Hyperbolic spectra of a graph and hyperolic embeddings

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1 general bla-bla-bla

The usual technique of (Euclidean) spectral graph embeddings (as taking eigenvectors of graph Laplacian) is described in Bonald's paper (see here: https://perso.telecom-paristech.fr/bonald/documents/spectral.pdf). However, it doesn't take into account different taxonomies of a graph. Different attempts for making graph embeddings into hyperbolic space (see definition of such thing down the text) was made earlier and can be googled. Our approach is based on taking eigenvectors of a special graph matrix. Some problems and hypotheses are formulated down the text.

2 Definition of hyperbolic space

For quadratic form $\mathbb{F}$ of signature $(n, 1)$ (i.e. n-dimensional eigenvectors-generated subspace with positive eigenvalues of its matrices and 1 eigenvector with negative eigenvalue) generates the following hyperboloid subvariety in $\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n,1}$ for later):

$$\mathbb{F}(x) = 1$$

is called n-dimensional hyperbolic space, its linear transformations are linear transformations of $\mathbb{R}^{n,1}$, preserving cone $\mathbb{F}(x) > 0$. Its' motions are linear transformations of $\mathbb{R}^{n,1}$, preserving values of form $\mathbb{F}$. It's subspaces are intersections of linear subspaces of $\mathbb{R}^{n,1}$ with $\{\mathbb{F}(x) = 1\}$. Distance between points A and B is $\rho(A, B) = \cosh^{-1}(\mathbb{F}(A, B))$, where $\cosh(z) = \frac{e^z + e^{-z}}{2}$ (and 1-dimensional supspaces are geodesic lines, i.e. lines of the shortest lengths between any its' points). Angle $\angle ABC$ is also easily computable (see this construction as Lorentz model).

Example 1. *For* $\mathbb{F}(x_0, x_1, x_2) = -x_0^2 + x_1^2 + x_2^2$ *we have usual hyperboloid*

3 Definition of Graph Hyperbolic Laplacian

Here is an intuition behind.

Intuition. Usual graph Laplacian is symmetric nonnegative-definite matrix and it easily can be interpreted as matrix of some Euclidean metric (for connected graph we even have only one kernel vector). So we add some "negative" vertex and connect it with others by "negative" edges. So, the degree of this additional vertex is negative and for others we have some decreasing of their original degrees. We get some quasi-Laplacian matrix with signature $(V(G) - 1, 1, 1)$ for connected graph (where $V(G)$ is a number of vertices).

Definition. For graph G denote by $L(G)$ matrix of the form $V(G) \times V(G)$ with elements of the form $L(G)_{ij} = -1$ iff vertices v_i and v_j share edge (otherwise $L(G)_{ij} = 0$) and $L(G)_{ii} = \deg v_i$. By *hyperbolic Laplacian* $HypLap(G)_{\alpha_1, \alpha_1, \ldots, \alpha_V}$ for positive α_i's with $\sum_i \alpha_i = 1$ we mean matrix of the form $(V(G) + 1) \times (V(G) + 1)$ and with entries like

$$HypLap(G)_{\alpha_1, \alpha_2, \ldots, \alpha_V} = \begin{pmatrix} -1 & \alpha_1 & \ldots & \alpha_i & \ldots & \alpha_V \\ \alpha_1 & L(G)_{11} - \alpha_1 & \ldots & L(G)_{1i} & \ldots & L(G)_{1V} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_i & L(G)_{i1} & \ldots & L(G)_{ii} - \alpha_i & \ldots & L(G)_{iV} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_V & L(G)_{V1} & \ldots & L(G)_{Vi} & \ldots & L(G)_{VV} - \alpha_V, \end{pmatrix}$$

where $v = V(G)$

Remark. Weights α_i may depend on properties of corresponding vertices v_i. For our first research purposes we will assume equal weights $\alpha_1 = \ldots = \alpha_V = k$ (however, some experiments with adjusting of weights is also interesting).

Lemma. *Matrix* $HypLap(G)_{\alpha_1, \alpha_2, \ldots, \alpha_V}$ *has kernel vector* $(1, 1, 1, \ldots, 1)$.

Proof. Direct computation. □

Theorem. *Assume graph G is connected.*
1) Matrix $HypLap(G)_{\alpha_1,\alpha_2,\dots,\alpha_V}$ has exactly 1-dimensional kernel subspace
2) Form $\mathbb{K}(x) = x^T HypLap(G)_{\alpha_1,\alpha_2,\dots,\alpha_V} x$ has signature $(V-1,1,1)$ (i.e. is representable in the form $y_1^2 + \dots y_{V-1}^2 - y_0^2$ for some linear transfromation $y = Ax$).

Proof. Follows from the proof of Gershgorin Circles Theorem (it is actually it's special case). There are $V-1$ positive eigenvalues and 1 negative □

4 Our construction and its properties

We consider $(V-1)$-dimensional hyperbolic space $\mathbb{K}(x) = x^T HypLap(G)_{\alpha_1,\alpha_2,\dots,\alpha_V} x = 1$ with $x \perp (1,1,\dots,1)$. Eigenvectors $y^{(i)}$ of $HypLap(G)_{\alpha_1,\alpha_2,\dots,\alpha_V}$ with positive eigenvalues form embedding of vertices of graph G with $v_j \mapsto (y_j^{(0)}, y_j^{(1)}, \dots, y_j^{(V)})$, which minimizes $\sum_{i,j|(v_i,v_j)-edge} \cosh(\rho(v_i,v_j))$ (easy comutation, since $\rho(A,B) = \cosh^{-1}\mathbb{K}(A,B)$).

5 Problem directions and formulations

5.1 NLP and other classification problems: graphs of taxonomies

Binary graphs have lot's of vertices and can be embedded in 2-dimensional hyperbolic space with nice properties link. Our hypothesis to check is that after orthogonal projection (in hyperbolic sense!) of binary graphs on some appropriate 2-dimensional subspace won't have any intersecting edges (or will have small enough of them). This direction is proposed by Ivan Oseledets.

5.2 Computer vision: detecting edges of objects on picture

Common edge detection algorithm (called "canny algorithm") on picture uses Laplacian of intensivity. This works not very well in case of shadows on the picture. There is a general technique of spectral clasterization (see works of Francis Bach on this topic). We can apply this modification of Laplacian for checking spectral clasterization on the picture. This project is approved on the Computer Vision course.