NAME: ____

PHYSICS 314 – Thermodynamics and Statistical Physics (Spring 2018)

Instructor: Josh Weber

Test #1

February 22, 2018

8:00 – 9:20 AM

Instructions:

1) Please remember to write your name on this page.
2) You will probably need to use a calculator capable of factorials and exponents.
3) **<u>Show all of your work and reasoning</u>** to maximize your chances for partial credit. *Correct answers that are not supported by work and reasoning will not receive full credit.*
4) Keep track of **units**, and give a reasonable number of **significant figures**.
5) Please ask me if any questions or instructions are unclear.
6) You may use one page of notes that is single-sided, hand-written, and written by you.

<u>Please staple this sheet onto the front of your exam when you are finished.</u>

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1. [6 pts]

A process brings *n* moles of an ideal gas from an initial state characterized by pressure P_i and volume V_i to a final state characterized by pressure P_f and volume V_f. Show that the variation in internal energy, ΔU, is given by the following expression. γ is the adiabatic exponent.

$$\Delta U = \frac{P_f V_f - P_i V_i}{\gamma - 1}$$

Start with the equipartition theorem.

$$\Delta U = \frac{f}{2} nR\Delta T = \frac{f}{2} nR(T_f - T_i)$$

Use Ideal Gas Law to replace T.

$$\Delta U = \frac{f}{2} nR\left(\frac{P_f V_f}{nR} - \frac{P_i V_i}{nR}\right)$$

$$\Delta U = \frac{f}{2}(P_f V_f - P_i V_i)$$

Use the definition of the adiabatic exponent.

$$\gamma = \frac{f + 2}{f}$$

$$\gamma = 1 + \frac{2}{f}$$

$$\gamma - 1 = \frac{2}{f}$$

$$\frac{1}{\gamma - 1} = \frac{f}{2} \qquad\qquad \Delta U = \frac{P_f V_f - P_i V_i}{\gamma - 1}$$

2. [6 pts – 2 pts each]

In your own words, define and explain the following concepts.

An answer of 2-4 sentences each, perhaps including some equations, is sufficient. If you use an equation, be sure to carefully define all variables. If there are qualifying conditions on a statement, be sure to include them.

a) The first law of thermodynamics

$$\Delta U = Q + W$$

The change in energy of a system is equal to the heat <u>added to</u> a system plus the work done <u>on</u> the system.

b) The second law of thermodynamics

$$S = k \ln \Omega$$

S = entropy, k = Boltzmann's constant, Ω = multiplicity

Entropy tends to increase.

Any large system in equilibrium will be found in the macrostate with the greatest entropy (aside from immeasurable fluctuations).

c) The fundamental assumption of statistical mechanics

in an isolated system in thermal equilibrium, all accessible microstates are equally probable.

3. [6 pts – 3 pts each]

Here is our class roster of 15 students.

First Name	Last Name
Allison	Bartz
Joe	Beggs
Kohei	Kotani
Hung	Le
Fengming	Li
Julie	Liu
Sarah	McCarthy
Jordan	Morris
Jill	Rix
Sarah	Ruiz
Patrick	Sheehan-Klenk
Zhiheng	Sheng
Daniel	Somorov
Olek	Yardas
Nathaniel	Zhu

Suppose I put all of your names in a hat, and then I pick one name at random for a first prize. I then put that name back in, and then I pull another name out for a second prize. I then put that name back in again, and repeat the process for a third prize. (Because I am putting the names back in each time, I am choosing from the full list of 15 students each time.)

a) What is the probability that I choose someone whose last name starts with an R for the first prize, someone whose last name starts with an L for the second prize, and someone whose last name starts with a Z for the third prize?
Answer as a percentage.

a) The probability of each event is independent of the others, so the probability of event 1 and event 2 and event 3 is the product of the probabilities. Count how many of each letter occur.

$$\left(\frac{2}{15}\right)\left(\frac{3}{15}\right)\left(\frac{1}{15}\right) = \frac{6}{15^3} = \frac{2}{1125} \approx 0.00177 \approx 0.18\%$$

b) *(This is a new scenario. The conditions in a) do not apply.)*
What is the probability that I choose your name for at least one of the three prizes?
Answer as a percentage.
Hint: What is the probability that I do NOT choose your name at all for any of the three prizes?

b) The probability of you not winning each time is $14/15$. Thus, the overall probability of you not winning any of the three times is given by that number cubed.

$$\left(\frac{14}{15}\right)\left(\frac{14}{15}\right)\left(\frac{14}{15}\right) = \frac{2744}{3375} \approx 81.3\%$$

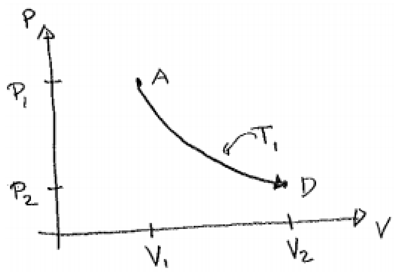
Thus, the overall chance of you winning is 100% - 81.3% = 18.7%.

4. [6 pts – 2 pts each]

Three processes performed on ideal gasses are shown below. In each case, indicate whether the work done *on* the system by the process shown is **positive, negative, or zero**. You do NOT need to find an expression for the amount of work done. ***Explain your reasoning.***

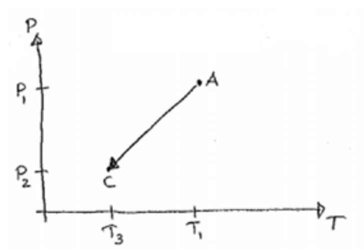
NOTE: Check the axes labels carefully! Part b) has different axes than parts a) and c).

a) *The process takes place at temperature T_1.*



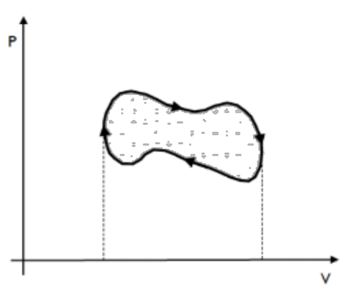
isotherm (expansion, W <0) ,

b) *The process proceeds along the straight line shown.*



NOTE AXES – (isochoric = W = 0)

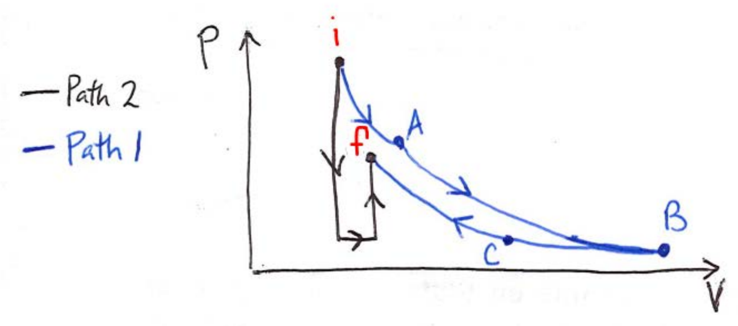
c) *The process is cyclic.*



Cycle (pressure higher when expanding, so W<0)

5. [6 pts]

The figure shows two paths that can be taken by an ideal gas from point i to point f.



Path 1 (in blue) consists of an isothermal expansion (from point i to A), then an adiabatic expansion (A to B), then an isothermal compression (B to C), and then an adiabatic compression (C to f).

The **magnitudes** of work done for a few of the sections are listed below.

i to A isotherm:	$\lvert W\rvert = 50\,J$
A to B adiabat:	$\lvert W\rvert = 40\,J$
B to C isotherm:	$\lvert W\rvert = 30\,J$
C to f adiabat:	$\lvert W\rvert = 25\,J$

Path 2 follows the path shown in black. What is the change in internal energy of the gas, ΔU, *if it follows Path **2*** from i to f? ***Explain your reasoning.***

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Since ΔE_{int} does not depend on the type of process,

$$\left(\Delta E_{\text{int}}\right)_{\text{path 2}} = \left(\Delta E_{\text{int}}\right)_{\text{path 1}}.$$

Also, since (for an ideal gas) it only depends on the temperature variable (so $\Delta E_{int} = 0$ for isotherms), then

$$\left(\Delta E_{\text{int}}\right)_{\text{path1}} = \sum\left(\Delta E_{\text{int}}\right)_{\text{adiabat}}.$$

Finally, since $Q = 0$ for adiabatic processes, then (for path 1)

$$\left(\Delta E_{\text{int}}\right)_{\text{adiabatic expansion}} = -W = -40\text{ J}$$
$$\left(\Delta E_{\text{int}}\right)_{\text{adiabatic compression}} = -W = -(-25)\text{ J} = 25\text{ J}.$$

Therefore, $\left(\Delta E_{\text{int}}\right)_{\text{path 2}} = -40\text{ J} + 25\text{ J} = -15\text{ J}$.

6. [12 pts – 6 pts each]

A two-state paramagnet has 5000 dipoles (spins).

Hint: I can think of two different ways of doing part a), but only one of those methods also works for part b). Be careful about the restrictions on any formulas you use.

a) Find the probability of having 2505 dipoles pointing up.
Give your answer as a percentage.

b) Find the probability of having 4995 dipoles pointing up.
My calculator could not handle this one as a percentage. Give your answer in the form of $e^{-\alpha}$, where α is a single number. (Hint: α is on the order of 1000.)

a)
For a two-state system, we have expression for the multiplicity that is a Gaussian distribution.

$$\Omega(N,x) \approx 2^N \sqrt{\frac{2}{N\pi}} e^{-\frac{2x^2}{N}}$$

x here is defined to be the difference from the most probable macrostate, *N/2*. NOTE: THIS ASSUMES THAT x << N, so it only works for part a).

For the current problem, *N* = 5000, and *x* = 2505 – 5000 / 2 = 5.

The probability is the multiplicity of the macrostate divided by the total multiplicity of the system. The total multiplicity for a two-state system is two to the power of (the number of dipoles).

$$\mathrm{Prob}(N = 5000, N_\uparrow = 2505) = \frac{\Omega(N = 5000, N_\uparrow = 2505)}{\Omega_{all}}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 2505) \approx \frac{2^{5000} \sqrt{\frac{2}{5000\pi}} e^{-\frac{2(5)^2}{5000}}}{2^{5000}}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 2505) \approx \sqrt{\frac{2}{5000\pi}} e^{-\frac{2(5)^2}{5000}}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 2505) \approx \sqrt{\frac{2}{5000\pi}} e^{-\frac{2(5)^2}{5000}}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 2505) \approx 1.1\%$$

b)

Note that we cannot use the Gaussian method here because the distance from the peak is not much smaller than the total number of dipoles.

The probability of getting the macrostate of 4995 spin up in 5000 flips is given by the number of microstates in that macrostate divided by the number of overall microstates.

$$\Omega(N = 5000, N_\uparrow = 4995) = \frac{N!}{N_\uparrow!\,(N - N_\uparrow)!}$$

$$\Omega(N = 5000, N_\uparrow = 4995) = \frac{5000!}{4995!\,(5000 - 4995)!}$$

$$\Omega(N = 5000, N_\uparrow = 4995) = \frac{5000!}{4995!\,5!}$$

The total number of microstates is the product of the number of microstates for each dipole (2).

$$\Omega_{all} = 2 \times 2 \times 2 \times \ldots = 2^{5000}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 4995) = \frac{\Omega(N = 5000, N_\uparrow = 4995)}{\Omega_{all}}$$

$$\mathrm{Prob}(N = 5000, N_\uparrow = 4995) = \frac{\frac{5000!}{4995!\,5!}}{2^{5000}}$$

Use the technique of taking the log and manipulating. (Later exponentiate to return to probability.)

$$\ln[\mathrm{Prob}(N = 5000, N_\uparrow = 4995)] = \ln\left(\frac{\frac{5000!}{4995!\,5!}}{2^{5000}}\right)$$

Use log properties.

$$\ln {}^{A}/_{B} = \ln A - \ln B$$

$$\ln A \times B = \ln A + \ln B$$

$$\ln[\mathrm{Prob}(N = 5000, N_\uparrow = 4995)] = \ln(5000!) - \ln(4995!) - \ln(5!) - \ln(2^{5000})$$

$$\ln A^B = B \ln A$$

$$\ln[\text{Prob}(N = 5000, N_\uparrow = 4995)] = \ln(5000!) - \ln(4995!) - \ln(5!) - 5000 \ln(2)$$

Use Stirling Approximation. $x! \approx x^x e^{-x}\sqrt{2\pi x}$ (Note the following equation takes up multiple lines.)

$$\begin{aligned}
&\ln[\text{Prob}(N = 5000, N_\uparrow = 4995)] \\
&\qquad \approx 5000 \ln(5000) - 5000 + \frac{1}{2}\ln(2\pi \times 5000) \\
&\qquad - \left[4995 \ln(4995) - 4995 + \frac{1}{2}\ln(2\pi \times 4995)\right] \\
&\qquad - \left[5 \ln(5) - 5 + \frac{1}{2}\ln(2\pi \times 5)\right] - 5000 \ln 2
\end{aligned}$$

$$\ln[\text{Prob}(N = 5000, N_\uparrow = 4995)] \approx 37591.1 - 37548.6 - 4.77 - 3465.74$$

$$\ln[\text{Prob}(N = 5000, N_\uparrow = 4995)] \approx -3427.92$$

Exponentiate to get the probability.

$$\text{Prob}(N = 5000, N_\uparrow = 4995) \approx e^{-3427.92}$$