Math 215.02, Spring 2018, Test Two Overview

What *not* to study

- No material from Chapter 1, except the concepts that are repeated in Chapters 2 or 3, will be on the test. Also, you will not be tested on Section 3.6.
- You will not have to derive formulas as in:
  - the proof of Proposition 2.1.1, pp. 44–46 (knowing the result and how to use it is quite important, though!)
  - the proofs of Proposition 2.3.13 p. 59 and Proposition 3.1.9 p. 82 (do not memorize formulas for Coord_α)
  - the proof of Proposition 3.1.10, pp. 83–84
  - the proof of Proposition 3.1.11, p. 86 (formula of the standard matrix of an orthogonal projection)
- You will not be asked to list all 10 items in Proposition 2.2.1, p. 47. You do need to know that $\mathbb{R}^2$ has these properties. Some of the properties will be in the definition list below.
- You will not be asked to give a formal proof of Theorem 2.3.10, p. 56, but you need to know how to apply the results.
- The material about how left inverses, right inverses, surjectivity, and injectivity are connected (p. 104–Proposition 3.3.9, p. 107) will not be on the test.

Best things to use for studying

- Class handouts and notes
- Problem sets and writing assignments
- Your own notes from reading the text

Definitions

As on Test 1, it is critical that you know the definitions precisely. We test definitions and their use because learning how to work with precise mathematical definitions is one of our major course goals. Here are the definitions that you can expect to be on the test.

Throughout, $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

- $\mathrm{Span}(\vec{u})$
- $\mathrm{Span}(\vec{u}, \vec{v})$
- a linear combination of $\vec{u}$ and $\vec{v}$
- a basis for $\mathbb{R}^2$
- The set $S \subseteq \mathbb{R}^2$ is closed under addition.
- The set $S \subseteq \mathbb{R}^2$ is closed under scalar multiplication.
- $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ is a linear transformation.
- $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ preserves addition.
- $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ preserves scalar multiplication.
- the standard basis for $\mathbb{R}^2$
- T is an injection (T is injective)
- T is a surjection (T is surjective)
- T is a bijection (T is bijective)
- $\mathrm{range}(T)$ (distinguish between vectors in the domain and codomain) (use set notation)
- $\mathrm{Null}(T)$ (distinguish between vectors in the domain and codomain) (use set notation)
- Suppose $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ is a linear transformation, and suppose α is a basis for $\mathbb{R}^2_{\text{domain}}$. What is the matrix $[T]_\alpha$?
- eigenvector of T
- eigenvalue of T
- characteristic polynomial of $[T]$
- additive inverse
- zero vector
- the formula for the inverse of $[T]$ (T is a bijective linear transformation)

Specific questions to answer to help prepare for the test

Throughout, $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

1. When is $\text{Span}(\vec{u}) = \text{Span}(\vec{u}, \vec{v})$? Note that you are being asked about the equality of two sets here.

2. Is $\text{Span}(\vec{u})$ closed under addition? scalar multiplication? Prove your claims or give specific counterexamples.

3. Is $\text{Span}(\vec{u}, \vec{v})$ closed under addition? scalar multiplication? Prove your claims or give specific counterexamples.

4. Is $\vec{0}$ always an element of $\text{Span}(\vec{u}, \vec{v})$? If so, explain why. If not, give a specific span that does not contain $\vec{0}$.

5. Can one vector span $\mathbb{R}^2$? Why or why not?

6. Explain how $\text{Span}(\vec{u}, \vec{v})$ is related to linear combinations of vectors in $\mathbb{R}^2$.

7. Suppose you are given a potential basis $\alpha = \left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$ for $\mathbb{R}^2$. How is the question of whether α is a basis for $\mathbb{R}^2$ connected to a system of linear equations in two variables? Explain. (You can't simply write "$ad - bc \neq 0$".)

8. Suppose $\text{Span}(\vec{u}) = \text{Span}(\vec{u}, \vec{v})$. What can you conclude about $\vec{v}$ and why? Explain.

9. Suppose $\alpha = \left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$ is a basis for $\mathbb{R}^2$. Suppose you are given a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Explain how to find $\text{Coord}_\alpha\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = [\begin{pmatrix} x \\ y \end{pmatrix}]_\alpha$ (different notation for the same thing) without simply using the formula on pages 59–60. That is, what does

$$\text{Coord}_\alpha\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} r \\ s \end{pmatrix}$$

mean relative to the basis α?

10. Give a concrete example of an injective linear transformation T; explain why T is injective.

11. Give a concrete example of an surjective linear transformation T; explain why T is surjective.

12. Give a concrete example of an non-injective linear transformation T; explain why T is not injective.

13. Give a concrete example of an non-surjective linear transformation T; explain why T is not surjective.

14. Suppose $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are two fixed vectors in $\mathbb{R}^2$. Explain how to use these two vectors to define a linear transformation $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$. (Prove that the function you define IS a linear transformation.)

15. Why does every linear transformation $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ have a standard matrix? How to you compute the standard matrix? How do you compute $T(\vec{v})$ using the standard matrix for T?

16. Suppose that $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ is a linear transformation. Show that $\text{Null}(T) = \{0\}$ if and only if T is injective.

17. Show that the composition of linear transformations is a linear transformation.

18. Is the composition of two surjective linear transformations (domain $\mathbb{R}^2$, codomain $\mathbb{R}^2$) necessarily surjective? Why or why not?

19. Is the composition of two injective linear transformations (domain $\mathbb{R}^2$, codomain $\mathbb{R}^2$) necessarily injective? Why or why not?

20. How does matrix multiplication relate to composition of linear transformations? Explain.

21. Suppose $T : \mathbb{R}^2_{\text{domain}} \to \mathbb{R}^2_{\text{codomain}}$ is a bijective linear transformation. Define T^{-1}; specify the domain and codomain of T^{-1}. Show that T^{-1} is a linear transformation.