Solutions to Problem Set 1

Problem 1: Since P contains the origin and both of $\vec{u}$ and $\vec{w}$, it follows that each of $\vec{u}$ and $\vec{w}$ are parallel to P. Therefore, to find a normal vector for P, we need to find a vector that is orthogonal to both $\vec{u}$ and $\vec{w}$. Computing the cross product, we find

$$\begin{aligned}
\vec{u} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ -7 & 1 & 4 \end{vmatrix} \\
&= \begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} \cdot \vec{i} - \begin{vmatrix} 2 & 1 \\ -7 & 4 \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} 2 & -3 \\ -7 & 1 \end{vmatrix} \cdot \vec{k} \\
&= (-12 - 1) \cdot \vec{i} - (8 + 7) \cdot \vec{j} + (2 - 21) \cdot \vec{k} \\
&= \langle -13, -15, -19 \rangle.
\end{aligned}$$

Thus, $\langle -13, -15, -19 \rangle$ is orthogonal to both $\vec{u}$ and $\vec{w}$, so works as a normal vector for the plane. Notice that any nonzero scalar multiple of this vector also works as a normal vector. As a result, we will instead take $\langle 13, 15, 19 \rangle$ so that we do not have to deal with negatives. Now given any point (x, y, z), we have that (x, y, z) is on P if and only if the vector $\langle x, y, z \rangle$ is parallel to the plane (since the plane contains the origin), which is if and only if

$$\langle 13, 15, 19 \rangle \cdot \langle x, y, z \rangle = 0.$$

Thus, (x, y, z) is on the plane if and only if

$$13x + 15y + 19z = 0.$$

It follows that we have an equation for P.

Problem 2a: Since L is the intersection of the two planes, a given point is on L exactly when it is on both planes. Thus, we want to check that when we plug the point into the two equations, the result in a true statement.

- For the point $(1, 0, 1)$, we have

  $$3 \cdot 1 + 4 \cdot 0 - 1 = 3 + 0 - 1 = 2,$$

  so $(1, 0, 1)$ is on the first plane. We also have

  $$1 - 2 \cdot 0 + 1 = 2,$$

  so since $2 \neq 4$, we see that $(1, 0, 1)$ is *not* on the second plane. Since $(1, 0, 1)$ is not on both planes, it is not on L.

- For the point $(1, 1, 5)$, we have

  $$3 \cdot 1 + 4 \cdot 1 - 5 = 3 + 4 - 5 = 2,$$

  so $(1, 1, 5)$ is on the first plane. We also have

  $$1 - 2 \cdot 1 + 5 = 4,$$

  so $(1, 1, 5)$ is on the second plane too. Since $(1, 1, 5)$ is on both planes, it is a point on L.

Problem 2b: To obtain a parametric description of L, we need both a point and a direction vector. We know from Problem 2a that $(1, 1, 5)$ is a point on L. We find a direction vector as follows. Since L is on both planes, it is parallel to both planes. Thus, any direction vector of L will be parallel to both planes, and hence orthogonal to normal vectors of both planes. We know that $\langle 3, 4, -1\rangle$ is a normal vector to $3x + 4y - z = 2$ and that $\langle 1, -2, 1\rangle$ is a normal vector to $x - 2y + z = 4$. Thus, the direction vectors of L will be the nonzero vectors that are orthogonal to both $\langle 3, 4, -1\rangle$ and $\langle 1, -2, 1\rangle$. We find an example of such a vector by taking the cross product:

$$\begin{aligned}
\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & -1 \\ 1 & -2 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix} \cdot \vec{i} - \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} \cdot \vec{k} \\
&= (4-2)\cdot \vec{i} - (3-(-1))\cdot \vec{j} + (-6-4)\cdot \vec{k} \\
&= \begin{pmatrix} 2 \\ -4 \\ -10 \end{pmatrix}.
\end{aligned}$$

Thus, $\langle 2, -4, -10\rangle$ could serve as a direction vector, as could any nonzero constant multiple. Now we obtain all points of L by starting with a given point like $(1, 1, 5)$, and adding to it all scalar multiples of $\langle 2, -4, -10\rangle$. Thus, a parametric equation for L is

$$\begin{array}{rcrcr} x & = & 1 & + & 2t \\ y & = & 1 & - & 4t \\ z & = & 5 & - & 10t. \end{array}$$

Problem 2c: A point (x, y, z) is on line if *there exists* $t \in \mathbb{R}$ such that the equations

$$\begin{array}{rcrcr} x & = & 1 & + & 2t \\ y & = & 1 & - & 4t \\ z & = & 5 & - & 10t \end{array}$$

are all true. Thus, to test if $(5, 2, 3)$ is on L, we are asking if there exists $t \in \mathbb{R}$ such that the equations

$$\begin{array}{rcrcr} 5 & = & 1 & + & 2t \\ 2 & = & 1 & - & 4t \\ 3 & = & 5 & - & 10t \end{array}$$

are all true. For $5 = 1 + 2t$ to be true, we must have $4 = 2t$, and hence $t = 2$. However, $t = 2$ does not work in the second equation because $1 - 4 \cdot 2 = -7$, which does not equal 2. Thus, there is no t making all of the equation true, and hence $(5, 2, 3)$ is not a point on L.

Problem 3a: Yes, the planes do intersect. The quick way to see this is to notice that two nonparallel planes in $\mathbb{R}^3$ must intersect, so it suffices to check that the given planes are not parallel. To do this, it suffices to show that the two normal vectors are not parallel, which is the same as showing that the two normal vectors are not multiples of each other. The normal vectors of the plane are $\langle 2, -3, 1\rangle$ and $\langle -4, 9, -2\rangle$. So we are asking if *there exists* $c \in \mathbb{R}$ with

$$c \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ -2 \end{pmatrix}.$$

We can rewrite this by asking if there exists $c \in \mathbb{R}$ such that $2c = -4$, $-3c = 9$, and $c = -2$ are all true. Now clearly the last equation requires that $c = -2$, but we have $-3 \cdot (-2) = 6$, which is not equal to 9. Thus, the two normal vectors are not parallel, so the two planes are not parallel, and hence the two planes must intersect.

By the way, it's also possible to show that the planes intersect by finding a point on both of them. One way to do this is to simplify your life by making an additional assumption. Can we find a point on both planes with z-coordinate equal to 0? In this case, we want to solve the following system of two equations:

$$\begin{array}{rcrcr} 2x & - & 3y & = & 7 \\ -4x & + & 9y & = & 3. \end{array}$$

Adding twice the first equation to the second, we conclude that $3y = 17$, which implies that $y = \frac{17}{3}$. Plugging this back in, we see that $x = 12$. Thus, $(12, \frac{17}{3}, 0)$ is a point on both planes, which you can check by plugging it into the equations.

Problem 3b: Given a point (x, y, z), it lies on the first line if *there exists* $t \in \mathbb{R}$ such that the equations

$$\begin{array}{rcrcr} x & = & -4 & + & 6t \\ y & = & 2 & + & t \\ z & = & 1 & + & 3t \end{array}$$

are all true. Similarly, given a point (x, y, z), it lies on the second line if *there exists* $t \in \mathbb{R}$ such that the equations

$$\begin{array}{rcrcr} x & = & 4 & + & 4t \\ y & = & 5 & - & t \\ z & = & 9 & - & 2t \end{array}$$

are all true. Notice that given the same point (x, y, z), it may lie on the first line with $t = 7$ as a witness, and it may simultaneously lie on the second line with $t = -3$ as a witness. In other words, the t's that arise might not be the same. Thus, what we are really asking is whether there exists $s, t \in \mathbb{R}$ such that the equations

$$\begin{array}{rcrcrcrcr} -4 & + & 6s & = & 4 & + & 4t \\ 2 & + & s & = & 5 & - & t \\ 1 & + & 3s & = & 9 & - & 2t \end{array}$$

are all true. Rearranging these, we are asking if there exists $s, t \in \mathbb{R}$ with

$$\begin{array}{rcrcr} 6s & - & 4t & = & 8 \\ s & + & t & = & 3 \\ 3s & + & 2t & = & 8. \end{array}$$

We now solve this system. Taking the third equation minus twice the second, we see that we must have $s = 2$. From here, the second equation tells us that we must have $t = 1$. This is the only possible solution, and plugging in $s = 2$ and $t = 1$ into the three equations, we quickly conclude that it makes all three equations true. Plugging in $t = 1$ into the second line, a common point on both lines is $(8, 4, 7)$ (this arises by plugging $t = 2$ into the first as well). Therefore, the lines do indeed intersect.

Problem 4a: The statement "There exists $x \in \mathbb{R}$ with $\sin x = \cos x$" is true. To see this, it suffices to give one example. We have $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and also $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, $\sin \frac{\pi}{4} = \cos \frac{\pi}{4}$, and this completes the argument.

Problem 4b: The statement "There exists $x \in \mathbb{R}$ with $\sin x = \cos x + 2$" is false. We argue this as follows. We know that

$$-1 \leq \sin x \leq 1$$

for all $x \in \mathbb{R}$, and also that

$$-1 \leq \cos x \leq 1$$

for all $x \in \mathbb{R}$. Thus, the only possibility for an $x \in \mathbb{R}$ to satisfy $\sin x = \cos x + 2$ is when $\cos x = -1$ and $\sin x = 1$. However, no such $x \in \mathbb{R}$ can possibly satisfy both of these because then we would have

$$\cos^2 x + \sin^2 x = (-1)^2 + 1^2 = 2,$$

contradicting the fact that $\cos^2 x + \sin^2 = 1$ for all $x \in \mathbb{R}$ (alternatively, you can just classify the values of x where each statement is true, and show that they do not overlap). Therefore, "There exists $x \in \mathbb{R}$ with $\sin x = \cos x + 2$" is false.

Problem 4c: The statement "There exists $m, n \in \mathbb{N}$ with $9m + 15n = 3$" is false. To prove this, we need to argue that "For all $m, n \in \mathbb{N}$, we have $9m + 15n \neq 3$" is true. Let $m, n \in \mathbb{N}$ be arbitrary, and consider the following three cases.

- *Case 1:* Suppose that $m = 0$ and $n = 0$. Since $9 \cdot 0 + 15 \cdot 0 = 0$, we have that $9m + 15n \neq 3$ in this case.
- *Case 2:* Suppose that $m \geq 1$. Since $n \geq 0$, we then have

$$\begin{aligned} 9m + 15n &\geq 9 \cdot 1 + 15 \cdot 0 \\ &= 9 \end{aligned}$$

  so $9m + 15n \neq 3$ in this case.
- *Case 2:* Suppose that $n \geq 1$. Since $m \geq 0$, we then have

$$\begin{aligned} 9m + 15n &\geq 9 \cdot 0 + 15 \cdot 1 \\ &= 15 \end{aligned}$$

  so $9m + 15n \neq 3$ in this case.

Notice that at least one of these three cases must be true, so we have shown that $9m+15n \neq 3$ unconditionally. Since m and n were arbitrary, it follows that we have shown that "For all $m, n \in \mathbb{N}$, we have $9m + 15n \neq 3$" is true. Therefore, our original statement is false.

Problem 4d: The statement "There exists $m, n \in \mathbb{Z}$ with $9m + 15n = 3$" is true. To see this, it suffices to give one example. We have

$$9 \cdot 2 + 15 \cdot (-1) = 18 - 15 = 3,$$

which completes the argument.

Problem 4e: The statement "For all $t \in \mathbb{R}$, we have $2\cos^4(3t) + 2\cos^2(3t) \cdot \sin^2(3t) - \cos(6t) = 1$" is true. We make use of the following two trigonometric identities:

- For all $x \in \mathbb{R}$, we have $\cos^2 x + \sin^2 x = 1$.

- For all $x \in \mathbb{R}$, we have $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$.

Given any arbitrary $t \in \mathbb{R}$, we have that $3t \in \mathbb{R}$, and so we can apply these identities as follows:

$$\begin{aligned}
2\cos^4(3t) + 2\cos^2(3t) \cdot \sin^2(3t) - \cos(6t) &= 2\cos^2(3t) \cdot [\cos^2(3t) + \sin^2(3t)] - \cos(6t) \\
&= 2\cos^2(3t) \cdot 1 - \cos(6t) \\
&= 2\cos^2(3t) - \cos(6t) \\
&= 2 \cdot \frac{1}{2}(1 + \cos(2 \cdot (3t))) - \cos(6t) \\
&= 1 + \cos(6t) - \cos(6t) \\
&= 1.
\end{aligned}$$

We have taken an arbitrary $t \in \mathbb{R}$ and shown that

$$2\cos^4(3t) + 2\cos^2(3t) \cdot \sin^2(3t) - \cos(6t) = 1$$

is true, so the result follows.

Problem 4f: The statement "For all $a \in \mathbb{R}$, we have $a^2 + 6a + 10 > 0$" is true. We argue this as follows. Let $a \in \mathbb{R}$ be arbitrary. We have

$$\begin{aligned}
a^2 + 6a + 10 &= a^2 + 6a + 9 + 1 \\
&= (a+3)^2 + 1 \\
&\geq 0 + 1 && \text{(since squares are nonnegative)} \\
&= 1.
\end{aligned}$$

Thus, we have $a^2 + 6a + 10 \geq 1$, and hence $a^2 + 6a + 10 > 0$. Since $a \in \mathbb{R}$ was arbitrary, the result follows.