Solutions to Problem Set 17

Problem 1: We use Proposition 4.2.14. Applying elementary row operations to the 3×3 matrix having these vectors as columns, we obtain

$$\begin{aligned}
\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_3) \\ \\ (R_1 \leftrightarrow R_3) \end{matrix} \\
&\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} && (-2R_1 + R_3) \\
&\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} && (-R_2 + R_3).
\end{aligned}$$

We have arrived an echelon form, and we notice that it has a leading entry in each row. Using Proposition 4.2.14, we conclude that

$$\text{Span}\left(\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \mathbb{R}^3.$$

Problem 2: Let $b_1, b_2, b_3 \in \mathbb{R}$ be arbitrary. By definition, we have

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right)$$

if and only if there exists $c_1, c_2, c_3 \in \mathbb{R}$ with

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

which is if and only if the following system has a solution:

$$\begin{array}{ccccccc}
 & & 3x_2 & + & x_3 & = & b_1 \\
x_1 & + & x_2 & + & x_3 & = & b_2 \\
5x_1 & - & x_2 & + & 3x_3 & = & b_2.
\end{array}$$

The augmented matrix of this system is

$$\begin{pmatrix} 0 & 3 & 1 & b_1 \\ 1 & 1 & 1 & b_2 \\ 5 & -1 & 3 & b_3 \end{pmatrix}.$$

Performing elementary row operations on this matrix, we obtain

$$\begin{aligned}
\begin{pmatrix} 0 & 3 & 1 & b_1 \\ 1 & 1 & 1 & b_2 \\ 5 & -1 & 3 & b_3 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & b_2 \\ 0 & 3 & 1 & b_1 \\ 5 & -1 & 3 & b_3 \end{pmatrix} && \begin{matrix} (R_1 \leftrightarrow R_2) \\ (R_1 \leftrightarrow R_2) \\ \end{matrix} \\
&\to \begin{pmatrix} 1 & 1 & 1 & b_2 \\ 0 & 3 & 1 & b_1 \\ 0 & -6 & -2 & -5b_2 + b_3 \end{pmatrix} && (-5R_1 + R_3) \\
&\to \begin{pmatrix} 1 & 1 & 1 & b_2 \\ 0 & 3 & 1 & b_1 \\ 0 & 0 & 0 & 2b_1 - 5b_2 + b_3 \end{pmatrix} && (2R_2 + R_3).
\end{aligned}$$

Using this, we claim that the above system has a solution if and only if $2b_1 - 5b_3 + b_3 = 0$. To see this, notice that if $2b_1 - 5b_3 + b_3 = 0$, then the last column of the above matrix in echelon form does not have a leading entry, so the system has a solution by Proposition 4.2.12. Conversely, if $2b_1 - 5b_3 + b_3 \neq 0$, then the last column of the above matrix in echelon form does have a leading entry, so the system does not have a solution by Proposition 4.2.12. Using this, we conclude that

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right)$$

if and only if $2b_1 - 5b_3 + b_3 = 0$.

Problem 3: We want to know whether there exists $c_1, c_2, c_3 \in \mathbb{R}$ with $g = c_1 f_1 + c_2 f_2 + c_3 f_3$. Recall that two functions are equal if and only if the give the same output on every possible input. Thus, we want to know whether there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $g(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$. In other words, does there exist $c_1, c_2 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have

$$x^3 + 8x^2 + 7 = c_1 \cdot (x^3 + 2x^2 + x) + c_2 \cdot (-3x^3 - 5x^2 + x + 2) + c_3 \cdot (x^2 - x + 1).$$

Expanding the right hand side and then recollecting terms having the same powers of x, we want to know if there exists $c_1, c_2, c_3 \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, we have

$$x^3 + 8x^2 + 0x + 7 = (c_1 - 3c_2)x^2 + (2c_1 - 5c_2 + c_3)x + (c_1 + c_2 - c_3)x + (2c_2 + c_3).$$

Using Proposition 4.2.18, this is true if and only if the following system of equations as a solution:

$$\begin{array}{rcrcrcr}
c_1 & - & 3c_2 & & & = & 1 \\
2c_1 & - & 5c_2 & + & c_3 & = & 8 \\
c_1 & + & c_2 & - & c_3 & = & 0 \\
 & & 2c_2 & + & c_3 & = & 7.
\end{array}$$

The augmented matrix of this system is

$$\begin{pmatrix} 1 & -3 & 0 & 1 \\ 2 & -5 & 1 & 8 \\ 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 7 \end{pmatrix}.$$

Performing elementary row operations on this matrix, we obtain

$$\begin{pmatrix}1 & -3 & 0 & 1\\ 2 & -5 & 1 & 8\\ 1 & 1 & -1 & 0\\ 0 & 2 & 1 & 7\end{pmatrix} \to \begin{pmatrix}1 & -3 & 0 & 1\\ 0 & 1 & 1 & 6\\ 0 & 4 & -1 & -1\\ 0 & 2 & 1 & 7\end{pmatrix} \qquad \begin{matrix}(-2R_1 + R_2)\\ (-R_1 + R_3)\end{matrix}$$

$$\to \begin{pmatrix}1 & -3 & 0 & 1\\ 0 & 1 & 1 & 6\\ 0 & 0 & -5 & -25\\ 0 & 0 & -1 & -5\end{pmatrix} \qquad \begin{matrix}(-4R_2 + R_3)\\ (-2R_2 + R_4)\end{matrix}$$

$$\to \begin{pmatrix}1 & -3 & 0 & 1\\ 0 & 1 & 1 & 6\\ 0 & 0 & -1 & -5\\ 0 & 0 & -5 & -25\end{pmatrix} \qquad \begin{matrix}(R_3 \leftrightarrow R_4)\\ (R_3 \leftrightarrow R_4)\end{matrix}$$

$$\to \begin{pmatrix}1 & -3 & 0 & 1\\ 0 & 1 & 1 & 6\\ 0 & 0 & -1 & -5\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad (-5R_3 + R_4).$$

Since the last column does not have a leading entry, we know that the system has a solution, and hence $g \in \text{Span}(f_1, f_2, f_3)$.

Although not necessary for this problem, we can also solve for c_1, c_2, c_3. The third line above tells us that $c_3 = 5$. Plugging into the second equation $c_2 + c_3 = 6$, we conclude that $c_2 = 1$. Plugging into the first equation $c_1 - 3c_2 = 1$, we conclude that $c_1 - 3 = 1$, and hence $c_1 = 4$. Therefore, we have $g = 4f_1 + 1f_2 + 5f_3$.

Problem 4: We want to know whether for all $a, b, c, d \in \mathbb{R}$, we can find $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$c_1 \begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix} + c_2 \begin{pmatrix}2 & 3\\ 7 & 2\end{pmatrix} + c_3 \begin{pmatrix}0 & 1\\ 2 & 6\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Let $a, b, c, d \in \mathbb{R}$ be arbitrary. Asking if there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$c_1 \begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix} + c_2 \begin{pmatrix}2 & 3\\ 7 & 2\end{pmatrix} + c_3 \begin{pmatrix}0 & 1\\ 2 & 6\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

is the same asking if there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$\begin{pmatrix}c_1 + 2c_2 & c_1 + 3c_2 + c_3\\ 2c_1 + 7c_2 + 2c_3 & 2c_2 + 6c_3\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

which is the same as asking if there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & & & = & a\\ c_1 & + & 3c_2 & + & c_3 & = & b\\ 2c_1 & + & 7c_2 & + & 2c_3 & = & c\\ & & 2c_2 & + & 6c_3 & = & d. \end{array}$$

The augmented matrix of this linear system is

$$\begin{pmatrix}1 & 2 & 0 & a\\ 1 & 3 & 1 & b\\ 2 & 7 & 2 & c\\ 0 & 2 & 6 & d\end{pmatrix}.$$

Performing elementary row operations on this matrix, we obtain

$$\begin{aligned}
\begin{pmatrix}1 & 2 & 0 & a\\ 1 & 3 & 1 & b\\ 2 & 7 & 2 & c\\ 0 & 2 & 6 & d\end{pmatrix} &\to \begin{pmatrix}1 & 2 & 0 & a\\ 0 & 1 & 1 & -a+b\\ 0 & 3 & 2 & -2a+c\\ 0 & 2 & 6 & d\end{pmatrix} && \begin{matrix}\\ (-R_1+R_2)\\ (-2R_1+R_3)\\ \\ \end{matrix}\\
&\to \begin{pmatrix}1 & 2 & 0 & a\\ 0 & 1 & 1 & -a+b\\ 0 & 0 & -1 & a-3b+c\\ 0 & 0 & 4 & 2a-2b+d\end{pmatrix} && \begin{matrix}\\ \\ (-3R_2+R_3)\\ (-2R_2+R_4)\end{matrix}\\
&\to \begin{pmatrix}1 & 2 & 0 & a\\ 0 & 1 & 1 & -a+b\\ 0 & 0 & -1 & a-3b+c\\ 0 & 0 & 0 & 6a-14b+4c+d\end{pmatrix} && \begin{matrix}\\ \\ \\ (4R_3+R_4).\end{matrix}
\end{aligned}$$

Notice that if $a = 0$, $b = 0$, $c = 0$, and $d = 1$, then this final matrix is

$$\begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Since there is a leading entry in the last column, we may use Proposition 4.2.12 to conclude that the corresponding system does not have a solution. Therefore

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \notin \text{Span}\left(\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}2 & 3\\ 7 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\ 2 & 6\end{pmatrix}\right),$$

and hence

$$\text{Span}\left(\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}2 & 3\\ 7 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\ 2 & 6\end{pmatrix}\right) \neq V.$$

Problem 5: First notice that $\{0\}$ and $\mathbb{R}$ are both subspaces of $\mathbb{R}$ by directly checking the three properties (these are trivial for $\{0\}$, and follow immediately for $\mathbb{R}$ because $\mathbb{R}$ is a vector space). We now argue that these are the only subspaces of $\mathbb{R}$. Let W be an arbitrary subspace of $\mathbb{R}$ with $W \neq \{0\}$. We can then fix $a \in W$ with $a \neq 0$. We have the following

- $W \subseteq \mathbb{R}$: This follows immediately from the definition of a subspace.
- $\mathbb{R} \subseteq W$: Let $b \in \mathbb{R}$ be arbitrary. We then have $b = \frac{b}{a} \cdot a$, so since $a \in W$, $\frac{b}{a} \in \mathbb{R}$, and W is closed under scalar multiplication, it follows that $b \in W$.

Therefore, $W = \mathbb{R}$. We have shown that if W is a subspace of $\mathbb{R}$ with $U \neq \{W\}$, then we must have that $W = \mathbb{R}$. In other words, $\{0\}$ and $\mathbb{R}$ are the only subspaces of $\mathbb{R}$.

Problem 6: Throughout this argument, let

$$S = \text{Span}\left(\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}, \begin{pmatrix}1\\ 0\\ -1\end{pmatrix}\right) \subseteq \mathbb{R}^3.$$

We show that $W = S$ by giving a double containment proof.

- We first show that $S \subseteq W$. Let $\vec{v} \in S$ be arbitrary. By definition, we can fix $c_1, c_2 \in \mathbb{R}$ with

$$\vec{v} = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

  We then have

$$\vec{v} = \begin{pmatrix} c_1 + c_2 \\ -c_1 \\ -c_2 \end{pmatrix}.$$

  Since $c_1 + c_2 + (-c_1) + (-c_2) = 0$, it follows that $\vec{v} \in W$. Since $\vec{v} \in S$ was arbitrary, we conclude that $S \subseteq W$.

- We now show that $W \subseteq S$. Let $\vec{w} \in W$ be arbitrary. By definition, we can fix $a_1, a_2, a_3 \in \mathbb{R}$ with

$$\vec{w} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

  and such that $a_1 + a_2 + a_3 = 0$. Notice then that $a_1 = -a_2 - a_3$, hence we have

$$\begin{aligned} (-a_2) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (-a_3) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} &= \begin{pmatrix} -a_2 \\ a_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -a_3 \\ 0 \\ a_3 \end{pmatrix} \\ &= \begin{pmatrix} -a_2 - a_3 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \vec{w}. \end{aligned}$$

  Since $-a_2 \in \mathbb{R}$ and $a_3 \in \mathbb{R}$, it follows that $\vec{w} \in S$. Since $\vec{v} \in W$ was arbitrary, we conclude that $W \subseteq S$.

Therefore, we have $W = S$.