Assignment: Problem Set 7

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Due Date: 02/21/2018

List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: In each of the following cases, determine if the given function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. If yes, explain why. If no, provide an explicit counterexample.

a. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} xy \\ x+y \end{pmatrix}$

Solution: T is not a linear transformation by definition because it does not preserve addition. Let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ be arbitrary, and $\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Note that:

$$\begin{aligned} T\left(\vec{v_1} + \vec{v_2}\right) = T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) =& T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) \\ =& \begin{pmatrix} (x_1 + x_2) \cdot (y_1 + y_2) \\ (x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \\ =& \begin{pmatrix} x_1y_1 + x_2y_2 + x_2y_1 + x_1y_2 \\ x_1 + y_1 + x_2 + y_2 \end{pmatrix}, \end{aligned}$$

whereas

$$\begin{aligned} T\left(\vec{v_1}\right) + T\left(\vec{v_2}\right) =& T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \\ = \begin{pmatrix} x_1y_1 \\ x_1 + y_1 \end{pmatrix} + \begin{pmatrix} x_2y_2 \\ x_2 + y_2 \end{pmatrix} \\ =& \begin{pmatrix} x_1y_1 + x_2y_2 \\ x_1 + y_1 + x_2 + y_2 \end{pmatrix} \end{aligned}$$

So we have that $T\left(\vec{v_1} + \vec{v_2}\right) \neq T\left(\vec{v_1}\right) + T\left(\vec{v_2}\right)$, and thus T does not conserve addition. Because $\vec{v_1}, \vec{v_2}$ were arbitrary, the result follows.

b. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y\sin^2(x^3) + y\cos^2(x^3) \\ y \end{pmatrix}$

Solution: T is not a linear transformation because T does not preserve scalar multiplication. Let $\vec{v} \in \mathbb{R}^2, r \in \mathbb{R}$ be arbitrary, and $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Note that:

$$\begin{aligned} T(r \cdot \vec{v}) = T(r \cdot \begin{pmatrix} x \\ y \end{pmatrix}) =& T(\begin{pmatrix} r \cdot x \\ r \cdot y \end{pmatrix}) \\ =& \begin{pmatrix} (ry)\sin^2((rx)^3) + (ry)\cos^2((rx)^3) \\ ry \end{pmatrix} \\ =& \begin{pmatrix} ry\sin^2(r^3x^3) + ry\cos^2(r^3x^3) \\ ry \end{pmatrix} \end{aligned}$$

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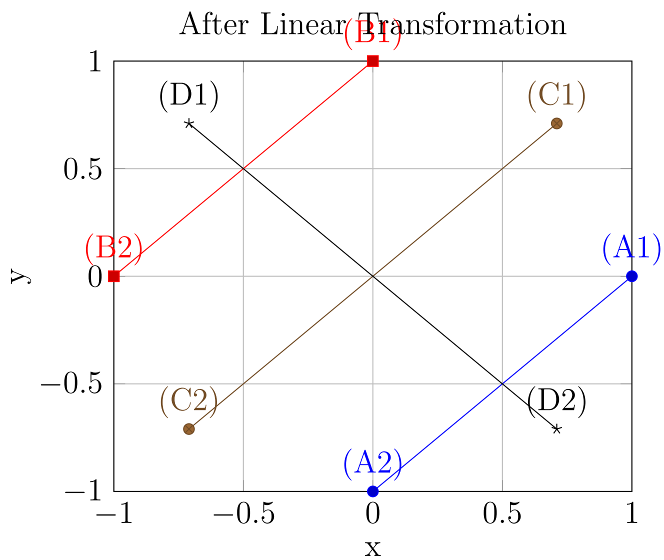
Problem 2: Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ -x \end{pmatrix}.$$

Plot the values of at least 4 points and where T sends them, and then use that to describe the action of T geometrically.

Solution: We plot choose 4 points $(1, 0), (0, 1), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and call them A1, B1, C1, and D1 respectfully. We plot where T sends them below, denoted by A2, B2, C2, and D2:



It seems to be the case that T is "flipping" points over the line $y = x$ about the origin. This is important because it means that points that lie on that line are also "flipped" but along the line and across the origin instead of some other point on the line.

Problem 3: Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}$$

is not injective.

Solution: We assume that T is injective. Note that:

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) &= \begin{pmatrix} (1) + 2(2) \\ 3(1) + 6(2) \end{pmatrix} \\ &= \begin{pmatrix} 1 + 4 \\ 3 + 12 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 15 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} T\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) &= \begin{pmatrix} (-1) + 2(3) \\ 3(-1) + 6(3) \end{pmatrix} \\ &= \begin{pmatrix} -1 + 6 \\ -3 + 18 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 15 \end{pmatrix}. \end{aligned}$$

So we have that $T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = T\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right)$, and note that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. However, we had assumed that T was injective, that is that whenever $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ satisfy $T(\vec{v_1}) = T(\vec{v_2})$, we have $\vec{v_1} = \vec{v_2}$. But we have just found two vectors in the domain of T that do not satisfy the definition of injective, so it must be the case that T is not injective.

Problem 4: Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ -5x + 3y \end{pmatrix}.$$

Show that

$$\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in \text{ range}(T)$$

by explicitly finding $\vec{v} \in \mathbb{R}^2$ with

$$T(\vec{v}) = \begin{pmatrix} -18 \\ 47 \end{pmatrix}.$$

Solution: If $\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in \text{ range}(T)$, then there exists a $\vec{v} \in \mathbb{R}^2$ with $T(\vec{v}) = \begin{pmatrix} -18 \\ 47 \end{pmatrix}$. So we solve for $\vec{v}$. We have:

$$T(\vec{v}) = T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ -5x + 3y \end{pmatrix} = \begin{pmatrix} -18 \\ 47 \end{pmatrix}$$

So we solve for $\begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} 2x - y \\ -5x + 3y \end{pmatrix} = \begin{pmatrix} -18 \\ 47 \end{pmatrix}$. We start with the first component of the vector:

$$\begin{aligned} 2x - y &= -18 \\ 2x + 18 &= y \end{aligned}$$

and we substitute this y into the other component,

$$\begin{aligned} -5x + 3(2x + 18) &= 47 \\ -5x + 6x + 54 &= 47 \\ x = 47 - 54 &= -7 \end{aligned}$$

We have found $x = -7$. Now we solve for y:

$$y = 2 * (-7) + 18 = -14 + 18 = 4$$

We check our answer:

$$\begin{aligned} T\left(\begin{pmatrix} -7 \\ 4 \end{pmatrix}\right) &= \begin{pmatrix} 2(-7) - (4) \\ -5(-7) + 3(4) \end{pmatrix} \\ &= \begin{pmatrix} -14 - 4 \\ 35 + 12 \end{pmatrix} \\ &= \begin{pmatrix} -18 \\ 47 \end{pmatrix} \end{aligned}$$

We have found a $\vec{v} \in \mathbb{R}^2$, namely $\begin{pmatrix} -7 \\ 4 \end{pmatrix}$, such that $T(\vec{v}) = \begin{pmatrix} -18 \\ 47 \end{pmatrix}$. Therefore, $\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in$ range(T).

Problem 5: Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear transformations. Show that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

Solution: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be arbitrary linear transformations. Let $\vec{v_1}, \vec{v_2}, \vec{w} \in \mathbb{R}^2$ be arbitrary vectors. Let $c \in \mathbb{R}$ be an arbitrary scalar. If $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then by the definition of a linear transformation, we have the following:

1. $(T \circ S)(\vec{v_1} + \vec{v_2}) = (T \circ S)(\vec{v_1}) + (T \circ S)(\vec{v_2})$

2. $(T \circ S)(c \cdot \vec{w}) = c \cdot (T \circ S)(\vec{w})$

We first show that *1.* is true.

$$\begin{aligned}
(T \circ S)(\vec{v_1} + \vec{v_2}) =& T(S(\vec{v_1} + \vec{v_2})) && \text{(By definition of function composition)} \\
=& T(S(\vec{v_1}) + S(\vec{v_2})) && \text{(By definition of linear transformation)} \\
=& T(S(\vec{v_1})) + T(S(\vec{v_2})) && \text{(By definition of linear transformation)} \\
=& (T \circ S)(\vec{v_1}) + (T \circ S)(\vec{v_2}) && \text{(By definition of function composition)}
\end{aligned}$$

Therefore, $(T \circ S)(\vec{v_1} + \vec{v_2}) = (T \circ S)(\vec{v_1}) + (T \circ S)(\vec{v_2})$, and the first condition is satisfied. Now we show that *2.* is true.

$$\begin{aligned}
(T \circ S)(c \cdot \vec{w}) =& T(S(c \cdot \vec{w})) && \text{(By definition of function composition)} \\
=& T(c \cdot S(\vec{w})) && \text{(By definition of linear transformation)} \\
=& c \cdot T(S(\vec{w})) && \text{(By definition of linear transformation)} \\
=& c \cdot (T \circ S)(\vec{w}) && \text{(By definition of function composition)}
\end{aligned}$$

Therefore, $(T \circ S)(c \cdot \vec{w}) = c \cdot (T \circ S)(\vec{w})$, and the second condition is satisfied. Both conditions have been satisfied, therfore the function $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

whereas

$$
\begin{aligned}
r \cdot T(\vec{v}) =& r \cdot \begin{pmatrix} y\sin^2(x^3) + y\cos^2(x^3) \\ y \end{pmatrix} \\
=& \begin{pmatrix} ry\sin^2(x^3) + ry\cos^2(x^3) \\ ry \end{pmatrix}
\end{aligned}
$$

So we have that $T(r \cdot \vec{v}) \neq r \cdot T(\vec{v})$, and thus T does not conserve scalar multiplication. Because $r, \vec{v}$ were arbitrary, the result follows.

c. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ 1 + y \end{pmatrix}$

Solution: T is not a linear transformation because T does not preserve scalar multiplication. Let $\vec{v} \in \mathbb{R}^2, r \in \mathbb{R}$ be arbitrary, and $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Note that:

$$
\begin{aligned}
T(r \cdot \vec{v}) = T(r \cdot \begin{pmatrix} x \\ y \end{pmatrix}) =& T(\begin{pmatrix} r \cdot x \\ r \cdot y \end{pmatrix}) \\
=& \begin{pmatrix} 2(rx) + 3(ry) \\ 1 + (ry) \end{pmatrix} \\
=& \begin{pmatrix} 2rx + 3ry \\ 1 + ry \end{pmatrix},
\end{aligned}
$$

whereas

$$
\begin{aligned}
r \cdot T(\vec{v}) =& r \cdot \begin{pmatrix} 2x + 3y \\ 1 + y \end{pmatrix} \\
=& \begin{pmatrix} r(2x + 3y) \\ r(1 + y) \end{pmatrix} \\
=& \begin{pmatrix} 2rx + 3ry \\ r + ry \end{pmatrix}.
\end{aligned}
$$

So we have that $T(r \cdot \vec{v}) \neq r \cdot T(\vec{v})$, and thus T does not conserve scalar multiplication. Because $r, \vec{v}$ were arbitrary, the result follows.