Finite Difference Method

CENG6202: Advanced Computational Methods in Geotechnical Engineering

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Introduction

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Several problems are difficult or imposible to solve using analytical or other simple methods for various reasons:

- The governing mathematical equation is nonlinear and can't be linearized without affecting the result.
- The computational domain is complex.
- The boundary conditions are of mixed types.
- The boundary conditions are time-dependent.
- The material property is non-homogeneous or anisotropic.

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FDM is based on approximations that allow replacing differential equations by **finite difference equations**.

In general, solution using the FDM involves three steps:

1. Divide the computational domain into a **grid** of nodes.
2. Approximate the governing differential equation by an equivalent finite difference equation with respect to the grid points.
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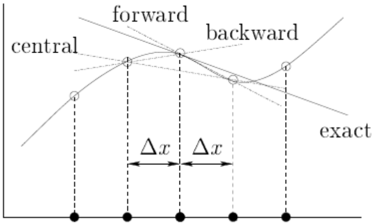

Forward Difference Equation:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

Central Difference Equation:

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

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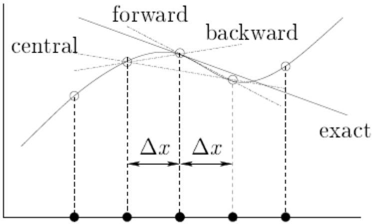

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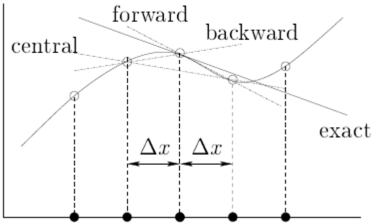

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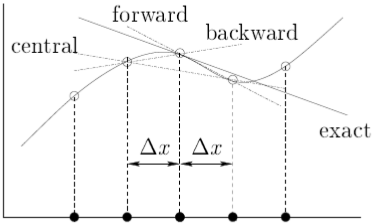

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The accuracy of the three approximations can be compared by looking at the Taylor series expansion of f.

Taylor series expansion of $f(x)$ at a point a:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^n(a) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

Series expansions at $x = a + h$ and $x = a - h$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$
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The orders of these approximations with respect to h:

$$f'(a) = \frac{f(a+h) - f(a)}{h} + \mathcal{O}(h)$$

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Higher order derivatives are derived in a similar way from the Taylor series expansions.

For example, the second derivative is given by

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} + \mathcal{O}(h^2)$$

This approximation is sometimes referred to as the **Symmetric Difference Equation**.

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Many physical quantities are functions of two or more variables.

Consider $f(t, x)$, a function of time and one-dimensional space. Parital derivative of f with respect to x for $a = x_\mathrm{o}$ and $h = \Delta x$:

$$f'(t, x_\mathrm{o}) = \frac{f(t, x_\mathrm{o} + \Delta x) - f(t, x_\mathrm{o})}{\Delta x}$$

For discrete values of $x = (x_1, x_2, \cdots, x_N)$, and discrete levels of $t = (t_\mathrm{o}, t_1, t_2, \cdots, t_n)$ the derivative at a point (t_n, x_i) is

$$f'(t_n, x_i) = \frac{f(t_n, x_{i+1}) - f(t_n, x_i)}{\Delta x} = \frac{f^n_{i+1} - f^n_i}{\Delta x}$$

Similarly, the partial derivative with respect to t:

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Groundwater Flow: General for confined aquifers

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right)=S_\mathrm{s}\frac{\partial h}{\partial t}$$

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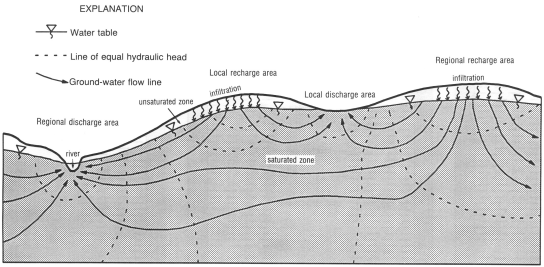
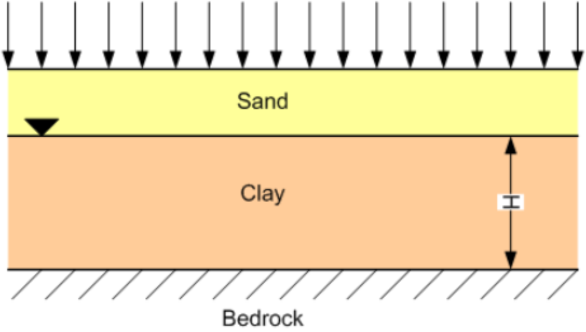


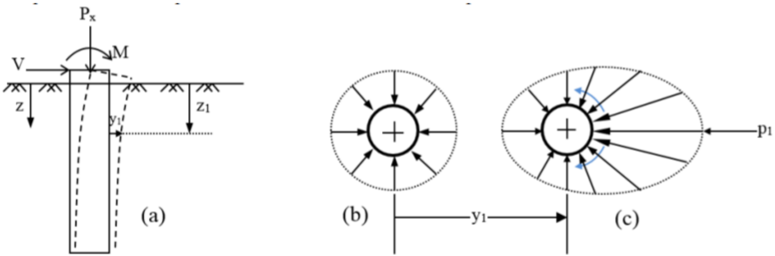
Consolidation:

$$\frac{\partial u}{\partial t} - c_{\mathrm{v}} \frac{\partial^2 u}{\partial z^2} = 0$$



Laterally Loaded Piles:

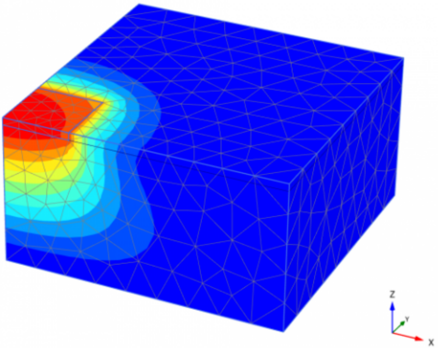
$$E_pI_p\frac{d^4y}{dx^4}+P_x\frac{d^2y}{dx^2}+E_{py}y-W=0$$



Coupled Deformation and Flow:

$$\nabla \cdot (\boldsymbol{\sigma}' + u\boldsymbol{I}) + \rho \boldsymbol{b} = \boldsymbol{0}$$

$$\nabla \cdot \dot{\boldsymbol{u}} + c\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{v}_{\mathrm{w}} = 0$$



Initial Conditions (ICs):

- Specify the values of field variables at the start of a simulation.
- Usually specified for time dependent variables at $t = 0$.

Boundary Conditions (BCs):

- Define the values of field variables or functions of field variables at the boundaries of the computational domain.
- Boundary types depend on spatial dimension:
  1D - Point, 2D - Edge, 3D - Face

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- Usually specified for time dependent variables at $t = 0$.

Boundary Conditions (BCs):

- Define the values of field variables or functions of field variables at the boundaries of the computational domain.
- Boundary types depend on spatial dimension:
  1D - Point, 2D - Edge, 3D - Face

Three main types of boundary conditions:

1. **Dirichlet BCs**: specify the value of the unknown function on a given boundary in the form
$$f = \bar{f} \quad \text{on } \Gamma_{\mathrm{D}}$$
where Γ_{D} is the boundary on which the condition is specified.
2. **Neumann BCs**: specify the normal derivative of the unknown function on a given boundary i.e.
$$\frac{\partial f}{\partial n} = q \quad \text{on } \Gamma_{\mathrm{N}}$$
where n is the normal to the boundary Γ_{N}.
3. **Robin BCs**: define a linear combination of Dirichlet and Neumann conditions in the form
$$\alpha f + \beta \frac{\partial f}{\partial n} = r \quad \text{on } \Gamma_{\mathrm{R}}$$
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Spatial Discretization: The computational domain over which we aim to solve the PDE is divided into a grid mesh with the required grid spacing. Depending on the spatial dimension of the problem, the grid required may be 1D, 2D or 3D.

Temporal Discretization: If there are time dependent quantities in the differential equation, the next step is to perform temporal discretization of these to obtain difference equations.

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$$Ax = b$$

which may be solved using the appropriate solution method.

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↪Steady-state Flow

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One-dimensional equivalent:

$$k \frac{\partial^2 h}{\partial x^2} = Q$$

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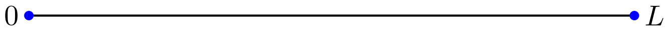
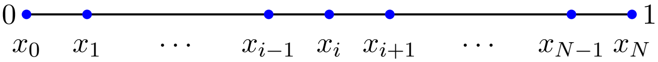


Spatial discretization: Divide the domain into a desired number of sub-domains, say N:



The *grid spacing* in this case is

$$\Delta x = \frac{1}{N}$$

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$$h(x_i) = h_i, \qquad i = 0, 1, \cdots, N$$

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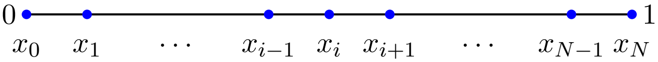

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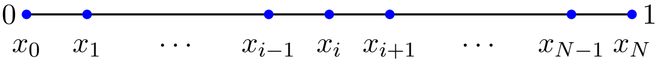

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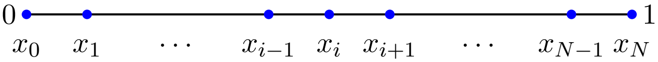

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Difference equation for partial derivative of hydraulic head:

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta x^2}$$

Discrete form of original equation, in terms of grid point values

$$k\,\frac{h_{i+1} - 2h_i + h_{i-1}}{(\Delta x)^2} = Q_i$$

Solution for $N = 2$:
Difference equation at internal grid point (assuming $Q_1 = 0$)

$$h_2 - 2h_1 + h_0 = 0$$

For known boundary values h_0 and h_2

$$h_1 = \frac{h_2 + h_0}{2}$$

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Solution for arbitrary N:

Difference equations at unknown nodes $i = 1, 2, \cdots, N-1$:

$$
\begin{aligned}
i &= 1 & k\frac{h_2 - 2h_1 + h_0}{(\Delta x)^2} &= Q_1 \\
i &= 2 & k\frac{h_3 - 2h_2 + h_1}{(\Delta x)^2} &= Q_2 \\
&\vdots & &\vdots \\
i &= N-1 & k\frac{h_N - 2h_{N-1} + h_{N-2}}{(\Delta x)^2} &= Q_{N-1}
\end{aligned}
$$

Solution for arbitrary N:

In matrix form:

$$\frac{k}{(\Delta x)^2}\begin{bmatrix} 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \end{bmatrix}\begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{N-1} \\ h_N \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \\ Q_{N-1} \end{Bmatrix}$$

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which is a linear equation system of the form

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Governing equation for one-dimensional consolidation:

$$\frac{\partial u}{\partial t} - c_{\mathrm{v}} \frac{\partial^2 u}{\partial z^2} = 0$$

Coefficient of consolidation

$$c_{\mathrm{v}} = \frac{k}{m_{\mathrm{v}} \gamma_{\mathrm{w}}}$$

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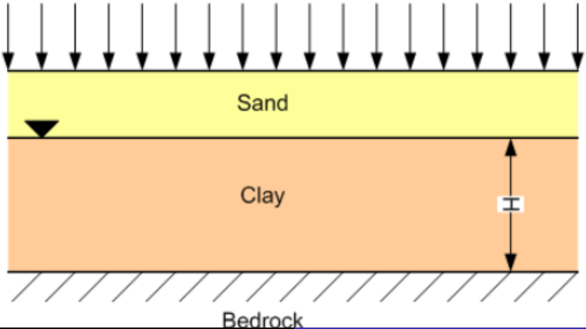
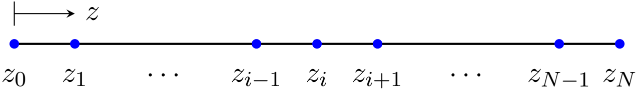


Spatial Discretization:
Consider a 1D domain sub-divided into N elements.



For a uniform grid spacing and domain depth/length d, we have

$$\Delta z = \frac{d}{N}$$

Temporal Discretization: For a simulation from time t_o to t_n, choosing a uniform time step Δt implies n time steps such that

$$t_n = n\Delta t$$

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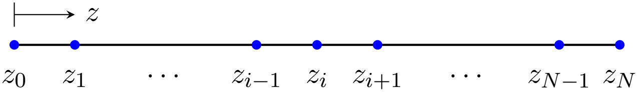

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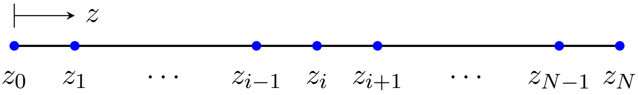

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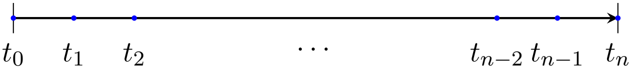

The excess pore pressure is a function of both time and space:

$$u = u(t, z)$$

The partial derivative with respect to time may be approximated as a difference equation in different ways, resulting in different solution methods.

1. Explicit Method: forward difference approximation in time.
2. Implicit Method: backward difference approximation in time.
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Forward difference approximation for time derivative

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Symmetric difference approximation for spatial derivative

$$\frac{\partial^2 u}{\partial z^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta z^2}$$

Final explicit finite difference equation

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Solving for pore water pressure at time step t_{n+1} gives

$$u_i^{n+1} = u_i^n + \kappa \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right)$$

where

$$\kappa = c_{\mathrm{v}} \frac{\Delta t}{\Delta z^2}$$

Stencil for the explicit method

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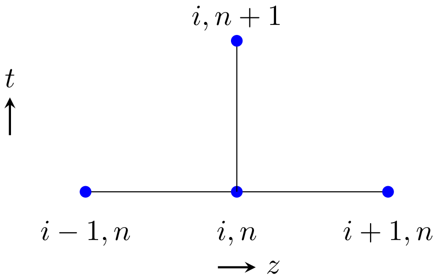

Stability and Convergence:

The explicit method is numerically stable and convergent when

$$\kappa \leq \frac{1}{2}$$

Thus, the time step used must satisfy

$$\Delta t \leq \frac{\Delta z^2}{2c_{\mathrm{v}}}$$

Accuracy:

First-order in time, $\mathcal{O}(\Delta t)$, and second-order in space, $\mathcal{O}(\Delta z^2)$.

Backward difference approximation for time derivative

$$\frac{\partial u}{\partial t} = \frac{u_i^n - u_i^{n-1}}{\Delta t}$$

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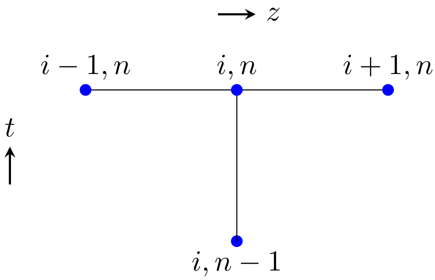

The pore pressure at node i at time step t_n depends on the pore pressure values of the neighboring nodes at the same time step.
$\Rightarrow$ We can not solve for u_i^n directly.

Solution for $N = 3$:

Assume the pore pressures at the boundary nodes are known.

Finite difference equations at unknown interior nodes

$$\frac{u_1^n - u_1^{n-1}}{\Delta t} - c_v \frac{u_2^n - 2u_1^n + u_0^n}{\Delta z^2} = 0$$
$$\frac{u_2^n - u_2^{n-1}}{\Delta t} - c_v \frac{u_3^n - 2u_2^n + u_1^n}{\Delta z^2} = 0$$

Equations to be solved for unknowns u_1^n and u_2^n

$$(1 + 2\kappa)u_1^n - \kappa u_2^n = \kappa u_0^n + u_1^{n-1}$$
$$-\kappa u_1^n + (1 + 2\kappa)u_2^n = \kappa u_3^n + u_2^{n-1}$$

In matrix form

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$$\frac{u_2^n - u_2^{n-1}}{\Delta t} - c_\mathrm{v}\frac{u_3^n - 2u_2^n + u_1^n}{\Delta z^2} = 0$$

Equations to be solved for unknowns u_1^n and u_2^n

$$(1+2\kappa)u_1^n - \kappa u_2^n = \kappa u_0^n + u_1^{n-1}$$
$$-\kappa u_1^n + (1+2\kappa)u_2^n = \kappa u_3^n + u_2^{n-1}$$

In matrix form

$$\begin{bmatrix} 1+2\kappa & -\kappa \\ -\kappa & 1+2\kappa \end{bmatrix} \begin{Bmatrix} u_1^n \\ u_2^n \end{Bmatrix} = \begin{Bmatrix} \kappa u_0^n + u_1^{n-1} \\ \kappa u_3^n + u_2^{n-1} \end{Bmatrix}$$

Solution for $N = 3$:

Assume the pore pressures at the boundary nodes are known.

Finite difference equations at unknown interior nodes

$$\frac{u_1^n - u_1^{n-1}}{\Delta t} - c_v \frac{u_2^n - 2u_1^n + u_0^n}{\Delta z^2} = 0$$
$$\frac{u_2^n - u_2^{n-1}}{\Delta t} - c_v \frac{u_3^n - 2u_2^n + u_1^n}{\Delta z^2} = 0$$

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$$\begin{bmatrix} 1 + 2\kappa & -\kappa \\ -\kappa & 1 + 2\kappa \end{bmatrix} \begin{Bmatrix} u_1^n \\ u_2^n \end{Bmatrix} = \begin{Bmatrix} \kappa u_0^n + u_1^{n-1} \\ \kappa u_3^n + u_2^{n-1} \end{Bmatrix}$$

Solution for arbitrary N:

Assume the pore pressures at the boundary nodes are known.

Finite difference equation

$$-\kappa u_{i-1}^{n} + (1+2\kappa)u_i^n - \kappa u_{i+1}^{n} = u_i^{n-1}$$

Solution for arbitrary N:

Assume the pore pressures at the boundary nodes are known.

Finite difference equation

$$-\kappa u_{i-1}^{n} + (1 + 2\kappa)u_i^n - \kappa u_{i+1}^n = u_i^{n-1}$$

Solution for arbitrary N:

Assume the pore pressures at the boundary nodes are known.

Finite difference equation

$$-\kappa u_{i-1}^{n} + (1+2\kappa)u_i^n - \kappa u_{i+1}^n = u_i^{n-1}$$

Finite difference equation at the unknown interior nodes

$$\begin{aligned}
i &= 1 && -\kappa u_0^n + (1+2\kappa)u_1^n - \kappa u_2^n = u_1^{n-1} \\
i &= 2 && -\kappa u_1^n + (1+2\kappa)u_2^n - \kappa u_3^n = u_2^{n-1} \\
&\vdots && \qquad\qquad\qquad \vdots \\
i &= N-2 && -\kappa u_{N-3}^n + (1+2\kappa)u_{N-2}^n - \kappa u_{N-1}^n = u_{N-2}^{n-1} \\
i &= N-1 && -\kappa u_{N-2}^n + (1+2\kappa)u_{N-1}^n - \kappa u_N^n = u_{N-1}^{n-1}
\end{aligned}$$

Solution for arbitrary N:

Assume the pore pressures at the boundary nodes are known.

Finite difference equation

$$-\kappa u_{i-1}^{n} + (1+2\kappa)u_i^n - \kappa u_{i+1}^n = u_i^{n-1}$$

In matrix form

$$\begin{bmatrix} -\kappa & 1+2\kappa & -\kappa & & & & & \\ & -\kappa & 1+2\kappa & -\kappa & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -\kappa & 1+2\kappa & -\kappa & \\ & & & & & -\kappa & 1+2\kappa & -\kappa \end{bmatrix} \begin{Bmatrix} {\color{blue}u_0^n} \\ u_1^n \\ u_2^n \\ \vdots \\ u_{N-2}^n \\ u_{N-1}^n \\ {\color{blue}u_N^n} \end{Bmatrix} = \begin{Bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ u_{N-2}^{n-1} \\ u_{N-1}^{n-1} \end{Bmatrix}$$

Solution for arbitrary N:

Assume the pore pressures at the boundary nodes are known.

Finite difference equation

$$-\kappa u_{i-1}^{n} + (1+2\kappa)u_i^n - \kappa u_{i+1}^{n} = u_i^{n-1}$$

Matrix form considering boundary conditions

$$\begin{bmatrix} 1+2\kappa & -\kappa & & & & \\ -\kappa & 1+2\kappa & -\kappa & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\kappa & 1+2\kappa & -\kappa \\ & & & & -\kappa & 1+2\kappa \end{bmatrix} \begin{Bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-2}^n \\ u_{N-1}^n \end{Bmatrix} = \begin{Bmatrix} \kappa u_0^n + u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ u_{N-2}^{n-1} \\ \kappa u_N^n + u_{N-1}^{n-1} \end{Bmatrix}$$

Stability and Convergence:

The implicit method is numerically stable and convergent.

Accuracy:

First-order in time, $\mathcal{O}(\Delta t)$, and second-order in space, $\mathcal{O}(\Delta z^2)$.

Central difference approximation for time derivative at $(t_{n+\frac{1}{2}}, z_i)$

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{2\frac{\Delta t}{2}} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Finite difference approximation for spatial derivative at $(t_{n+\frac{1}{2}}, z_i)$

$$\frac{\partial^2 u}{\partial z^2} = \frac{u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}}{\Delta z^2}$$

Assume for all i that

$$u_i^{n+\frac{1}{2}} \approx \frac{1}{2}\left(u_i^n + u_i^{n+1}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{1}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta z^2}\right)$$

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Final finite difference equation

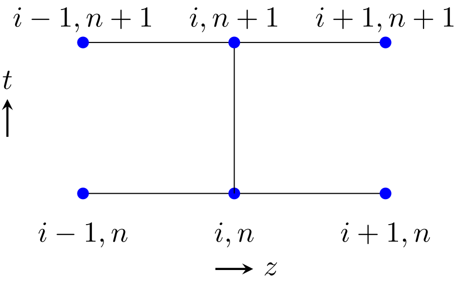
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{c_\mathrm{v}}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta z^2}\right) = 0$$

Stencil for the Crank-Nicolson method

Final finite difference equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{c_\mathrm{v}}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta z^2}\right) = 0$$

Stencil for the Crank-Nicolson method



Simplified finite difference equation with unknowns on the LHS

$$-\frac{\kappa}{2}u_{i-1}^{n+1} + (1+\kappa)u_i^{n+1} - \frac{\kappa}{2}u_{i+1}^{n+1} = \frac{\kappa}{2}u_{i-1}^{n} + (1-\kappa)u_i^{n} + \frac{\kappa}{2}u_{i+1}^{n}$$

Solution for arbitrary N:

Simplified finite difference equation with unknowns on the LHS

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Solution for arbitrary N:

Finite difference equation at the unknown interior nodes

$$\begin{aligned}
-\frac{\kappa}{2}u_0^{n+1} + (1+\kappa)u_1^{n+1} - \frac{\kappa}{2}u_2^{n+1} &= \frac{\kappa}{2}u_0^{n} + (1-\kappa)u_1^{n} + \frac{\kappa}{2}u_2^{n} \\
-\frac{\kappa}{2}u_1^{n+1} + (1+\kappa)u_2^{n+1} - \frac{\kappa}{2}u_3^{n+1} &= \frac{\kappa}{2}u_1^{n} + (1-\kappa)u_2^{n} + \frac{\kappa}{2}u_3^{n} \\
\vdots \qquad\qquad & \vdots \qquad\qquad \vdots \\
-\frac{\kappa}{2}u_{N-3}^{n+1} + (1+\kappa)u_{N-2}^{n+1} - \frac{\kappa}{2}u_{N-1}^{n+1} &= \frac{\kappa}{2}u_{N-3}^{n} + (1-\kappa)u_{N-2}^{n} + \frac{\kappa}{2}u_{N-1}^{n} \\
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\end{aligned}$$

Simplified finite difference equation with unknowns on the LHS

$$-\frac{\kappa}{2}u_{i-1}^{n+1} + (1+\kappa)u_i^{n+1} - \frac{\kappa}{2}u_{i+1}^{n+1} = \frac{\kappa}{2}u_{i-1}^{n} + (1-\kappa)u_i^{n} + \frac{\kappa}{2}u_{i+1}^{n}$$

Solution for arbitrary N:

In matrix form

$$\begin{bmatrix} 1+\kappa & -\frac{\kappa}{2} & & & & \\ -\frac{\kappa}{2} & 1+\kappa & -\frac{\kappa}{2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{\kappa}{2} & 1+\kappa & -\frac{\kappa}{2} \\ & & & & -\frac{\kappa}{2} & 1+\kappa \end{bmatrix} \begin{Bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-2}^{n+1} \\ u_{N-1}^{n+1} \end{Bmatrix} = \begin{Bmatrix} b_1 + \frac{\kappa}{2}u_0^{n+1} \\ b_2 \\ \vdots \\ b_{N-2} \\ b_{N-1} + \frac{\kappa}{2}u_N^{n+1} \end{Bmatrix}$$

where

$$b_i = \frac{\kappa}{2}u_{i-1}^{n} + (1-\kappa)u_i^{n} + \frac{\kappa}{2}u_{i+1}^{n}$$

Consider a 1 m thick soil subjected to a surcharge load of 50 kPa.

Coefficient of consolidation:

$$c_v = 2 \times 10^{-6}\ \mathrm{m}^2/\mathrm{s}$$

Dimensionless time

$$T = \frac{c_v t}{H^2}$$

Solve for the excess pore pressure as a function of time using the explicit, implicit and Crank-Nicolson methods.

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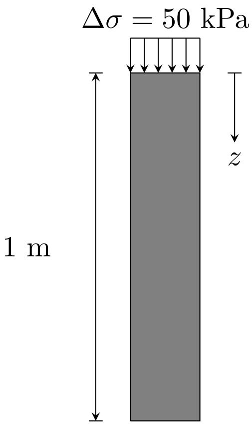

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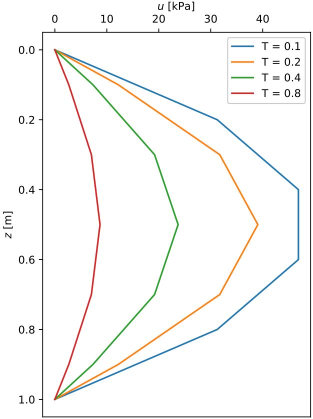
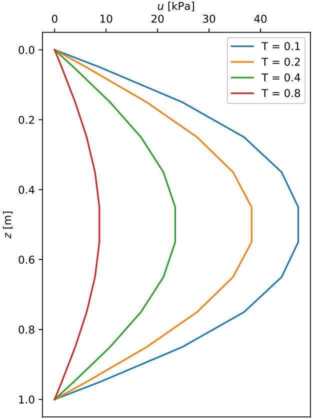
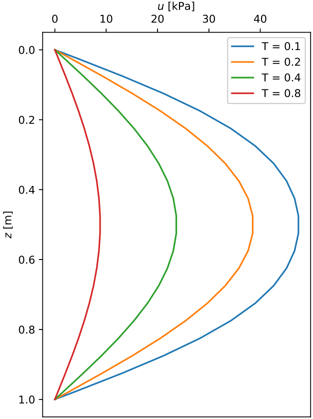


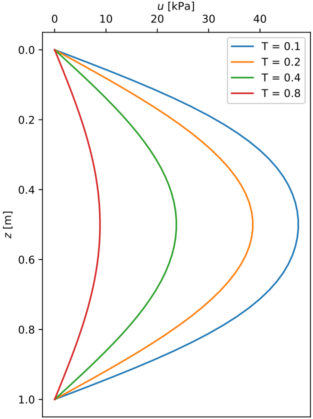
Explicit: $N = 10, \Delta t = 2500$



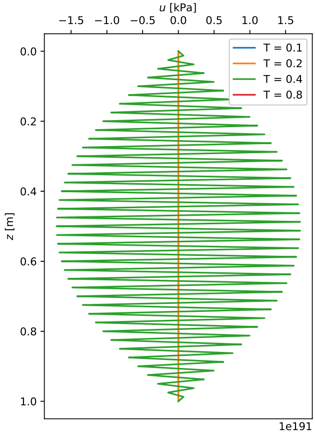
Explicit: $N = 20, \Delta t = 645$



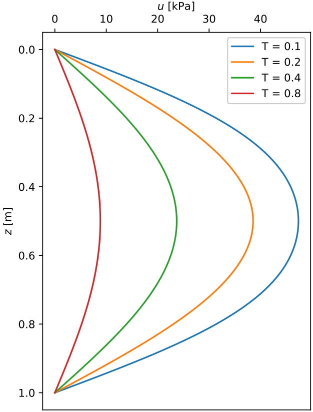
Explicit: $N = 40, \Delta t = 156$



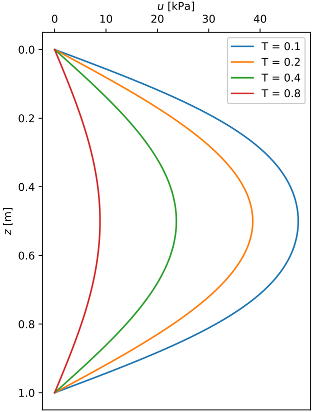
Explicit: $N = 80, \Delta t = 39$



Explicit: $N = 80, \Delta t = 50$



Implicit: $N = 80, \Delta t = 200$



Crank-Nicolson: $N = 80, \Delta t = 100$

Contents

Two-dimensional Problems

Steady-state groundwater flow in two-dimensions: The governing differential equation is

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} = Q$$

For an isotropic material and neglecting source/sink terms

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

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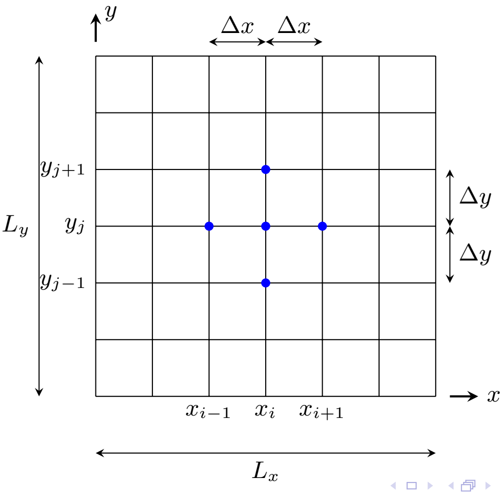
Spatial Discretization:

N_x and N_y elements with grid spacing of Δx and Δy, respectively.



For a uniform grid we have

$$\Delta x = \Delta y$$

Grid spacings are calculated from

$$\Delta x = \frac{L_x}{N_x} \quad \text{and} \quad \Delta y = \frac{L_y}{N_y}$$

Notation for hydrauclic head at point (x_i, j_j)

$$h(x_i, y_j) = h_{i,j}$$

for $i = 0, 1, \cdots, N_x$ and $j = 0, 1, \cdots, N_y$.

Boundary conditions are specified on edges. For example

$$h_{0,j} = h_{N,j} = \bar{h}_1 \quad \text{and} \quad h_{i,0} = h_{i,N} = \bar{h}_2$$

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Finite difference approximations for spatial derivatives

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{\Delta x^2}$$
$$\frac{\partial^2 h}{\partial y^2} = \frac{h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{\Delta y^2}$$

Based on the governing equation, we get

$$\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{\Delta x^2} + \frac{h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{\Delta y^2} = 0$$

For a uniform grid where $\Delta x = \Delta y$

$$h_{i,j-1} + h_{i-1,j} - 4h_{i,j} + h_{i+1,j} + h_{i,j+1} = 0$$

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Solution for $N_x = N_y = 4$:

Consider a $1\ \mathrm{m} \times 1\ \mathrm{m}$ grid divided into 4 elements.

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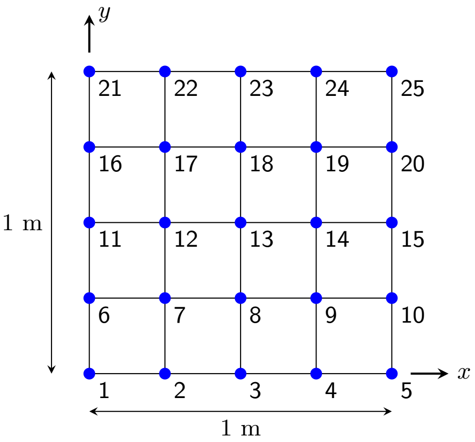


Assume that the boundary conditions are

$$
\begin{aligned}
h &= 10 && \text{for } y = 1 \\
h &= 0 && \text{for } x = 0, x = 1 \text{ and } y = 0
\end{aligned}
$$

Finite difference equations for nodes 7, 8 and 9

$$
\begin{aligned}
-4h_7 + h_8 + h_{12} &= 0 \\
h_7 - 4h_8 + h_9 + h_{13} &= 0 \\
h_8 - 4h_9 + h_{14} &= 0
\end{aligned}
$$

Finite difference equations for nodes 12, 13 and 14

$$
\begin{aligned}
h_7 - 4h_{12} + h_{13} + h_{17} &= 0 \\
h_8 + h_{12} - 4h_{13} + h_{14} + h_{18} &= 0 \\
h_9 + h_{13} - 4h_{14} + h_{19} &= 0
\end{aligned}
$$

Assume that the boundary conditions are

$$
\begin{aligned}
h &= 10 && \text{for } y = 1 \\
h &= 0 && \text{for } x = 0, x = 1 \text{ and } y = 0
\end{aligned}
$$

Finite difference equations for nodes 7, 8 and 9

$$
\begin{aligned}
-4h_7 + h_8 + h_{12} &= 0 \\
h_7 - 4h_8 + h_9 + h_{13} &= 0 \\
h_8 - 4h_9 + h_{14} &= 0
\end{aligned}
$$

Finite difference equations for nodes 12, 13 and 14

$$
\begin{aligned}
h_7 - 4h_{12} + h_{13} + h_{17} &= 0 \\
h_8 + h_{12} - 4h_{13} + h_{14} + h_{18} &= 0 \\
h_9 + h_{13} - 4h_{14} + h_{19} &= 0
\end{aligned}
$$

Assume that the boundary conditions are

$$
\begin{aligned}
h &= 10 \qquad \text{for } y = 1 \\
h &= 0 \qquad \text{for } x = 0, x = 1 \text{ and } y = 0
\end{aligned}
$$

Finite difference equations for nodes 7, 8 and 9

$$
\begin{aligned}
-4h_7 + h_8 + h_{12} &= 0 \\
h_7 - 4h_8 + h_9 + h_{13} &= 0 \\
h_8 - 4h_9 + h_{14} &= 0
\end{aligned}
$$

Finite difference equations for nodes 12, 13 and 14

$$
\begin{aligned}
h_7 - 4h_{12} + h_{13} + h_{17} &= 0 \\
h_8 + h_{12} - 4h_{13} + h_{14} + h_{18} &= 0 \\
h_9 + h_{13} - 4h_{14} + h_{19} &= 0
\end{aligned}
$$

Finite difference equations for nodes 17, 18 and 19

$$h_{12} - 4h_{17} + h_{18} = -10$$
$$h_{13} + h_{17} - 4h_{18} + h_{19} = -10$$
$$h_{14} + h_{18} - 4h_{19} = -10$$

Assembling the equations into a matrix form gives

$$\left[\begin{array}{ccc|ccc|ccc} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{array}\right] \begin{Bmatrix} h_7 \\ h_8 \\ h_9 \\ h_{12} \\ h_{13} \\ h_{14} \\ h_{17} \\ h_{18} \\ h_{19} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ -10 \\ -10 \end{Bmatrix}$$

Finite difference equations for nodes 17, 18 and 19

$$\begin{aligned} h_{12} - 4h_{17} + h_{18} &= -10 \\ h_{13} + h_{17} - 4h_{18} + h_{19} &= -10 \\ h_{14} + h_{18} - 4h_{19} &= -10 \end{aligned}$$

Assembling the equations into a matrix form gives

$$\left[\begin{array}{ccc|ccc|ccc} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{array}\right] \begin{Bmatrix} h_7 \\ h_8 \\ h_9 \\ h_{12} \\ h_{13} \\ h_{14} \\ h_{17} \\ h_{18} \\ h_{19} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ -10 \\ -10 \end{Bmatrix}$$

The final equation system is of the form

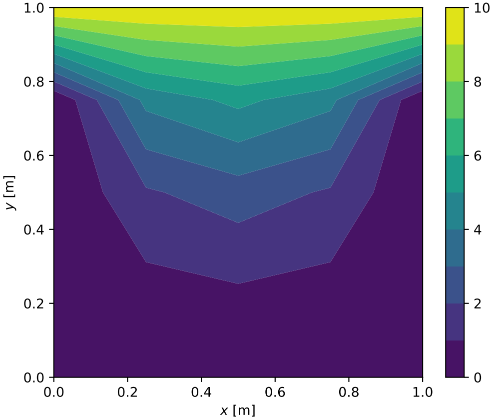
$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

where

$$\boldsymbol{A} = \left[\begin{array}{c|c|c} \boldsymbol{B} & \boldsymbol{I} & \mathbf{O} \\ \hline \boldsymbol{I} & \boldsymbol{B} & \boldsymbol{I} \\ \hline \mathbf{O} & \boldsymbol{I} & \boldsymbol{B} \end{array}\right]$$

and

$$\boldsymbol{B} = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Solution for $Nx = Ny = 4$.

Solution for an $N_x \times N_y$ grid:

Assume that the grid is uniform in both directions i.e. $\Delta x = \Delta y$.
Number of grid points in the computational domain:

$$(N_x + 1) \times (N_y + 1)$$

Coefficient matrix $\boldsymbol{A}$ with a similar structure:

$$\boldsymbol{A} = \left[\begin{array}{c|c|c|c|c} \boldsymbol{B} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hline \boldsymbol{I} & \boldsymbol{B} & \boldsymbol{I} & \ddots & \vdots \\ \hline \boldsymbol{0} & \boldsymbol{I} & \ddots & \ddots & \boldsymbol{0} \\ \hline \vdots & \ddots & \ddots & \boldsymbol{B} & \boldsymbol{I} \\ \hline \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{B} \end{array}\right]$$

Size of matrix $\boldsymbol{A}$:

$$(N_x - 1)^2 \times (N_y - 1)^2$$

Solution for an $N_x \times N_y$ grid:

Assume that the grid is uniform in both directions i.e. $\Delta x = \Delta y$.

Number of grid points in the computational domain:

$$(N_x + 1) \times (N_y + 1)$$

Coefficient matrix $\boldsymbol{A}$ with a similar structure:

$$\boldsymbol{A} = \left[\begin{array}{c|c|c|c|c} \boldsymbol{B} & \boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \boldsymbol{I} & \boldsymbol{B} & \boldsymbol{I} & \ddots & \vdots \\ \hline \mathbf{0} & \boldsymbol{I} & \ddots & \ddots & \mathbf{0} \\ \hline \vdots & \ddots & \ddots & \boldsymbol{B} & \boldsymbol{I} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{I} & \boldsymbol{B} \end{array}\right]$$

Size of matrix $\boldsymbol{A}$:

$$(N_x - 1)^2 \times (N_y - 1)^2$$

Solution for an $N_x \times N_y$ grid:

Assume that the grid is uniform in both directions i.e. $\Delta x = \Delta y$. Number of grid points in the computational domain:

$$(N_x + 1) \times (N_y + 1)$$

Coefficient matrix A with a similar structure:

$$A = \left[\begin{array}{c|c|c|c|c} B & I & \mathbf{0} & \cdots & \mathbf{0} \\ \hline I & B & I & \ddots & \vdots \\ \hline \mathbf{0} & I & \ddots & \ddots & \mathbf{0} \\ \hline \vdots & \ddots & \ddots & B & I \\ \hline \mathbf{0} & \cdots & \mathbf{0} & I & B \end{array}\right]$$

Size of matrix A:

$$(N_x - 1)^2 \times (N_y - 1)^2$$

Solution for an $N_x \times N_y$ grid:

Assume that the grid is uniform in both directions i.e. $\Delta x = \Delta y$.
Number of grid points in the computational domain:

$$(N_x + 1) \times (N_y + 1)$$

Coefficient matrix $\boldsymbol{A}$ with a similar structure:

$$\boldsymbol{A} = \left[\begin{array}{c|c|c|c|c} \boldsymbol{B} & \boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \boldsymbol{I} & \boldsymbol{B} & \boldsymbol{I} & \ddots & \vdots \\ \hline \mathbf{0} & \boldsymbol{I} & \ddots & \ddots & \mathbf{0} \\ \hline \vdots & \ddots & \ddots & \boldsymbol{B} & \boldsymbol{I} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{I} & \boldsymbol{B} \end{array}\right]$$

Size of matrix $\boldsymbol{A}$:

$$(N_x - 1)^2 \times (N_y - 1)^2$$

Matrices $\boldsymbol{B}$ and $\boldsymbol{I}$:

$$\boldsymbol{B} = \begin{bmatrix} -4 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -4 & 1 \\ 0 & \cdots & 0 & 1 & -4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Sizes of matrices $\boldsymbol{B}$ and $\boldsymbol{I}$:

$$(N_x - 1) \times (N_y - 1)$$

RHS vector $\boldsymbol{b}$:

$$\boldsymbol{b} = \{0 \quad 0 \quad \cdots \quad 0 \quad -10 \quad \cdots \quad -10\}^{\mathsf{T}}$$

Matrices $\boldsymbol{B}$ and $\boldsymbol{I}$:

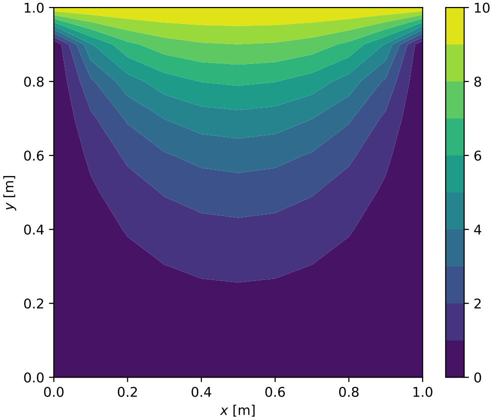
$$\boldsymbol{B} = \begin{bmatrix} -4 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -4 & 1 \\ 0 & \cdots & 0 & 1 & -4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Sizes of matrices $\boldsymbol{B}$ and $\boldsymbol{I}$:

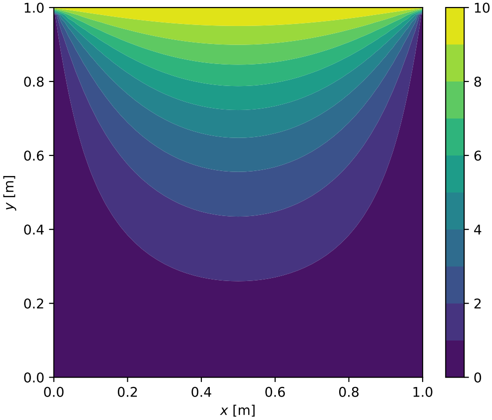
$$(N_x - 1) \times (N_y - 1)$$

RHS vector $\boldsymbol{b}$:

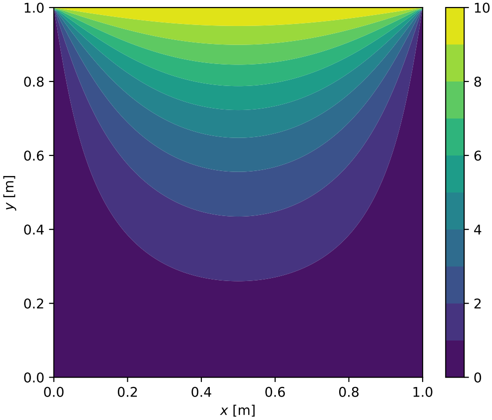
$$\boldsymbol{b} = \{0 \quad 0 \quad \cdots \quad 0 \quad -10 \quad \cdots \quad -10\}^\mathsf{T}$$



Solution for $Nx = Ny = 10$.



Solution for $Nx = Ny = 50$.



Solution for $Nx = Ny = 100$.