Homework Assignment 1

PHYSICS 314 - THERMODYNAMICS & STATISTICAL PHYSICS (Spring 2018)

Due Friday, February 2nd, by noon, Noyce 1135

I cannot award full credit for work that I am unable to read or follow. For my benefit and for yours, please:

- *Write neatly*
- *Show and EXPLAIN all steps*
- *Make diagrams large and clearly-labeled*

You are welcome to collaborate with others on this assignment. However, the work you turn in should be your own. Please cite collaborators and outside sources. See the syllabus for details.

Regardless of the number of parts, all homework problems are weighted equally. Regardless of the number of questions, all homework assignments are weighted equally.

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1) In the Fahrenheit scale, water freezes at a temperature of 32° F and boils at a temperature of 212° F.

   a) Derive an equation to convert from Fahrenheit to Celsius and one to convert back in the opposite direction. The two scales are linear.

   Since both scales are linear, we can write the Fahrenheit temperature (F) in terms of the Celsius temperature (C) using $F = mC + b$, where *m* and *b* are constants to be determined.

   Water freezes at C = 0 and F = 32.

$$32 = m(0) + b$$

$$32 = b$$

   Water boils at C = 100 and F =212. Use the result for b.

$$212 = m100 + 32$$

$$180 = m100$$

$$\frac{9}{5} = m$$

   Combine these results.

$$F = \frac{9}{5}C + 32$$

   Now solve for C.

$$F - 32 = \frac{9}{5}C$$

$$\frac{5}{9}F - \frac{160}{9} = C$$

   b) The normal body temperature of a human is 98.6° F. What is this temperature in Celsius? In kelvin?

   Use the result from a).

$$\frac{5}{9}F - \frac{160}{9} = C$$

$$\frac{5}{9}(98.6) - \frac{160}{9} = C$$

$$C = 37^oC$$

The Kelvin scale is also linear.

$$K = xC + b$$

The degree separation on the Celsius and Kevin scales are the same, so x=1. Also, absolute zero (0 K) is -273.15°C.

$$K = C + 273.15$$

Plug in the Celsius body temperature.

$$K = 37 + 273.15$$

$$K = 310.15\ K$$

2) See equation 1.40 in Schroeder (pg. 26). For an ideal gas undergoing adiabatic compression, determine the relationships between the following parameters. Each answer should be in terms of the adiabatic exponent, γ, and a constant.

a) V and T

Start with equation 1.40.

$$V^\gamma P = c_1$$

c_1 is a constant.

Use the ideal gas law to substitute in for P.

$$V^\gamma \left(\frac{nRT}{V}\right) = c_1$$

$$nRV^{\gamma-1}T = c_1$$

nR is a constant, so we can divide both sides by nR.

$$V^{\gamma-1}T = c_2$$

c_2 is a new constant.

$$c_2 = \frac{c_1}{nR}$$

b) P and T

Start with equation 1.40.

$$V^\gamma P = c_1$$

Use the ideal gas law to substitute in for V.

$$\left(\frac{nRT}{P}\right)^\gamma P = c_1$$

$$(nR)^\gamma T^\gamma P^{1-\gamma} = c_1$$

nR^γ is a constant, so we can divide both sides by nR^γ.

$$T^\gamma P^{1-\gamma} = c_3$$

c_3 is a new constant.

$$c_3 = \frac{c_1}{(nR)^\gamma}$$

3) Thermal Expansion

a) For a liquid, the fractional increase in volume per unit change in temperature (with fixed pressure) is called the *thermal expansion coefficient*, β.

$$\beta \equiv \frac{\Delta V/_V}{\Delta T}$$

V is the volume. T is the temperature. Δ is a (technically *infinitesimal)* change.

When the temperature of liquid mercury increases by one kelvin, its volume increase by one part in 5500. Thus, $\beta_{Hg} = {}^{1}/_{5500K}$. (This value actually changes a small amount with temperature, but the variation is less than a percent over a couple hundred degrees in the temperature range of interest, so it can be ignored.)

A mercury thermometer uses this thermal expansion to measure temperature. Suppose such a thermometer has a bulb at the bottom that is full of mercury. Most of the mercury is contained in the bulb, but as this mercury expands, a very small amount travels up a narrow tube that is marked with temperatures.

Suppose that the bulb at the bottom contains 60 mm^3. Also suppose that a 1 degree ***Celsius*** temperature change changes the height of the mercury in the cylindrical tube by 8.5 mm. What is the radius of the narrow inner tube?

Rearrange the definition of the thermal expansion coefficient.

$$\beta = \frac{\Delta V/_V}{\Delta T}$$

$$\Delta V = \beta \times V \times \Delta T$$

This ΔV occurs in the narrow tube, so we write it in terms of the fixed cross-sectional area ($A=\pi r^2$) and the change in height (*h*).

$$\Delta V = \pi r^2 \times \Delta h = \beta \times V \times \Delta T$$

Solve for r and plug in the known values.

$$r = \sqrt{\frac{\beta \times V \times \Delta T}{\pi \times \Delta h}}$$

$$r = \sqrt{\frac{{}^{1}/_{5500K} \times 60\ mm^3 \times 1K}{\pi \times 8.5mm}}$$

$$r \approx 0.02mm$$

b) Suppose that now that we have two new thermometers. Their dimensions are unknown. However, they are identical, except that one is filled with mercury and one is filled with the mercury-alternative toluene. For a temperature change of 0.5 degrees Celsius, the height of the mercury changes by 4 mm. By how much does the height of the toluene change?

$$\beta_{toluene} = 0.001K^{-1}$$

As in a), we can use the definition of thermal expansion.

$$\Delta V = \pi r^2 \times \Delta h = \beta \times V \times \Delta T$$

Note that Δh is proportional to β. (*r*, *V*, and *ΔT* are constant between the two identical thermometers.)

$$\frac{\Delta h_{Hg}}{\beta_{Hg}} = \frac{\Delta h_{toluene}}{\beta_{toluene}}$$

Solve for the height change in toluene.

$$\Delta h_{toluene} = \frac{\Delta h_{Hg}}{\beta_{Hg}} \beta_{toluene}$$

$$\Delta h_{toluene} = \frac{4mm}{{}^{1}/_{5500K}} 0.001K^{-1}$$

$$\Delta h_{toluene} = 22mm$$

c) For a solid, the *linear thermal expansion coefficient*, α, is the fractional increase in length per degree change in temperature.

$$\alpha \equiv \frac{{}^{\Delta L}/_{L}}{\Delta T}$$

A dial thermometer uses a coiled metal strip made out of two different metals laminated together. Qualitatively explain in a few sentences how this could be used as a thermometer.

The thermal expansion coefficients of the two metals are different. When the temperature changes, one expands more than the other. This causes a stress which causes the coils to coil further (or less), and thus to turn. The temperature is thus indicated by how far the coil turns.

d) Show that the volume thermal expansion coefficient of a solid is *approximately* equal to the sum of its linear expansion coefficients in three directions.

$$\beta = \alpha_x + \alpha_y + \alpha_z$$

Hint: Consider a rectangular prism. Choose one of the (at least) two ways to do this. (1) Consider a differential form of the definition of the thermal expansion coefficients. (2) Calculate the change in volume from the prism expanding in each of the three dimensions, ignoring very small contributions.

(1)

Use differential forms of the expansion definitions. (The derivatives are partial derivatives with pressure held constant.)

$$\beta = \frac{1}{V}\frac{dV}{dT}$$

$$\alpha = \frac{1}{L}\frac{dL}{dT}$$

Evaluate the first derivative using the volume of a prism.

$$\beta = \frac{1}{V}\frac{d}{dT}(L_x L_y L_z)$$

Use the product rule.

$$\beta = \frac{1}{V}\left[\frac{dL_x}{dT}(L_y L_z) + \frac{dL_y}{dT}(L_x L_z) + \frac{dL_z}{dT}(L_y L_x)\right]$$

Use the definition of the linear expansion to replace the derivative terms.

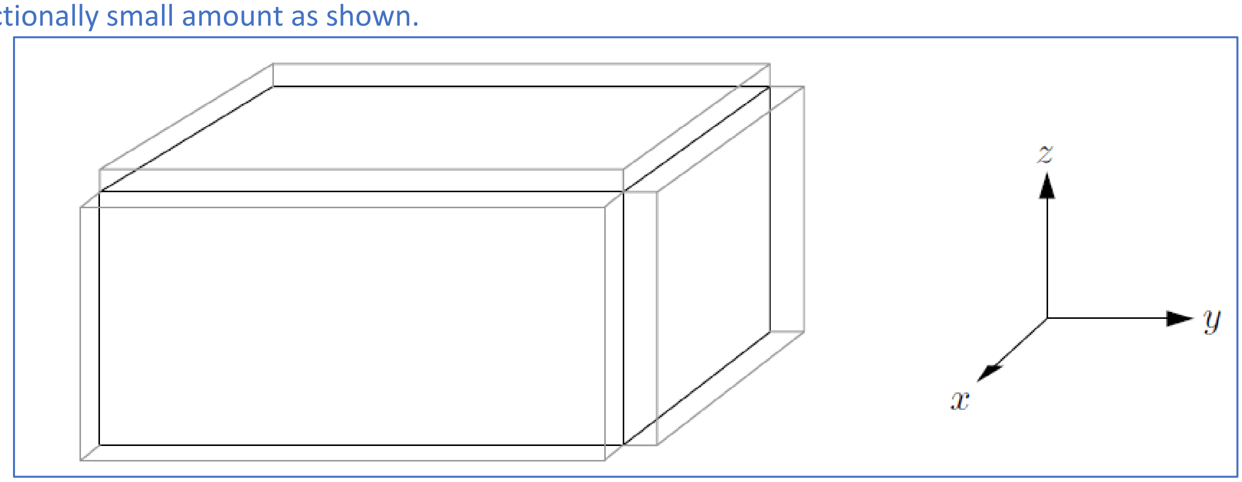
$$\beta = \frac{1}{V}\left[\alpha_x L_x (L_y L_z) + \alpha_y L_y (L_x L_z) + \alpha_z L_z (L_y L_x)\right]$$

$$\beta = \frac{1}{V}\left[\alpha_x + \alpha_y + \alpha_z\right] L_x L_y L_z$$

$$\beta = \left[\alpha_x + \alpha_y + \alpha_z\right]$$

(2)

Use a rectangular prism of sides L_x, L_y, and L_z. As the temperature increases, the volume increases by a fractionally small amount as shown.



Note that we've ignored the strips along the edges. Since the change in length is very small, the change in volume from these strips contains the product of two very small terms, and thus is extremely small. Thus we will ignore their contribution to the volume change. With this simplification, we can approximate the volume.

$$\Delta V = (\Delta L_x) \times L_y \times L_z + (\Delta L_y) \times L_x \times L_z + (\Delta L_z) \times L_y \times L_x$$

Now rewrite the ΔL terms using the definition of linear expansion.

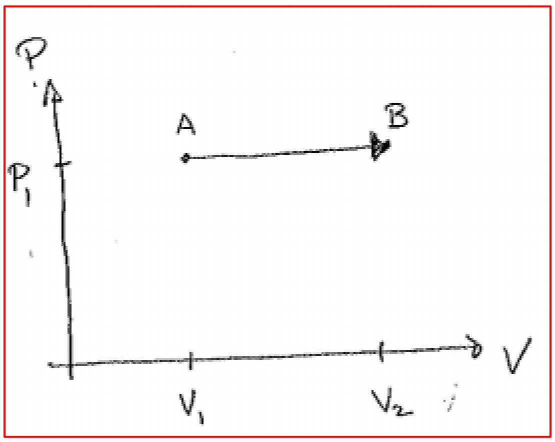
$$\Delta V = (\alpha_x L_x \Delta T) \times L_y \times L_z + (\alpha_y L_y \Delta T) \times L_x \times L_z + (\alpha_z L_z \Delta T) \times L_y \times L_x$$

$$\Delta V = (\alpha_x + \alpha_y + \alpha_z) \times L_x \times L_y \times L_z \times \Delta T$$

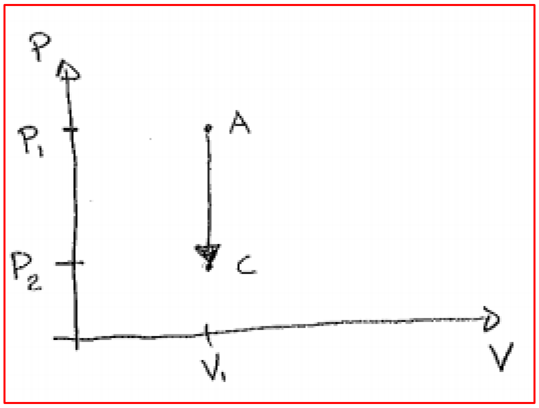
$$\Delta V = (\alpha_x + \alpha_y + \alpha_z) \times V \times \Delta T$$

Compare this to the expression from part a). This is the volume expansion definition with $\beta = (\alpha_x + \alpha_y + \alpha_z)$.

4) Consider one mole of an ideal gas.
   a) Depict each of the following processes on a separate *P*–*V* diagram.
      i) An *isobaric* process (one that takes place at a constant pressure) carrying the gas from an initial state *A* to a final state *B*.

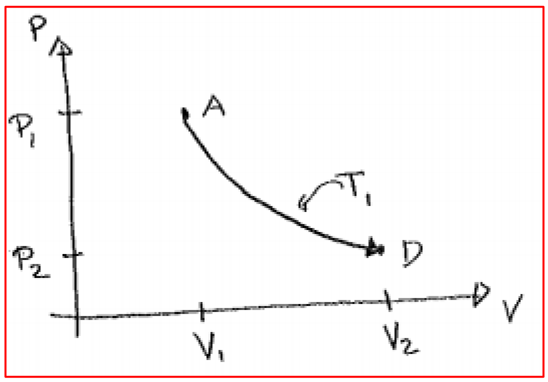


      ii) An *isochoric* process (one that takes place at a constant volume) carrying the gas from an initial state *A* to a final state *C*.

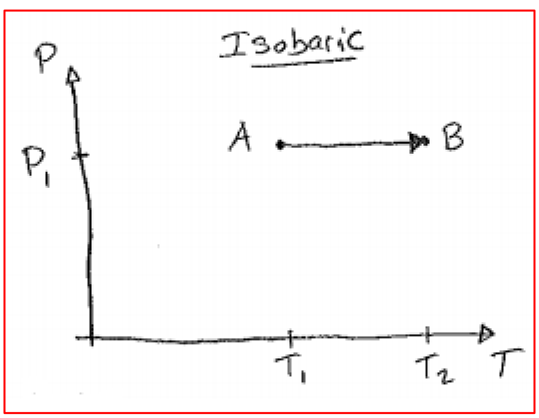


iii) An *isothermal* process (one that takes place at a constant temperature) carrying the gas from an initial state *A* to a final state *D.*

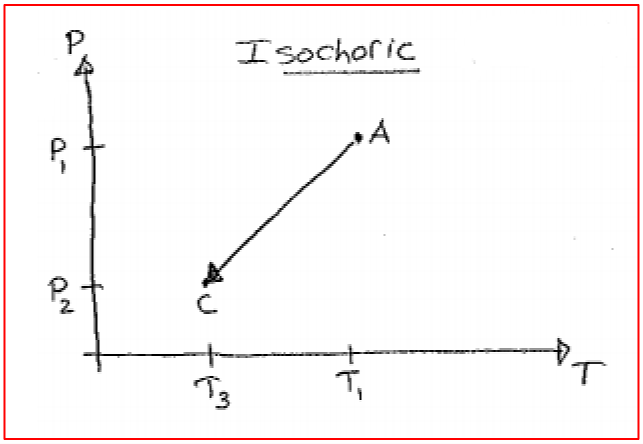
By the ideal gas law, with constant T, $P \propto \frac{1}{V}$.

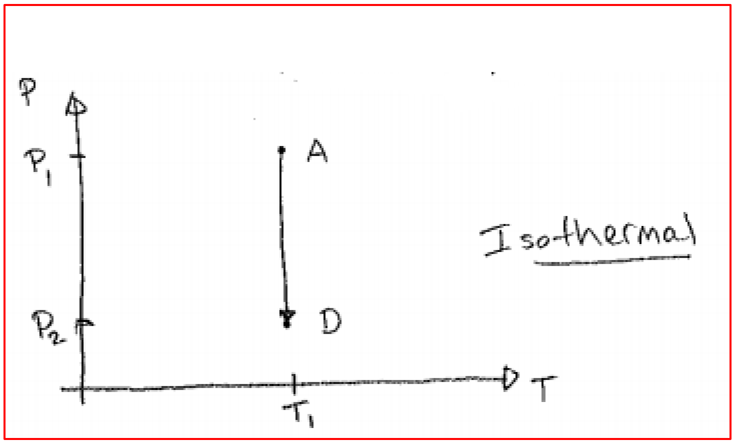


b) Depict these same three processes (isobaric, isochoric, and isothermal) on separate *P–T* diagrams.



By the ideal gas law, with constant V, $P \propto T$.





5) A horizontal cylinder, closed off by a piston, contains a monatomic ideal gas (adiabatic exponent $\gamma = 5/3$) with pressure P_0. Its initial volume is V_0 and its initial temperature is T_0. You wish to reduce the volume of the gas by a factor of 2, and you consider a few different ways of doing so.
   a) Calculate the work done on the system if you reduce the volume by a factor of 2 with a process that is:
      i) Isothermal

$$W = -\int_{V_o}^{V_o/2} PdV$$

Replace P using the ideal gas law.

$$W = -\int_{V_o}^{V_o/2} \frac{nRT_0}{V} dV$$

Temperature is constant. Pull out constants.

$$W = -nRT_0 \int_{V_o}^{V_o/2} \frac{dV}{V}$$

$$W = -nRT_0 \ln V|_{V_0}^{V_o/2}$$

$$W = -nRT_0 \left(\ln\left[\frac{V_0}{2}\right] - \ln[V_0]\right)$$

Use log rules to simplify.

$$W = -nRT_0 \ln\left[\frac{V_0/2}{V_0}\right]$$

$$W = -nRT_0 \ln\left[\frac{1}{2}\right]$$

Use log rules to simplify and ideal gas law to simplify.

$$W = nRT_0 \ln[2] = P_0 V_0 \ln[2]$$

ii) Isobaric

$$W = -\int_{V_o}^{V_o/2} P dV$$

Pressure is constant.

$$W = -P_0 \int_{V_o}^{V_o/2} dV$$

$$W = -P_0 V|_{V_0}^{V_o/2}$$

$$W = -P_0 V|_{V_0}^{V_o/2}$$

$$W = -P_0 \left({}^{V_0}\!/_2 - V_0\right)$$

$$W = \frac{P_0 V_0}{2}$$

iii) Adiabatic

The process is adiabatic, so Q =0. Thus, by the first law, W = ΔU.

Calculate U using the equipartition theorem, $U = \frac{f}{2} NkT$. f = 3 for a monoatomic ideal gas.

$$W = \Delta U = U_f - U_i$$
$$W = \frac{3}{2} NkT_f - \frac{3}{2} NkT_i$$

Use the ideal gas law to write this in terms of P and V.

$$W = \frac{3}{2} P_f V_f - \frac{3}{2} P_0 V_0$$

$$W = \frac{3}{2} P_f \frac{V_0}{2} - \frac{3}{2} P_0 V_0$$

$$W = \frac{3}{4} P_f V_0 - \frac{3}{2} P_0 V_0$$

Need to get expression for the final pressure. Use the fact about adiabats from equation 1.40.

$$V^{\gamma} P = constant$$

$$V_f{}^{\gamma} P_f = V_0{}^{\gamma} P_0$$

Sub in known final volume and solve for final pressure.

$$\left(\frac{V_0}{2}\right)^{\gamma} P_f = V_0{}^{\gamma} P_0$$

$$P_f = 2^{\gamma} P_0$$

Plug in the given adiabatic constant (or calculate it with $\gamma = \frac{f+2}{f}$).

$$P_f = 2^{5/3} P_0$$

Plug this into the expression for work above.

$$W = \frac{3}{4} 2^{5/3} P_0 V_0 - \frac{3}{2} P_0 V_0$$

$$W = \left(3 \times 2^{-1/3} - \frac{3}{2}\right) P_0 V_0$$

b) Rank the three different processes in order of increasing work done on the system.

Compare answers from part a).

$$\ln 2 \approx 0.69$$

$$\frac{1}{2} = 0.5$$

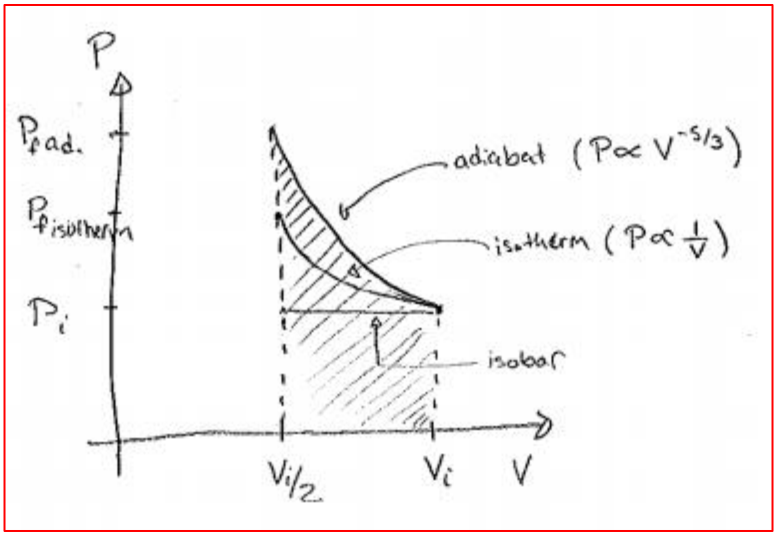
$$\left(3 \times 2^{-1/3} - \frac{3}{2}\right) \approx 0.88$$

$$W_{adiabat} > W_{isotherm} > W_{isobar}$$

The adiabatic process requires the most work.

c) Sketch each of these three processes on a single P-V diagram, and comment on how this illustrates the result of part b).

The isobar is at constant pressure. The isotherm has constant T, so by the ideal gas law, $P \propto \frac{1}{V}$. From equation 1.40 from the text, for the adiabat $P \propto V^{5/3}$.



The work done on the system is given by the following integral.

$$W = -\int_{V_o}^{V_o/2} PdV$$

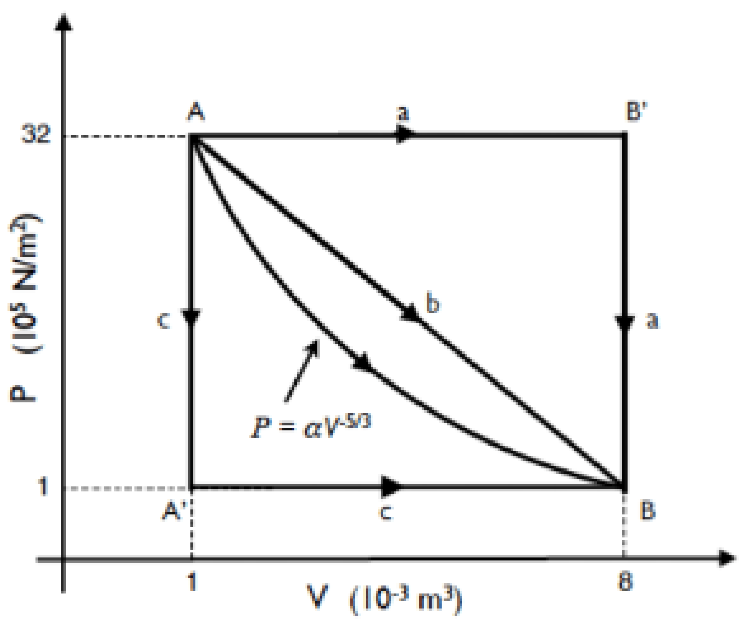
This is just the area under the curve.* The largest area under the curve is the adiabat, then the isotherm, then the isobar. This matches our answer from b).

*Wait! Why isn't the opposite of the area under the curve? Careful! There are two negative signs here. There is the explicit one out front, and then also another implied one because we are integrating from right to left.

6) In a process A → B in which no heat is exchanged with the environment, the pressure P of a certain amount of gas is found to change with its volume V according to the relation:

$$P = \alpha V^{-5/3}$$

where α is a constant. This process, along with three other processes (a, b, and c) that take the gas from state A to state B, is shown in the diagram below.



Now consider the other three processes shown. Find the work done on this system ***and*** the net heat absorbed by this system in each case. *Note the labels on the diagram; give numerical answers.*

We are told that in the process that follows $P = \alpha V^{-5/3}$, no heat is transferred. (The process is adiabatic.) Since Q = 0, the first law yields $W = \Delta U$. (Note: This is true only for the adiabatic path, not in general.) Use this to calculate ΔU. Because all points have the same initial and final states, they have the same ΔU. (ΔU is path independent.)

$$\Delta U = W = -\int_A^B P dV$$

Plug in the expression for pressure for the adiabat.

$$\Delta U = -\int_A^B \alpha V^{-5/3} dV$$

$$\Delta U = \alpha \frac{3}{2} V^{-2/3} \Big|_{V_A}^{V_B}$$

$$\Delta U = \alpha \frac{3}{2} \left(V_B^{-2/3} - V_A^{-2/3}\right)$$

To calculate α, use any state's pressure-volume pair. (Here we use A.)

$$P = \alpha V^{-5/3}$$

$$\alpha = P_A V_A^{5/3}$$

Insert this above.

$$\Delta U = \frac{3}{2} P_A V_A^{5/3} \left(V_B^{-2/3} - V_A^{-2/3}\right)$$

Use the values from the plot.

$$\Delta U = \frac{3}{2}\left(32 \times 10^5 \, {}^{N}\!/_{m^2}\right)(1 \times 10^{-3} m^3)^{5/3}\left[(8 \times 10^{-3} m^3)^{-2/3} - (1 \times 10^{-3} m^3)^{-2/3}\right]$$

$$\Delta U = -3600 \, J$$

Recall that this is path independent; it is true for parts a-c.

a) The system is expanded from its original to its final volume, with heat being added to maintain a constant pressure. The volume is then kept constant, and heat is extracted to reduce the pressure to 10^5 N m^{-2}.

First find the work. Break this process up into two steps.

$$W = -\int_A^B P dV$$

$$W = -\int_A^{B'} PdV + -\int_{B'}^{B} PdV$$

Note that the volume is constant from B' to B, so the second integral is zero.

$$W = -\int_A^{B'} PdV$$

For the first integral, P is constant.

$$W = -P_A \int_A^{B'} dV$$

$$W = -P_A V|_{V_A}^{V_{B'}}$$

$$W = -P_A(V_{B'} - V_A)$$

$$W = -\left(32 \times 10^5 \,{}^N\!/_{m^2}\right)[(8 \times 10^{-3}m^3) - (1 \times 10^{-3}m^3)]$$

$$W = -22{,}400J$$

Use the first law to find the heat.

$$\Delta U = W + Q$$

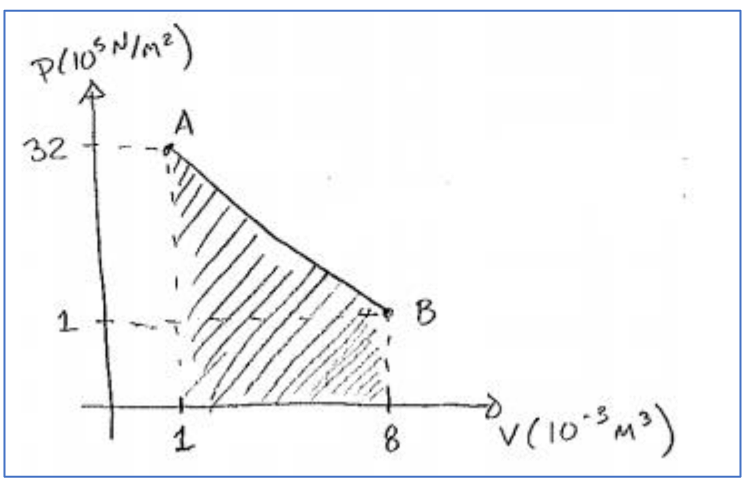
$$Q = -W + \Delta U$$

Plug in the W from above and the path independent ΔU.

$$Q = 22{,}400J - 3600J$$

$$Q = 18{,}800J$$

b) The volume is increased, and heat is supplied to cause the pressure to decrease linearly with volume.



The work is given by the opposite of the area under the curve. Rather than bothering with integration (which would work), just calculate the area by adding the triangle to the rectangle.

$$W = -\int_{A}^{B} PdV$$

$$W = -\left[\frac{1}{2}(7 \times 10^{-3} m^3)\left(31 \times 10^5 \,{}^{N}\!/_{m^2}\right) + (7 \times 10^{-3} m^3)\left(1 \times 10^5 \,{}^{N}\!/_{m^2}\right)\right]$$

$$W = -11{,}550J$$

Again use the first law to find the heat. ΔU is the same as above because it is path independent.

$$Q = -W + \Delta U$$

$$Q = 11{,}550J - 3600J$$

$$Q = 7950J$$

c) The two steps of process a) are performed in the opposite order.

First find the work. Break this process up into two steps.

$$W = -\int_{A}^{B} PdV$$

$$W = -\int_{A}^{A'} PdV + -\int_{A'}^{B} PdV$$

Note that the volume is constant from A' to A, so the first integral is zero.

$$W = -\int_{A'}^{B} PdV$$

For the remaining integral, P is constant.

$$W = -P_B \int_{A'}^{B} dV$$

$$W = -P_B V|_{V_{A'}}^{V_B}$$

$$W = -P_B(V_B - V_{A'})$$

$$W = -\left(1 \times 10^5 \,{}^{N}\!/_{m^2}\right)[(8 \times 10^{-3} m^3) - (1 \times 10^{-3} m^3)]$$

$$W = -700J$$

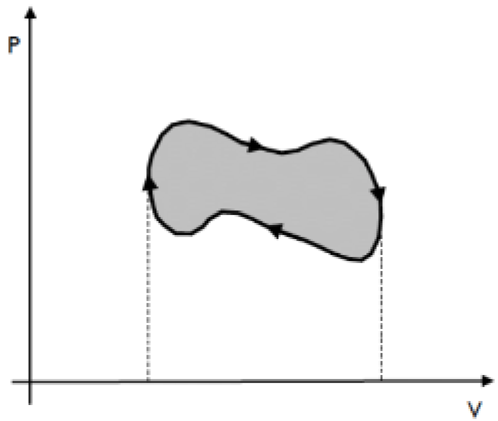
Again use the first law to find the heat. ΔU is the same as above because it is path independent.

$$Q = -W + \Delta U$$

$$Q = 700J - 3600J$$

$$Q = -2{,}900J$$

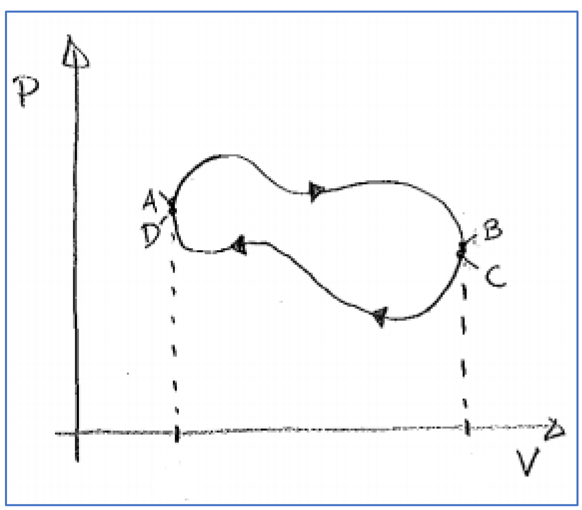
7) A system undergoes the process that appears in the P-V diagram below. Such a process is called 'cyclic' since the system ends up in a final state that is identical to its initial state; it is a closed loop. Show that the work done ***by*** the system is given by the area contained within the closed curve. Use a combination of labeled sketches, words, and symbolic math to explain.



First, note that it is the work done BY this system. This is the opposite of we are generally asked for (the work done ON the system). This gives us an additional negative sign.

$$W_{cycle} = \int PdV$$

The integral is taken over the whole cycle in the direction shown. Break the cycle down into two sections as shown.



Note that point A and point D are actually the same point. Point B and C are also the same point. They are written like this to emphasize the two paths. Call A to B the higher pressure upper path (with increasing volume). Call C to D the lower pressure lower path (with decreasing volume).

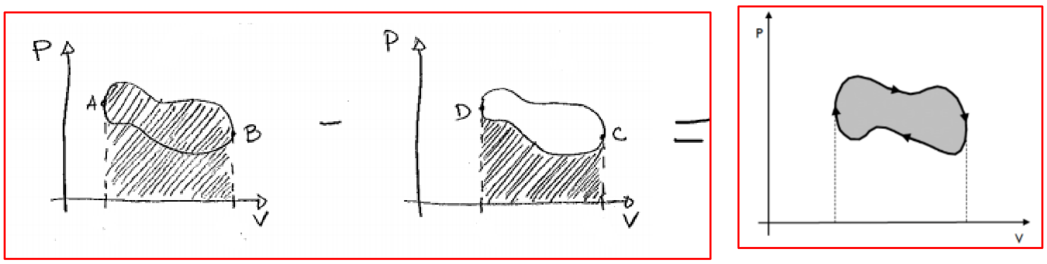
$$W_{cycle} = \int_A^B PdV + \int_C^D PdV$$

The first integral is just the area under the upper path from A to B (because the volume increases along the path). However, because the second path decreases in volume, it is the *opposite* of the area underneath the lower path. To translate this into an area under the curve, can rewrite the second integral, switching the bounds of integration.

$$W_{cycle} = \int_A^B PdV - \int_D^C PdV$$

Now that these are both area under the curves, the statement can be seen graphically.

$W_{cycle} =$



8) List <u>*three*</u> main ideas from this homework assignment. For example, you could write a few-sentence explanation of a concept, or list an equation and explain the variables and in what circumstances the equation applies.
The goal is for you to review and to reflect on the big picture. Think about what you might want to remember when you look back at this homework before the test. I hope that this will be useful for your studying. I am not looking for anything specific here; you will be graded on effort and completion.