Solutions to Written Assignment 2

Problem 1a: Yes, f is injective. To prove this, we need to show that the statement

"For all $m, n \in \mathbb{Z}$, if $f(m) = f(n)$, then $m = n$"

is true. Let $m, n \in \mathbb{Z}$ be arbitrary with $f(m) = f(n)$. By definition of f, we then have that

$$(3m^2 - 77, 5m + 6) = (3n^2 - 77, 5n + 6).$$

Since these two ordered pairs are equal, we know that their first coordinates are equal and their second coordinates are equal. Looking at second coordinates, we conclude that $5m + 6 = 5n + 6$ (we could also conclude that $3m^2 - 77 = 3n^2 - 77$, but that is irrelevant to our argument). Subtracting 6 from both sides, it follows that $5m = 5n$. Diving both sides by 5, we conclude that $m = n$.

We have taken arbitrary $m, n \in \mathbb{Z}$ with $f(m) = f(n)$, and shown that in this we must have $m = n$. Therefore, f is injective.

Problem 1b: No, f is not surjective. Recall that f is not surjective if the statement

"For all $(k, m) \in \mathbb{Z}^2$, there exists $n \in \mathbb{Z}$ with $f(n) = (k, m)$"

is false, which is the same as showing that its negation

"There exists $(k, m) \in \mathbb{Z}^2$ such that for all $n \in \mathbb{Z}$, we have $f(n) \neq (k, m)$"

is true. In order to verify this, we just need to provide an example of such a (k, m) (with justification). Consider the ordered $(-80, 0) \in \mathbb{Z}^2$. Now if $n \in \mathbb{Z}$ is arbitrary, then $f(n) = (3n^2 - 77, 5n + 6)$, and

$$\begin{aligned} 3n^2 - 77 &\geq 3 \cdot 0 - 77 \\ &= -77, \end{aligned}$$

so the first coordinate of $f(n)$ will be at least -77. Since the first coordinate of $(-80, 0)$ is not at least -77, we conclude that $f(n) \neq (-80, 0)$ for all $n \in \mathbb{Z}$.

Problem 2: To prove that $g \circ f \colon A \to C$ is surjective, we need to show that the statement

"For all $c \in C$, there exists $a \in A$ with $(g \circ f)(a) = c$"

is true. To prove this statement, let $c \in C$ be arbitrary. Since $g \colon B \to C$ is surjective, we can fix $b \in B$ with $g(b) = c$. Since $f \colon A \to B$ is surjective, we can fix $a \in A$ with $f(a) = b$. We then have

$$\begin{aligned} (g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c. \end{aligned}$$

Thus, we shown that there does exist $a \in A$ with $(g \circ f)(a) = c$. Since $c \in C$ was arbitrary, the result follows.

Problem 3: We first determine $f(1)$. Notice that

$$\begin{aligned} 5 &= f(2) \\ &= f(2 \cdot 1) \\ &= f(2) \cdot f(1) && \text{(by assumption)} \\ &= 5 \cdot f(1) \end{aligned}$$

so we have $5 = 5 \cdot f(1)$. Dividing both sides by 5, we conclude that $f(1) = 1$. Now notice that

$$\begin{aligned} f(6) &= f(2 \cdot 3) \\ &= f(2) \cdot f(3) && \text{(by assumption)} \\ &= 5 \cdot 7 \\ &= 35. \end{aligned}$$

Next, we have

$$\begin{aligned} 1 &= f(1) && \text{(from above)} \\ &= f\left(6 \cdot \frac{1}{6}\right) \\ &= f(6) \cdot f\left(\frac{1}{6}\right) && \text{(by assumption)} \\ &= 35 \cdot f\left(\frac{1}{6}\right) && \text{(from above)} \end{aligned}$$

so we conclude that $1 = 35 \cdot f(\frac{1}{6})$. Dividing both sides by 35, it follows that $f(\frac{1}{6}) = \frac{1}{35}$.