1 Recap

1.1 Interesting vectors

Eigenvectors are those for which a given transformation acts like scalar multiplication. That is, for a linear transformation encoded in a matrix A, the eigenvectors are those for which there is a non-zero scalar, λ, with the property that

$$A\vec{x} = \lambda\vec{x} \tag{1}$$

See discussion and proofs associated w/ 3.5.3-3.5.6

1.2 Examples

Example 1. Find the characteristic polynomial and real eigenvalues of

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix}$$

when does this have Null $\neq \{0\}$?

when

$$(5-\lambda)(5-\lambda) - 3 \cdot 3 = 0$$

$$\Rightarrow (5-\lambda)^2 = 9$$

$$\Rightarrow 5-\lambda = 3 \quad \text{or} \quad 5-\lambda = -3$$

$$\Rightarrow \lambda = 2 \quad \text{or} \quad \lambda = 8$$

Example 2. Find the characteristic polynomial and real eigenvalues of

$$A = \begin{pmatrix} 7 & 2 \\ -2 & 3 \end{pmatrix}$$

$$(A - \lambda I)$$

$$(7-\lambda)(3-\lambda)+4 = 0$$

$$21 - 10\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$(\lambda - 5)^2 = 0 \quad \text{or} \quad \lambda = 5$$

2 Exploration of some Theory

Let X be the set of all 2×2 matrices and define $f : X \to \mathbb{R}$ by

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc \tag{2}$$

Exercise 1. Given matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

Show that $f(AB) = f(A)f(B)$.

$$(AB) = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

$$f(A) \cdot f(B) = (ad - bc)(wz - yx)$$

$$= adwz - adyx - bcwz + bzyx$$

$$f(AB) = (aw + by)(cx + dz) - (ax + bz)(cw + dy)$$

$$\vdots$$

$$= adwz - adyx - bcwz + bzyx$$

Exercise 2. If A is invertible, show that

$$f(A^{-1}) = \frac{1}{f(A)}$$

By exercise 1 $f(I) = 1 = f(A \cdot A^{-1}) = f(A) \cdot f(A^{-1})$

$\Rightarrow f(A) = \dfrac{1}{f(A^{-1})}$ (possible because $f(A^{-1}) \neq 0$)

Exercise 3. Explain how the function, f, is related to the characteristic polynomial of a matrix.

$f(A - \lambda I)$ = characteristic polynomial of A.

Exercise 4. Suppose P is an invertible matrix and A and B are matrices with the property

$$A = PBP^{-1} \quad (1)$$

Show that A and B have the same characteristic polynomial and hence the same eigenvalues.

$$\begin{aligned}
f(A - \lambda I) &= f(PBP^{-1} - \lambda I(PP^{-1})) && \text{((1) and det of } P^{-1} \text{ and 3.2.8 \#4)} \\
&= f(PBP^{-1} - P(\lambda I)P^{-1}) && \text{(3.2.8 \#3,4 and 3.2.6 \#6)} \\
&= f(P(B - \lambda I)P^{-1}) && \text{(3.2.6 \# 4 and 5)} \\
&= f(P)f(B - \lambda I)f(P^{-1}) && \text{(Exercise 1)} \\
&= f(B - \lambda I) && \text{(Exercise 2)}
\end{aligned}$$

So, by Exercise 3 we have shown the result.

For Next Time

- Finish this worksheet
- Finish Reading Section 3.5
- Go through the review sheet for the exam.