Finite Element Method

CENG6202: Advanced Computational Methods in Geotechnical Engineering

Yared Worku, PhD

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Introduction

The Finite Element Method (FEM) is the most widely used computational method in science and engineering.

FEM first used in geotechnical engineering in the 1960s and developed considerably in the decades that followed.

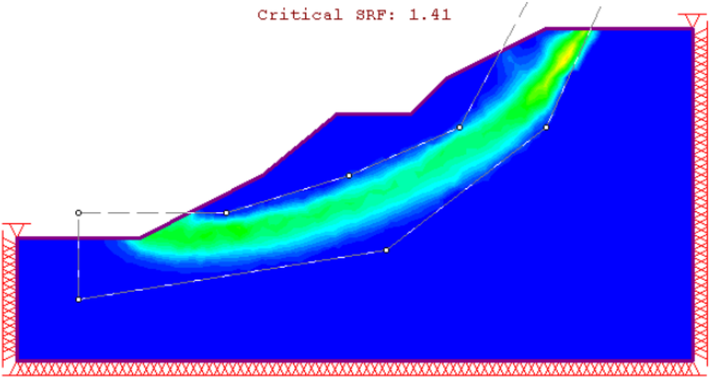
Several commercial finite element software packages exist today.

The numerical algorithms and the soil constitutive models are hidden behind user interfaces. $\Rightarrow$ Danger of being a Black Box!

FEM is a poweful computational method the can be applied to solve complex geotechnical problems that are difficult or impossible to solve using other methods.

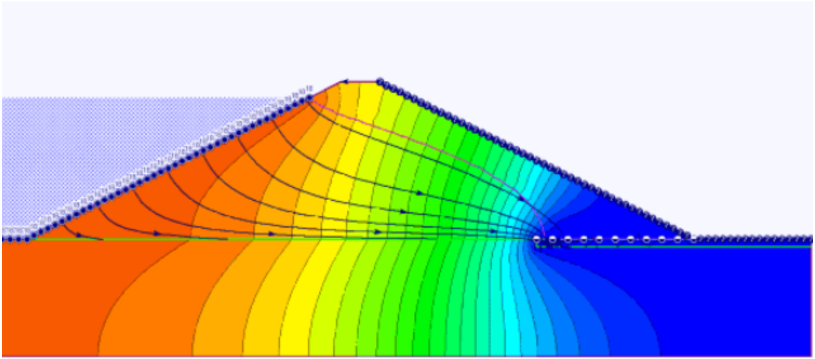
Like all numerical methods, FEM is an approximate solution method but the accuracy of the approximation can be excellent if properly applied.

FEM can be applied to various geotechnical engineering problems.



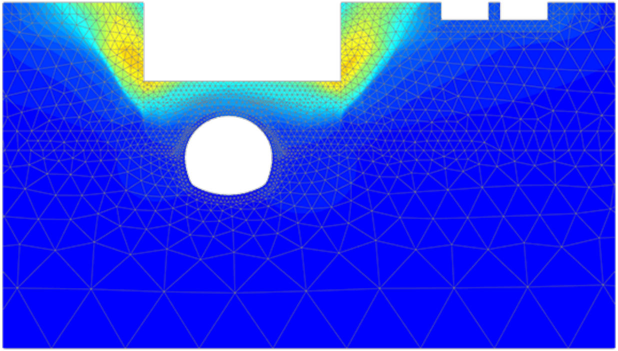
Slope stability analysis

FEM can be applied to various geotechnical engineering problems.



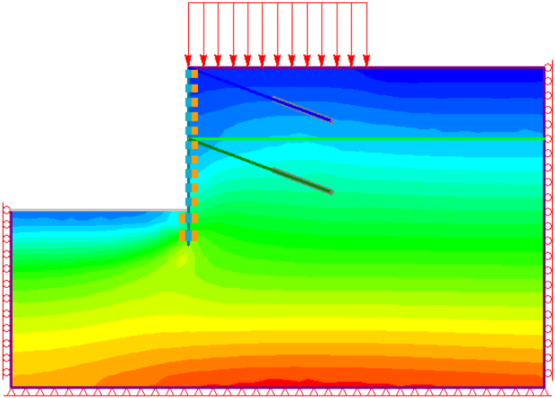
Seepage through embankment dams

FEM can be applied to various geotechnical engineering problems.



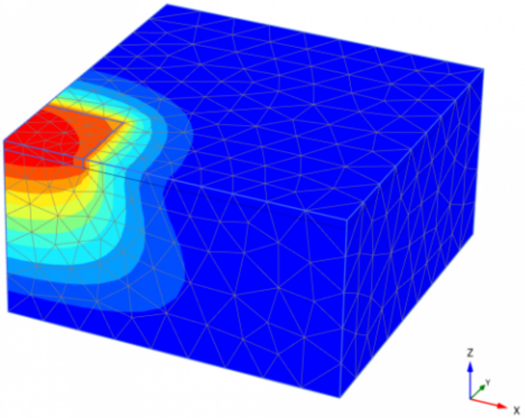
Excavation above a tunnel and next to a foundation

FEM can be applied to various geotechnical engineering problems.



Sheet pile walls and anchors

FEM can be applied to various geotechnical engineering problems.



Bearing capacity of foundations

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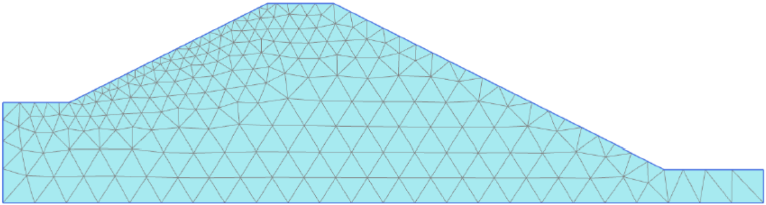
Basic Principles

Main steps of FEM:

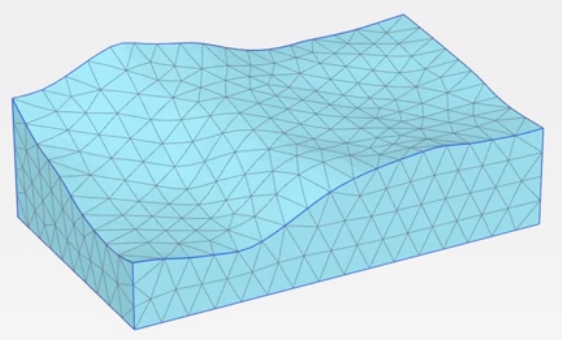
1. Discretize the domain: the computational domain is divided into a finite number of elements.
2. Choose interpolation functions: select functions that are used to interpolate field variables.
3. Define element properties: element matrices defining the physical problem are established.
4. Assemble element equations: element equations are assembled to form a global matrix equation, while imposing BCs.
5. Solve the global system: the assembled equations are solved for the unknown field variables using the appropriate solution method.
6. Compute secondary solutions: secondary field variables that are functions of the primary variables are computed.

Dividing the domain into a finite number of elements creates a finite element mesh.



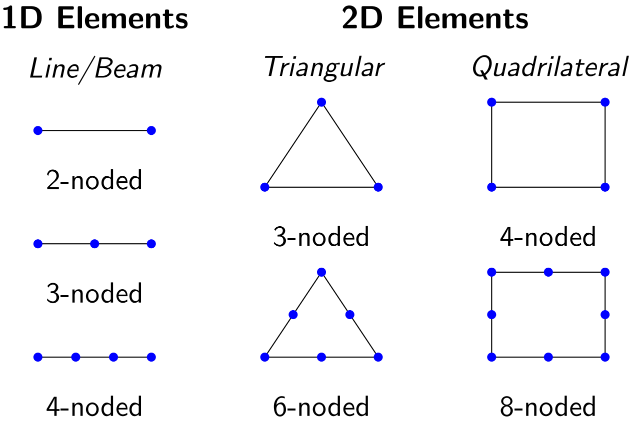
2D finite element mesh. © PLAXIS BV.

Dividing the domain into a finite number of elements creates a finite element mesh.



3D finite element mesh. © PLAXIS BV.

Various types of elements are used to discretize the domain.



Degrees of freedom are located at the nodes.

Shape functions for 1D finite elements are polynomial of the form

$$N_i(x) = a_1 + a_2 x + a_3 x^2 + \cdots$$

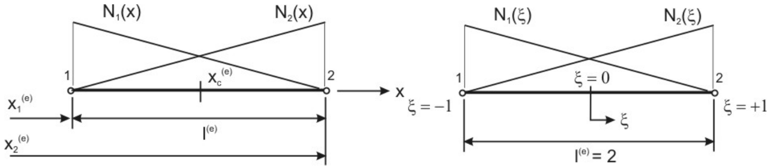
Shape function N_i always take a value of 1 node i and 0 at other nodes i.e.

$$N_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The polynomial degree of the shape function depends on the number of nodes of the element.

Linear Line Elements:

Linear 1D elements have two nodes and the corresponding shape functions may be expressed in terms of global or local coordinates.



The local coordinate is given by

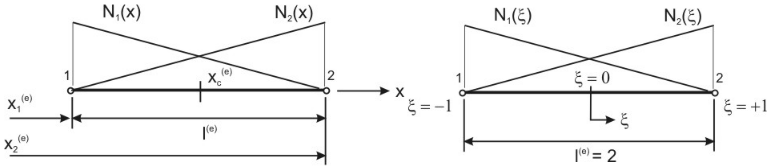
$$\xi = 2\frac{x - x_c}{l^e}$$

The shape functions are defined by

$$N_i(\xi) = \prod_{j=1(j\neq i)}^{n} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j}\right)$$

Linear Line Elements:

Linear 1D elements have two nodes and the corresponding shape functions may be expressed in terms of global or local coordinates.



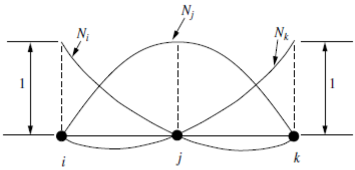
The local coordinate is given by

$$\xi = 2\frac{x - x_c}{l^e}$$

The two shape functions are

$$N_1(\xi) = \frac{1}{2}(1 - \xi) \qquad \text{and} \qquad N_2(\xi) = \frac{1}{2}(1 + \xi)$$

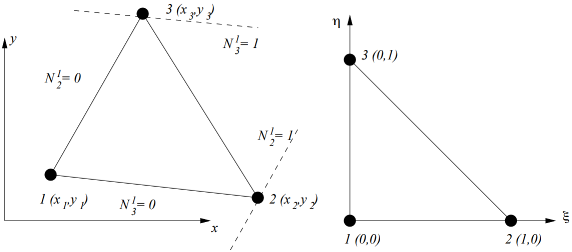
Quadratic Line Elements:
Have three nodes and three shape functions.



The shape functions are

$$N_1(\xi) = \frac{1}{2}\xi(1-\xi)$$
$$N_2(\xi) = (1+\xi)(1-\xi)$$
$$N_3(\xi) = \frac{1}{2}\xi(1+\xi)$$

Linear Triangular Elements:
Have three nodes, one at each vertex, and three shape functions.
The shape functions are more conveniently written in terms of local coordinates.

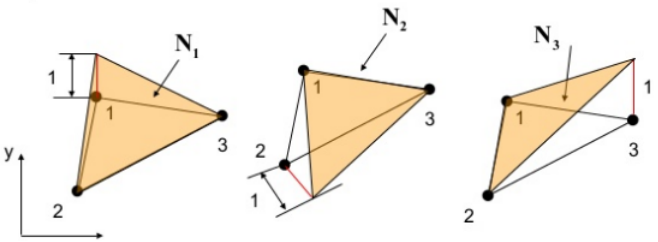


The shape functions take the form

$$N(\xi, \eta) = c_1 + c_2\xi + c_3\eta$$

The three shape functions for a linear triangular element are

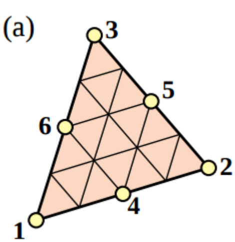
$$N_1(\xi,\eta) = 1 - \xi - \eta$$
$$N_2(\xi,\eta) = \xi$$
$$N_3(\xi,\eta) = \eta$$



Property of the shape functions

$$\sum_{i=1}^{3} N_i = 1$$

Quadratic Triangular Elements:
Have 6 nodes, one at each vertex and one at each mid-side.

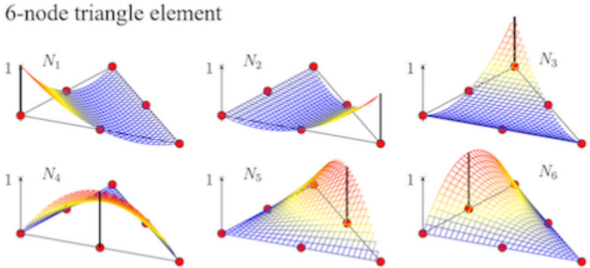


The quadratic shape functions in terms of the local coordinate system take the form

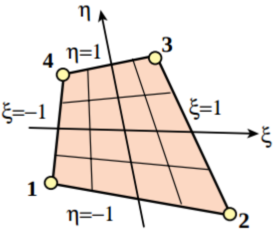
$$N(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2$$

The six shape functions in terms of the local coordinates are

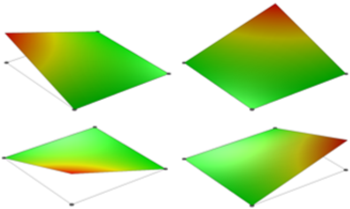
$$
\begin{aligned}
N_1(\xi,\eta) &= (1-\xi-\eta)(1-2\xi-2\eta)\\
N_2(\xi,\eta) &= \xi(2\xi-1)\\
N_3(\xi,\eta) &= \eta(2\eta-1)\\
N_4(\xi,\eta) &= 4\xi(1-\xi-\eta)\\
N_5(\xi,\eta) &= 4\xi\eta\\
N_6(\xi,\eta) &= 4\eta(1-\xi-\eta)
\end{aligned}
$$



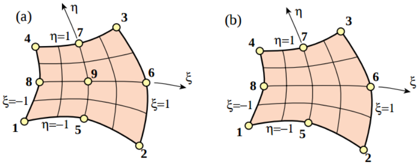
Bilinear Quadrilateral Elements: 4 nodes and 4 shape functions.

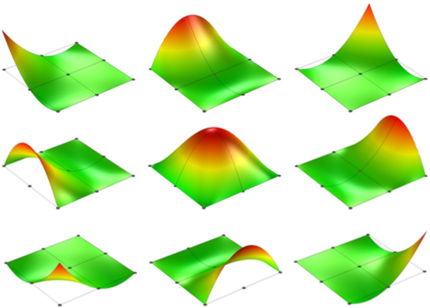


$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$



Biquadratic Quadrilateral Elements:



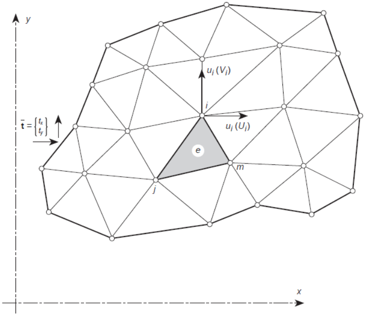


Displacements:

For a general three-dimensional problem, the global displacement vector is

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Example: 2D finite element mesh with nodal displacements



The local displacement vector within an element is

$$\boldsymbol{u}^e = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

The nodal displacement vector at node i is

$$\boldsymbol{a}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

The nodal displacement vector for all nodes of the element (a three-noded triangular element in this case)

$$\boldsymbol{a}^e = \{u_i \;\; v_i \;\; u_j \;\; v_j \;\; u_m \;\; v_m\}^\mathsf{T}$$

The interpolation of the diplacement vector for the element using the shape functions is performed as

$$\boldsymbol{u}^e = \boldsymbol{N}\boldsymbol{a}^e$$

where $\boldsymbol{N}$ in this case is given by

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Thus, we have

$$\boldsymbol{u}^e = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Strains and Stresses:
For example, for a plane strain problem

$$\boldsymbol{\varepsilon}^e = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

This may be written as

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\mathcal{L}} \boldsymbol{N} \boldsymbol{a}^e = \boldsymbol{B} \boldsymbol{a}^e$$

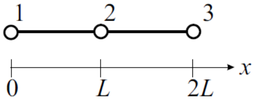
Stresses are computed based on the constitutive relationship:

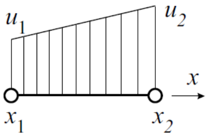
$$\boldsymbol{\sigma} = \boldsymbol{D} \boldsymbol{\varepsilon}^e$$

Galerkin's Method: Consider the ordinary differential equation

$$a\frac{d^2u}{dx^2} + b = 0, \qquad 0 \le x \le 2L$$

We aim to solve this on a 1D domain with two linear elements





Boundary conditions

$$u = 0 \quad \text{at } x = 0$$
$$u = r \quad \text{at } x = 2L$$

The variable u is interpolated using the shape functions as

$$u = \boldsymbol{N}\boldsymbol{u}^e = N_1 u_1 + N_2 u_2$$

Using this in the differential equation gives

$$a\frac{d^2}{dx^2}\boldsymbol{N}\boldsymbol{u}^e + b = \zeta$$

Galerkin's method minimizes the residual ζ by multiplying the equation by the shape functions and equating to zero i.e.

$$\int_{x_1}^{x_2} \boldsymbol{N}^\intercal a\frac{d^2}{dx^2}\boldsymbol{N}\boldsymbol{u}^e dx + \int_{x_1}^{x_2} \boldsymbol{N}^\intercal b dx = 0$$

Performing integration by parts results in

$$\int_{x_1}^{x_2} \frac{d\boldsymbol{N}^\intercal}{dx} a \frac{d\boldsymbol{N}}{dx} dx \boldsymbol{u}^e - \int_{x_1}^{x_2} \boldsymbol{N}^\intercal b dx - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a\frac{du}{dx}|_{x=x_2} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} a\frac{du}{dx}|_{x=x_1} = 0$$

In a compact form

$$\boldsymbol{k}^e \boldsymbol{u}^e = \boldsymbol{f}^e$$

where

$$\boldsymbol{k}^e = \int_{x_1}^{x_2} \frac{d\boldsymbol{N}^\mathsf{T}}{dx} a \frac{d\boldsymbol{N}}{dx} dx$$

$$\boldsymbol{f}^e = \int_{x_1}^{x_2} \boldsymbol{N}^\mathsf{T} b dx + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} a \frac{du}{dx}|_{x=x_2} - \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} a \frac{du}{dx}|_{x=x_1}$$

In general, the element matrices in the form of integrals are computed through numerical integration since analytical expressions are difficult to obtain for practical problems.

After the element matrices and vectors, $\boldsymbol{k}^e$ and $\boldsymbol{f}^e$, are computed for all elements, they are assembled into a global equation system of the form

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

where

$$\boldsymbol{K} = \sum_{e}^{N_e} \boldsymbol{k}^e$$

$$\boldsymbol{F} = \sum_{e}^{N_e} \boldsymbol{f}^e$$

The assembly is performed by considering the global node numbering and interacting nodes.

Contents

Constitutive Models

For a realistic simulation of geotechnical engineering problems, the real behavior of the soil must be captured by the constitutive model.

Various constitutive models exist which fall in the categories of *elastic*, *elastic-perfectly plastic*, *elastoplastic*, *viscoplastic* or *creep* models.

Constitutive models introduce a number of material parameters that are required as inputs in finite element analyses.

Understanding the theoretical formulation of soil constitutive models is important to perform realistic simulations.

A general incremental stress-strain relationship

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\sigma_z \\ \Delta\tau_{xy} \\ \Delta\tau_{yz} \\ \Delta\tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_y \\ \Delta\varepsilon_z \\ \Delta\gamma_{xy} \\ \Delta\gamma_{yz} \\ \Delta\gamma_{zx} \end{Bmatrix}$$

which can be written in a compact form as

$$\Delta\boldsymbol{\sigma} = \boldsymbol{D}\Delta\boldsymbol{\varepsilon}$$

In terms of effective stresses

$$\Delta\boldsymbol{\sigma}' = \boldsymbol{D}'\Delta\boldsymbol{\varepsilon}$$

Linear elastic behavior can be expressed in terms of two independent material parameters:

1. Young's modulus E and
2. Poisson's ratio ν

Linear elastic behavior can be expressed in terms of two independent material parameters:

1. Young's modulus E and
2. Poisson's ratio ν

Constitutive matrix: Effective stress formulation:

$$\boldsymbol{D}' = \frac{E'}{(1+\nu')(1-2\nu')}\begin{bmatrix} 1-\nu' & \nu' & \nu' & 0 & 0 & 0 \\ \nu' & 1-\nu' & \nu' & 0 & 0 & 0 \\ \nu' & \nu' & 1-\nu' & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu'}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu'}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu'}{2} \end{bmatrix}$$

Linear elastic behavior can be expressed in terms of two independent material parameters:

1. Young's modulus E and
2. Poisson's ratio ν

Constitutive matrix: Total stress formulation:

$$\boldsymbol{D}_n = \frac{E_n}{(1+\nu_n)(1-2\nu_n)}\begin{bmatrix} 1-\nu_n & \nu_n & \nu_n & 0 & 0 & 0 \\ \nu_n & 1-\nu_n & \nu_n & 0 & 0 & 0 \\ \nu_n & \nu_n & 1-\nu_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu_n}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu_n}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu_n}{2} \end{bmatrix}$$

In geomechanics, the shear and bulk moduli, G and K are usually preferred to describe the property of soil in the elastic range. For an effective stress formulation

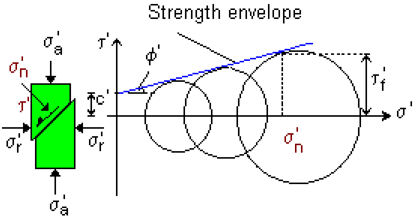
$$G = \frac{E'}{2(1+\nu')} \quad \text{and} \quad K' = \frac{E'}{3(1-2\nu')},$$

The undrained bulk modulus is

$$K_n = \frac{E_n}{3(1-2\nu_n)}$$

The Mohr-Coulomb model is the most widely used soil constitutive model in geotechnical finite element analysis.

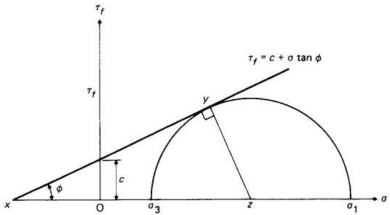
It is developed based on the Mohr-Coulomb failure criterion, simply known as the Coulomb criterion.



The tangent line is known as the Coulomb failure criterion and is expressed as

$$\tau_{\mathrm{f}} = c' + \sigma_{\mathrm{nf}} \tan \phi'$$

The Coulomb criterion may also be expressed in terms of axial and radial stresses.



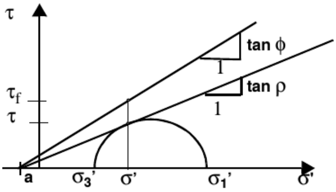
The failure criterion may be written as

$$\sigma_1' - \sigma_3' = 2c' \cos\phi' + (\sigma_1' - \sigma_3') \sin\phi'$$

The yield function based on the failure criterion is

$$F = \sigma_1' - \sigma_3' - 2c' \cos\phi' - (\sigma_1' - \sigma_3') \sin\phi'$$

During loading, the mobilized friction angle increases until failure.



A factor of safety is usually introduced as

$$FS = \frac{\tau}{\tau_f} = \frac{\tan \rho}{\tan \phi'}$$

Failure criterion in terms of principal stresses, for a certain mobilized friction

$$\frac{1}{2}(\sigma_1' - \sigma_3') = \left(\frac{c'}{\tan\phi'} + \frac{1}{2}(\sigma_1' + \sigma_3')\right)\sin\rho$$

which may be rearranged as

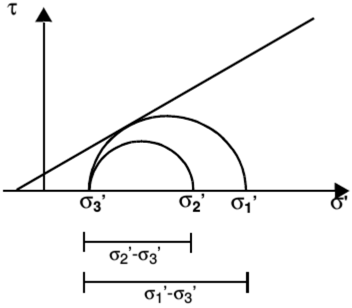
$$\left(\sigma_1' + \frac{c'}{\tan\phi'}\right) = N\left(\sigma_3' + \frac{c'}{\tan\phi'}\right)$$

where

$$N = \frac{1+\sin\rho}{1-\sin\rho}$$

Sometimes the Mohr-Coulomb failure criterion is expressed in terms of the mean stress p' and the deviatoric stress q.

$$p' = \frac{1}{3}(\sigma_1' + \sigma_2' + \sigma_3')$$
$$q = \sigma_1' - \sigma_3'$$



Introducing a parameter b

$$b = \frac{\sigma_2' - \sigma_3'}{\sigma_1' - \sigma_3'}$$

Major and minor principal stresses

$$\sigma_1' = p' + \frac{2-b}{3}q$$
$$\sigma_3' = p' - \frac{1+b}{3}q$$

The failure criterion is now written as

$$\left(p' + \frac{2-b}{3}q + \frac{c'}{\tan\phi'}\right) = N\left(p' - \frac{1+b}{3}q + \frac{c'}{\tan\phi'}\right)$$

In terms of the mean and deviatoric stresses is given by

$$q = M(p' + \frac{c'}{\tan\phi'})$$

where

$$M = \frac{3(N-1)}{3+(1+b)(N-1)}$$

Yield function

$$F = q - M(p' + \frac{c'}{\tan\phi'})$$

Parameters of the Mohr-Coulomb model

Parameter	**Symbol**
Young's modulus	E'
Poisson's ratio	ν'
Cohesion	c'
Friction angle	ϕ'
Dilatancy angle	ψ

Cam-Clay model is one of the first critical-state models developed to capture the behavior of soft soils.

Developed further into the Modified Cam-Clay (MCC) model.

The most important aspects of soil behavior described by the MCC model are

- strength
- volume change during shearing (compression or dilatancy)
- critical state at which soil elements distort unlimitedly without changes in stress or volume

The MCC model captures the behavior of soft soils realistically.

In critical state soil mechanics, the state of a soil is described by three parameters: the effective mean stress p', the deviatoric stress q and the specific volume v.

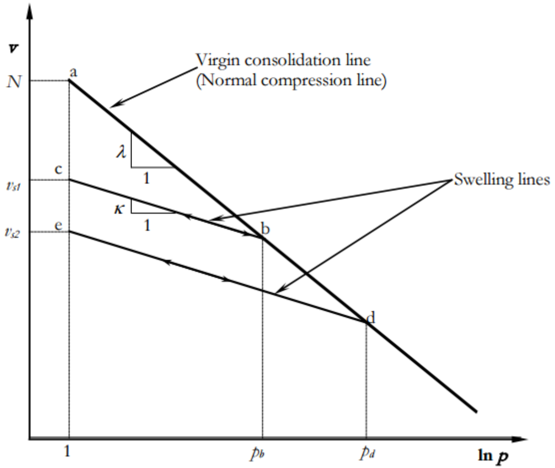
The mean and deviatoric stresses are defined as

$$p' = \frac{1}{3}(\sigma_1' + \sigma_2' + \sigma_3')$$
$$q = \sqrt{\frac{1}{2}\left[(\sigma_1' - \sigma_2')^2 + (\sigma_2' - \sigma_3')^2 + (\sigma_3' - \sigma_1')^2\right]}$$

The specific volume is defined as

$$v = 1 + e$$

Normal Compression and Swelling Lines:



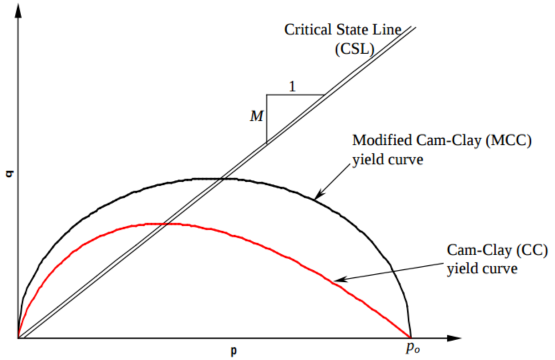
The normal compression line is defined by the equation

$$v = N - \lambda \ln p'$$

The unloading-reloading lines are defined by the equation

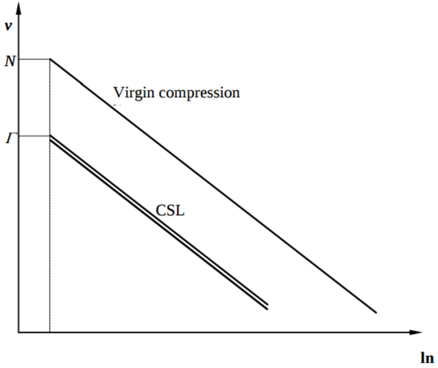
$$v = v_{\mathrm{s}} - \kappa \ln p'$$

Critical State Line:
Continued loading reaches a state where further shearing can occur without any changes in stress or volume. ⇒ Critical State!



The slope of the critical state line is a parameter M dependent on the type of soil.

Location of critical state line on a v-$\ln p'$ plane



Relationship between N and Γ:

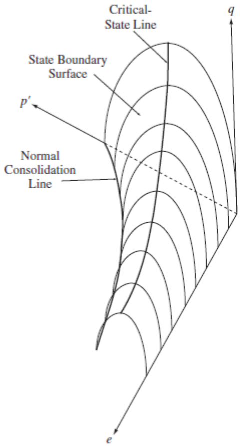
$$\Gamma = N - (\lambda - \kappa)\ln 2$$

Yield Function:
The yield function for the MCC model is given by

$$F = \frac{q^2}{p'^2} + M^2\left(1 - \frac{p'_o}{p'}\right)$$

The parameter p'_o is known as the pre-consolidation pressure and controls the size of the yield surface.

In a three-dimensional v-p'-q space, the yield surface defined by the MCC model is known as the state boundary surface.



Material Parameters:
In geotechnics, the shear modulus G and the bulk modulus K are preferred to describe the behavior of the material in the elastic range. The bulk modulus is dependent on the mean stress and is calculated as

$$K = \frac{vp'}{\kappa}$$

Analysis using the MCC model requires specification of either the shear modulus G or the Poisson's ratio ν.

If G is known, ν is calculated from

$$\nu = \frac{3K - 2G}{2G + 6K}$$

If ν is given, then G can be determined from

$$G = \frac{3(1 - 2\nu)}{2(1 + \nu)} K$$

The initial state of consolidation must also be specified.

This may be accomplished by defining the pre-consolidation pressure p'_{o}. An alternative is to define what is known as the *over-consolidation ratio* (OCR), which is given by

$$OCR = \frac{p'_{\mathrm{o}}}{p'}$$

Parameter	Symbol
Shear modulus *or* Poisson's ratio	G ν
Pre-consolidation pressure *or* Over-consolidation ratio	p'_o OCR
Specific volume of NC line at unit pressure *or* Specific volume of CSL at unit pressure	N Γ
Slope of CSL in q-p' plane Slope of NC line in v-$\ln p'$ plane Slope of unloading-reloading line in v-$\ln p'$	M λ κ

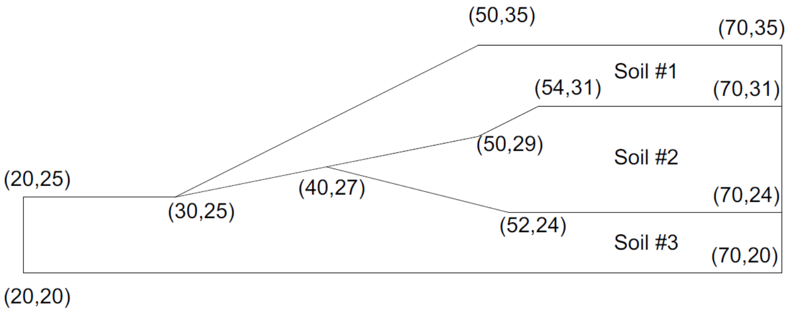
Material parameters of the MCC model.

Contents

Creating a finite element model and performing simulations:

- Creating a geometry
- Discretizing the geometry/Creating a FE mesh
- Selecting constitutive models and defining material parameters
- Assigning material parameters
- Creating structural elements
- Defining initial and boundary conditions
- Selecting calculation settings
- Solving the problem
- Interpreting the results
- Extracting and reporting results

Slope geometry



Material properties

	Elastic Modulus (MN/m^2)	C (KN/m^2)	Φ ($^\circ$)	Density (KN/m^2)
Soil #1	5000	0	38.0	19.5
Soil #2	5000	5.3	23.0	19.5
Soil #3	5000	7.2	20.0	19.5