Solutions to Problem Set 19

Problem 1a: We use Proposition 4.2.14 and Proposition 4.3.3. Applying elementary row operations to the corresponding matrix, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 8 \\ 2 & 0 & 9 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} && \begin{matrix} (-3R_1 + R_2) \\ (-2R_1 + R_3) \end{matrix} \\
&\rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} && \begin{matrix} (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3). \end{matrix}
\end{aligned}$$

Since this last matrix is in echelon form, and had a leading entry in each row and each column, we can use Proposition 4.2.14 and Proposition 4.3.3 to conclude that α is a basis for $\mathbb{R}^3$.

Problem 1b: We want to find the unique choice of $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix},$$

so we want to solve a linear system. Applying row operations to the augmented matrix, we obtain

$$\begin{aligned}
\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & -3 & 8 & 5 \\ 2 & 0 & 9 & -5 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 5 & -7 \end{pmatrix} && \begin{matrix} (-3R_1 + R_2) \\ (-2R_1 + R_3) \end{matrix} \\
&\rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 5 & -7 \\ 0 & 0 & 2 & 2 \end{pmatrix} && \begin{matrix} (R_2 \leftrightarrow R_3) \\ (R_2 \leftrightarrow R_3). \end{matrix}
\end{aligned}$$

The last line tells us that $2c_3 = 2$, so $c_3 = 1$. The middle line gives the equation

$$2c_2 + 5c_3 = -7.$$

Since $c_3 = 1$, we must have $2c_2 + 5 = -7$, so $2c_2 = -12$, and hence $c_2 = -6$. Now the first line tells us that

$$c_1 - c_2 + 2c_3 = 1.$$

Plugging in our values of c_2 and c_3, we conclude that

$$c_1 - (-6) + 2 \cdot 1 = 1,$$

so $c_1 = -7$. Therefore, we have

$$(-7) \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + (-6) \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix},$$

and hence

$$\left[\begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix} \right]_\alpha = \begin{pmatrix} -7 \\ -6 \\ 1 \end{pmatrix}.$$

Problem 2a: We use Theorem 4.4.2. Let $h \in \mathcal{P}_3$ be arbitrary, and fix $b_0, b_1, b_2, b_3 \in \mathbb{R}$ such that $h(x) = b_3x^3 + b_2x^2 + b_1x + b_0$. We want to show that there exists a unique choice of $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with

$$c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 = g.$$

Since functions are equal when they give the same output on every input, this is equivalent to asking whether there exists a unique choice of $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1 \cdot x^3 + c_2 \cdot (x^3 + x^2) + c_3 \cdot (x^3 + x^2 + x) + c_4 \cdot (x^3 + x^2 + x + 1) = b_3x^3 + b_2x^2 + b_1x + b_0.$$

for all $x \in \mathbb{R}$. Expanding the left-hand side and collection terms, this is the same as asking whether there exists a unique choice of $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$(c_1 + c_2 + c_3 + c_4) \cdot x^3 + (c_2 + c_3 + c_4) \cdot x^2 + (c_3 + c_4) \cdot x + c_4 = b_3x^3 + b_2x^2 + b_1x + b_0$$

for all $x \in \mathbb{R}$. Since polynomial functions are equal exactly when the corresponding coefficients are equal, this is the same as asking whether the system

$$\begin{array}{ccccccccc} c_1 & + & c_2 & + & c_3 & + & c_4 & = & b_3 \\ & & c_2 & + & c_3 & + & c_4 & = & b_2 \\ & & & & c_3 & + & c_4 & = & b_1 \\ & & & & & & c_4 & = & b_0 \end{array}$$

has a unique solution. The corresponding augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & b_3 \\ 0 & 1 & 1 & 1 & b_2 \\ 0 & 0 & 1 & 1 & b_1 \\ 0 & 0 & 0 & 1 & b_0 \end{pmatrix}.$$

Notice that this matrix is in echelon form. Furthermore, since there is a leading entry in each row, and a leading entry in every column but the last, we know from Proposition 4.2.12 that there exists a unique solution no matter what b_0, b_1, b_2, b_3 are. Therefore, α is a basis $\mathcal{P}_3$ by Theorem 4.4.2.

Problem 2b: Following the logic in part a with $b_3 = 3$, $b_2 = 7$, $b_1 = 7$, and $b_0 = -2$, we want to solve the following linear system:

$$\begin{array}{ccccccccc} c_1 & + & c_2 & + & c_3 & + & c_4 & = & 3 \\ & & c_2 & + & c_3 & + & c_4 & = & 7 \\ & & & & c_3 & + & c_4 & = & 7 \\ & & & & & & c_4 & = & -2. \end{array}$$

The last equation tells us immediately that $c_4 = -2$. From here, the third equation gives $c_3 - 2 = 7$, so $c_3 = 9$. Now plugging these into the second equation, we see that

$$c_2 + 9 + (-2) = 7,$$

so $c_2 = 0$. Finally, the first equation says that

$$c_1 + 0 + 9 + (-2) = 3,$$

so $c_1 = -4$. Therefore, we have

$$(-4) \cdot f_1 + 0 \cdot f_2 + 9 \cdot f_3 + (-2) \cdot f_4 = g,$$

and hence

$$[g]_\alpha = \begin{pmatrix} -4 \\ 0 \\ 9 \\ -2 \end{pmatrix}.$$

Problem 3a: We first check that α is linearly independent. Let $c_1, c_2 \in \mathbb{R}$ be arbitrary with $c_1 f_1 + c_2 f_2 = 0$. We then have that

$$c_1 \cdot (x^2 - 4) + c_2 \cdot (x - 2) = 0$$

for all $x \in \mathbb{R}$, so

$$c_1 \cdot x^2 + c_2 \cdot x + (-4c_1 - 2c_2) = 0x^2 + 0x + 0$$

for all $x \in \mathbb{R}$. Since polynomial functions are equal precisely when the coefficients are equal, we conclude that $c_1 = 0$ and $c_2 = 0$. Therefore, (f_1, f_2) is linearly independent.

We now check that $\text{Span}(f_1, f_2) = W$. To do this, we give a double containment argument.

- $\text{Span}(f_1, f_2) \subseteq W$: Let $g \in \text{Span}(f_1, f_2)$ be arbitrary. Fix $c_1, c_2 \in \mathbb{R}$ with $g = c_1 f_1 + c_2 f_2$. We then have

$$\begin{aligned} g(2) &= (c_1 f_1 + c_2 f_2)(2) \\ &= c_1 f_1(2) + c_2 f_2(2) \\ &= c_1 \cdot (2^2 - 4) + c_2 \cdot (2 - 2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0, \end{aligned}$$

  so $g \in W$.

- $W \subseteq \text{Span}(f_1, f_2)$. Let $g \in W$ be arbitrary. Since $g \in \mathcal{P}_2$, we can fix $a, b, c \in \mathbb{R}$ with $g(x) = ax^2 + bx + c$. Since $g \in W$, we have $g(2) = 0$, so $4a + 2b + c = 0$. Subtracting $4a + 2b$ from both sides, we conclude that $c = -4a - 2b$. Now notice that

$$\begin{aligned} a \cdot (x^2 - 4) + b \cdot (x - 2) &= ax^2 + bx + (-4a - 2b) \\ &= ax^2 + bx + c \end{aligned}$$

  for all $x \in \mathbb{R}$, so $af_1 + bf_2 = g$. Since $a, b \in \mathbb{R}$, it follows that $g \in \text{Span}(f_1, f_2)$.

Combining these two containments, we conclude that $\text{Span}(f_1, f_2) = W$.

Since (f_1, f_2) is linearly independent and $\text{Span}(f_1, f_2) = W$, it follows that α is a basis of W. Furthermore, we know that $\dim(W) = 2$ because the dimension is the number of elements in any basis.

Problem 3b: First notice that

$$\begin{aligned} g(2) &= 2 \cdot 2^2 - 7 \cdot 2 + 6 \\ &= 8 - 14 + 6 \\ &= 0, \end{aligned}$$

so $g \in W$. We need to find the unique $c_1, c_2 \in \mathbb{R}$ with $g = c_1 f_1 + c_2 f_2$, i.e. with

$$2x^2 - 7x + 6 = c_1 \cdot (x^2 - 4) + c_2 \cdot (x - 2)$$

for all $x \in \mathbb{R}$. Expanding the right-hand side, we want to find the unique $c_1, c_2 \in \mathbb{R}$ with $g = c_1 f_1 + c_2 f_2$, i.e. with

$$2x^2 - 7x + 6 = c_1 x^2 + c_2 x + (-4c_1 - 2c_2)$$

for all $x \in \mathbb{R}$. Looking at the coefficients of x^2 and x, we must have $c_1 = 2$ and $c_2 = -7$. Notice that in this case, we have

$$-4c_1 - 2c_2 = -8 + 14 = 6,$$

so these values do indeed work. Therefore, we have

$$[g]_\alpha = \begin{pmatrix} 2 \\ -7 \end{pmatrix}.$$

Problem 4: Let

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

We claim that α is a basis for W.

We first check that α is linearly dependent. Let $c_1, c_2, c_3 \in \mathbb{R}$ be arbitrary with

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We then have

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so we must have $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Therefore, α is linearly independent.

We now check that $\text{Span}(\alpha) = W$ by given a double-containment proof.

- $\text{Span}(\alpha) \subseteq W$: Let $A \in \text{Span}(\alpha)$ be arbitrary. Fix $c_1, c_2, c_3 \in \mathbb{R}$ with

$$A = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

  We then have

$$A = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}.$$

  Notice that the $(2,1)$ entry equals the $(1,2)$ entry, so $A \in W$.

- $W \subseteq \text{Span}(\alpha)$. Let $A \in W$ be arbitrary. By definition of W, we can fix $a, b, d \in \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

  Notice that we then have

$$A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

  so $A \in \text{Span}(\alpha)$.

Combining these two containments, we conclude that $\text{Span}(\alpha) = W$.

Since α is linearly independent and $\text{Span}(\alpha) = W$, it follows that α is a basis of W. Furthermore, we know that $\dim(W) = 3$ because the dimension is the number of elements in any basis.

Problem 5: In Problem 2 on Problem Set 18, we showed that

$$(-2) \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} + (-4) \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix},$$

and hence

$$\begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right).$$

Using Proposition 4.4.4, it follows that

$$\text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -8 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right) = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right),$$

and hence

$$W = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right).$$

Let

$$\alpha = \left(\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 9 \\ 5 \end{pmatrix} \right).$$

We just noticed that $\text{Span}(\alpha) = W$. To show that α is linearly independent, we apply elementary row operations to the corresponding matrix:

$$\begin{pmatrix} 0 & 2 & 6 \\ 1 & 0 & -1 \\ 3 & 2 & 9 \\ -1 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 6 \\ 3 & 2 & 9 \\ -1 & -1 & 5 \end{pmatrix} \qquad \begin{matrix} (R_1 \leftrightarrow R_2) \\ (R_2 \leftrightarrow R_2) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 12 \\ 0 & -1 & 4 \end{pmatrix} \qquad \begin{matrix} (\frac{1}{2} \cdot R_2) \\ (-3R_1 + R_2) \\ (R_1 + R_4) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 7 \end{pmatrix} \qquad \begin{matrix} (\frac{1}{2} \cdot R_2) \\ (-2R_1 + R_2) \\ (R_2 + R_4) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{pmatrix} \qquad (\tfrac{1}{6} \cdot R_3)$$

$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad (-7R_3 + R_4).$$

Notice that every column of this last matrix in echelon form has a leading entry, so Proposition 4.3.3 implies that α is linearly independent.

Since α is linearly independent and $\text{Span}(\alpha) = W$, it follows that α is a basis of W.