# Solutions to Written Assignment 9

**Problem 1:** Let $U$ and $W$ be subspaces of $\mathbb{R}^6$ with $\dim(U) = 4$ and $\dim(W) = 3$. Since $\dim(U) = 4$, we can fix a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ of $U$. Since $\dim(W) = 3$, we can fix a basis $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$ of $W$. We then have that $(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ is linearly independent (because it is a basis of $U$) and $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$ is linearly independent (because it is a basis of $W$). However,

$$(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{w}_1, \vec{w}_2, \vec{w}_3)$$

is a sequence of 7 vectors in $\mathbb{R}^6$, so is linearly dependent by Corollary 4.3.5. Using the contrapositive of Problem 3b on Written Assignment 8, we conclude that we must have

$$\operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4) \cap \operatorname{Span}(\vec{w}_1, \vec{w}_2, \vec{w}_3) \neq \{\vec{0}\}.$$

Now by definition of a basis, we have $U = \operatorname{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ and $W = \operatorname{Span}(\vec{w}_1, \vec{w}_2, \vec{w}_3)$, so it follows that $U \cap W \neq \{\vec{0}\}$.

**Problem 2:** To show that $(T(\vec{u}_1), T(\vec{u}_2), \dots, T(\vec{u}_n))$ is linearly independent, we need to take arbitrary constants $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$c_1 \cdot T(\vec{u}_1) + c_2 \cdot T(\vec{u}_2) + \cdots + c_n \cdot T(\vec{u}_n) = \vec{0}$$

is true, and show that we have $c_i = 0$ for all $i \in \{1, 2, \dots, n\}$.

Let $c_1, c_2, \dots, c_n \in \mathbb{R}$ be arbitrary with

$$c_1 \cdot T(\vec{u}_1) + c_2 \cdot T(\vec{u}_2) + \cdots + c_n \cdot T(\vec{u}_n) = \vec{0}.$$

Since $T$ is a linear transformation, we know that

$$c_1 \cdot T(\vec{u}_1) + c_2 \cdot T(\vec{u}_2) + \cdots + c_n \cdot T(\vec{u}_n) = T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n),$$

so we can conclude that

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n) = \vec{0}.$$

We also know that $T(\vec{0}) = \vec{0}$ from Proposition 5.1.4, so we have

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n) = T(\vec{0}).$$

Since $T$ is injective, we conclude that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{0}.$$

Now we also know that $(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n)$ is linear independent, so since we know that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{0},$$

we can conclude that $c_i = 0$ for each $i \in \{1, 2, \dots, n\}$.