Exam 1

MAT215 - Spring 2018

- All work must be your own.
- Do not start until instructed to do so.
- Scrap paper is available at the front of the room.
- A proof attempt outline template is available at the front of the room. You may use this to help you organize your ideas, or as a reference for doing so, but you must transfer any work that you want scored for credit onto your main exam pages, or plain scrap pages that you turn in with your exam.
- Only use one side, of any page, for work you wish to have scored for credit. If you use scrap paper, write your name on the back of each page and turn it in with your exam.
- You are expected to remain in the exam room until you have completed your exam.
- No notes, or other references, are permitted for use during the exam.
- You are expected to refrain from glancing at other students' exams, and to be reasonably careful that your responses are not visible to others. (This includes work on scrap pages.)
- Everything except for the writing utensils you will be using to complete the exam must be secured in your bag, and kept at the front of the room.
- Please, be sure that your phone is silenced completely (no vibrate mode), and no alarms are scheduled to go off during the exam.
- Phones, smart watches, or any device that might be used to communicate with others must be disabled, or completely inaccessible, during the exam.

Name: Key

1 Definitions - 6 points

For each of the following, state the definition as precisely as possible. No partial credit will be given. These statements should be identical, or nearly identical, to the statements given in the textbook. Proposition or theorem statements that provide conditions which are equivalent to the formal definition will not earn credit.

1. Define *compostion of functions.*

If $f: A \to B$ and $g: B \to C$ are functions we define the composition of g and f to be the function

$$g \circ f : A \to C$$

defined by letting $g \circ f(a) = g(f(a))$ for all $a \in A$.

2. Define *surjective function.*

A function $f: A \to B$ is surjective if for all $b \in B$ there exists an $a \in A$ with $f(a) = b$.

3. Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. Define $Span(\vec{u}_1, \vec{u}_2)$.

$Span(\vec{u}_1, \vec{u}_2) = \{ c_1\vec{u}_1 + c_2\vec{u}_2 : c_1, c_2 \in \mathbb{R} \}$

2 Short Answers - 6 points

1. Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, and say

$$\vec{u}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

Consider the statement: If for all $\vec{v} \in \mathbb{R}^2$, there exist $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$, then $a_1b_2 - a_2b_1 \neq 0$

a) Give the negation of the statement:

$\forall \vec{v} \in \mathbb{R}^2 \ \exists c_1, c_2 \in \mathbb{R}$ w/ $\vec{v} = c_1\vec{u} + c_2\vec{u}$

<u>and</u> $a_1b_2 - a_2b_1 = 0$.

b) Give the contrapositive of the statement:

If $a_1b_2 - a_2b_1 = 0$ then $\exists \vec{v} \in \mathbb{R}^2$ such that $\vec{v} \neq c_1\vec{u}_1 + c_2\vec{u}_2 \ \forall c_1, c_2 \in \mathbb{R}$

2. Replace **(A)**,**(B)**,**(C)**, and **(D)** below with appropriate phrases so that the result is a correct proof of the statement "If $a, b \in \mathbb{Z}$ are both odd, then $a + b$ is even":

Let $a, b \in \mathbb{Z}$ be two arbitrary odd integers. Since a is odd, we can **(A)**. Since b is odd, we can **(B)**. Now notice that $a + b =$ **(C)**. Since **(D)** $\in \mathbb{Z}$, we conclude that $a + b$ is an even integer. Since $a, b \in \mathbb{Z}$ were arbitrary, the result follows.

(A) fix k with $a = 2k+1$ (def of odd)

(B) fix ℓ with $b = 2\ell + 1$ (def of odd)

(C) $2(k+\ell+1)$

(D) $k+\ell+1$

3 Proofs - 12 points

Prove the following statements. You must formally justify each of your claims by stating the definition or result from which it follows. Do not refer to results by their number in the text, state them completely.

3.1

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be arbitraty and let $S = Span(\vec{u}_1, \vec{u}_2)$. Prove that

a) $\vec{0} \in S$

b) For all $\vec{v}_1, \vec{v}_2 \in S$, we have that $\vec{v}_1 + \vec{v}_2 \in S$

c) For all $\vec{v} \in S$ and all $d \in \mathbb{R}$, we have $d \cdot \vec{v} \in S$

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be arbitrary and let $S = Span(\vec{u}_1, \vec{u}_2)$.

a) Notice that $0 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \in S$ by definition of span. Also $0 \cdot \vec{u}_1 + 0 \vec{u}_2 = \vec{0} + \vec{0} = \vec{0}$ by properties of vector addition and scaler multiplication. Thus $\vec{0} \in S$.

b) Let $\vec{v}_1, \vec{v}_2 \in S$ be arbitrary. By the def. of span, there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ w/

$$\vec{v}_1 = a_1 \vec{u}_1 + a_2 \vec{u}_2 \quad \text{and} \quad \vec{v}_2 = b_1 \vec{u}_1 + b_2 \vec{u}_2$$

Now $\vec{v}_1 + \vec{v}_2 = (a_1 + b_1)\vec{u}_1 + (a_2 + b_2)\vec{u}_2$

and $(a_1 + b_1), (a_2 + b_2) \in \mathbb{R}$ so $\vec{v}_1 + \vec{v}_2 \in S$.

c) Let $\vec{v} \in S$ and $d \in \mathbb{R}$ be arbitrary. By the def of span, there exist $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2$. Notice that $d\vec{v} = (dc_1)\vec{u}_1 + (dc_2)\vec{u}_2$ and $dc_1, dc_2 \in \mathbb{R}$. Thus $d\vec{v} \in S$, by def. of span.

Definition. A function $T : \mathbb{R}^2 \to \mathbb{R}^2$ where, for all $c_1, c_2 \in \mathbb{R}$ and all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, we have that

1. $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
2. $T(c_1 \cdot \vec{v}_1) = c_1 \cdot T(\vec{v}_1)$

is called a *linear transformation.*

3.2

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear transformations. Show that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. (*Hint:* You do not need to use any properties of linear transformations aside from the definition.)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^2$ be arbitrary linear transformations.

For all $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ we have

$$\begin{aligned} T\circ S(\vec{v}_1 + \vec{v}_2) &= T(S(\vec{v}_1 + \vec{v}_2)) && \text{def. of composition} \\ &= T(S(\vec{v}_1) + S(\vec{v}_2)) && \text{because } S \text{ is a lin. tran.} \\ &= T(S(\vec{v}_1)) + T(S(\vec{v}_2)) && \text{because } T \text{ is a lin. tran.} \\ &= T\circ S(\vec{v}_1) + T\circ S(\vec{v}_2) && \text{def of composition} \end{aligned}$$

So $T\circ S$ preserves vector addition.

Now, for all $\vec{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} T\circ S(c\vec{v}) &= T(S(c\cdot\vec{v})) && \text{def. of composition} \\ &= T(c\cdot S(\vec{v})) && \text{because } S \text{ is a lin. tran.} \\ &= c\,T(S(\vec{v})) && \text{because } T \text{ is a lin. tran.} \\ &= c\, T\circ S(\vec{v}) && \text{def of composition} \end{aligned}$$

So $T\circ S$ preserves scalar multiplication and we are done.