Assignment: Problem Set 23

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: None

- Problem 5: None

- Problem 6: Not Applicable

Problem 1: Calculate

$$\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix}.$$

Solution: We use the following facts to calculate $\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix}$:

- If B is obtained from A by interchanging two rows, then $\det(B) = -\det(A)$.
- If B is obtained from A by multiplying a row by c, then $\det(B) = c \cdot \det(A)$.
- If B is obtained from A by row combination, then $\det(B) = \det(A)$.

So we get

$$\begin{aligned}
\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix} &= \begin{vmatrix} 3 & 0 & -2 & 1 \\ 1 & 0 & 4 & -1 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix} && \begin{matrix} \\ -R_1 + R_2 \\ \\ \\ \end{matrix} \\
&= \begin{vmatrix} 0 & 0 & -14 & 4 \\ 1 & 0 & 4 & -1 \\ 0 & 2 & 12 & 2 \\ 0 & 0 & -9 & 2 \end{vmatrix} && \begin{matrix} -3R_2 + R_1 \\ \\ 5R_2 + R_3 \\ -3R_2 + R_4 \end{matrix} \\
&= \begin{vmatrix} 0 & 0 & 0 & \frac{8}{9} \\ 1 & 0 & 4 & -1 \\ 0 & 2 & 12 & 2 \\ 0 & 0 & -9 & 2 \end{vmatrix} && \begin{matrix} -\frac{14}{9}R_4 + R_1 \\ \\ \\ \\ \end{matrix} \\
&= \begin{vmatrix} 0 & 0 & -9 & 2 \\ 0 & 2 & 12 & 2 \\ 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & \frac{8}{9} \end{vmatrix} && \begin{matrix} R_4 \leftrightarrow R_1 \\ R_3 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3 \\ R_1 \leftrightarrow R_4 \end{matrix} \\
&= (-1)\begin{vmatrix} 1 & 0 & 4 & -1 \\ 0 & 2 & 12 & 2 \\ 0 & 0 & -9 & 2 \\ 0 & 0 & 0 & \frac{8}{9} \end{vmatrix} && \begin{matrix} R_3 \leftrightarrow R_1 \\ \\ R_1 \leftrightarrow R_3 \\ \\ \end{matrix} \\
&= (-1)(1 \cdot 2 \cdot (-9) \cdot \frac{8}{9}) && \text{(By Proposition 5.3.10)} \\
&= (-1)(2 \cdot (-8)) = 16
\end{aligned}$$

So $\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ -5 & 2 & -8 & 7 \\ 3 & 0 & 3 & -1 \end{vmatrix} = 16.$

Problem 2: Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5.$$

Find, with explanation, the value of

$$\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix}.$$

Solution: We use the following facts to calculate $\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix}$:

- If B is obtained from A by interchanging two rows, then $\det(B) = -\det(A)$.
- If B is obtained from A by multiplying a row by c, then $\det(B) = c \cdot \det(A)$.
- If B is obtained from A by row combination, then $\det(B) = \det(A)$.

We perform elementary row operations on $\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix}$:

$$\begin{aligned}
\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix} &= (-1)\begin{vmatrix} 2g+3a & 2h+3b & 2i+3c \\ -d & -e & -f \\ a & b & c \end{vmatrix} && \begin{matrix} R_2 \leftrightarrow R_1 \\ R_1 \leftrightarrow R_2 \end{matrix} \\
&= (-1)(-1)\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \end{vmatrix} && \begin{matrix} R_3 \leftrightarrow R_1 \\ R_1 \leftrightarrow R_3 \end{matrix} \\
&= (-1)(-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 2g+3a & 2h+3b & 2i+3c \end{vmatrix} && -R_2 \\
&= (-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 2g & 2h & 2i \end{vmatrix} && -3R_1 + R_3 \\
&= (-1)(2)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} && \tfrac{1}{2}R_3 \\
&= -(2)(5) = -10
\end{aligned}$$

So $\begin{vmatrix} -d & -e & -f \\ 2g+3a & 2h+3b & 2i+3c \\ a & b & c \end{vmatrix} = -10.$

Problem 3: Show that

$$\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = 0$$

for all $a, b, c, x, y \in \mathbb{R}$.

Solution: Let $a, b, c, x, y \in \mathbb{R}$ be arbitrary. Consider $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix}$. For a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, we have that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - gf) + c(dh - ge)$, so it follows that

$$\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} =$$

$$=a((b+x)(c+y) - (b+y)(c+x)) - b((a+x)(c+y) - (a+y)(c+x))$$
$$+c((a+x)(b+y) - (a+y)(b+x))$$
$$=a(b+x)(c+y) - a(b+y)(c+x) - b(a+x)(c+y) + b(a+y)(c+x) + c(a+x)(b+y) - c(a+y)(b+x)$$
$$=(ac + ay - ca - cy)(b+x) + (ba + by - ab - ay)(c+x) + (cb + cy - bc - by)(a+x)$$
$$=(ay - cy)(b+x) + (by - ay)(c+x) + (cy - by)(a+x)$$
$$=ayb + ayx - cyb - cyx + byc + byx - ayc - ayx + cya + cyx - bya - byx$$
$$=ayb - bya + ayx - ayx + byc - cyb + cyx - cyx + byx - byx + cya - ayc$$
$$=0 + 0 + 0 + 0 + 0 + 0 = 0$$

So $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = 0$. Because $a, b, c, x, y \in \mathbb{R}$ were arbitrary, the result follows.

Problem 4: Given $c \in \mathbb{R}$, consider the matrix

$$A_c = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}.$$

a. Use a cofactor expansion to compute $\det(A_c)$.

Solution: We find the cofactors C_{11}, C_{12}, C_{13}. By Definition 5.3.13, we have that

$$\begin{aligned} C_{11} =& (-1)^{1+1} \cdot \begin{vmatrix} 9 & c \\ c & 3 \end{vmatrix} = ((9)(3) - c^2 = 27 - c^2 \\ C_{12} =& (-1)^{1+2} \cdot \begin{vmatrix} 1 & c \\ 1 & 3 \end{vmatrix} = (-1)(3 - c) = c - 3 \\ C_{13} =& (-1)^{1+3} \cdot \begin{vmatrix} 1 & 9 \\ 1 & c \end{vmatrix} = c - 9 \end{aligned}$$

By Theorem 5.3.4, we have that

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} =& 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13} \\ =& 27 - c^2 + c - 3 + c - 9 = 15 + 2c - c^2 \end{aligned}$$

So $\det A_c = 15 + 2c - c^2$.

b. Find all values of c such that A_c is invertible. Explain.

Solution: By Corollary 5.3.11, A_c is invertible if and only if $\det(A_c) \neq 0$. So A_c is invertible if and only if $15 + 2c - c^2 \neq 0$. Using the quadratic equation, we find that $15 + 2c - c^2 = 0$ for $c = -3$ and $c = 5$. So A_c is invertible for all $c \in \mathbb{R} \setminus \{-3, 5\}$.

Problem 5: Find a basis for the eigenspace of the matrix

$$\begin{pmatrix} 1 & 4 & 1 \\ 6 & 6 & 2 \\ -3 & -4 & -3 \end{pmatrix}$$

corresponding to $\lambda = -2$.

Solution: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by letting $[T] = \begin{pmatrix} 1 & 4 & 1 \\ 6 & 6 & 2 \\ -3 & -4 & -3 \end{pmatrix}$. By Proposition 5.4.2, the eigenspace of $[T]$ corresponding to $\lambda = -2$ is the set $W = \{\vec{v} \in \mathbb{R}^3 : T(\vec{v}) = -2\vec{v}\}$. So we want to find all eigenvectors $\vec{v}$ of $[T]$ corresponding to eigenvalue -2. Let $\vec{v} \in \mathbb{R}^3$ be arbitrary and nonzero, and fix $x, y, z \in \mathbb{R}$ with $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Suppose that $T(\vec{v}) = -2\vec{v}$. By Definition 5.3.1, we have that $\vec{v}$ is an eigenvector of T corresponding to $\lambda = -2$. Notice that we have

$$\begin{aligned}
\vec{0}_{\mathbb{R}^3} =& T(\vec{v}) - (-2)\vec{v} \\
=& [T]\vec{v} + 2\vec{v} \\
=& ([T] + 2I_{\mathbb{R}^3})\vec{v} = \left(\begin{pmatrix} 1 & 4 & 1 \\ 6 & 6 & 2 \\ -3 & -4 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
=& \begin{pmatrix} 3 & 4 & 1 \\ 6 & 8 & 2 \\ -3 & -4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
=& \begin{pmatrix} 3x + 4y + z \\ 6x + 8y + 2z \\ -3x - 4y - z \end{pmatrix}
\end{aligned}$$

Notice that this is a linear system in the variables x, y, z, so to find $\vec{v}$ we just need to solve the system for x, y, z. We construct the augmented matrix and perform Gaussian Elimination:

$$\begin{pmatrix} 3 & 4 & 1 & 0 \\ 6 & 8 & 2 & 0 \\ -3 & -4 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} -2R_1 + R_2 \\ R_1 + R_3 \end{matrix}$$

We obtain the solution $z = -3x - 4y$. Introducing parameters $t = x, s = y$ we construct the solution set of the system: $\left\{ \begin{pmatrix} t \\ 0 \\ -3t \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ -4s \end{pmatrix} : t, s \in \mathbb{R} \right\} = \text{Span}\left(\left(\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \right) \right)$. So $W = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} \in \text{Span}\left(\left(\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \right) \right) \right\} = \text{Span}\left(\left(\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \right) \right)$.

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We now show that $\left(\begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\-4\end{pmatrix}\right)$ is linearly independent, and therefore a basis for Span $\left(\begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\-4\end{pmatrix}\right)$. Consider the 3×2 matrix with $\begin{pmatrix}1\\0\\-3\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-4\end{pmatrix}$ as its columns. Performing Gaussian Elimination on this matrix, we get

$$\begin{pmatrix}1 & 0\\0 & 1\\-3 & -4\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix} \begin{matrix}\\ \\ 3R_1 + 4R_2 + R_3\end{matrix}$$

Notice that there is a leading entry in every column, so by Proposition 4.3.3 $\left(\begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\-4\end{pmatrix}\right)$ is linearly independent. Therefore, $\left(\begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\-4\end{pmatrix}\right)$ is a basis for the eigenspace of $[T]$ corresponding to $\lambda = -2$.