Problem Set 13: Due Monday, April 2

Problem 1: Explain why a 2×2 matrix A is invertible if and only if 0 is not an eigenvalue of A.

Problem 2: Define a sequence of numbers as follows. Let $g_0 = 0$, $g_1 = 1$, and $g_n = \frac{1}{2}(g_{n-1} + g_{n-2})$ for all $n \in \mathbb{N}$ with $n \geq 2$. In other words, if $n \geq 2$, then the n^{th} term of the sequence is the average of the two previous terms. Notice that if

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix},$$

then

$$A \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix} = \begin{pmatrix} g_{n+2} \\ g_{n+1} \end{pmatrix}$$

for all $n \in \mathbb{N}$, so

$$A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} g_{n+1} \\ g_n \end{pmatrix}$$

for all $n \in \mathbb{N}$.
a. Find an invertible matrix P and a diagonal matrix D with $A = PDP^{-1}$.
b. Find a general formula for g_n.
c. As n gets large, the values of g_n approach a fixed number. Find that number.
Hint: Look at Example 3.5.21.

Problem 3: Show that $\det(AB) = \det(A) \cdot \det(B)$ for all 2×2 matrices A and B.
Note: Intuitively, if the linear transformation with standard matrix B distorts area by a factor of s, and the linear transformation with standard matrix A distorts area by a factor of r, then the *composition* of these linear transformations will distort area by a factor of rs (because matrix multiplication corresponds to function composition), with appropriate signs. Although it is possible to make this geometric sketch precise by using arguments similar to the ones at the end of Section 3.6, you should give a computational argument in this problem by just using the formula for the determinant.

Problem 4: Show that if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Hint: Start with the fact that $AA^{-1} = I$, and use the previous problem.

Problem 5: Show that if $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and α is a basis of $\mathbb{R}^2$, then $\det([T]_\alpha) = \det([T])$. Thus, although we might obtain different matrices when we represent T with respect to different bases, the resulting matrices will all have the same determinant.

Problem 6: Given a 2×2 matrix A and an $r \in \mathbb{R}$, what is the relationship between $\det(r \cdot A)$ and $\det(A)$? Explain.