Worksheet 15 - Even More Eigenvalues and Eigenvectors

1 Practice and Applications

Exercise 1. Consider the matrix

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

(a) Find the characteristic polynomial of A from ws14 char poly $= (x-2)(x-8)$

(b) Find the real eigenvalues of A so, $\lambda_1 = 2$ and $\lambda_2 = 8$ are our eigenvalues

(c) For each real eigenvalue, find an associated eigenvector.

$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 5x + 3y = 2x \text{ and} \\ 3x + 5y = 2y \end{matrix} \Rightarrow x = -y \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 2$.

similarly

$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 5x + 3y = 8x \\ 3x + 5y = 8y \end{matrix}$ and $\Rightarrow x = y \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = 8$.

Exercise 2. Let A the matrix in Exercise 1.

1. Verify that the eigenvectors you found in Exercise 1 form a basis, α, for $\mathbb{R}^2$.
2. Compute $[A]_\alpha$.

$\alpha = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ is a basis by 2.3.10

Using 3.4.7 and 3.3.14 we get.

$[A]_\alpha = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$

Exercise 3. Corollary 3.5.20 guarantees that any matrix with two distinct eigenvalues is diagonalizable. Explain why you would expect this to be the case based on our geometric interpretation of eigenvalues and eigenvectors.

Let $\lambda_1, \lambda_2, \vec{u}_1$, and $\vec{u}_2$ be eigenvalues and vectors for A.

I.e. $A\vec{u}_1 = \lambda_1\vec{u}_1$ and $A\vec{u}_2 = \lambda_2\vec{u}_2$.

In $\alpha = (\vec{u}_1, \vec{u}_2)$ coordinates $[\vec{u}_1]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[\vec{u}_2]_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

thus, as A scales $\vec{u}_1$ and $\vec{u}_2$, we expect

$[A]_\alpha$ to scale $[\vec{u}_1]_\alpha$ and $[\vec{u}_2]_\alpha$. So, we expect $[A]_\alpha = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

and, in fact, by WS14 Exercise 4, we expect $[A]_\alpha = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Definition. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define the *transpose of* A, denoted A^T, by

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Exercise 4. Are the eigenvalues of A the same as the eigenvalues of A^T? If so, why? If not, give a counterexample.

Yes. They have the same characteristic polynomial:

$$(a-\lambda)(d-\lambda) - cb = (a-\lambda)(b-\lambda) - bc$$

Exercise 5. Are the eigenvectors of A the same as the eigenvectors of A^T? If so, why? If not, give a counterexample.

No.

$\begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 5 and 2 w/ eigenvectors

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 5 and 2 w/ eigenvectors

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Exercise 6. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the property that $a + b = c + d$.

(a) Prove that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) Find all eigenvalues of A.

for $\lambda = a+b = c+d$

use the quadratic formula

$$\lambda = \frac{1}{2}\left((a+d) \pm \sqrt{a^2 + 4bc - 2ad + d^2}\right)$$

Exercise 7. Let A be the 2×2 matrix

$$A = \begin{pmatrix} .4 & 1-c \\ .6 & c \end{pmatrix}$$

a) • What are the eigenvalues of A? (They will depend on c).

b) • Show that A only has one eigenvector if $c = 1.6$.

c) • When $c = 0.8$, then A^n will approach what matrix as n becomes very large?

a) $\lambda_1 = c - 3/5 \quad \lambda_2 = 1$

b) the char. poly will be $(\lambda - 1)^2$, so only one eigenvalue

c) $\lambda_1 = 1 \quad \lambda_2 = 1/5 \quad$ w/ $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $\alpha = \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ and $P = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{-4}\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}$

$[A]_\alpha = P^{-1}AP \Rightarrow [A]_\alpha^n = P^{-1}A^n P$

$[A]_\alpha^n = \begin{pmatrix} 1 & 0 \\ 0 & 1/5 \end{pmatrix}^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as n gets large

$\Rightarrow A^n \to P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1}$ as n gets large $\Rightarrow A^n \to \begin{pmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{pmatrix}$

Exercise 8. Look back at some examples and compare the sum of the eigenvalues of a given matrix to the sum of the entries along the diagonal of the matrix. Compare the product of eigenvalues to the product of entries along the diagonal. What might you conjecture to be true? (Try and prove your conjecture(s).)

D.I.Y.

For Next Time

- Prepare for the exam
- Send me questions you have from your review by Wednesday morning