2 Short Answers - 6 points

Precisely describe (using proper terminology) how we can apply results from this class to describe the solution sets of the systems of equations encoded in the following augmented matrices:

- $A = \left(\begin{array}{cccccc|c} 1 & 0 & 1 & 4 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Because the matrix is in echelon form and it has a leading entry in the last column, we know that the system of linear equations that it encodes is <u>inconsistent</u>.

- $B = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{array}\right)$

Because the matrix is in echelon form and has a leading entry in every column except for the last, we know the system of linear equations that it encodes is consistent and has a unique solution.

- $C = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right)$

The matrix is in echelon form and the last column has no leading entries but there are other columns w/out leading entries. So, the encoded system is consistent and has infinitely many solutions.

3 Proofs - 12 points

Prove the following statements. You must formally justify each of your claims by stating the definition or result from which it follows. Do not refer to results by their number in the text, state them completely.

3.1

Let V be a vector space and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w} \in V$. Show that the following are equivalent:

(a) $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}) = \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$

(b) $\vec{w} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$

(a) $\Rightarrow$ (b) Suppose $\mathrm{Span}(u_i\text{'s}, w) = \mathrm{Span}(u_i\text{'s})$, then

$\vec{w} = \sum_{i=1}^{n} 0 \cdot \vec{u}_i + 1\vec{w}$ so $\vec{w} \in \mathrm{Span}(\vec{u}_1, \ldots, \vec{u}_n, \vec{w})$

thus, $\vec{w} \in \mathrm{Span}(\vec{u}_1, \ldots, \vec{u}_n)$ ∎

(b) $\Rightarrow$ (a) Suppose $\vec{w} \in \mathrm{Span}(\vec{u}_1, \ldots, \vec{u}_n)$ (i.e. $\vec{w} = \sum_{i=1}^{n} s_i \vec{u}_i$ $s_i \in \mathbb{R}$)

"$\supseteq$"

$\vec{x} \in \mathrm{Span}(u_i\text{'s}) \Rightarrow \exists d_i$ w/ $\sum_{i=1}^{n} d_i \vec{u}_i = \vec{x} = \sum_{i=1}^{n} d_i u_i + 0 \cdot \vec{w}$

so $\vec{x} \in \mathrm{Span}(\vec{u}_i\text{'s}, \vec{w})$, and $\mathrm{span}(u_i\text{'s}) \subseteq \mathrm{Span}(u_i\text{'s}, \vec{w})$

"$\subseteq$"

Let let $\vec{y} = \left(\sum_{i=1}^{n} c_i \vec{u}_i\right) + d\vec{w}$, $c_i, d \in \mathbb{R}$, be arbitrary.

then $\vec{y} = \sum_{i=1}^{n} c_i u_i + d\left(\sum_{j=1}^{n} s_j \vec{u}_j\right)$

$= \sum_{i=1}^{n} (c_i + d s_i) \vec{u}_i \in \mathrm{Span}(u_1, \ldots, u_n)$

So $\mathrm{span}(\vec{u}_1, \ldots, \vec{u}_n, \vec{w}) \subseteq \mathrm{Span}(\vec{u}_1, \ldots, \vec{u}_n)$ ∎

3.2

Let V be a vector space and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{w} \in V$. Suppose that $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ is linearly independent and that $(\vec{u}_1 + \vec{w}, \vec{u}_2 + \vec{w}, \ldots, \vec{u}_k + \vec{w})$ is linearly dependent. Show that $\vec{w} \in \operatorname{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$.

Notice that $\vec{w} \neq 0$, otherwise $\vec{u}_i + \vec{w} = \vec{u}_i$ for all $i \in \{1, 2, \ldots k\}$ and then $(\vec{u}_1 + \vec{w}, \vec{u}_2 + w, \ldots, \vec{u}_k + w) = (u_1, u_2, \ldots, u_k)$ and cannot be both linearly independent and linearly dependent.

$(\vec{u}_1 + \vec{w}, \ldots, u_k + \vec{w})$ linearly dependent $\Rightarrow \exists c_1, c_2, \ldots, c_k \in \mathbb{R}$ w/ $c_i \neq 0$, for some $i \in \{1, 2, \ldots, k\}$, such that

$$c_1(\vec{u}_1 + \vec{w}) + c_2(\vec{u}_2 + \vec{w}) + \cdots + c_k(\vec{u}_k + w) = \vec{0}$$

Thus

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k + (c_1 + c_2 + \cdots + c_k)\vec{w} = \vec{0}$$

so

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k = (-c_1 - c_2 - \cdots - c_k)\vec{w}.$$

If $-c_1 - c_2 - \cdots - c_k = 0$ then $(-c_1 - \cdots - c_k)\vec{w} = \vec{0}$, and thus $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k = \vec{0}$.

But, $(u_1, u_2, \ldots, u_k)$ is linearly independent so this only happens when $c_i = 0$ for all $i \in \{1, 2, \ldots k\}$ contradicting our assumptions.

So $-c_1 - c_2 - \cdots - c_k = d \neq 0$ and

$$\frac{c_1}{d}\vec{u}_1 + \frac{c_2}{d}\vec{u}_2 + \cdots + \frac{c_k}{d}\vec{u}_k = \omega$$

So $\vec{w} \in \operatorname{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$.

∎

4 Bonus - 1 Point

Consider the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 5 & -3 & -8 & -3 & 6 & 9 & 1 & -1 & -1 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & -5 & 9 & 3 & -9 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 4 & -9 & -1 & -1 & -1 \\ -2 & 0 & -2 & 2 & 0 & -6 & 2 & -2 & 0 & 8 & 0 & -6 & 5 & -1 & 9 \\ 0 & 2 & -7 & 9 & 4 & -9 & -3 & -1 & -6 & -2 & -1 & -9 & 7 & -5 & 3 \\ 0 & 0 & 0 & -5 & -7 & 9 & -2 & -9 & 3 & -8 & 8 & 6 & 4 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 6 & 7 & 9 & 7 & -3 \\ 0 & 0 & 0 & 0 & 0 & -7 & 0 & -8 & -7 & -3 & -7 & -3 & 4 & -1 & 6 \\ 0 & 0 & -2 & -1 & -7 & -7 & -3 & -6 & -5 & 2 & 5 & -7 & 3 & -4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -5 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -2 & 1 & 8 & 7 & -3 & 5 & -5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 2 & 9 & 8 & -1 \end{pmatrix}$$

$\Rightarrow -4x+3y=-5$
$-3y=-6$

$\Rightarrow y=2$ and $x=\frac{11}{4}$

$$B = \begin{pmatrix} 0 & -5 & 3 & -3 & 5 & 9 & -5 & 5 & 0 & 6 & 0 & 2 & -4 & 1 & -4 \\ 0 & 0 & 0 & 0 & -5 & -3 & -9 & 7 & 1 & 5 & -5 & 5 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 6 & -3 & 5 & -7 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 7 & -2 & 1 & 8 & 4 \\ 0 & 0 & 1 & -4 & -4 & -3 & 6 & 5 & 2 & 9 & -1 & -6 & -2 & -1 & -4 \\ -6 & 2 & -9 & -7 & 4 & -1 & 5 & 3 & 9 & 6 & 0 & 9 & -2 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 5 & 1 & -6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -7 & 6 & 8 & -1 & -9 & 8 & -6 & 8 & 3 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & 1 & 9 & -6 & 1 & -6 & 4 & 4 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -6 & 8 & 8 & 8 & -6 & -9 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 7 & -3 & -2 & -3 & 9 & -8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 9 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & -3 \end{pmatrix}$$

$\Rightarrow -5x+y=-3$
$2y=4$

$\Rightarrow y=2$
$x=1 \neq \frac{11}{4}$

Is A row-equivalent to B? Justify your answer.

as the solution sets cannot be the same, if we regard these as augmented matrices, then they cannot be row equivalent because row operations preserve solution sets.