# Problem Set 8

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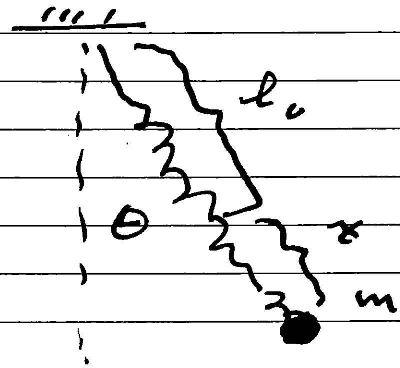


a)

$$T = \frac{1}{2} m \left[ (l_0 + x)^2 \dot{\theta}^2 + \dot{x}^2 \right]$$

$$U = - mg \cos\theta \, (l_0 + x) + \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m \left[ (l_0 + x)^2 \dot{\theta}^2 + \dot{x}^2 \right] + mg \cos\theta \, (l_0 + x) + - \frac{1}{2} k x^2$$

$\underline{\underline{\theta}}$

$$\frac{\partial L}{\partial \theta} = - mg \sin\theta \, (l_0 + x)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m (l_0 + x)^2 \dot{\theta}$$

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{\theta}} \right) = 2m (l_0 + x) \dot{x} \dot{\theta} + m (l_0 + x)^2 \ddot{\theta}$$

So

$$2m(l_0+x)\dot{x}\dot{\theta} + m(l_0+x)^2\ddot{\theta} =$$
$$- mg \sin\theta \, (l_0+x)$$

$$\frac{2\dot{x}\dot{\theta}}{l_0+x} + \ddot{\theta} = -\frac{g}{l_0+x}\sin\theta$$

$$\boxed{\ddot{\theta} = -\frac{g}{l_0+x}\sin\theta - \frac{2\dot{x}\dot{\theta}}{l_0+x}} \quad (1)$$

$\underline{\underline{x}}$

$$\frac{\partial L}{\partial x} = m(l_0+x)\dot{\theta}^2 + mg\cos\theta - kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

So

$$m\ddot{x} = m(l_0+x)\dot{\theta}^2 + mg\cos\theta - kx$$

$$\boxed{\ddot{x} = (l_0+x)\dot{\theta}^2 + \cancel{m}g\cos\theta - \frac{k}{m}x} \quad (2)$$

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b) equilibrium (stable) $\dot{x} = 0$ $\dot{\theta} = 0$

(1) $\Rightarrow$ $\sin\theta = 0$ $\Rightarrow$ $\theta = 0$ stable

(2) $\Rightarrow$ $g\cos\theta - \frac{k}{m}x = 0$

From (1) $\cos\theta = +1$

so $g = \frac{k}{m}x$ or $x_0 = \frac{m}{k}g$

equi' x value.

Sub. into (1) & (2) keeping only 1st order terms. $x = x_0 + x'$

(1) $\rightarrow$ $\ddot{\theta} = -\frac{g}{\ell_0 + x_0 + x'}\theta \rightarrow \boxed{\ddot{\theta} = \frac{-g}{\ell_0 + x_0}\theta}$

(2) $\rightarrow$ $\ddot{x}' = \cancel{\ldots} - \frac{k}{m}(x_0 + x') + g$

$\ddot{x}' = -\frac{k}{m}x_0 + g - \frac{k}{m}x'$

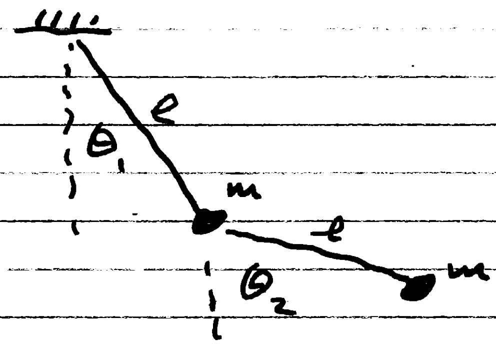
$\boxed{\ddot{x}' = -\frac{k}{m}x'}$

c)

Thus the motion is independ - that of a pendulum & a mass on a spring.

# Double Pendulum

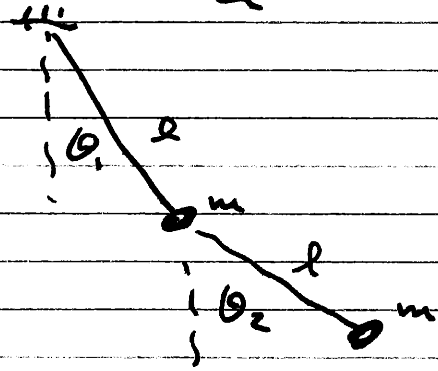
Consider the double pendulum shown below




a) Obtain the Lagrangian for this system. Simplify this expression as much as possible. Assuming both $\theta_1$ & $\theta_2$ small, obtain an expression for the Lagrangian correct to second order.

b) Use this approximate Lagrangian to obtain equations of motion for the system. Find the normal modes of the system.

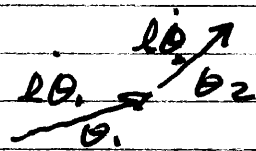
# Double Pendulum




Kinetic energy –

upper mass $\quad \frac{1}{2} m l^2 \dot{\theta}_1^2$

lower mass.




$$v^2 = (l\dot{\theta}_1 \cos\theta_1 + l\dot{\theta}_2 \cos\theta_2)^2 + (l\dot{\theta}_1 \sin\theta_1 + l\dot{\theta}_2 \sin\theta_2)^2$$

$$= l^2 \left[ \dot{\theta}_1^2 \cos^2\theta_1 + 2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2 + \dot{\theta}_2^2 \cos^2\theta_2 + \dot{\theta}_1^2 \sin^2\theta_1 + 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2 + \dot{\theta}_2^2 \sin^2\theta_2 \right]$$

$$= l^2 \left[ \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \sin\theta_2 \right]$$

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$$v^2 = l^2\left[\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2 \cos(\theta_2 - \theta_1)\right]$$

$$U = -mgl\cos\theta_1 - mgl(\cos\theta_1 + \cos\theta_2)$$

$$L = \tfrac{1}{2}ml^2\dot\theta_1^2 + \tfrac{1}{2}ml^2\left[\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)\right]$$
$$+2mgl\cos\theta_1 + mgl\cos\theta_2$$

This is without approx. at this pt.

For second order approx -

Note $\cos\theta \cong 1 - \frac{1}{2}\theta^2$

$$L \cong \tfrac{1}{2}ml^2\dot\theta_1^2 + \tfrac{1}{2}ml^2\left[\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\right]$$
$$- mgl\theta_1^2 - \tfrac{1}{2}mgl\theta_2^2$$

$$= ml^2\dot\theta_1^2 + \tfrac{1}{2}ml^2(\dot\theta_2^2 + 2\dot\theta_1\dot\theta_2$$
$$- mgl\theta_1^2 - \tfrac{1}{2}mgl\theta_2^2)$$

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$\underline{\underline{\theta_1}}$

$$\frac{\partial L}{\partial \theta_1} = -2mgl\,\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2ml^2\dot{\theta}_1 + ml^2\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2$$

$\Rightarrow$

$$-2mgl\,\theta_1 = 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2$$

$$-2\frac{g}{l}\theta_1 = 2\ddot{\theta}_1 + \ddot{\theta}_2$$

$$\boxed{\ddot{\theta}_1 = -\frac{g}{l}\theta_1 - \frac{1}{2}\ddot{\theta}_2}$$

or

$$-\ddot{\theta}_2 = 2\ddot{\theta}_1 + 2\frac{g}{l}\theta_1$$

$$\boxed{-\frac{1}{2}\ddot{\theta}_2 = \ddot{\theta}_1 + \frac{g}{l}\theta_1} \quad (1)$$

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$\theta_2$

$=$

$$\frac{\partial L}{\partial \theta_2} = -mgl\,\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2}ml^2\left(2\dot{\theta}_2 + 2\dot{\theta}_1\right) = ml^2(\dot{\theta}_2 + \dot{\theta}_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = ml^2(\ddot{\theta}_2 + \ddot{\theta}_1)$$

so

$$ml^2(\ddot{\theta}_2 + \ddot{\theta}_1) = -mgl\,\theta_2$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 = -\frac{g}{l}\theta_2$$

$$\boxed{\ddot{\theta}_2 + \frac{g}{l}\theta_2 = -\ddot{\theta}_1} \quad (2)$$

Normal modes

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix} e^{i\omega t}$$

Eq. (1) $\Rightarrow$ $$\frac{\omega^2}{2}B = \left(-\omega^2 + \frac{g}{l}\right)A$$

$$A\left(-\omega^2 + \frac{g}{l}\right) = \frac{\omega^2}{2}B \quad (3)$$

Eq (2) =>

$$\left(-w^2 + \frac{g}{l}\right) B = w^2 A \qquad (4)$$

mult. (3) & (4)

$$\left(-w^2 + \frac{g}{l}\right)^2 = \left(\frac{w^4}{2}\right)$$

$$\left(-w^2 + \frac{g}{l}\right) = \pm \frac{w^2}{\sqrt{2}}$$

$$\frac{g}{l} = w^2\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

define $\frac{g}{l} \equiv w_0^2$

$$w^2 = \frac{w_0^2}{1 \pm \frac{1}{\sqrt{2}}}$$

These are the freq. of the normal modes.

From eq. (3)

$$A\left[-\frac{w_0^2}{1 \pm \frac{1}{\sqrt{2}}} + w_0^2\right] = \frac{w_0^2}{1 \pm \frac{1}{\sqrt{2}}} B$$

$$A\left[-1+\left(1\pm\frac{1}{\sqrt{2}}\right)\right]=B$$

$$\pm\frac{1}{\sqrt{2}}A = B$$

So one mode has $\theta_1$ & $\theta_2$ in phase - the other has them out of phase -