Assignment: Written Assignment 4

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and that $\vec{u_1}, \vec{u_2} \in \mathbb{R}^2$. Show that if $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$, then $\text{Span}(T(\vec{u_1}), T(\vec{u_2})) = \mathbb{R}^2$.

Hint: You need only show that $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u_1}), T(\vec{u_2}))$, as the reverse containment is immediate. Start by taking an arbitrary $\vec{w} \in \mathbb{R}^2$. To show that $\vec{w} \in \text{Span}(T(\vec{u_1}), T(\vec{u_2})$, what do you need to do?

Solution: We have that $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$. Applying Theorem 2.3.10, it follows that that there does not exist $n \in \mathbb{R}$ with $n\vec{u_1} = \vec{u_2}$. Because T is surjective, it follows from Corollary 3.3.5 that T is bijective, that is, that there is exactly one $\vec{v} \in \mathbb{R}^2$ for every $\vec{u} \in \mathbb{R}^2$ with $\vec{u} = T(\vec{v})$. Combining this with the fact that that there does not exist $n \in \mathbb{R}$ with $n\vec{u_1} = \vec{u_2}$, it must be the case that that there does not exist $m \in \mathbb{R}$ with $mT(\vec{u_1}) = T(\vec{u_2})$ (if we assume that it were the case, this would imply $m\vec{u_1} = \vec{u_2}$ because T is bijective, but we know that there does not exist $n \in \mathbb{R}$ with $n\vec{u_1} = \vec{u_2}$, so we have a contradiction and so the previous statement about T follows). Applying Theorem 2.3.10, it follows that $\text{Span}(T(\vec{u_1}), T(\vec{u_2})) = \mathbb{R}^2$.

Problem 2: Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and that $\vec{u}, \vec{w} \in \mathbb{R}^2$. Show that if $\vec{w} \notin \operatorname{Span}(\vec{u})$, then $T(\vec{w}) \notin \operatorname{Span}(T(\vec{u}))$.

Solution: Let $\vec{u}, \vec{w} \in \mathbb{R}^2$ be arbitrary with the property that $\vec{w} \notin \operatorname{Span}(\vec{u})$. Because $\vec{w} \notin \operatorname{Span}(\vec{u})$, by definition there does not exist a $c \in \mathbb{R}$ with $c \cdot \vec{u} = \vec{w}$. We assume that there exists a surjective linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\vec{w}) \in \operatorname{Span}(T(\vec{u}))$. Because $T(\vec{w}) \in \operatorname{Span}(T(\vec{u}))$, we can fix a $d \in \mathbb{R}$ such that $T(\vec{w}) = d \cdot T(\vec{u})$ (by definition of Span). We conclude that $T(\vec{w}) = T(d \cdot \vec{u})$ (by the definition of linear transformation). Because T is surjective, it follows from Corollary 3.3.5 that T is bijective, that is, that for there is exactly one $\vec{v} \in \mathbb{R}^2$ for every $\vec{u} \in \mathbb{R}^2$ with $\vec{u} = T(\vec{v})$. So it must be the case that $\vec{w} = d \cdot \vec{u}$. But we have defined $\vec{u}, \vec{w}$ such that that there does not exist a $c \in \mathbb{R}$ with $c \cdot \vec{u} = \vec{w}$. Our assumption that there exists a surjective linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\vec{w}) \in \operatorname{Span}(T(\vec{u}))$ has led us to a contradiction, so it must be the case that $T(\vec{w}) \notin \operatorname{Span}(T(\vec{u}))$.

Problem 3: In this problem, we determine which 2×2 matrices commute with *every* 2×2 matrix.

a. Show that if $r \in \mathbb{R}$ and we let
$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$
then $AB = BA$ for every 2×2 matrix B.

Solution: Let B be an arbitrary 2×2 matrix, and fix $a, b, c, d \in \mathbb{R}$ with $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that:

$$\begin{aligned}
AB &= \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} && \text{(By definition of } A, B\text{)} \\
&= \begin{pmatrix} ra + 0 \cdot c & rb + 0 \cdot d \\ 0 \cdot a + rc & 0 \cdot b + rd \end{pmatrix} && \text{(By definition 3.2.1)} \\
&= \begin{pmatrix} ra + 0 & rb + 0 \\ 0 + rc & 0 + rd \end{pmatrix} = \begin{pmatrix} ar + 0 & br + 0 \\ 0 + cr & 0 + dr \end{pmatrix} \\
&= \begin{pmatrix} ar + 0 & 0 + br \\ cr + 0 & 0 + dr \end{pmatrix} = \begin{pmatrix} ar + b \cdot 0 & a \cdot 0 + br \\ cr + d \cdot 0 & c \cdot 0 + dr \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} && \text{(By definition 3.2.1)} \\
&= BA
\end{aligned}$$

Therefore, $AB = BA$. Because B was an arbitrary 2×2 matrix, the result follows.

b. Suppose that A is a 2×2 matrix with the property that $AB = BA$ for every 2×2 matrix B. Show that there exists $r \in \mathbb{R}$ such that
$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Solution: Let A, B be arbitrary 2×2 matrices, and fix $a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2} \in \mathbb{R}$ such that $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$, $B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$. We assume that that $AB = BA$, so we get:

$$\begin{aligned}
AB &= BA \\
\Rightarrow \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} &= \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} && \text{(By definition of } A, B\text{)} \\
\Rightarrow \begin{pmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{pmatrix} &= \begin{pmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} \end{pmatrix} && \text{(By definition 3.2.1)}
\end{aligned}$$

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Let $a_{1,1} = a_{2,2} = r, a_{1,2} = a_{2,1} = 0$, where $r \in \mathbb{R}$ is some fixed constant. So $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$. We test to see if $AB = BA$ is true

$$\begin{aligned}\begin{pmatrix} rb_{1,1} + 0b_{2,1} & rb_{1,2} + 0b_{2,2} \\ 0b_{1,1} + rb_{2,1} & 0b_{1,2} + rb_{2,2} \end{pmatrix} &= \begin{pmatrix} b_{1,1}r + b_{1,2}0 & b_{1,1}0 + b_{1,2}r \\ b_{2,1}r + b_{2,2}0 & b_{2,1}0 + b_{2,2}r \end{pmatrix} \\ \Rightarrow \begin{pmatrix} rb_{1,1} & rb_{1,2} \\ rb_{2,1} & rb_{2,2} \end{pmatrix} &= \begin{pmatrix} b_{1,1}r & b_{1,2}r \\ b_{2,1}r & b_{2,2}r \end{pmatrix},\end{aligned}$$

and it is!. We have found an $r \in \mathbb{R}$ such that $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ and has the property that $AB = BA$ for every 2×2 matrix B.