Assignment: Written Assignment 5

Name: Oleksandr Yardas

Due Date: 03/12/2018

List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Is it always possible to find a basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha \neq [T]$? Either prove this is true, or give a counterexample (with justification).

Solution: We assume that is is always possible to find a basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha \neq [T]$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary linear transformation. Consider the case in which T is the linear transformation with standard matrix $[T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that $[T] = [id] = I$ by Definition 3.2.7. Let $\alpha = (\vec{u_1}, \vec{u_2})$ be an arbitrary basis of $\mathbb{R}^2$, and fix $a, b, c, d \in \mathbb{R}$ with $\vec{u_1} = \begin{pmatrix} a \\ c \end{pmatrix}, \vec{u_2} = \begin{pmatrix} b \\ d \end{pmatrix}$. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Applying Proposition 3.4.7, we have that

$$\begin{aligned} [T]_\alpha = P^{-1}[T]P &= P^{-1}IP && \text{(By definition of } [T]\text{)} \\ &= P^{-1}P && \text{(By Proposition 3.2.8)} \\ &= I && \text{(By definition)} \end{aligned}$$

So $[T]_\alpha = I$ for any basis α, and so it follows that, in this specific case, $[T]_\alpha = [T]$ for any basis α. We assumed that it is always possible to find a basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha \neq [T]$ for an arbitrary linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, however we have found a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T]_\alpha = [T]$ for *any* basis α. This contradicts our assumption, so it must be the case that is it not *always* possible to find a basis $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ such that $[T]_\alpha \neq [T]$ for an arbitrary linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Problem 2: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $\alpha = (\vec{u_1}, \vec{u_2})$ and $\beta = (\vec{w_1}, \vec{w_2})$ be bases of $\mathbb{R}^2$. Show that there exists an invertible 2×2 matrix R with $[T]_\beta = R^{-1} \cdot [T]_\alpha \cdot R$, and explicitly describe how to calculate R.

Solution: Fix $a, b, c, d, e, f, g, h \in \mathbb{R}$ with $\vec{u_1} = \begin{pmatrix} a \\ c \end{pmatrix}, \vec{u_2} = \begin{pmatrix} b \\ d \end{pmatrix}, \vec{w_1} = \begin{pmatrix} e \\ g \end{pmatrix}, \vec{w_2} = \begin{pmatrix} f \\ h \end{pmatrix}$, and let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. We have that $\alpha = (\vec{u_1}, \vec{u_2})$ and $\beta = (\vec{w_1}, \vec{w_2})$ are bases of $\mathbb{R}^2$, so by definition of basis, we have that $Span(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$ and $Span(\vec{w_1}, \vec{w_2}) = \mathbb{R}^2$. Applying Theorem 2.3.10, it follows that $ad - bc \neq 0$ and $eh - fg \neq 0$, and so by Proposition 3.3.16, we conclude that P and Q are invertible and have unique inverses which, by definition, are denoted by P^{-1} and Q^{-1} respectively. By Proposition 3.4.7, we have $[T]_\alpha = P^{-1}[T]P$ and $[T]_\beta = Q^{-1}[T]Q$. We want to show that there exists an invertible 2×2 matrix R with $[T]_\beta = R^{-1} \cdot [T]_\alpha \cdot R$, so we will need to express $[T]_\beta$ in terms of R^{-1}, R, and $[T]_\alpha$. We do this as follows: We first solve for $[T]$ in terms of P^{-1}, P, and $[T]_\alpha$. We start with the equation $[T]_\alpha = P^{-1}[T]P$. Taking the matrix product with P, we get

$$\begin{aligned} P[T]_\alpha = PP^{-1}[T]P &= I[T]P && \text{(By definition of inverse)} \\ &= [T]P && \text{(By Propositon 3.2.8)} \end{aligned}$$

We then take the matrix product with P^{-1}, giving $P[T]_\alpha P^{-1} = [T]PP^{-1}$. The right hand side simplifies to $[T]I$ (by the definition of inverse), which then further simplifies to $[T]$ (by Proposition 2.3.8). So we have that $[T] = P[T]_\alpha P^{-1}$. Substituting for $[T]$ in $[T]_\beta = Q^{-1}[T]Q$, we get

$$\begin{aligned} [T]_\beta &= Q^{-1}(P[T]_\alpha P^{-1})Q \\ &= (Q^{-1}P)[T]_\alpha(P^{-1}Q) && \text{(By Proposition 3.2.6)} \end{aligned}$$

Recall that P and Q are invertible. By Proposition 3.1.18, it follows that P^{-1} and Q^{-1} are invertible. Notice that $(Q^{-1}P) = (Q)^{-1}(P^{-1})^{-1} = (P^{-1}Q)^{-1}$ by Proposition 3.3.18. So we can rewrite our equation as $[T]_\beta = (P^{-1}Q)^{-1}[T]_\alpha(P^{-1}Q)$. Letting $P^{-1}Q = R$, we get $[T]_\beta = (R)^{-1}[T]_\alpha(R) = R^{-1}[T]_\alpha R$. $P^{-1}Q$ is invertible, so R is invertible. We conclude that, for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $\alpha = (\vec{u_1}, \vec{u_2})$ and $\beta = (\vec{w_1}, \vec{w_2})$ are bases of $\mathbb{R}^2$, and fixing $a, b, c, d, e, f, g, h \in \mathbb{R}$ with $\vec{u_1} = \begin{pmatrix} a \\ c \end{pmatrix}, \vec{u_2} = \begin{pmatrix} b \\ d \end{pmatrix}, \vec{w_1} = \begin{pmatrix} e \\ g \end{pmatrix}, \vec{w_2} = \begin{pmatrix} f \\ h \end{pmatrix}$ and defining two $2{\times}2$ matrices P and Q by letting $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, there exists an invertible $2{\times}2$ matrix $R = P^{-1}Q$ with $[T]_\beta = R^{-1}[T]_\alpha R$. We if we know the explicit values for α and β we can calculate R by the definition of matrix multiplication, that is, if we know the explicit values of $a, b, c, d, e, f, g, h \in \mathbb{R}$, then we know the explicit value of P^{-1} (given by Proposition 3.3.16) and the explicit value of Q, and we can compute $R = P^{-1}Q = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} de - bg & df - bh \\ -ce + ag & -cf + ah \end{pmatrix} = R$

Problem 3: Given two 2×2 matrices A and B, write $A \sim B$ to mean that there exists a 2×2 invertible matrix P with $B = P^{-1}AP$.

Cultural Aside: Using Problem 2 along with our work in class, it follows that $A \sim B$ if and only if A and B are both representations of a common linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, but with respect to possibly different coordinates. In this problem, you are proving that $\sim$ is something called an *equivalence relation*, a concept that you will see repeatedly throughout your mathematical journey.

a. Show that $A \sim A$ for all 2×2 A.

Solution: Let A be an arbitrary 2×2 matrix. We assume that $A \nsim A$ for all 2×2 A, and it follows from the definition of $\sim$ that for all 2×2 matrices P, we have $A \neq P^{-1}AP$. Consider the case in which $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that $1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$, so P is indeed invertible. By Proposition 3.3.16, P has a unique inverse P^{-1} given by $P^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -0 \\ -0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P$. So by assumption, we have that $A \neq PAP$. Notice that $P = I$ by definition 3.3.16. Applying Proposition 3.2.8, we have that $IA = A$ and $AI = A$, so it follows that $PA = A$ and $AP = A$. So we have that $(PA)P = AP = A$. By our previous equation $A \neq PAP$, we conclude that $A \neq A$. This is clearly a contradiction, as for any particular 2×2 matrix A, it is always the case that $A = A$. So it must be that case that our assumption that $A \nsim A$ for all 2×2 A is false, and so it must indeed be the case that $A \sim A$ for all 2×2 matrices A, that is, that there exists a 2×2 invertible matrix P with $A = P^{-1}AP$ for all 2×2 matrices A. Because A was arbitrary, the result follows.

b. Show that if A and B are 2×2 matrices with $A \sim B$, then $B \sim A$.

Solution: Let A, B be arbitrary 2×2 matrices such that $A \sim B$. By definition of $A \sim B$, there exists a 2×2 invertible matrix P with $B = P^{-1}AP$. We can manipulate this equation by taking the matrix product with P yielding $PB = PP^{-1}AP = IAP$ (by the definition of inverse matrix). It follows that $PB = AP$ (by Proposition 3.2.8). Taking the matrix product with P^{-1}, we get $PBP^{-1} = APP^{-1} = AI = A$. We conclude that $PBP^{-1} = A$. Because P is invertible, P^{-1} is invertible, and it follows that $(P^{-1})^{-1} = P$ (by Proposition 3.3.18), and we rewrite our equation as $A = (P^{-1})^{-1}B(P^{-1})$. Letting $P^{-1} = Q$, we rewrite our equation as $A = Q^{-1}BQ$. Because P^{-1} is invertible, Q is invertible, and so $A = Q^{-1}BQ$ satisfies the definition of $B \sim A$. Because A, B were arbitrary, the result follows.

c. Show if A, B and C are 2×2 with both $A \sim B$ and $B \sim C$, then $A \sim C$.

Solution: Let A, B, C be arbitrary 2×2 matrices with $A \sim B$ and $B \sim C$. By definition of $\sim$, there exist invertible 2×2 matrices P and Q with $B = P^{-1}AP$ and $C = Q^{-1}BQ$.

Substituting for B into the second equation, we get

$$C = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) \qquad \text{(By Proposition 3.2.6)}$$

Notice that $Q^{-1}P^{-1} = (PQ)^{-1}$ by Proposition 3.3.18, so we can rewrite the previous equation as $C = (PQ)^{-1}A(PQ)$. Letting $PQ = R$, we rewrite our equation as $C = (R)^{-1}A(R) = R^{-1}AR$. P and Q are both invertible, so by Proposition 3.3.18, R is invertible, and so $C = R^{-1}AR$ satisfies the definition of $A \sim C$. Because A, B, C were arbitrary, the result follows.

PAGE 2 OF 2 FOR PROBLEM 3