Solutions to Problem Set 22

Problem 1: Applying elementary rows operations to the matrix augmented by I_2, we obtain

$$\begin{aligned}
\begin{pmatrix} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix} && ((1/3)\cdot R_1) \\
&\to \begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & -1/3 & -5/3 & 1 \end{pmatrix} && (-5R_1 + R_2) \\
&\to \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & -1/3 & -5/3 & 1 \end{pmatrix} && (2R_2 + R_1) \\
&\to \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{pmatrix} && ((-3)\cdot R_2).
\end{aligned}$$

Therefore, our matrix is invertible and we have

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}.$$

Problem 2: We know that $\dim(\mathcal{P}_4) = 5$ because $(1, x, x^2, x^3, x^4)$ is a basis of $\mathcal{P}_4$. We also know that $\dim(V) = 4$ because

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

is a basis of V. Since $\dim(\mathcal{P}_4) > \dim(V)$, we can use Corollary 5.2.15 to immediately conclude that there is no injective linear transformation $T\colon \mathcal{P}_4 \to V$.

Problem 3a: Applying elementary rows operations to the matrix augmented by I_3, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{pmatrix} && (R_1 + R_3) \\
&\to \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} && (-R_2 + R_3) \\
&\to \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} && ((-1)\cdot R_3) \\
&\to \begin{pmatrix} 1 & 1 & 0 & 4 & -3 & 3 \\ 0 & 2 & 0 & 4 & -3 & 4 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} && (-3R_3 + R_1) \\
& && (-4R_3 + R_2) \\
&\to \begin{pmatrix} 1 & 1 & 0 & 4 & -3 & 3 \\ 0 & 1 & 0 & 2 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} && ((1/2)\cdot R_2) \\
&\to \begin{pmatrix} 1 & 0 & 0 & 2 & -3/2 & 1 \\ 0 & 1 & 0 & 2 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} && (-R_2 + R_1).
\end{aligned}$$

Therefore, our matrix is invertible and we have

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3/2 & 1 \\ 2 & -3/2 & 2 \\ -1 & 1 & -1 \end{pmatrix}.$$

Problem 3b: Applying elementary rows operations to the matrix augmented by I_3, we obtain

$$\begin{aligned}
\begin{pmatrix} 0 & 4 & 4 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 4 & 1 & 0 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_2) \\ (R_1 \leftrightarrow R_2) \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 4 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & -3 & 1 \end{pmatrix} && \begin{matrix} \\ \\ (-3R_1 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & -3 & 1 \\ 0 & 4 & 4 & 1 & 0 & 0 \end{pmatrix} && \begin{matrix} \\ (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 6 & -2 \end{pmatrix} && \begin{matrix} \\ \\ (-2R_2 + R_3). \end{matrix}
\end{aligned}$$

Since the 3×3 matrix on the left is in echelon form but does not have a leading entry in each row, we conclude that our matrix

$$\begin{pmatrix} 0 & 4 & 4 \\ 1 & -2 & 0 \\ 3 & -4 & 2 \end{pmatrix}$$

is not invertible.

Problem 3c: Applying elementary rows operations to the matrix augmented by I_3, we obtain

$$\begin{aligned}
\begin{pmatrix} 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_3) \\ \\ (R_1 \leftrightarrow R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 3/2 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & -2 & 0 & -1/2 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \end{pmatrix} && \begin{matrix} ((1/2) \cdot R_1) \\ ((-1/2) \cdot R_2) \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 3/2 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & -2 & 0 & -1/2 & 0 \\ 0 & 0 & 7 & 1 & 1/2 & 0 \end{pmatrix} && \begin{matrix} \\ \\ (-R_2 + R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 3/2 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & -2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{pmatrix} && \begin{matrix} \\ \\ ((1/3) \cdot R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 3/2 & 0 & 1/7 & 1/14 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{pmatrix} && \begin{matrix} (R_3 + R_1) \\ (2R_3 + R_2) \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 0 & -2/7 & 17/28 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{pmatrix} && \begin{matrix} ((-3/2) \cdot R_2 + R_1). \\ \\ \end{matrix}
\end{aligned}$$

Therefore, our matrix is invertible and we have

$$\begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} -2/7 & 17/28 & 1/2 \\ 2/7 & -5/14 & 0 \\ 1/7 & 1/14 & 0 \end{pmatrix} = \frac{1}{28} \cdot \begin{pmatrix} -8 & 17 & 14 \\ 8 & -10 & 0 \\ 4 & 2 & 0 \end{pmatrix}.$$

Problem 4: We provide a counterexample. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that $A \cdot A = I_2$ and $B \cdot B = I_2$, so both A an B are invertible. However, we have

$$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, = 0$$

which is not invertible because $C \cdot 0 = 0 \neq I_2$ for all 2×2 matrices C.

Problem 5a: Notice that A is a 2×3 matrix. Since $3 > 2$, we can use Corollary 5.2.18 to immediately conclude that A does not have a left inverse. Alternatively, we can attack this problem directly. Suppose that $a, b, c, d, e, f \in \mathbb{R}$ are such that

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have

$$\begin{pmatrix} a & b & a \\ c & d & c \\ e & f & e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Looking at the first row alone, we see that we must have both $a = 1$ and $a = 0$. This is a contradiction, so

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

does not have a left inverse.

Problem 5b: Notice that for any $a \in \mathbb{R}$, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \\ 1 - a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since there are infinitely many $a \in \mathbb{R}$, the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

has infinitely many right inverses.