Worksheet 23 - Coordinates, and Solving Equations Involving Linear Transformations

Definition. Suppose that V is a vector space and that $\alpha = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ is a basis of V. We define a function

$$Coord_\alpha \colon V \to \mathbb{R}^n$$

as follows. Given $\vec{v} \in V$, let $c_1, c_2, \ldots, c_n \in \mathbb{R}$ be the unique values such that $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$, and define

$$Coord_\alpha(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

We call this vector the *coordinates of $\vec{v}$ relative to α*. We also use the notation $[\vec{v}]_\alpha$ for $Coord_\alpha(\vec{v})$.

Definition. Let $T \colon V \to W$ be a linear transformation, let $\alpha = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ be a basis for V, and let $\beta = (\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ be a basis for W. We then define the matrix of T relative to α and β to be the $m \times n$ matrix where the i^{th} column is $[T(\vec{u}_i)]_\beta$. We denote this matrix by $[T]_\alpha^\beta$.

Exercise 1. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation with the following properties:

$$T\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) = \begin{pmatrix}2\\1\\1\end{pmatrix}; \quad T\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\2\\0\end{pmatrix}; \quad T\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}4\\5\\1\end{pmatrix}; \quad T\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\7\\-1\end{pmatrix}.$$

Consider the bases for $\mathbb{R}^4$ and $\mathbb{R}^3$ given by:

$$\alpha = \left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\\0\end{pmatrix}, \begin{pmatrix}1\\3\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\0\end{pmatrix}\right)$$

$$\beta = \left(\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right)$$

(a) Compute $[T]_\alpha^\beta$.

(b) Find the solution set for the equation $T(\vec{v}) = \vec{0}$.

(c) If $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$ are the standard basis vectors in $\mathbb{R}^3$ then find solutions sets for the following equations:

(i) $T(\vec{x}) = \vec{e}_1$

(ii) $T(\vec{y}) = \vec{e}_2$

(iii) $T(\vec{z}) = \vec{e}_3$

Exercise 2. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with the following properties:

$$T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}2\\1\\0\end{pmatrix}; \qquad T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\2\\0\end{pmatrix}; \qquad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}4\\5\\1\end{pmatrix}.$$

(a) Find the solution set for the equation $T(\vec{v}) = \vec{0}$.

(b) If $\vec{e}_1, \vec{e}_2$, and $\vec{e}_3$ are the standard basis vectors in $\mathbb{R}^3$ then find solutions sets for the following equations:

   (i) $T(\vec{x}) = \vec{e}_1$

   (ii) $T(\vec{y}) = \vec{e}_2$

   (iii) $T(\vec{z}) = \vec{e}_3$

(c) Is T injective?

(d) Is T surjective?

(e) Find a vector $\vec{v} \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$ such that $T(\vec{v}) = \lambda\vec{v}$.

Exercise 3. How do your observations in the preceding exercises compare with our observations for linear transformations in $\mathbb{R}^2$? (There is a lot to say here, and this is a strong candidate for a short answer question on the final exam.)