PHYS 600: Homework 2

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Problem 1 Friedmann Equation II

We wish to derive

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{1}$$

from the equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{R_0^2 a^2} \tag{2}$$

$$0 = \dot{\rho} + 3H(\rho + P) \tag{3}$$

Begin by taking the time derivative on both sides of Eq. 2:

$$\begin{aligned}
2H\frac{a\ddot{a} - \dot{a}^2}{a^2} &= \frac{8\pi G}{3}\dot{\rho} - 2H\frac{\kappa}{R_0^2 a^2} && \left(H = \frac{\dot{a}}{a}\right) \\
2H\frac{\ddot{a}}{a} - 2H^3 &= -3H\frac{8\pi G}{3}(\rho + P) - 2H\frac{\kappa}{R_0^2 a^2} && \textit{(Eq. 3)} \\
\frac{\ddot{a}}{a} - H^2 &= -\frac{8\pi G}{3}\rho - \frac{4\pi G}{3}\rho - 3\frac{4\pi G}{3}P - \frac{\kappa}{R_0^2 a^2} \\
\frac{\ddot{a}}{a} - \cancel{H^2} &= -\cancel{H^2} - \frac{4\pi G}{3}\rho - 3\frac{4\pi G}{3}P && \textit{(Eq. 2)} \\
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P)
\end{aligned}$$

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Problem 2 Cosmological Dimming

We saw in class that the angular diameter distance goes as $(1+z)^{-1}$, which implies that angular size goes as $(1+z)$. Additionally, we note that the bolometric luminosity L scales as $(1+z)^{-2}$, where the two factors of $(1+z)$ are due to cosmological redshift and hubble drag, respectively.

By definition, the bolometric surface brightness of an object, I_{e} is given by

$$I_{\mathrm{e}} = \frac{L}{4\pi r^2},$$

where L is the intrinsic bolometric luminosity and $r$is the radius of the object. Using the scalings above, we then get for the observed surface brightness:

$$\begin{aligned} I_{\mathrm{o}} &= \frac{L(1+z)^{-2}}{4\pi\left(r(1+z)\right)^2} \\ &= I_{\mathrm{e}}(1+z)^{-4} \end{aligned}$$

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Problem 3 Magnitudes and K-corrections

- $$\begin{aligned} m &= -2.5\log\left[\frac{f}{f_0}\right] \\ &= -2.5\log\left[\frac{f}{f(10\text{ pc})}\frac{f(10\text{ pc})}{f_0}\right] \\ &= -2.5\log\left[\frac{f(10\text{ pc})}{f_0}\right] - 2.5\log\left[\frac{f}{f(10\text{ pc})}\right] \\ &= M - 2.5\log\left[\frac{10\text{ pc}}{D_{\mathrm{L}}(z)}\right]^2 \\ &= M + 5\log\left[\frac{D_{\mathrm{L}}(z)}{10\text{ pc}}\right] \\ &= M + \mathrm{DM}(z) \end{aligned}$$

- In general, flux S is related to bolometric luminosity L by

  $$S = \frac{L}{4\pi D_{\mathrm{L}}^2}$$

  For a flux in the frequency interval $(\nu, \nu + \mathrm{d}\nu)$, we must look not at the energy band being observed, but the emitted energy band, which is given by $\nu(1+z)$, hence the intrinsic luminosity band we are interested in is $L_{\nu(1+z)}$, and we must multiply by the ratio $L_{\nu(1+z)}/L_{\nu}$. Finally, we add a factor of $(1+z)$ to account for the fact that the photons have redshifted, so we need to convert back to the restframe. So we get:

  $$S_{\nu} = (1+z)\frac{L_{\nu(1+z)}}{L_{\nu}}\frac{L_{\nu}}{4\pi D_{\mathrm{L}}^2}$$

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Problem 4 A Static Universe

- Begin by differentiating Friedmann I with respect to time:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\dot{a}}{a}\right)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{8\pi G}{3}\left(\rho_\mathrm{M}+\rho_\Lambda\right)-\frac{k}{a^2}\right] \\
2\frac{\ddot{a}}{a}\frac{\dot{a}}{a} &= \frac{8\pi G}{3}\left(\dot{\rho_\mathrm{M}}+\dot{\rho_\Lambda}\right)-\left(-2k\frac{\dot{a}}{a^3}\right) \\
2\frac{\ddot{a}}{a}\frac{\dot{a}}{a} &= \frac{8\pi G}{3}\left[-3\frac{\dot{a}}{a}\left(\rho_\mathrm{M}+\cancelto{0}{P_\mathrm{M}}+\rho_\Lambda-\rho_\Lambda\right)\right]+2k\frac{\dot{a}}{a^3} \\
\implies \frac{\ddot{a}}{a} &= -4\pi G\rho_\mathrm{M}+\frac{k}{a^2}
\end{aligned}$$

  We used the continuity equation with $P_\mathrm{M}=0$ and $P_\Lambda=-\rho_\Lambda$ in the third line.

- We wish the solve the following system of equations for Λ and k:

$$\begin{aligned}
0 &= \frac{8\pi G}{3}\rho_\mathrm{M}+\frac{\Lambda}{3}-\frac{k}{a^2} \\
0 &= -4\pi G\rho_\mathrm{M}+\frac{k}{a^2}
\end{aligned}$$

  Add them:

$$\begin{aligned}
0 &= -\frac{4\pi G}{3}\rho_\mathrm{M}+\frac{\Lambda}{3} \\
\implies \Lambda &= 4\pi G\rho_\mathrm{M} \\
\implies k &= 4\pi G a^2\rho_\mathrm{M} \\
&= a^2\Lambda
\end{aligned}$$

  Since $k>0$, this is a closed universe.

- Substituting into the second Friedmann equation, we get

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\delta a(t)\right] &= -4\pi G\rho_\mathrm{M}(t)a(t)+\frac{k}{a(t)} \\
&= -\Lambda\left[1-3\delta a(t)\right]\left[1+\delta a(t)\right]+\Lambda\left[1+\delta a(t)\right] \\
&= \Lambda\left[2\delta a(t)-1\right]+\Lambda\left[1+\delta a(t)\right] \\
\delta\ddot{a}(t) &= 3\Lambda\delta a(t) \\
\implies \delta a(t) &= Ae^{\sqrt{3\Lambda}t}+Be^{-\sqrt{3\Lambda}t}
\end{aligned}$$

  Applying the initial conditions $\delta a(t_0)=\delta a_0$ and $\delta\dot{a}(t_0)=0$, we get $A=\frac{1}{2}\delta a_0 e^{-\sqrt{3\Lambda}t_0}$ and $B=\frac{1}{2}\delta a_0 e^{\sqrt{3\Lambda}t_0}$. Thus

$$\delta a(t) = \frac{\delta a_0}{2}\left[e^{\sqrt{3\Lambda}(t-t_0)}+e^{-\sqrt{3\Lambda}(t-t_0)}\right]$$

Since $\Lambda > 0$ ($\Longleftarrow$ $\rho_{\mathrm{M}} > 0$), the perturbation grows exponentially and the solution is unstable.

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Problem 5 Redshift Drift

Start with the definition of redshift as advised:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_0}\left[1+z\right] &= \frac{\mathrm{d}}{\mathrm{d}t_0}\left[\frac{a(t_0)}{a(t_1)}\right] \\
\frac{\mathrm{d}z}{\mathrm{d}t_0} &= \frac{1}{a(t_1)}\frac{\mathrm{d}a(t_0)}{\mathrm{d}t_0} - \frac{a(t_0)}{a(t_1)^2}\frac{\mathrm{d}a(t_1)}{\mathrm{d}t_1}\frac{\mathrm{d}t_1}{\mathrm{d}t_0} \\
&= \frac{a_0}{a(t_1)}\frac{\dot{a}_0}{a_0} - \frac{a(t_0)}{a(t_1)}\frac{\dot{a}(t_1)}{a(t_1)}\frac{\mathrm{d}t_1}{\mathrm{d}t_0} \\
&= (1+z)H_0 - (1+z)H(t_1)(1+z)^{-1} \\
&= (1+z)H_0 - H(t_1)
\end{aligned}$$

We have used $\mathrm{d}t_1/\,\mathrm{d}t_0 = (1+z)^{-1}$, which is just the time dilation formula, and be derived using dimensional analysis:

$$\begin{aligned}
1+z &= \frac{\lambda_0}{\lambda_1} \\
\Longrightarrow\ 1+z &= \frac{\nu_1}{\nu_0}
\end{aligned}$$

but by definition of frequency (and by virtue of its units) $\nu \propto \frac{1}{\mathrm{d}t}$, so we have

$$1+z = \frac{\mathrm{d}t_0}{\mathrm{d}t_1}.$$

Assuming a matter-dominated flat universe, we have $H(z) = H_0(1+z)^{3/2}$. So at $z=1$,

$$\begin{aligned}
\left.\frac{\mathrm{d}z}{\mathrm{d}t_0}\right|_{z=1} &= \left(2 - 2^{3/2}\right)H_0 \\
&\approx -0.828H_0 \\
&= -82.8h\frac{\mathrm{km}}{\mathrm{s\ Mpc}} \\
&\approx -8.462^{-11}h\frac{1}{\mathrm{yr}}
\end{aligned}$$

According to the paper cited in the problem set, Keck spectra in 2003 could measure z to a precision of 10^{-5}. So we would need to wait 10^6 years before we would be able to detect redshift drift for sources at $z=1$ using that technology.

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