y

Problem Set #4

2-4

$$\frac{dn}{dt} = \frac{nN}{\tau} - \frac{n}{\tau_0}$$

$$\frac{dN}{dt} = R - \frac{nN}{\tau} - \frac{N}{\tau_s}$$

a) $$\frac{dn}{dt} = 0 \Rightarrow \frac{n_0 N_0}{\tau} - \frac{n_0}{\tau_0} = 0$$

$$\frac{N_0}{\tau} = \frac{1}{\tau_0} \Rightarrow \boxed{N_0 = \frac{\tau}{\tau_0}} \quad (1)$$

$$\frac{dN}{dt} = 0 \Rightarrow R - \frac{n_0 N_0}{\tau} - \frac{N_0}{\tau_s} = 0$$

from (1) $$R - \frac{n_0}{\tau}\frac{\tau}{\tau_0} - \frac{\tau}{\tau_0 \tau_s} = 0$$

$$R - \frac{\tau}{\tau_0 \tau_s} = \frac{n_0}{\tau_0}$$

$$\boxed{R\tau_0 - \frac{\tau}{\tau_s} = n_0} \quad (2)$$

Sub. $N = N_0 + \Delta N$

$n = n_0 + \Delta n$

$$\frac{d\Delta n}{dt} = \frac{(n_0 + \Delta n)(N_0 + \Delta N)}{\tau} - \frac{n_0 + \Delta n}{\tau_0}$$

$$\frac{d\,\Delta n}{dt} = \frac{n_0 N_0}{\tau} + \frac{n_0 \Delta N}{\tau} + \frac{N_0 \Delta n}{\tau} - \frac{n_0}{\tau_0} - \frac{\Delta n}{\tau_0}$$

(drop $\Delta N \Delta n$ term)

$$\frac{d\,\Delta n}{dt} = \frac{n_0}{\tau}\left(\frac{\tau}{\tau_0}\right) + \frac{n_0 \Delta N}{\tau} + \frac{N_0 \Delta n}{\tau} - \frac{n_0}{\tau_0} + \frac{\Delta n}{\tau_0}$$

$$\frac{d\,\Delta n}{dt} = \frac{n_0}{\tau}\Delta N + \frac{N_0}{\tau}\Delta n - \frac{\Delta n}{\tau_0}$$

$$= \frac{n_0}{\tau}\Delta N + \frac{1}{\tau_0}\Delta n - \frac{\Delta n}{\tau_0}$$

$$\boxed{\frac{d\,\Delta n}{dt} = \frac{n_0}{\tau}\Delta N} \quad (3)$$

$$\frac{d\,\Delta N}{dt} = R - \frac{(n_0 + \Delta n)(N_0 + \Delta N)}{\tau} - \frac{N_0 + \Delta N}{\tau_s}$$

$$\frac{d\,\Delta N}{dt} = R - \frac{n_0 N_0}{\tau} - \frac{n_0 \Delta N}{\tau} - \frac{N_0 \Delta n}{\tau} - \frac{N_0}{\tau_s} - \frac{\Delta N}{\tau_s}$$

$$= R - \left(R\tau_0 - \frac{\tau}{\tau_s}\right)\frac{\tau}{\tau_0}\frac{1}{\tau} - \left(R\tau_0 - \frac{\tau}{\tau_s}\right)\frac{1}{\tau}\Delta N$$

$$- \frac{\tau}{\tau_0}\frac{\Delta n}{\tau} - \frac{\tau}{\tau_0}\frac{1}{\tau_s} - \frac{1}{\tau_s}\Delta N$$

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$$\frac{d\Delta N}{dt} = \cancel{R} - \cancel{R} + \cancel{\frac{I}{\tau_s \tau_0}} - \left(\frac{R\tau_0}{\tau} - \cancel{\frac{1}{\tau_s}}\right)\Delta N$$

$$- \frac{\Delta n_0}{\tau_0} - \cancel{\frac{\tau}{\tau_0 \tau_s}} - \cancel{\frac{1}{\tau_s}\Delta N}$$

$$\boxed{\frac{d\Delta N}{dt} = -\frac{R\tau_0}{\tau}\Delta N - \frac{\Delta n}{\tau_0}} \quad (4)$$

Take $\frac{d}{dt}$ of 4

$$\frac{d^2\Delta N}{dt^2} = -\frac{R\tau_0}{\tau}\frac{d\Delta N}{dt} - \frac{1}{\tau_0}\frac{d\Delta n}{dt}$$

use (3)

$$\frac{d^2\Delta N}{dt^2} = -\frac{R\tau_0}{\tau}\frac{d\Delta N}{dt} - \frac{1}{\tau_0}\frac{n_0}{\tau}\Delta N$$

This is a damped HO. Thus ΔN will osc. with $\omega_0^2 = \frac{1}{\tau_0 \tau} n_0$ and approach $\Delta N = 0$ or $N \to N_0$

So ΔN is something like

$$\Delta N = e^{-t/2\tau'} \cos(\omega_0 t)$$

From (4) we see that $\frac{d\Delta N}{dt}$ also does $\Longrightarrow$ so does Δn.
Approaches n_0, N_0

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soln. $x = a e^{\lambda t}$

where

$$\lambda = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} \equiv -\frac{\gamma}{2} \pm \frac{\gamma'}{2}$$

Define $\lambda_1 = \frac{\gamma}{2} - \frac{\gamma'}{2}$

$\lambda_2 = \frac{\gamma}{2} + \frac{\gamma'}{2}$

$$x = A e^{-\lambda_1 t} + B e^{-\lambda_2 t}$$

Put in initial Cond.

$$x(0) = A + B$$

$$\dot{x} = -\lambda_1 A e^{-\lambda_1 t} - \lambda_2 B e^{-\lambda_2 t}$$

$$\dot{x}(0) = -\lambda_1 A - \lambda_2 B$$

$$\frac{\dot{x}(0)}{\lambda_1} = -A - \frac{\lambda_2}{\lambda_1} B$$

So

$$x(0) + \frac{\dot{x}(0)}{\lambda_1} = B - \frac{\lambda_2}{\lambda_1} B$$

$$\frac{x(0) + \dot{x}(0)/\lambda_1}{\left(1 - \frac{\lambda_2}{\lambda_1}\right)} = B$$

$$B = \frac{\lambda_1 x(0) + \dot{x}(0)}{\lambda_1 - \lambda_2}$$

Now $\quad A = x(0) - B$

$$A = x(0) - \frac{\lambda_1 x(0) + \dot{x}(0)}{\lambda_1 - \lambda_2}$$

$$= \frac{x(0)(\lambda_1 - \lambda_2) - \lambda_1 x(0) - \dot{x}(0)}{\lambda_1 - \lambda_2}$$

$$= \frac{-\lambda_2 x(0) - \dot{x}(0)}{\lambda_1 - \lambda_2}$$

So

$$A = \frac{\lambda_2 x(0) + \dot{x}(0)}{\lambda_2 - \lambda_1}$$

$$B = -\frac{(\lambda_1 x(0) + \dot{x}(0))}{\lambda_2 - \lambda_1}$$

$$x = \frac{\lambda_2 x(0) + \dot{x}(0)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} +$$

$$- \frac{(\lambda_1 x(0) + \dot{x}(0))}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$\dot{x} = -\lambda_1 \frac{\lambda_2 x(0) + \dot{x}(0)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} +$$

$$+ \lambda_2 \frac{(\lambda_1 x(0) + \dot{x}(0))}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$\dot{x} = x(0)\left\{ -\lambda_1 \frac{\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \lambda_2 \frac{\left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right)}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} \right\}$$

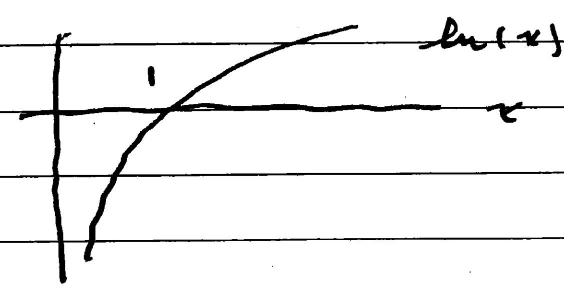
when is $\dot{x} = 0$

$$+\lambda_1\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right) e^{-\lambda_1 t} = \lambda_2\left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right) e^{-\lambda_2 t}$$

$$e^{(\lambda_2-\lambda_1)t} = \frac{\lambda_2\left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right)}{\lambda_1\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)}$$

$$\underbrace{(\lambda_2 - \lambda_1)}_{pos} t = \ln\left\{ \frac{\lambda_2\left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right)}{\lambda_1\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)} \right\}$$

~~thus~~



For this to have a soln. $\left\{ \frac{\lambda_2\left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right)}{\lambda_1\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)} \right\}$ must be > 1

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Two cases. both $\lambda_2 \left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right)$ &

$\lambda_1 \left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)$ both pos or

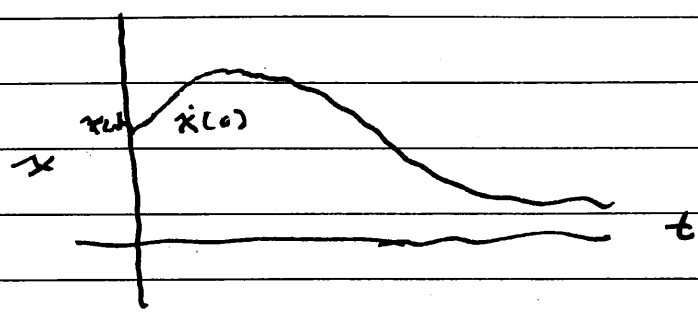
both neg.

Both pos.

$$\lambda_2 \left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right) > \lambda_1 \left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right)$$

$$\lambda_2 \frac{\dot{x}(0)}{x(0)} > \lambda_1 \frac{\dot{x}(0)}{x(0)}$$

~~or~~ There is a soln. if $\frac{\dot{x}(0)}{x(0)}$ pos.



both neg.

$$\lambda_2 \left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right) < \lambda_1 \left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right) < 0$$

$$\lambda_2 \frac{\dot{x}(0)}{x(0)} < \lambda_1 \frac{\dot{x}(0)}{x(0)}$$

Only true if $\frac{\dot{x}(0)}{x(0)}$ neg.

both neg $\Rightarrow$ $\frac{\dot{x}(0)}{x(0)}$ must be neg.

$$\frac{\dot{x}(0)}{x(0)} = -y$$

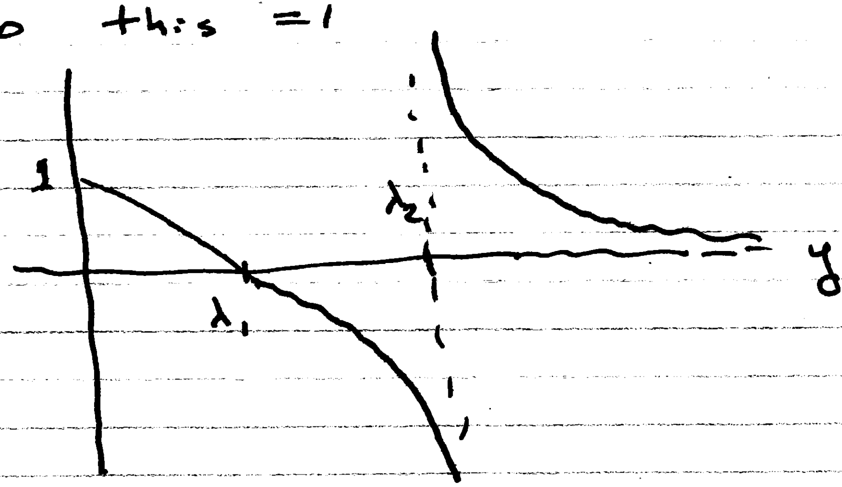
$$t = \frac{1}{\lambda_2 - \lambda_1} \ln \left\{ \frac{\lambda_2(\lambda_1 - y)}{\lambda_1(\lambda_2 - y)} \right\}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \ln \underbrace{\left\{ \frac{\lambda_2(y - \lambda_1)}{\lambda_1(y - \lambda_2)} \right\}}_{\text{want this to be } > 1}$$

Note at $y = \lambda_1$ it is zero - at

$y = \lambda_2$ it goes to $-\infty$ / $+\infty$

at $y = 0$ this $= 1$



So must have $y > \lambda_2$

or $\frac{\dot{x}(0)}{x(0)} < -\lambda_2$

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For $x = 0$

$$x = +\frac{\lambda_2 x(0) + \dot{x}(0)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1 x(0) + \dot{x}(0)}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

For $x = 0$

$$\lambda_2 x(0) + \dot{x}(0) e^{-\lambda_1 t} = \lambda_1 x(0) + \dot{x}(0) e^{-\lambda_2 t}$$

$$\left(\lambda_2 + \frac{\dot{x}(0)}{x(0)}\right) e^{-\lambda_1 t} = \left(\lambda_1 + \frac{\dot{x}(0)}{x(0)}\right) e^{-\lambda_2 t}$$

$$e^{(\lambda_2 - \lambda_1) t} = \frac{\lambda_1 + \frac{\dot{x}(0)}{x(0)}}{\lambda_2 + \frac{\dot{x}(0)}{x(0)}}$$

$$(\lambda_2 - \lambda_1) t = \ln \left[\frac{\lambda_1 + \frac{\dot{x}(0)}{x(0)}}{\lambda_2 + \frac{\dot{x}(0)}{x(0)}} \right]$$

For $\frac{\dot{x}(0)}{x(0)}$ pos this is always < 1

so no soln. For $\frac{\dot{x}(0)}{x(0)}$ neg. $\frac{\dot{x}(0)}{x(0)} = -y$

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{\lambda_1 - y}{\lambda_2 - y}\right) = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{y - \lambda_1}{y - \lambda_2}\right)$$

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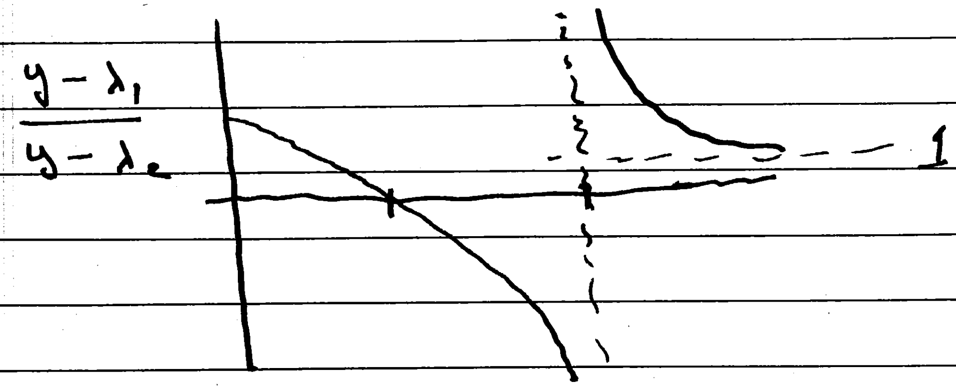


when is this $= 1$?

$$\frac{y-\lambda_1}{y-\lambda_2} = 1 \Rightarrow y-\lambda_1 = y-\lambda_2$$

never

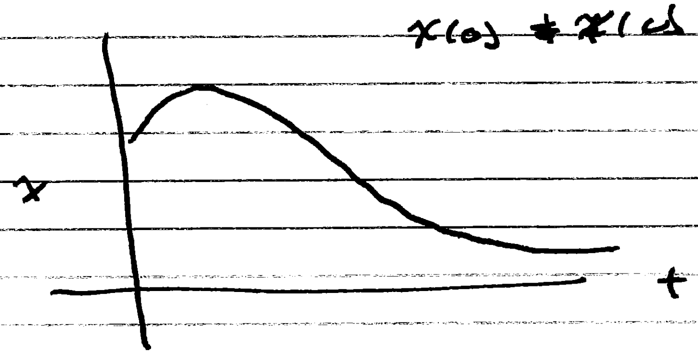
So for $\frac{y-\lambda_1}{y-\lambda_2}$ must have $y > \lambda_2$

$$\Rightarrow \frac{\dot{x}(0)}{x(0)} < -\lambda_2$$

So what have we learned - If $x(0)$ & $\dot{x}(0)$ are positive x will exceed $x(0)$ at some time, but it never reaches $x=0$. If $\frac{\dot{x}(0)}{x(0)}$ is neg. it will reach $x=0$ if $\frac{\dot{x}(0)}{x(0)} < -\lambda_2$

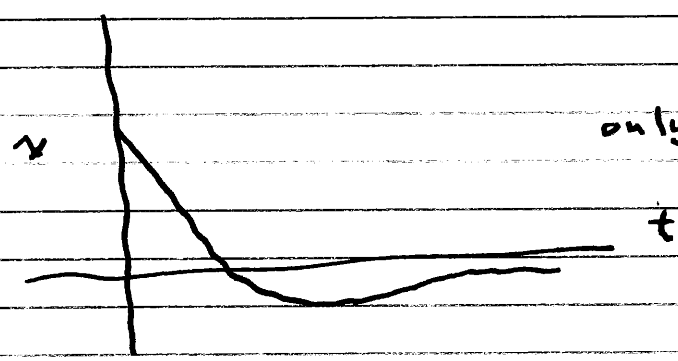
Behaviour.

$x(0)$ & $\dot{x}(0)$ pos.



only one max/min

For $x(0)$ pos & $\dot{x}(0)$ sufficiently neg.



only one crossing.