Solutions to Problem Set 11

Problem 1a: Since $5 \cdot 1 - 2 \cdot 3 = -1$ is nonzero, we can use Theorem 2.3.10 to conclude that $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$. Therefore, $\alpha = (\vec{u}_1, \vec{u}_2)$ is a basis of $\mathbb{R}^2$.

Problem 1b: We have

$$\begin{aligned} T(\vec{u}_1) &= \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 30 - 21 \\ 20 - 15 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 5 \end{pmatrix}. \end{aligned}$$

Now to determine $[T(\vec{u}_1)]_\alpha$, we want to find $c_1, c_2 \in \mathbb{R}$ with

$$T(\vec{u}_1) = c_1 \vec{u}_1 + c_2 \vec{u}_2,$$

i.e. with

$$\begin{pmatrix} 9 \\ 5 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, we want to solve the following system of equations:

$$\begin{array}{rcrcr} 5x & + & 2y & = & 9 \\ 3x & + & y & = & 5. \end{array}$$

Either by working through the algebra, or by applying Proposition 2.1.1, we see that there is a unique solution, namely $(1, 2)$. In other words, we have

$$T(\vec{u}_1) = 1 \cdot \vec{u}_1 + 2 \cdot \vec{u}_2,$$

It follows that

$$[T(\vec{u}_1)]_\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Problem 1c: We have

$$\begin{aligned} T(\vec{u}_2) &= \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 12 - 7 \\ 8 - 5 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 3 \end{pmatrix}. \end{aligned}$$

Now to determine $[T(\vec{u}_2)]_\alpha$, we want to find $c_1, c_2 \in \mathbb{R}$ with

$$T(\vec{u}_2) = c_1 \vec{u}_1 + c_2 \vec{u}_2,$$

i.e. with

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In this case, it is immediate that

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, we have

$$T(\vec{u}_2) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2,$$

so

$$[T(\vec{u}_2)]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Problem 1d: The matrix $[T]_\alpha$ has $[T(\vec{u}_1)]_\alpha$ in the first column and $[T(\vec{u}_2)]_\alpha$ in the second column. Thus

$$[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

Problem 2: Let

$$P = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}.$$

Using Proposition 3.3.16, we then have

$$\begin{aligned} P^{-1} &= \frac{1}{5 \cdot 1 - 2 \cdot 3} \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} \\ &= (-1) \cdot \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}. \end{aligned}$$

Thus, by Proposition 3.4.7, we have

$$\begin{aligned} [T]_\alpha &= P^{-1} \cdot [T] \cdot P \\ &= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

Problem 3a: We have

$$\begin{aligned} T(\vec{v}) &= \begin{pmatrix} 6 & -7 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 - 14 \\ 4 - 10 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ -6 \end{pmatrix}. \end{aligned}$$

Now to determine $[T(\vec{v})]_\alpha$, we want to find $c_1, c_2 \in \mathbb{R}$ with

$$T(\vec{v}) = c_1\vec{u}_1 + c_2\vec{u}_2,$$

i.e. with

$$\begin{pmatrix} -8 \\ -6 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, we want to solve the following system of equations:

$$\begin{array}{rcrcr} 5x & + & 2y & = & -8 \\ 3x & + & y & = & -6. \end{array}$$

Either by working through the algebra, or by applying Proposition 2.1.1, we see that there is a unique solution, namely $(-4, 6)$. In other words, we have

$$T(\vec{v}) = (-4) \cdot \vec{u}_1 + 6 \cdot \vec{u}_2,$$

It follows that

$$[T(\vec{v})]_\alpha = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$$

Problem 3b: to determine $[\vec{v}]_\alpha$, we want to find $c_1, c_2 \in \mathbb{R}$ with

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2,$$

i.e. with

$$\begin{pmatrix} -8 \\ -6 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, we want to solve the following system of equations:

$$\begin{array}{rcrcr} 5x & + & 2y & = & 1 \\ 3x & + & y & = & 2. \end{array}$$

Either by working through the algebra, or by applying Proposition 2.1.1, we see that there is a unique solution, namely $(3, -7)$. It follows that

$$[\vec{v}]_\alpha = \begin{pmatrix} 3 \\ -7 \end{pmatrix}.$$

Therefore, using Proposition 3.4.3, we have

$$\begin{aligned} [T(\vec{v})]_\alpha &= [T]_\alpha \cdot [\vec{v}]_\alpha \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 7 \\ 6 + 0 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 6 \end{pmatrix}. \end{aligned}$$

Problem 4: Let

$$P = \begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix}.$$

Using Proposition 3.3.16, we have

$$\begin{aligned} P^{-1} &= \frac{1}{(-4)\cdot 4 - 9\cdot(-2)}\begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} \\ &= \frac{1}{2}\cdot\begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix}. \end{aligned}$$

Appealing to Proposition 3.4.7, we conclude that

$$\begin{aligned} [T]_\alpha &= P^{-1}\cdot[T]\cdot P \\ &= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -\frac{9}{2} \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -16 & 35 \\ -14 & 32 \end{pmatrix} \\ &= \begin{pmatrix} 31 & -74 \\ 12 & -29 \end{pmatrix}. \end{aligned}$$

Problem 5: Let A and B be 2×2 matrices and assume that $A\vec{v} = B\vec{v}$ for all $\vec{v}\in\mathbb{R}^2$. Fix $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$ with

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Applying our assumption in the case of $\vec{e}_1$, we know that $A\vec{e}_1 = B\vec{e}_1$. Since

$$A\vec{e}_1 = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \qquad \text{and} \qquad B\vec{e}_1 = \begin{pmatrix} a_2 \\ c_2 \end{pmatrix},$$

we conclude that $a_1 = a_2$ and $c_1 = c_2$. Applying our assumption in the case of $\vec{e}_2$, we know that $A\vec{e}_2 = B\vec{e}_2$. Since

$$A\vec{e}_2 = \begin{pmatrix} b_1 \\ d_1 \end{pmatrix} \qquad \text{and} \qquad B\vec{e}_2 = \begin{pmatrix} b_2 \\ d_2 \end{pmatrix},$$

we conclude that $b_1 = b_2$ and $d_1 = d_2$. Putting this all together, it follows that $A = B$.