Mathematical Preliminaries

CENG6202: Advanced Computational Methods in Geotechnical Engineering

Yared Worku, PhD

Fall Semester 2019

Contents

1. Vectors
2. Matrices
3. Linear System of Equations

Contents

1 Vectors

2 Matrices

3 Linear System of Equations

Vectors

A vector is a column of numbers. For example, vector $\boldsymbol{a}$ of *size* n is written as

$$\boldsymbol{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$$

The *transpose* of vector $\boldsymbol{a}$:

$$\boldsymbol{a}^\mathsf{T} = \{a_1 \; a_2 \; \cdots \; a_n\}$$

The *magnitude* or *norm* of vector $\boldsymbol{a}$ is defined as

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Vectors

A vector is a column of numbers. For example, vector $\boldsymbol{a}$ of *size* n is written as

$$\boldsymbol{a} = \left\{ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \right\}$$

The *transpose* of vector $\boldsymbol{a}$:

$$\boldsymbol{a}^\intercal = \{a_1 \; a_2 \; \cdots \; a_n\}$$

The *magnitude* or *norm* of vector $\boldsymbol{a}$ is defined as

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

A vector is a column of numbers. For example, vector $\boldsymbol{a}$ of *size* n is written as

$$\boldsymbol{a} = \left\{ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \right\}$$

The *transpose* of vector $\boldsymbol{a}$:

$$\boldsymbol{a}^\mathsf{T} = \{a_1 \; a_2 \; \cdots \; a_n\}$$

The *magnitude* or *norm* of vector $\boldsymbol{a}$ is defined as

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Vectors

A vector is a column of numbers. For example, vector $\boldsymbol{a}$ of *size* n is written as

$$\boldsymbol{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$$

The *transpose* of vector $\boldsymbol{a}$:

$$\boldsymbol{a}^{\mathsf{T}} = \{a_1 \; a_2 \; \cdots \; a_n\}$$

The *magnitude* or *norm* of vector $\boldsymbol{a}$ is defined as

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

The *dot product* (also called *scalar product*) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, each of size n, is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The dot product can also be written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\intercal} \boldsymbol{b} = \{a_1 \; a_2 \; \cdots \; a_n\} \left\{ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right\} = \sum_{i=1}^{n} a_i b_i$$

The *dot product* (also called *scalar product*) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, each of size n, is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The dot product can also be written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^\top \boldsymbol{b} = \{a_1 \; a_2 \; \cdots \; a_n\} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} = \sum_{i=1}^{n} a_i b_i$$

The *dot product* (also called *scalar product*) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, each of size n, is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The dot product can also be written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathsf{T}} \boldsymbol{b} = \{a_1 \ a_2 \ \cdots \ a_n\} \left\{ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right\} = \sum_{i=1}^{n} a_i b_i$$

The product $\boldsymbol{ab}^\intercal$, however, results in a matrix, i.e.

$$\boldsymbol{ab}^\intercal = \left\{ \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix} \right\} \{b_1 \; b_2 \; \cdots \; b_n\} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix}$$

The *cross product* of three-dimensional vectors $\boldsymbol{a}^\intercal = \{a_1 \; a_2 \; a_3\}$ and $\boldsymbol{b}^\intercal = \{b_1 \; b_2 \; b_3\}$ is a vector (say $\boldsymbol{c}$) and is defined as

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta \, \boldsymbol{n}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{n}$ is a unit normal vector ($\|\boldsymbol{n}\| = 1$) to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.

The elements of vector $\boldsymbol{c}$ are

$$\boldsymbol{c} = \left\{ \begin{array}{c} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{array} \right\}$$

The *cross product* of three-dimensional vectors $\boldsymbol{a}^\mathsf{T} = \{a_1 \ a_2 \ a_3\}$ and $\boldsymbol{b}^\mathsf{T} = \{b_1 \ b_2 \ b_3\}$ is a vector (say $\boldsymbol{c}$) and is defined as

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin\theta \, \boldsymbol{n}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{n}$ is a unit normal vector ($\|\boldsymbol{n}\| = 1$) to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.

The elements of vector $\boldsymbol{c}$ are

$$\boldsymbol{c} = \left\{ \begin{array}{c} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{array} \right\}$$

Contents

1. Vectors
2. Matrices
3. Linear System of Equations

Matrices

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$

Symmetric matrix: Square and $a_{ij} = a_{ji}$

Skew matrix: Square and $a_{ij} = -a_{ji}$

Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

Matrices

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$
Symmetric matrix: Square and $a_{ij} = a_{ji}$
Skew matrix: Square and $a_{ij} = -a_{ji}$
Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

Matrices

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$

Symmetric matrix: Square and $a_{ij} = a_{ji}$

Skew matrix: Square and $a_{ij} = -a_{ji}$

Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$

Symmetric matrix: Square and $a_{ij} = a_{ji}$

Skew matrix: Square and $a_{ij} = -a_{ji}$

Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$
Symmetric matrix: Square and $a_{ij} = a_{ji}$
Skew matrix: Square and $a_{ij} = -a_{ji}$
Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

Matrices

A matrix $\boldsymbol{A}$ with n rows and m columns is written as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The *transpose* of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Square matrix: $n = m$
Symmetric matrix: Square and $a_{ij} = a_{ji}$
Skew matrix: Square and $a_{ij} = -a_{ji}$
Skew-symmetric matrix: Square, $a_{ii} = 0$ and $a_{ij} = -a_{ji}$

Upper and Lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Strictly upper and strictly lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U}' = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L}' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

Upper and Lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Strictly upper and strictly lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U}' = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L}' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

Upper and Lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Strictly upper and strictly lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U}' = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L}' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

Upper and Lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Strictly upper and strictly lower triangular decompositions of $\boldsymbol{A}$:

$$\boldsymbol{U}' = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L}' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

Addition, subtraction, scalar multiplication and transposition:

1. $\boldsymbol{A} \pm \boldsymbol{B} = \boldsymbol{B} \pm \boldsymbol{A}$
2. $(\boldsymbol{A} \pm \boldsymbol{B})^\intercal = \boldsymbol{A}^\intercal \pm \boldsymbol{B}^\intercal$
3. $(\boldsymbol{A}^\intercal)^\intercal = \boldsymbol{A}$
4. $(s\boldsymbol{A})^\intercal = s(\boldsymbol{A}^\intercal)$

Addition, subtraction, scalar multiplication and transposition:

1. $\boldsymbol{A} \pm \boldsymbol{B} = \boldsymbol{B} \pm \boldsymbol{A}$
2. $(\boldsymbol{A} \pm \boldsymbol{B})^\intercal = \boldsymbol{A}^\intercal \pm \boldsymbol{B}^\intercal$
3. $(\boldsymbol{A}^\intercal)^\intercal = \boldsymbol{A}$
4. $(s\boldsymbol{A})^\intercal = s(\boldsymbol{A}^\intercal)$

Addition, subtraction, scalar multiplication and transposition:

1. $\boldsymbol{A} \pm \boldsymbol{B} = \boldsymbol{B} \pm \boldsymbol{A}$
2. $(\boldsymbol{A} \pm \boldsymbol{B})^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} \pm \boldsymbol{B}^{\mathsf{T}}$
3. $(\boldsymbol{A}^{\mathsf{T}})^{\mathsf{T}} = \boldsymbol{A}$
4. $(s\boldsymbol{A})^{\mathsf{T}} = s(\boldsymbol{A}^{\mathsf{T}})$

Addition, subtraction, scalar multiplication and transposition:

1. $\boldsymbol{A} \pm \boldsymbol{B} = \boldsymbol{B} \pm \boldsymbol{A}$
2. $(\boldsymbol{A} \pm \boldsymbol{B})^\intercal = \boldsymbol{A}^\intercal \pm \boldsymbol{B}^\intercal$
3. $(\boldsymbol{A}^\intercal)^\intercal = \boldsymbol{A}$
4. $(s\boldsymbol{A})^\intercal = s(\boldsymbol{A}^\intercal)$

Matrix-Vector multiplication:

$$\boldsymbol{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\boldsymbol{y}^\intercal \boldsymbol{A} = \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\}$$

Matrix-Vector multiplication:

$$\boldsymbol{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\boldsymbol{y}^{\top}\boldsymbol{A} = \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\}$$

Matrix-Vector multiplication:

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\boldsymbol{y}^\top \boldsymbol{A} = \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\}$$

Matrix-Vector multiplication:

$$\boldsymbol{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} = \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\}$$

Matrix-Vector multiplication:

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\boldsymbol{y}^\intercal \boldsymbol{A} = \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\}$$

Matrix-Vector multiplication:

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{Bmatrix}$$

Vector-Matrix multiplication:

$$\begin{aligned} \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} &= \{y_1 \quad y_2 \cdots y_n\} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \\ &= \{y_1 a_{11} + \cdots + y_n a_{n1} \quad \cdots \quad y_1 a_{1m} + \cdots + y_n a_{nm}\} \end{aligned}$$

Bilinear form:

$$\boldsymbol{y}^\intercal \boldsymbol{A}\boldsymbol{x} \rightarrow \text{Scalar}$$

Quadratic form:

$$\boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{x} \rightarrow \text{Scalar}$$

A matrix $\boldsymbol{A}$ is called positive-definite if

$$\boldsymbol{z}^\intercal \boldsymbol{A}\boldsymbol{z} > 0 \qquad \forall \boldsymbol{z} \neq 0$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

Matrix-Matrix multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1m}b_{m1} & \cdots & a_{11}b_{1p} + \cdots + a_{1m}b_{mp} \\ a_{21}b_{11} + \cdots + a_{2m}b_{m1} & \cdots & a_{21}b_{1p} + \cdots + a_{2m}b_{mp} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \cdots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1p} + \cdots + a_{nm}b_{mp} \end{bmatrix}$$

The *inverse* of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Some properties of inverse:

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{\intercal})^{-1} = (\boldsymbol{A}^{-1})^{\intercal}$
3. $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

The *inverse* of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Some properties of inverse:

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{\intercal})^{-1} = (\boldsymbol{A}^{-1})^{\intercal}$
3. $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

The *inverse* of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Some properties of inverse:

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{\intercal})^{-1} = (\boldsymbol{A}^{-1})^{\intercal}$
3. $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

The *inverse* of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Some properties of inverse:

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{\intercal})^{-1} = (\boldsymbol{A}^{-1})^{\intercal}$
3. $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

The *inverse* of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Some properties of inverse:

1. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $(\boldsymbol{A}^{\intercal})^{-1} = (\boldsymbol{A}^{-1})^{\intercal}$
3. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{AB}| = |\boldsymbol{A}| |\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^{\intercal}| = |\boldsymbol{A}|$
3. $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^n |\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|AB| = |A| |B|$ for square matrices A and B of the same size.
2. $|A^{\mathsf{T}}| = |A|$
3. $|A^{-1}| = 1/|A|$
4. $|sA| = s^n |A|$ for an $n \times n$ matrix A.
5. $|I| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{AB}| = |\boldsymbol{A}| |\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^{\intercal}| = |\boldsymbol{A}|$
3. $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^n |\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k}\,|\boldsymbol{A}_{ik}|\,, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^{\mathsf{T}}| = |\boldsymbol{A}|$
3. $\left|\boldsymbol{A}^{-1}\right| = 1/\,|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^{n}\,|\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{AB}| = |\boldsymbol{A}| |\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^{\mathsf{T}}| = |\boldsymbol{A}|$
3. $\left|\boldsymbol{A}^{-1}\right| = 1/|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^n |\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{AB}| = |\boldsymbol{A}| |\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^\mathsf{T}| = |\boldsymbol{A}|$
3. $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^n |\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

The *determinant* of a matrix $\boldsymbol{A}$ is denoted by $\det \boldsymbol{A}$ or $|\boldsymbol{A}|$ and is given by

$$|\boldsymbol{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} |\boldsymbol{A}_{ik}|, \quad k = 1, \cdots, n$$

Some properties of determinants:

1. $|\boldsymbol{AB}| = |\boldsymbol{A}| |\boldsymbol{B}|$ for square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size.
2. $|\boldsymbol{A}^{\mathsf{T}}| = |\boldsymbol{A}|$
3. $\left|\boldsymbol{A}^{-1}\right| = 1/|\boldsymbol{A}|$
4. $|s\boldsymbol{A}| = s^n |\boldsymbol{A}|$ for an $n \times n$ matrix $\boldsymbol{A}$.
5. $|\boldsymbol{I}| = 1$ for an identity matrix of any size.

A block matrix or partitioned matrix is a matrix that is composed of submatrices:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

A block diagonal matrix is a square diagonal matrix in which the diagonal elements are square matrices of any size and the off-diagonal elements are zero or zero matrices of any size:

$$P = \begin{bmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{bmatrix}$$

where A and B are square matrices and $\mathbf{O}$ is a zero matrix.

A block matrix or partitioned matrix is a matrix that is composed of submatrices:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

A block diagonal matrix is a square diagonal matrix in which the diagonal elements are square matrices of any size and the off-diagonal elements are zero or zero matrices of any size:

$$P = \begin{bmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{bmatrix}$$

where A and B are square matrices and $\mathbf{O}$ is a zero matrix.

A block matrix or partitioned matrix is a matrix that is composed of submatrices:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

A block diagonal matrix is a square diagonal matrix in which the diagonal elements are square matrices of any size and the off-diagonal elements are zero or zero matrices of any size:

$$P = \begin{bmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{bmatrix}$$

where A and B are square matrices and $\mathbf{O}$ is a zero matrix.

Contents

1. Vectors
2. Matrices
3. Linear System of Equations

Linear System of Equations

The final problem to be solved in computational science and engineering usually takes the form:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Homogeneous system of equations ($\boldsymbol{b} = \mathbf{0}$):

$$\boldsymbol{Ax} = \mathbf{0}$$

Given a matrix $\boldsymbol{A}$ of size $m \times n$ and a vector $\boldsymbol{b}$ of size m:

- If $n < m$, the system is called over-determined and there is no solution.
- If $n > m$, the system is under-determined and the equations may be solved as far as possible.
- If $m = n$ and $\boldsymbol{A}$ is non-singular, a direct solution can be obtained.

Linear System of Equations

The final problem to be solved in computational science and engineering usually takes the form:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Homogeneous system of equations ($\boldsymbol{b} = \mathbf{0}$):

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$$

Given a matrix $\boldsymbol{A}$ of size $m \times n$ and a vector $\boldsymbol{b}$ of size m:

- If $n < m$, the system is called over-determined and there is no solution.
- If $n > m$, the system is under-determined and the equations may be solved as far as possible.
- If $m = n$ and $\boldsymbol{A}$ is non-singular, a direct solution can be obtained.

Linear System of Equations

The final problem to be solved in computational science and engineering usually takes the form:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Homogeneous system of equations ($\boldsymbol{b} = \mathbf{0}$):

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$$

Given a matrix $\boldsymbol{A}$ of size $m \times n$ and a vector $\boldsymbol{b}$ of size m:

- If $n < m$, the system is called over-determined and there is no solution.
- If $n > m$, the system is under-determined and the equations may be solved as far as possible.
- If $m = n$ and $\boldsymbol{A}$ is non-singular, a direct solution can be obtained.

The final problem to be solved in computational science and engineering usually takes the form:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Homogeneous system of equations ($\boldsymbol{b} = \mathbf{0}$):

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$$

Given a matrix $\boldsymbol{A}$ of size $m \times n$ and a vector $\boldsymbol{b}$ of size m:

- If $n < m$, the system is called over-determined and there is no solution.
- If $n > m$, the system is under-determined and the equations may be solved as far as possible.
- If $m = n$ and $\boldsymbol{A}$ is non-singular, a direct solution can be obtained.

Linear System of Equations

The final problem to be solved in computational science and engineering usually takes the form:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Homogeneous system of equations ($\boldsymbol{b} = \mathbf{0}$):

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$$

Given a matrix $\boldsymbol{A}$ of size $m \times n$ and a vector $\boldsymbol{b}$ of size m:

- If $n < m$, the system is called over-determined and there is no solution.
- If $n > m$, the system is under-determined and the equations may be solved as far as possible.
- If $m = n$ and $\boldsymbol{A}$ is non-singular, a direct solution can be obtained.

Direct Methods: Examples

- Gaussian Elimination
- LU Factorization

Iterative Methods: Examples

- Jacobi Iteration
- Gauss-Seidel Iteration
- Successive Over-Relaxation (SOR)

Direct Methods: Examples

- Gaussian Elimination
- LU Factorization

Iterative Methods: Examples

- Jacobi Iteration
- Gauss-Seidel Iteration
- Successive Over-Relaxation (SOR)

Direct Methods: Examples

- Gaussian Elimination
- LU Factorization

Iterative Methods: Examples

- Jacobi Iteration
- Gauss-Seidel Iteration
- Successive Over-Relaxation (SOR)

Elementary row operations on augmented matrix $\boldsymbol{A}|\boldsymbol{b}$ to get a simpler system $\boldsymbol{A}'|\boldsymbol{b}'$:

$$\boldsymbol{A}|\boldsymbol{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \boldsymbol{A}'|\boldsymbol{b}' = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Equivalent linear system:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{Bmatrix}$$

Elementary row operations on augmented matrix $\boldsymbol{A}|\boldsymbol{b}$ to get a simpler system $\boldsymbol{A}'|\boldsymbol{b}'$:

$$\boldsymbol{A}|\boldsymbol{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \boldsymbol{A}'|\boldsymbol{b}' = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Equivalent linear system:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{Bmatrix}$$

Elementary row operations on augmented matrix $\boldsymbol{A}|\boldsymbol{b}$ to get a simpler system $\boldsymbol{A}'|\boldsymbol{b}'$:

$$\boldsymbol{A}|\boldsymbol{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \boldsymbol{A}'|\boldsymbol{b}' = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Equivalent linear system:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{Bmatrix}$$

Elementary row operations on augmented matrix $\boldsymbol{A}|\boldsymbol{b}$ to get a simpler system $\boldsymbol{A}'|\boldsymbol{b}'$:

$$\boldsymbol{A}|\boldsymbol{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \boldsymbol{A}'|\boldsymbol{b}' = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Equivalent linear system:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{Bmatrix}$$

Elementary row operations on augmented matrix $\boldsymbol{A}|\boldsymbol{b}$ to get a simpler system $\boldsymbol{A}'|\boldsymbol{b}'$:

$$\boldsymbol{A}|\boldsymbol{b} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Longrightarrow \boldsymbol{A}'|\boldsymbol{b}' = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Equivalent linear system:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{Bmatrix}$$

Solution for x_n from the n^{th} row as

$$x_n = \frac{b'_n}{a'_{nn}}$$

Remaining unknowns by back substitution:

$$x_i = \frac{1}{a'_{ii}}\left(b'_i - \sum_{j=i+1}^{n} a'_{ij}x_j\right)$$

Solution for x_n from the n^{th} row as

$$x_n = \frac{b'_n}{a'_{nn}}$$

Remaining unknowns by back substitution:

$$x_i = \frac{1}{a'_{ii}}\left(b'_i - \sum_{j=i+1}^{n} a'_{ij}x_j\right)$$

Coefficient matrix $\boldsymbol{A}$ first decomposed into lower and upper triangular matrices:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

such that

$$\boldsymbol{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Elements of $\boldsymbol{L}$ and $\boldsymbol{U}$ can be calculated from:

$$l_{ij} = \frac{1}{u_{ii}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right)$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

Coefficient matrix $\boldsymbol{A}$ first decomposed into lower and upper triangular matrices:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

such that

$$\boldsymbol{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Elements of $\boldsymbol{L}$ and $\boldsymbol{U}$ can be calculated from:

$$l_{ij} = \frac{1}{u_{ii}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right)$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}$$

Coefficient matrix $\boldsymbol{A}$ first decomposed into lower and upper triangular matrices:

$$\boldsymbol{A} = \boldsymbol{LU}$$

such that

$$\boldsymbol{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Elements of $\boldsymbol{L}$ and $\boldsymbol{U}$ can be calculated from:

$$l_{ij} = \frac{1}{u_{ii}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right)$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ may now be written as

$$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$$

Two simpler systems to solve:

$$\boldsymbol{Ly} = \boldsymbol{b}$$
$$\boldsymbol{Ux} = \boldsymbol{y}$$

Solutions:

$$y_1 = \frac{b_1}{l_{11}} \qquad y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right) \qquad i = 2, \cdots, n$$

$$x_n = \frac{y_n}{u_{nn}} \qquad x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}x_j\right) \qquad i = n-1, \cdots, 1$$

The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ may now be written as

$$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$$

Two simpler systems to solve:

$$\boldsymbol{Ly} = \boldsymbol{b}$$
$$\boldsymbol{Ux} = \boldsymbol{y}$$

Solutions:

$$y_1 = \frac{b_1}{l_{11}} \qquad y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right) \qquad i = 2, \cdots, n$$

$$x_n = \frac{y_n}{u_{nn}} \qquad x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}x_j\right) \qquad i = n-1, \cdots, 1$$

The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ may now be written as

$$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$$

Two simpler systems to solve:

$$\boldsymbol{Ly} = \boldsymbol{b}$$
$$\boldsymbol{Ux} = \boldsymbol{y}$$

Solutions:

$$y_1 = \frac{b_1}{l_{11}} \qquad y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j\right) \qquad i = 2, \cdots, n$$

$$x_n = \frac{y_n}{u_{nn}} \qquad x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right) \qquad i = n-1, \cdots, 1$$

The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ may now be written as

$$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$$

Two simpler systems to solve:

$$\boldsymbol{Ly} = \boldsymbol{b}$$
$$\boldsymbol{Ux} = \boldsymbol{y}$$

Solutions:

$$y_1 = \frac{b_1}{l_{11}} \qquad y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right) \qquad i = 2, \cdots, n$$

$$x_n = \frac{y_n}{u_{nn}} \qquad x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}x_j\right) \qquad i = n-1, \cdots, 1$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for x on the LHS based on assumed/previous x on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ now written as

$$\left(\boldsymbol{D} + \boldsymbol{L}'\right)\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \boldsymbol{L}'\right)^{-1}\left(\boldsymbol{b} - \boldsymbol{U}'\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for x on the LHS based on assumed/previous x on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \le e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1 - \omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|x^{(k+1)} - x^{(k)}\right\| \leq e$$

Coefficient matrix $\boldsymbol{A}$ is decomposed such that

$$\boldsymbol{A} = \boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'$$

Scalar factor ω is introduced to the linear system

$$\omega\left(\boldsymbol{L}' + \boldsymbol{D} + \boldsymbol{U}'\right)\boldsymbol{x} = \omega\boldsymbol{b}$$

$$\Rightarrow \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)\boldsymbol{x} = \omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}$$

Solve for $\boldsymbol{x}$ on the LHS based on assumed/previous $\boldsymbol{x}$ on the RHS:

$$\boldsymbol{x}^{(k+1)} = \left(\boldsymbol{D} + \omega\boldsymbol{L}'\right)^{-1}\left(\omega\boldsymbol{b} - \left[\omega\boldsymbol{U}' + (\omega - 1)\boldsymbol{D}\right]\boldsymbol{x}^{(k)}\right)$$

Forward substitution expression of iteration procedure

$$x_i^{(k+1)} = (1 - \omega)x^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

Perform iteration until an error criterion is satisfied:

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq e$$