Problem Set 15: Due Monday, April 9

Problem 1: Recall that $\mathcal{P}$ is the vector space of all polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$. Let W be the subset of $\mathcal{P}$ consisting of those polynomials that have a nonnegative constant term (i.e. the constant terms is greater than or equal to 0). Is W a subspace of $\mathcal{P}$? Either prove or give a counterexample.

Problem 2: Let $V = \mathbb{R}^4$. Write down a system of four equations in three unknowns such that

$$\begin{pmatrix} 1 \\ 7 \\ 0 \\ 6 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ -8 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right)$$

if and only if the system has a solution.

Problem 3: Let V be the vector space of all 2×2 matrices. Show that

$$\begin{pmatrix} -2 & 7 \\ -1 & -9 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix} \right).$$

Problem 4: Let $\mathcal{D}$ be the vector space of all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $f_1\colon \mathbb{R} \to \mathbb{R}$ be the function $f_1(x) = \sin^2 x$ and let $f_2\colon \mathbb{R} \to \mathbb{R}$ be the function $f_2(x) = \cos^2 x$. Finally, let $W = \text{Span}(f_1, f_2)$, and notice that W is a subspace of $\mathcal{D}$. Determine, with explanation, whether the following functions are elements of W.
a. The function $g_1\colon \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = 3$.
b. The function $g_2\colon \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = x^2$.
c. The function $g_3\colon \mathbb{R} \to \mathbb{R}$ given by $g_3(x) = \sin x$.
d. The function $g_4\colon \mathbb{R} \to \mathbb{R}$ given by $g_4(x) = \cos 2x$.

Problem 5: Let V be a vector space, and let W be a subspace of V. Recall that

$$V \backslash W = \{\vec{v} \in V : \vec{v} \notin W\},$$

i.e. $V \backslash W$ is the set of elements of V that are *not* in W. Is $V \backslash W$ always a subspace of V? Sometimes a subspace of V? Never a subspace of V? Explain.

Problem 6: Let $\mathcal{F}$ be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Let W be the set of all even functions, i.e.

$$W = \{f \in \mathcal{F} : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}.$$

Is W a subspace of $\mathcal{F}$? Either prove or give a counterexample.