Solutions to Problem Set 7

Problem 1a: The function

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} xy \\ x+y \end{pmatrix}$$

is not a linear transformation. To see this, notice that

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) &= \begin{pmatrix} 1 \cdot 1 \\ 1+1 \end{pmatrix} + \begin{pmatrix} 1 \cdot 2 \\ 1+2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 5 \end{pmatrix} \end{aligned}$$

while

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) \\ &= \begin{pmatrix} 2 \cdot 3 \\ 2+3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 5 \end{pmatrix}. \end{aligned}$$

Since

$$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \neq T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right),$$

it follows that T is not a linear transformation.

Problem 1b: The function

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y\sin^2(x^3) + y\cos^2(x^3)) \\ y \end{pmatrix}$$

is a linear transformation. To see this, notice that for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned} T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \begin{pmatrix} y\sin^2(x^3) + y\cos^2(x^3)) \\ y \end{pmatrix} \\ &= \begin{pmatrix} y \cdot (\sin^2(x^3) + \cos^2(x^3)) \\ y \end{pmatrix} \\ &= \begin{pmatrix} y \\ y \end{pmatrix} \\ &= \begin{pmatrix} 0 \cdot x + 1 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix}. \end{aligned}$$

Therefore, T is a linear transformation by Proposition 2.4.3.

Problem 1c: The function

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y \\ 1 + y \end{pmatrix}$$

is not a linear transformation. To see this, notice that

$$\begin{aligned} T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 2\cdot 0 + 3\cdot 0 \\ 1+0 \end{pmatrix} + \begin{pmatrix} 2\cdot 1 + 3\cdot 0 \\ 1+0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

while

$$\begin{aligned} T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 2\cdot 1 + 3\cdot 0 \\ 1+0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

Since

$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \neq T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right),$$

it follows that T is not a linear transformation. Alternatively, we can simply notice that $T(\vec{0}) \neq \vec{0}$ from our above calculation, so T is not a linear transformation by Proposition 2.4.2.

Problem 2: We have the following:

- $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
- $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.
- $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.
- $T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Building off of the last example, notice that for any $x \in \mathbb{R}$, we have

$$T\left(\begin{pmatrix} x \\ -x \end{pmatrix}\right) = \begin{pmatrix} x \\ -x \end{pmatrix}.$$

In other words, T fixes every point on the line $y = -x$. Also, for every $x \in \mathbb{R}$, we have

$$T\left(\begin{pmatrix} x \\ x \end{pmatrix}\right) = \begin{pmatrix} -x \\ -x \end{pmatrix},$$

so T reflects every point on the perpendicular line $y = x$ across the origin. Looking at the first two examples above in conjunction with this, it starts to look like T reflects every point across the line $y = -x$.

Problem 3: Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}.$$

Notice that

$$\begin{aligned} T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 + 2 \cdot 0 \\ 3 \cdot 0 + 6 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, to show that T is not injective, it suffices to find a nonzero $\vec{v} \in \mathbb{R}^2$ with $T(\vec{v}) = \vec{0}$. Notice that

$$\begin{aligned} T\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) &= \begin{pmatrix} 2 + 2 \cdot (-1) \\ 3 \cdot 2 + 6 \cdot (-1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Since we have

$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = T\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right),$$

but

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

it follows that T is not injective.

Problem 4: We want to find $x, y \in \mathbb{R}$ with

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -18 \\ 47 \end{pmatrix},$$

which is the same as finding $x, y \in \mathbb{R}$ with

$$\begin{pmatrix} 2x - y \\ -5x + 3y \end{pmatrix} = \begin{pmatrix} -18 \\ 47 \end{pmatrix}.$$

Equivalently, we want to find a solution to the following system:

$$\begin{array}{rcrcr} 2x & - & y & = & -18 \\ -5x & + & 3y & = & 47. \end{array}$$

Since $2 \cdot 3 - (-1) \cdot (-5) = 1$, which is nonzero, we can use Proposition 2.1.1 to conclude that there is a unique solution, which is

$$\left(\frac{3 \cdot (-18) - (-1) \cdot 47}{1}, \frac{2 \cdot 47 - (-5) \cdot (-18)}{1}\right) = (-7, 4).$$

Thus, we have

$$T\left(\begin{pmatrix} -7 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} -18 \\ 47 \end{pmatrix},$$

and hence

$$\begin{pmatrix} -18 \\ 47 \end{pmatrix} \in \text{range}(T).$$

Problem 5: We first check that $T \circ S$ preserves addition. Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ be arbitrary. We have

$$\begin{aligned}
(T \circ S)(\vec{v}_1 + \vec{v}_2) &= T(S(\vec{v}_1 + \vec{v}_2)) && \text{(by definition)} \\
&= T(S(\vec{v}_1) + S(\vec{v}_2)) && \text{(since S is a linear transformation)} \\
&= T(S(\vec{v}_1)) + T(S(\vec{v}_2)) && \text{(since T is a linear transformation)} \\
&= (T \circ S)(\vec{v}_1) + (T \circ S)(\vec{v}_2) && \text{(by definition).}
\end{aligned}$$

Therefore, the function $T \circ S$ preserves addition.

We now check that $T \circ S$ preserves scalar multiplication. Let $\vec{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ be arbitrary. We have

$$\begin{aligned}
(T \circ S)(c \cdot \vec{v}) &= T(S(c \cdot \vec{v})) && \text{(by definition)} \\
&= T(c \cdot S(\vec{v})) && \text{(since S is a linear transformation)} \\
&= c \cdot T(S(\vec{v})) && \text{(since T is a linear transformation)} \\
&= c \cdot (T \circ S)(\vec{v}) && \text{(by definition).}
\end{aligned}$$

Therefore, the function $T \circ S$ preserves scalar multiplication as well. It follows that $T \circ S$ is a linear transformation.