Solutions to Problem Set 20

Problem 1: Let

$$\alpha = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 0 \\ 1 \end{pmatrix} \right).$$

By definition of W, we know that $\text{Span}(\alpha) = W$. We now show that α is linear independent by appealing to Proposition 4.3.3. Applying elementary row operations to the corresponding matrix, we obtain:

$$\begin{aligned}
\begin{pmatrix} 0 & 4 & 7 \\ 0 & 5 & 8 \\ 1 & 2 & 0 \\ 3 & 7 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 8 \\ 0 & 4 & 7 \\ 3 & 7 & 1 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_3) \\ \\ (R_1 \leftrightarrow R_3) \\ \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 8 \\ 0 & 4 & 7 \\ 0 & 1 & 1 \end{pmatrix} && \begin{matrix} \\ \\ \\ (-3R_1 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 8 \\ 0 & 4 & 7 \end{pmatrix} && \begin{matrix} \\ (R_2 \leftrightarrow R_4) \\ \\ (R_2 \leftrightarrow R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} && \begin{matrix} \\ \\ (-5R_2 + R_3) \\ (-4R_2 + R_4) \end{matrix} \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} && \begin{matrix} \\ \\ \\ (-R_3 + R_4) \end{matrix}
\end{aligned}$$

This last matrix is in echelon form and has a leading entry in each column, so Proposition 4.3.3 tells is that α is linearly independent. Since we also know that $\text{Span}(\alpha) = W$, we can conclude that

$$\alpha = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis of W. By definition, $\dim(W)$ is the number of elements in any basis, so $\dim(W) = 3$.

Problem 2: We first solve the following system of equations:

$$\begin{array}{ccccccccc} 2a & & & - & c & & & = & 0 \\ & & b & + & c & - & d & = & 0. \end{array}$$

The augmented matrix of this system is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

Notice that this matrix is already in echelon form, and that the third and fourth columns do not have leading entries. We therefore introduce parameters for the variables c and d, say $c = s$ and $d = t$. We can now solve the second equation $b + c - d = 0$ for b in terms of s and t to get

$$\begin{aligned} b &= -c + d \\ &= -s + t \end{aligned}$$

We can also solve the first equation $2a - c = 0$ for a in terms of s and t to get

$$\begin{aligned} a &= \frac{1}{2} \cdot c \\ &= \frac{1}{2} \cdot s. \end{aligned}$$

Thus, the solution set as a subset of $\mathbb{R}^4$ is

$$\left\{ \begin{pmatrix} \frac{1}{2}s \\ -s + t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Arranging these in a matrix according to our a, b, c, d, we conclude that

$$W = \left\{ \begin{pmatrix} \frac{1}{2} \cdot s & -s + t \\ s & t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

which we can rewrite as

$$W = \left\{ s \cdot \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

In other words, we have

$$W = \operatorname{Span}\left(\begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

We now check that

$$\left(\begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

is linearly independent. Let $c_1, c_2 \in \mathbb{R}$ be arbitrary with

$$c_1 \cdot \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We then have

$$\begin{pmatrix} \frac{1}{2} \cdot c_1 & -c_1 + c_2 \\ c_1 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

hence $c_1 = 0$ and $c_2 = 0$ (by looking at the second row). Therefore,

$$\left(\begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

is linearly independent. Since it also spans W from above, we know that is is a basis of W. Finally, we can conclude that $\dim(W) = 2$ because we have found a basis with 2 elements.

Problem 3: We first check that T preserves addition. Let $f_1, f_2 \in \mathcal{P}_1$ be arbitrary. Fix $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned} T(f_1 + f_2) &= T((a_1 x + b_1) + (a_2 x + b_2)) \\ &= (a_1 + a_2)x + (b_1 + b_2) \\ &= \begin{pmatrix} (a_1 + a_2) - (b_1 + b_2) \\ b_1 + b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 + a_2 - b_2 \\ b_1 + b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 - b_2 \\ b_2 \end{pmatrix} \\ &= T(a_1 x + b_1) + T(a_2 x + b_2) \\ &= T(f_1) + T(f_2). \end{aligned}$$

Therefore, T preserves addition.

We next check that T preserves scalar multiplication. Let $f \in \mathcal{P}_1$ and $c \in \mathbb{R}$ be arbitrary. Fix $a, b \in \mathbb{R}$ with $f(x) = ax + b$. We have

$$\begin{aligned} T(c \cdot f) &= T(c \cdot (ax + b)) \\ &= T((ca)x + (cb)) \\ &= \begin{pmatrix} ca - cb \\ cb \end{pmatrix} \\ &= c \cdot \begin{pmatrix} a - b \\ b \end{pmatrix} \\ &= c \cdot T(ax + b) \\ &= c \cdot T(f). \end{aligned}$$

Therefore, T preserves scalar multiplication.

Since T preserves both addition and scalar multiplication, it follows that T is a linear transformation.

Problem 4a: Notice that for any $x, y, z \in \mathbb{R}$, we have

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1x + (-1)y + 0z \\ 1x + 0y + 1z \\ 0x + 1y + 1z \end{pmatrix}$$

Therefore, T is a linear transformation by Proposition 5.1.2.

Problem 4b: We have

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) &= \begin{pmatrix} 1 - 1 \\ 1 + (-1) \\ 1 + (-1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Problem 4c: Notice that we also have

$$T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

either by direction calculation or because T is a linear transformation. Therefore, we have

$$T\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) = T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right).$$

Since

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

it follows that T is not injective.

Problem 5a: We first check that T preserves addition. Let $A_1, A_2 \in V$ be arbitrary. Fix $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$ with

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

We have

$$\begin{aligned} T(A_1 + A_2) &= T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}\right) \\ &= 2 \cdot (a_1 + a_2) - (d_1 + d_2) \\ &= 2a_1 + 2a_2 - d_1 - d_2 \\ &= 2a_1 - d_1 + 2a_2 - d_2 \\ &= T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \\ &= T(A_1) + T(A_2). \end{aligned}$$

Therefore, T preserves addition.

We next check that T preserves scalar multiplication. Let $A \in V$ and $r \in \mathbb{R}$ be arbitrary. Fix $a, b, c, d \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have

$$\begin{aligned} T(r \cdot A) &= T\left(r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}\right) \\ &= 2(ra) - (rd) \\ &= r \cdot (2a - d) \\ &= r \cdot T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right). \end{aligned}$$

Therefore, T preserves scalar multiplication.

Since T preserves both addition and scalar multiplication, it follows that T is a linear transformation.

Problem 5b: Let $r \in \mathbb{R}$ be arbitrary. Notice that

$$\begin{aligned} T\left(\begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}\right) &= 2 \cdot 0 - (-r) \\ &= r, \end{aligned}$$

so we have an input that produces r. Since $r \in \mathbb{R}$ was arbitrary, we conclude that T is surjective.