Waves and Turbulence

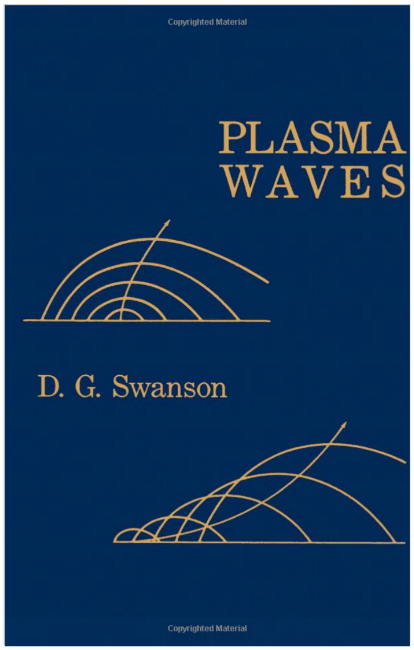
by S.J. Diem with cited contributions

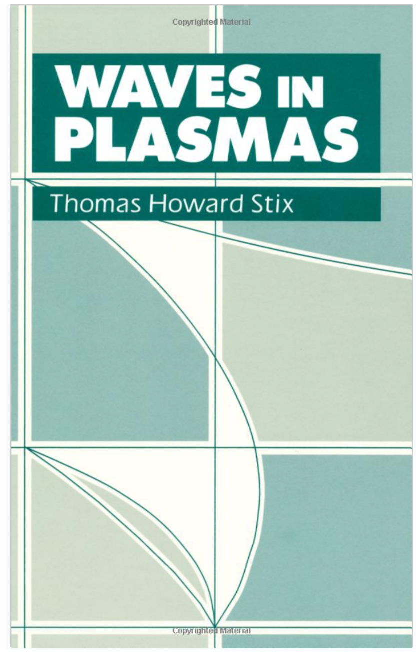
Presented at:

SULI One Week Course, PPPL

June 10th, 2019

Waves References

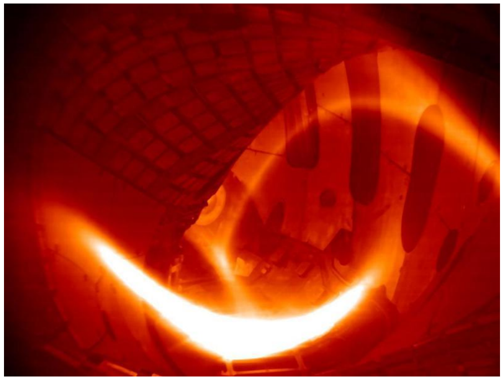




Plasmas support wide variety of wave phenomena

- Waves found naturally in plasmas
  - Instabilities, fluctuations, wave-induced transport
- Waves can deliver energy-momentum in plasma
  - Heating, current drive, particle acceleration
  - Mode stabilization, plasma confinement, α-channeling
- Waves can be used in plasma diagnostics
  - Interferometry, reflectometry, Faraday rotation, Thomson scattering

Photo of aurora: Senior Airman Joshua Strang



First W7-X plasma, IPP, Greifswald

Plasmas support wide variety of wave phenomena

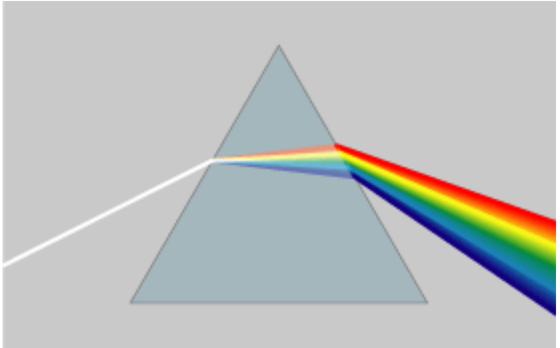
- How do we describe waves in plasmas?
- What can the dispersion relation tell us?
- Examples of waves and what we can do with them

Plasmas support wide variety of wave phenomena

- **How do we describe waves in plasmas?**
- What can the dispersion relation tell us?
- Examples of waves and what we can do with them

How are waves in plasmas described?

- Wave characteristics can change based on surroundings
- Dispersion relation describes relationship between wavelength and frequency of wave, $\omega(k)$



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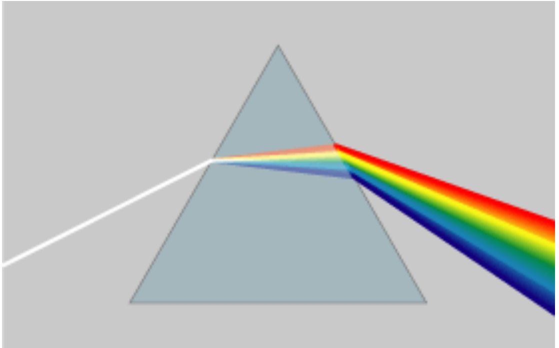

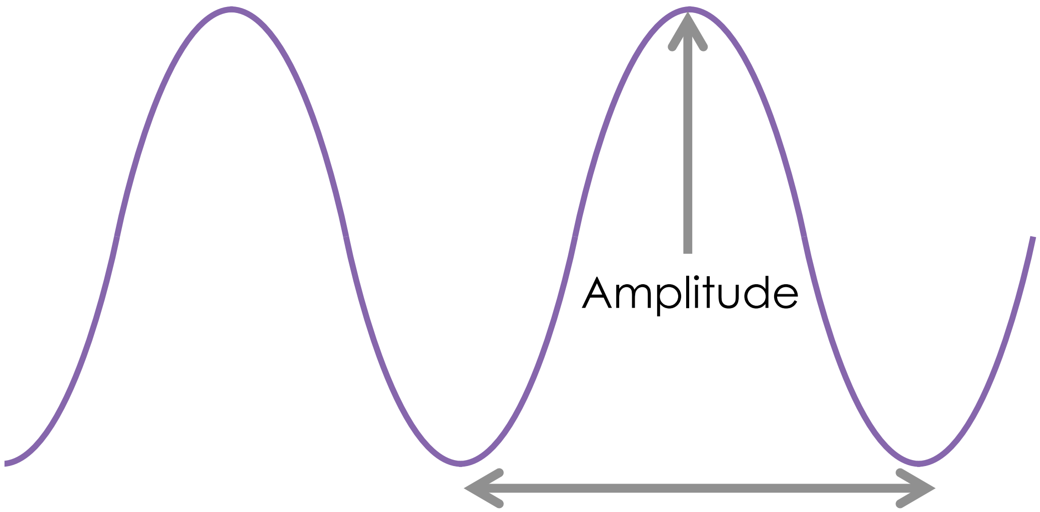

Wavenumber: $k=2\pi/\lambda$

Angular frequency: $\omega=2\pi f$

Phase velocity: $v_p=\omega/k$

How are waves in plasmas described?

- Plasmas respond to magnetic fields, plasmas conduct electricity
- Process to derive cold plasma dispersion relation:

  Step 1: Determine assumptions

  Step 2: Fourier analyze Maxwell's equations to obtain wave equation

  Step 3: Obtain dielectric tensor, relates plasma current to electric field

  Step 4: Combine all of the above to yield the dispersion relation, $\omega(k)$
- This process only results in waves in plasmas
  - No resulting instabilities because there are no sources
  - Provides basic framework for how more complex dispersion relations are derived

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Step 1: Assumptions

- Plasma is homogenous in space
- Uniform magnetic field (no gradients or curvature), anisotropic
- Cold, infinite plasma
  - T_e=T_i=0 : motionless without waves, zero gyroradius, no thermal effects

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Step 2: Fourier analyze Maxwell's equations, obtain wave equation

Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Faraday's law of induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

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$$\vec{\nabla} \times \vec{B} = \boxed{\mu_0 \vec{j}} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}} = \boxed{\mu_0 \frac{\partial \vec{D}}{\partial t}}$$

Plasma current

Vacuum displacement

D = electric displacement, accounts for the effects of free and bound charges in materials

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Where: $$\mu_0 \frac{\partial \vec{D}}{\partial t} \equiv \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\overline{\overline{K}} \bullet \vec{E}\right)$$

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Dielectric tensor – will be derived shortly…
Contains all of the plasma response

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Faraday's law of induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Apply Fourier analysis in space and time: $\vec{E}, \vec{B} \approx \exp\left(i\vec{k} \bullet \vec{r} - i\omega t\right)$

Ampere's law

$$i\vec{k} \times \vec{B} = -i\omega\mu_0\varepsilon_0\overline{\overline{K}} \bullet \vec{E}$$

Faraday's law of induction

$$i\vec{k} \times \vec{E} = i\omega\vec{B}$$

Step 2: Fourier analyze Maxwell's equations, obtain wave equation

Ampere's law

$$\vec{k} \times \vec{B} + \omega \mu_0 \varepsilon_0 \bar{\bar{K}} \bullet \vec{E} = 0$$

Faraday's law of induction

$$\frac{1}{\omega} \vec{k} \times \vec{E} = \vec{B}$$

Simplify using: $\frac{1}{c^2} = \mu_0 \varepsilon_0$

Index of refraction: $\vec{n} = \frac{c\vec{k}}{\omega}$

$$\boxed{\vec{n} \times \vec{n} \times \vec{E} + \bar{\bar{K}} \bullet E = 0}$$

For **K** = 1, get vacuum waves

How are waves in plasmas described?

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Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \overline{\overline{K}} \bullet E = 0$

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D = electric displacement, accounts for the effects of free and bound charges in materials

Where:

$$\mu_0 \frac{\partial \vec{D}}{\partial t} \equiv \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\overline{\overline{K}} \bullet \vec{E}\right)$$

Dielectric tensor – will be derived shortly…
Contains all of the plasma physics

Use the above to relate the electric field to the dielectric tensor and plasma current:

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\overline{\overline{K}} \bullet \vec{E}\right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Fourier analyze $\Rightarrow$

$$\boxed{\overline{\overline{K}} \bullet \vec{E} = \vec{E} + \frac{i}{\omega \varepsilon_0} \vec{j}}$$

Step 3: Dielectric tensor $\vec{n}\times\vec{n}\times\vec{E}+\overline{\overline{K}}\bullet E=0$

$$\overline{\overline{K}}\bullet\vec{E}=\vec{E}+\frac{i}{\omega\varepsilon_0}\vec{j}$$

We know how to describe current carried by a charge:

$$\vec{j}=\sum_s n_s q_s \vec{v}_s$$

Use single particle equation of motion to find velocity:

$$n_s m_s \frac{d\vec{v}_s}{dt}=n_s q_s\left(\vec{E}+\vec{v}_x\times\vec{B}\right)$$

Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \overline{\overline{K}} \bullet E = 0$

Use single particle equation of motion to find velocity:

$$n_s m_s \frac{d\vec{v}_s}{dt} = n_s q_s \left(\vec{E} + \vec{v}_x \times \vec{B} \right)$$

Use the following assumptions to solve for velocity components:

Apply Fourier analysis in space and time: $f\left(\vec{r}, t\right) = f \exp\left(i\vec{k} \bullet \vec{r} - i\omega t\right)$

Linearize equations $f = f_0 + f_1 + \ldots\ldots$ and $f_0 >> f_1$

Choose: $\vec{B} = B_0 \hat{z}$ and $\vec{E} = \vec{E}_1 = E_x \hat{x} + E_z \hat{z}$

Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \overline{\overline{K}} \bullet E = 0$

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Choose: $\vec{B} = B_0 \hat{z}$ and $\vec{E} = \vec{E}_1 = E_x \hat{x} + E_z \hat{z}$

Solve equations of motion to get v(E)

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Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \overline{\overline{K}} \bullet E = 0$

$$v_x = \frac{q}{\omega m} \frac{iE_x - \frac{\Omega_c}{\omega} E_y}{1 - \frac{\Omega_c^2}{\omega^2}}$$

$$v_y = \frac{q}{\omega m} \frac{iE_y + \frac{\Omega_c}{\omega} E_x}{1 - \frac{\Omega_c^2}{\omega^2}}$$

$$v_z = \frac{iq}{\omega m} E_z$$

Where cyclotron frequency defined as:

$$\Omega_c = \frac{qB_0}{m}$$

Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \overline{\overline{K}} \bullet E = 0$

$$\boxed{\begin{aligned} v_x &= \frac{q}{\omega m} \frac{iE_x - \frac{\Omega_c}{\omega} E_y}{1 - \frac{\Omega_c^2}{\omega^2}} \\ v_y &= \frac{q}{\omega m} \frac{iE_y + \frac{\Omega_c}{\omega} E_x}{1 - \frac{\Omega_c^2}{\omega^2}} \end{aligned}}$$

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Goes to infinity as wave frequency approaches cyclotron frequency

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Unaffected by background magnetic field

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Take these expressions for velocity, put back into dielectric tensor:

$$\vec{j} = \sum_s n_s q_s \vec{v}_s$$

$$\bar{\bar{K}} \bullet \vec{E} = \vec{E} + \frac{i}{\omega \varepsilon_0} \vec{j}$$

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$$\bar{\bar{K}}\bullet\vec{E}=\vec{E}+\frac{i}{\omega\varepsilon_0}\vec{j}$$

$$\bar{\bar{K}}\bullet\vec{E}=\begin{bmatrix}K_{11} & K_{12} & K_{13}\\ K_{21} & K_{22} & K_{23}\\ K_{31} & K_{32} & K_{33}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}$$

Step 3: Dielectric tensor $\vec{n} \times \vec{n} \times \vec{E} + \bar{\bar{K}} \bullet E = 0$

$$\bar{\bar{K}} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_{cs}^2}$$

$$D = \sum_s \frac{\Omega_{cs}\omega_{ps}^2}{\omega\left(\omega^2 - \Omega_{cs}^2\right)}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

$$\omega_{ps}^2 = \frac{q^2 n}{m}$$

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Step 4: Combine to obtain dispersion relation

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$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_{cs}^2}$$

$$D = \sum_s \frac{\Omega_{cs}\omega_{ps}^2}{\omega\left(\omega^2 - \Omega_{cs}^2\right)}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

Let θ be the angle between B_0 and **n**

Step 4: Combine to obtain dispersion relation

$$\vec{n} \times \vec{n} \times \vec{E} + \bar{\bar{K}} \bullet E = 0$$

$$\begin{bmatrix} S - n^2 \cos^2\theta & -iD & n^2 \cos\theta \sin\theta \\ iD & S - n^2 & 0 \\ n^2 \cos\theta \sin\theta & 0 & P - n^2 \sin^2\theta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

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For non-trivial solutions:

$$\det[\ ] = 0$$

Will give the dispersion relation to relate $n(\omega)$ or $\omega(k)$ or $\omega(k,\theta)$

Step 4: Combine to obtain dispersion relation

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Will give the dispersion relation to relate n(ω) or ω(k) or ω(k,θ)

Can write the solution in the convenient Appleton-Hartree Form:

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

$$R = S + D = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega + \frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

$$L = S - D = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega - \frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

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$$\vec{n}\times\vec{n}\times\vec{E}+\bar{\bar{K}}\bullet E=0$$

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$$R=S+D=1-\sum_s\frac{\omega_{ps}^2}{\omega\left(\omega+\frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

$$L=S-D=1-\sum_s\frac{\omega_{ps}^2}{\omega\left(\omega-\frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

Plasmas support wide variety of wave phenomena

- How do we describe waves in plasmas?
- **What can the dispersion relation tell us?**
- Examples of waves and what we can do with them

Dispersion relation contains lots of information

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General condition for **resonance** occurs for: $n^2 \to \infty \quad \lambda \to 0$

$$\tan^2\theta = -P/S$$

Waves resonate with particle motion

Dispersion relation contains lots of information

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

General condition for **resonance** occurs for: $n^2 \to \infty \quad \lambda \to 0$

$$\tan^2\theta = -P/S$$ Waves resonate with particle motion

General condition for **cutoff** occurs for: $n \to 0 \quad \lambda \to \infty$

$$PRL = 0$$ Waves will not propagate

Plasmas support wide variety of wave phenomena

- How do we describe waves in plasmas?
- What can the dispersion relation tell us?
- **Examples of waves and what we can do with them**

Waves in cold plasma dispersion relation

- Propagation parallel to B_0, $\theta = 0$
  - P=0, plasma oscillations
  - $n^2 = R$
  - $n^2 = L$
- Propagation perpendicular to B_0, $\theta = \pi/2$
  - $n^2=P$
  - $n^2=RL/S$

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

Waves in cold plasma dispersion relation

- Propagation parallel to B_0, $\theta = 0$
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- Propagation perpendicular to B_0, $\theta = \pi/2$
  - $n^2=P$
  - $n^2=RL/S$

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

Whistler waves

- Assumptions:
  - Wave propagates along magnetic field: $\theta = 0$
  - Look at root of dispersion relation: $n^2 = R$
  - Consider frequency range: $\Omega_{ci} << \omega << \Omega_{ce} \sim \omega_{pe}$

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega + \frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega - \Omega_{ce}\right)} - \frac{\omega_{pi}^2}{\omega\left(\omega + \Omega_{ci}\right)}$$

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$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega - \Omega_{ce}\right)} - \frac{\omega_{pi}^2}{\omega\left(\omega + \Omega_{ci}\right)} \approx 1 + \frac{\omega_{pe}^2}{\omega\Omega_{ce}} - \frac{\omega_{pi}^2}{\omega^2}$$

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  - Wave propagates along magnetic field: $\theta = 0$
  - Look at root of dispersion relation: $n^2 = R$
  - Consider frequency range: $\Omega_{ci} << \omega << \Omega_{ce} \sim \omega_{pe}$

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega + \frac{q_s}{|q_s|}\Omega_{cs}\right)}$$

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega\left(\cancel{\omega} - \Omega_{ce}\right)} - \frac{\omega_{pi}^2}{\omega\left(\omega + \cancel{\Omega_{ci}}\right)} \approx \cancel{1} + \frac{\omega_{pe}^2}{\omega\Omega_{ce}} - \cancel{\frac{\omega_{pi}^2}{\omega^2}} \approx \frac{\omega_{pe}^2}{\omega\Omega_{ce}}$$

Whistler waves

- Assumptions:
  - Wave propagates along magnetic field: $\theta = 0$
  - Look at root of dispersion relation: $n^2 = R$
  - Consider frequency range: $\Omega_{ci} << \omega << \Omega_{ce} \sim \omega_{pe}$

$$\boxed{n^2 \approx \frac{\omega_{pe}^2}{\omega\Omega_{ce}}}$$

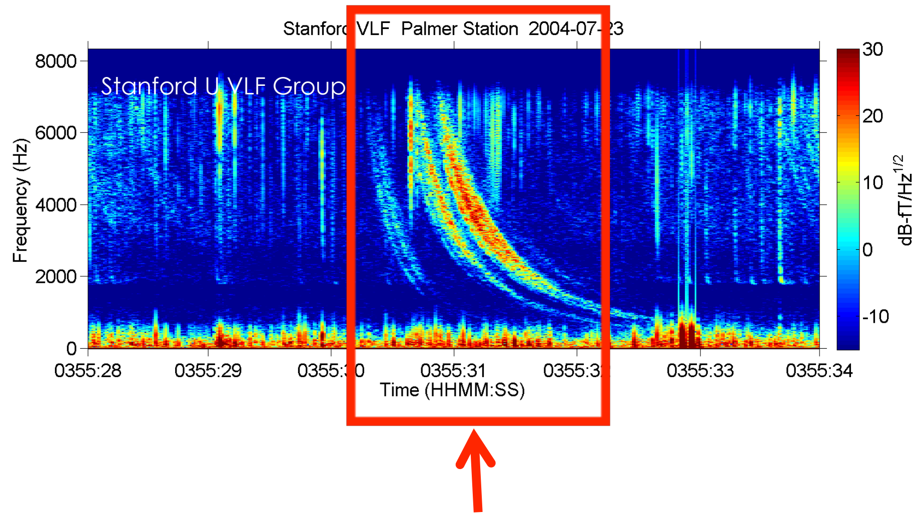
$$v_p = \frac{c}{n} = c\sqrt{\frac{\omega\Omega_{ce}}{\omega_{pe}^2}}$$ Phase velocity

$$v_g = \frac{d\omega}{dk} = \frac{2kc^2\Omega_{ce}}{\omega_{pe}^2} = 2v_p$$ Group velocity

$v_p, v_g \propto \omega$ High frequencies to propagate faster along B

Whistler waves found in magnetosphere

- Originally observed by radio/telephone operators in WWI/II
- Lightning strikes excite broad range of radio frequency waves in magnetosphere
- Some whistlers born at strike site, propagate along earth's dipole field
- Because of dispersion, higher frequency waves go faster than lower frequency: higher frequency at front of wave packet

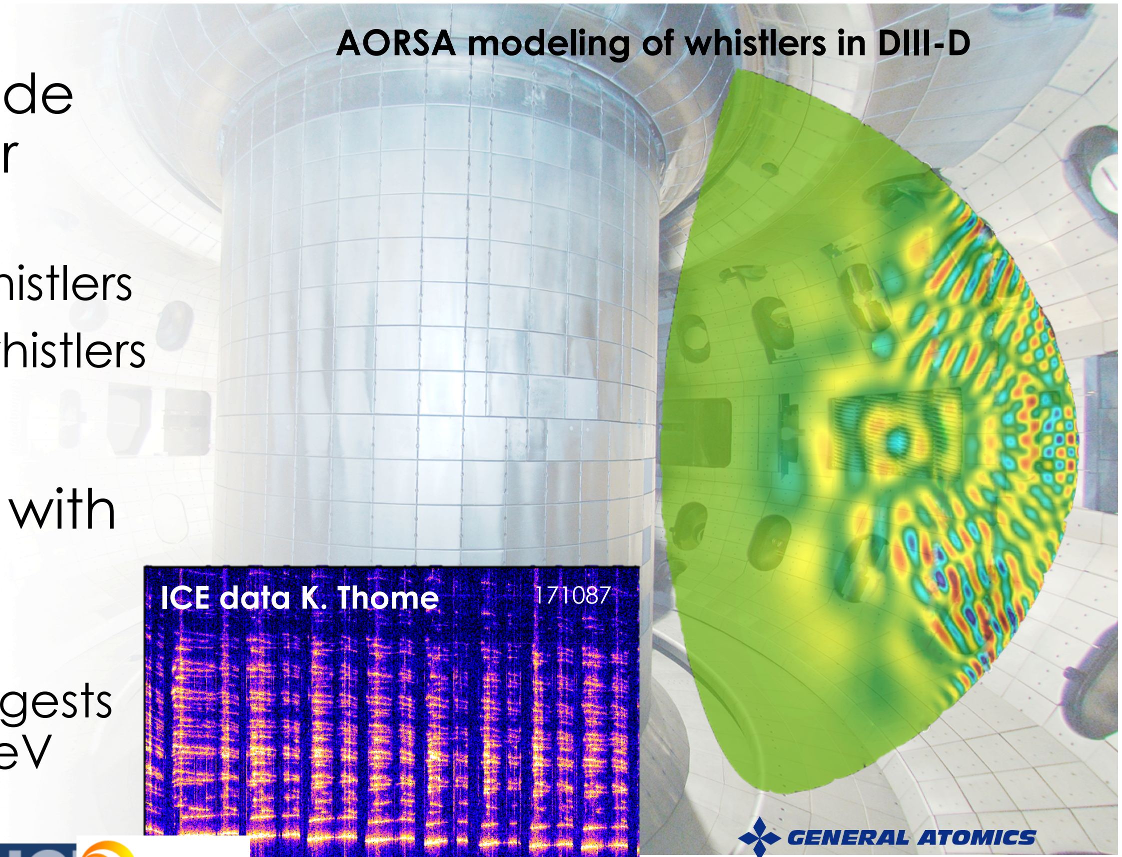


Whistlers heard as a descending tone

Whislter waves observed in tokamaks

- Runaway electrons provide driving energy for whistler waves
  - Increasing B suppresses whistlers
  - Decreasing B enhances whistlers

- Observed more whistlers with increased intensity in measured hard x-rays
  - Dispersion relationship suggests electron energy ~10-15 MeV



K. Thome RSI (2018)

D. Spong PRL (2018)

Waves in cold plasma dispersion relation

- Propagation parallel to B_0, $\theta = 0$
    - P=0, plasma oscillations
    - $n^2 = R$
    - $n^2 = L$
- **Propagation perpendicular to B_0, $\theta = \pi/2$**
    - **$n^2=P$**
    - **$n^2=RL/S$**

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

Electron cyclotron range of frequency waves provide heating and drive current

- Consider electron cyclotron (EC) frequency range, there are two solutions to cold plasma dispersion relation:
- Ordinary mode (O-mode):
  - E is parallel to B
  - Independent of B
  - Depends on n_e

$$n_\perp^2(\omega) = P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

- Extraordinary mode (X-mode):
  - E is perpendicular to B
  - Depends on B, n_e

$$n_\perp^2(\omega) = \frac{RL}{S}$$

Electron cyclotron range of frequency waves provide heating and drive current

- Consider electron cyclotron (EC) frequency range, there are two solutions to cold plasma dispersion relation:
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  - Depends on n_e

Cutoff when $\omega=\omega_{pe}$

$$n_\perp^2(\omega) = P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

- Extraordinary mode (X-mode):
  - E is perpendicular to B
  - Depends on B, n_e

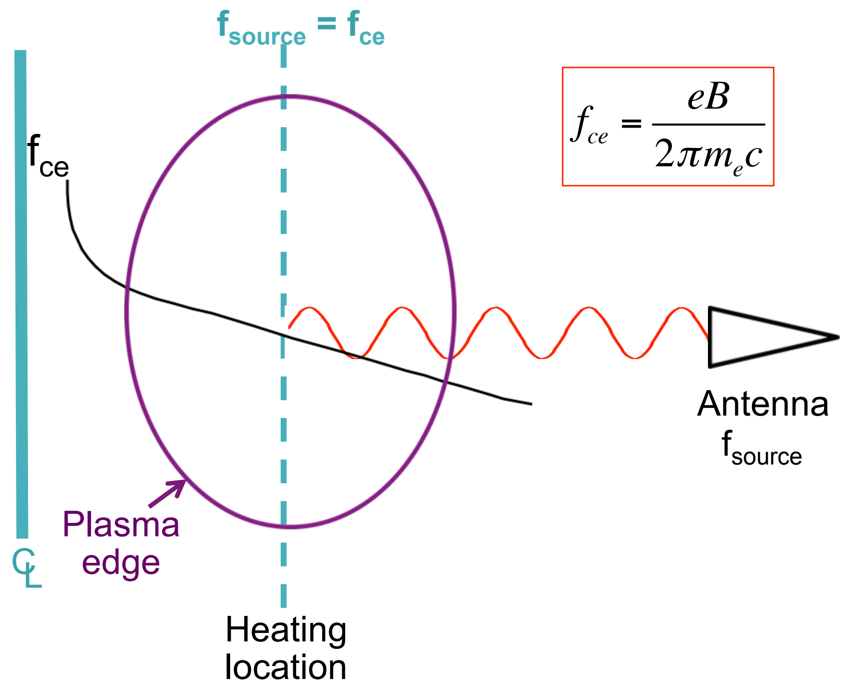
Cutoff when R=0 or L=0
"R and L cutoffs"

$$n_\perp^2(\omega) = \frac{RL}{S}$$

Resonant when S=0
"Upper Hybrid Resonance"

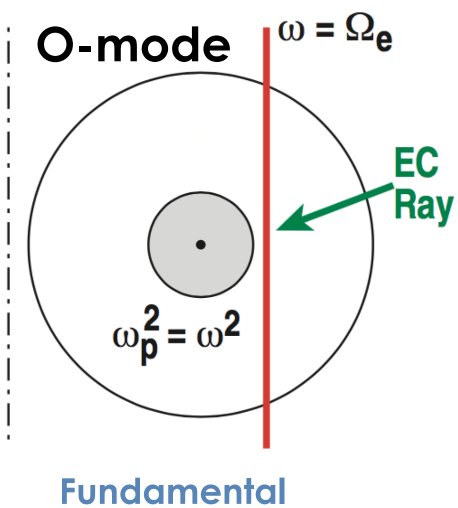
How can O-modes and X-modes have resonances?

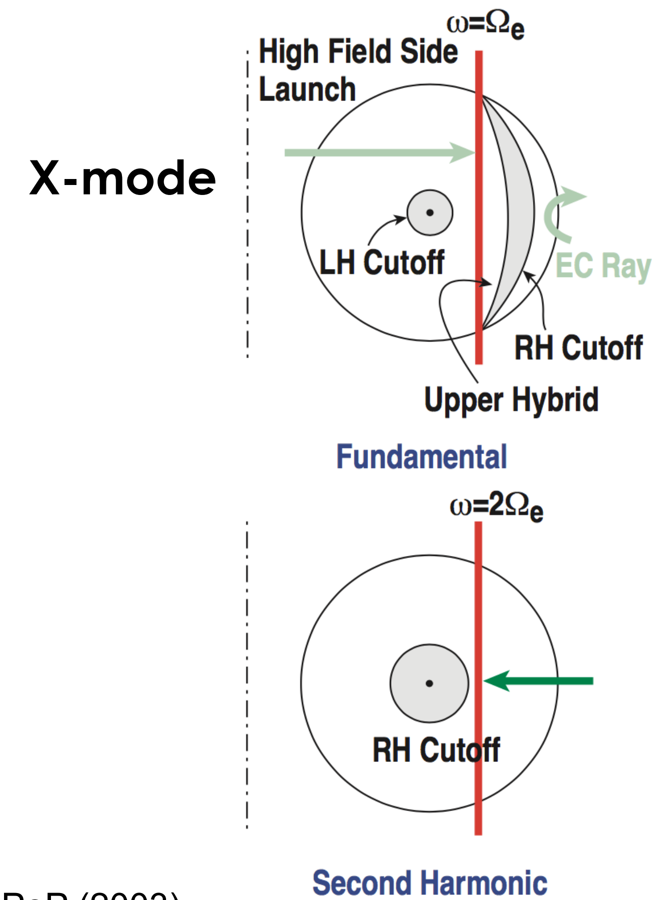
- Hot plasma dispersion relation has resonances
  - O-mode resonant at Ω_C, X-mode resonant at $n\Omega_C$
  - Same cutoffs as cold plasma dispersion relations
- Launched RF waves absorbed near cyclotron resonance
  - Tuned to either electron or ion cyclotron motion
  - RF source frequency can be chosen to heat precise radius
  - For tokamaks, $B_t \propto 1/R$



EC waves provide localized heating/current drive

- Can provide:
  - Electron heating
  - Current profile control, sustainment
  - Control of MHD activity

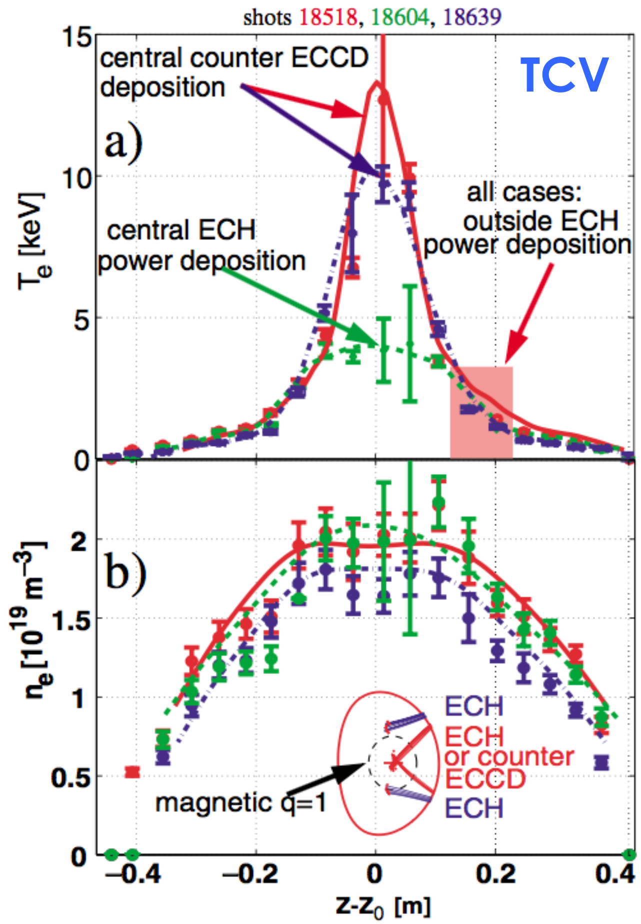




R. Prater PoP (2003)

EC waves provide localized heating/current drive

- Many examples of ECH/ECCD in tokamaks and other confinement devices
  - Large scale, high performance devices depend on waves for heating
- ECH/ECCD can provide current profile tailoring in TCV
  - Improve central electron energy confinement
  - Stabilize MHD modes



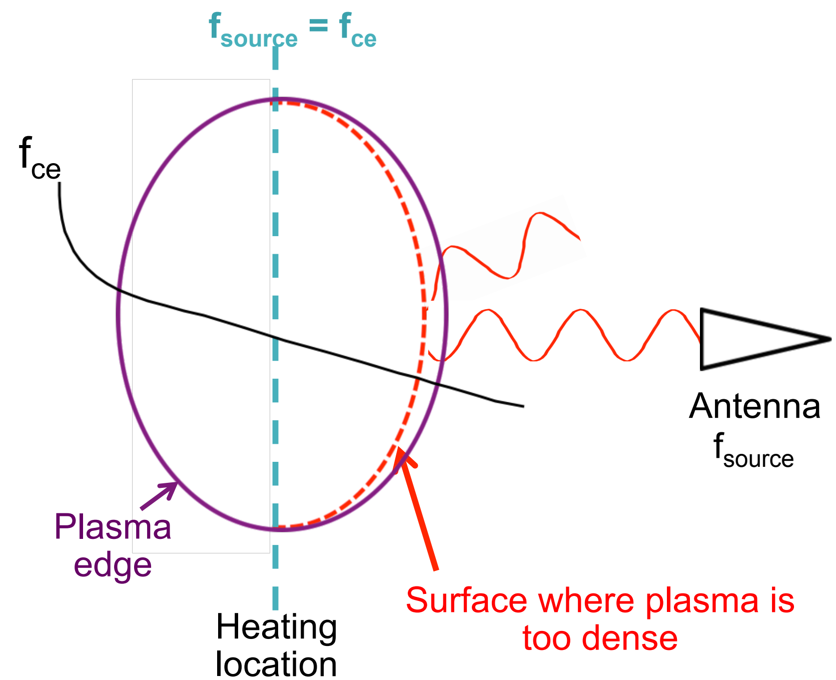
Z.A. Pietrzyk PRL 86, 8 (2001)

Electron cyclotron wave injection provides plasma heating and current drive – in certain conditions

- If plasma is too dense, O-mode & X-mode reflected near plasma edge
  - Happens in spherical tokamaks and stellarators

$$\omega_{source} > \omega_{pe}$$

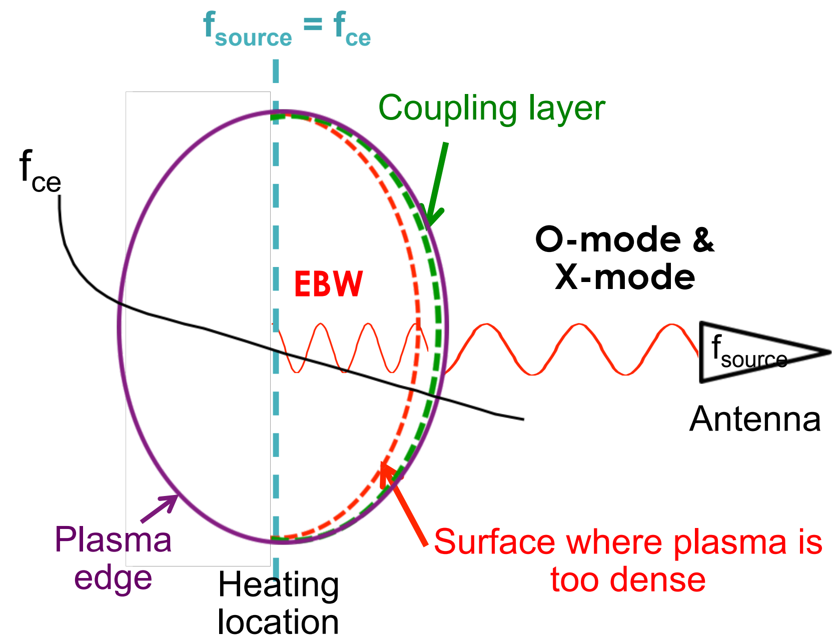
- Alternative heating method required



Electron Bernstein waves can travel in high density plasmas

- Electron Bernstein Waves (EBW) can only travel inside the plasma
  - Wave moves due to coherent motion of charged particles

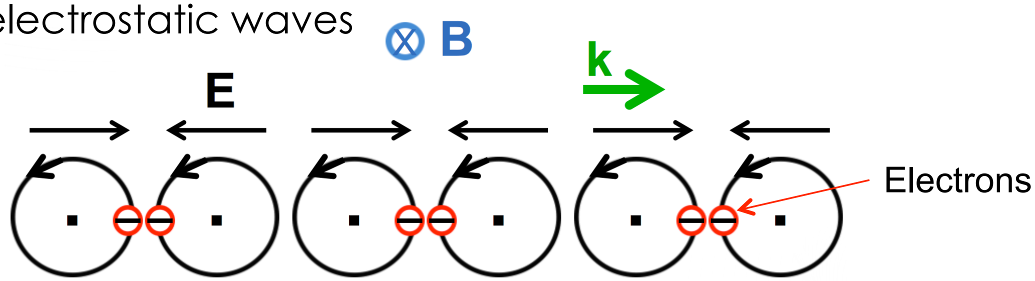
- Can only couple to EBW by launching O- or X-modes



Oak Ridge National Laboratory

Electron Bernstein waves can propagate in overdense plasmas

- Electron Bernstein waves (EBW) are hot plasma waves:
  - Perpendicularly propagating, $k_{||}=0$
  - Do not experience a density cutoff in the plasma
  - Longitudinal, electrostatic waves



- Cannot propagate in vacuum -> must launch O- or X-mode to mode couple to EBW

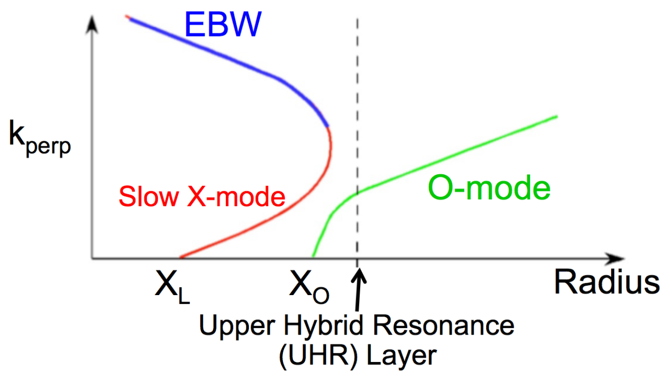
$$1-2\sum_s \frac{4\pi n_s m_s c^2}{\lambda B_0^2}\left[\sum_s e^{-\lambda} I_n(\lambda)\frac{n^2}{\left(\omega/\Omega\right)^2-n^2}\right]=0 \quad \text{Where: } \lambda=\frac{k_\perp^2 \kappa T_\perp}{m\Omega^2}$$

- As wave frequency approaches EC harmonic, $\omega=n\Omega_C$, wave is strongly absorbed

EM waves can couple to EBW at conversion layer before reflection at density cutoff

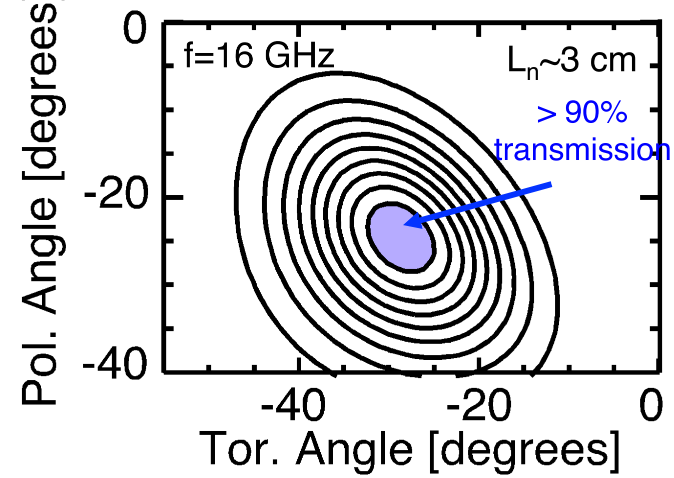
- EBW coupling efficiency depends on plasma parameters at conversion layer:
  - Density gradient
  - Magnetic field pitch
- Requires oblique launch of O-mode

O-X-EBW Mode Conversion

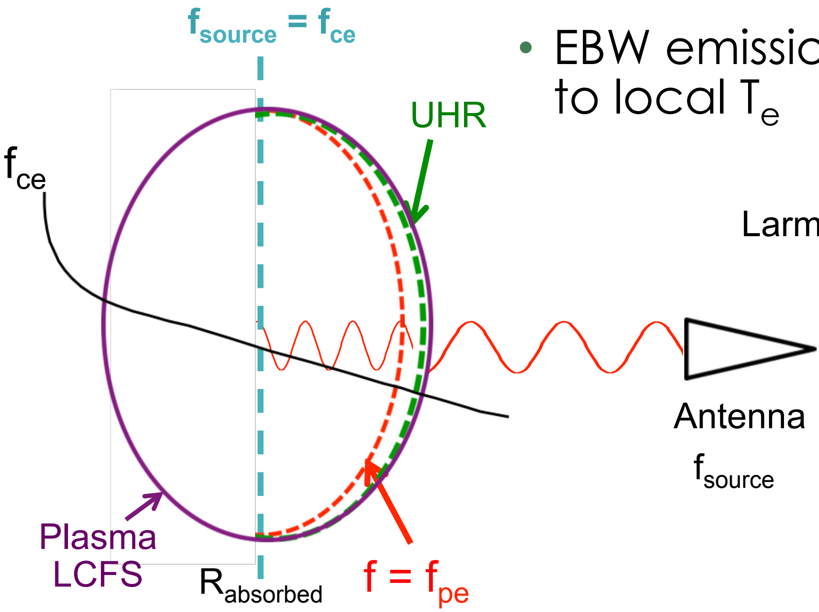


$$\omega_{UHR} = \sqrt{\omega_{pe}^2 + \Omega_{ce}^2}$$

Calculated O-X-EBW transmission window for NSTX



EBW emission can be used to measure temperature



- EBW emission at blackbody levels, proportional to local T_e

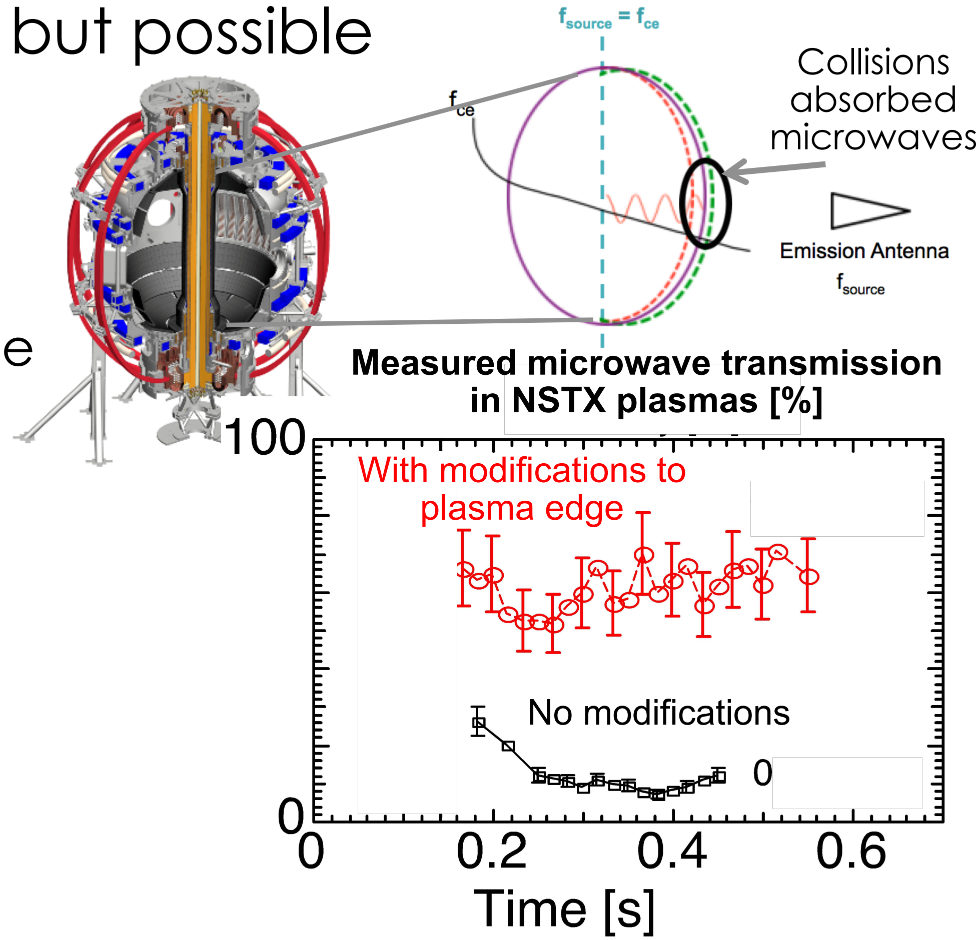
Larmor formula: $$I_\omega = \frac{\omega^2 k_B T_{rad}}{8\pi^3 c^2}$$

$$Transmission = \frac{T_{rad}(EBE)}{T_e(\text{Thomson})}$$

- Physics of O-X-EBW injection and EBW-X-O emission are symmetric, assuming no parasitic effects
- Measured T_{rad} = local T_e provided EBW-X-O conversion efficiency known: $f_{ce} \sim 1/R$ ➔ radial localization

Coupling microwave power into high density fusion plasmas can be difficult – but possible

- Plasma naturally emits microwaves from cyclotron resonance location
- Assumed physics of microwave emission from high density plasmas same as launching
  - Measurements on NSTX didn't agree with predictions
  - Plasma edge had too many collisions, absorbed microwaves
- ***Unexpected results present opportunities***



Waves in cold plasma dispersion relation

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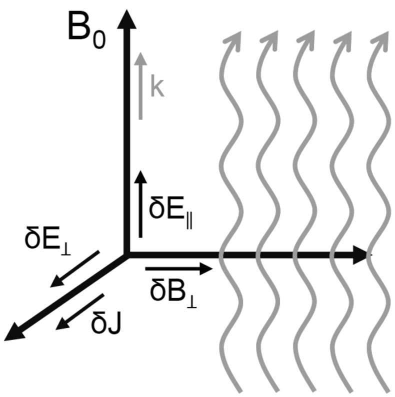
Low frequency MHD waves – Alfvén waves

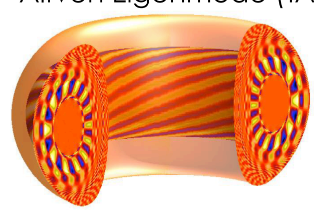
- Very low frequency waves ($\omega << \Omega_{CI}$)
- MHD wave where ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines
  - Linearize MHD equations to obtain shear Alfvén
  - EM waves that propagate along magnetic field lines



$$\omega = k_{||} v_A$$

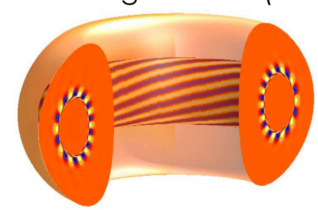
$$v_A = \frac{B}{\sqrt{\mu_0 n_i m_i}}$$

Toroidicity Induced Alfvén Eigenmode (TAE)



$$\omega_{TAE} \sim \frac{v_A}{2R}\left(\frac{n}{m+1/2}\right) \propto \frac{B}{\sqrt{n_e}}$$

Reversed Shear Alfvén Eigenmode (RSAE)

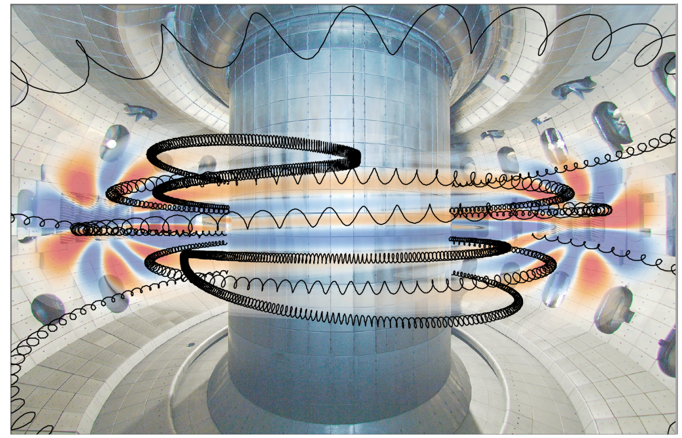


$$\omega_{RSAE} \sim \frac{v_A}{R}\left(\frac{m - nq_{min}}{q_{min}}\right) \propto \frac{1}{q_{min}} \frac{B}{\sqrt{n_e}}$$

C. Collins AAPS (2018)

Alfvén eigenmodes (AE) can cause fast-ion transport

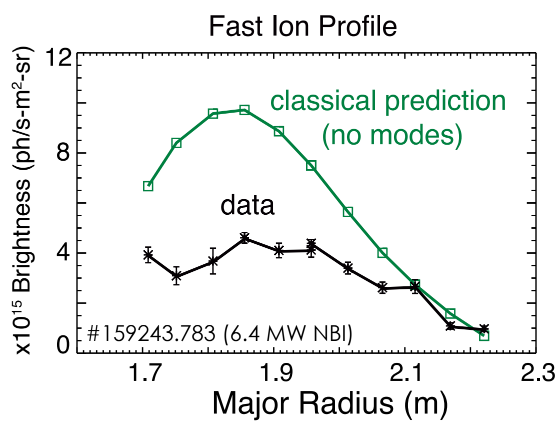
- Fast ions created through NBI, ion cyclotron resonance heating, or fusion reactions



- AEs are MHD instabilities driven by wave particle interactions

- In DIII-D, high beam power can drive strong AE activity, causing fast-ion profile to flatten

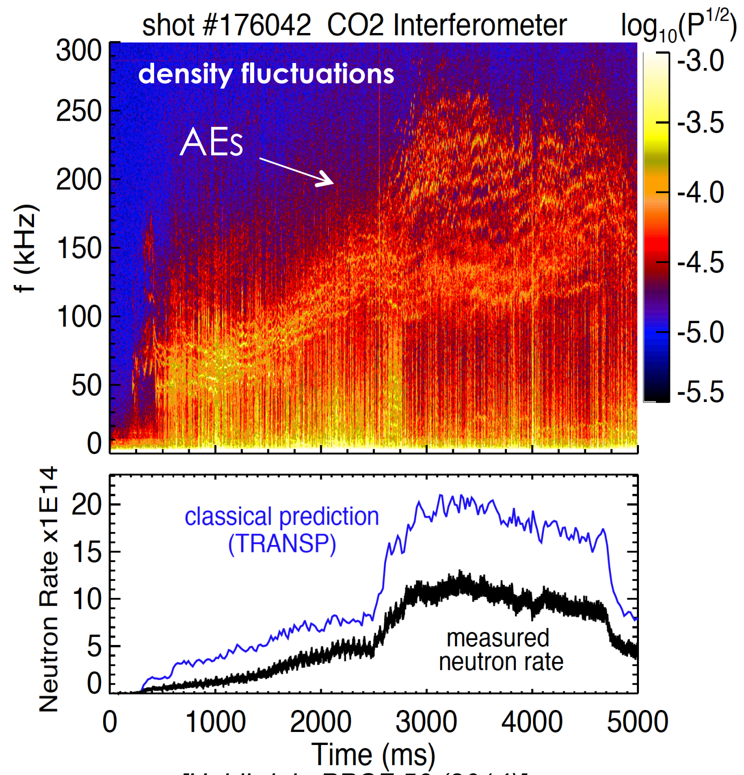
[Heidbrink et al., PRL 99, 245002 (2007)]



C. Collins AAPS (2018)

Fast-ion transport can reduce fusion performance and lead to losses that damage fusion reactor walls

- AEs cause transport that can:
  - Reduce absorbed beam heating power
  - Reduce current drive
  - Reduce achievable β_N (fusion power $\propto$ $(\beta_N)^2$)
  - Cause fast ion losses that damage walls
- A 'sea' of AEs are predicted to be unstable in ITER
- Important questions:
  - When is transport significant?
  - What can we do to control AE transport



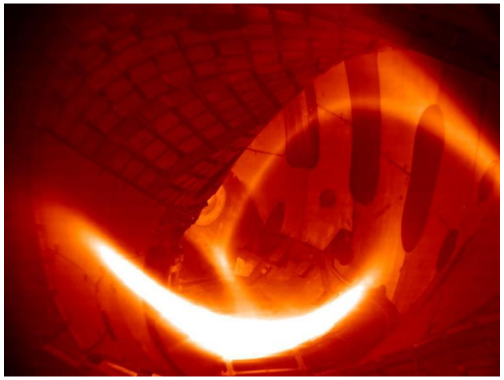
[Heidbrink, PPCF 56 (2014)]

[Holcomb, PoP 22 (2015)]

Plasmas support wide variety of wave phenomena

- Waves found naturally in plasmas
  - Described by dispersion relation
- Waves can deliver energy-momentum in plasma
- Waves can be used in plasma diagnostics
- Waves can drive turbulence…

Photo pf aurora: Senior Airman Joshua Strang



First W7-X plamsa, IPP, Greifswald

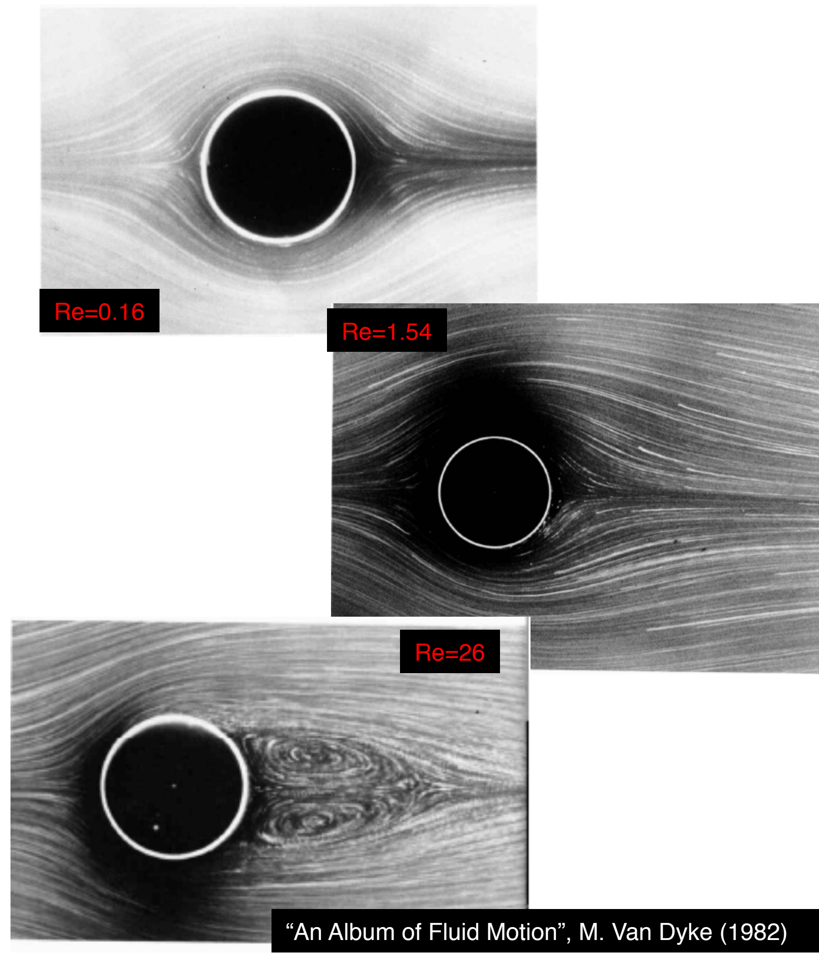
Turbulence – References

- See previous lectures by Saskia Mordjick and Troy Carter - http://suli.pppl.gov
- Greg Hammett has a lot of great introductory material to fusion, tokamaks, drift waves, ITG turbulence, gyrokinetics, etc… (*w3.pppl.gov/~hammett)*
- Greg Hammett & Walter Guttenfelder gave five 90 minute lectures on turbulence at the 2018 Graduate Summer School (*gss.pppl.gov*)
- Transport & Turbulence reviews:
  - Liewer, Nuclear Fusion (1985)
  - Wootton, Phys. Fluids B (1990)
  - Carreras, IEEE Trans. Plasma Science (1997)
  - Wolf, PPCF (2003)
  - Tynan, PPCF (2009)
  - ITER Physics Basis (IPB), Nuclear Fusion (1999)
  - Progress in ITER Physics Basis (PIPB), Nuclear Fusion (2007)

References from W. Guttenfelder

What is turbulence?

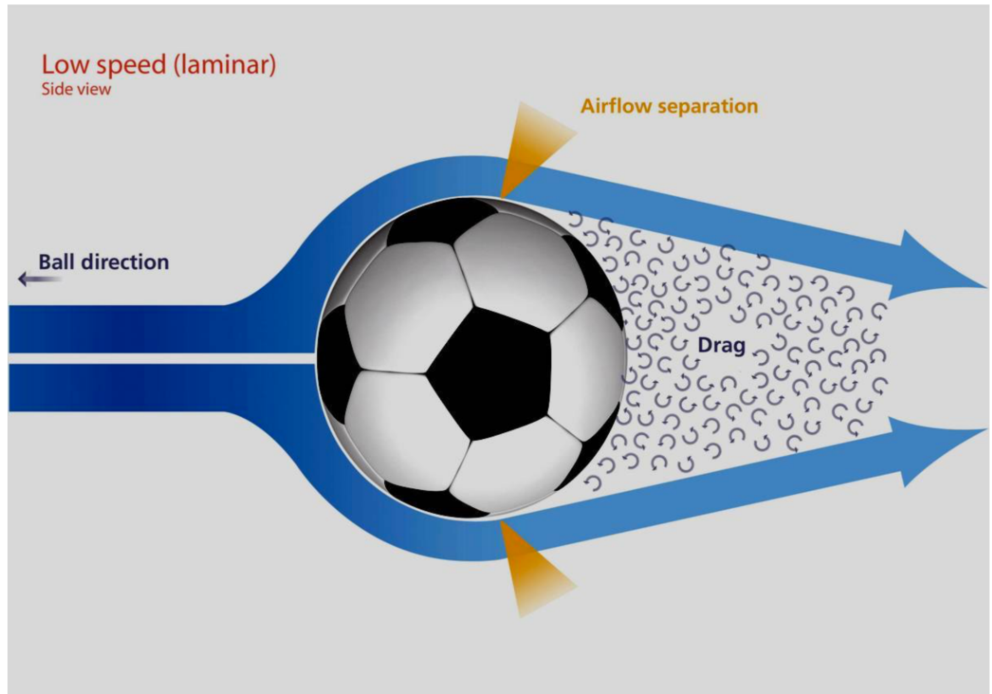
- Turbulence is fluid motion characterized by chaotic changes in pressure and flow velocity
  - Irregular: treated statistically
  - Diffusive: available supply of energy accelerates mixing of fluids
- Turbulence spans wide range of spatial and temporal scales
- Turbulence is not a property of the *fluid*, it's a feature of the *flow*
- Examples of turbulence?



"An Album of Fluid Motion", M. Van Dyke (1982)

Turbulence effects soccer ball performance – low speed

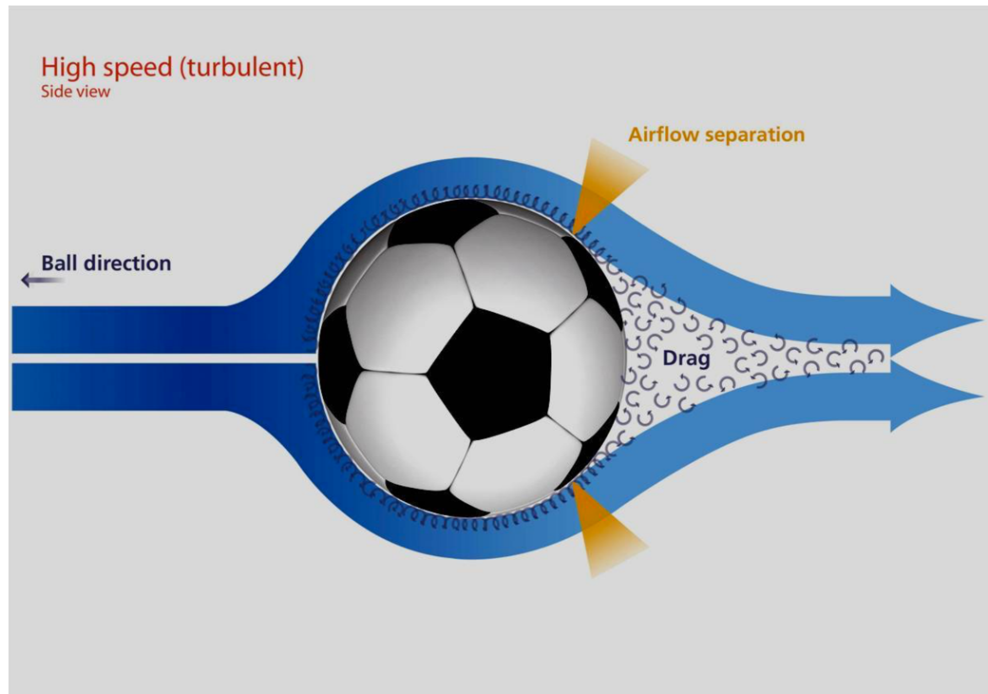
- At low speeds, laminar airflow regime
- Boundary layer separates early
- Large wake created with high drag on ball



Reference: A.L. Kiratidis & D.B. Leinweber, European Journal of Physics, Vol. 39, #3 (2018)

Turbulence effects soccer ball performance – high speed

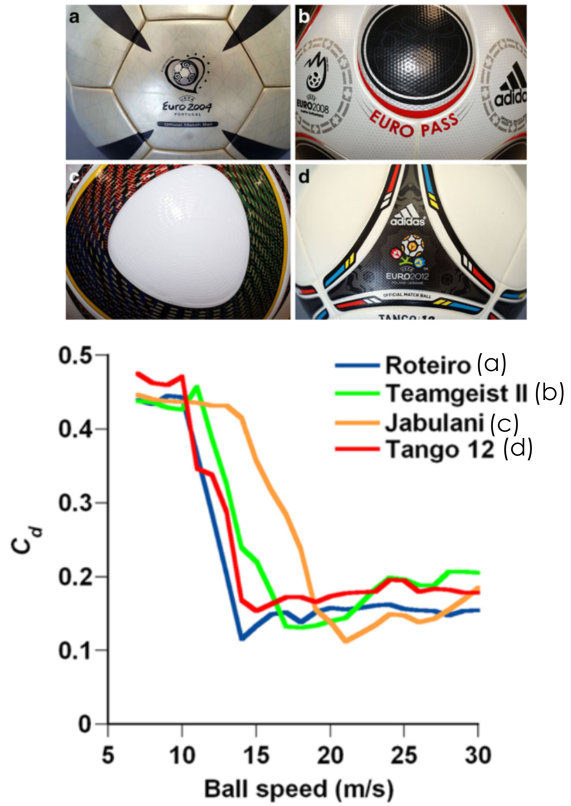
- At high speeds, turbulent airflow regime
- Boundary layer separates late
- Smaller wake created, lower drag on ball



Reference: A.L. Kiratidis & D.B. Leinweber, European Journal of Physics, Vol. 39, #3 (2018)

Parameter modifications can affect turbulence

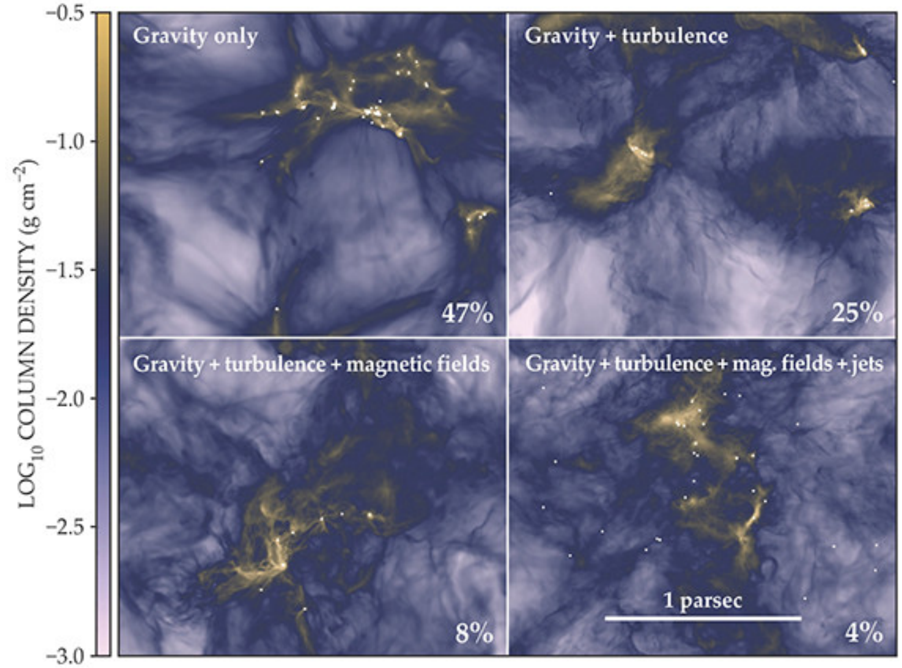
- Rougher surfaced soccer balls lead to more predictable flight
  - Affects speed at which flow transitions from laminar to turbulent around ball
- What this tells us about turbulence:
  - Turbulence can dramatically effect outcomes/performance
  - Not accounting for turbulence can lead to unexpected behaviors
  - Can find knobs to “tune” turbulence to take advantage of it



T. Asai & K. Seo SpringerPlus (2013)

Turbulence affects star formation

- Turbulence kicks around gas, making it harder for gravity to collapse clouds
- Turbulence is supersonic, experiences shocks/strong local compressions necessary to seed gravitational collapse
- Kick-starts star formation in localized regions of the cloud



Published in: Christoph Federrath; *Physics Today* **71,** 38-42 (2018)
DOI: 10.1063/PT.3.3947

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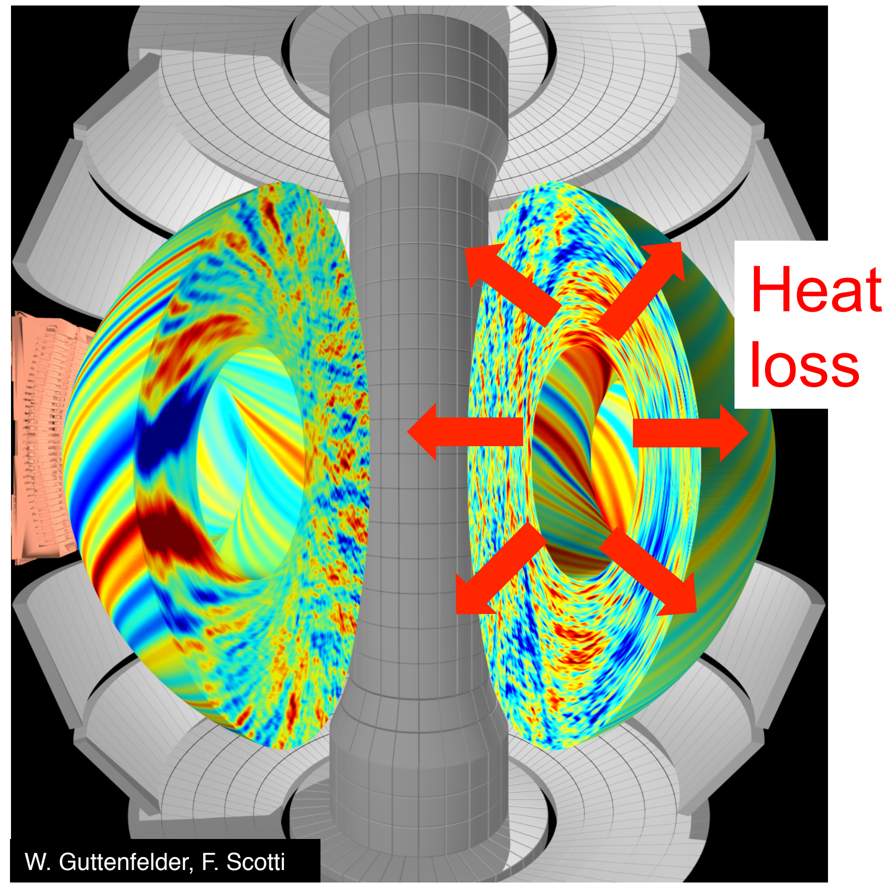
Sufficient energy confinement in magnetic fusion energy devices to reach ignition

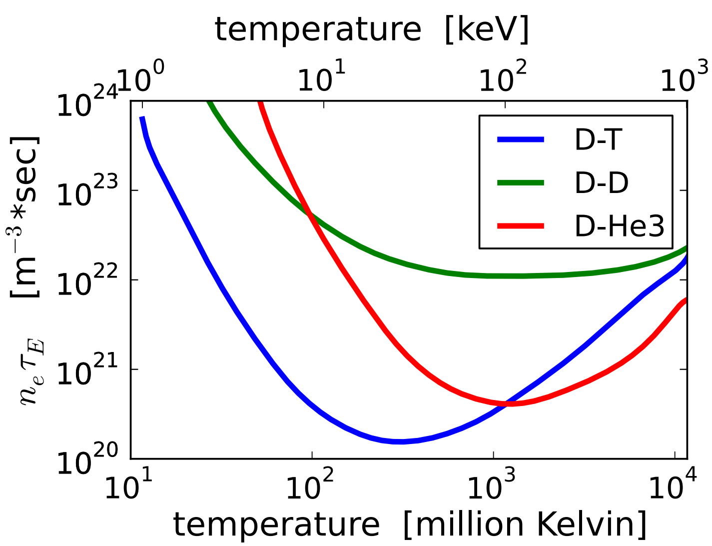
- Sustained fusion reactions require enough particles **(density)** that are energetic enough **(temperature)** and collide often enough **(confinement time)**

- Confinement is not perfect, devices can leak heat at a significant rate



Triple product, Lawson criterion, determines ignition

- Require power losses > input power
  - Depends on density, temperature, confinement time



Temperature = 150 Million C

Pressures = 2-4 atm

Need τ_E = 1-2 s

$$\tau_{C,collisions} \sim \frac{1}{D_{collisions}} \sim 100 \text{ s}$$

Oak Ridge National Laboratory

Triple product, Lawson criterion, determines ignition

- Require power losses > input power
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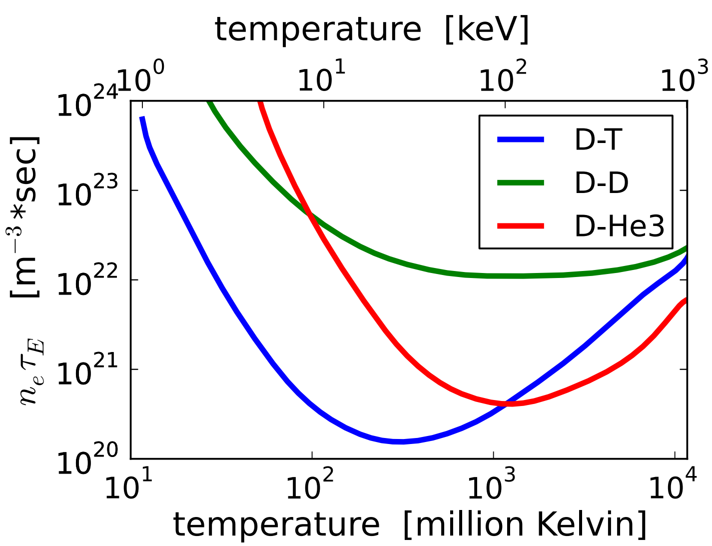


Temperature = 150 Million C

Pressures = 2-4 atm

Need τ_E = 1-2 s

$$\tau_{C,collisions} \sim \frac{1}{D_{collisions}} \sim 100 \text{ s}$$

$$\tau_{C,\text{experimental}} \sim 0.1 \text{ s}$$

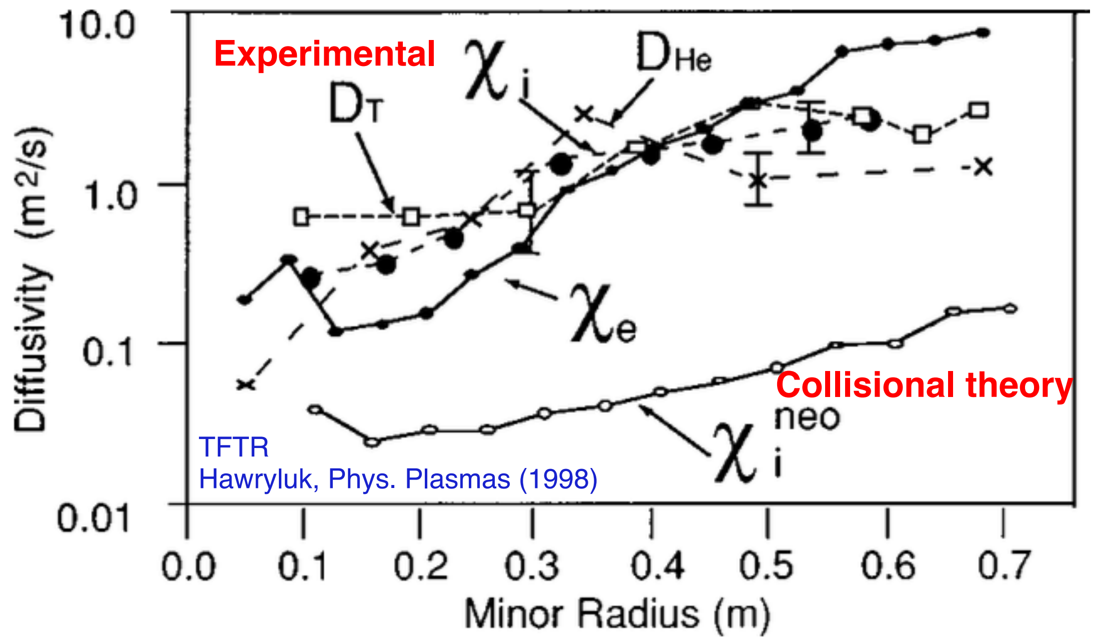
Inferred experimental transport larger than classical theory – extra "anomalous" contribution

- Turbulent diffusion coefficient orders of magnitude larger than collisional (neo-classical) diffusion

$$\text{confinement time} \sim \frac{1}{D}$$

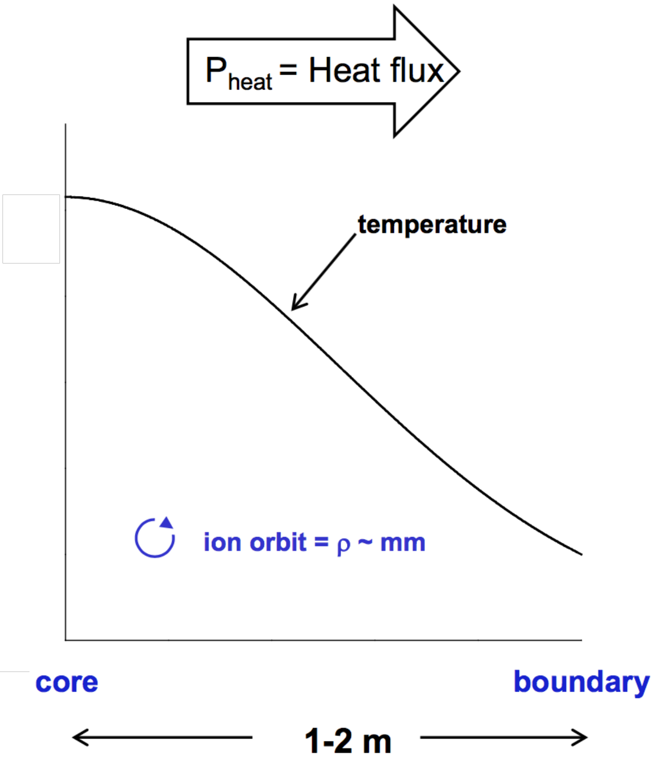
- Results in lower than expected energy confinement

Transport coefficients for TFTR supershot



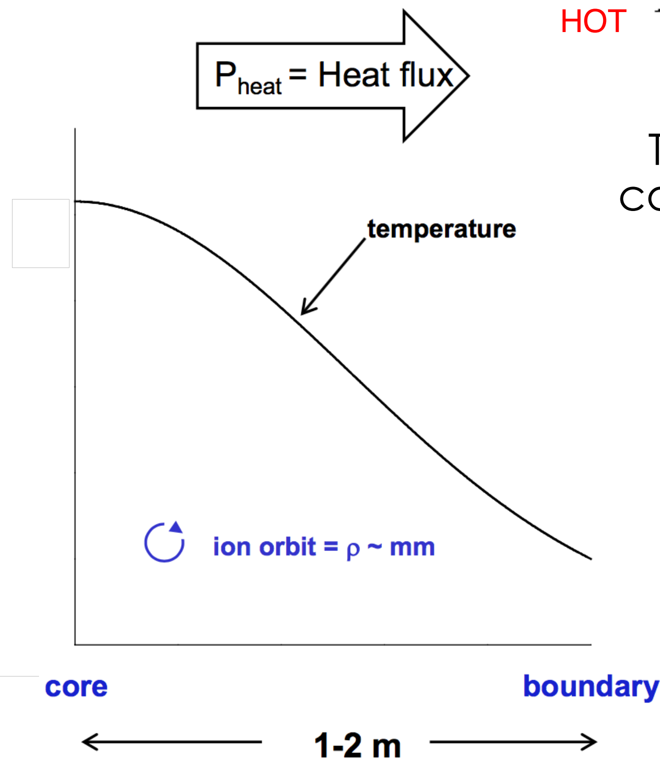
Diffusion by collisions will try to relax gradients

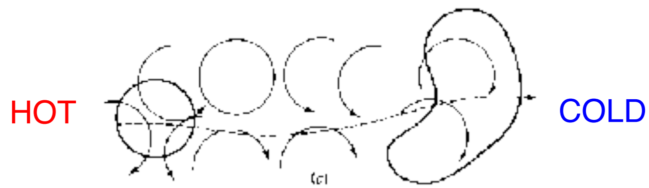
$$\tau_{C,collisions} \sim \frac{1}{D_{collisions}} \sim 100\ \mathrm{s}$$



From W. Guttenfelder

Increasing gradients eventually cause small scale instability -> turbulence

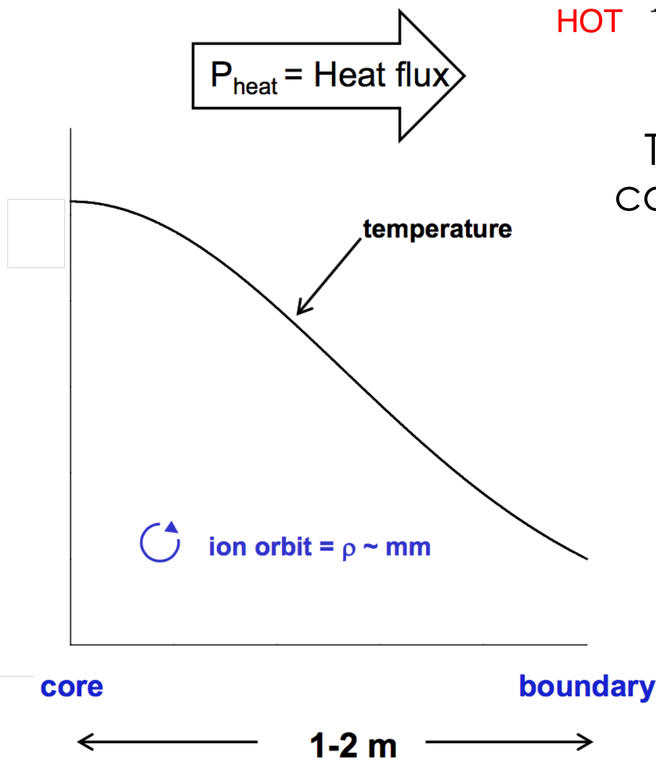


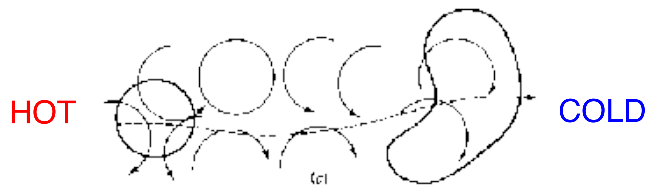


Turbulence mixing hot core with cool boundary plasma

From W. Guttenfelder

Increasing gradients eventually cause small scale instability -> turbulence





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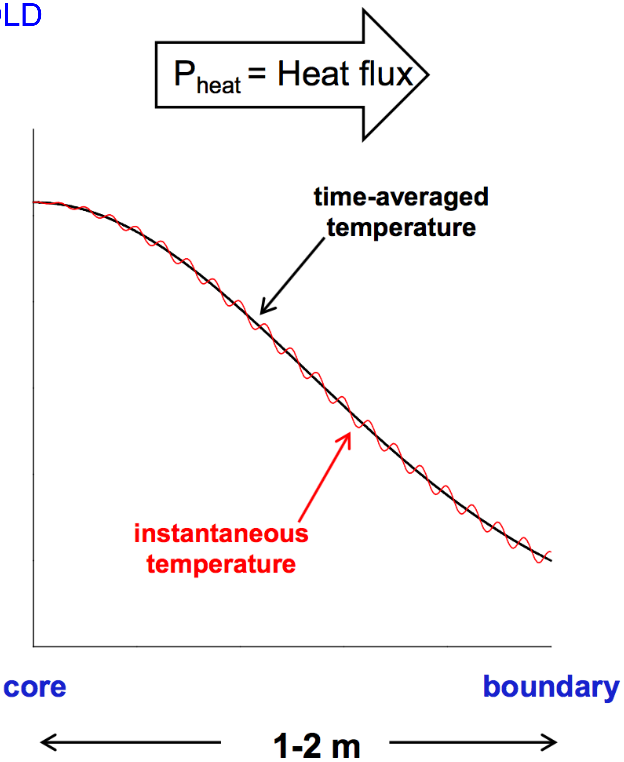
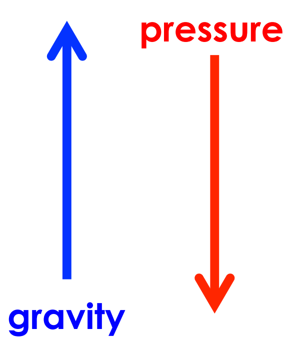
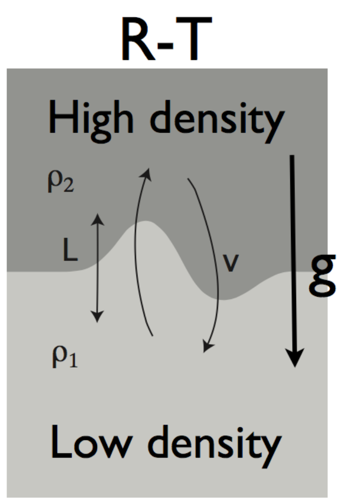


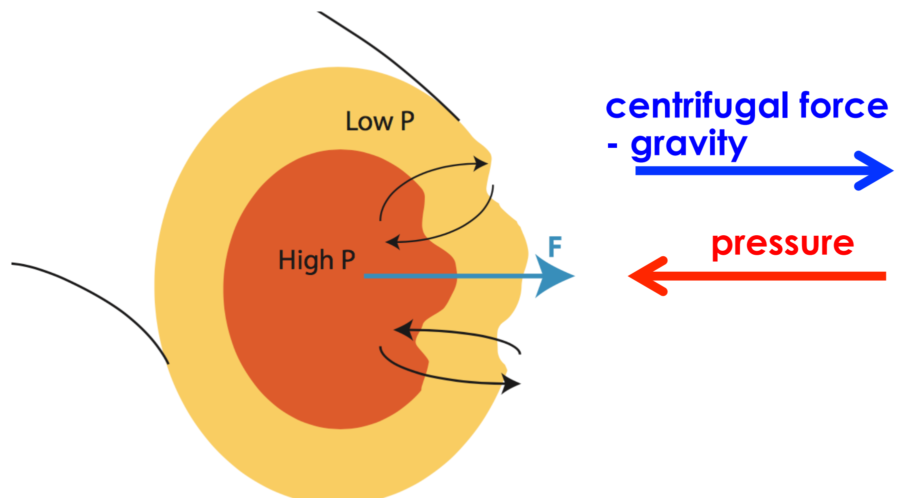
From W. Guttenfelder

Instabilities and turbulence driven by thermal energy gradients

- Perturbations that mix hot core plasma and cold edge plasma can release free energy
- Interchange drive is important (analogous to Rayleigh-Taylor)
- Effective gravity provided by magnetic field gradient/curvature







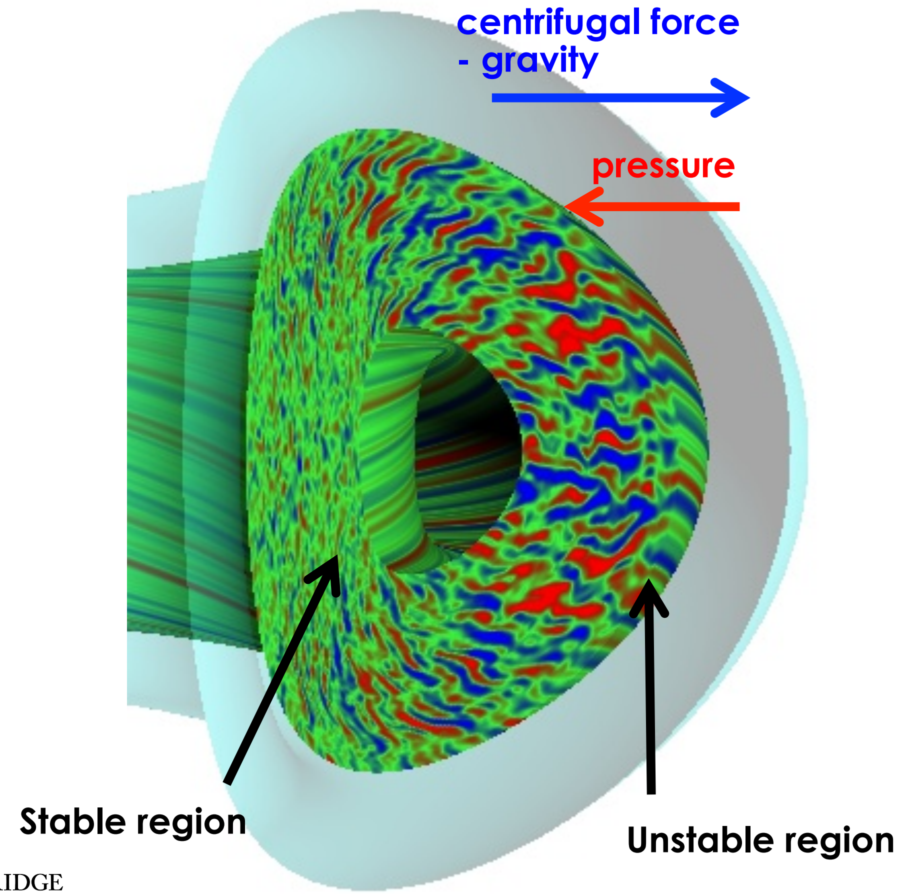
Simulation of turbulence in a tokamak

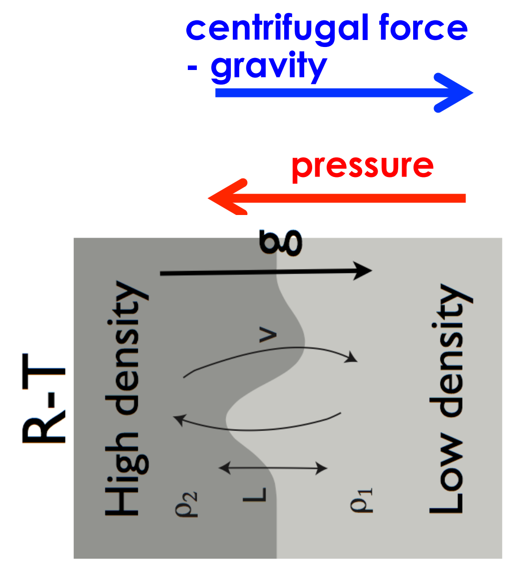
Code: **GYRO**

Authors: Jeff Candy and Ron Waltz

Gyrokinetic simulation by Jeff Candy, Ron Waltz (GA) https://w3.pppl.gov/~hammett/viz/viz.html

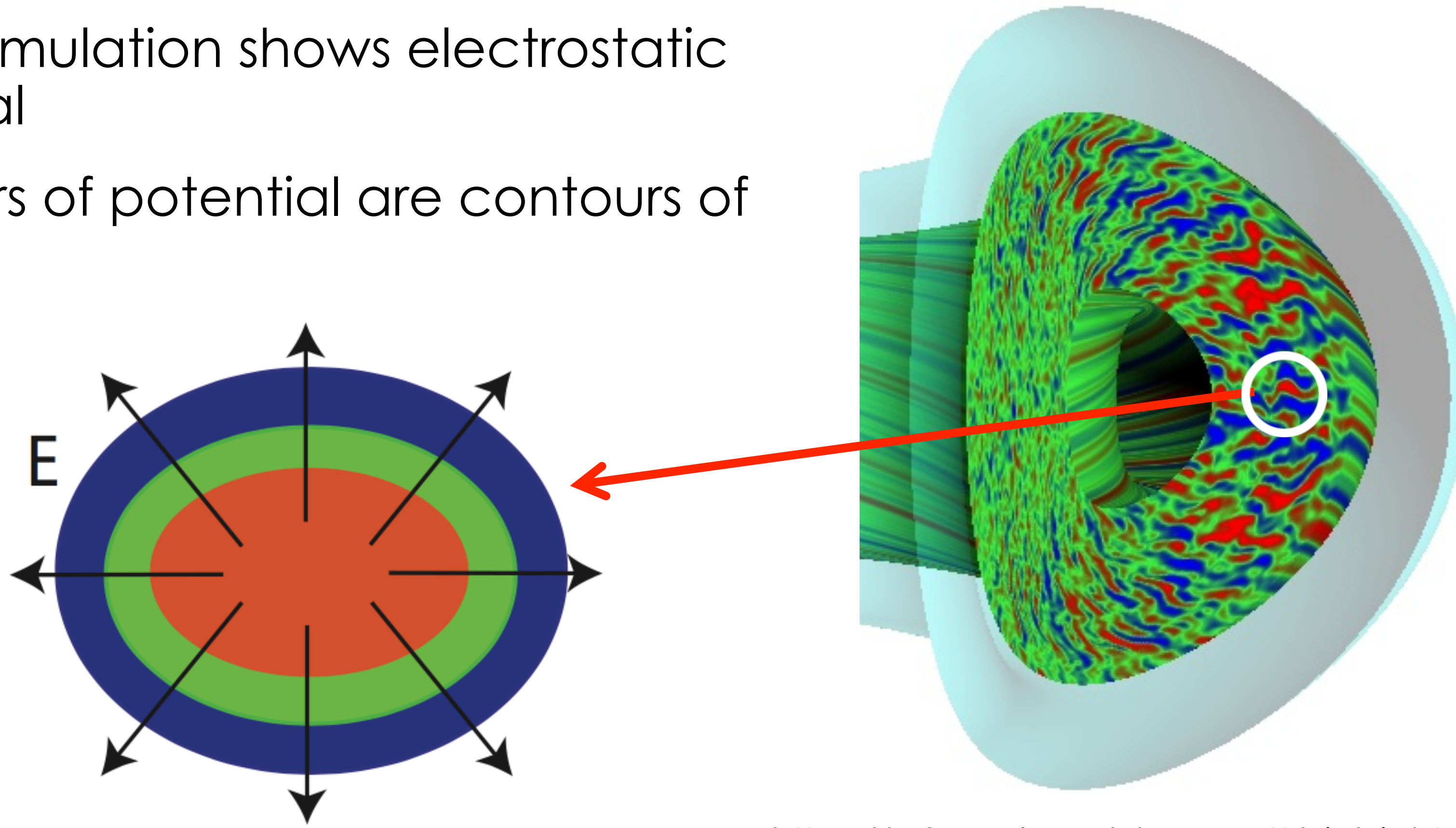
Inertial force in toroidal field acts like an effective gravity





Turbulent transport by ‘eddies’

- GYRO simulation shows electrostatic potential
- Contours of potential are contours of ExB flow



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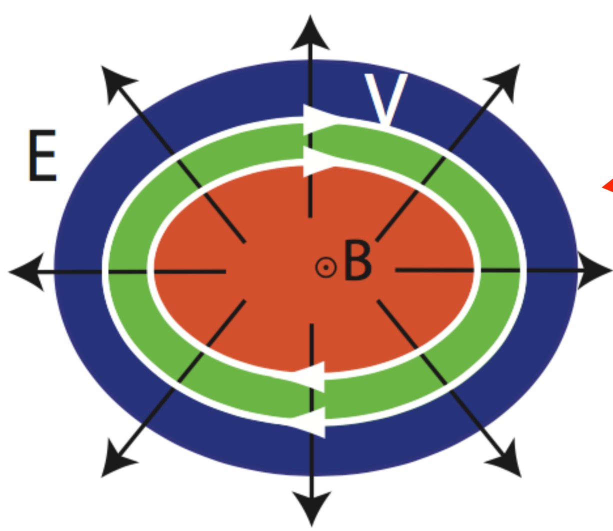
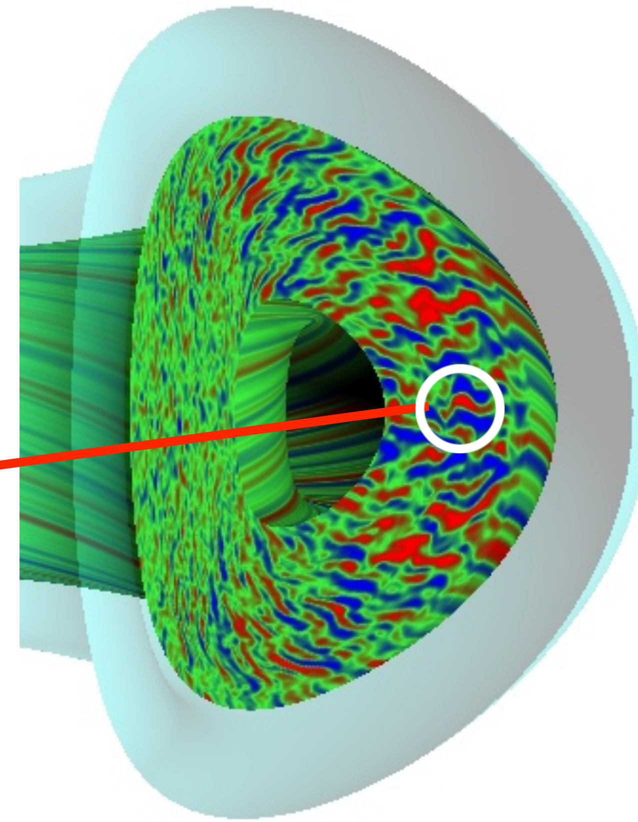
Turbulent transport by ‘eddies’

- Contours of potential are contours of ExB flow
- Eddies create E-field, combined with B-field results in circulation

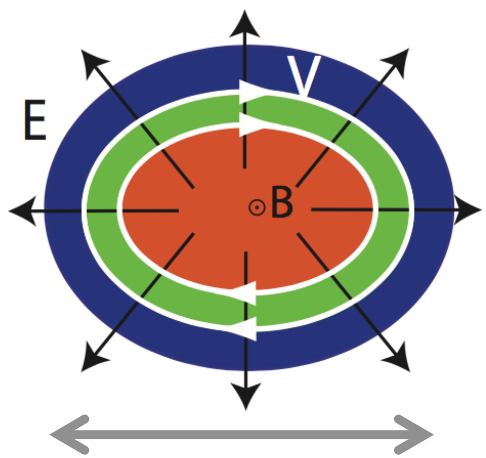




https://w3.pppl.gov/~hammett/viz/viz.html

Diffusion increases as temperature increases, limits temperature gradients

- Turbulent diffusion is a random walk by eddy de-correlation



$$D \sim \frac{(\Delta x)^2}{\Delta t} \sim \frac{L_C^2}{\tau_C}$$

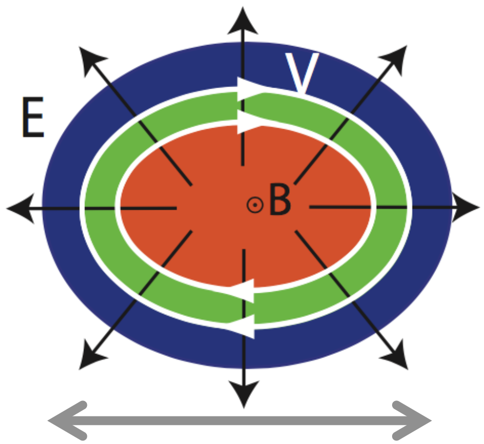
Eddy size

Eddy "turnover" time

$$\tau_C \sim \frac{L_C}{v} \qquad v \sim \frac{E}{B} \sim \frac{\phi}{L_C}\frac{1}{B}$$

$$\boxed{D \sim \frac{\phi}{B} \sim \frac{T}{B}}$$

Diffusion increases as temperature increases, limits temperature gradients



$$\boxed{D \sim \frac{\phi}{B} \sim \frac{T}{B}}$$

$$D_{classical} \sim \rho^2 \nu \sim T^{-1/2}$$ Classical diffusion

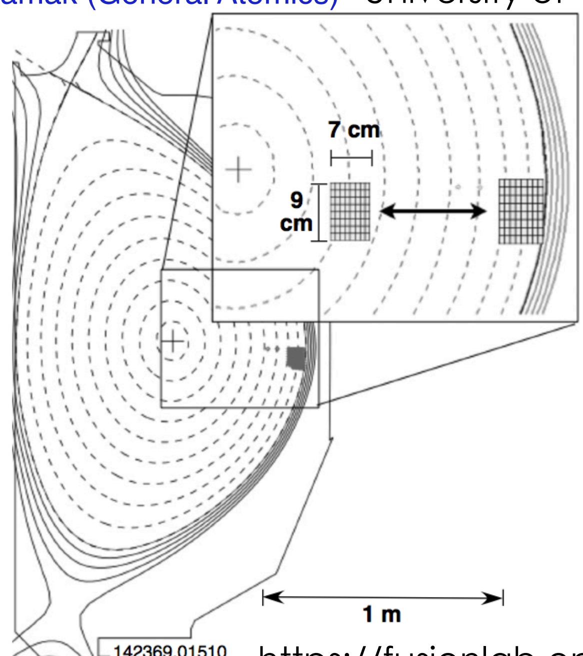
$$\left(\nu \sim T^{-3/2}\right)$$

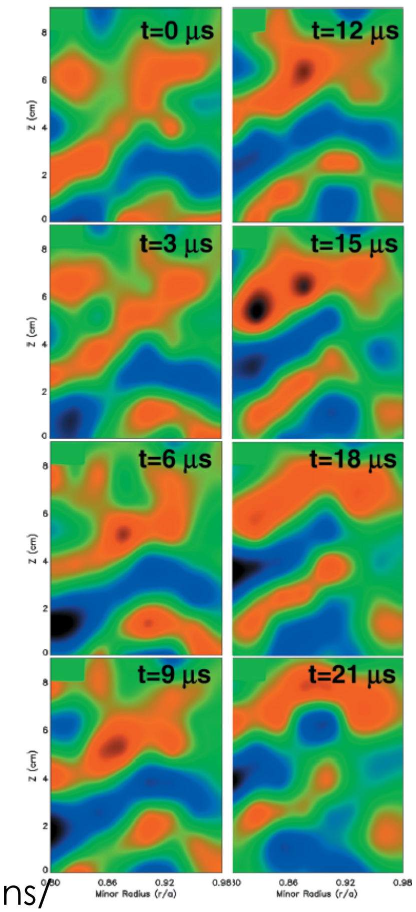
- Collisional (classical) diffusion weaker as plasma gets hotter
  - As T increases (more heating power), confinement degrades
  - Opposite turbulent transport
- Controlling ***size*** and ***correlation time*** of eddies controls confinement

Turbulence is observed in magnetically confined plasmas

- Beam Emission Spectroscopy (BES) provides 2D image of turbulence in tokamaks

DIII-D tokamak (General Atomics) University of Wisconsin



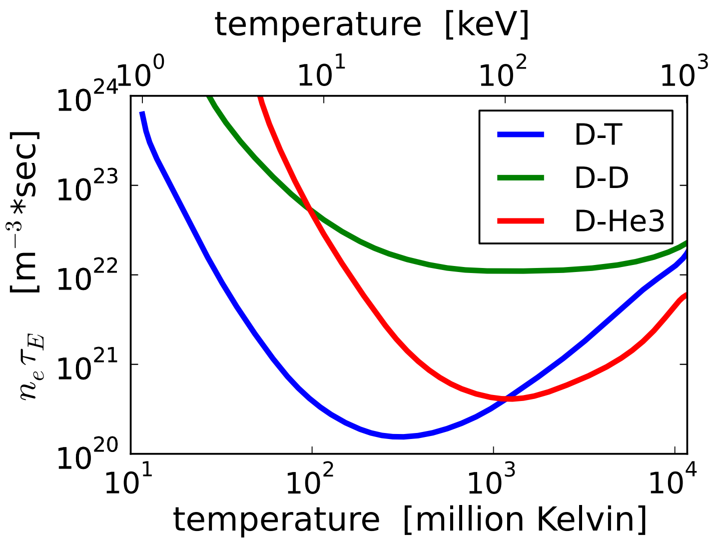


Measured density fluctuations

https://fusionlab.ep.wisc.edu/publications/

Turbulence determines confinement, ignition in tokamaks

- Triple product, Lawson criterion, determines ignition
  - Power losses > input power depends on density, temperature, confinement time



$$\tau_{C,collisions} \sim \frac{1}{D_{collisions}} \sim 100 \text{ s}$$

$$\tau_{C,\text{experimental}} \sim 0.1 \text{ s}$$

$$\boxed{\tau_{C,turbulence} \sim \frac{1}{D_{turbulenc}} \sim 0.1 \text{ s}}$$

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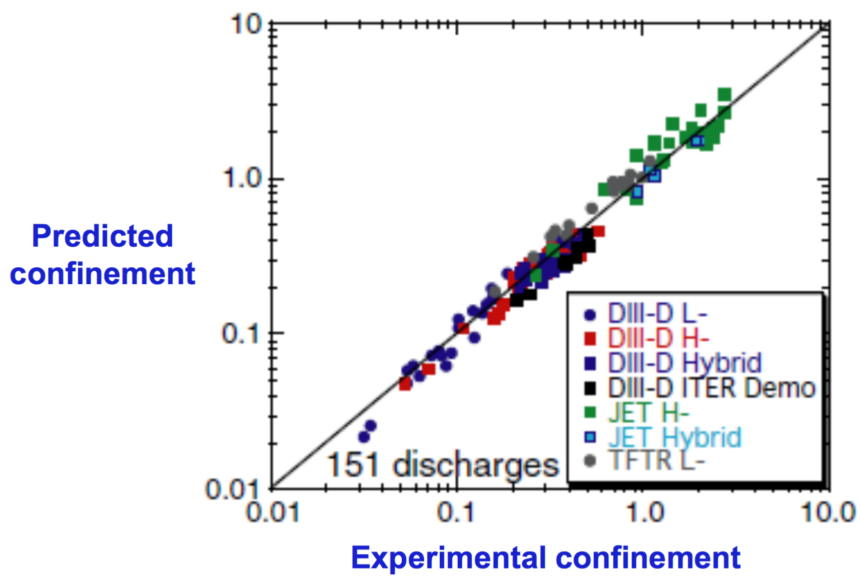

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Kinsey, 2010

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OAK RIDGE National Laboratory

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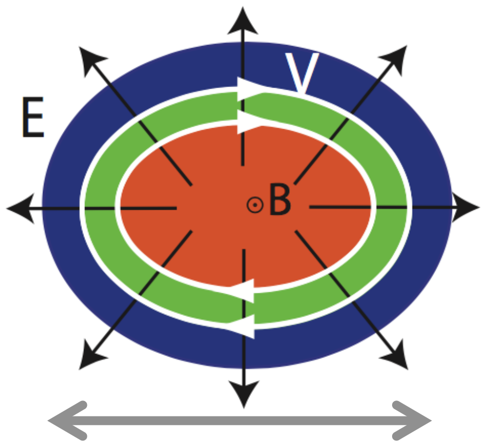

$L_c \sim 10\rho_s$

$$\boxed{D \sim \frac{\phi}{B} \sim \frac{T}{B}}$$

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$$\left(\nu \sim T^{-3/2}\right)$$

Classical diffusion

- Collisional (classical) diffusion weaker as plasma gets hotter
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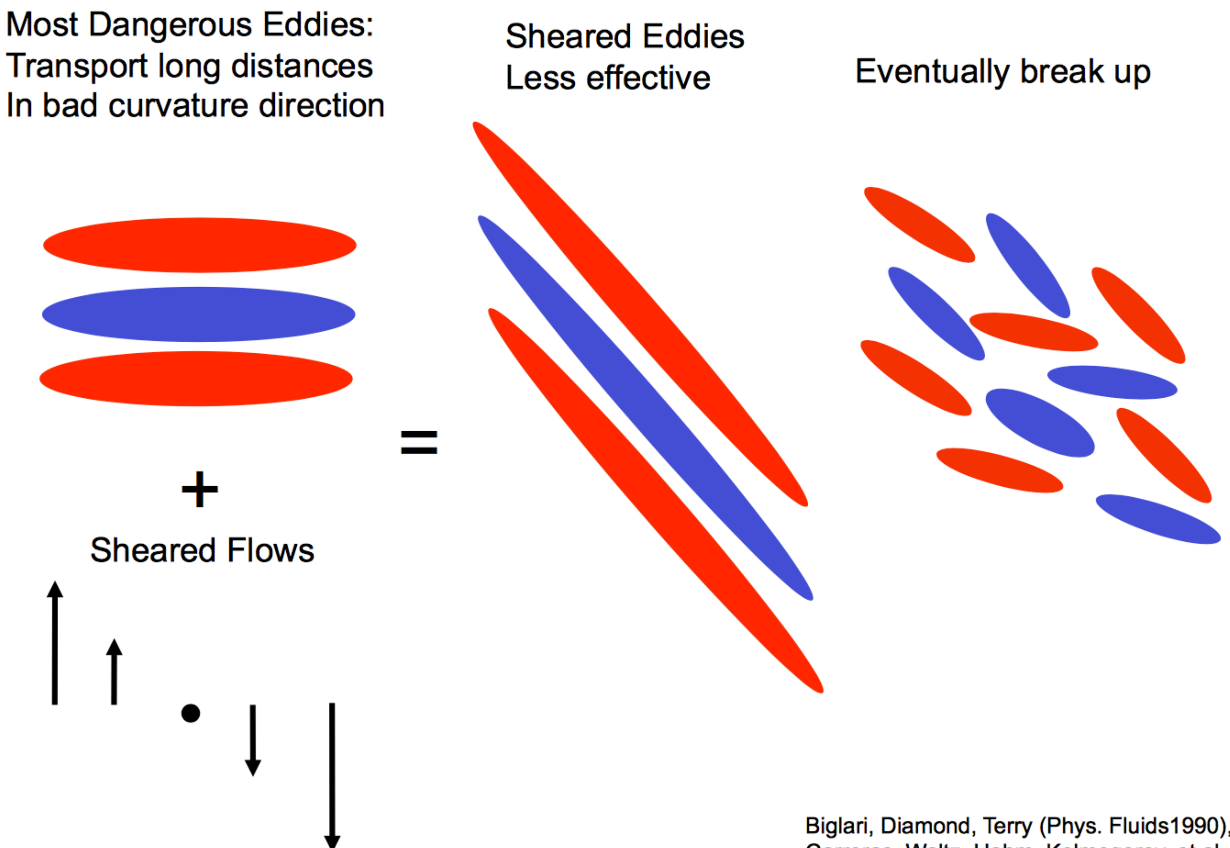
- Controlling ***size*** and ***correlation time*** of eddies controls confinement

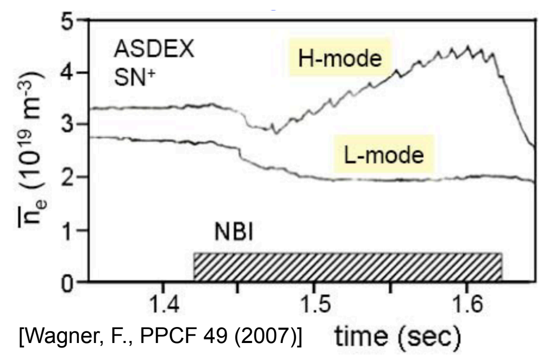
By changing background flow, can tilt and break eddies



Biglari, Diamond, Terry (Phys. Fluids1990),
Carreras, Waltz, Hahm, Kolmogorov, et al.

Plasma can self organize into a 'high' confinement state

- As input power increased further, spontaneous transition to "high" confinement regime discovered in 1982



[Wagner, F., PPCF 49 (2007)]

Wagner, Phys. Rev. Lett. 49, 1408 (1982)

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- Insulated transport barrier in edge formed
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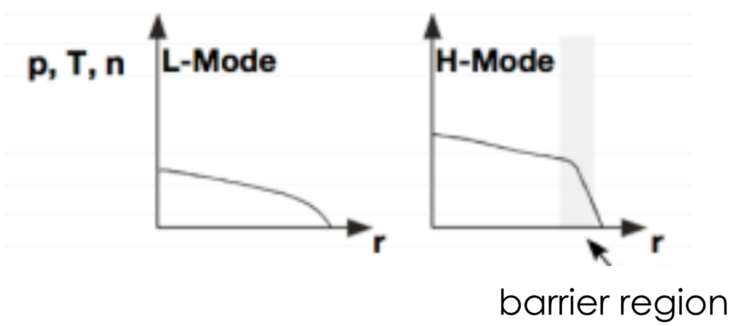
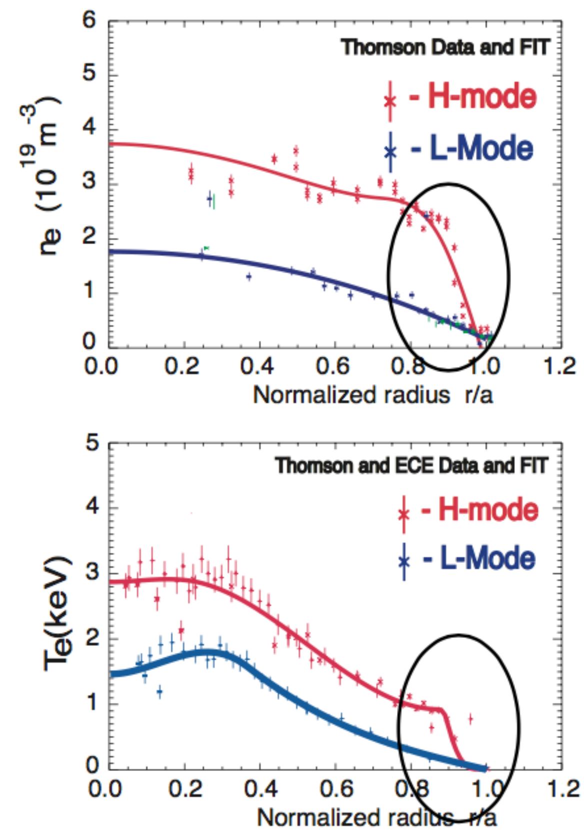
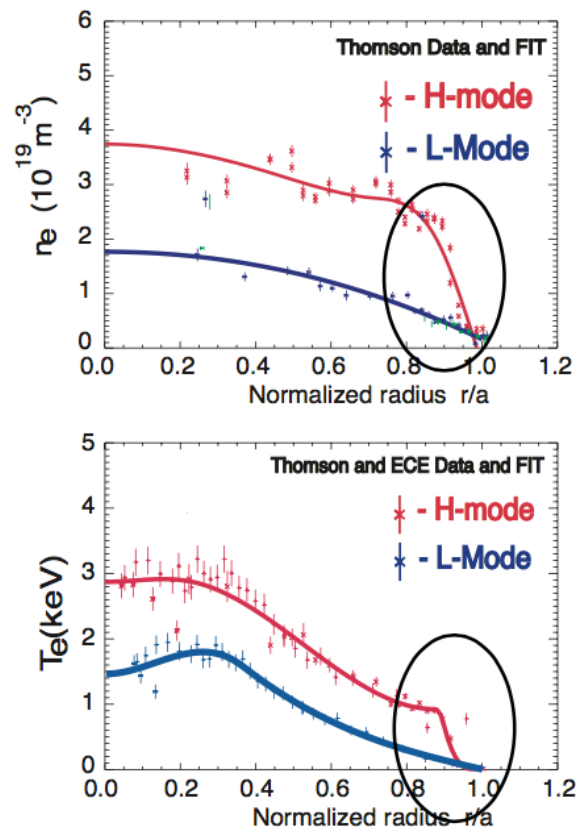




Data from DIII-D

Plasma can self organize into a ‘high’ confinement state

- As input power increased further, spontaneous transition to “high” confinement regime discovered in 1982
- Insulated transport barrier in edge formed
  - Steepened gradients
- Transport barrier forms by suppression of turbulence
  - Strong, localized cross-field flow (rotation) observed in barrier region

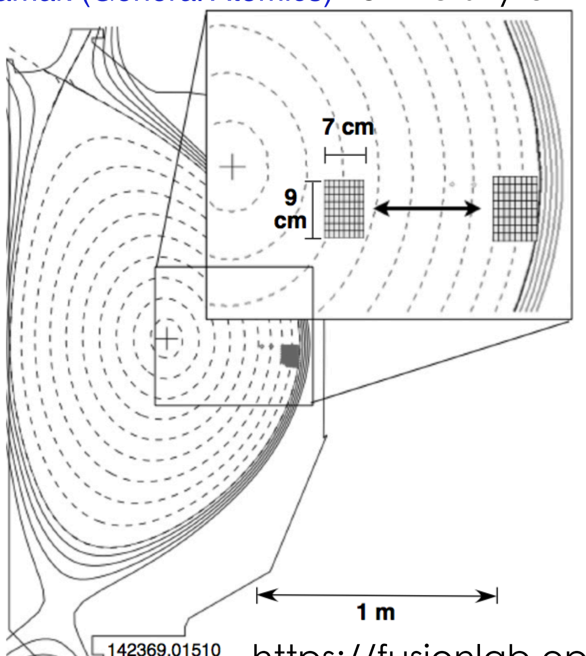


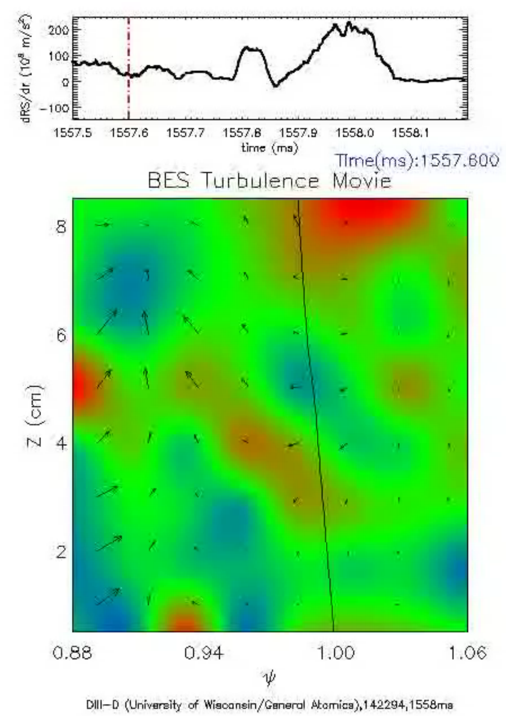
Data from DIII-D

BES measurements show fast turbulence and flow response during L-H transition

- Beam Emission Spectroscopy (BES) provides 2D image of turbulence in tokamaks

DIII-D tokamak (General Atomics) University of Wisconsin





Measured density fluctuations

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Z.Yan, et al PRL 112 (2014)

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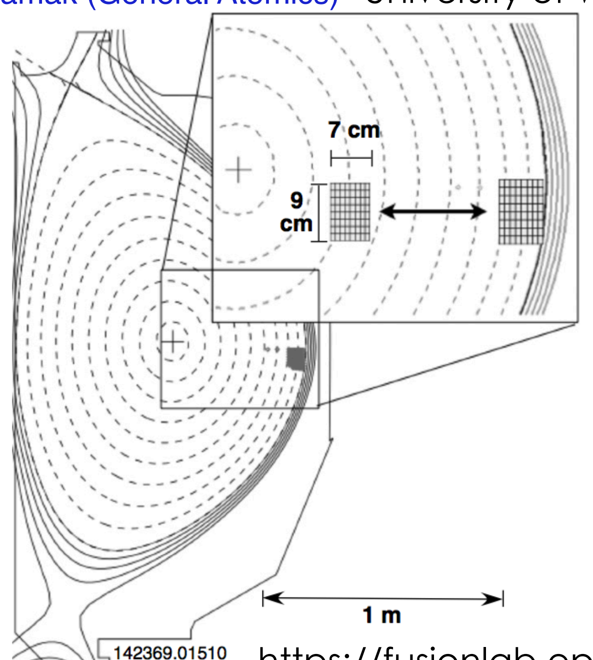
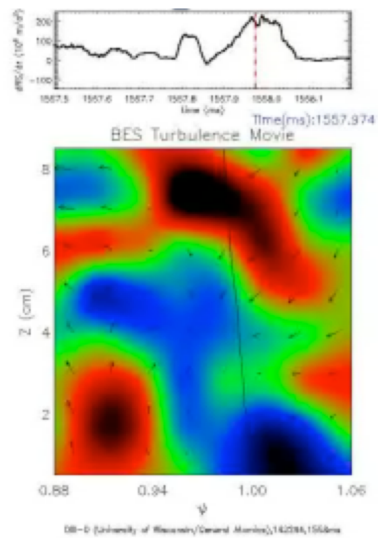
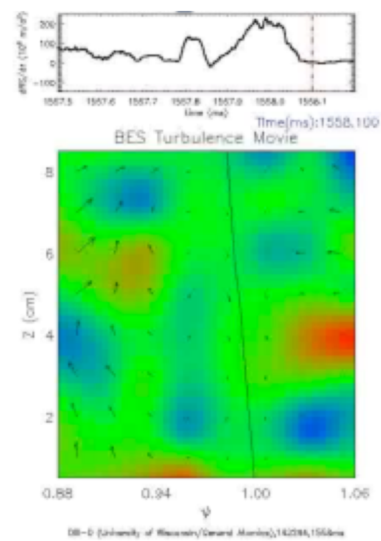




Before L-H transition



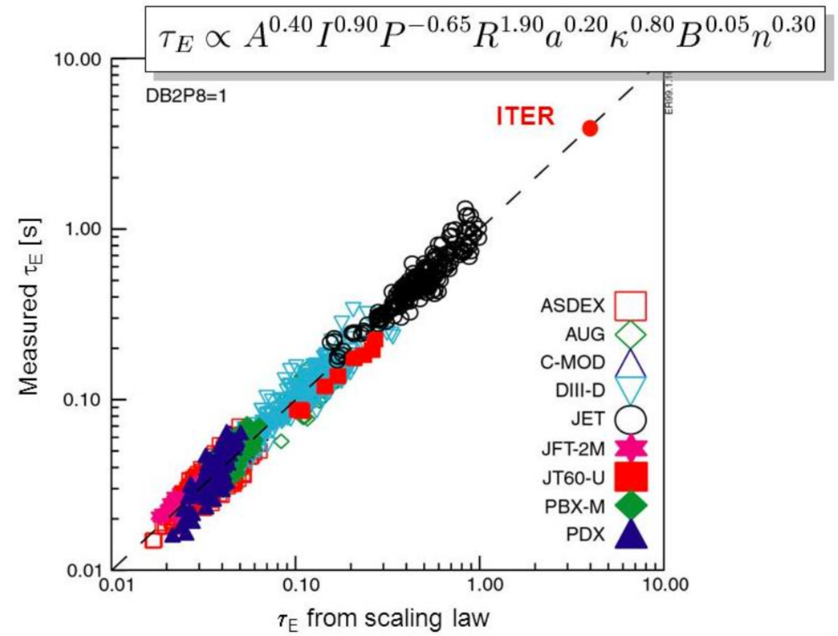
After L-H transition

https://fusionlab.ep.wisc.edu/publications/

Z.Yan, et al PRL 112 (2014)

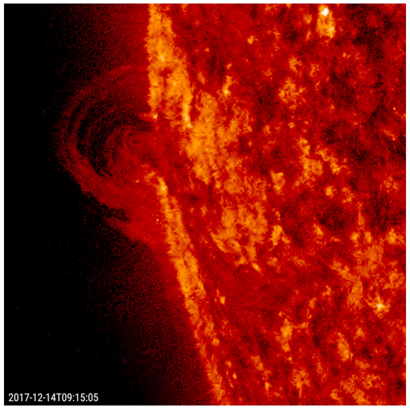
H-mode has been fundamental to progress in fusion, but still poorly understood

- Important advances in understanding changes in turbulence and turbulent transport in H-mode, but a lot of work remains
  - Mechanism for H-mode trigger?
  - What determines height of "pedestal"?
  - What sets transport in H-mode?....
- Rely on projections using empirical transport scaling laws



Concepts of turbulence to remember

- Turbulence is fluid motion characterized by chaotic changes in pressure and flow velocity
- Turbulence is a critical element in determining performance and size of fusion plasmas
  - Turbulence causes transport larger than collisional transport
  - Sheared flow can help reduce heat loss



https://sdo.gsfc.nasa.gov/gallery