

The reason we want to be able to communicate between coordinate systems:

> The standard basis we have chosen is completely arbitrary! There is no reason to expect that it sets up a good model for any real world problem.

Example 1. Suppose a shower has two knobs:

- Hot water, $\vec{w}_1$
- Cold water, $\vec{w}_2$

Here, we can describe any setting as $c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2$. We could even use the standard basis define

$$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So that the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is the setting with the how water set to 2 and the cold water set to 3.

This system allows us to describe how to get to every setting, but it is not always how we think about making adjustments in this scenario. When we are adjusting shower settings, under these conditions, we might want to think in terms of the following:

- I want to change the pressure, but maintain the current temperature.
- I want to change the temperature, but maintain the current pressure.

As such, we might consider a different basis, $\alpha = (\vec{p}, \vec{w})$, with:

$$\vec{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\vec{t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We can still describe every possible setting using $d_1 \cdot \vec{p} + d_2 \cdot \vec{t}$, but the interpretation of the coordinates are different. In this basis, the setting $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is the one with a water pressure of 2 and a temperature of 3. Whenever you change the temperature or pressure using knobs that describe hot and cold, you are performing a change of coordinates in your head.

You can imagine all sorts of situations where you can make changes to several settings (inputs) that have more than one effect on the outcome (output). The ability to change coordinates means we can frame the description of changes in our settings (change our coordinates) so that they more readily correspond to the way we want to think about the problem.

2 Practice and Applications

Exercise 1. In Example 1, we can describe settings in either of these ways:

$$\begin{aligned} \text{Hot/Cold: } & c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2 \\ \text{Pressure/Temperature: } & d_1 \cdot \vec{p} + d_2 \cdot \vec{t} \end{aligned}$$

(a) If the hot/cold setting is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, what is corresponding the pressure/temperature setting?

(b) If the pressure/temperature is $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$, what is the corresponding hot/cold setting?

Exercise 2. In the future, you simply enter your preferred shower setting on a keypad, get in, and the water starts in that setting. However, different companies use different bases in their programming.

- Why is it important that you know what basis the company used before entering your settings and getting in?

- Suppose the hot/cold basis is the standard basis. If you know the basis for your shower at home, ψ, and you find yourself about to use a shower using a different basis, ϕ, use the terminology of change of coordinates to describe how you will decide setting to enter in this new shower.

Exercise 3. A novice shower programmer make defines the settings for their model in terms of

$$v_1 = 2 \cdot \vec{w}_1 + 4 \cdot \vec{w}_2$$
$$v_2 = -1 \cdot \vec{w}_1 - 2 \cdot \vec{w}_2$$

The programmer finds a focus group, gives them the basis, and ask them to try out the prototype shower and give feedback. What problem is likely to show up when the members of the focus group try to get the settings they want? Give a specific example to illustrate the problem. How can the programmer be sure to address the problem?

Exercise 4. Is it possible that anyone in the focus group from Exercise 3 doesn't have any problems with the prototype shower? Explain.

For Next Time

- Finish this worksheet
- Read Section 3.5, through Example 3.5.10

More Practice

Exercise 5. Answer the following:

(a) Is there a 2×2 matrix A with

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot A = I? \tag{1}$$

(b) Is there a 2×2 matrix B with

$$B \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = I? \tag{2}$$

What can you say about A, B, or $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What results in the book explain your findings?

2.1 A Special Function

Let A be the set of all 2×2 matrices, and suppose $f : A \to \mathbb{R}$ is a function with the following properties:

1. $f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$

2. $f\left(\begin{pmatrix} a & a \\ b & b \end{pmatrix}\right) = 0$ for all $a, b \in \mathbb{R}$

3. $f\left(\begin{pmatrix} a & xb \\ c & xd \end{pmatrix}\right) = x \cdot f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left(\begin{pmatrix} xa & b \\ xc & d \end{pmatrix}\right)$ for all $a, b, c, d, x \in \mathbb{R}$

4. $f\left(\begin{pmatrix} a+w & b \\ c+v & d \end{pmatrix}\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + f\left(\begin{pmatrix} w & b \\ v & d \end{pmatrix}\right)$ and
   $f\left(\begin{pmatrix} a & b+w \\ c & d+v \end{pmatrix}\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + f\left(\begin{pmatrix} a & w \\ c & v \end{pmatrix}\right)$ for all $a, b, d, c, w, v \in \mathbb{R}$

Exercise 6. Show that a function with the properties described above also has the property that

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -f\left(\begin{pmatrix} b & a \\ d & c \end{pmatrix}\right)$$

Exercise 7. Suppose f is a function with the properties described above. Show that if A is invertible, then $f(A) \neq 0$.

Exercise 8. Is the converse of the statement is Exercise 7 true? Prove or give a counterexample.