PHYS 600: Homework 4

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Problem 1 Recombination

1. The process we are interested in here is recombination of protons and electrons into neutral hydrogen. i.e., the reaction

$$e^- + p^+ \leftrightarrow \mathrm{H} + \gamma$$

tending to the right as the Universe cools. Recombination occurred at $z \sim 1100$ (around the redshift of the CMB), so we can treat the species as non-relativistic. We then have, for each species of interest,

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$$

where T is the temperature of the baryon-photon fluid with which these particles are all in equilibrium. In order to eliminate chemical potential from the equation, we evaluate:

$$\begin{aligned}
\frac{n_\mathrm{H}}{n_e n_p} &= \frac{g_\mathrm{H}}{g_e g_p}\left(\frac{m_\mathrm{H}}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} e^{\frac{\cancel{\mu_\mathrm{H} - (\mu_e + \mu_p)} - m_\mathrm{H} + m_e + m_p}{T}} \\
\implies \frac{n_\mathrm{H}}{n_e^2} &= \left(\frac{m_\mathrm{H}}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} e^{E_I/T} \\
&= \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{E_I/T} \qquad (1)
\end{aligned}$$

where we have used the fact that $\mu_\mathrm{H} = \mu_e + \mu_p$, $g_\mathrm{H} = 4$, $m_p \approx m_\mathrm{H}$, $g_e = g_p = 2$, $m_e + m_p - m_\mathrm{H} = 13.6\ \mathrm{eV} \equiv E_I$ (the ionization energy of hydrogen), and $n_e = n_p$ in a neutral universe.

Now, we shift our focus to finding an expression for X_e, the free electron fraction defined by $X_e \equiv n_e/(n_p + n_\mathrm{H})$. If we assume that protons and hydrogen comprises all of the baryons in the Universe at this point (i.e., we ignore the abundance of helium), then we have

$$n_p + n_e = n_b = \eta n_\gamma$$

$$= \eta \frac{\zeta(3)}{\pi^2} 2T^3$$

where η is the baryon-to-photon ratio, we have made use of the number density formula for relativistic species, and the fact that $g_\gamma = 2$. But now we can make use of Eq. 1 and write:

$$\begin{aligned}
\frac{1 - X_e}{X_e^2} &= \frac{(1 - n_e)n_b}{n_e^2} \\
&= \frac{n_{\mathrm{H}}}{n_e^2} n_b \\
&= \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{E_I/T},
\end{aligned}$$

the Saha equation.

2. The Saha equation is a quadratic equation for X_e of the form $AX_e + BX_e + C = 0$, with $B = 1$, $C = -1$, and

$$A = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{E_I/T}.$$

We then have, from the quadratic formula

$$X_e = \frac{-1 \pm \sqrt{1 + 4A}}{2A}$$

and we choose the $+$ sign, since X_e must be a real number and $A > 0$ for all T.

To get our expression to be in terms of z, we use the relation $T = T_0(1 + z)$, where T_0 is the temperature of the CMB today. See Fig. 1. The code to generate this plot can be found in the appendix.

3. We use a bisection method to invert our function $X_e(z)$. See the appendix. We find that $\boxed{z(X_e = 0.1) \approx 1259}$ and $\boxed{z(X_e = 0.5) \approx 1377}$. As can be seen from the plot, the transition for a universe with almost all electrons free to one in which almost none are happens over a relatively short span of redshifts.

4. For a single component universe, we have (see lecture notes)

$$\begin{aligned}
t &= \frac{1}{H_0} \frac{2}{3(1 + w)} a^{\frac{3}{2}(1+w)} \\
&= \frac{1}{H_0} \frac{2}{3(1 + w)} \left(\frac{1}{1 + z} \right)^{\frac{3}{2}(1+w)}
\end{aligned}$$

For matter, $w \approx 0$, and we get that $\boxed{t(z = 1259) \approx 208,000 \text{ years}}$ (for $h = 0.7$).

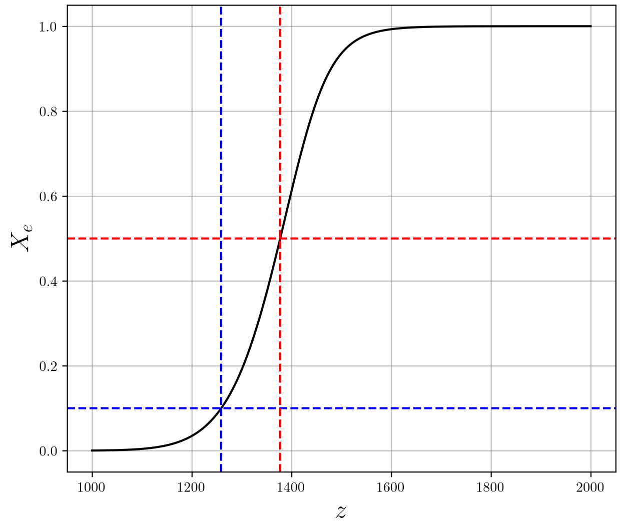

Figure 1: The free electron fraction as a function of redshift around the epoch of reionization. The redshifts at which $X_e = 0.1$ and 0.5 are indicated.

5. We seek to solve

$$\begin{aligned}
\Gamma_T(z) &= H(z) \\
n_e(z)\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} && \text{(matter domination)} \\
X_e(z)n_b(z)\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} \\
X_e(z)\eta\frac{\zeta(3)}{\pi^2}2T_0^3(1+z)^3\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} && (n_b \text{ formula above}) \\
\implies X_e(z)(1+z)^{3/2} &= \frac{H_0\pi^2\sqrt{\Omega_{\mathrm{m},0}}}{2\eta\zeta(3)\sigma_T T_0^3} \\
&\approx 215.592 && \text{(see calculation in appendix)}
\end{aligned}$$

We use a numerical bisection method to invert this equation for z and we find that $\boxed{z_{\text{dec}} \approx 1114}$.

Using the same calculations as above, we get $\boxed{t_{\text{dec}} \approx 250,000 \text{ years}}$ and $X_e(z_{\text{dec}}) \approx \boxed{0.0058}$. In the words of Baumann, "a large degree of neutrality is necessary before the universe becomes transparent to photons."

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Problem 2 "What-if" BBN

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Problem 3 Freeze-in DM

(a) hi

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