Assignment: Written Assignment 7

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List Your Collaborators:

- Problem 1: None

- Problem 2: None

- Problem 3: None

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: In our definition of the elementary row operation known as "row combination", we replace row i by the sum of itself and a multiple of a different row j (where different means that $j \neq i$). Suppose then that we consider the operation where we take a row i and replace it by the sum of itself and a multiple of row i. Do we necessarily preserve the solution set of the system by doing this? As always, you must explain if your answer is yes, or you must provide a specific counterexample (with justification) if your answer is no.

Solution: Notice that taking a row i and replacing it with the sum of itself and a multiple of row i has the same effect as multiplying row i by a constant. So if the multiple of row i is such that the sum of i and the multiple of row i gives a nonzero row, then this operation is identical to rescaling which is an elementary row operation. However, this is not a property of the new operation, so it is possible to have the multiple of row i be such that the sum of i and the multiple of row i gives a zero row, which would be the same as multiplying row i by 0, and this would change the solution set of the system. Therefore we do not necessarily preserve the solution set of the system by doing this operation.

Problem 2: Show that for all $a, b, c \in \mathbb{R}$, the matrices

$$\begin{pmatrix} 4 & 2 & 1 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$$

are not row equivalent, i.e. there does not exists a sequence of elementary row operations that turns the first matrix into the second matrix.

Solution: Let $a, b, c \in \mathbb{R}$ be arbitrary. Let $A = \begin{pmatrix} 4 & 2 & 1 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$. We can view A and B as matrices encoding linear systems in the variables (x, y). We assume that the solution set of A is equal to the solution set of B. By Corollary 4.2.5, there exists a finite sequence of row operations that we can apply to A to obtain B and vice versa, and so by definition A is row equivalent to B. We find the solution set of B by using Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix} \begin{matrix} \\ 2R_1 + R_2 \\ -R_1 + R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{2}R_2 + R_1 \\ \frac{1}{2}R_2 \\ -R_2 + R_3 \end{matrix}$$

We have a new system, C

$$x = \frac{1}{2}$$
$$y = \frac{3}{2}$$

This system has the solution set $S_C = \{(\frac{1}{2}, \frac{3}{2})\}$. By Corollary 4.2.5, this solution set is equal to the solution set of the original system, S_B. Therefore, $S_B = \{(\frac{1}{2}, \frac{3}{2})\}$. By assumption, $S_B = S_A$, and it follows from the definition of set equality that $S_B \subseteq S_A$.

PAGE 1 OF 2 FOR PROBLEM 2

Problem 3: Let V be a vector space. Suppose that U and W are both subspaces of V. We showed in Written Assignment 6 that $U \cup W$ might not be a subspace of V. Instead, let

$$U + W = \{\vec{v} \in V : \text{ There exists } \vec{u} \in U \text{ and } \vec{w} \in W \text{ with } \vec{v} = \vec{u} + \vec{w}\}.$$

That is, $U + W$ is the set of all vectors in V that can be written as the sum of an element of U and an element of W. Show that $U + W$ is a subspace of V.

Solution: Let $\vec{v} \in U + W$ be arbitrary. By definition, we can fi$\vec{u} \in U$ and $\vec{w} \in W$ such that $\vec{v} = \vec{u} + \vec{w}$. Notice that $U \subseteq V$ and $W \subseteq V$ by assumption. Because $\vec{u} \in U$ and $U \subseteq V$, it follows that $\vec{u} \in V$ by definition of subset. Similarly, because $\vec{w} \in W$ and $W \subseteq V$, it follows that $\vec{w} \in V$ by definition of subset. Because V is a vector space and $\vec{u}, \vec{w} \in V$, by Property 1 of vector spaces, it follows that $\vec{u} + \vec{w} \in V$. Because $\vec{v} = \vec{u} + \vec{w}$, it follows that $\vec{v} \in V$. Because $\vec{v} \in U + W$ was arbitrary, it follows that $U + W \subseteq V$. We now check that $U + W$ is indeed a subspace of V. If $U + W$ is a subspace of V, then $U + W$ has the following properties as laid out in Definition 4.1.12:

1. $\vec{0} \in U + W$
2. For all $\vec{v_1}, \vec{v_2} \in U + W$, we have that $\vec{v_1} + \vec{v_2} \in U + W$
3. For all $\vec{v} \in U + W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{v} \in U + W$

We check all three properties:

1. Because U and W are both subspaces of V, by definition we have that $\vec{0} \in U$ and $\vec{0} \in W$. By definition of $\vec{0}$, we have that $\vec{0} + \vec{0} = \vec{0}$. So by definition of $U + W$, it follows that $\vec{0} \in U + W$. So the first property is satisfied.

2. Let $\vec{v_1}, \vec{v_2} \in U + W$ be arbitrary. Because $\vec{v_1}, \vec{v_2} \in U + W$, by definition we can fix $\vec{u_1}, \vec{u_2} \in W$ and $\vec{w_1}, \vec{w_2} \in W$ with $\vec{u_1} + \vec{w_1} = \vec{v_1}$ and $\vec{u_2} + \vec{w_2} = \vec{v_2}$. Notice that $\vec{v_1} + \vec{v_2} = (\vec{u_1} + \vec{w_1}) + (\vec{u_2} + \vec{w_2}) = (\vec{u_1} + \vec{w_1}) + (\vec{w_2} + \vec{u_2})$ (By Property 3 of vector spaces) $=$ $(\vec{u_1} + \vec{u_2}) + (\vec{w_1} + \vec{w_2})$ (By Property 4 of vector spaces). Because $\vec{u_1}, \vec{u_2} \in U$ and U is a subspace of V, by definition of subspace we have that $\vec{u_1} + \vec{u_2} \in U$. Similarly, $\vec{w_1}, \vec{w_2} \in W$ and W is a subspace of V, so by definition of subspace we have that $\vec{w_1} + \vec{w_2} \in W$. $\vec{v_1} + \vec{v_2} = (\vec{u_1} + \vec{u_2}) + (\vec{w_1} + \vec{w_2})$, so it follows from the definition of $U + W$ that $(\vec{v_1} + \vec{v_2}) \in U + W$. Because $\vec{v_1}, \vec{v_2} \in U + W$ were arbitrary, we have that $\vec{v_1} + \vec{v_2} \in U + W$ for all $\vec{v_1}, \vec{v_2} \in U + W$, thus the second property is satisfied.

3. Let $\vec{v} \in U + W$ be arbitrary, and let $r \in \mathbb{R}$ be arbitrary. Because $\vec{v} \in U + W$, we can fix $\vec{u} \in U$ and $\vec{w} \in W$ such that $\vec{v} = \vec{u} + \vec{w}$. Notice that $r \cdot \vec{v} = r \cdot (\vec{u} + \vec{w}) = r \cdot \vec{u} + r \cdot \vec{w}$ (By Property 7 of vector spaces). Because U and W are both subspaces of V, it follows from Property 3 that $r \cdot \vec{u} \in U$ and $r \cdot \vec{w} \in W$. Because $r \cdot \vec{v} = r \cdot \vec{u} + r \cdot \vec{w}$, it follows from the definition of $U + W$, that $r \cdot \vec{v} \in U + W$. Because $\vec{v} \in U + W$ and $r \in \mathbb{R}$ were arbitrary, we have that $r \cdot \vec{v} \in U + W$ for all $\vec{v} \in U + W$ and all $r \in \mathbb{R}$, thus the third property is satisfied.

We have shown that $U + W$ has all three properties of a subspace of V, therefore, $U + W$ is indeed a subspace of V.

Notice that plugging in $(x, y) = (\frac{1}{2}, \frac{3}{2})$ into system A yields:

$$4(\frac{1}{2}) + 2(\frac{3}{2}) = 2 \qquad \rightarrow \qquad 2 + 3 = 2 \tag{1}$$

$$a(\frac{1}{2}) - 1(\frac{3}{2}) = 0 \qquad \rightarrow \qquad \frac{a-3}{2} = 0 \tag{2}$$

$$b(\frac{1}{2}) + c(\frac{3}{2}) = 5 \qquad \rightarrow \qquad \frac{b+3c}{2} = 5 \tag{3}$$

So gives $5 = 2$, $\frac{a-3}{2} = 0$, and $\frac{b+3c}{2} = 5$. Our assumption that $S_B = S_A$ has led to a contradiction, namely that $5 = 2$, and thus is must be the case that $S_B \neq S_A$. By Corollary 4.2.5, there does not exist a finite sequence of row operations that we can apply to A to obtain B and vice versa, and so by definition A is not row equivalent to B. Because a, b, c were arbitrary, the result follows.