# Homework Assignment 3

PHYSICS 314 - THERMODYNAMICS & STATISTICAL PHYSICS (Spring 2018)

***Due Friday, February 16th, by noon, Noyce 1135***

---

*I cannot award full credit for work that I am unable to read or follow. For my benefit and for yours, please:*

- *Write neatly*
- *Show and EXPLAIN all steps*
- *Make diagrams large and clearly-labeled*

*You are welcome to collaborate with others on this assignment. However, the work you turn in should be your own. Please cite collaborators and outside sources. See the syllabus for details.*

*Regardless of the number of parts, all homework problems are weighted equally. Regardless of the number of questions, all homework assignments are weighted equally.*

---

1) A box is separated by a partition that divides its volume into the ratio 3 : 1. The larger portion of the box contains 1,000 molecules of Ne gas. The smaller portion contains 100 molecules of He gas. A small hole is punctured in the partition, and one waits until equilibrium is attained.
   a) Find the mean number of molecules of each type on either side of the partition.
   b) After a long time, what is the probability of finding 1,000 molecules of Ne gas in the larger portion and 100 molecules of He gas in the smaller portion (*i.e.*, the same distribution as in the initial system)?

2) Use Sterling's approximation to show that the multiplicity of an Einstein solid, for any large values of N (number of oscillators) and q (number of energy units) is roughly equal to the expression shown below.

$$\Omega \approx \frac{\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N}{\sqrt{2\pi q \frac{q+N}{N}}}$$

(The square root in the denominator in the final expression shown above is merely large. Often it can be neglected.) *Hint: Do not neglect the* $\sqrt{2\pi N}$ *in Stirling's approximation, and first show the following equality.*

$$\Omega = \frac{N}{q+N} \frac{(q+N)!}{q!\,N!}$$

3) For a single large two-state paramagnet, the multiplicity function is very sharply peaked about $N_\uparrow = N/2$. Use Stirling's approximation to estimate the height of the peak in the multiplicity function, that is, to approximate $\Omega\left(N_\uparrow = \frac{N}{2}\right)$. *Do not neglect the* $\sqrt{2\pi N}$ *in Stirling's approximation.* For full credit, simplify to the following form. (Each box represents a single number or variable.) *Hint: Use the* ln *manipulation technique for very large numbers*.

$$\Omega\left(N_\uparrow = \frac{N}{2}\right) \approx \blacksquare^{\blacksquare} \sqrt{\frac{\blacksquare}{\blacksquare\,\blacksquare}}$$

4) *This is a continuation of problem 3).* Follow the steps to derive a formula for the multiplicity function in the vicinity of the peak, in terms of $x \equiv N_\uparrow - (N/2)$.
   a) First show that the multiplicity can be written as shown.

$$\Omega(N_\uparrow) \approx \frac{N^N}{\left(\left({N}/{2}\right)^2 - x^2\right)^{N/2} \left({N}/{2} + x\right)^x \left({N}/{2} - x\right)^{-x}} \sqrt{\frac{N}{2\pi\left[\left({N}/{2}\right)^2 - x^2\right]}}$$

b) To deal with these numbers, take the natural log of the multiplicity. Show the following.

$$\ln \Omega(N_\uparrow) \approx N \ln N - \frac{N}{2} \ln\left[\left({N}/{2}\right)^2 - x^2\right] - x \ln[({N}/{2} + x)] + x \ln[({N}/{2} - x)] + \frac{1}{2} \ln\left(\frac{N}{2\pi}\right)$$
$$- \frac{1}{2} \ln\left(\left({N}/{2}\right)^2 - x^2\right)$$

c) In the vicinity of the peak, $N \gg x$. Use this fact to rewrite some of the terms in the multiplicity. Show the following.

$$\ln\left[\left({N}/{2}\right)^2 - x^2\right] \approx 2 \ln\left[\frac{N}{2}\right] - \left(\frac{2x}{N}\right)^2$$

$$\ln[{N}/{2} \pm x] \approx \ln[{N}/{2}] \pm \frac{2x}{N}$$

d) Plug the approximations from c) into the multiplicity term from b) and get an expression for the multiplicity. Check that your formula agrees with your answer to problem 3) when x = 0.

e) What is the half-width half-max of the multiplicity function from b)? (That is, at what value of x will it reach ½ its maximum value?)

5) The mathematics of the previous problems can also be applied to a one-dimensional random walk – a journey consisting of N steps, all the same size, each chosen randomly to be either forward or backward.
   a) Where are you most likely to find yourself after the end of a very long random walk? (Did you assume N was even? If so, what happens if N is odd?)
   b) Suppose you take a random walk of 10,000 steps. What is the half-width half-max (in steps) of the multiplicity function of possible ending positions for your random journey? This gives you an idea of where you can expect to end up. (Follow Schroeder's advice and ignore the weird unit problem.)

6) A good example of a random walk in nature is the diffusion of a molecule through a gas. The average step length is then referred to as the mean free path (l), given by Schroeder in Equation 1.62:

$$l = \frac{1}{4\pi r^2} \frac{V}{N}$$

(where r is the radius of the molecule in question). Assume this model is correct, and treat the situation as one-dimensional, neglecting any small numerical factors that might arise from the varying step size and the multi-dimensional nature of the path. Assume the gas is a diatomic ideal gas.
   a) Volume *V* and number of molecules *N* are sometimes tough to estimate in an open area. Find the mean free path in terms of the molecular radius *r*, the temperature *T*, and the pressure *P*.
   b) Estimate the average speed of a gas molecule in terms of temperature *T*, number of degrees of freedom *f*, and molecular mass *m*.
   c) Find the average time between collisions $t_{avg}$ in terms of *f*, *k*, *T*, *m*, *r*, and *P*.
   d) Show that the following expression is a (half-width half-max) estimate of the expected net displacement *D* of an air molecule traveling through air in a time Δt.

$$D = \frac{1}{r}\sqrt{\frac{\Delta t \ln 2}{8\pi P}}\left(\frac{f}{m}\right)^{1/4} k^{3/4}T^{3/4}$$

e) Assume T = 298 K and atmospheric pressure. Assume an air molecule has $r \approx 1.5 \times 10^{-10}\ m$ and $m = 46 \times 10^{-27}\ kg$. Let the total time be one second. Find a numerical answer for the expression in d).

7) A coin is flipped 400 times. Find the probability (as a percentage) of getting 215 heads in these 400 flips. Use the following two methods:
   *a)* Use Stirling's approximation for the factorials in the equation for Ω(n). *Hint: The $\sqrt{2\pi N}$ term cannot be neglected here.*
   *b)* Approximate the probability distribution for number of heads as a Gaussian distribution. *Hint: Use the result from 4d).*
   *c)* Do you answers match? *Hint: They should!*

8) List *three* main ideas from this homework assignment. For example, you could write a few-sentence explanation of a concept, or list an equation and explain the variables and in what circumstances the equation applies.
   *The goal is for you to review and to reflect on the big picture. Think about what you might want to remember when you look back at this homework before the test. I hope that this will be useful for your studying. I am not looking for anything specific here; you will be graded on effort and completion.*