PHYS 600: Homework 4

Yarone Tokayer

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Problem 1 Recombination

1. The process we are interested in here is recombination of protons and electrons into neutral hydrogen. i.e., the reaction

$$e^- + p^+ \leftrightarrow \mathrm{H} + \gamma$$

tending to the right as the Universe cools. Recombination occurred at $z \sim 1100$ (around the redshift of the CMB), so we can treat the species as non-relativistic. We then have, for each species of interest, i,

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}}$$

where T is the temperature of the baryon-photon fluid with which these particles are all in equilibrium. In order to eliminate chemical potential from the equation, we evaluate:

$$\begin{aligned}
\frac{n_\mathrm{H}}{n_e n_p} &= \frac{g_\mathrm{H}}{g_e g_p}\left(\frac{m_\mathrm{H}}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} e^{\frac{\cancel{\mu_\mathrm{H}} - \cancel{(\mu_e + \mu_p)} - m_\mathrm{H} + m_e + m_p}{T}} \\
\implies \frac{n_\mathrm{H}}{n_e^2} &= \left(\frac{m_\mathrm{H}}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} e^{E_I/T} \\
&= \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{E_I/T} \qquad (1)
\end{aligned}$$

where we have used the fact that $\mu_\mathrm{H} = \mu_e + \mu_p$, $g_\mathrm{H} = 4$, $m_p \approx m_\mathrm{H}$, $g_e = g_p = 2$, $m_e + m_p - m_\mathrm{H} = 13.6$ eV $\equiv E_I$ (the ionization energy of hydrogen), and $n_e = n_p$ in a neutral universe.

Now, we shift our focus to finding an expression for X_e, the free electron fraction defined by $X_e \equiv n_e/(n_p + n_\mathrm{H})$. If we assume that protons and hydrogen comprises all of the baryons in the Universe at this point (i.e., we ignore the abundance of helium), then we have

$$n_p + n_\mathrm{H} = n_b = \eta n_\gamma$$

$$= \eta \frac{\zeta(3)}{\pi^2} 2T^3$$

where η is the baryon-to-photon ratio, we have made use of the number density formula for relativistic species, and the fact that $g_\gamma = 2$. But now we can make use of Eq. 1 and write:

$$\begin{aligned}
\frac{1 - X_e}{X_e^2} &= \frac{(1 - n_e)n_b}{n_e^2} \\
&= \frac{n_{\mathrm{H}}}{n_e^2} n_b \\
&= \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{E_I/T},
\end{aligned}$$

the Saha equation.

2. The Saha equation is a quadratic equation for X_e of the form $AX_e + BX_e + C = 0$, with $B = 1$, $C = -1$, and

$$A = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{E_I/T}.$$

We then have, from the quadratic formula

$$X_e = \frac{-1 \pm \sqrt{1 + 4A}}{2A}$$

and we choose the + sign, since X_e must be a real number and $A > 0$ for all T.

To get our expression to be in terms of z, we use the relation $T = T_0(1 + z)$, where T_0 is the temperature of the CMB today. See Fig. 1. The code to generate this plot can be found in the appendix.

3. We use a bisection method to invert our function $X_e(z)$. See the appendix. We find that $\boxed{z(X_e = 0.1) \approx 1259}$ and $\boxed{z(X_e = 0.5) \approx 1377}$. As can be seen from the plot, the transition from a universe with almost all free electrons free to one in which there are almost none are happens over a relatively short span of redshifts.

4. For a single component universe, we have (see lecture notes)

$$\begin{aligned}
t &= \frac{1}{H_0} \frac{2}{3(1 + w)} a^{\frac{3}{2}(1+w)} \\
&= \frac{1}{H_0} \frac{2}{3(1 + w)} \left(\frac{1}{1 + z} \right)^{\frac{3}{2}(1+w)}
\end{aligned}$$

For matter, $w \approx 0$, and we get that $\boxed{t(z = 1259) \approx 208{,}000 \text{ years}}$ (for $h = 0.7$).

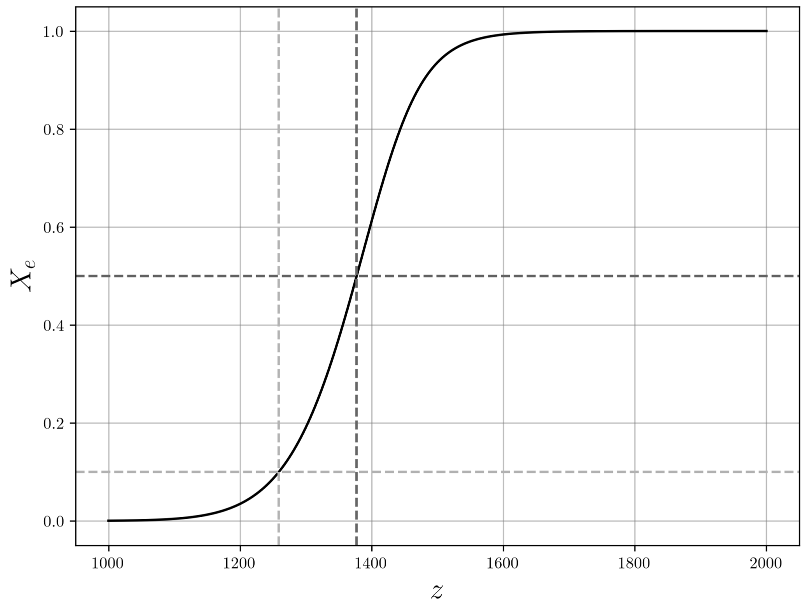

Figure 1: The free electron fraction as a function of redshift around the epoch of reionization. The redshifts at which $X_e = 0.1$ and 0.5 are indicated.

5. We seek to solve

$$\begin{aligned}
\Gamma_T(z) &= H(z) \\
n_e(z)\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} && \text{(matter domination)} \\
X_e(z)n_b(z)\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} \\
X_e(z)\eta\frac{\zeta(3)}{\pi^2}2T_0^3(1+z)^3\sigma_T &= H_0\sqrt{\Omega_{\mathrm{m},0}(1+z)^3} && \text{(n_b formula above)} \\
\implies X_e(z)(1+z)^{3/2} &= \frac{H_0\pi^2\sqrt{\Omega_{\mathrm{m},0}}}{2\eta\zeta(3)\sigma_T T_0^3} \\
&\approx 215.592 && \text{(see calculation in appendix)}
\end{aligned}$$

We use a numerical bisection method to invert this equation for z and we find that $\boxed{z_{\mathrm{dec}} \approx 1114}$.

Using the same calculations as above, we get $\boxed{t_{\mathrm{dec}} \approx 250{,}000 \text{ years}}$ and $X_e(z_{\mathrm{dec}}) \approx \boxed{0.0058}$. In the words of Baumann, "a large degree of neutrality is necessary before the universe becomes transparent to photons."

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Problem 2 "What-if" BBN

- Here, instead of the combination of protons and electrons into hydrogen, we consider the decay of neutrons into protons:

$$
\begin{aligned}
n + \nu_e &\leftrightarrow p^+ + e^- \\
n + e^+ &\leftrightarrow p^+ + \bar{\nu}_e.
\end{aligned}
$$

  As before, we have for each species,

$$
n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{\frac{\mu_i - m_i}{T}}
$$

  Assuming that the chemical potentials of electrons and neutrinos are negligible, and using the face that $g_n = g_p$, we have

$$
\begin{aligned}
\frac{n_n}{n_p} &= \left(\frac{m_n}{m_p} \right)^{3/2} e^{\frac{-(m_n - m_p)}{T}} \\
&\approx e^{\frac{-(m_n - m_p)}{T}} \\
&\equiv e^{-\frac{Q}{T}},
\end{aligned}
$$

  where we have approximated $m_n/m_p \approx 1$, and defined $Q \equiv m_n - m_p$.

  Defining the neutron fraction $X_n \equiv n_n/(n_n + n_p)$, we have

$$
\begin{aligned}
X_n^{-1} &= \frac{n_n}{n_n} + \frac{n_p}{n_n} \\
&= 1 + e^{\frac{Q}{T}} \\
\implies X_n &= \frac{1}{1 + e^{\frac{Q}{T}}} \\
&= \frac{e^{-\frac{Q}{T}}}{1 + e^{-\frac{Q}{T}}}.
\end{aligned}
$$

- For a freezeout temperature of 0.8 MeV, we get a freezeout abundance of $\sim \boxed{0.16}$. See appendix for the calculation.

- If every two neutrons ends up in a helium-4 atom, then

$$
\begin{aligned}
Y_P &= \frac{4 n_{\mathrm{He}}}{n_{\mathrm{H}}} \\
&= \frac{2 n_n}{n_p} \\
&= \frac{2 n_n}{n_n + n_p} \frac{n_n + n_p}{n_p}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2n_n}{n_n + n_p}\frac{n_n + n_p}{1 - n_n} \\
&= \frac{2X_n}{1 - X_n} \\
&\approx \boxed{0.39}.
\end{aligned}
$$

- Here we perform the same calculations as above, but with $Q = -2.6$ MeV (double the actual difference). We get $\boxed{X_n \approx 0.037}$ and $\boxed{Y_P \approx 0.078}$.

....

Problem 3 Freeze-in DM

(a) As instructed we begin with the Boltzmann equation for freeze-in:

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}na^3 = 2\Gamma h(t) n_{\sigma,\mathrm{eq}}(t)$$

We define $Y \equiv n/T^3$, which implies that, for the LHS,

$$
\begin{aligned}
\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}na^3 &= \frac{T^3}{T_0^3}\frac{\mathrm{d}}{\mathrm{d}t}n\frac{T_0^3}{T^3} \\
&= T^3\frac{\mathrm{d}Y}{\mathrm{d}t} \\
\implies \frac{\mathrm{d}Y}{\mathrm{d}t} &= 2\Gamma h(t) Y_{\sigma,\mathrm{eq}}(t)
\end{aligned}
$$

Now, to switch variables from t to x, we note that the epoch of interest is radiation domination, during which $a \propto \sqrt{t}$. Using the fact that $x \propto T^{-1} \propto a$, we have

$$\mathrm{d}x \propto t^{-1/2}\,\mathrm{d}t \propto x^{-1}\,\mathrm{d}t \implies \frac{\mathrm{d}}{\mathrm{d}t} \propto x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}$$

So we have

$$
\begin{aligned}
x^{-1}\frac{\mathrm{d}Y}{\mathrm{d}x} &= \text{const.} \times 2\Gamma h(x) Y_{\sigma,\mathrm{eq}}(x) \\
\implies &\boxed{\frac{\mathrm{d}Y}{\mathrm{d}x} = \lambda_1 x h(x) Y_{\mathrm{eq}}(x)}
\end{aligned}
$$

(b) First we need an expression for $Y_{\mathrm{eq}}(x)$. In general for Fermions, we have the equilibrium numberpye density as

$$n = \frac{g}{2\pi^2}T^3 I(x) \qquad\qquad I(x) = \int_0^\infty \mathrm{d}\xi \frac{\xi^2}{\exp\left[\sqrt{\xi^2 + x^2}\right] + 1}$$

$$\implies Y_{\mathrm{eq}} = \frac{g}{2\pi^2}I(x)$$

Since we are looking for order of magnitude results, we can look at the case of $g = 1$. See plot in Fig. 2

(c) Notice that in the "freeze in" scenario, abundance *decreases* with the interaction rate Γ, while in the "freeze out" scenario, the opposite is true.

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A Python code

Phys 600: HW 4

Yarone Tokayer

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In [1]:
```
# Install packages

#from astropy.cosmology import FlatLambdaCDM
from astropy import units as u

import numpy as np
from scipy.special import zeta

import matplotlib.pyplot as plt

from tqdm import tqdm
```

In [2]:
```
plt.rcParams.update({
    "text.usetex": True,
    "font.family": "Helvetica"
})
```

Problem 1

1.2: Figure for $X_e(z)$

In [3]:
```
# Constants

Tcmb0 = 0.235 * u.meV # CMB temperature today
eta = 6e-10 # Baryons-to-photon fraction (Baumann)
E_I = 13.6 * u.eV # Ionization energy of hydrogen
m_e = 0.510998950 * u.MeV # Mass of electron (Wikipedia)
```

In [4]:
```
def x_e(z):
    '''
    Function to compute the electron fraction as a function of redshift

    Inputs:
    z - Redshift

    Returns:
    x_e - free electron fraction at that redshift
    '''

    T = Tcmb0 * ( 1 + z )

    a = (2 * zeta(3) / np.pi ** 2) * eta * ((2 * np.pi * T / m_e) ** (3/2)) * np.exp(E_I / T)

    x_e = ((-1 + np.sqrt(1 + 4*a)) / (2 * a)).to(u.dimensionless_unscaled).value

    return x_e
```

1.3: Invert to find $z(X_e)$

In [5]:
```
def x_e_to_z(x_e_target, tolerance=1e-6):
    '''
    Function to find redshift (z) given the free electron fraction (x_e) using a bisection method.

    Inputs:
    x_e_target - Free electron fraction
    tolerance - Tolerance for the bisection method convergence

    Returns:
    z - Redshift corresponding to the given lookback time
    '''

    # Initialize the search interval for z
    z_low, z_high = 1e2, 1e4

    # Perform the bisection search
    while z_high - z_low > tolerance:
        z_mid = (z_low + z_high) / 2
        x_e_mid = x_e(z_mid)
        if x_e_mid > x_e_target:
            z_high = z_mid
        else:
            z_low = z_mid

    # The bisection method has converged; return the redshift
    return z_mid
```

Our two points of interest:

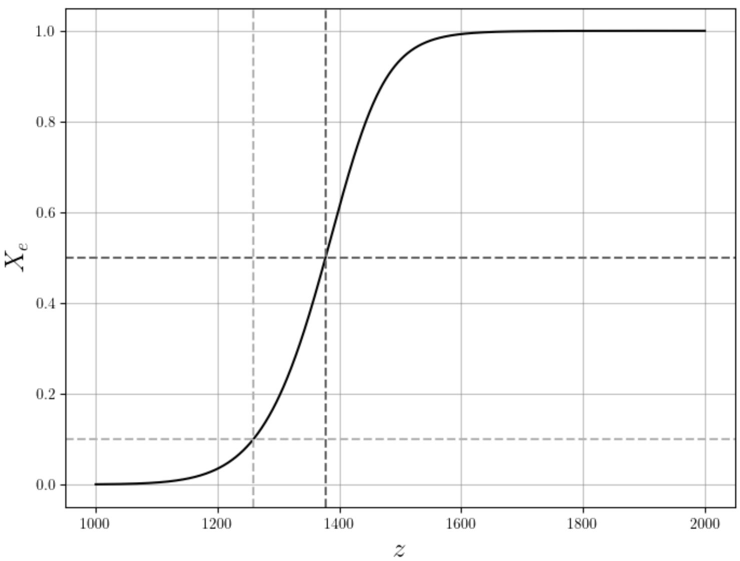
In [6]:
```
x_e_to_z(0.1), x_e_to_z(0.5)
```

Out[6]:
```
(1258.9799694891553, 1377.2434625017922)
```

Now we can plot everything:

In [7]:
```
# Redshift range to calculate

z = np.linspace(1000, 2000, 1000)
```

In [8]:
```
fig, ax = plt.subplots(1, 1, figsize=(8,6))

ax.plot(z, x_e(z), color='black')
ax.axhline(y=0.5, color=(0.4,0.4,0.4), ls='--')
ax.axhline(y=0.1, color=(0.7,0.7,0.7), ls='--')
ax.axvline(x=x_e_to_z(0.5), color=(0.4,0.4,0.4), ls='--')
ax.axvline(x=x_e_to_z(0.1), color=(0.7,0.7,0.7), ls='--')

ax.set_axisbelow(True)
ax.xaxis.grid(color='gray', alpha=0.5, linestyle='-')
ax.yaxis.grid(color='gray', alpha=0.5, linestyle='-')

ax.set_xlabel('$z$', fontsize=18)
ax.set_ylabel(r'$ X_e$', fontsize=18)

# ax.legend(fontsize=14)

fig.savefig('/Users/yaronetokayer/Yale Drive/Classes/PHYS 600/phys600 hw/phys600 hw 4/x_e_plot.png',
          dpi=300, bbox_inches='tight')
```



1.4: Age of the universe

In [9]:
```
h_0 = 70 * u.km / u.s / u.Mpc
```

In [10]:
```
((1 / h_0) * (2/3) * (1 / (1 + x_e_to_z(0.1)))**(3/2)).to(u.kyr)
```

Out[10]: 208.21491 kyr

1.5: z at decoupling

Evaluate the target value for $X_e(z)(1+z)^{3/2}$ (see write-up)

In [11]:
```
# Constants
Om0 = 0.3; sigma_T = 2e-3 * u.MeV**-2
```

In [12]:
```
# Need to convert h_0 to natural units
h_0_nat = h_0 * u.GeV * 6.5821e-25 * u.s
```

In [13]:
```
target = (h_0_nat * np.pi**2 * np.sqrt(Om0) / (2 * eta * zeta(3) * sigma_T * Tcmb0**3)).to(u.dimensionless_unscaled)
target
```

Out[13]: 215.59192

Use a bisection method to solve for z:

In [14]:
```
tolerance=1e-6

# Initialize the search interval for z
z_low, z_high = 1e2, 1e4

# Perform the bisection search
while z_high - z_low > tolerance:
    z_dec = (z_low + z_high) / 2
    x_e_mid = x_e(z_dec) * (1 + z_dec)**(3/2)
    if x_e_mid > target:
        z_high = z_dec
    else:
        z_low = z_dec

# The bisection method has converged; return the redshift
z_dec
```

Out[14]: 1113.8222924259026

The age of the universe at this redshift:

In [15]:
```
((1 / h_0) * (2/3) * (1 / (1 + z_dec))**(3/2)).to(u.kyr)
```

Out[15]: $250.17782 \; \mathrm{kyr}$

The free electron fraction at this redshift:

In [16]:
```
x_e(z_dec)
```

Out[16]: 0.005791939562935705

Problem 2

Assuming the "freezeout" temperature is 0.8 MeV, estimate the freeze out abundance of neutrons (X_n). We do this using our formula for X_n:

In [17]:
```
def x_n(t, q=1.3 * u.MeV):
    '''
    Function to compute the neutron fraction as a function of temperature

    Inputs:
    t - Temperatute (astropy natural (energy) units)
    q - Neutron mass minus the proton mass (astropy natural (energy) units)

    Returns:
    x_n - neutron fraction at that temperature
    '''

    ratio = (q / t).to(u.dimensionless_unscaled).value

    x_n = np.exp( -ratio ) / (1 + np.exp( -ratio ) )

    return x_n
```

In [18]:
```
x_n_fo = x_n(0.8 * u.MeV)
x_n_fo
```

Out[18]: 0.1645164628965632

Mass fraction of helium-4 at this abundance:

In [19]:
```
2 * x_n_fo / ( 1 - x_n_fo )
```

Out[19]: 0.3938233504083882

Now for $Q = 2.6$ MeV:

In [20]:
```
x_n(0.8 * u.MeV, q=2.6 * u.MeV), 2 * x_n(0.8 * u.MeV, q=2.6 * u.MeV) / ( 1 - x_n(0.8 * u.MeV, q=2.6 * u.MeV) )
```

Out[20]: (0.037326887344129464, 0.07754841566344403)

Problem 3

Plot Y_{eq} and Y

Functions:

In [21]:
```
def y_eq(x):
    '''
    Function to calculate Y_eq as a function of dimensionless time x=n/T^3

    Inputs:
```

```
    x - A single value or a NumPy array of dimensionless time values

    Returns:
    y_eq - A single value or a NumPy array of Y_eq values
    '''

    # Ensure that x is a NumPy array for vectorized calculations
    x = np.asarray(x)

    xi = np.linspace(0, 500, 10000)  # xi values to integrate over

    if x.size == 1:  # Scalar input
        integrand = xi**2 / (np.exp(np.sqrt(xi**2 + x**2)) + 1)
        y_eq_value = (1 / (2 * np.pi**2)) * np.trapz(integrand, x=xi)
        return y_eq_value

    else:  # Array input
        # Expand dimensions of x and xi to allow broadcasting
        x_expanded = x[:, np.newaxis]
        xi_expanded = xi[np.newaxis, :]

        integrand = xi_expanded**2 / (np.exp(np.sqrt(xi_expanded**2 + x_expanded**2)) + 1)
        y_eq_values = (1 / (2 * np.pi**2)) * np.trapz(integrand, x=xi, axis=1)

        return y_eq_values
```

In [22]:
```
def y(x, lam=1e-6, x0=0.01, y0=1e-20):
    '''
    Function to compute Y for freeze in DM

    Inputs:
    x - A single value or a NumPy array of dimensionless time values
    lam - Lambda parameter (default: 1e-6)
    x0 - Initial value of x (default: 0.01)
    y0 - Initial value of Y (default: 1e-20)

    Returns:
    y - A single value or a NumPy array of Y values
    '''

    # Ensure that x is a NumPy array for vectorized calculations
    x = np.asarray(x)

    if x.size == 1:
        x_array = np.linspace(x0, x, 1000)
        h = x_array / (x_array + 2)
        integrand = lam * x_array * h * y_eq(x_array)
        delta_y = np.trapz(integrand, x=x_array)
        return y0 + delta_y
    else:
        y_values = np.empty(len(x))
        for i, x_value in tqdm(enumerate(x), total=len(x)):
            x_array = np.linspace(x0, x_value, 1000)
            h = x_array / (x_array + 2)
            integrand = lam * x_array * h * y_eq(x_array)
            delta_y = np.trapz(integrand, x=x_array)
            y_value = y0 + delta_y
            y_values[i] = y_value
        return np.array(y_values)
```

Compute:

In [23]:
```
x0 = 0.01
xf = 100
x = np.geomspace(x0, xf, 1000)
lams = [1e-6, 1e-8, 1e-10]
ys = []

for lam in lams:
    ys.append( y(x, lam=lam, x0=x0) )
```

```
100%|████████████████████████████████████████| 1000/1000 [01:09<00:00, 14.34it/s]
100%|████████████████████████████████████████| 1000/1000 [01:10<00:00, 14.24it/s]
100%|████████████████████████████████████████| 1000/1000 [01:08<00:00, 14.66it/s]
```

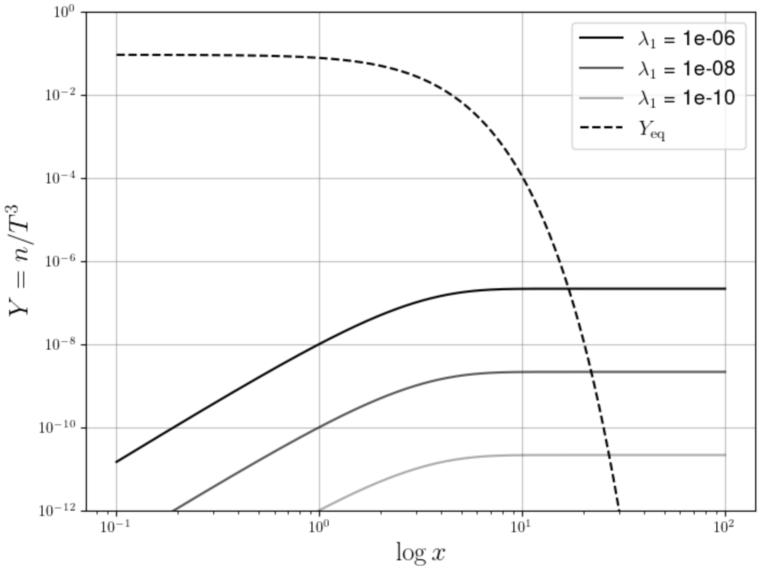
In [24]:
```
y_eq_plot = y_eq(x)
```

Truncate arrays for plotting:

In [25]:
```
mask = x >= 0.1
```

In [26]:
```
fig, ax = plt.subplots(1, 1, figsize=(8,6))

for i, y in enumerate(ys):
    ax.plot(x[mask], y[mask], color='black',
            label=r'$\lambda_1$ = ' + str(lams[i]), alpha = 1 - i / len(ys))

ax.plot(x[mask], y_eq_plot[mask], color='black', ls='--', label=r'$Y_\mathrm{eq}$')

ax.set_axisbelow(True)
ax.xaxis.grid(color='gray', alpha=0.5, linestyle='-')
ax.yaxis.grid(color='gray', alpha=0.5, linestyle='-')
```

```python
ax.set_xscale('log')
ax.set_yscale('log')

ax.set_ylim([1e-12,1])

ax.set_xlabel(r'$\log x$', fontsize=18)
ax.set_ylabel(r'$Y=n/T^3$', fontsize=18)

ax.legend(fontsize=14)

fig.savefig('/Users/yaronetokayer/Yale Drive/Classes/PHYS 600/phys600 hw/phys600 hw 4/y_plot.png',
          dpi=300, bbox_inches='tight')
```



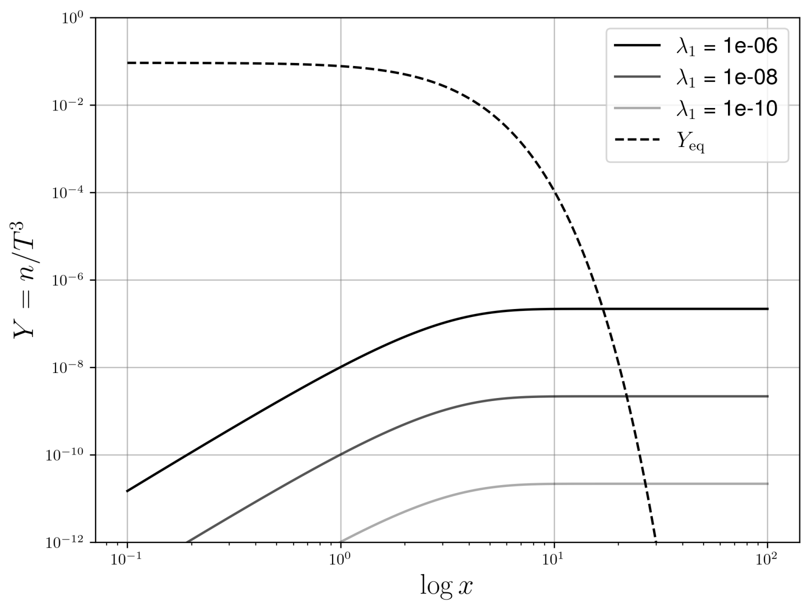


Figure 2: The free electron fraction as a function of redshift around the epoch of reionization. The redshifts at which $X_e = 0.1$ and 0.5 are indicated.