Hyperbolic embedding's curvature computation proposal

Yaroslav Abramov

1 Idea

Let's sample small enough balls (of small radius p) in our data. Their volume in hyperbolic geometry is proportional to $p^2/c^{(n-2)/a}$ (see Appendix) and on the same time is proportional to density of distribution multilpied by number of points inside a ball. Hence we can sample curvature from this notion.

2 Appendix: volume of a ball in hyperbolic space

Denote by $Vol(B_R^{H^n})$ volume of a radius R ball in n-dimensional hyperbolic space and by $Vol(B_R^{E^n})$ volume of a radius R ball in n-dimensional Euclidean space.

Let's compute volume of a ball of radius R in H^n. Consider as a model for hyperbolic space Poincare ball model of radius a, then sectional curvature equals to $\frac{-1}{a^2}$ and metric tensor equals to $ds^2 = 4\frac{\sum_i dx_i^2}{(1-||x||^2/a^2)^2}$.

Put our ball of radius R in center of our model. Then in this model it will become a ball with radius $\rho(R)$. Let's first compute $\rho(R)$. We know that lengths of lines are integrals of $\sqrt{ds^2}$, so

$$R = \int_0^1 \frac{2\rho(R)dt}{1-\frac{\rho(R)^2t^2}{a^2}} = \rho(R)\int_0^1 (\frac{1}{1-\rho(R)t/a} + \frac{1}{1+\rho(R)t/a})dt =$$

$$= a(-\ln(1-\rho(R)t/a) + \ln(1+\rho(R)t/a) \mid_{t=0}^{t=1} = a\ln(\frac{1+\rho(R)/a}{1-\rho(R)/a}),$$

hence $(1-\rho(R)/a)\exp(R/a) = (1+\rho(R)/a)$ and

$$\rho(R) = a\frac{1+\exp(R/a)}{1-\exp(R/a)}.$$

Now let's compute volume by formula $\int_\Omega \sqrt{\det g_{ij}}dx_1\dots dx_n$, where $ds^2 = \sum_{i,j} g_{ij}dx_idx_j$. It equals to

$$Vol(B_R^{H^n}) = \int_{B_{\rho(R)}(0)} \frac{2^n dx_1\dots dx_n}{(1-||x||^2/a^2)^n},$$

and after change into polar coordinates

$$x_i = r\sin\psi_1\dots\sin\psi_{i-2}\cos\psi_{i-1},$$

where $\det Jacobian = \det(\frac{\partial x_i}{\partial r}, \frac{\partial x_i}{\partial \psi_j}) = (-1)^{n-2}r^{n-1}\sin^{n-2}\psi_1\sin^{n-2}\psi_2\dots\sin\psi_{n-1}$, we have:

$$Vol(B_R^{H^n}) = \int_0^{\rho(R)} \frac{2^n r^{n-1}dr}{(1-r^2/a^2)^n}\int_0^{pi} d\psi_1\int_0^{2\pi} d\psi_2\dots\int_0^{2\pi} d\psi_{n-1}|\sin^{n-2}\psi_1\sin^{n-2}\psi_2\dots\sin\psi_{n-1}| =$$

$$= nVol(B_1^{E^n})\int_0^{\rho(R)} \frac{2^n r^{n-1}dr}{(1-r^2/a^2)^n} = n2^nVol(B_1^{E^n})\int_0^{\rho(R)} \frac{r^{n-1}dr}{(1-r^2/a^2)^n} =$$

$$= C_n\int_0^{\rho(R)} dr\frac{r^{n-1}}{(1-r^2/a^2)^n} = C_na^n\int_0^{\rho(R)} d(r/a)\frac{(r/a)^{n-1}}{(1-r^2/a^2)^n} =$$

$$= C_na^n\int_0^{\rho(R)/a} dl\frac{l^{n-1}}{(1-l^2)^n} = C_na^n/4\int_0^{\rho(R)/a} d(\frac{1}{1-l} + \frac{1}{1+l})\frac{l^{n-2}}{(1-l^2)^{n-2}} =$$

$$= C_n a^n/2^n \int_0^{\rho(R)/a} d(\frac{1}{1-l} + \frac{1}{1+l})(\frac{1}{1-l} - \frac{1}{1+l})^{n-2} =$$

$$= C_n a^n/2^n \int_0^{\rho(R)/a} d(\frac{1}{1-l} + \frac{1}{1+l})((\frac{1}{1-l} + \frac{1}{1+l})^2 - \frac{4}{(1-l)(1+l)})^{(n-2)/2} =$$

$$= C_n a^n/2^n \int_0^{\rho(R)/a} d(\frac{1}{1-l} + \frac{1}{1+l})((\frac{1}{1-l} + \frac{1}{1+l})^2 - 2(\frac{1}{1-l} + \frac{1}{1+l}))^{(n-2)/2} =$$

$$= C_n a^n/2^n \int\limits_2^{\frac{2}{1-\rho(R)^2/a^2}} ds(s^2-2s)^{(n-2)/2} = C_n a^n/2^n \int\limits_2^{\frac{2}{1-\rho(R)^2/a^2}} d(s-1)((s-1)^2-1)^{(n-2)/2} =$$

$$= C_n a^n/2^n \int\limits_1^{\frac{2}{1-\rho(R)^2/a^2}-1} dt(t^2-1)^{(n-2)/2} = C_n a^n/2^n \int\limits_0^{acosh(\frac{2}{1-\rho(R)^2/a^2}-1)} d(coshu)(cosh^2u-1)^{(n-2)/2} =$$

$$= C_n a^n/2^n \int\limits_0^{acosh(\frac{1+\rho(R)^2/a^2}{1-\rho(R)^2/a^2})} du(sinhu)^{n-1} = C_n a^n/2^n \int\limits_0^{acosh(\frac{1+\exp(2R/a)}{2\exp(R/a)})} du((\exp(u)-\exp(-u))/2)^{n-1} =$$

$$= D_n a^n \int_0^{R/a} du((\exp(u)-\exp(-u))/2)^{n-1} =$$

(by Newton-Leibnitz)

$$= a^n\Theta(R/a((\exp(R/a)-\exp(-R/a))/2)^{n-1}) =$$

(by Taylor)

$$= a^n\Theta(R^2/a^2) = a^{n-2}\Theta(R^2)$$

as asymptotics on small R

3 Computing probability that 2 points inside a hyperbolic ball are close enough

We need first to compute the volume of a "lense" $\{x = (x_1, x_2, \dots, x_n) \in B_R^{H^n}(0) \mid x_1 > b \, in \, Poincare \, model\}$.

This volume equals to

$$\int_b^{\rho(R)} dx_1 \int_{-\sqrt{\rho(R)^2-x_1^2}}^{+\sqrt{\rho(R)^2-x_1^2}} dx_2 \int_{-\sqrt{\rho(R)^2-x_1^2-x_2^2}}^{+\sqrt{\rho(R)^2-x_1^2-x_2^2}} dx_3 \dots \int_{-\sqrt{\rho(R)^2-x_1^2-\dots-x_{n-1}^2}}^{+\sqrt{\rho(R)^2-x_1^2-\dots-x_{n-1}^2}} dx_n \frac{2^n}{(1-|x|^2/a^2)^n}$$

Again, let's make a change of coordinates, but "only partly polar": if $i > 1$ then

$$x_i = L \sin\varphi_2 \dots \sin\varphi_{i-2}\cos\varphi_{i-1}$$

Then that volume equals to

$$\int_b^{\rho(R)} dx_1 \int_0^{\sqrt{\rho(R)^2-x_1^2}} dL \int_0^{2\pi} d\varphi_2 \int_0^{2\pi} d\varphi_3 \dots \int_0^{2\pi} d\varphi_{n-1} |\sin^{n-2}\varphi_2 \sin^{n-3}\varphi_3 \dots \sin\varphi_{n-1}| \frac{2^n L^{n-2}}{(1-\frac{x_1^2+L^2}{a^2})^n} =$$

$$= (n-1)Vol(B_1^{E^{n-1}}) \int\limits_b^{\rho(R)} dx_1 \int\limits_0^{\sqrt{\rho(R)^2-x_1^2}} \frac{2^n L^{n-2} dL}{(1-\frac{x_1^2+L^2}{a^2})^n} =$$

$$= (n-1)2^n Vol(B_1^{E^{n-1}})a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} dy_1 \int\limits_0^{\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} \frac{S^{n-2}dS}{(1-y_1^2-S^2)^n} =$$

(Here C_{n-1} is the same as in formulas for hyperbolic volume in previous section)

$$= 2C_{n-1}a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^n} \int\limits_0^{\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} \frac{S^{n-2}dS}{(1-(\frac{S}{\sqrt{1-y_1^2}})^2)^n} =$$

$$= 2C_{n-1}a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} \frac{T^{n-2}dT}{(1-T^2)^n} =$$

$$= 2C_{n-1}a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} \frac{T^{n-3}}{(1-T^2)^{n-3}} \frac{TdT}{(1-T^2)^3} =$$

$$= C_{n-1}a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} \frac{T^{n-3}}{(1-T^2)^{n-3}} \frac{dT^2}{(1-T^2)^3} =$$

$$= C_{n-1}a^n \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} (\frac{T}{1-T^2})^{n-3} d\frac{-1/2}{(1-T^2)^2} =$$

$$= \frac{-C_{n-1}a^n}{8} \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} (\frac{1}{2(1-T)} - \frac{1}{2(1+T)})^{n-3} d(\frac{1}{1-T} + \frac{1}{1+T})^2 =$$

$$= \frac{-C_{n-1}a^n}{2^n} \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_0^{\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}} ((\frac{1}{1-T}+\frac{1}{1+T})^2 - 2(\frac{1}{1-T}+\frac{1}{1+T}))^{\frac{n-3}{2}} d(\frac{1}{1-T}+\frac{1}{1+T})^2 =$$

$$S(\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2}) = \frac{2}{1-(1-y_1^2)((\frac{\rho(R)}{a})^2-y_1^2)} = \frac{2}{1-\frac{\rho(R)^2}{a^2}+y_1^2(1+\frac{\rho(R)^2}{a^2})-y_1^4}$$

$$= \frac{1}{1-(1-y_1^2)((\frac{1+\exp(R/a)}{1-\exp(R/a)})^2-y_1^2)} =$$

$$= \frac{(1-\exp(R/a))^2}{1-(1-y_1^2)((1+\exp(R/a))^2-y_1^2(1-\exp(R/a))^2)} =$$

$$= \frac{1-2\exp(R/a)+\exp(2R/a)}{-y_1^2(1+\exp(R/a))^2-y_1^4(1-\exp(R/a))^2+y_1^2(1-\exp(R/a))^2} =$$

$$= \frac{1-2\exp(R/a)+\exp(2R/a)}{-4y_1^2\exp(R/a)-y_1^4(1-2\exp(R/a)+\exp(2R/a))} = \frac{coshR/a-1}{-2y_1^2-y_1^4(coshR/a-1)}$$

$$= \frac{-C_{n-1}a^n}{2^n} \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_{S(0)}^{S(\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2})} ((S-1)^2-1)^{\frac{n-3}{2}} dS^2 =$$

$$= \frac{-C_{n-1}a^n}{2^n} \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_{0}^{acosh(S(\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2})-1)} (cosh^2U-1)^{\frac{n-3}{2}} d(coshU+1)^2 =$$

$$acosh(S(T)-1) = acosh(\frac{2}{1-T^2}-1) = acosh(\frac{1+T^2}{1-T^2}) = \ln(\frac{1+T^2}{1-T^2} + \sqrt{(\frac{1+T^2}{1-T^2})^2-1}) = \ln(\frac{1+2T+T^2}{1-T^2})$$

$$T = \frac{1-\exp(-f)}{1+\exp(-f)} = tanh(f/2)$$

$$acosh(S(T)-1) = acosh(\frac{1+T^2}{1-T^2}) = acosh(cosh(f)) = f$$

$$f = 2arctanh(T) = \ln(\frac{1+T}{1-T})$$

$$acosh(S(\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2})-1) = acosh(\frac{\frac{\rho(R)^2}{a^2} - y_1^2(1+\frac{\rho(R)^2}{a^2}) + y_1^4}{1-\frac{\rho(R)^2}{a^2} + y_1^2(1+\frac{\rho(R)^2}{a^2}) - y_1^4}) =$$

$$= \ln(\frac{\frac{\rho(R)^2}{a^2} - y_1^2(1+\frac{\rho(R)^2}{a^2}) + y_1^4 + \sqrt{(\frac{\rho(R)^2}{a^2} - y_1^2(1+\frac{\rho(R)^2}{a^2}) + y_1^4)^2 - (1-\frac{\rho(R)^2}{a^2} + y_1^2(1+\frac{\rho(R)^2}{a^2}) - y_1^4)^2}}{1-\frac{\rho(R)^2}{a^2} + y_1^2(1+\frac{\rho(R)^2}{a^2}) - y_1^4}) =$$

$$= \frac{-C_{n-1}a^n}{2^{n-1}} \int\limits_{\frac{b}{a}}^{\frac{\rho(R)}{a}} \frac{dy_1}{(1-y_1^2)^{n/2+1}} \int\limits_{0}^{acosh(S(\sqrt{1-y_1^2}\sqrt{(\frac{\rho(R)}{a})^2-y_1^2})-1)} sinh^{n-2}U(coshU+1)dU =$$

$$m = k\ln(\frac{1+n}{1-n})$$

$$1+n = e^{m/k}(1-n)$$

$$n = \frac{e^{m/k}-1}{e^{m/k}+1}$$

$$\frac{1}{1-n^2} = \frac{cosh(m/k)}{2}$$

$$(1-n^2)-(G^2-n^2) = 1-G^2$$

$$C = acosh(\sqrt{\frac{1-n^2}{1-G^2}})$$

$$T = \frac{1-G^2}{2}cosh2C$$