Worksheet 17 - Row Operations

1 Introduction

The matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1}$$

is equivalent to the system of two linear equations with two unknowns written as

$$ax_1 + bx_2 = y_1 \tag{2}$$
$$cx_1 + dx_2 = y_2 \tag{3}$$

We already have established the machinery that we need to use matrix operations to solve this system of equations, but in order to generalize this idea two more equations and more unknowns it will be helpful to look at the problem from another point of view.

In particular, before we knew anything about matrices, we might have tried to solve this system of equations using only the following *elementary row operations*:

1. swapping the order of the equations
2. adding equation one to equation two
3. adding equation one to equation one
4. multiplying equation one by a scalar
5. multiplying equation two by a scalar

Consider the first operation: swapping the order of the equations. The result of swapping (2) and (3) is

$$\begin{aligned} cx_1 + dx_2 &= y_2 \\ ax_1 + bx_2 &= y_1 \end{aligned}$$

This may seem pretty useless, for now, but I promise it will come in handy some day. Notice, also, that the matrix equation that encodes this system is slightly different than (1), and is given by

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \tag{4}$$

2 Applications and Practice

Exercise 1. Find a 2×2 matrix, A_1 such that

$$A_1 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

and

$$A_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}$$

Exercise 2. The matrix you found in Exercise 1, when multiplied on the left of both sides of Equation 1, results in Equation 4. As a result, we say that A_1 encodes the first elementary row operation. Find 2×2 matrices A_2, A_3, A_4, A_5 that encode the remaining elementary row operations.

Exercise 3. What do the matrices you found in Exercises 1 and 2 do if you multiply them on the right of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

instead of the left?

Exercise 4. Solve the following system of linear equations. First, by using the elementary row operations, one at a time, carefully listing each step. Then by applying the corresponding matrices you found above, in the correct order, to the left and right side of the matrix equation that encodes this system.

$$\begin{aligned} 3x_1 + 4x_2 &= 10 \\ 1x_1 + 8x_2 &= 5, \end{aligned}$$

Exercise 5. How might you define multiplication of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the left of a 2×3 matrix

$$\begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix}$$

so that the matrices you defined above will encode the same elementary row operations for a system of two equations in three variables? (Hint: Think of each column of the matrix as a vector.)

Exercise 6. What issues arise if you try to define a similar operation multiplying on the right?

Now, consider the system of linear equations in three variables given by

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 5 \qquad (5) \\ x_1 + x_2 + 4x_3 &= 6. \qquad (6) \end{aligned}$$

If one multiplies (5) by -1 and adds the result to (6), then multiplies (5) by -1 again, the result is

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 5 \qquad (7) \\ x_3 &= 1. \qquad (8) \end{aligned}$$

Exercise 7. How many solutions does system of two equations, (7) and (8), in three variables have? Express this as a vector or the span of a vector or vectors in $\mathbb{R}^3$, or the translation by a vector of a span of a vector or vectors in $\mathbb{R}^3$.

Exercise 8. Write the matrix equation that encodes (5) and (6). Use the multiplication you defined in Exercise 5 and the matrices from Exercise 2 to perform the operations that gave us (7) and (8).