1 The Basic Idea

If we have a linear transformation that describes a phenomena, it may be informative to know which vectors are pointing in the same[1]direction before and after the transformation. That is, for which vectors does our linear transformation act the same as scalar multiplication? In other words:

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then for which $\vec{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ do we have

$$T(\vec{v}) = \lambda\vec{v}$$

Or, using matrix notation, if $[T] = A$, then for which $\vec{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ do we have

$$A\vec{v} = \lambda\vec{v}$$

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the primary method we have for finding eigenvalues of A is by determining which values of λ result in

$$\text{Null}(A - \lambda \cdot I) \neq \{0\}$$

which is equivalent to finding the zeros of the *characteristic polynomial*, i.e. solving

$$(a - \lambda)(d - \lambda) - bc = 0.$$

Finally, if λ_1 is an eigenvalue, then

$$\text{Null}(A - \lambda_1 \cdot I) = \text{Span}(\vec{v}_1)$$

for some vector $\vec{v}_1$, and the vectors in this span are the eigenvectors associated to λ_1.

2 Practice and Applications

Exercise 1. Given

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of M, M^2, M^{-1}, and $M + 4 \cdot I$.

Using Corollary 3.5.5 and Def 3.5.6 we find the eigenvalues for each matrix by finding the roots of the characteristic polynomial. Then, we can solve for our eigenvectors algebraically

[1] Here we mean "same" to be "parallel". That is, $\vec{v}$ and $-\vec{v}$ point in the same direction.

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow M - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}$$

$\text{Null}\left(\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}\right) \neq \{0\}$ when $(2-\lambda)(2-\lambda) - (-1)(-1) = 0$

(i.e. when $ad - bc = 0$ by Theorem 3.3.3)

$$(2-\lambda)^2 - 1 = 0 \Rightarrow (2-\lambda)^2 = 1$$

$$\Rightarrow 2-\lambda = 1 \text{ or } 2-\lambda = -1$$

$$\Rightarrow \lambda = 1 \text{ or } \lambda = 3$$

So $\lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues for M.

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2x - y = x$ and $-x + 2y = y$

$\Rightarrow x = y$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_1$ is an eigenvector associated to $\lambda_1 = 1$.

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2x - y = 3x$ and $-x + 2y = 3y$

$\Rightarrow x = -y$ so $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector associated to $\lambda_2 = 3$.

Similarly we can verify that

M^2 has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 9$ associated to eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

M^{-1} has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/3$ associated to eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

$M + 4I$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 7$ associated to eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

Exercise 2. Given

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \tag{1}$$

find the eigenvalues and eigenvectors of A.

Again, following the procedure above, we find

A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$ associated to eigenvectors $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

Exercise 3. If A has eigenvalues λ_1 and λ_2 associated to eigenvectors $\vec{v}_1$ and $\vec{v}_2$, repectively, then what are the eigenvalues and eigenvectors of $A \cdot A$? Prove your claim and illustrate with an example. (Hint: You only need to use the definition of linear transformation.)

If $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$ w/ $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda x$ $(*)$

then $A^2\vec{x} \underset{\substack{3.2.5 \\ \#4}}{=} A(A\vec{x}) \underset{(*)}{=} A(\lambda\vec{x}) \underset{\substack{3.2.5 \\ \#2}}{=} \lambda(A\vec{x}) \underset{(*)}{=} \lambda(\lambda\vec{x}) \underset{\substack{2.2.1 \\ \#9}}{=} \lambda^2\vec{x}$.

So, the eigenvalues of A^2 are the squares of the eigenvalues of A, and the eigenvectors are the same.

Exercise 4. Combining your observations from Exercise 2 and 3, what can you say about the eigenvalues and eigenvectors of the following matrix?:

$$B = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^{99}$$

It probably has eigenvalues 1 and $\left(\frac{1}{2}\right)^{99}$ associated to the eigenvectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

The matrix in Equation (1) is in the form of a *Markov* matrix. These are used to model probabilistic changes in the state of a phenomena. If a phenomena has states that change probabilistically, we can model that process in the following way:

- Set up a matrix with the entries representing the probabilities of each of the following:

$$\begin{pmatrix} \text{Start in State 1, remain in State 1} & \text{Start in State 2, change to State 1} \\ \text{Start in State 1, change to State 2} & \text{Start in State 2, remain in State 2} \end{pmatrix}$$

- Input a vector of the form

$$\begin{pmatrix} \text{\% of observations currently in State 1} \\ \text{\% of observations currently in State 2} \end{pmatrix} \tag{2}$$

Exercise 5. If the states in the Markov matrix,A, from Equation (1) are

- State 1: A person lives in Los Angeles
- State 2: A person lives in New York

and the probabilities refer to the change in state over the course of a year, then:

(a) What are the observations being referred to in Equation (2) in this context?

The people living in NYC or LA.

(The sum of these is the total number of observations)

(b) How would you interpret the output after applying A to an input vector?

$$\begin{pmatrix} \text{\% of total population in LA after a year} \\ \text{\% of total population in NYC after a year} \end{pmatrix}$$

(c) How would you interpret your observations from Exercise 4 in this context? (What do the eigenvalues and eigenvectors mean?)

If 60% of people live in LA and 40% live in NYC, then the populations will remain stable.

For Next Time

- Finish this worksheet
- Read through Example 3.5.16