Solutions to Problem Set 21

Problem 1a: In order to compute $[T]_{\alpha}^{\varepsilon_2}$, we need to compute each of $[T(x^2)]_{\varepsilon_2}$, $[T(x)]_{\varepsilon_2}$, and $[T(1)]_{\varepsilon_2}$. We have

$$\begin{aligned} T(x^2) &= \begin{pmatrix} 0^2 \\ 2^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned}$$

so

$$[T(x^2)]_{\varepsilon_2} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

We also have

$$\begin{aligned} T(x) &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ &= 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned}$$

so

$$[T(x)]_{\varepsilon_2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and

$$\begin{aligned} T(1) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned}$$

so

$$[T(1)]_{\varepsilon_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It follows that

$$[T]_{\alpha}^{\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

Problem 1b: In order to compute $[T]_{\alpha}^{\beta}$, we need to compute each of $[T(x^2)]_{\beta}$, $[T(x)]_{\beta}$, and $[T(1)]_{\beta}$. We have

$$\begin{aligned} T(x^2) &= \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-2) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

so

$$[T(x^2)]_{\beta} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

We also have

$$
\begin{aligned}
T(x) &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
&= 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\end{aligned}
$$

so

$$[T(x)]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and

$$
\begin{aligned}
T(1) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\end{aligned}
$$

so

$$[T(1)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It follows that

$$[T]_\alpha^\beta = \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 0 \end{pmatrix}.$$

Problem 2: We have

$$
\begin{aligned}
T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Therefore,

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 3: Consider the 4×5 matrix with the given 5 vectors as columns:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 3 & 6 & 9 & 3 & -7 \\ 0 & 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 7 & -3 \end{pmatrix}.$$

Applying elementary row operations to this matrix, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 3 & 6 & 9 & 3 & -7 \\ 0 & 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 7 & -3 \end{pmatrix} &\to \begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -5 & -5 & 5 & 5 \end{pmatrix} && \begin{matrix} \\ (-3R_1 + R_2) \\ \\ (-2R_1 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} && \begin{matrix} \\ \\ \\ (5R_3 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} && \begin{matrix} \\ (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3) \\ \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 3 & 1 & -4 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} && \begin{matrix} \\ \\ \\ (-R_3 + R_4). \end{matrix}
\end{aligned}$$

Notice that this last matrix is in echelon form, and that the first, second, and fifth columns have leading entries. Using Proposition 5.2.10, we conclude that

$$\left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ -7 \\ 0 \\ -3 \end{pmatrix} \right)$$

is a basis for W. We have found a basis for W with 3 vectors, so $\dim(W) = 3$.

Problem 4a:

$$[T] = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix}$$

Performing elementary row operations on this matrix, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -3 \end{pmatrix} && \begin{matrix} \\ (-3R_1 + R_2) \\ (R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \begin{matrix} \\ \\ (-2R_2 + R_3). \end{matrix}
\end{aligned}$$

Now we know form Proposition 5.2.3 that $\text{range}(T)$ is the span of the columns of $[T]$. Since that last matrix above is in echelon form, we may combine this with Proposition 5.2.10 to conclude that

$$\left(\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} \right)$$

is a basis for range(T). Alternatively, we can use Corollary 5.2.4 to conclude that T is surjective, so range$(T) = \mathbb{R}^3$, and hence

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis for range(T).

To determine Null(T), we want to find the values of $x, y, z, w \in \mathbb{R}$ such that

$$T\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we want to find the values of $x, y, z, w \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 9 & -5 \\ -1 & 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In other words, we want to solve a homogeneous linear systems whose augmented matrix is given by $[T]$ together with a column of 0's on the right. Omitting this column of 0's and looking at the above computation, we notice the following. First, the last line tells us that $w = 0$. Since there is no leading entry in the third column, we introduce a parameter $z = t$. The second line now tells us that $y + 3z - 2w = 0$, so $y = -3z = -3t$. Finally, the first line tells us that $x + 2z - w = 0$, so $x = -2z = -2t$. Therefore, we have

$$\text{Null}(T) = \left\{ \begin{pmatrix} -2t \\ -3t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \cdot \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

It follows that

$$\text{Null}(T) = \text{Span}\left(\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right).$$

Now the sequence of one vector

$$\left(\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right)$$

is linearly independent because the vector is nonzero (see the discussion on p. 181). Therefore,

$$\left(\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right)$$

is a basis for Null(T).

Problem 4b: Since we found a basis for range(T) consisting of 3 elements, we have rank$(T) = 3$. Similarly, since we found a basis for Null(T) consisting of 1 element, we have nullity$(T) = 1$. Notice that rank(T) + nullity$(T) = 4 = \dim(\mathbb{R}^4)$, as the Rank-Nullity Theorem tells us must be the case.

Problem 5: We first show that range$(T) = \mathcal{P}_3$, i.e. that range(T) is the set of all polynomial functions of degree at most 3.

- We first show that every element of range$(T) \subseteq \mathcal{P}_3$. Let $g \in$ range(T) be arbitrary. Fix $f \in \mathcal{P}_5$ with $T(f) = g$, so $f'' = g$. Since $f \in \mathcal{P}_5$, we can fix $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$ such that

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

  for all $x \in \mathbb{R}$. We then have

$$f'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

  for all $x \in \mathbb{R}$, and hence

$$f''(x) = 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2$$

  for all $x \in \mathbb{R}$. Since $g = f''$, we conclude that g has degree at most 3, and so $g \in \mathcal{P}_3$.

- We now show that $\mathcal{P}_3 \subseteq$ range(T). Let $g \in \mathcal{P}_3$ be arbitrary. Fix $a_0, a_1, a_2, a_3 \in \mathbb{R}$ such that

$$g(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

  for all $x \in \mathbb{R}$. Define $f \colon \mathbb{R} \to \mathbb{R}$ by letting

$$f(x) = \frac{a_3}{20} \cdot x^5 + \frac{a_2}{12} \cdot x^4 + \frac{a_1}{6} \cdot x^3 + \frac{a_0}{2} \cdot x^2$$

  for all $x \in \mathbb{R}$ and notice that $f \in \mathcal{P}_5$. We have

$$f'(x) = \frac{a_3}{4} \cdot x^4 + \frac{a_2}{3} \cdot x^3 + \frac{a_1}{2} \cdot x^2 + a_0x$$

  for all $x \in \mathbb{R}$, and hence

$$f''(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

  for all $x \in \mathbb{R}$. Thus, we have $f'' = g$, and so $T(f) = g$. It follows that range$(T) \subseteq \mathcal{P}_3$.

Combining the two containments, we have shown that range$(T) = \mathcal{P}_3$. Since $\dim(\mathcal{P}_3) = 4$ (because $(1, x, x^2, x^3))$ is a basis for $\mathcal{P}_3$), we conclude that rank$(T) = 4$.

Now notice that $\dim(\mathcal{P}_5) = 6$, so since rank(T) + nullity$(T) = \dim(\mathcal{P}_5)$ by the Rank-Nullity Theorem, we must have $4 +$ nullity$(T) = 6$. Therefore, nullity$(T) = 2$. Alternatively, one can argue directly that Null$(T) = \mathcal{P}_1$ to prove this.