Assignment: Problem Set 12

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: None

Problem 1: Explain why a 2×2 matrix A is invertible if and only if 0 is not an eigenvalue of A.

Solution: Let A be the 2×2 matrix that is the standard matrix of some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, and fix $a, b, c, d \in \mathbb{R}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notice that

$$\begin{aligned} Null(T) =& \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\} \\ =& \{\vec{v} \in \mathbb{R}^2 : [T]\vec{v} = \vec{0}\} && \text{(By Proposition 3.1.4)} \\ =& \{\vec{v} \in \mathbb{R}^2 : A\vec{v} = \vec{0}\} && \text{(By definiton of } A\text{)} \\ =& Null(A) \end{aligned}$$

So $Null(T) = Null(A)$. Now let $\lambda \in \mathbb{R}$ be arbitrary. By Corollary 3.5.5, λ is an eigenvalue of A if and only if $Null(A - \lambda I) \neq \{\vec{0}\}$.

We first show that if A is invertible, then 0 is not an eigenvalue of A. Suppose that A is invertible, so by Proposition 3.3.16, $ad - bc \neq 0$. $ad - bc \neq 0$, so by Theorem 3.3.3, $Null(T) = \{\vec{0}\}$, and it follows that $Null(A) = \{\vec{0}\}$. Consider the case where $\lambda = 0$. We then have that $Null(A - \lambda I) = Null(A - 0I) = Null(A - 0) = Null(A)$. $Null(A) = \{\vec{0}\}$, so $Null(A - 0I) = \{\vec{0}\}$, and it follows from Corollary 3.5.5 that 0 is not an eigenvalue of A. Therefore, if A is invertible, 0 is not an eigenvalue of A.

We now prove the converse, that if 0 is not an eigenvalue of A, then A is invertible. We prove the contrapositive: If A is not invertible, then 0 is an eigenvalue of A. Suppose that A is not invertible. By Proposition 3.3.16, $ad - bc = 0$. Now let $\lambda \in \mathbb{R}$ be arbitrary. By Corollary 3.5.5, λ is an eigenvalue of A if and only if $Null(A - \lambda I) \neq \{\vec{0}\}$. Consider the case where $\lambda = 0$. We then have that $Null(A - \lambda I) = Null(A - 0I) = Null(A - 0) = Null(A)$. So 0 is an eigenvalue of A if and only if $Null(A) \neq \{\vec{0}\}$. We have two cases for A:

1. All of a, b, c, d equal 0, that is, that $A = 0$. Applying Theorem 3.3.3, there exists a nonzero $\vec{u} \in \mathbb{R}^2$ with $Null(T) = Span(\vec{u})$, so it follows that $Null(A) = Span(\vec{u})$. So $Null(A) \neq \{\vec{0}\}$, and 0 is an eigenvalue of A.

2. $ad - bc = 0$ and at least one of a, b, c, d is nonzero. Applying Theorem 3.3.3, we have $Null(T) = \mathbb{R}^2$, and it follows that $Null(A) = \mathbb{R}^2$. So $Null(A) \neq \{\vec{0}\}$, and 0 is an eigenvalue of A.

These two cases exhaust all possibilities, so it must be the case that if A is not invertible, then 0 is an eigenvalue of A. Because we have proven the contrapositive, the original statement must also be true. Therefore, if 0 is not an eigenvalue of A, then A is invertible. We have proven the implication and its converse to both be true, therefore, a 2×2 matrix A is invertible if and only if 0 is not an eigenvalue of A.

Problem 2: Define a sequence of numbers as follows. Let $g_0 = 0$, $g_1 = 1$, and $g_n = \frac{1}{2}(g_{n-1} + g_{n-2})$ for all $n \in \mathbb{N}$ with $n \geq 2$. Notice that if

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

then

$$A\begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} = \begin{pmatrix} g_{n+2} \\ g_{n+1} \end{pmatrix}$$

for all $n \in \mathbb{N}$, so

$$A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix}$$

for all $n \in \mathbb{N}$.

a. Find an invertible matrix P and a diagonal matrix D with $A = PDP^{-1}$.

Solution: Let $\alpha = (\vec{u_1}, \vec{u_2})$ be an arbitrary basis of $\mathbb{R}^2$, and fix $a, b, c, d \in \mathbb{R}$ with $\vec{u_1} = \begin{pmatrix} a \\ c \end{pmatrix}, \vec{u_2} = \begin{pmatrix} b \\ d \end{pmatrix}$ Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and define T such that $[T] = A$. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By definition of basis, $Span(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$, so by Theorem 2.3.10, $ad - bc \neq 0$. It follows that P is invertible by Proposition 3.3.16. Applying Proposition 3.4.7, we have that $[T]_\alpha = P^{-1}AP$. It follows that $A = P[T]_\alpha P^{-1}$. By Proposition 3.5.14, $[T]_\alpha$ is a diagonal matrix if and only if $\vec{u_1}, \vec{u_2}$ are eigenvectors of A, that is to say, that the eigenvectors of A form a basis of $\mathbb{R}^2$ $\alpha = (\vec{u_1}, \vec{u_2})$. We first find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned} (0.5 - \lambda)(0 - \lambda) - (1)(0.5) &= 0 \\ \lambda^2 - 0.5\lambda - 0.5 &= 0 \\ (\lambda - 1)(\lambda + 0.5) &= 0 \end{aligned}$$

So A has distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.5$. We now find the eigenvectors

We now find eigenvectors of A corresponding to eigenvalues 1 and -0.5, that is, we find the value of the vectors $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ that satisfy

$$\left(\begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix} - 1I\right)\vec{v_1} = \vec{0} \qquad \text{and} \qquad \left(\begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix} + 0.5I\right)\vec{v_2} = \vec{0}$$

Letting, $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we get,

$$\begin{pmatrix} -0.5 & 0.5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0.5 \\ 1 & 0.5 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

which become

Problem 3: Show that $\det(AB) = \det(A)\cdot\ \det(B)$ for all 2×2 matrices A and B. *Note:* Intuitively, if the linear transformation with standard matrix B distorts area by a factor of s, and the linear transformation with standard matrix A distorts area by a factor of r, then the *composition* of these linear transformations will distort area by a factor of rs (because matrix multiplication corresponds to function composition), with appropriate signs. Although it is possible to make this geometric sketch precise by using arguments similar to the ones at the end of Section 3.6, you should give a computational argument in this problem by just using the formula for the determinant.

Solution: Let A, B be arbitrary 2×2 matrices, and fix $a, b, c, d, e, f, g, h \in \mathbb{R}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Let $C = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$. By definition of the determinant, $\det(A) = ad - bc$, and $\det(B) = eh - gf$. So

$$\begin{aligned} \det(A)\cdot\det(B) =& (ad-bc)(eh-gf) \\ =& adeh - adgf - bceh + bcgf = adeh + bgcf - cebh - dgaf \end{aligned}$$

. So $\det(A)\cdot\det(B) = adeh + bgcf - cebh - dgaf$. Notice that

$$\begin{aligned} \det(C) = \det(AB) =& (ae+bg)(cf+dh) - (ce+dg)(af+bh) \\ =& \cancel{aecf} + aedh + bgcf + \cancel{bgdh} \\ & - \cancel{ceaf} - cebh - dgaf - \cancel{dgbh} \\ =& aedh + bgcf - cebh - dgaf = \det(A)\cdot\det(B) \end{aligned}$$

Therefore, $\det(AB) = \det(A)\cdot\ \det(B)$. Becase, A, B were arbitrary, the result follows.

Problem 4: Show that if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Hint: Start with the fact that $AA^{-1} = I$, and use the previous problem.

Solution: Let A be an arbitrary invertible 2×2 matrix, and fix $a, b, c, d \in \mathbb{R}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Proposition 3.3.16, $ad - bc \neq 0$, and A had a unique inverse, denoted A^{-1}, given by $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. By definition, $\det(A) = ad - bc = \frac{(ad-bc)^2}{ad-bc}$. Note that $\det(A^{-1}) = \frac{da}{(ad-bc)^2} - \frac{bc}{(ad-bc)^2} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{\left(\frac{(ad-bc)^2}{ad-bc}\right)} = \frac{1}{\det(A)}$. Because A was arbitrary, the result follows.

We can also show this as follows: Let A be an arbitrary invertible 2×2 matrix. Because A is invertible, there exists a unique inverse of A denoted by A^{-1}, for which $AA^{-1} = I$ and $A^{-1}A = I$. So $\det(AA^{-1}) = \det(I)$. By the previous problem, $\det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$, so it follows that $\det(A) \cdot \det(A^{-1}) = \det(I)$. Manipulating, we get $\det(A^{-1}) = \frac{\det(I)}{\det(A)}$. Notice that $\det(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(0) = 1$, so $\det(A^{-1}) = \frac{1}{\det(A)}$. Because A was arbitrary, the result follows.

Problem 5: Show that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and α is a basis of $\mathbb{R}^2$, then $\det([T]_\alpha) = \det([T])$. Thus, although we might obtain different matrices when we represent T with respect to different bases, the resulting matrices will all have the same determinant.

Solution: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by fixing $e, f, g, h \in \mathbb{R}$ such that $[T] = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, and let $\alpha = \left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$ be an arbitrary basis of $\mathbb{R}^2$, and let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By similar reasoning as in problem 2, we have that P is invertible. By Proposition 3.4.7, we then have that

$$[T]_\alpha = P^{-1}[T]P$$

So $\det([T]_\alpha) = \det(P^{-1}[T]P) = \det(P^{-1}) \cdot \det([T]) \cdot \det(P)$ (By the result of Problem 3) $= \cancel{\frac{1}{\det(P)}} \cdot \det([T]) \cdot \cancel{\det(P)}$ (By the result of Problem 4) $= \det([T])$. So $\det([T]_\alpha) = \det([T])$. Because α, T were arbitrary, the result follows.

Problem 6: Given a 2×2 matrix A and an $r \in \mathbb{R}$, what is the relationship between $\det(r \cdot A)$ and $\det(A)$? Explain.

Solution: Let A be an arbitrary 2×2 matrix, and let $r \in \mathbb{R}$ be arbitrary. Fix $a, b, c, d \in \mathbb{R}$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notice that $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, and that $\det(r \cdot A) = \begin{vmatrix} ra & rb \\ rc & rd \end{vmatrix} = r^2(ad - bc) = r^2 \cdot \det(A)$. So $\det(r \cdot A) = r^2 \cdot \det(A)$. The determinant of a matrix gives the signed area distortion factor of the linear transformation on vectors $\vec{v}, \vec{u} \in \mathbb{R}^2$ that form a parallelogram, with that matrix as the standard matrix of the linear transformation. Scaling the linear transformation by a factor of r scales the magnitude of the transformed vectors by a factor of r, and it follows that the area of the transformed parallelogram is scaled by r^2, so it must be the case that the area distortion factor is also scaled by r^2.

$$\begin{pmatrix} -0.5x_1 + 0.5y_1 \\ x_1 - y_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_2 + 0.5y_2 \\ x_2 + 0.5y_2 \end{pmatrix}$$

Letting $x_1 = 1, y_1 = 1, x_2 = 0.5, y_2 = -1$, we get

$$\begin{pmatrix} -0.5 + 0.5 \\ -0.5 + 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \text{and} \qquad \begin{pmatrix} 0.5 - 0.5 \\ 0.5 - 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ an eigenvector of A corresponding to eigenvalue 1, and $\begin{pmatrix} 0.5 \\ -1 \end{pmatrix}$ an eigenvector of A corresponding to eigenvalue -0.5. Notice that $(1)(-1) - (0.5)(1) = -1 - 0.5 = -1.5 \neq 0$, so by Theorem 2.3.10, $Span\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}\right) = \mathbb{R}^2$, so $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}\right)$ is a basis of $\mathbb{R}^2$. Letting $D = [T]_\beta$, where $\beta = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}\right)$, we have that D is a diagonal matrix, and $P = \begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix}$. It follows that $P^{-1} = -\frac{2}{3}\begin{pmatrix} -1 & -0.5 \\ -1 & 1 \end{pmatrix}$. We compute $[T]_\alpha$ using Proposition 3.4.7.

$$\begin{aligned}
[T]_\alpha = P^{-1}AP &= -\frac{2}{3}\begin{pmatrix} -1 & -0.5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix} \\
&= -\frac{2}{3}\begin{pmatrix} -1 & -0.5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (0.5)(1) + (0.5)(1) & (0.5)(0.5) + (0.5)(-1) \\ (1)(1) + (0)(1) & (1)(0.5) + (0)(-1) \end{pmatrix} \\
&= -\frac{2}{3}\begin{pmatrix} -1 & -0.5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -0.25 \\ 1 & 0.5 \end{pmatrix} \\
&= -\frac{2}{3}\begin{pmatrix} (-1)(1) + (-0.5)(1) & (-1)(-0.25) + (-0.5)(0.5) \\ (-1)(1) + (1)(1) & (-1)(-0.25) + (1)(0.5) \end{pmatrix} \\
&= -\frac{2}{3}\begin{pmatrix} -1.5 & 0 \\ 0 & 0.75 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}
\end{aligned}$$

So $D = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}$, and it is clear to see that this matrix is diagonal. Therefore, for $A = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}$, we have that $A = PDP^{-1}$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}$ is diagonal, $P = \begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix}$ is invertible.

b. Find a general formula for g_n.

Solution: Recall that $A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix}$, so g_n is dot product of the matrix-vector product $A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Notice that

$$A^2 = AA = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1}$$
$$A^3 = A(A^2) = PDP^{-1}PD^2P^{-1} = PDID^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

so $A^n = A(A^{n-1}) = PDP^{-1}PD^{n-1}P^{-1} = PDID^{n-1}P^{-1} = PDD^{n-1}P^{-1} = PD^nP^{-1}$. So $A^n = PD^nP^{-1}$ Notice that for a diagonal matrix $M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $M^2 = MM = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} (a)(a)+(0)(0) & (a)(0)+(0)(d) \\ (0)(a)+(d)(0) & (0)(0)+(d)(d) \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}$, and $M^3 = M(M^2) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} = \begin{pmatrix} (a)(a^2)+(0)(0) & (a)(0)+(0)(d^2) \\ (0)(a^2)+(d)(0) & (0)(0)+(d)(d^2) \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & d^3 \end{pmatrix}$, so it must be the case that $M^n = M(M^{n-1}) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a^{n-1} & 0 \\ 0 & d^{n-1} \end{pmatrix} = \begin{pmatrix} (a)(a^{n-1})+(0)(0) & (a)(0)+(0)(d^{n-1}) \\ (0)(a^{n-1})+(d)(0) & (0)(0)+(d)(d^{n-1}) \end{pmatrix} = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix}$. It follows that

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-0.5)^n \end{pmatrix}\left(-\frac{2}{3}\right)\begin{pmatrix} -1 & -0.5 \\ -1 & 1 \end{pmatrix} \\ &= \left(-\frac{2}{3}\right)\begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} (1)(-1)+(0)(-1) & (1)(-0.5)+(0)(1) \\ (0)(-1)+(-0.5)^n(-1) & (0)(-0.5)+(-0.5)^n(1) \end{pmatrix} \\ &= \left(-\frac{2}{3}\right)\begin{pmatrix} 1 & 0.5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -0.5 \\ -(-0.5)^n & (-0.5)^n \end{pmatrix} \\ &= -\frac{2}{3}\begin{pmatrix} (1)(-1)+(0.5)(-(-0.5)^n) & (1)(-0.5)+(0.5)(-0.5)^n \\ (1)(-1)+(-1)(-(-0.5)^n) & (1)(-0.5)+(-1)(-0.5)^n \end{pmatrix} \\ &= -\frac{2}{3}\begin{pmatrix} (-0.5)^{n+1}-1 & -0.5+0.5(-0.5)^n \\ -1+(-0.5)^n & -0.5-(-0.5)^n \end{pmatrix} \end{aligned}$$

So $A^n = -\frac{2}{3}\begin{pmatrix} (-0.5)^{n+1}-1 & -0.5+0.5(-0.5)^n \\ -1+(-0.5)^n & -0.5-(-0.5)^n \end{pmatrix}$. It follows that

$$\begin{aligned} \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} = A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= -\frac{2}{3}\begin{pmatrix} (-0.5)^{n+1}-1 & -0.5+0.5(-0.5)^n \\ -1+(-0.5)^n & -0.5-(-0.5)^n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= -\frac{2}{3}\begin{pmatrix} (-0.5)^{n+1}-1 \\ (-0.5)^n-1 \end{pmatrix} \end{aligned}$$

and so $g_n = \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{2}{3}((-0.5)^n - 1) = \frac{2(-1)^n}{-3(2)^n} + \frac{2}{3} = \frac{(-1)^{n-1}}{3(2)^{n-1}} + \frac{2}{3} = \frac{1}{3(-2)^{n-1}} + \frac{2}{3}$. Therfore, $g_n = \frac{2}{3} + \frac{1}{3(-2)^{n-1}}$.

c. As n gets large, the values of g_n approach a fixed number. Find that number.

Solution: As $n \to \infty$, $|\frac{1}{3(-2)^{n-1}}| \to 0$, so as $n \to \infty$, $g_n \to \frac{2}{3}$.