Mechanics **Spring 2018**
Exam #1 **Prof. Case**

1. A body of mass m moving along the x axis is subject to one of two forces,

$F = ax^3$ or $F = at^3$

You are to find $x(t)$ where the initial position is x_0 and the initial velocity is v_0. In one case the calculation is easy while the other is rather difficult. Solve the easier problem.

$F = at^3$ is the easy one.

$m\ddot{x} = at^3$

$\ddot{x} = \frac{at^3}{m}$

$\int_0^t \ddot{x}\,dt' = \int_0^t \frac{at'^3}{m} = \frac{a}{m}\left(\frac{t'^4}{4}\right|_0^t = \frac{a}{4m}t^4.$

$\dot{x}(t) - \underbrace{\dot{x}(0)}_{v_0} = \frac{a}{4m}t^4$

$\dot{x}(t) = \frac{a}{4m}t^4 + v_0$

$\int_0^t \dot{x}\,dt = \int_0^t \left(\frac{a}{4m}t'^4 + v_0\right)dt' = \frac{a}{4m}\frac{t^5}{5} + v_0 t$

$x(t) - x_0 = \frac{a}{20m}t^5 + v_0 t$

$\boxed{x(t) = \frac{a}{20m}t^5 + v_0 t + x_0}$

2. Given the three vectors

$$\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$$
$$\vec{B} = -6\hat{i} - 4\hat{j} + 2\hat{k}$$
$$\vec{C} = \hat{i} - 2\hat{j} - \hat{k}$$

find two that are perpendicular and two that are parallel or antiparallel.

Looking at $\vec{A}$ & $\vec{B}$ one sees. $\vec{B} = -2\vec{A}$

So $\vec{A}$ & $\vec{B}$ are anti-parallel.

Take $\vec{A} \cdot \vec{C} = 3 - 4 + 1 = 0 = AC\cos(\theta)$

$\angle$ between

So A & C are $\perp$

$B \cdot C = -6 + 8 - 2 = 0$

B & C are also $\perp$

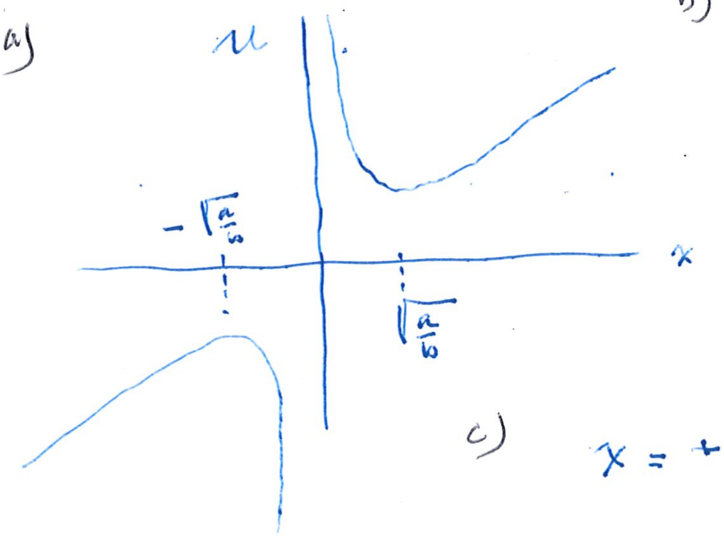
3. A particle of mass m moves in a potential given by

$$U = \frac{a}{x} + bx$$

where a and b are positive constants.

a. Make a sketch of the potential
b. Find point(s) of equilibrium.
c. Which are stable?
d. Find the frequency for oscillation for motion near the point(s) of stable equilibrium.

a)



equilibrium ↓

b) $$-F = +\frac{dU}{dx} = -\frac{a}{x^2} + b = 0$$

$$-a + x^2 b = 0$$

$$x^2 = \frac{a}{b}$$

$$x = \pm\sqrt{\frac{a}{b}}$$

c) $x = +\sqrt{\frac{a}{b}}$ is stable (a min.)

For osc near $x = \sqrt{\frac{a}{b}}$ Take $\left.\frac{d^2U}{dx^2}\right|_{x=\sqrt{\frac{a}{b}}}$

effective spring const ↓

d) $$k = \frac{d^2U}{dx^2} = +2ax^{-3} \quad \text{at } x = \sqrt{\frac{a}{b}}$$

$$k = 2a\left(\frac{b}{a}\right)^{3/2} = 2\frac{b^{3/2}}{a^{1/2}}$$

$$\omega_0^2 = \frac{k}{m} = \frac{2b^{3/2}}{m a^{1/2}} \qquad \omega_0 = \sqrt{\frac{2b^{3/2}}{m a^{1/2}}}$$

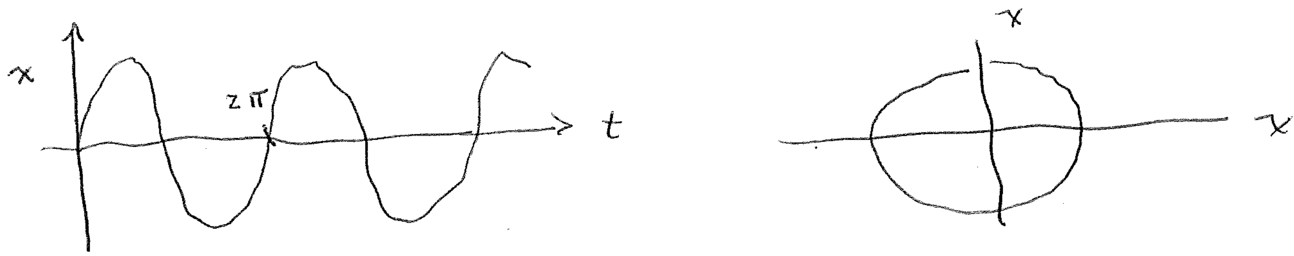
There are other ways to do pt. d).

4. Draw sketches of solutions for each of the following equations. Show solutions as x vs t and as phase space plots. In all cases assume that the system starts out at $x=0, \dot{x}=+1$.

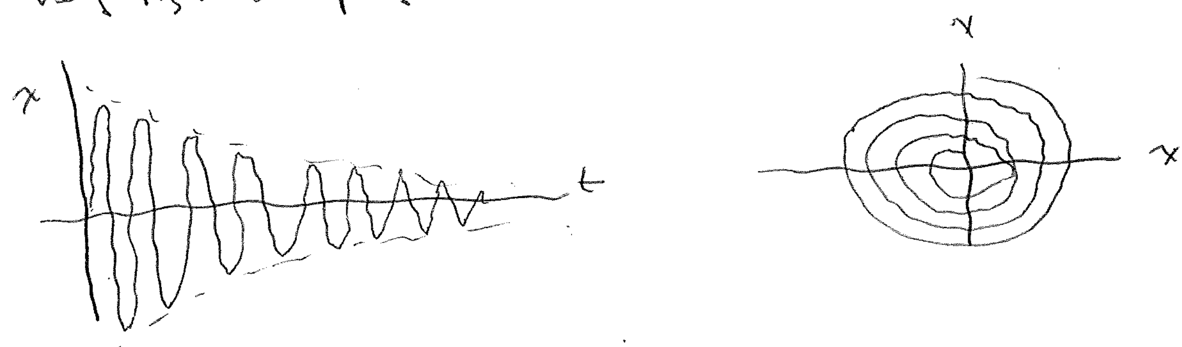
   a. $\ddot{x}=-x$
   b. $\ddot{x}=-x-.001\dot{x}$
   c. $\ddot{x}=-x-30\dot{x}$
   d. $\ddot{x}=-x+\cos(1.01t)$

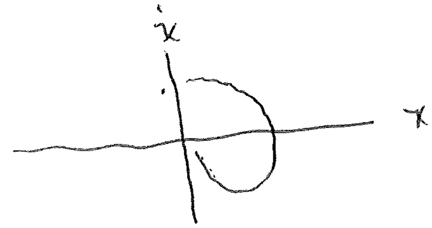
   The idea is to be roughly quantitative without actually calculating numbers.

a)

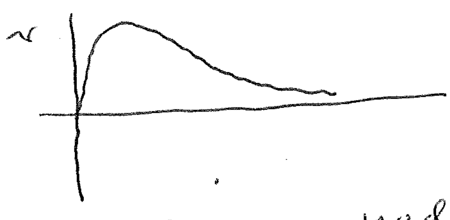


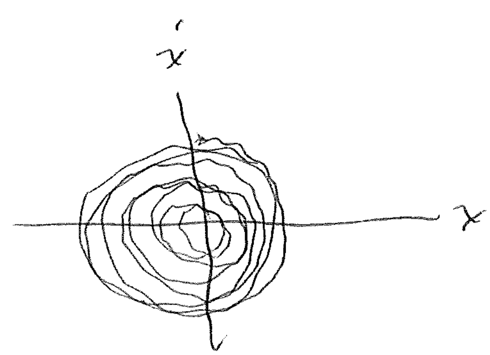
b) very light damping





c) heavy damping





d) Driven - no damping - close to resonance