Assignment: Problem Set 20

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Working in $\mathbb{R}^4$, let

$$W = \text{Span}\left(\begin{pmatrix}0\\0\\1\\3\end{pmatrix}, \begin{pmatrix}4\\5\\2\\7\end{pmatrix}, \begin{pmatrix}7\\8\\0\\1\end{pmatrix}\right).$$

Explain why $\dim(W) = 3$.

Solution: Let $\alpha = \left(\begin{pmatrix}0\\0\\1\\3\end{pmatrix}, \begin{pmatrix}4\\5\\2\\7\end{pmatrix}, \begin{pmatrix}7\\8\\0\\1\end{pmatrix}\right)$. Notice that $\begin{pmatrix}0\\0\\1\\3\end{pmatrix}, \begin{pmatrix}4\\5\\2\\7\end{pmatrix}, \begin{pmatrix}7\\8\\0\\1\end{pmatrix} \in \mathbb{R}^4$, so by Proposition 4.1.16 W is a subspace of $\mathbb{R}^4$. Let A be the 4×3 matrix with the elements of α as its columns. Performing Gaussian Elimination to obtain an echelon form of A, we get:

$$\begin{aligned}
\begin{pmatrix}0 & 4 & 7\\0 & 5 & 8\\1 & 2 & 0\\3 & 7 & 1\end{pmatrix} &\to \begin{pmatrix}0 & 4 & 7\\0 & 5 & 8\\1 & 2 & 0\\0 & 1 & 1\end{pmatrix} \begin{matrix}\\ \\ \\ -3R_3 + R_4\end{matrix}\\
&\to \begin{pmatrix}0 & 0 & 3\\0 & 0 & 3\\1 & 2 & 0\\0 & 1 & 1\end{pmatrix} \begin{matrix}-4R_4 + R_1\\ -5R_4 + R_2\\ \\ \\ \end{matrix}\\
&\to \begin{pmatrix}0 & 0 & 3\\0 & 0 & 0\\1 & 2 & 0\\0 & 1 & 1\end{pmatrix} \begin{matrix}\\ -R_1 + R_2\\ \\ \\ \end{matrix}\\
&\to \begin{pmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 0 & 3\\0 & 0 & 0\end{pmatrix} \begin{matrix}R_3 \leftrightarrow R_1\\ R_4 \leftrightarrow R_2\\ R_1 \leftrightarrow R_3\\ R_2 \leftrightarrow R_4\end{matrix}
\end{aligned}$$

Notice that there is a leading entry in every column of the echelon matrix, so by Proposition 4.3.3, α is linearly independent. Because $\text{Span}(\alpha) = W$, it follows from Definition 4.4.1 that α is a basis for W. Because α has 3 elements, it follows from Definition 4.4.9 that $\dim(W) = 3$.

Problem 2: Let V be the vector space of all 2×2 matrices. Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : 2a - c = 0 \text{ and } b + c - d = 0 \right\}.$$

It turns out that W is a subspace for V (no need to show this). Find a basis for W, and determine $\dim(W)$.
Hint: First try to write W as the span of some elements of V by solving the system of equations.

Solution: Notice that we can use the rules that define W to rewrite $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : 2a = c \text{ and } b + 2a = d \right\}$, which we can then rewrite as $W = \left\{ \begin{pmatrix} a & b \\ 2a & b + 2a \end{pmatrix} \in V : a, b \in \mathbb{R} \right\}$. Let $A \in W$ be arbitrary, and fix $x, y \in \mathbb{R}$ with $A = \begin{pmatrix} x & y \\ 2x & y + 2x \end{pmatrix}$. Let $\alpha = \left(\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$. Notice that

$$A = \begin{pmatrix} x & y \\ 2x & y + 2x \end{pmatrix} = x \cdot \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} + y \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

so $A \in \text{Span}(\alpha)$. Because $A \in W$ was arbitrary, it follows that $W \subseteq \text{Span}(\alpha)$. Now let $B \in \text{Span}(\alpha)$ be arbitrary. By definition of Span we can fix $a, b \in \mathbb{R}$ with $B = a \cdot \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Notice that

$$B = a \cdot \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 2a & b + 2a \end{pmatrix},$$

so $B \in W$. Because $B \in \text{Span}(\alpha)$ was arbitrary, it follows that $\text{Span}(\alpha) \subseteq W$. Because $W \subseteq \text{Span}(\alpha)$ and $\text{Span}(\alpha) \subseteq W$, it follows that $\text{Span}(\alpha) = W$. Now suppose that $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \right)$. By definition of Span, we can fix $c \in \mathbb{R}$ with $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = c \cdot \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 2c & 2c \end{pmatrix}$. Because matrices are equal exactly when their entries are equal, we have that $0 = c, 1 = 0, 0 = 2c, 1 = 2c$. The second equation does not depend on c and immediately gives us a contradiction, so it must be the case that $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin \text{Span}\left(\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \right)$. Notice that the sequence $\left(\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \right)$ is linearly independent, so By Proposition 4.4.12 it follows that α is linearly independent. Because $\text{Span}(\alpha) = W$ and α is linearly independent, it follows from Definition 4.4.1 that α is a basis for W. Because α has two elements, it follows from Definition 4.4.9 that $\dim(W) = 2$.

Problem 3: Define $T : \mathcal{P}_1 \to \mathbb{R}^2$ by letting

$$T(a + bx) = \begin{pmatrix} a - b \\ b \end{pmatrix}.$$

Show that T is a linear transformation.

Solution: Let $f_1, f_2 \in \mathcal{P}_1$ be arbitrary, and fix $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $f_1(x) = a_1 + b_1 x$ and $f_2(x) = a_2 + b_2 x$ for all $x \in \mathbb{R}$. Notice that

$$\begin{aligned} T(f_1 + f_2) &= T(a_1 + b_1 x + a_2 + b_2 x) \\ &= T((a_1 + a_2) + (b_1 + b_2)x) \\ &= \begin{pmatrix} (a_1 + a_2) - (b_1 + b_2) \\ (b_1 + b_2) \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 + a_2 - b_2 \\ b_1 + b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 - b_2 \\ b_2 \end{pmatrix} = T(f_1) + T(f_2) \end{aligned}$$

So $T(f_1 + f_2) = T(f_1) + T(f_2)$. Because $f_1, f_2 \in \mathcal{P}_1$ were arbitrary, it follows that $T(f_1 + f_2) = T(f_1) + T(f_2)$ for all $f_1, f_2 \in \mathcal{P}_1$.

Now let $r \in \mathbb{R}$ be arbitrary, and let $f \in \mathcal{P}_1$ be arbitrary and fix $p, q \in \mathbb{R}$ with $f(x) = p + qx$ for all $x \in \mathbb{R}$. Notice that

$$\begin{aligned} T(r \cdot f) &= T(r \cdot (p + qx)) \\ &= T((rp) + (rq)x) \\ &= \begin{pmatrix} rp - rq \\ rq \end{pmatrix} \\ &= r \cdot \begin{pmatrix} p - q \\ q \end{pmatrix} = r \cdot T(f) \end{aligned}$$

So $T(r \cdot f) = r \cdot T(f)$. Because $f \in \mathcal{P}_1$ and $r \in \mathbb{R}$ were arbitrary, it follows that $T(r \cdot f) = r \cdot T(f)$ for all $f \in \mathcal{P}_1$ and all $r \in \mathbb{R}$.

We have shown that $T(f_1 + f_2) = T(f_1) + T(f_2)$ for all $f_1, f_2 \in \mathcal{P}_1$ and that $T(r \cdot f) = r \cdot T(f)$ for all $f \in \mathcal{P}_1$ and all $r \in \mathbb{R}$, so it follows from Definition 5.1.1 that $T : \mathcal{P}_1 \to \mathbb{R}^2$ is indeed a linear transformation.

Problem 4: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y \\ x + z \\ y + z \end{pmatrix}.$$

a. Explain why T is a linear transformation.

Solution: Fix $a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, a_{3,1}, a_{3,2}, a_{3,3} \in \mathbb{R}$ with $a_{1,1} = a_{2,1} = a_{2,3} = a_{3,2} = a_{3,3} = 1, a_{1,3} = a_{2,2} = a_{3,1} = 0, a_{1,2} = -1$. Let $x_1, x_2, x_3 \in \mathbb{R}$ be arbitrary. Notice that

$$\begin{aligned} \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 \end{pmatrix} &= \begin{pmatrix} 1 \cdot x_1 + -1 \cdot x_2 + 0 \cdot x_3 \\ 1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 - x_2 \\ x_1 + x_3 \\ x_2 + x_3 \end{pmatrix} = T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right). \end{aligned}$$

Because $x_1, x_2, x_3 \in \mathbb{R}$ were arbitrary, we can express T as the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ by letting $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 \end{pmatrix}$ for all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. By Proposition 5.1.2, it follows that T is a linear transformation.

b. Give an example of a nonzero $\vec{v} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{0}$.

Solution: Let $\vec{p} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Notice that

$$T(\vec{p}) = \begin{pmatrix} 1 - 1 \\ 1 + (-1) \\ 1 + (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}.$$

So $T(\vec{p}) = \vec{0}$.

c. Show that T is not injective.

Solution: Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. By Definition 1.6.7, if S is injective, then for all $\vec{a}, \vec{b} \in \mathbb{R}$, if $S(\vec{a}) = S(\vec{b})$, then $\vec{a} = \vec{b}$. Taking the contrapositive we obtain the following statement: If there exist $\vec{a}, \vec{b} \in \mathbb{R}^3$ with $S(\vec{a}) = S(\vec{b})$ and $\vec{a} \neq \vec{b}$, then S is not injective. We showed above that, letting $\vec{p} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $T(\vec{p}) = \vec{0}$. Because T is a linear transformation, by Proposition 5.1.4 $T(\vec{0}) = \vec{0}$. So $T(\vec{p}) = T(\vec{0})$. Notice that $\vec{p} \neq \vec{0}$. It follows from the statement above that T is not injective.

Problem 5: Let V be the vector space of all 2×2 matrices. Define $T : V \to \mathbb{R}$ by letting

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 2a - d.$$

a. Show that T is a linear transformation.

Solution: Let $A_1, A_2 \in V$ be arbitrary, and fix $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$ with $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Notice that

$$\begin{aligned}
T(A_1 + A_2) =& T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \\
=& T\left(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}\right) && \text{(By Definition 5.1.13)} \\
=& 2(a_1 + a_2) - (d_1 + d_2) && \text{(By Definition of } T\text{)} \\
=& 2a_1 + 2a_2 - d_1 - d_2 \\
=& (2a_1 - d_1) + (2a_2 - d_2) \\
=& T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = T(A_1) + T(A_2).
\end{aligned}$$

So $T(A_1 + A_2) = T(A_1) + T(A_2)$. Because $A_1, A_2 \in V$ were arbitrary, it follows that $T(A_1 + A_2) = T(A_1) + T(A_2)$ for all $A_1, A_2 \in V$.

Now let $r \in \mathbb{R} bearbitrary$. Let $A \in V$ be arbitrary, and fix $w, x, y, z \in \mathbb{R}$ with $A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Notice that

$$\begin{aligned}
T(r \cdot A) =& T\left(r \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}\right) \\
=& T\left(\begin{pmatrix} rw & rx \\ ry & rz \end{pmatrix}\right) \\
=& 2(rw) - rz && \text{(By Definition of } T\text{)} \\
=& r \cdot (2w - z) \\
=& r \cdot T\left(\begin{pmatrix} w & x \\ y & z \end{pmatrix}\right) = r \cdot T(A).
\end{aligned}$$

So $T(r \cdot A) = r \cdot T(A)$. Because $A \in V$ and $r \in \mathbb{R}$ were arbitrary, it follows that $T(r \cdot A) = r \cdot T(A)$ for all $A \in V$ and all $r \in \mathbb{R}$.

We have shown that $T(A_1 + A_2) = T(A_1) + T(A_2)$ for all $A_1, A_2 \in V$ and that $T(r \cdot A) = r \cdot T(A)$ for all $A \in V$ and all $r \in \mathbb{R}$, so it follows from Definition 5.1.1 that $T : V \to \mathbb{R}$ is indeed a linear transformation.

PAGE 1 OF 2 FOR PROBLEM 5

b. Show that T is surjective.

Solution: Let $S : V \to \mathbb{R}$ be a linear transformation. By Definition 1.6.7, if for all $c \in \mathbb{R}$, there exists $M \in V$ such that $S(M) = c$, then S is surjective. Let $r \in \mathbb{R}$ be arbitrary. Letting $b, c \in \mathbb{R}$, notice that $\begin{pmatrix} r & b \\ c & r \end{pmatrix} \in V$ and that $T(A) = T\left(\begin{pmatrix} r & b \\ c & r \end{pmatrix}\right) = 2r - r = r$. Thus, we have shown the existence of an $A \in V$ such that $T(A) = r$. Because $r \in \mathbb{R}$ was arbitrary, it follows that for all $r \in \mathbb{R}$, there exists $A \in V$ such that $T(A) = r$. It follows from the above statement that T is surjective.