Assignment: Writing Assignment 2

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Consider the function $f : \mathbb{Z} \to \mathbb{Z}^2$ given by $f(n) = (3n^2 - 77, 5n + 6)$.

a. Is f injective? Justify your answer carefully.

Solution: We assume that f is injective. Let $a, b \in \mathbb{Z}$ be arbitrary. By definition of injective, whenever $a, b \in \mathbb{Z}$ satisfy $f(a) = f(b)$, we have that $a = b$. So we get

$$(3a^2 - 77, 5a + 6) = (3b^2 - 77, 5b + 6)$$

We will look at each element of the ordered pair separately. For the second element of the ordered pair, we have:

$$\begin{aligned} 5a + 6 &= 5b + 6 \\ 5a &= 5b \\ a &= b \end{aligned}$$

We must now show that this is also the case for the first element of the ordered pair in order for f to satisfy the definition of injective. For the first element of the ordered pair, we have:

$$\begin{aligned} 3a^2 - 77 &= 3b^2 - 77 \\ 3a^2 &= 3b^2 \\ a^2 &= b^2 \end{aligned}$$

By definition, for all $z \in \mathbb{Z}$, $z^2 \geq 0$. So for our arbitrary $a, b \in \mathbb{Z}$ with $a^2 = b^2$, we conclude there are two solutions: $a = b$, $a = -b$. But we assumed that f was injective, and this assumption has led to a conclusion that contradicts the definition of injective. Because a, b were arbitrary, it must be the case that f is not injective.

b. Is f surjective? Justify your answer carefully.

Solution: We assume that f is surjective. By the definition of surjective, for all $(x, y) \in \mathbb{Z}^2$, there exists a $n \in \mathbb{Z}$ such that $f(n) = (x, y)$. Let $(x, y) = (-80, 0)$. So we get:

$$3n^2 - 77 = -80 \text{ and } 5n + 6 = -6$$

Solving for n, we find that:

$$\begin{aligned} 3n^2 &= -3 \text{ and } 5n = -6 \\ n^2 &= -1 \text{ and } n = -\frac{6}{5} \\ n &= \pm\sqrt{-1} \text{ and } n = -\frac{6}{5} \end{aligned}$$

We conclude that for $(x, y) = (-80, 0)$, we have $n = +\sqrt{-1}$, $n = -\sqrt{-1}$ for the first element and $n = -\frac{6}{5}$ for the second element. Note that for all of these values, $n \notin \mathbb{Z}$. Also note that we have two values of n for the first element, and completely different value of n for the second element. But we assumed that f was surjective, that is, that for all $(x, y) \in \mathbb{Z}^2$, there exists **a** $n \in \mathbb{Z}$ such that $f(n) = (x, y)$, however we have found an $(x, y) \in \mathbb{Z}^2$, $(-80, 0)$, such that for all $n \in \mathbb{Z}$, $f(n) \neq (-80, 0)$, so it must be the case that f is not surjective.

Problem 2: Suppose that A, B, and C are sets and that both $f : A \to B$ and $g : B \to C$ are surjective functions. Show that the function $g \circ f : A \to C$ is surjective.
Hint: You are trying to prove that the function $g \circ f : A \to C$ is surjective. Following the guidelines before the proof of Proposition 1.6.9, you should start by taking an arbitrary $c \in C$. With this c in hand, you goal is to build an $a \in A$ with $(g \circ f)(a) = c$.

Solution: Let $c \in C$ be arbitrary. g is surjective, so for any $c \in C$, there exists a $b \in B$ such that $g(b) = c$. f is surjective, so for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Notice that for each $b \in B$ with $g(b) = c$, there is an $a \in A$ with $f(a) = b$. So we can say that for every $c \in C$, there exists an $a \in A$ such that $g(f(a)) = c$. This function satisfies the definition of surjective. By the definition of function composition, $(g \circ f)(a) = g(f(a))$ for all $a \in A$. So $g(f(a)) = (g \circ f)(a)$ is surjective.

Problem 3: Suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. Suppose that $f(2) = 5$ and $f(3) = 7$. What is $f(\frac{1}{6})$? Explain. *Hint:* What can you say about $f(1)$?

Solution: By the definition of f, we have

$$\begin{aligned} f(2) = f(4 \cdot \frac{1}{2}) &= f(4) \cdot f(\frac{1}{2}) \\ \frac{f(2)}{f(4)} &= f(\frac{1}{2}) \\ \frac{f(2)}{f(2) \cdot f(2)} &= f(\frac{1}{2}) \\ \frac{1}{f(2)} &= f(\frac{1}{2}) \end{aligned}$$

and

$$\begin{aligned} f(3) = f(9 \cdot \frac{1}{3}) &= f(9) \cdot f(\frac{1}{3}) \\ \frac{f(3)}{f(9)} &= f(\frac{1}{3}) \\ \frac{f(3)}{f(3) \cdot f(3)} &= f(\frac{1}{3}) \\ \frac{1}{f(3)} &= f(\frac{1}{3}) \end{aligned}$$

Notice that

$$\begin{aligned} f(\frac{1}{6}) = f(\frac{1}{2} \cdot \frac{1}{3}) &= f(\frac{1}{2}) \cdot f(\frac{1}{3}) \\ &= \frac{1}{f(2)} \cdot \frac{1}{f(3)} \\ &= \frac{1}{5} \cdot \frac{1}{7} \\ &= \frac{1}{35} \end{aligned}$$

So $f(\frac{1}{6}) = \frac{1}{35}$.