1b) Prove by induction that the sum of all elements in the m^{th} row of the triangle is 2^m.

First, we will define $a_{m,n}$ term such $a_{m,n} = a_{m-1,n} + a_{m-1,n-1}$ where m is the row and n the column

$1 = 2^0$
$1+1 = 2^1$
$1+2+1 = 2^2$
$1+3+3+1 = 2^3$
$1+4+6+4+1 = 2^4$

Prove by Induction:

$m=0$

$2^0 = 1$ and the sum of line 0 is indeed 1

Induction step:

Assuming that $a_{m,0}+a_{m,1}+\dots+a_{m,n} = \sum_{k=0}^{n} a_{m,k} = 2^m$ is correct,

we need to prove $a_{m,0}+a_{m,1}+\dots+a_{m,n}+a_{m+1,n} = \sum_{k=0}^{n+1} a_{m+1,k} = 2^{m+1}$

Proof:

$$\sum_{k=0}^{n+1} a_{m+1,k} = a_{m+1,0} + a_{m+1,1} + a_{m+1,2} + \dots + a_{m+1,n} + a_{m+1,n+1}$$

using def of pascal triangle ($a_{i,j} = a_{i-1,j} + a_{i-1,j-1}$) $= a_{m,0} + a_{m,-1} + a_{m,1} + a_{m,0} + a_{m,2} + a_{m,1} + \dots + a_{m,n} + a_{m,n-1} + a_{m,n+1} + a_{m,n}$

simplify: $= a_{m,-1} + 2a_{m,0} + 2a_{m,1} + \dots + 2a_{m,n} + a_{m,n+1}$

simplify $= 0 + 2(a_{m,0} + a_{m,1} + \dots + a_{m,n}) + 0$

$= 2 \cdot \sum_{k=0}^{n} a_{m,k}$

using the induction hypo $= 2 \cdot 2^m = 2^{m+1}$ □

1c) Prove that:

$$(i,0)+(i+1,1)+\dots+(i+j,j)=(i+j+1,j) \quad \forall\, i,j\in\mathbb{N}$$

example $i=1$, $j=3$

$(i,0)+(i+1,1]+(i+2,2)+(i+3,3)=$

$=(1,0)+(1+1,1)+(1+2,2)+(1+3,3)=$

$=(1,0)+(2,1)+(3,2)+(4,3)=$

$= 1 + 2 + 3 + 4 = 10$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Proof by induction: (j is the variable while i is constant)

base case: $j=0$

$(i,0)=1$ for any i, and also

$(i+j+1,j)=(i+0+1,0)=(i+1,0)=1$ for any i. ☺

Induction step: Assuming $\sum_{k=0}^{j}(i+k,k)=(i+j+1,j)$ is true

need to prove $\sum_{k=0}^{j+1}(i+k,k)=(i+j+2,j+1)$

proof: $\sum_{k=0}^{j+1}(i+k,k)=\underbrace{(i,0)+(i+1,1)+\dots+(i+j,j)}+(i+j+1,j+1)=$

using induction $=$ $(i+j+1,j)$ $+$ $(i+j+1,j+1)$

using definition of pascal triangle ($a_{m,n}=a_{m-1,n}+a_{m-1,n-1}$) $=$ $(i+j+2,j+1)$ $\square$