Assignment: Written Assignment 8

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: Alicia Ledesma-Alonso

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Let V be a vector space, and let $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_m} \in V$. Assume that $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n})$ is linearly dependent. Show that $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_m})$ is linearly dependent.

Solution: Let $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_m} \in V$ be arbitrary. By definition, a sequence $(\vec{v_1}, \ldots, \vec{v_n})$ is linearly dependent if there exists $c_1, \ldots, c_n \in \mathbb{R}$ with $c_1\vec{u_1} + \cdots + c_n\vec{u_n} = \vec{0}$ such that at least one c_i is nonzero. By assumption, we have that $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n})$ is linearly dependent, so by definition we can fix $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} = \vec{0}$ with at least one nonzero a_i. Now fix $b_1, b_2, \ldots, b_m \in \mathbb{R}$ with $b_1 = b_2 = \cdots = b_m = 0$. Notice that

$$
\begin{aligned}
&a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} \\
&\quad + b_1\vec{w_1} + b_2\vec{w_2} + \cdots + b_m\vec{w_m} = a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} + \vec{0} + \vec{0} + \cdots + \vec{0} \quad \text{(By Proposition 4.1.11)} \\
=&a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} + \vec{0} \\
=&\vec{0}
\end{aligned}
$$

We conclude that $a_1\vec{u_1} + a_2\vec{u_2} + \cdots + a_n\vec{u_n} + b_1\vec{w_1} + b_2\vec{w_2} + \cdots + b_n\vec{w_n} = \vec{0}$. Because there is at least one nonzero a_i, by definition the sequence $(\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \ldots, \vec{w_n})$ is linearly dependent.

Problem 2: Let V be a vector space and let $\vec{u}, \vec{v}, \vec{w} \in V$. Assume that $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent. Show that $(\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w})$ is linearly independent.
Hint: Think carefully about how to start your argument. Remember that you want to prove that $(\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w})$ is linearly independent, which is a "for all'" statement.

Solution: Let $\vec{u}, \vec{v}, \vec{w} \in V$ be arbitrary and suppose that $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent. By definition, for all $x, y, z \in \mathbb{R}$, if $x\vec{u} + y\vec{v} + z + \vec{w} = \vec{0}$, then $x = y = z = 0$. Let $a, b, c \in \mathbb{R}$ be arbitrary, and suppose that $a(\vec{u}+\vec{v}) + b(\vec{u}+\vec{w}) + c(\vec{v}+\vec{w}) = \vec{0}$. Notice that

$$\begin{aligned} a(\vec{u}+\vec{v}) + b(\vec{u}+\vec{w}) + c(\vec{v}+\vec{w}) =& a\vec{u} + a\vec{v} + b\vec{u} + b\vec{w} + c\vec{v} + c\vec{w} \quad \text{(By Property 8 of vector spaces.)} \\ =& a\vec{u} + b\vec{u} + a\vec{v} + c\vec{v} + b\vec{w} + c\vec{w} \\ =& (a+b)\vec{u} + (a+c)\vec{v} + (b+c)\vec{w} \quad \text{(By Property 8 of vector spaces)} \end{aligned}$$

So $a(\vec{u}+\vec{v}) + b(\vec{u}+\vec{w}) + c(\vec{v}+\vec{w}) = (a+b)\vec{u} + (a+c)\vec{v} + (b+c)\vec{w}$. Because $a(\vec{u}+\vec{v}) + b(\vec{u}+\vec{w}) + c(\vec{v}+\vec{w}) = \vec{0}$ (by assumption), it follows that $(a+b)\vec{u} + (a+c)\vec{v} + (b+c)\vec{w} = \vec{0}$. Because $(\vec{u}, \vec{v}, \vec{w})$ is linearly independent, it must be the case that $(a+b) = 0$, $(a+c) = 0$, and $(b+c) = 0$. We have the following a system of linear equations in the variables a, b, c:

$$\begin{array}{ccccccc} 1a & + & 1b & + & 0c & = & 0 \\ 1a & + & 0b & + & 1c & = & 0 \\ 0a & + & 1b & + & 1c & = & 0 \end{array}$$

We use Gaussian elimination to find the solution set of this system:

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \\ -R_3 + R_2 \\ \\ \end{array} \\ &\to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} R_2 + R_1 \\ \\ \\ \end{array} \\ &\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{2}R_1 \\ -\frac{1}{2}R_1 + R_2 \\ \\ \end{array} \\ &\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ R_2 + R_3 \end{array} \end{aligned}$$

Notice that the above matrix is in echelon form and that there are no leading entries in the last column, so by Proposition 4.1.12 we conclude that there is a unique solution to the system, which is $(a, b, c) = (0, 0, 0)$, so the solution set of the system is $S = \{(0, 0, 0)\}$. Because a, b, c were arbitrary, it follows that for all $a, b, c \in \mathbb{R}$, if $a(\vec{u}+\vec{v}) + b(\vec{u}+\vec{w}) + c(\vec{v}+\vec{w}) = \vec{0}$ then $a = b = c = 0$. This satisfies the definition of linearly independent sequence. Therefore $(\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w})$ is linearly independent.

Problem 3: Let V be a vector space, and let $\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \dots, \vec{w_m} \in V$. Assume that both $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ and $(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$ are linearly independent.

a. Give an example of this situation where $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$ is linearly dependent.

Solution: Consider the following sequences of vectors in $\mathbb{R}^2$:

$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$$

and

$$\left(\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right)$$

Notice that $\begin{pmatrix}1\\0\end{pmatrix} \notin \text{Span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$ and that $\begin{pmatrix}0\\1\end{pmatrix} \notin \text{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$. Applying Proposition 4.3.2, we conclude that $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$ is linearly independent. By similar reasoning, we conclude that $\left(\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right)$ is linearly independent. Now consider the seqence $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right)$. Notice that we have 4 vectors in $\mathbb{R}^2$. Applying Corollary 4.3.5, we conclude that $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right)$ is linearly dependent.

b. Assume also that

$$\text{Span}(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}) \cap \text{Span}(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m}) = \{\vec{0}\}.$$

Show that $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$ is linearly independent.

Solution: If $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}, \vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$ is linearly independent, then by definition, for all $c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_m \in \mathbb{R}$, if $c_1\vec{u_1} + \cdots + c_n\vec{u_n} + d_1\vec{w_1} + \cdots + d_m\vec{w_m} = \vec{0}$, then $c_1 = c_2 = \cdots = c_n = d_1 = d_2 \cdots = d_m = 0$. Let $c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_m \in \mathbb{R}$ be arbitrary. Suppose that $c_1\vec{u_1} + \cdots + c_n\vec{u_n} + d_1\vec{w_1} + \cdots + d_m\vec{w_m} = \vec{0}$. Subtracting all the $d_j\vec{w_j}$ terms from the left hand side, we get $c_1\vec{u_1} + \cdots + c_n\vec{u_n} = -d_1\vec{w_1} - \cdots - d_m\vec{w_m}$. Suppose that $\text{Span}(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}) \cap \text{Span}(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m}) = \{\vec{0}\}$. By definition of set intersection, we get $\{\vec{v} \in V : \vec{v} \in \text{Span}(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}) \text{ and } \vec{v} \in \text{Span}(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m})\} = \{\vec{0}\}$, that is, $\vec{0}$ is the only vector that is a linear combination of $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ that is also a linear combination of $(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$. So by definition of linear combination, $c_1\vec{u_1} + \cdots + c_n\vec{u_n} = -d_1\vec{w_1} - \cdots - d_m\vec{w_m}$ implies that $c_1\vec{u_1} + \cdots + c_n\vec{u_n} = -d_1\vec{w_1} - \cdots - d_m\vec{w_m} = \vec{0}$. Because $(\vec{u_1}, \vec{u_2}, \dots, \vec{u_n})$ and $(\vec{w_1}, \vec{w_2}, \dots, \vec{w_m})$ are both linearly independent, we have that for all $c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_m \in \mathbb{R}$ that $c_1 = c_2 = \cdots = c_n = d_1 = d_2 = \cdots = d_m = 0$. Because $c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_m$ were arbitrary, the result follows.