# IE221 PROBABILITY

# DISTRIBUTION COMPARISON AND LIMITS OF THE STRONG LAW OF LARGE NUMBERS AND THE CENTRAL LIMIT THEOREM

**Fall 2024–2025**

**Teamwork Group 22**

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## 1.INTRODUCTION

The Strong Law of Large Numbers (SLLN) and the Central Limit Theorem (CLT) are two fundamental results in probability theory that justify the use of sample averages and normal approximations in statistical analysis. These theorems play a crucial role in many practical applications such as simulation, risk analysis, statistical inference, and Monte Carlo methods.

Although SLLN and CLT are often presented as general and powerful results, their validity relies on specific assumptions, including independence, identical distribution, and the existence of moments. In practice, data may originate from skewed or heavy-tailed distributions, where convergence behavior can be slow, unstable, or may not occur at all.

In the earlier phases of this project (Team Work 1 and Team Work 2), SLLN and CLT were experimentally verified using the Uniform(0,1) distribution under ideal conditions. These initial experiments served as a baseline example and provided a clear reference point for later comparisons.

The aim of Team Work 5 (TW3) is to extend this experimental framework to different probability distributions with varying tail behaviors and moment properties, and to investigate the boundaries of SLLN and CLT. By analyzing the Uniform, Exponential, Pareto, and Cauchy distributions, this project explores when these theorems work, when they fail, and why such differences arise.

An important focus of this study is the comparison between different modes of convergence. While SLLN guarantees almost sure convergence along individual sample paths, CLT describes convergence in distribution, focusing on the shape of the distribution of sums rather than on single realizations.

Furthermore, this project emphasizes the practical interpretation of theoretical assumptions. Heavy-tailed distributions such as Pareto and Cauchy demonstrate that commonly assumed conditions, such as finite variance, may be violated in realistic settings.

Monte Carlo simulation is used as the primary methodological tool throughout the analysis. Cumulative mean plots are employed to study almost sure convergence for SLLN, while histograms and Normal Q-Q plots of standardized sums are used to assess convergence to normality under CLT.

## 2. DISTRIBUTIONS AND THEIR PROPERTIES

In this study, five probability distributions are analyzed to investigate the convergence behavior of the Strong Law of Large Numbers (SLLN) and the Central Limit Theorem (CLT) under different distributional assumptions. These distributions are deliberately chosen to represent a wide range of characteristics, including boundedness, skewness, tail heaviness, and the existence of moments, all of which directly influence convergence properties. This diversity allows for a meaningful comparison between ideal theoretical conditions and more challenging, non-ideal cases that may arise in practice.

The existence of finite mean and variance plays a central role in the theoretical validity of SLLN and CLT. While SLLN requires only the existence of the expected value, CLT additionally requires finite variance. By selecting distributions that satisfy, partially satisfy, or violate these assumptions, the limits of both theorems can be systematically examined through simulation. This approach makes it possible to clearly identify which assumptions are essential for each theorem and how their violation affects empirical convergence behavior.

### 2.1 Uniform Distribution U(0,1)

$$f(x) = 1, 0 \leq x \leq 1$$
$$E[X] = 0.5, \mathrm{Var}(X) = \frac{1}{12}$$

The Uniform distribution is bounded, symmetric, and has finite moments of all orders. Because extreme values cannot occur, sample averages tend to stabilize quickly. For this reason, the Uniform distribution serves as a benchmark case in this study and provides a clear reference for ideal convergence behavior under both SLLN and CLT.

### 2.2 Exponential Distribution (λ = 1)

$$f(x) = e^{-x}, x \geq 0$$
$$E[X] = 1, \mathrm{Var}(X) = 1$$

The Exponential distribution is right-skewed and unbounded but remains light-tailed, with all moments finite. Compared to the Uniform distribution, larger fluctuations in sample averages are expected at small sample sizes due to skewness. Nevertheless, both SLLN and CLT are theoretically valid, allowing the effect of asymmetry on convergence speed to be observed.

### 2.3 Pareto Distribution (α = 3)

$$f(x) = \frac{3}{x^4}, x \geq 1$$
$$E[X] = 1.5, \mathrm{Var}(X) = 0.75$$

The Pareto distribution with α = 3 is heavy-tailed, meaning that large observations occur with non-negligible probability. Although both the mean and variance are finite, extreme values can significantly influence the sample mean. This distribution is particularly useful for illustrating how heavy tails slow down convergence even when theoretical assumptions are satisfied.

### 2.4 Pareto Distribution (α = 1.5)

$$f(x) = \frac{1.5}{x^{2.5}}, x \geq 1$$
$$E[X] = 3, \mathrm{Var}(X) = \infty$$

For α = 1.5, the Pareto distribution exhibits extremely heavy tails. While the expected value exists, the variance is infinite, violating a key assumption of CLT. This distribution provides an important example in which SLLN is theoretically valid, but CLT fails, highlighting the fundamental difference between the two theorems.

### 2.5 Cauchy Distribution

$$f(x) = \frac{1}{\pi(1 + x^2)}, x \in \mathbb{R}$$

The Cauchy distribution represents an extreme case of heavy-tailed behavior. Neither the expected value nor the variance exists, making it unsuitable for both SLLN and CLT. This distribution serves as a clear counterexample and emphasizes the necessity of checking theoretical assumptions before applying probabilistic results.

## 3. METHODOLOGY

Monte Carlo simulation is used as the primary methodological approach in this project, as it allows abstract convergence concepts to be examined empirically. All random variables are generated independently and according to their specified distributions, ensuring that the independence and identical distribution assumptions required by both the Strong Law of Large Numbers and the Central Limit Theorem are satisfied.

For the SLLN analysis, a single long sequence of observations is generated for each distribution, with the sample size chosen to be sufficiently large ($n \geq 10{,}000$). The cumulative sample mean is computed at each step and plotted as a function of the number of observations. This representation provides a clear visualization of almost sure convergence and allows differences in convergence speed and stability across distributions to be directly observed.

For the CLT analysis, repeated sampling is required in order to study convergence in distribution. For each distribution and each selected sample size ($n = 5$, 30, and 100), $m = 1000$ independent replications are generated. The resulting sums are standardized using the theoretical mean and variance when they exist. In cases where the variance does not exist, the standardized sums are still examined to illustrate the failure of the normal approximation.

Histograms of the standardized sums and corresponding Normal Q-Q plots are then used to evaluate convergence toward the standard normal distribution. These complementary visualization tools make it possible to assess both the central behavior and tail behavior of the distributions, providing deeper insight into the rate and quality of convergence. This methodology enables a direct comparison between theoretical expectations and empirical outcomes across all distributions considered in the study.

## 4. RESULTS: STRONG LAW OF LARGE NUMBERS (SLLN)

In this section, the Strong Law of Large Numbers is examined for each distribution using cumulative sample mean plots. The objective is to observe whether the sample mean converges to the theoretical expected value and to assess the stability and speed of this convergence. Differences in convergence behavior are interpreted in relation to distributional properties such as boundedness, skewness, and tail heaviness.

### 4.1 Uniform Distribution

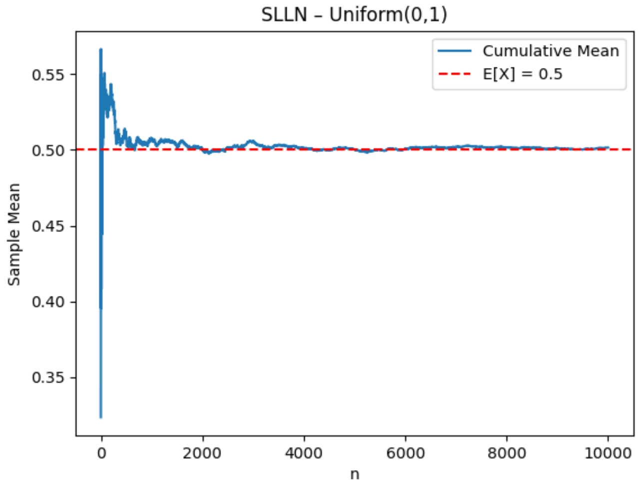


***Figure 1:*** *Cumulative sample mean plot for the Uniform(0,1) distribution.*

The cumulative mean for the Uniform(0,1) distribution converges rapidly and smoothly to the theoretical mean value of 0.5. Initial fluctuations are visible for small sample sizes, but these diminish quickly as the number of observations increases. After a relatively small number of samples, the running mean stabilizes and remains close to the expected value.

This behavior represents a clear empirical demonstration of almost sure convergence. Because the Uniform distribution is bounded, extreme values cannot occur, leading to fast and stable convergence of the sample mean.

### 4.2 Exponential Distribution

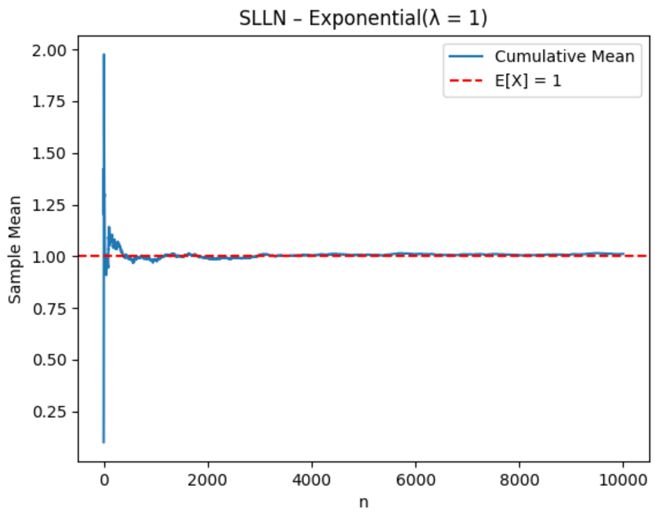


***Figure 2:*** *Cumulative sample mean plot for the Exponential($\lambda$ = 1) distribution.*

For the Exponential($\lambda = 1$) distribution, the cumulative mean converges to the theoretical mean of 1. Compared to the Uniform case, the convergence path exhibits larger fluctuations, particularly at early stages of the simulation. These fluctuations are caused by the right-skewed nature of the distribution, where occasional large observations temporarily increase the sample mean.

Despite this variability, the cumulative mean eventually stabilizes, confirming that SLLN holds for skewed distributions as long as the expected value exists. This result illustrates that skewness affects the rate of convergence but not its validity.

### 4.3 Pareto Distribution ($\alpha = 3$)

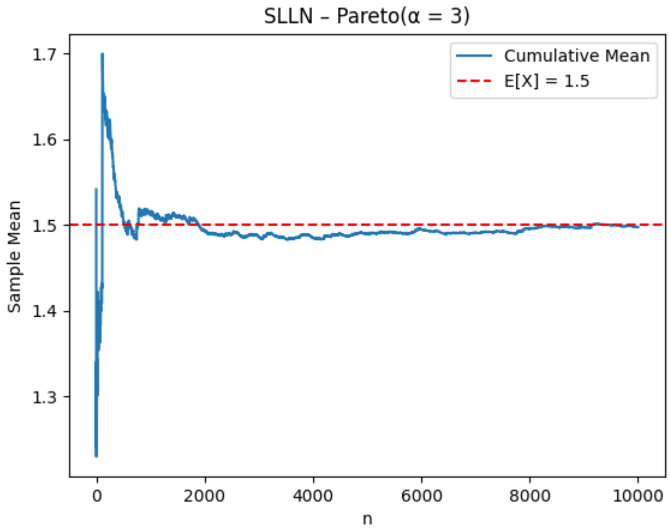


***Figure 3:*** *Cumulative sample mean plot for the Pareto distribution with $\alpha = 3$.*

The cumulative mean for the Pareto distribution with $\alpha = 3$ converges toward the theoretical mean of 1.5 but does so in an irregular manner. Even at large sample sizes, sudden jumps in the running mean are observed due to rare but extremely large values.

Although both the mean and variance of this distribution are finite, its heavy-tailed nature significantly slows down convergence. This example highlights the practical limitation of SLLN: while convergence is guaranteed theoretically, it may be unstable and require very large sample sizes in practice.

### 4.4 Pareto Distribution (α = 1.5)

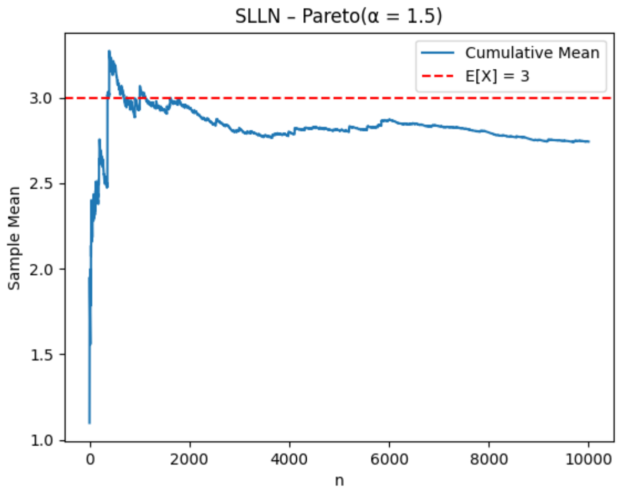


***Figure 4:*** *Cumulative sample mean plot for the Pareto distribution with α = 1.5.*

For α = 1.5, the Pareto distribution exhibits highly unstable behavior. The cumulative mean fluctuates dramatically throughout the simulation and does not settle within a narrow range, even for large numbers of observations.

Although the expected value exists, the infinite variance causes extreme observations to dominate the sample mean for extended periods. This result demonstrates that SLLN may hold in theory but be of limited practical usefulness when convergence is extremely slow or erratic.

### 4.5 Cauchy Distribution

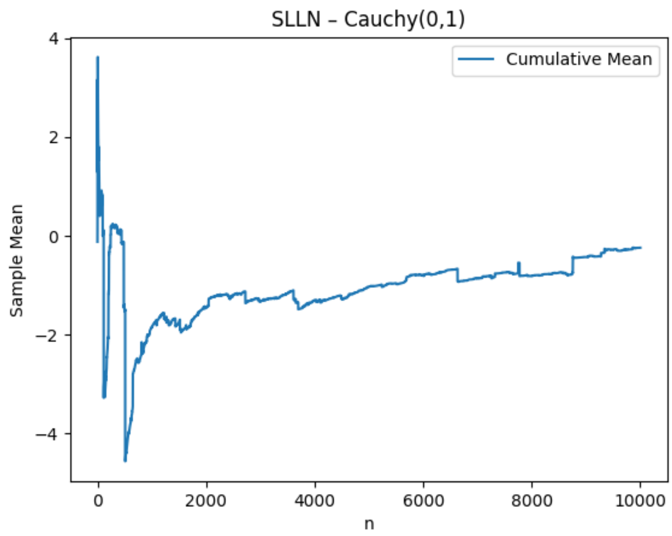


***Figure 5:*** *Cumulative sample mean plot for the Cauchy distribution.*

The cumulative mean for the Cauchy distribution shows no sign of convergence. Large and persistent fluctuations are observed throughout the simulation, with no stabilization around a fixed value.

This behavior directly reflects the fact that the Cauchy distribution has no defined expected value. Since the fundamental assumption of SLLN is violated, the theorem does not apply, making the Cauchy distribution a clear counterexample.

## 5. RESULTS: CENTRAL LIMIT THEOREM (CLT)

This section examines the validity of the Central Limit Theorem for each distribution by analyzing the distributions of standardized sums. Histograms and Normal Q-Q plots are used to assess convergence toward the standard normal distribution for increasing sample sizes.

### 5.1 Uniform Distribution

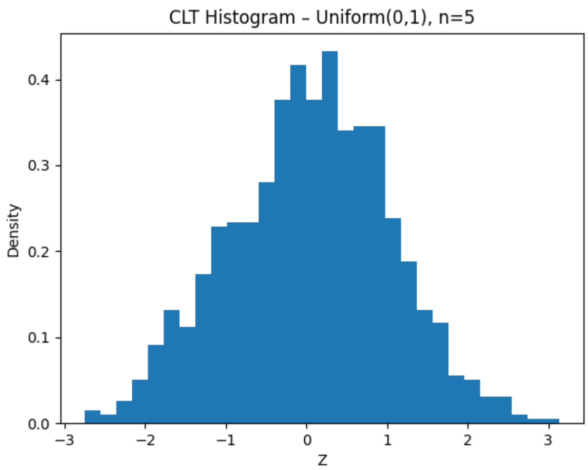


***Figure 6:*** *Histogram of standardized sums for the Uniform(0,1) distribution with n = 5.*

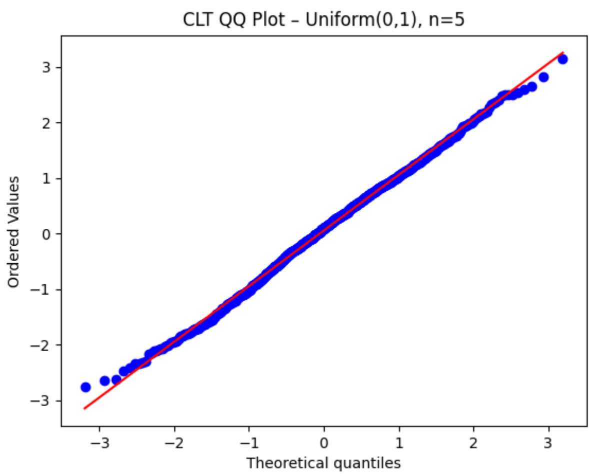


***Figure 7:*** *Normal Q-Q plot of standardized sums for the Uniform(0,1) distribution with n = 5.*

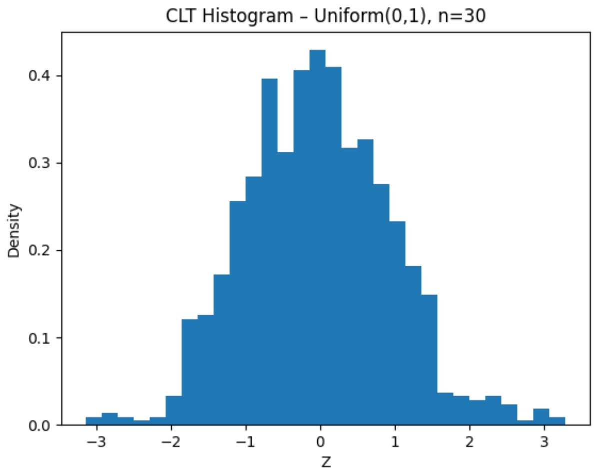


***Figure 8:*** *Histogram of standardized sums for the Uniform(0,1) distribution with n = 30.*

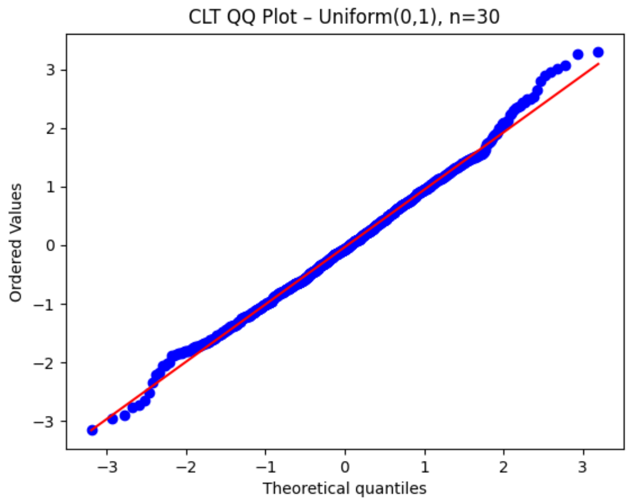


***Figure 9:*** *Normal Q-Q plot of standardized sums for the Uniform(0,1) distribution with n = 30.*

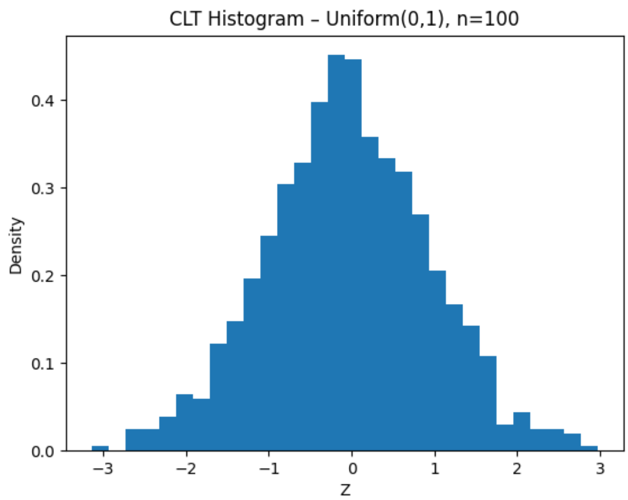


***Figure 10:*** *Histogram of standardized sums for the Uniform(0,1) distribution with n = 100.*

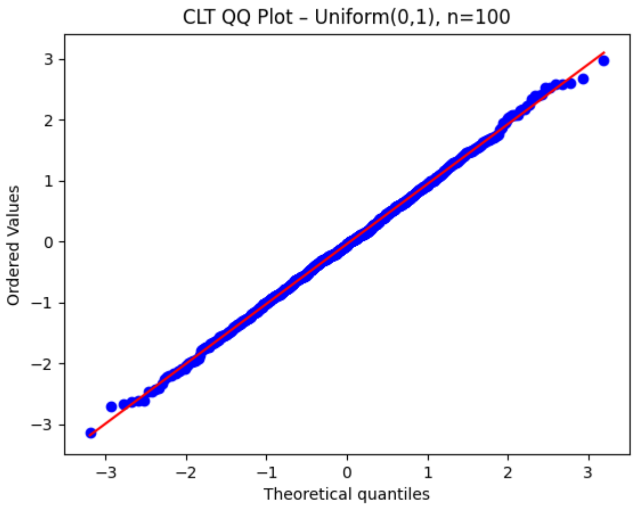


***Figure 11:*** *Normal Q-Q plot of standardized sums for the Uniform(0,1) distribution with n = 100.*

For small sample sizes (n = 5), the histogram of standardized sums deviates noticeably from the normal distribution, and the Q-Q plot shows curvature at the tails. As the sample size increases to n = 30 and n = 100, the histogram becomes increasingly symmetric and closely matches the standard normal density.

The Q-Q plots confirm this convergence, with points aligning closely along the reference line for larger n. This behavior represents the classical form of CLT under ideal conditions.

### 5.2 Exponential Distribution

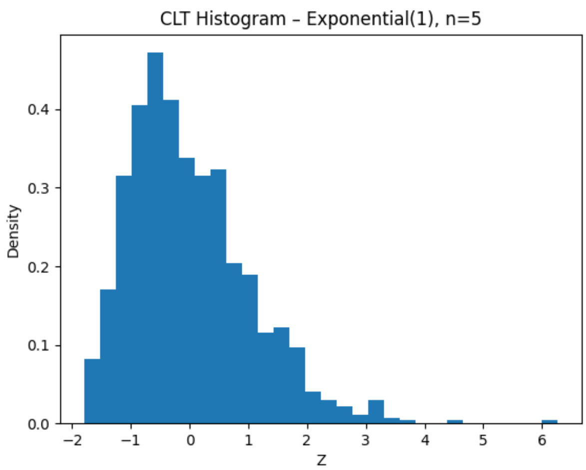


***Figure 12:*** *Histogram of standardized sums for the Exponential(λ = 1) distribution with n = 5.*

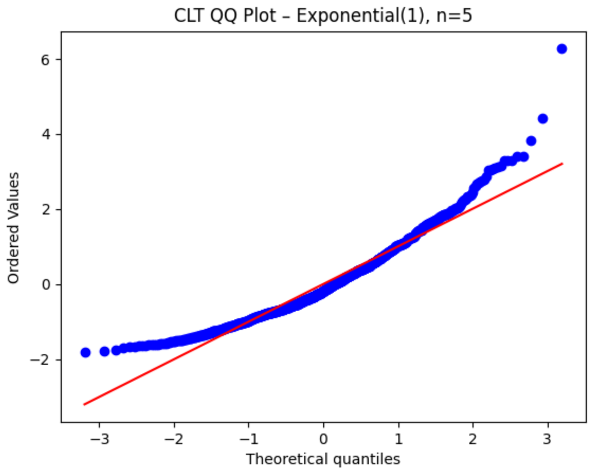


***Figure 13:*** *Normal Q-Q plot of standardized sums for the Exponential(λ = 1) distribution with n = 5.*

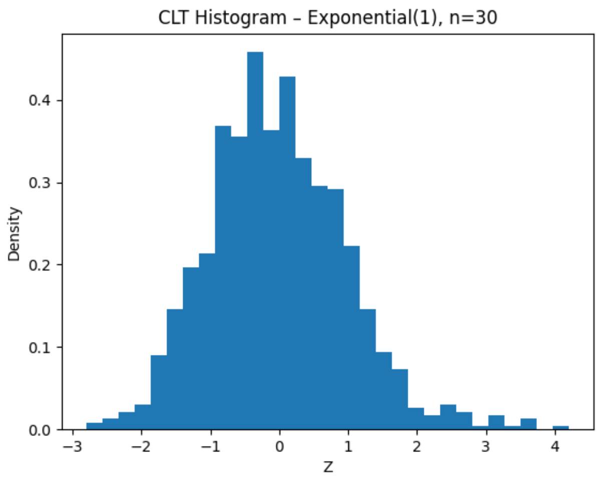


***Figure 14:*** *Histogram of standardized sums for the Exponential(λ = 1) distribution with n = 30.*

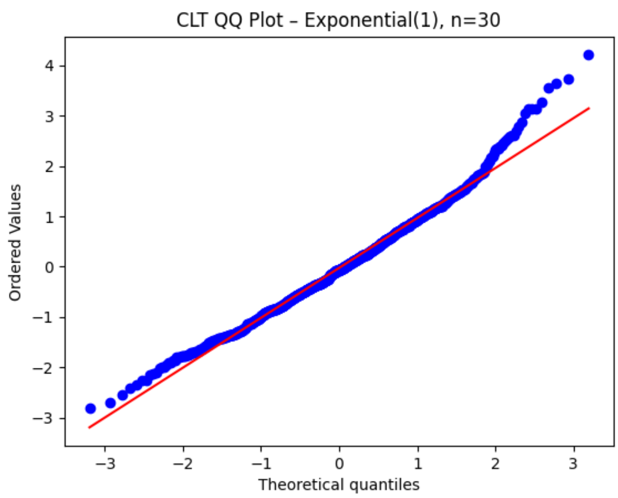


***Figure 15:*** *Normal Q-Q plot of standardized sums for the Exponential(λ = 1) distribution with n = 30.*

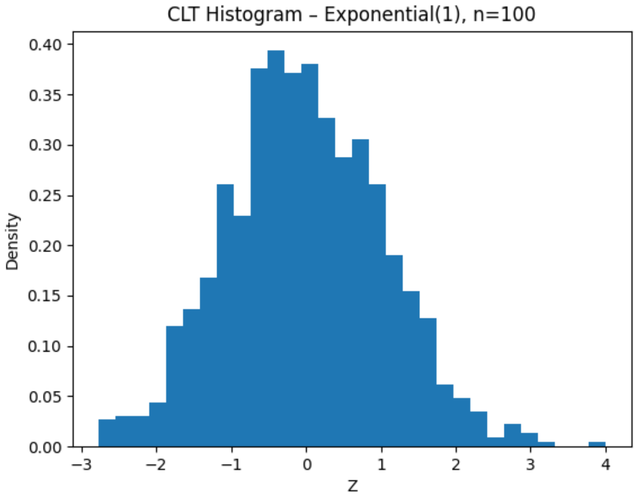


***Figure 16:*** *Histogram of standardized sums for the Exponential(λ = 1) distribution with n = 100.*

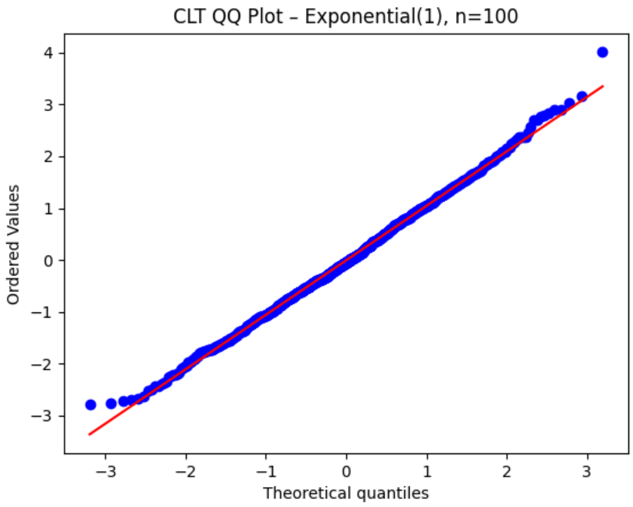


***Figure 17:*** *Normal Q-Q plot of standardized sums for the Exponential(λ = 1) distribution with n = 100.*

Due to skewness, convergence to normality is slower for the Exponential distribution. At n = 5, the histogram remains visibly skewed, and the Q-Q plot shows deviations from linearity, particularly in the upper tail.

As the sample size increases, the effect of skewness diminishes. For n = 100, the standardized sums closely resemble a normal distribution, confirming that CLT holds despite asymmetry when variance is finite.

### 5.3 Pareto Distribution (α = 3)

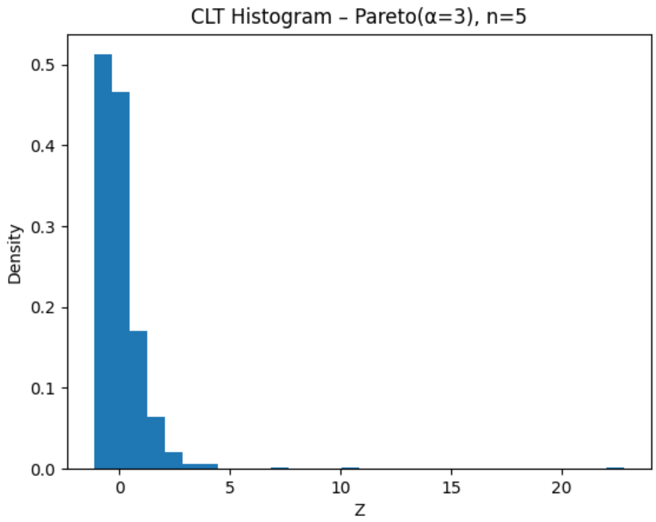


***Figure 18:*** *Histogram of standardized sums for the Pareto distribution with α = 3 and n = 5.*

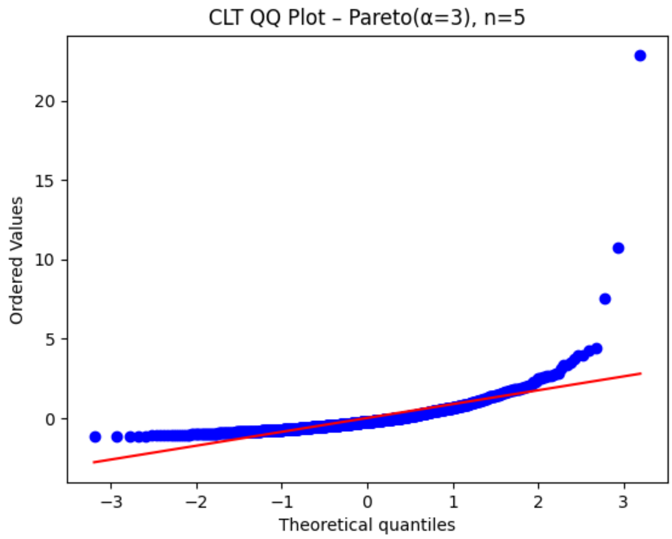


***Figure 19:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 3 and n = 5.*

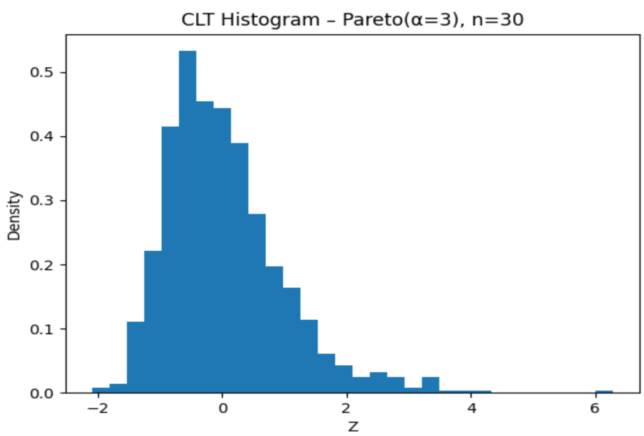


***Figure 20:*** *Histogram of standardized sums for the Pareto distribution with α = 3 and n = 30.*

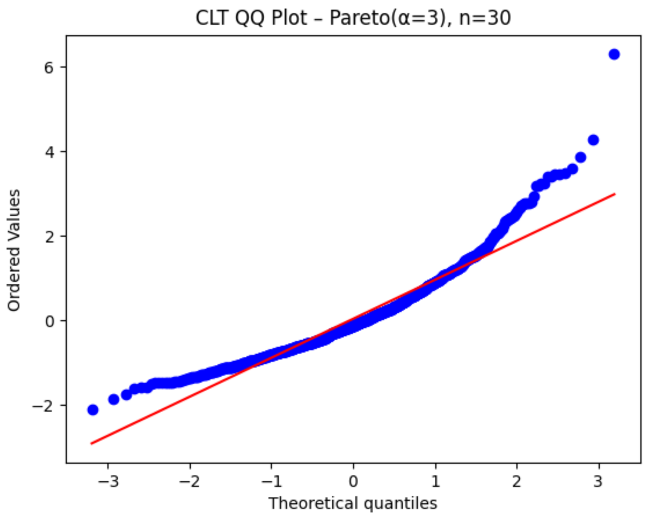


***Figure 21:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 3 and n = 30.*

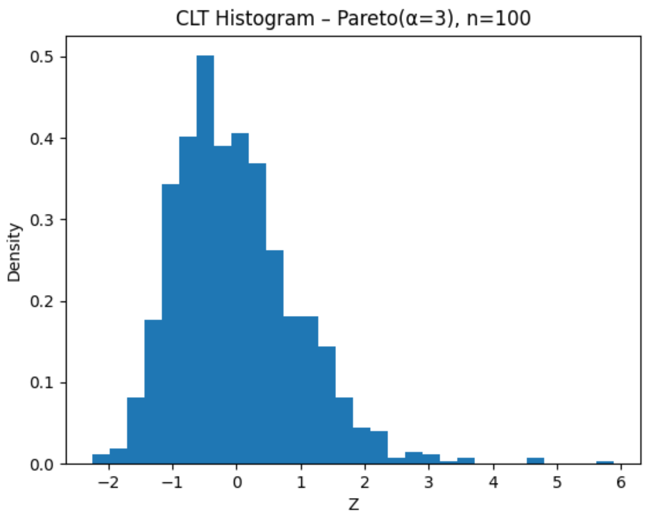


***Figure 22:*** *Histogram of standardized sums for the Pareto distribution with α = 3 and n = 100.*

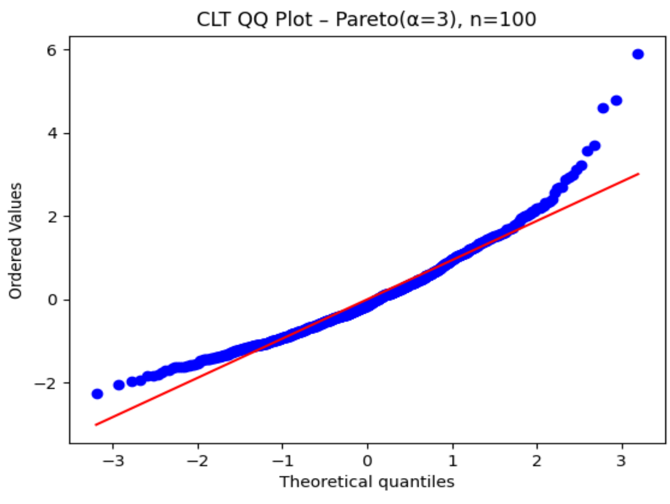


***Figure 23:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 3 and n = 100.*

For the Pareto distribution with α = 3, convergence to normality is noticeably slower. Even at n = 30, the histogram exhibits heavy tails, and the Q-Q plot shows deviations at the extremes.

At n = 100, the central part of the distribution becomes approximately normal, but tail deviations persist. This indicates that heavy-tailed distributions require much larger sample sizes for the normal approximation to become reliable.

### 5.4 Pareto Distribution (α = 1.5)

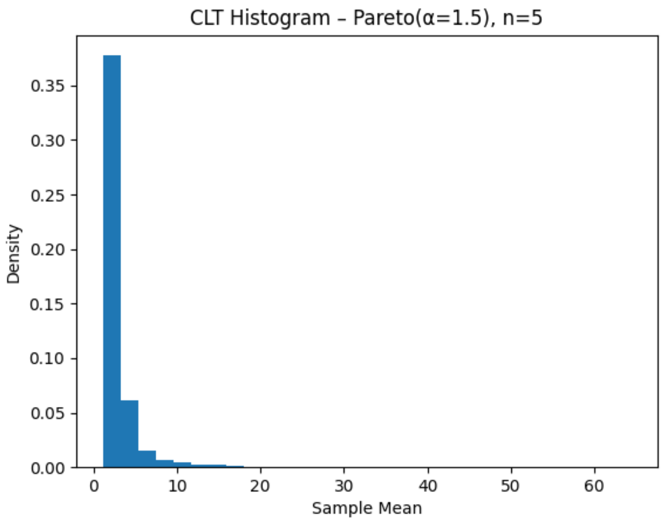


***Figure 24:*** *Histogram of standardized sums for the Pareto distribution with α = 1.5 and n = 5.*

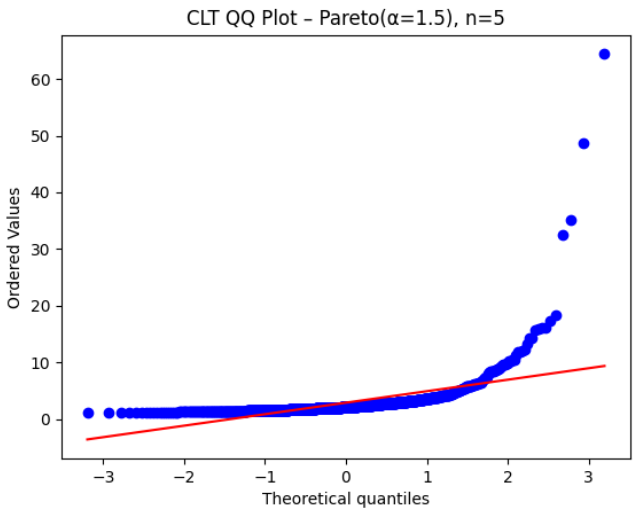


***Figure 25:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 1.5 and n = 5.*

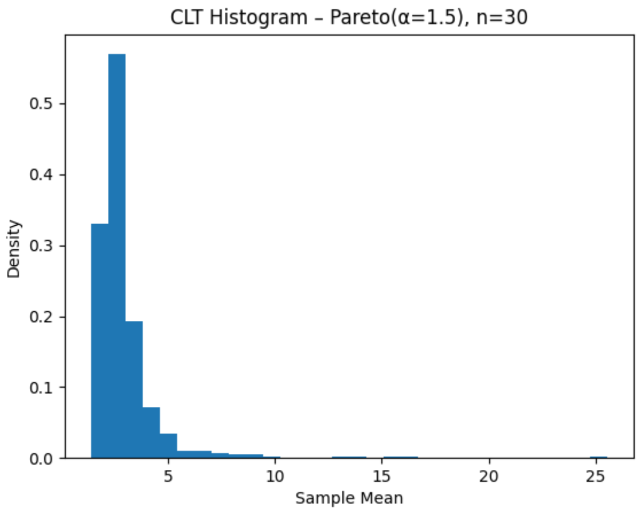


***Figure 26:*** *Histogram of standardized sums for the Pareto distribution with α = 1.5 and n = 30.*

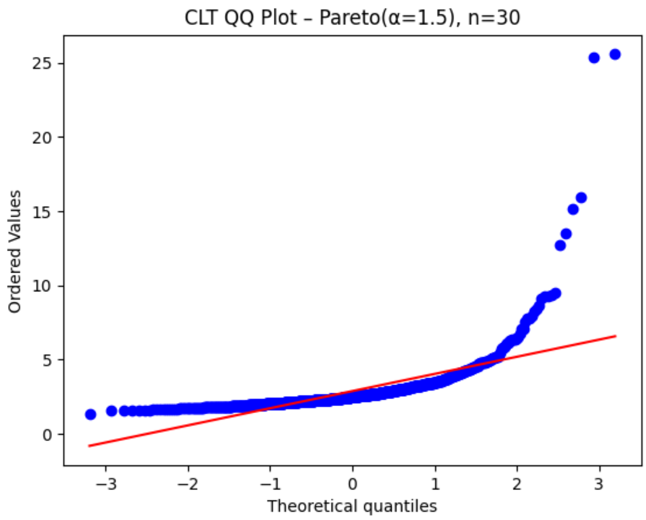


***Figure 27:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 1.5 and n = 30.*

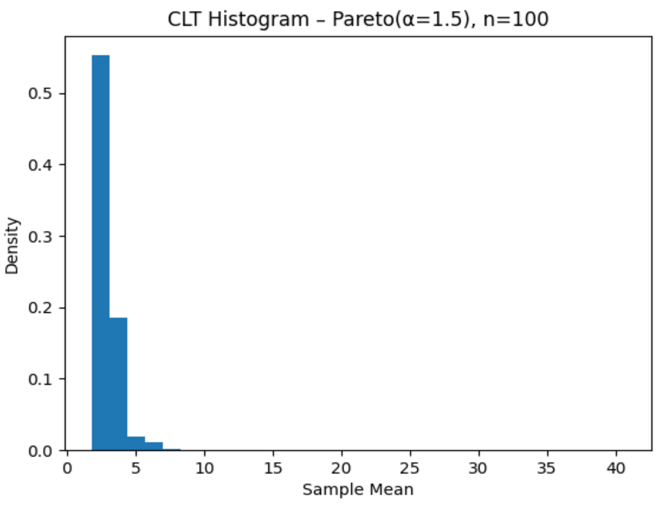


***Figure 28:*** *Histogram of standardized sums for the Pareto distribution with α = 1.5 and n = 100.*

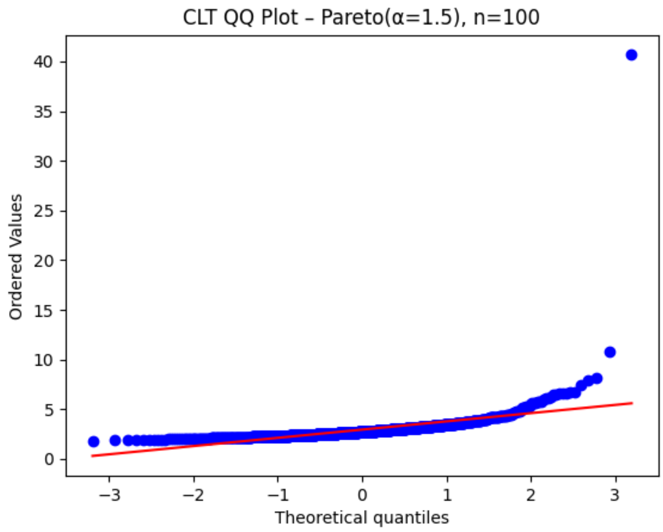


***Figure 29:*** *Normal Q-Q plot of standardized sums for the Pareto distribution with α = 1.5 and n = 100.*

The standardized sums for the Pareto distribution with α = 1.5 do not converge to a normal distribution for any sample size considered. The histograms remain irregular, and the Q-Q plots show severe departures from linearity across the entire range.

This behavior is explained by the infinite variance of the distribution, which violates a key assumption of CLT. As a result, the normal approximation fails completely.

### 5.5 Cauchy Distribution

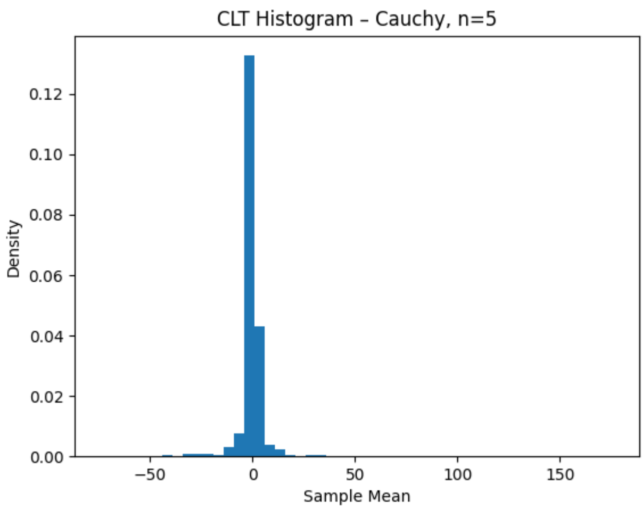


***Figure 30:*** *Histogram of standardized sums for the Cauchy distribution with n = 5.*

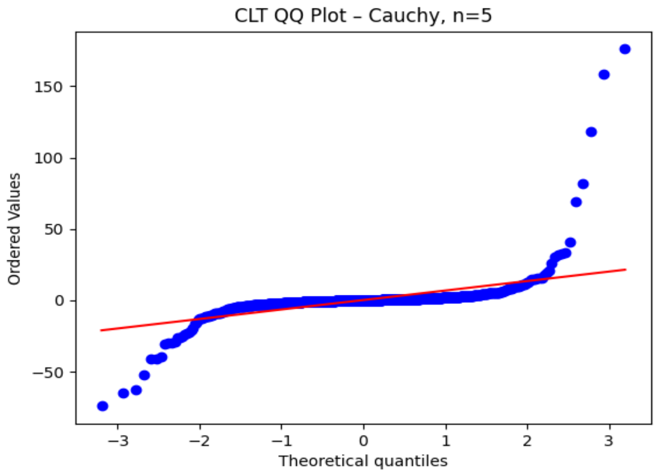


***Figure 31:*** *Normal Q-Q plot of standardized sums for the Cauchy distribution with n = 5.*

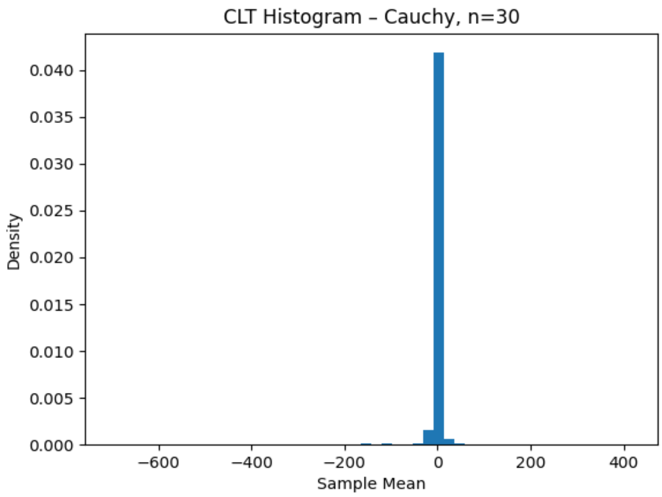


***Figure 32:*** *Histogram of standardized sums for the Cauchy distribution with n = 30.*

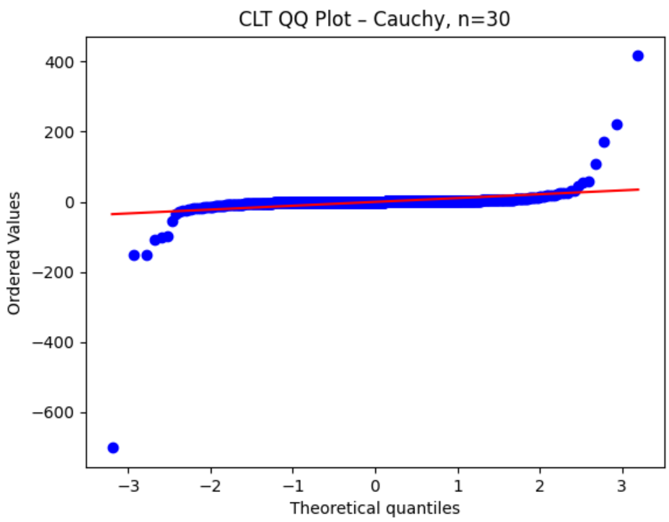


***Figure 33****: Normal Q-Q plot of standardized sums for the Cauchy distribution with n = 30.*

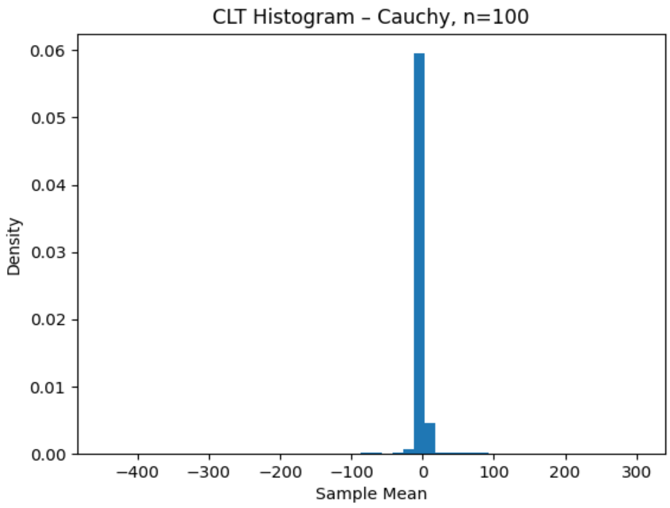


***Figure 34:*** *Histogram of standardized sums for the Cauchy distribution with n = 100.*

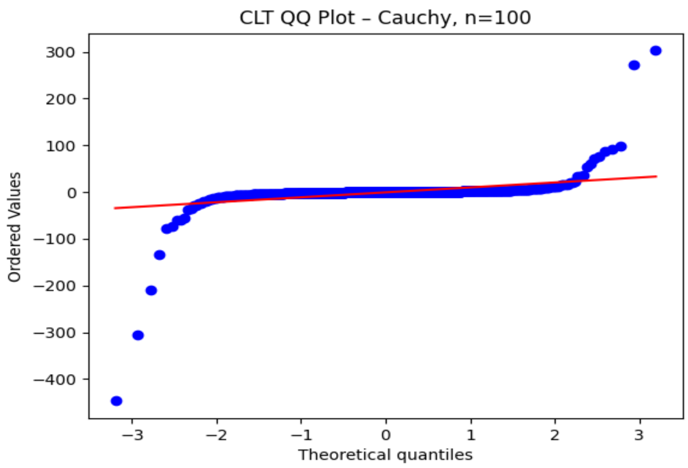


***Figure 35:*** *Normal Q-Q plot of standardized sums for the Cauchy distribution with n = 100.*

For the Cauchy distribution, neither the histograms nor the Q-Q plots show any indication of convergence to normality. Extreme variability persists even for large sample sizes, and the Q-Q plots display dramatic deviations from the reference line.

This confirms that CLT does not apply when neither the mean nor the variance exists, making the Cauchy distribution an extreme counterexample.

## 6. CONCLUSION

This project experimentally examined the Strong Law of Large Numbers and the Central Limit Theorem using Monte Carlo simulation across a variety of probability distributions. By analyzing distributions with different tail behaviors and moment properties, the study demonstrated that convergence behavior depends critically on whether the theoretical assumptions of each theorem are satisfied.

The results show that the Strong Law of Large Numbers holds whenever the expected value exists; however, the speed and stability of convergence vary significantly across distributions. For bounded and light-tailed distributions such as the Uniform and Exponential distributions, convergence is rapid and stable. In contrast, for heavy-tailed distributions such as the Pareto distribution, convergence becomes slower and more irregular, even when the mean is finite. In the case of the Cauchy distribution, where the expected value does not exist, the sample mean fails to converge entirely.

The Central Limit Theorem was observed to require stricter conditions than SLLN. Distributions with finite variance exhibited convergence toward the normal distribution as sample size increased, although the rate of convergence differed depending on skewness and tail heaviness. For distributions with infinite variance, such as the Pareto distribution with $\alpha = 1.5$ and the Cauchy distribution, no convergence to normality was observed, confirming the necessity of the finite variance assumption.

A key insight of this study is that SLLN and CLT describe fundamentally different types of convergence. While SLLN concerns almost sure convergence along individual sample paths, CLT describes convergence in distribution across repeated experiments. This distinction explains why SLLN may hold while CLT fails for certain distributions, and highlights the importance of selecting appropriate analytical tools based on the underlying data-generating process.

Overall, the findings emphasize that theoretical probabilistic results should not be applied blindly in practice. Before relying on sample averages or normal approximations, it is essential to verify whether the assumptions of the underlying theorems are satisfied. Monte Carlo simulation proves to be an effective approach for revealing both the power and the limitations of classical results in probability theory.