Instructions

Sit with your group at one table. Put everything away in your bags except for a pen or pencil. All notes, worksheets, books, phones, computers, etc. should be out of sight.

1 Definitions

Definition 1.1. The *vectors* in $\mathbb{R}^2$ are the elements of the set

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

We define the *sum of vectors* in $\mathbb{R}^2$ by

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

We define *scalar multiplication* of a vector by

$$d \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} da \\ db \end{pmatrix}$$

for all $d \in \mathbb{R}$

Notation. We use the notation $\vec{w}$ to indicate that w is a vector. We use the notation $\vec{0}$ to denote the vector whose entries are all zero.

Definition 1.2. Let $\vec{u}$ be a vector in $\mathbb{R}^2$. We define the *span of* $\vec{u}$ to be the following subset of $\mathbb{R}^2$:

$$Span(\vec{u}) = \{c \cdot \vec{u} : c \in \mathbb{R}\}$$

Definition 1.3. Let $\vec{u}_1, \vec{u}_2, \vec{v} \in \mathbb{R}^2$. We say that $\vec{v}$ is a *linear combination* of $\vec{u}_1$ and $\vec{u}_2$ if there exists $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$.

Definition 1.4. Let $\vec{u}_1, \vec{u}_2, \in \mathbb{R}^2$. We define the *span* of $\vec{u}_1$ and $\vec{u}_2$ to be the subset of $\mathbb{R}^2$ made up of all linear combinations of $\vec{u}_1$ and $\vec{u}_2$. That is, we define

$$Span(\vec{u}_1, \vec{u}_2) = \{c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 : c_1, c_2 \in \mathbb{R}\}$$

2 Results

Proposition 1. *For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$, and all $c, d \in \mathbb{R}$, we have:*

1. $\vec{v} + \vec{w} \in \mathbb{R}^2$
2. $c \cdot \vec{v} \in \mathbb{R}^2$
3. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
4. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
5. $\vec{v} + \vec{0} = \vec{v}$
6. $\vec{v} + (-1 \cdot \vec{v}) = 0$
7. $c \cdot (\vec{v} + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w}$
8. $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$
9. $c \cdot (d \cdot \vec{v}) = (cd) \cdot \vec{v}$
10. $1 \cdot \vec{v} = \vec{v}$

Proposition 2. *Let $\vec{u} \in \mathbb{R}^2$ be arbitrary, and let $S = Span(\vec{u})$. We have the following*

1. $\vec{0} \in S$
2. *For all $\vec{v}_1, \vec{v}_2 \in S$, we have $\vec{v}_1 + \vec{v}_2 \in S$*
3. *For all $d \in \mathbb{R}$ and $\vec{v} \in S$, we have $d \cdot \vec{v} \in S$*

Proposition 3. *For all $\vec{u} \in \mathbb{R}^2$, we have $Span(\vec{u}) \neq \mathbb{R}^2$*

Proposition 4. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ be arbitrary, and let $S = Span(\vec{u}_1, \vec{u}_2)$. We have the following:*

1. $\vec{0} \in S$
2. *For all $\vec{v}_1, \vec{v}_2 \in S$, we have $\vec{v}_1 + \vec{v}_2 \in S$*
3. *For all $d \in \mathbb{R}$ and $\vec{v} \in S$, we have $d \cdot \vec{v} \in S$*

Proposition 5. *For all $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ we have $Span(\vec{u}_1) \subseteq Span(\vec{u}_1, \vec{u}_2)$.*

Proposition 6. *Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. $Span(\vec{u}_1, \vec{u}_2) = Span(\vec{u}_1)$ if and only if $\vec{u}_2 \in Span(\vec{u}_1)$.*

Proposition 7. *For all $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ we have that $Span(\vec{u}_1, \vec{u}_2) = Span(\vec{u}_2, \vec{u}_1)$.*

3 Exercise

Starting Point

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$, and say

$$\vec{u}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

Statements

A: $Span(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$

B: $\vec{u}_1$ and $\vec{u}_2$ are both not zero, and $\vec{u}_2$ is not in $Span(\vec{u}_1)$.

C: $\vec{u}_1$ and $\vec{u}_2$ are both not zero, and $\vec{u}_1$ is not in $Span(\vec{u}_2)$.

D: $a_1 b_2 \neq a_2 b_1$

E: For any $\vec{v} \in \mathbb{R}$ there is a unique pair, $c_1, c_2 \in \mathbb{R}$ with $\vec{v} = c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2$