Problem Set # 2

1-9 $\vec{F} = qE_0 \cos(\omega t)\,\hat{y}$

$\vec{v}(0) = \vec{v}_0$

$\vec{r}(0) = 0$

$\vec{F} = m\vec{a}$

Note: vel. comp. in the $\hat{x}$ & $\hat{z}$ directions are const. v_{0x} & v_{0z}.

y component:

$$m\frac{dv_y}{dt} = qE_0 \cos(\omega t)$$

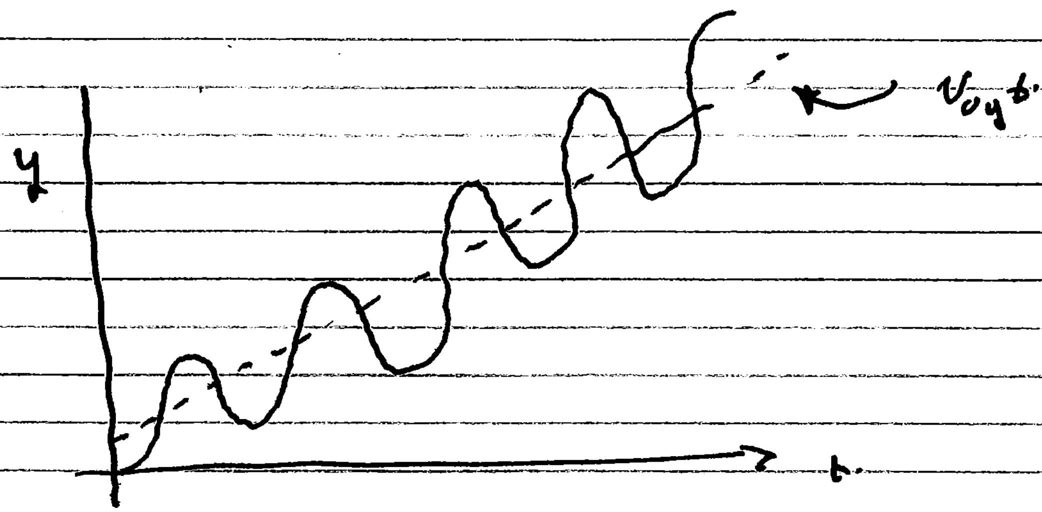
$$\frac{dv_y}{dt} = \frac{qE_0}{m}\cos(\omega t)$$

$$v_y = \frac{qE_0}{m\omega}\sin(\omega t) + v_{0y}$$

$$y = -\frac{qE_0}{m\omega^2}\cos(\omega t) + v_{0y}t + y_0 \qquad y_0 = \frac{qE_0}{m\omega^2}$$

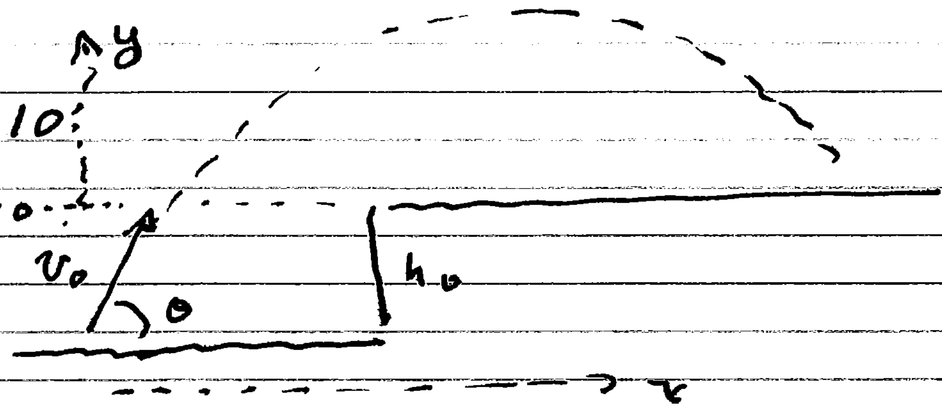
$$y = -\frac{qE_0}{m\omega^2}\cos(\omega t) + \frac{qE_0}{m\omega^2} + v_{0y}t$$

$$y = \left(1 - \cos(\omega t)\right)\frac{qE_0}{m\omega^2} + v_{0y}t$$



y
t
$v_{0y}t$

1-10:



$$\vec{r} = \frac{1}{2}\vec{g}\,t^2 + \vec{v}_0 t + \vec{r}_0$$

y comp:

$$y = -\frac{1}{2} g t^2 + v_0 \sin\theta\, t - h_0$$

Reaches $y=0$ when

$$0 = -\frac{1}{2} g t^2 + v_0 \sin\theta\, t - h_0$$

$$\frac{1}{2} g t^2 - v_0 \sin\theta + h_0 = 0$$

$$t = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta - 4\left(\frac{1}{2}g\right) h_0}}{g}$$

Want larger time.

$$t = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta - 2 g h_0}}{g}$$

x comp:

$$x = v_0 \cos\theta \, t$$

$$x = v_0 \cos\theta \left[\frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta - 2gh_0}}{g} \right]$$

$$= \frac{v_0^2 \cos\theta \sin\theta + v_0 \cos\theta \sqrt{v_0^2 \sin^2\theta - 2gh_0}}{g}$$

$$x = \frac{v_0^2 \cos\theta \sin\theta}{g} \left[1 + \sqrt{1 - \frac{2gh_0}{v_0^2 \sin^2\theta}} \right]$$

Approx

$$\sqrt{1+\varepsilon} \simeq 1 + \tfrac{1}{2}\varepsilon$$

$$x = \frac{v_0^2 \cos\theta \sin\theta}{g} \left(2 - \frac{gh_0}{v_0^2 \sin^2\theta} \right)$$

Note: $\cos\theta \sin\theta = \frac{1}{2} \sin(2\theta)$

$$x = \frac{v_0^2}{2g} \sin(2\theta) \left(2 - \frac{gh_0}{v_0^2 \sin^2\theta} \right)$$

max. for θ

$$0 = \frac{dx}{d\theta} = \frac{v_0^2}{2g}\, 2\cos(2\theta)\left(2 - \frac{g h_0}{v_0^2 \sin^2\theta}\right) +$$

$$\frac{v_0^2}{2g} \sin(2\theta)\left(-\frac{g h_0}{v_0^2}\left(-2\sin^{-3}\theta\right)\cos\theta\right)$$

$$0 = 2\cos(2\theta)\left(2 - \frac{g h_0}{v_0^2 \sin^2\theta}\right) +$$

$$+ \sin(2\theta)\left(\frac{2 g h_0}{v_0^2} \sin^{-3}\theta \cos\theta\right)$$

If $h_0 = 0 \Rightarrow 2\cos(2\theta) = 0$

$$\Rightarrow 2\theta = 90^\circ,\ \theta = 45^\circ$$

So

$$0 = 4\cos(2\theta) + \frac{g h_0}{v_0^2}\left[\underbrace{-\frac{2\cos(2\theta)}{v_0^2 \sin^2\theta} + \frac{2\sin(2\theta)\cos\theta}{\sin^3\theta}}_{\text{Put } \theta = 45^\circ \text{ in here}}\right]$$

$$-\frac{2\cos(90)}{v_0^2 \sin^2 45} + \frac{2\sin(90)\cos 45}{\sin^3 45} =$$

#

$$= 0 + \frac{2(1)^{1}/\sqrt{2}}{1/2^{3/2}}$$

$$= \frac{2}{\sqrt{2}}\, 2^{3/2} = 4$$

so

$$0 = \cancel{4} \cos(2\theta) + \frac{g h_0}{v_0^2}\, 4$$

$$0 = \cos(2\theta) + \frac{g h_0}{v_0^2}\, \cancel{4}$$

$$\theta = \frac{\pi}{4} + \delta$$

$$\cos(2\theta) = \cos\left(\frac{\pi}{2} + 2\delta\right)$$

$$\cong \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)(2\delta)$$

$$= 0 - 2\delta$$

So

$$0 = -2\delta + \frac{g h_0}{v_0^2}\, \cancel{4}$$

$$\delta = \frac{g h_0}{2 v_0^2} \quad \text{(radians)}$$

$$\theta \cong \frac{\pi}{4} + \frac{g h_0}{2 v_0^2}$$

1-16

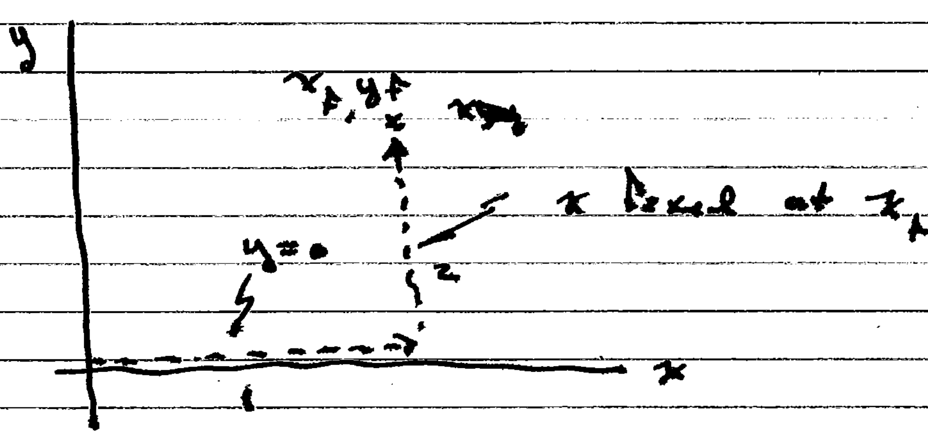
a) Check $\vec{\nabla} \times \vec{F}$

$$\vec{F} = c(x\hat{x} + y\hat{y})\, e^{-\alpha(x^2+y^2)}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

a) $$\left(0,\ 0,\ \underbrace{\frac{\partial\left(y e^{-\alpha(x^2+y^2)}\right)}{\partial x} - \frac{\partial\left(x e^{-\alpha(x^2+y^2)}\right)}{\partial y}}_{-2\alpha x y e^{-\alpha(x^2+y^2)} + 2\alpha y x e^{-\alpha(x^2+y^2)}} \right)$$

so $\bar{\nabla} \times \bar{F} = 0 \Rightarrow U$ exists.



$$\int F \cdot dr = \int_1 F_x\,dx + \int_2 F_y\,dy$$

$$F_x(x, y=0) = c\,x\,e^{-\alpha x^2}$$

$$\int_0^{x_f} F_x\,dx = c\int_0^{x_f} x\,e^{-\alpha x^2}\,dx = \frac{-c}{2\alpha}\,e^{-\alpha x^2}\Big|_0^{x_f}$$

$$= -\frac{c}{2\alpha}\left(e^{-\alpha x_f^2} - 1\right)$$

$$F_y(x_f, y) = c\,y\,e^{-\alpha x_f^2}\,e^{-\alpha y^2}$$

$$\int_0^{y_f} F_y\,dy = c\,e^{-\alpha x_f^2}\int y\,e^{-\alpha y^2}\,dy$$

$$= -\frac{c\,e^{-\alpha x_f^2}}{2\alpha}\left(e^{-\alpha y^3}\Big|_0^{y_f}\right.$$

$$= -\frac{c\,e^{-\alpha x_f^2}}{2\alpha}\left(e^{-\alpha y_f^2} - 1\right)$$

So

$$U(x_f, y_f) - U(0,0) =$$

$$\frac{c}{2\alpha}\left(e^{-\alpha x_f^2} - 1\right) + \frac{c}{2\alpha}\,e^{-\alpha x_f^2}\left(e^{-\alpha y_f^2} - 1\right)$$

$$\mathcal{U}(x_f, y_f) - \mathcal{U}(0,0) =$$

$$\frac{c}{2\alpha}\left[e^{-\alpha x_f^2} + e^{-\alpha(x_f^2+y_f^2)} - e^{-\alpha x_f^2}\right] + $$

$$-\frac{c}{2\alpha}$$

$$\mathcal{U}(x_f, y_f) - \mathcal{U}(0,0) = \frac{c}{2\alpha} e^{-\alpha(x_f^2+y_f^2)} +$$

$$-\frac{c}{2\alpha}.$$

b) $$F = c(y\hat{x} - x\hat{y})e^{-\alpha(x^2+y^2)}$$

$$\vec{\nabla}\times\vec{F} = \left(0,\ 0,\ -e^{-\alpha(x^2+y^2)} - y\frac{\partial}{\partial x}e^{-\alpha(x^2+y^2)}\right.$$

$$\left. - e^{-\alpha(x^2+y^2)} - x\frac{\partial}{\partial y}e^{-\alpha(x^2+y^2)}\right)$$

$$\neq 0$$

$\Rightarrow$ no potential energy

c) $\vec{F} = c\,(x\hat{x} + y\hat{y})\,e^{-\beta(x^2+y^2)^{1/2}}$

$$= c\,\vec{r}\,e^{-\beta r}$$

In polar coor.

$$\vec{\nabla} U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}$$

so U, if it exists, would be

$$F_r = -\frac{\partial U}{\partial r}$$

$$c\,r\,e^{-\beta r} = -\frac{\partial U}{\partial r}$$

$$dU = -c\,r\,e^{-\beta r}\,dr$$

From Tables

$$U(r_f) - U(0) = -c\int_0^{r_f} r\,e^{-\beta r}\,dr$$

$$U(r_f) - U(0) = -c\left(\frac{e^{-\beta r}}{\beta^2}(-\beta r - 1)\right)\Bigg|_0^{r_f}$$

$$U(r_f) - U(0) = -c\left(\frac{e^{-\beta r_f}}{\beta^2}(-\beta r_f - 1) - \frac{1}{\beta^2}(-1)\right)$$

$$= \frac{c}{\beta^2}\left(1 + e^{-\beta r_f}(\beta r_f + 1)\right)$$

d) $\vec{F} = c\,(x^3 y\,\hat{x} + x y^2\,\hat{y} + z\,\hat{z})$

$$\vec{\nabla} \times \vec{F} = (0,\ 0,\ y^2 - x^3) \neq 0$$

Potential energy does not exist

1-23.

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F}\cdot\vec{r} = m\frac{d\vec{v}}{dt}\cdot\vec{r}$$

Integrate over one period τ

$$\int_0^\tau \vec{F}\cdot\vec{r}\,dt = m\int_0^\tau \frac{d\vec{v}}{dt}\cdot\vec{r}\,dt$$

Note:

$$\frac{d\vec{v}}{dt}\cdot\vec{r} = \frac{d}{dt}(\vec{v}\cdot\vec{r}) - \vec{v}\cdot\underbrace{\frac{d\vec{r}}{dt}}_{=\,\vec{v}}$$

(equivalent to integration by parts)

$$\int_0^\tau \vec{F}\cdot\vec{r}\,dt = m\left\{\int_0^\tau \frac{d}{dt}(\vec{v}\cdot\vec{r}) - \int_0^\tau v^2\,dt\right\}$$

$$\int_0^\tau \vec{F}\cdot\vec{r}\,dt = m\left\{\underbrace{\vec{v}\cdot\vec{r}\,\Big|_0^\tau}_{=\,0} - \int_0^\tau mv^2\,dt\right.$$

[periodic motion]

$$\int_0^{\tau} \vec{F}\cdot\vec{r}\,dt = -\int_0^{\tau} m v^2\,dt$$

Note $KE = \frac{1}{2} m v^2$

$$\int_0^{\tau} KE\,dt = \tau \underbrace{\langle KE \rangle}_{\text{ave of KE.}}$$

So

$$\boxed{\langle \vec{F}\cdot\vec{r} \rangle = -2\langle KE \rangle}$$

For $\quad U = c r^n$

$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial r}\hat{r} = -c n r^{n-1}\hat{r}$$

$$= -c n r^{n-2}\vec{r}$$

$$\vec{F}\cdot\vec{r} = -c n r^{n-2}\,\vec{r}\cdot\vec{r} = -c n r^n$$

$$= -nU$$

So

$$\langle \vec{F}\cdot\vec{r} \rangle = -n\langle U \rangle$$

$$\Rightarrow \boxed{n\langle U \rangle = 2\langle KE \rangle}$$