# Assignment: Writing Assignment 3

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## List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

**Problem 1:** Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - y \\ x + y \end{pmatrix}.$$

Is $T$ injective? Justify your answer carefully.

*Solution:* Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$ be arbitrary vectors. If $T$ is injective, then whenever $T(\vec{v}) = T(\vec{w})$, we have that $\vec{v} = \vec{w}$. That is, $T$ is injective if whenever $T(\vec{v}) = T(\vec{w})$ implied that $\vec{v} = \vec{w}$. We assume that $T$ is injective. So we have:

$$\begin{aligned} T(\vec{v}) =& T(\vec{w}) \\ \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} =& \begin{pmatrix} w_1 - w_2 \\ w_1 + w_2 \end{pmatrix} && \text{(by the definition of } T) \end{aligned}$$

We solve for $v_1, v_2$. Taking the first component, we manipulate to express $v_1$ in terms of $v_2, w_1, w_2$:

$$\begin{aligned} v_1 - v_2 &= w_1 - w_2 \\ v_1 &= w_1 - w_2 + v_2, \end{aligned}$$

and substitute for $v_1$ in the bottom component:

$$\begin{aligned} v_1 + v_2 =& w_1 + w_2 \\ (w_1 - w_2 + v_2) + v_2 =& w_1 + w_2 \\ w_1 - w_2 + 2v_2 =& w_1 + w_2 \\ 2v_2 =& w_1 + w_2 - (w_1 - w_2) \\ 2v_2 =& 2w_2 \\ v_2 =& w_2 \end{aligned}$$

So $v_2 = w_2$. Substituting back in for $v_2$ in the top component, we have:

$$\begin{aligned} v_1 - (w_2) =& w_1 - w_2 \\ v_1 =& w_1 \end{aligned}$$

Our assumption has lead us to the conclusion that $v_1 = w_1, v_2 = w_2$. So $\vec{v} = \vec{w}$, therefore $T$ is injective by definition. Because $\vec{v}, \vec{w}$ were arbitrary, the result follows.

**Problem 2:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$$\text{range}(T) = \{\vec{w} \in \mathbb{R}^2 : \text{There exists } \vec{v} \in \mathbb{R}^2 \text{ with } \vec{w} = T(\vec{v})\}.$$

Notice that $\vec{0} \in \text{range}(T)$ because we know that $T(\vec{0}) = \vec{0}$ by Proposition 2.4.2.

a. Show that if $\vec{w_1}, \vec{w_2} \in \text{range}(T)$, then $\vec{w_1} + \vec{w_2} \in \text{range}(T)$.

*Solution:* Let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ be arbitrary. We fix vectors $\vec{w_1}, \vec{w_2}, \vec{v} \in \mathbb{R}^2$ by letting $T(\vec{v_1}) = \vec{w_1}, T(\vec{v_2}) = \vec{w_2}, \vec{v_1} + \vec{v_2} = \vec{v}$. So $\vec{w_1}, \vec{w_2} \in \text{range}(T)$ by definition. Notice that:

$$\begin{aligned} T(\vec{v}) &= T(\vec{v_1} + \vec{v_2}) \\ &= T(\vec{v_1}) + T(\vec{v_2}) && \text{(by definition of linear transformation)} \\ &= \vec{w_1} + \vec{w_2} \end{aligned}$$

So $T(\vec{v}) = \vec{w_1} + \vec{w_2}$. Because, $\vec{w_1} + \vec{w_2} \in \mathbb{R}^2$, $\vec{w_1} + \vec{w_2} \in \text{range}(T)$ by definition. Because $\vec{v_1}, \vec{v_2}$ were arbitrary, the result follows.

b. Show that if $\vec{w} \in \text{range}(T)$ and $c \in \mathbb{R}$, then $c\vec{w} \in \text{range}(T)$.

*Solution:* Let $\vec{v} \in \mathbb{R}^2, c \in \mathbb{R}$ be arbitrary. We fix a vector $\vec{w} \in \mathbb{R}^2$ by letting $T(\vec{v}) = \vec{w}$. So $\vec{w} \in \text{range}(T)$ by definition. Notice that:

$$\begin{aligned} T(c\vec{v}) &= c \cdot T(\vec{v}) && \text{(by definition of linear transformation)} \\ &= c \cdot \vec{w} \\ &= c\vec{w} \end{aligned}$$

So $T(c\vec{v}) = c\vec{w}$. Because, $c\vec{w} \in \mathbb{R}^2$, $c\vec{w} \in \text{range}(T)$ by definition. Because $\vec{w}, c$ were arbitrary, the result follows.

**Problem 3:** We defined linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, but we can also define them from $\mathbb{R}$ to $\mathbb{R}$ as follows. A linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ with both of the following properties:

- $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
- $f(c \cdot x) = c \cdot f(x)$ for all $c, x \in \mathbb{R}$.

a. Let $r \in \mathbb{R}$. Show that the function $g_r : \mathbb{R} \to \mathbb{R}$ given by $g_r(x) = rx$ is a linear transformation.

*Solution:* We check to see if $g_r : \mathbb{R} \to \mathbb{R}$ given by $g_r(x) = rx$ satisfies the above conditions. Let $x, y \in \mathbb{R}$ be arbitrary. Note that:

$$\begin{aligned} g_r(x + y) =& r(x + y) && \text{(by definition of } g_r\text{)} \\ =& rx + ry \\ =& g_r(x) + g_r(y) && \text{(by definition of } g_r\text{)} \end{aligned}$$

So for arbitrary $x, y \in \mathbb{R}$, we have $g_r(x+y) = g_r(x)+g_r(y)$. So $g_r$ satisfies the first condition. Now we test the second condition. Let $c \in \mathbb{R}$ be arbitrary. Note that:

$$\begin{aligned} g_r(c \cdot x) =& r(c \cdot x) && \text{(by definition of } g_r\text{)} \\ =& crx \\ =& c \cdot (rx) \\ =& c \cdot g_r(x) && \text{(by definition of } g_r\text{)} \end{aligned}$$

So for arbitrary $c \in \mathbb{R}$, we have $g_r(c \cdot x) = c \cdot g_r(x)$. So $g_r$ satisfies the second condition. We have shown both conditions to be satisfied for $g_r : \mathbb{R} \to \mathbb{R}$ given by $g_r(x) = rx$, so $g_r$ is a linear transformation by definition.

b. Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both linear transformations, and $f(1) = g(1)$, then $f = g$.

*Solution:* Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary linear transformations. Let $c \in \mathbb{R}$ be arbitrary. Notice that:

$$\begin{aligned} f(c) = f(c \cdot 1) =& c \cdot f(1) && \text{(by definition of linear transformation from } \mathbb{R} \to \mathbb{R}\text{)} \\ =& c \cdot g(1) && \text{(by assumption)} \\ =& g(c) && \text{(by definition of linear transformation from } \mathbb{R} \to \mathbb{R}\text{)} \end{aligned}$$

So for arbitrary $c \in \mathbb{R}$, $f(c) = g(c)$. So $f = g$. Because $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ were arbitrary linear transformations, the result follows.