Problem Set 6: Due Monday, February 19

Note: In Problems 1 and 4, please underline or write in a different color the parts that go into the blanks.

Problem 1: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement following statement: If $\vec{u}, \vec{w} \in \mathbb{R}^2$ and $\vec{w} \in \text{Span}(\vec{u})$, then $\text{Span}(\vec{w}) \subseteq \text{Span}(\vec{u})$.

Let $\vec{v} \in \text{Span}(\vec{w})$ be arbitrary. Since $\vec{w} \in \text{Span}(\vec{u})$, we can ________________. Since $\vec{v} \in \text{Span}(\vec{w})$, we can ________________. Now notice that $\vec{v} =$________________. Since ________________$\in \mathbb{R}$, we conclude that $\vec{v} \in \text{Span}(\vec{u})$. Since $\vec{v} \in \text{Span}(\vec{w})$ was arbitrary, the result follows.

Problem 2: Given $\vec{u} \in \mathbb{R}^2$, is the set $\text{Span}(\vec{u})$ always closed under componentwise multiplication? In other words, if

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \text{Span}(\vec{u}) \qquad \text{and} \qquad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \text{Span}(\vec{u}),$$

must it be the case that

$$\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} \in \text{Span}(\vec{u})?$$

Either argue that this is always true, or provide a specific counterexample (with justification).

Problem 3: Let $\vec{u}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, let $\vec{u}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, and let $\alpha = (\vec{u}_1, \vec{u}_2)$.

a. Show that $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, so $\alpha = (\vec{u}_1, \vec{u}_2)$ is a basis for $\mathbb{R}^2$.

b. Find the coordinates of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ relative to α. In other words, calculate $Coord_\alpha\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$.

c. Find the coordinates of $\begin{pmatrix} 8 \\ 17 \end{pmatrix}$ relative to α. In other words, calculate $Coord_\alpha\left(\begin{pmatrix} 8 \\ 17 \end{pmatrix}\right)$.

In each part, briefly explain how you carried out your computation.

Problem 4: In this problem we work through the proof of Proposition 2.9 in the notes, which says the following: Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. The following are equivalent.

1. $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$.
2. $\vec{u}_2 \in \text{Span}(\vec{u}_1)$.

Fill in the blanks below with appropriate phrases so that the result is a correct proof:

We first show that 1 implies 2. Assume then that 1 is true, so assume that $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$. Notice that $\vec{u}_2 =$________________. Since ________________$\in \mathbb{R}$, it follows that $\vec{u}_2 \in \text{Span}(\vec{u}_1, \vec{u}_2)$. Since $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$, we conclude that ________________.

We now show that 2 implies 1. Assume then that 2 is true, so assume that $\vec{u}_2 \in \text{Span}(\vec{u}_1)$. By definition, we can ________________. To show that $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$, we give a double containment proof.

- Using Proposition ________________, we know immediately that $\text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2)$.
- We now show that $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$. Let $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2)$ be arbitrary. By definition we can ________________. Notice that $\vec{v} =$________________. Since ________________$\in \mathbb{R}$, it follows that $\vec{v} \in \text{Span}(\vec{u}_1)$. Since $\vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2)$ was arbitrary, we conclude that $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$.

Since we haves shown both $\text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2)$ and $\text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1)$, we conclude that $\text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1)$.