Solutions to Written Assignment 4

Problem 1: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be such that $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$. We need to show that $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) = \mathbb{R}^2$. Notice that $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) \subseteq \mathbb{R}^2$ immediately from the definition.

We now show that $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$. Let $\vec{w} \in \mathbb{R}^2$ be arbitrary. Since T is surjective, we can fix $\vec{v} \in \mathbb{R}^2$ with $T(\vec{v}) = \vec{w}$. Now $\vec{v} \in \mathbb{R}^2$ and we know that $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, so we conclude that $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2)$. Therefore, we can fix $c_1, c_2 \in \mathbb{R}$ with

$$\vec{v} = c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2.$$

Applying T to both sides and using the fact that T is a linear transformation, we conclude that

$$\begin{aligned} T(\vec{v}) &= T(c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2) \\ &= c_1 \cdot T(\vec{u}_1) + c_2 \cdot T(\vec{u}_2). \end{aligned}$$

Since $T(\vec{v}) = \vec{w}$, it follows that

$$\vec{w} = c_1 \cdot T(\vec{u}_1) + c_2 \cdot T(\vec{u}_2).$$

Since $c_1, c_2 \in \mathbb{R}$, we conclude that $\vec{w} \in \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$. Since $\vec{w} \in \mathbb{R}^2$ was arbitrary, it follows that $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$.

We have shown that both $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$ are true, so it follows that $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) = \mathbb{R}^2$.

Problem 2: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be an injective linear transformation and let $\vec{u}, \vec{w} \in \mathbb{R}^2$. We want to prove that if $\vec{w} \notin \text{Span}(\vec{u})$, then $T(\vec{w}) \notin \text{Span}(T(\vec{u}))$. We accomplish this by proving the contrapositive. That is, we show that if $T(\vec{w}) \in \text{Span}(T(\vec{u}))$, then $\vec{w} \in \text{Span}(\vec{u})$.

Suppose then that $T(\vec{w}) \in \text{Span}(T(\vec{u}))$. By definition, we can fix $c \in \mathbb{R}$ with $T(w) = c \cdot T(\vec{u})$. Since T is linear transformation, we have $c \cdot T(\vec{u}) = T(c \cdot \vec{u})$, hence $T(\vec{w}) = T(c \cdot \vec{u})$. Now $\vec{w} \in \mathbb{R}^2$, $c \cdot \vec{u} \in \mathbb{R}^2$ and T is injective, so we can conclude that $\vec{w} = c \cdot \vec{u}$. Since $c \in \mathbb{R}$, it follows that $\vec{w} \in \text{Span}(\vec{u})$.

Problem 3a: Suppose that $r \in \mathbb{R}$ and we let

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Consider an arbitrary 2×2 matrix B and fix $a, b, c, d \in \mathbb{R}$ with

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then have

$$AB = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

and

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

Therefore $AB = BA$. Since B was an arbitrary 2×2 matrix, we conclude that $AB = BA$ for all 2×2 matrices B.

Problem 3b: Suppose that A is a 2×2 matrix with the property that $AB = BA$ for every 2×2 matrix B. Fix $a, b, c, d \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using our hypothesis in the special case when

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying out these matrices, we see that

$$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

Since these matrices are equal, we conclude that $b = 0$ and $c = 0$, hence

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

With this in mind, we again use our hypothesis but now with the matrix

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We then conclude that

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Multiplying out these matrices, we see that

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

Since these matrices are equal, we conclude that $a = d$, hence

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

We conclude that A is of the desired form (by letting $r = a$).