MAT215 Exam 3

Olek Yardas

TOTAL POINTS

24 / 24

QUESTION 1

Definitions 6 pts

1.1 Subspace 2 / 2

✓ + **2 pts** Correct

+ **0 pts** Incorrect or not precise enough

1.2 Echelon form 2 / 2

✓ + **2 pts** Correct

+ **0 pts** Incorrect or not precise enough

1.3 Basis 2 / 2

✓ + **2 pts** Correct

+ **0 pts** Incorrect or not precise enough

QUESTION 2

2 Short Answer 6 / 6

✓ + **1.25 pts** **Noted that each of the matrices are in echelon form.**

✓ + **2 pts** **Described the role of leading entries in determinging the natures of the solutions sets.**

✓ + **2 pts** **Identified the inconsistent system, and the consistent systems. Also, the sizes of the solutions sets.**

✓ + **0.75 pts** **Used proper notation and terminology.**

+ **0 pts** Omitted.

QUESTION 3

Proofs 12 pts

3.1 Proposition 4.4.4. 6 / 6

✓ + **6 pts** Correct

+ **5 pts** This is a good piece of work, yet there are some mathematical errors, some writing errors, or some lack of detail that needs addressing.

+ **4 pts** There is some good intuition here, but there is at least one serious flaw.

+ **2.5 pts** Major flaws or omissions that need to be addressed.

+ **0 pts** Not enough to score.

3.2 Linear independence and dependence 6 / 6

✓ + **6 pts** Correct

+ **5 pts** This is a good piece of work, yet there are some mathematical errors, some writing errors, or some lack of detail that needs addressing.

+ **4 pts** There is some good intuition here, but there is at least one serious flaw.

+ **2.5 pts** Major flaws or omissions that need to be addressed.

+ **0 pts** Omitted or not enough correct progress to score.

QUESTION 4

4 Bonus 0 / 0

+ **1 pts** Correct and sufficiently justified.

✓ + **0 pts** **Omitted or not enough to score.**

Exam 3

MAT215 - Spring 2018

- All work must be your own.
- Do not start until instructed to do so.
- Scrap paper is available at the front of the room.
- Only use one side, of any page, for work you wish to have scored for credit. If you use scrap paper, write your name on the back of each page and turn it in with your exam.
- You are expected to remain in the exam room until you have completed your exam.
- Notes, or other references, are NOT permitted for use during the exam.
- You are expected to refrain from glancing at other students' exams, and to be reasonably careful that your responses are not visible to others. (This includes work on scrap pages.)
- Everything except for the writing utensils you will be using to complete the exam must be secured in your bag, and kept at the front of the room.
- Please, be sure that your phone is silenced completely (no vibrate mode), and no alarms are scheduled to go off during the exam.
- Phones, smart watches, or any device that might be used to communicate with others must be disabled, or completely inaccessible, during the exam.

Name: Oleksandr Vardes

1 Definitions - 6 points

For each of the following, state the definition as precisely as possible. No partial credit will be given. These statements should be identical, or nearly identical, to the statements given in the textbook. Proposition or theorem statements that provide conditions which are equivalent to the formal definition will not earn credit.

1. Let V be a vector space. Define *subspace of* V.

A subspace of V is a subset $W \subseteq V$ with the following properties:

- $\vec{0} \in W$
- For all $\vec{w_1}, \vec{w_2} \in W$, we have $\vec{w_1} + \vec{w_2} \in W$
- For all $\vec{w} \in W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{w} \in W$

2. Define *leading entry* and *echelon form.*

A leading entry of a row in a matrix is the entry that is the leftmost ~~non~~zero element in its row.

A matrix is said to be in echelon form if the following properties are true:

- All zero rows are below nonzero rows
- For each nonzero row (except the first row), the leading entry of that row is to the right of the leading entry in the row above it.

3. Let V be a vector space. Define *basis of* V.

A basis of V is a sequence of vectors $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ of V such that both of the following are true:

- $\mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) = V$
- $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ is linearly independent.

2 Short Answers - 6 points

Precisely describe (using proper terminology) how we can apply results from this class to describe the solution sets of the systems of equations encoded in the following augmented matrices:

- $A = \begin{pmatrix} 1 & 0 & 1 & 4 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

All leading entries are to the right of the leading entry in the row above it (except for the first row) and there are no zero rows, so by definition this matrix is in echelon form. Notice that this is a leading entry in the last column. Applying Prop. 4.2.12, we conclude the system encoded by this matrix is inconsistent, and therefore has no solution. Letting the solution set be S_1, it follows that $S_1 = \{\}$

6

2

- $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$

There are no zero rows and all leading entries are to the right of the leading entry in the row above it, so by definition this matrix is in echelon form. Notice that there is a leading entry in every column except for the last one, so by Prop 4.2.12, it follows that the system encoded by this matrix is consistent and has a unique solution.

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~~6~~

3

- $C = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$

There are no zero rows and all leading entries are to the right of the leading entry in the row above it, so by definition this matrix is in echelon form. Notice that there are no leading entries in the 2nd, 4th and last columns, so by proposition 4.2.12 the system encoded by this matrix is consistent and has infinite solutions. For each column that contains no leading entry (except for the last one) we can introduce parameters to vary them through $\mathbb{R}$:

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3 Proofs - 12 points

Prove the following statements. You must formally justify each of your claims by stating the definition or result from which it follows. Do not refer to results by their number in the text, state them completely.

3.1

Let V be a vector space and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w} \in V$. Show that the following are equivalent:

(a) $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}) = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$

(b) $\vec{w} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$

Let V be a vector space, and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w} \in V$ be arbitrary.

$b \Rightarrow a$

Suppose that $\vec{w} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$. By definition of Span, we can fix $c_1, c_2, \ldots, c_n \in \mathbb{R}$ with $\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_n\vec{u}_n$. Let $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$ be arbitrary, so by definition of Span we can fix $d_1, d_2, \ldots, d_n, d_{n+1} \in \mathbb{R}$ with

$\vec{v} = d_1\vec{u}_1 + d_2\vec{u}_2 + \ldots + d_n\vec{u}_n + d_{n+1}\vec{w}$. Notice that

$$\begin{aligned}
\vec{v} &= d_1\vec{u}_1 + d_2\vec{u}_2 + \ldots + d_n\vec{u}_n + d_{n+1}\vec{w} \\
&= d_1\vec{u}_1 + d_2\vec{u}_2 + \ldots + d_n\vec{u}_n + d_{n+1}(c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_n\vec{u}_n) && \text{(by def. of } \vec{w}\text{)} \\
&= d_1\vec{u}_1 + d_2\vec{u}_2 + \ldots + d_n\vec{u}_n + (d_{n+1}c_1)\vec{u}_1 + (d_{n+1}c_2)\vec{u}_2 + \ldots + (d_{n+1}c_n)\vec{u}_n && \text{(by properties of vector spaces)} \\
&= (d_1 + d_{n+1}c_1)\vec{u}_1 + (d_2 + d_{n+1}c_2)\vec{u}_2 + \ldots + (d_n + d_{n+1}c_n)\vec{u}_n
\end{aligned}$$

$(d_1 + d_{n+1}c_1), (d_2 + d_{n+1}c_2), \ldots, (d_n + d_{n+1}c_n) \in \mathbb{R}$, so $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ by definition of Span. Because $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$ was arbitrary, it follows that $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$. Now let $\vec{q} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ be arbitrary. By definition of Span we can fix $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\vec{q} = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n$. Notice that $\vec{q} = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n + 0 \cdot \vec{w}$, so $\vec{q} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$. Because $\vec{q} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ was arbitrary, it follows that $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$. We have proven both containments, so $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$.

$a \Rightarrow b$

Now suppose that $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$. Notice that $\vec{w} = 0 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 + \ldots + 0 \cdot \vec{u}_n + 1 \cdot \vec{w}$, so $\vec{w} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w})$. Because $\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n, \vec{w}) = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$, it follows that $\vec{w} \in \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$.

We have shown both implications, so the result follows.

3.2

Let V be a vector space and let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{w} \in V$. Suppose that $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ is linearly independent and that $(\vec{u}_1 + \vec{w}, \vec{u}_2 + \vec{w}, \ldots, \vec{u}_k + \vec{w})$ is linearly dependent. Show that $\vec{w} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$.

Let $\alpha = (\vec{u}_1+\vec{w}, \vec{u}_2+\vec{w}, \ldots, \vec{u}_k+\vec{w})$, let $\beta = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$

Fix $c_1, c_2, \ldots, c_k \in \mathbb{R}$ with $(\vec{u}_1+\vec{w})c_1 + (\vec{u}_2+\vec{w})c_2 + \ldots + (\vec{u}_k+\vec{w})c_k = \vec{0}$.

Notice that

$$\begin{aligned}\vec{0} &= (\vec{u}_1+\vec{w})c_1 + (\vec{u}_2+\vec{w})c_2 + \ldots + (\vec{u}_k+\vec{w})c_k \\ &= \vec{u}_1c_1 + \vec{w}c_1 + \vec{u}_2c_2 + \vec{w}c_2 + \ldots + \vec{u}_kc_k + \vec{w}c_k \quad \text{(by properties of vector spaces)} \\ &= \vec{u}_1c_1 + \vec{u}_2c_2 + \ldots + \vec{u}_kc_k + (c_1+c_2+\ldots+c_k)\vec{w} \quad \text{(by properties of vector spaces)}\end{aligned}$$

Subtracting $(c_1+c_2+\ldots+c_k)\vec{w}$ from both sides, we get

$-(c_1+c_2+\ldots+c_k)\vec{w} = \vec{u}_1c_1 + \vec{u}_2c_2 + \ldots + \vec{u}_kc_k$ ①

Because α is linearly dependent, by definition there exist $i \in \{1,2,\ldots,k\}$ such that at least one c_i is nonzero. ~~Consider the case where $\vec{w} = \vec{0}$. We then have that $-(c_1+c_2+\ldots+c_k)\vec{0} = \vec{0} = \vec{u}_1c_1 + \vec{u}_2c_2 + \ldots + \vec{u}_kc_k$. Because β is linearly independent, by definition $c_1 = c_2 = \ldots = c_k = 0$. Our assumption that $\vec{w} = \vec{0}$ has lead to a contradiction, so it must be the case that $\vec{w} \neq \vec{0}$.~~

~~Now~~ Consider the case where $-(c_1+c_2+\ldots+c_k) = 0$. We then have that $0 \cdot \vec{w} = \vec{0} = \vec{u}_1c_1 + \vec{u}_2c_2 + \ldots + \vec{u}_kc_k$. Because β is linearly independent, by definition we have that $c_1 = c_2 = \ldots = c_k = 0$. But we also have that there must exist at least one nonzero c_i. Our assumption that $-(c_1+c_2+\ldots+c_k) = 0$ has lead to a contradiction, so it must be the case that $-(c_1+c_2+\ldots+c_k) \neq 0$. Letting $s = \sum_{i=1}^{k} c_i$, we have $-s \neq 0$. Dividing by $-s$ on both sides of equation ①, we get $\vec{w} = \frac{c_1}{-s}\vec{u}_1 + \frac{c_2}{-s}\vec{u}_2 + \ldots + \frac{c_k}{-s}\vec{u}_k$.

$\frac{c_1}{-s}, \frac{c_2}{-s}, \ldots, \frac{c_k}{-s} \in \mathbb{R}$, so by definition of span, $\vec{w} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$.

4 Bonus - 1 Point

Consider the following matrices:

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 5 & -3 & -8 & -3 & 6 & 9 & 1 & -1 & -1 & 1 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & -5 & 9 & 3 & -9 & 3 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 4 & -9 & -1 & -1 & -1 \\
-2 & 0 & -2 & 2 & 0 & -6 & 2 & -2 & 0 & 8 & 0 & -6 & 5 & -1 & 9 \\
0 & 2 & -7 & 9 & 4 & -9 & -3 & -1 & -6 & -2 & -1 & -9 & 7 & -5 & 3 \\
0 & 0 & 0 & -5 & -7 & 9 & -2 & -9 & 3 & -8 & 8 & 6 & 4 & -3 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 6 & 7 & 9 & 7 & -3 \\
0 & 0 & 0 & 0 & 0 & -7 & 0 & -8 & -7 & -3 & -7 & -3 & 4 & -1 & 6 \\
0 & 0 & -2 & -1 & -7 & -7 & -3 & -6 & -5 & 2 & 5 & -7 & 3 & -4 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -5 & -1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 3 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -2 & 1 & 8 & 7 & -3 & 5 & -5 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 2 & 9 & 8 & -1
\end{pmatrix}$$

$$B = \begin{pmatrix}
0 & -5 & 3 & -3 & 5 & 9 & -5 & 5 & 0 & 6 & 0 & 2 & -4 & 1 & -4 \\
0 & 0 & 0 & 0 & -5 & -3 & -9 & 7 & 1 & 5 & -5 & 5 & 1 & 6 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 6 & -3 & 5 & -7 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 7 & -2 & 1 & 8 & 4 \\
0 & 0 & 1 & -4 & -4 & -3 & 6 & 5 & 2 & 9 & -1 & -6 & -2 & -1 & -4 \\
-6 & 2 & -9 & -7 & 4 & -1 & 5 & 3 & 9 & 6 & 0 & 9 & -2 & -2 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 5 & 1 & -6 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & -7 & 6 & 8 & -1 & -9 & 8 & -6 & 8 & 3 & 0 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 7 & 1 & 9 & -6 & 1 & -6 & 4 & 4 & 2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -6 & 8 & 8 & 8 & -6 & -9 & 5 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 7 & -3 & -2 & -3 & 9 & -8 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 9 & 2 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & -3
\end{pmatrix}$$

Is A row-equivalent to B? Justify your answer.

FROM Pg 3

The system encoded by this matrix is

$1x_1 + 2x_2 + 3x_3 + 4x_4 = 5$ (1)

$2x_2 + 3x_3 + 4x_4 = 5$ (2)

$3x_3 + 4x_4 = 5$ (3)

$4x_4 = 5.$ (4)

Back substituting, we can solve uniquely for each variable:

(2) $x_4 = \frac{5}{4}$, so (3) becomes $3x_3 + 5 = 5$, so $3x_3 = 0 \to x_3 = 0$

(2) becomes $2x_2 + 0 + 5 = 5 \to 2x_2 = 0 \to x_2 = 0$

(1) becomes $x_1 + 0 + 0 + 5 = 5 \to x_1 = 0$

So the solution set of the system is $\{(0, 0, 0, \frac{5}{4})\}$

~~The system encoded by this matrix is~~

~~$x_1 + 2x_2 + x_4 = 1$~~

~~$x_3 + x_4 = 3$~~

(3) We parametrize $x_2 = s$, $x_3 = t$ so solution set is $\left\{\begin{pmatrix}1\\0\end{pmatrix} + s\begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} : s, t \in \mathbb{R}\right\}$

$-4t$

~~$S_3 = \{(x_1, t, s, x_4) : x_1 = -2 + s - 2t, x_4 = 3 - s, t, s \in \mathbb{R}\}$~~