Assignment: Written Assignment 6

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List Your Collaborators:

- Problem 1: Dan

- Problem 2: None

- Problem 3: Not applicable

- Problem 4: Not Applicable

- Problem 5: Not Applicable

- Problem 6: Not Applicable

Problem 1: Let V be a vector space. Prove each of the following. For this problem, be very explicit and mention which property and/or result you are using in each step of your argument.

a. Suppose that $c \in \mathbb{R}$ and $\vec{v} \in V$ are such that $c \cdot \vec{v} = \vec{0}$. Show that either $c = 0$ or $\vec{v} = \vec{0}$.

Solution: Let $\vec{v} \in V$ and let $c \in \mathbb{R}$ be arbitrary such that $c \cdot \vec{v} = \vec{0}$.

Suppose that $c \neq 0$. We can then fix a $d \in \mathbb{R}$ such that $\frac{1}{c} = d$. Notice that

$$\begin{aligned} d \cdot \vec{0} = d \cdot (c \cdot \vec{v}) &= (dc) \cdot \vec{v} && \text{(By Property 9)} \\ &= (\frac{1}{c}c) \cdot \vec{v} && \text{(By definition of } d\text{)} \\ &= (1) \cdot \vec{v} \\ &= \vec{v} && \text{(By Property 10)} \end{aligned}$$

So we have that $d \cdot \vec{0} = \vec{v}$. Because $d \in \mathbb{R}$, by Proposition 4.1.11.2, $d \cdot \vec{0} = \vec{0}$. Since $d \cdot \vec{0} = \vec{v}$, it follows that $\vec{0} = \vec{v}$. So $\vec{0} = \vec{v}$.

Now suppose that $\vec{v} \neq \vec{0}$. Because $\vec{v} \in V$, by Proposition 4.1.11.1, $0 \cdot \vec{v} = \vec{0}$. Because $c \cdot \vec{v} = \vec{0}$ and $\vec{v} \neq \vec{0}$, there is only value of c that satisfies $c \cdot \vec{v} = \vec{0}$ for all (nonzero) $\vec{v} \in V$, which is $c = 0$. So $c = 0$.

Now suppose that $\vec{v} = \vec{0}, c = 0$. It follows that $c \cdot \vec{v} = 0$ by Proposition 4.1.11.

These three cases cover all possible values of $\vec{v}, c$. Because $\vec{v} \in V, c \in \mathbb{R}$ were arbitrary with the aforementioned property, the result follows.

b. Suppose that $c, d \in \mathbb{R}$ and $\vec{v} \in V$ are such that $c \cdot \vec{v} = d \cdot \vec{v}$. Show that if $\vec{v} \neq \vec{0}$, then $c = d$.

Solution: Let $\vec{v} \in V$ be nonzero, and let $c, d \in \mathbb{R}$ be arbitrary. Suppose that $c \cdot \vec{v} = d \cdot \vec{v}$. By Proposition 4.1.9 and Definition 4.1.10, $d \cdot \vec{v} + (-d \cdot \vec{v}) = \vec{0}$. Because $c \cdot \vec{v} = d \cdot \vec{v}$, it follows that $c \cdot \vec{v} + (-d \cdot \vec{v}) = \vec{0}$ So we have

$$\begin{aligned} \vec{0} = c \cdot \vec{v} + (-d \cdot \vec{v}) &= c \cdot \vec{v} + (-1) \cdot (d \cdot \vec{v}) && \text{(By Proposition 4.1.11.3)} \\ &= c \cdot \vec{v} + (-1 \cdot d) \cdot \vec{v} && \text{(By Property 9)} \\ &= c \cdot \vec{v} + (-d) \cdot \vec{v} \\ &= (c - d) \cdot \vec{v} && \text{(By Property 8)} \end{aligned}$$

So $\vec{0} = (c - d) \cdot \vec{v}$. Because $\vec{v} \neq \vec{0}$, by our result in part a, we have that $c - d = 0$. It follows that $c = d$. Because $\vec{v} \in V$ was nonzero and $c, d \in \mathbb{R}$ were arbitrary, the result follows.

Problem 2: Let V be a vector space. Suppose that U and W are both subspaces of V.

a. Let $U \cap W$ be the intersection of U and W, i.e. $U \cap W = \{\vec{v} \in V : \vec{v} \in U \text{ and } \vec{v} \in W\}$. Show that $U \cap W$ is a subspace of V.

Solution: Notice that $U \subseteq V$ and $W \subseteq V$, so by definition of set intersection, it follows that $U \cap W \subseteq V$. We now check that $U \cap W$ is indeed a subspace of V. If $U \cap W$ is a subspace of V, then $U \cap W$ has the following properties as laid out in Definition 4.1.12:

1. $\vec{0} \in U \cap W$
2. For all $\vec{v_1}, \vec{v_2} \in U \cap W$, we have that $\vec{v_1} + \vec{v_2} \in U \cap W$
3. For all $\vec{v} \in U \cap W$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{v} \in U \cap W$

We check all three properties:

1. Because U and W are both subspaces of V, by definition we have that $\vec{0} \in U$ and $\vec{0} \in W$. So by definition of set intersection, it follows that $\vec{0} \in U \cap W$. So the first property is satisfied.

2. Let $\vec{u}, \vec{w} \in U \cap W$ be arbitrary. Because $\vec{u}, \vec{w} \in U \cap W$, by definition of set intersection it follows that $\vec{u}, \vec{w} \in U$ and $\vec{u}, \vec{w} \in W$. Because U and W are both subspaces of V, it follows from Property 2 that $\vec{u} + \vec{w} \in U$ and $\vec{u} + \vec{w} \in W$. So by definition of set intersection, we have that $\vec{u} + \vec{w} \in U \cap W$. Because $\vec{u}, \vec{w} \in U \cap W$ were arbitrary, we have that $\vec{u} + \vec{w} \in U \cap W$ for all $\vec{u}, \vec{w} \in U \cap W$, thus the second property is satisfied. the result follows.

3. Let $\vec{v} \in U \cap W$ be arbitrary, and let $r \in \mathbb{R}$ be arbitrary. Because $\vec{v} \in U \cap W$, by definition of set intersection it follows that $\vec{v} \in U$ and $\vec{v} \in W$. Because U and W are both subspaces of V, it follows from Property 3 that $r \cdot \vec{v} \in U$ and $r \cdot \vec{v} \in W$. So by definition of set intersection, we have that $r \cdot \vec{v} \in U \cap W$. Because $\vec{v} \in U \cap W$ and $r \in \mathbb{R}$ were arbitrary, we have that $r \cdot \vec{v} \in U \cap W$ for all $\vec{v} \in U \cap W$ and all $r \in \mathbb{R}$, thus the third property is satisfied.

We have shown that $U \cap W$ has all three properties of a subspace of V, therefore, $U \cap W$ is indeed a subspace of V.

b. Let $U \cup W$ be the union of U and W, i.e. $U \cup W = \{\vec{v} \in V : \vec{v} \in U \text{ or } \vec{v} \in W\}$. By constructing an explicit example, show that $U \cup W$ need *not* be a subspace of V.

Solution: Let $V = \mathcal{F}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Addition and scalar multiplication are defined on V as follows:

1. $f(x) + g(x) = (f + g)(x)$ for all $x \in \mathbb{R}$

2. $r \cdot f(x) = (r \cdot f)(x)$ for all $x \in \mathbb{R}$

c. Show that if a vector space has more than 1 element, then it must have infinitely many elements.

Solution: Let V be a vector space with more than one element. By definition of vector spaces, one of these elements must be a vector that acts as $\vec{0}$. From our treatment of set theory we know that the other element must be unique, that is that the other element must be nonzero. We denote this element by $\vec{v}$. Let $c \in \mathbb{R}$ be arbitrary. By Property 2 of vector spaces, $c \cdot \vec{v} \in V$. Because $c \in \mathbb{R}$ was arbitrary, there are an infinite number of unique $c \cdot \vec{v} \in V$. Therefore, V must have infinitely many elements.

Let $U_e \subseteq V$ be the subset of V consisting of all even functions. Let $W_o \subseteq V$ be the subset of V consisting of all odd functions.

- We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if for all $x \in \mathbb{R}$, we have that $f(x) = f(-x)$.

- We say that a function $g : \mathbb{R} \to \mathbb{R}$ is *odd* if, for all $x \in \mathbb{R}$, we have that $g(x) = -g(-x)$.

We now check that U_e is indeed a subspace of V. If U_e is a subspace of V, then U_e has the following properties as laid out in Definition 4.1.12:

1. There exists a $\vec{u_0} \in U_e$ such that $\vec{u} + \vec{u_0} = \vec{u}$ for all $\vec{u} \in U_e$.
2. For all $\vec{u_1}, \vec{u_2} \in U_e$, we have $\vec{u_1} + \vec{u_2} \in U_e$
3. For all $\vec{u} \in U_e$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{u} \in U_e$

We check all three properties

1. We define a function $u_0 : \mathbb{R} \to \mathbb{R}$ by letting $u_0(x) = 0$ for all $x \in \mathbb{R}$. Notice that $u_0(x) = 0 = u_0(-x)$, so by definition u_0 is an even function, and it follows that $u_0 \in U_e$. Now let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary even function, so $u \in U_e$ by definition. Notice that $u(x) + u_0(x) = u(x) + 0 = u(x)$. Because $u \in U_e$ was arbitrary, it follows that $u(x) + u_0(x) = u(x)$ for all $u \in U_e$. So u_0 satisfies the definition of $\vec{u_0}$, and thus the first property is satisfied.

2. Let $u_1 : \mathbb{R} \to \mathbb{R}$, $u_2 : \mathbb{R} \to \mathbb{R}$ be arbitrary even functions. So $u_1, u_2 \in U_e$ by definition. Because u_1 is even, we have that $u_1(x) = u_1(-x)$ for all $x \in \mathbb{R}$. Similarly, u_2 is even, so $u_2(x) = u_2(-x)$ for all $x \in \mathbb{R}$. Notice that $(u_1 + u_2)(x) = u_1(x) + u_2(x) = u_1(-x) + u_2(-x) = (u_1 + u_2)(-x)$ for all $x \in \mathbb{R}$. So $(u_1 + u_2)(x) = (u_1 + u_2)(-x)$ for all $x \in \mathbb{R}$, so by definition $u_1 + u_2$ is an even function, and it follows that $u_1 + u_2 \in U_e$. Since $u_1, u_2 \in U$ were arbitrary, we have that $u_1 + u_2 \in U_e$ for all $u_1, u_2 \in U_e$, and thus the second property is satisfied.

3. Let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary even function, so $u \in U_e$ by definition. Because u is even, we have that $u(x) = u(-x)$ for all $x \in \mathbb{R}$. Now let $r \in \mathbb{R}$ be arbitrary. Notice that $(r \cdot u)(x) = r \cdot u(x) = r \cdot u(-x) = (r \cdot u)(-x)$ for all $x \in \mathbb{R}$. So $(r \cdot u)(x) = (r \cdot u)(-x)$ for all $x \in \mathbb{R}$, so by definition, $r \cdot u$ is an even function, and it follows that $r \cdot u \in U_e$. Since $u \in U_e$ and $r \in \mathbb{R}$ were arbitrary, we have that $r \cdot u \in U_e$ for all $u \in U_e$ and all $r \in \mathbb{R}$, and thus the third property is satisfied.

We have shown that U_e has all three properties of a subspace of V, therefore, U_e is indeed a subspace of V.

We now check that W_o is indeed a subspace of V. If W_o is a subspace of V, then W_o has the following properties as laid out in Definition 4.1.12:

1. There exists a $\vec{w_0} \in W_o$ such that $\vec{w} + \vec{w_0} = \vec{w}$ for all $\vec{w} \in W_o$.
2. For all $\vec{w_1}, \vec{w_2} \in W_o$, we have $\vec{w_1} + \vec{w_2} \in W_o$
3. For all $\vec{w} \in W_o$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{w} \in W_o$

We check all three properties

1. We define a function $w_0 : \mathbb{R} \to \mathbb{R}$ by letting $w_0(x) = 0$ for all $x \in \mathbb{R}$. Notice that $w_0(x) = 0 = (-1) \cdot 0 = (-1) \cdot w_0(-x) = -w_0(x)$(By Proposition 4.1.11.3), so by definition w_0 is an odd function, and it follows that $w_0 \in W_o$. Now let $w : \mathbb{R} \to \mathbb{R}$ be an arbitrary odd function, so $w \in W_o$ by definition.Notice that $w(x) + w_0(x) = w(x) + 0 = w(x)$. Because $w \in W_o$ was arbitrary, it follows that $w(x) + w_0(x) = w(x)$ for all $w \in W_o$. So w_0 satisfies the definition of $\vec{w_0}$, and thus the first property is satisfied.

2. Let $w_1 : \mathbb{R} \to \mathbb{R}$, $w_2 : \mathbb{R} \to \mathbb{R}$ be arbitrary odd functions. So $w_1, w_2 \in W_o$ by definition. Because w_1 is odd, we have that $w_1(x) = -w_1(-x)$ for all $x \in \mathbb{R}$. Similarly, w_2 is odd, so $w_2(x) = -w_2(-x)$ for all $x \in \mathbb{R}$. Notice that $(w_1 + w_2)(x) = w_1(x) + w_2(x) = -w_1(-x) + -w_2(-x) = -(w_1 + w_2)(-x)$ for all $x \in \mathbb{R}$. So $(w_1 + w_2)(x) = -(w_1 + w_2)(-x)$ for all $x \in \mathbb{R}$, so by definition $w_1 + w_2$ is an odd function, and it follows that $w_1 + w_2 \in W_o$. Since $w_1, w_2 \in W_o$ were arbitrary, we have that $w_1 + w_2 \in W_o$ for all $w_1, w_2 \in W_o$, and thus the second property is satisfied.

3. Let $w : \mathbb{R} \to \mathbb{R}$ be an arbitrary odd function, so $w \in W_o$ by definition. Because w is odd, we have that $w(x) = -w(-x)$ for all $x \in \mathbb{R}$. Now let $r \in \mathbb{R}$ be arbitrary. Notice that $(r \cdot w)(x) = r \cdot w(x) = r \cdot -w(-x) = (r \cdot (-1)) \cdot w(-x) = ((-1) \cdot r) \cdot w(-x) = (-1) \cdot (r \cdot w)(-x) = -(r \cdot w)(-x)$ for all $x \in \mathbb{R}$. So $(r \cdot w)(x) = -(r \cdot w)(-x)$ for all $x \in \mathbb{R}$, so by definition, $r \cdot w$ is an odd function, and it follows that $r \cdot w \in W_o$. Since $w \in W_o, r \in \mathbb{R}$ were arbitrary, we have that $r \cdot w \in W_o$ for all $w \in W_o$ and all $r \in \mathbb{R}$, and thus the third property is satisfied.

We have shown that W_o has all three properties of a subspace of V, therefore, W_o is indeed a subspace of V.

So U_e and W_o are both subspaces of V. We now consider the union of U_e and W_o, that is, we want to check if $U_e \cup W_o$ is a subspace of V. Notice that $U_e \subseteq V$ and $W_o \subseteq V$, so by definition of set union, it follows that $U_e \cup W_o \subseteq V$. If $U_e \cup W_o$ is a subspace of V, then $U_e \cup W_o$ has the following properties as laid out in Definition 4.1.12:

1. There exists a $\vec{v_0} \in U_e \cup W_o$ such that $\vec{v} + \vec{v_0} = \vec{w}$ for all $\vec{v} \in U_e \cup W_o$.
2. For all $\vec{v_1}, \vec{v_2} \in U_e \cup W_o$, we have that $\vec{v_1} + \vec{v_2} \in U_e \cup W_o$
3. For all $\vec{v} \in U_e \cup W_o$ and all $c \in \mathbb{R}$, we have $c \cdot \vec{v} \in U_e \cup W_o$

Consider the second property:

2. Let $u : \mathbb{R} \to \mathbb{R}$ be an arbitrary nonzero even function, and let $w : \mathbb{R} \to \mathbb{R}$ be an arbitrary nonzero odd function. So by definition, $u \in U_e$ and $w \in W_o$, thus it follows from the definition of set union that $u, w \in U_e \cup W_o$. Because u is even, we have that $u(x) = u(-x)$ for all $x \in \mathbb{R}$. Because w is odd, we have that $w(x) = -w(-x)$ for all $x \in \mathbb{R}$. Notice that $(u + w)(x) = u(x) + w(x) = u(-x) + (-w(-x)) = u(-x) - w(-x)$. So $(u+w)(x) = u(-x)-w(-x)$ for all $u, w \in U_e \cup W_o$. Notice that $u(-x)-w(-x)$ can be not be expressed as an even function nor as an odd function, and so it follows that $u(-x) - w(-x)$ is neither even nor odd. So $u(-x) - w(-x) \notin U_e$ and $u(-x) - w(-x) \notin W_o$ (this also follows from the definitions of U_e and W_o) so by definition, we have that $u(-x) - w(-x) \notin U_e \cup W_o$. Because $(u + w)(x) = u(-x) - w(-x)$, it follows that $u + w \notin U_e \cup W_o$. Therfore $U \cup W$ does not satisfy the second property.

We have that $U_e \cup W_o$ does not satisfy the second property, therefore $U_e \cup W_o$ is *not* a subspace of V. It follows that for arbitrary subspaces U, W of an arbitrary vector space V, it need not be the case that $U \cup W$ is also a subspace of V.

PAGE 4 OF 4 FOR PROBLEM 2