Assignment: Problem Set 14

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Let $V = \mathbb{R}^3$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix}.$$

Show that there is no element of V that serves as $\vec{0}$. That is, show that there does not exist $\vec{z} \in V$ such that $\vec{v} + \vec{z} = \vec{v}$ for all $\vec{v} \in V$.

Solution: We assume that V is a vector space, with addition and scalar multiplication defined as above, but we rewrite the operators as $\oplus$ and $\odot$ so as to avoid confusion. Because V is a vector space, all 10 properties stated in Definition 4.1.1 are true, in addition to all of the propositions that follow from the Definition 4.1.1. Consider Property 5. Let $\vec{v} \in V$ be arbitrary. Since $\vec{v} \in V$, by Property 5 there must exist a $\vec{z} \in V$ with $\vec{v} \oplus \vec{z} = \vec{v}$ for all $\vec{v} \in V$. If there exists a $\vec{w} \in V$ that does not satisfy this property, it would follow that there is no element of V that serves as a $\vec{z}$ for all $\vec{v} \in V$. It would also follow that our assumption of V being a vector space is false. Consider $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in V$. Notice that $\vec{w} \oplus \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \vec{w}$. Because $\vec{v} \in V$ was arbitrary, we conclude that there is no $\vec{z} \in V$ for which $\vec{w} \oplus \vec{z} = \vec{w}$. We have found a $\vec{w} \in V$ for which Property 5 does not hold, so it must be the case that there does not exist any $\vec{z} \in V$ for which $\vec{v} \oplus \vec{z} = \vec{v}$ for all $\vec{v} \in V$. Because Property 5 does not hold, and it also follows that V is not a vector space.

Problem 2: Let $V = \mathbb{R}^2$, but with the following operations:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ a_2 \end{pmatrix}.$$

Also, let

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Show that V is not a vector space by explicitly finding a counterexample to one of the 10 properties.

Solution: Consider Property 8:

For all $\vec{v} \in V$ and all $c, d \in \mathbb{R}$, we have $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$. Let $\vec{v} \in V$ be arbitrary, and fix $x, y \in \mathbb{R}$ such that $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Notice that

$$\begin{aligned} (1 + 0) \cdot \vec{v} = 1 \cdot \vec{v} =& 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ =& \begin{pmatrix} 1 \cdot x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

and that

$$\begin{aligned} 1 \cdot \vec{v} + 0 \cdot \vec{v} =& 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + 0 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ =& \begin{pmatrix} 1 \cdot x \\ y \end{pmatrix} + \begin{pmatrix} 0 \cdot x \\ y \end{pmatrix} \\ =& \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix} \end{aligned}$$

So we have that $(1 + 0) \cdot \vec{v} \neq 1 \cdot \vec{v} + 0 \cdot \vec{v}$. Because $1, 0 \in \mathbb{R}$, we have found an explicit counterexample to Property 8. So V does not follow Property 8, and it follows that V is not a vector space.

Problem 3: Let V be a vector space. Show that $\vec{u}+(\vec{v}+\vec{w}) = \vec{w}+(\vec{v}+\vec{u})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$. Carefully state what property you are using in every step of your argument.

Solution: Let $\vec{u}, \vec{v}, \vec{w} \in V$ be arbitrary. Consider the sum $\vec{u} + (\vec{v} + \vec{w})$. Notice that

$$\begin{aligned}
\vec{u} + (\vec{v} + \vec{w}) &= (\vec{u} + \vec{v}) + \vec{w} && \text{(By Property 4)} \\
&= \vec{w} + (\vec{u} + \vec{v}) && \text{(By Property 3)} \\
&= \vec{w} + (\vec{v} + \vec{u}) && \text{(By Property 3)}
\end{aligned}$$

So $\vec{u} + (\vec{v} + \vec{w}) = \vec{w} + (\vec{v} + \vec{u})$. Because $\vec{u}, \vec{w}, \vec{v}$ were arbitrary, the result follows.

Problem 4: Let V be a vector space. Recall that, given $\vec{v} \in V$, we defined $-\vec{v}$ to be the unique $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$. Moreover, given $\vec{v}, \vec{w} \in V$, we defined $\vec{v} - \vec{w}$ to mean $\vec{v} + (-\vec{w})$. Prove each of the following, and carefully state what property and/or result you are using every step of your arguments.

a. Show that $-(\vec{v} + \vec{w}) = (-\vec{v}) + (-\vec{w})$ for all $\vec{v}, \vec{w} \in V$.

Solution: Let $\vec{v}, \vec{w} \in V$ be arbitrary. Consider $-(\vec{v} + \vec{w})$. Notice that

$$\begin{aligned}
-(\vec{v} + \vec{w}) &= (-1) \cdot (\vec{v} + \vec{w}) && \text{(By Proposition 4.1.11.3)} \\
&= (-1) \cdot \vec{v} + (-1) \cdot \vec{w} && \text{(By Property 7)} \\
&= (-\vec{v}) + (-\vec{w}) && \text{(By Proposition 4.1.11.3)}
\end{aligned}$$

So we have that $-(\vec{v} + \vec{w}) = (-\vec{v}) + (-\vec{w})$. Becasue $\vec{v}, \vec{w} \in V$ were arbitrary, the result follows.

b. Show that $c \cdot (\vec{v} - \vec{w}) = c \cdot \vec{v} - c \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in V$ and all $c \in \mathbb{R}$.

Solution: Let $\vec{v}, \vec{w} \in V, c \in \mathbb{R}$ be arbitrary. Consider $c \cdot (\vec{v} - \vec{w})$. Notice that

$$\begin{aligned}
c \cdot (\vec{v} - \vec{w}) &= c \cdot (\vec{v} + (-\vec{w})) && \text{(By Definition 4.1.10.2)} \\
&= c \cdot \vec{v} + c \cdot (-\vec{w}) && \text{(By Property 7)} \\
&= c \cdot \vec{v} + c \cdot ((-1) \cdot \vec{w}) && \text{(By Proposition 4.1.11.3)} \\
&= c \cdot \vec{v} + (c \cdot (-1)) \cdot \vec{w} && \text{(By Property 9)} \\
&= c \cdot \vec{v} + ((-1) \cdot c) \cdot \vec{w} && \\
&= c \cdot \vec{v} + (-1) \cdot (c \cdot \vec{w}) && \text{(By Property 9)} \\
&= c \cdot \vec{v} + (-(c \cdot \vec{w})) && \text{(By Proposition 4.1.11.3)} \\
&= c \cdot \vec{v} - (c \cdot \vec{w}) && \text{(By Definition 4.1.10.2)} \\
&= c \cdot \vec{v} - c \cdot \vec{w} &&
\end{aligned}$$

So we have that $c \cdot (\vec{v} - \vec{w}) = c \cdot \vec{v} - c \cdot \vec{w}$. Because $\vec{v}, \vec{w} \in V, c \in \mathbb{R}$ were arbitrary, the result follows.

Problem 5: Show that
$$\left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$$
is a subspace of $\mathbb{R}^3$.

Solution: Let $W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$. Notice that $W \subseteq \mathbb{R}^3$. Let $\vec{v}, \vec{w} \in W$ be arbitrary, and let $r \in \mathbb{R}$ be arbitrary. Fix $a, b, c, x, y, z \in \mathbb{R}$ such that $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. If W is a subspace of $\mathbb{R}^3$, W obeys the properties laid out in Definition 4.1.12, that is, W obeys

1. $\vec{0} \in W$
2. For all $\vec{w_1}, \vec{w_2} \in W$, we have $\vec{w_1} + \vec{w_2} \in W$
3. For all $\vec{w} \in W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{w} \in W$

Notice that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, and that $0 + 0 + 0 = 0$. So we have that $\vec{0} \in W$, and the first property is satisfied.

Notice that $\vec{v} + \vec{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$, and that $(a + x) + (b + y) + (z + c) = a+x+b+y+c+z = a+b+c+x+y+z = 0+0 = 0$, so $\vec{v}+\vec{w} \in W$. Since $\vec{v}, \vec{w} \in W$ were arbitrary, we have that $\vec{v} + \vec{w} \in W$ for all $\vec{v}, \vec{w} \in W$, so the second property is satisfied.

Notice that $r \cdot \vec{v} = r \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \end{pmatrix}$, and that $ra + rb + rc = r(a + b + c) = r(0) = 0$, so $r \cdot \vec{v} \in W$. Since $\vec{v} \in W, r \in \mathbb{R}$ were arbitrary, we have that $r \cdot \vec{v} \in W$ for all $\vec{v} \in W, r \in \mathbb{R}$, so the third property is satisfied.

All three properties defining a subspace have been satisfied, so we conclude that W is indeed a subspace of $\mathbb{R}^3$.