Solutions to Problem Set 14

Problem 1: We need to argue that no vector serves as a zero vector. We prove this by contradiction. Suppose instead that such a vector does exist, and fix $\vec{z} \in V$ such that $\vec{v} + \vec{z} = \vec{v}$ for all $\vec{v} \in V$. Fix $a, b, c \in \mathbb{R}$ with

$$\vec{z} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

In particular, we then must have that

$$\vec{z} + \vec{z} = \vec{z},$$

hence

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Therefore, we must have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and hence

$$\vec{z} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

However, notice that

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \vec{z} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

so

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \vec{z} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore,

$$\vec{z} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

does not actually serve as a zero vector. We have reached a contradiction, so it follows that no element of V serves as $\vec{0}$.

Problem 2: We argue that Property 8 is not true by providing a specific counterexample. Notice that

$$\begin{aligned}(1+2)\cdot\begin{pmatrix}5\\4\end{pmatrix} &= 3\cdot\begin{pmatrix}5\\4\end{pmatrix}\\ &= \begin{pmatrix}3\cdot 5\\4\end{pmatrix}\\ &= \begin{pmatrix}15\\4\end{pmatrix},\end{aligned}$$

while

$$\begin{aligned}1\cdot\begin{pmatrix}5\\4\end{pmatrix}+2\cdot\begin{pmatrix}5\\4\end{pmatrix} &= \begin{pmatrix}1\cdot 5\\4\end{pmatrix}+\begin{pmatrix}2\cdot 5\\4\end{pmatrix}\\ &= \begin{pmatrix}5\\4\end{pmatrix}+\begin{pmatrix}10\\4\end{pmatrix}\\ &= \begin{pmatrix}15\\8\end{pmatrix}.\end{aligned}$$

Therefore

$$(1+2)\cdot\begin{pmatrix}5\\4\end{pmatrix}\neq 1\cdot\begin{pmatrix}5\\4\end{pmatrix}+2\cdot\begin{pmatrix}5\\4\end{pmatrix}.$$

Problem 3: Let $\vec{u},\vec{v},\vec{w}\in V$ be arbitrary. We have

$$\begin{aligned}\vec{u}+(\vec{v}+\vec{w}) &= (\vec{v}+\vec{w})+\vec{u} && \text{(by Property 3)}\\ &= (\vec{w}+\vec{v})+\vec{u} && \text{(by Property 3)}\\ &= \vec{w}+(\vec{v}+\vec{u}) && \text{(by Property 4).}\end{aligned}$$

Problem 4a: Let $\vec{v},\vec{w}\in V$ be arbitrary. We have

$$\begin{aligned}(\vec{v}+\vec{w})+((-\vec{w})+(-\vec{v})) &= \vec{v}+(\vec{w}+((-\vec{w})+(-\vec{v}))) && \text{(by Property 4)}\\ &= \vec{v}+((\vec{w}+(-\vec{w}))+(-\vec{v})) && \text{(by Property 4)}\\ &= \vec{v}+(\vec{0}+(-\vec{v})) && \text{(by definition of } -\vec{w})\\ &= \vec{v}+(-\vec{v}) && \text{(by Proposition 4.1.7)}\\ &= \vec{0} && \text{(by definition of } -\vec{v}).\end{aligned}$$

We have shown that $(-\vec{w})+(-\vec{v})$ is the additive inverse of $\vec{v}+\vec{w}$, so by definition it follows that

$$-(\vec{v}+\vec{w}) = (-\vec{w})+(-\vec{v}).$$

Using the fact that $(-\vec{w})+(-\vec{v}) = (-\vec{v})+(-\vec{w})$ by Property 3, we conclude that $-(\vec{v}+\vec{w}) = (-\vec{v})+(-\vec{w})$.

Alternatively, one can use the fact that $(-1)\cdot\vec{v} = -\vec{v}$ for all $\vec{v}\in V$ by Proposition 4.1.11, together with Propoerty 7.

Problem 4b: Let $\vec{v}, \vec{w} \in V$ and $c \in \mathbb{R}$ be arbitrary. We have

$$\begin{aligned}
c \cdot (\vec{v} - \vec{w}) &= c \cdot (\vec{v} + (-\vec{w})) && \text{(by definition)} \\
&= c \cdot \vec{v} + c \cdot (-\vec{w}) && \text{(by Property 7)} \\
&= c \cdot \vec{v} + c \cdot ((-1) \cdot \vec{w}) && \text{(by Proposition 4.1.11)} \\
&= c \cdot \vec{v} + (c \cdot (-1)) \cdot \vec{w} && \text{(by Property 9)} \\
&= c \cdot \vec{v} + ((-1) \cdot c) \cdot \vec{w} && \text{(by the commutative law in } \mathbb{R}\text{)} \\
&= c \cdot \vec{v} + (-1) \cdot (c \cdot \vec{w}) && \text{(by Property 9)} \\
&= c \cdot \vec{v} + (-(c \cdot \vec{w})) && \text{(by Proposition 4.1.11)} \\
&= c \cdot \vec{v} - c \cdot \vec{w} && \text{(by definition).}
\end{aligned}$$

Problem 5: Let

$$W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}.$$

To check that W is a subspace of $\mathbb{R}^3$, we need to check the three properties.

- First notice that we have
$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$$
because $0 + 0 + 0 = 0$.

- Let $\vec{w}_1, \vec{w}_2 \in W$ be arbitrary. By definition of W, we can fix $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ with
$$\vec{w}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \text{and} \qquad \vec{w}_2 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$
and such that both $a_1 + a_2 + a_3 = 0$ and $b_1 + b_2 + b_3 = 0$. Now we have
$$\vec{w}_1 + \vec{w}_2 = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix},$$
and also
$$\begin{aligned}
(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$
Therefore, $\vec{w}_1 + \vec{w}_2 \in W$.

- Let $\vec{w} \in W$ and $c \in \mathbb{R}$ be arbitrary. By definition of W, we can fix $a_1, a_2, a_3 \in \mathbb{R}$ with
$$\vec{w} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and such that $a_1 + a_2 + a_3 = 0$. Now we have

$$c \cdot \vec{w} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix},$$

and also

$$\begin{aligned} ca_1 + ca_2 + ca_3 &= c \cdot (a_1 + a_2 + a_3) \\ &= c \cdot 0 \\ &= 0. \end{aligned}$$

Therefore, $c\vec{w} \in W$.

We have shown that $\vec{0} \in W$, that W is closed under addition, and that W is closed under scalar multiplication. It follows that W is a subspace of $\mathbb{R}^3$.