Solutions to Written Assignment 5

Problem 1: No, it is not always possible. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^2$. Using Proposition 3.1.7, we know that T is a linear transformation and that

$$[T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\alpha = (\vec{u}_1, \vec{u}_2)$ be an arbitrary basis of $\mathbb{R}^2$. Notice that we have

$$\begin{aligned} T(\vec{u}_1) &= \vec{u}_1 \\ &= 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2, \end{aligned}$$

so

$$[T(\vec{u}_1)]_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly, we have

$$\begin{aligned} T(\vec{u}_2) &= \vec{u}_2 \\ &= 0 \cdot \vec{u}_1 + 1 \cdot \vec{u}_2, \end{aligned}$$

so

$$[T(\vec{u}_2)]_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that

$$[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, whenever $\alpha = (\vec{u}_1, \vec{u}_2)$ is a basis of $\mathbb{R}^2$, we have $[T]_\alpha = [T]$.

Problem 2: Let P be the 2×2 matrix with $\vec{u}_1$ in the first column and with $\vec{u}_2$ in the second column. By Proposition 3.4.7, we have that P is invertible and that

$$[T]_\alpha = P^{-1} \cdot [T] \cdot P.$$

Let Q be the 2×2 matrix with $\vec{w}_1$ in the first column and with $\vec{w}_2$ in the second column. Using Proposition 3.4.7 again, we have that Q is invertible and that

$$[T]_\beta = Q^{-1} \cdot [T] \cdot Q.$$

Multiplying both sides of the first equation on the left by P, we conclude that

$$P \cdot [T]_\alpha = [T] \cdot P.$$

Multiplying both sides of this new equation on the right by P^{-1}, we conclude that

$$P \cdot [T]_\alpha \cdot P^{-1} = [T].$$

Plugging this expression for $[T]$ into the second equation above involving $[T]_\beta$, it follows that

$$[T]_\beta = Q^{-1}P \cdot [T]_\alpha \cdot P^{-1}Q.$$

Since P is an invertible matrix, we can use Proposition 3.3.18 to conclude that P^{-1} is an invertible matrix and $(P^{-1})^{-1} = P$. Since P^{-1} and Q are both invertible, we can use Proposition 3.3.18 again to conclude that $P^{-1}Q$ is invertible and that

$$\begin{aligned}(P^{-1}Q)^{-1} &= Q^{-1}(P^{-1})^{-1} \\ &= Q^{-1}P.\end{aligned}$$

Therefore, we have

$$[T]_\beta = (P^{-1}Q)^{-1} \cdot [T]_\alpha \cdot P^{-1}Q.$$

Thus, we may let $R = P^{-1}Q$.

Problem 3a: Let A be an arbitrary 2×2 matrix. Since $II = I$, we have that I is invertible and that $I^{-1} = I$. Therefore, we have

$$\begin{aligned}A &= IAI \\ &= I^{-1}AI.\end{aligned}$$

It follows that $A \sim A$.

Problem 3b: Let A and B be two arbitrary 2×2 matrices with $A \sim B$. Since $A \sim B$, we can fix an invertible 2×2 matrix P with

$$B = P^{-1}AP.$$

Multiplying both sides of this equality on the left by P, we conclude that

$$PB = AP.$$

Multiplying both sides of this equality of the right by P^{-1}, we then have that

$$PBP^{-1} = A.$$

Using Proposition 3.3.18, we know that P^{-1} is invertible and that $(P^{-1})^{-1} = P$. Therefore, we have

$$\begin{aligned}A &= PBP^{-1} \\ &= (P^{-1})^{-1}BP^{-1}.\end{aligned}$$

Since P^{-1} is an invertible 2×2 matrix, it follows that $B \sim A$.

Problem 3c: Let A, B, and C be arbitrary 2×2 matrices with $A \sim B$ and $B \sim C$. Since $A \sim B$, we can fix an invertible 2×2 matrix P with

$$B = P^{-1}AP.$$

Since $B \sim C$, we can fix an invertible 2×2 matrix Q with

$$C = Q^{-1}BQ.$$

Plugging our first expression for B into the second equation, we have

$$C = Q^{-1}P^{-1}APQ.$$

Now since P and Q are both invertible, we can use Proposition 3.3.18 to conclude that PQ is invertible and that $(PQ)^{-1} = Q^{-1}P^{-1}$. Therefore, we have

$$\begin{aligned} C &= Q^{-1}P^{-1}APQ \\ &= (PQ)^{-1}APQ \end{aligned}$$

Since PQ is an invertible 2×2 matrix, it follows that $A \sim C$.