PHYS 600 : HW 5

I : Growth of Matter Perturbations - Matter and Radiation

The equation for the growth of a matter perturbation in linear theory above the Jean's scale is given by

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m\delta_m\,.$$

In what follows, we will assume that the Universe is only made of matter and radiation, and is flat. Furthermore, we will ignore any fluctuations in the radiation, although it will contribute to the expansion rate (H).

- Show that this above evolution equation can be written as

$$\frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2}\,.$$

- Define $y = a/a_{eq}$, where a_{eq} is the scale factor at matter-radiation equality. Show that the Hubble parameter can be written as

$$H(y) = \frac{A}{y^2}\sqrt{1+y}\,.$$

  Work out A in terms of physical constants and the energy density at equality.
- Substitute this into the equation for growth and show that it can be written as

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} = \frac{3}{2}\frac{\delta_m}{y(1+y)}\,.$$

- Verify (by eg. direct substitution, you may use Mathematica) that the solutions to this equation are

$$\delta_m \propto 1 + \frac{3}{2}y$$

  and

$$\delta_m \propto \left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}\,.$$

- Check to see that the growing modes at early and late times match your expectation (for radiation and matter domination respectively).

II: Spherical Collapse

In class, we used the following parametric solution for a spherical overdensity of mass M and energy E :

$$r(\theta) = A(1 - \cos\theta)$$

and

$$t(\theta) = B(\theta - \sin\theta)$$

with

$$A = \frac{GM}{2|E|}$$

and $A^3 = GMB^2$. Verify (either by solving or substituting) this solution.

III: Equality Scale

As we saw in class, the equality scale sets the turnover in the matter power spectrum. Show that the equality scale is given by

$$k_{eq} = a_{eq}H_{eq} = H_0\sqrt{\frac{2\Omega_M}{a_{eq}}}\,.$$

Estimate this scale for a cosmology roughly like the one we live in.

IV: A Study in Simulations

This problem has you explore the results of an N-body simulation, and compare it to the various aspects of the class so far. For this, we will use one of the Quijote simulations `https://quijote-simulations.readthedocs.io/en/latest/`.

We will use the fiducial cosmology, which is a flat ΛCDM cosmology with $\Omega_M = 0.3175$, $\Omega_b = 0.049$, $h = 0.6711$, $n_s = 0.9624$, and $\sigma_8 = 0.834$.

On Canvas (and our Github repository), you will find a directory with the following data. All of these are simple text files. There is a header with the column information as well.

- `linear_pk.txt` : The linear theory power spectrum at $z = 0$.
- `Pk_m_z=<z>.txt` : The non-linear power spectrum at different redshifts ($z = 0$, $z = 0.5$, $z = 1$ and $z = 2$) The columns here are k and $P(k)$, just as in the linear power spectrum.
- `halo_z=<z>.txt` : The halos in the simulation at different redshifts ($z = 0$, and $z = 1$). This is just a list of halo masses, including a virial mass estimate, as well as 200b and 200c estimates.

Orienting Yourself : The Linear Power Spectrum

- Start by making a plot of the linear power spectrum at $z = 0$. You should plot these on log scales on both axes.
- Verify that at large scales (low k), the power spectrum is given by

$$P(k) \propto k^{n_s}$$

  which is the primordial power spectrum (since $T(k) = 1$ at large scales). Note the slight deviation from the Harrison-Zel'dovich spectrum ($n_s = 1$).
- Estimate k_{eq} from the plot. Compare this to your expectation (see previous problem).
- Compute the transfer function $T(k)$ from the power spectrum and plot it. Normalize it to 1 at large scales. Compare the shape to what we calculated in class.
- Calculate σ_8 from the linear power spectrum. Compare this to the value given above.

Non-linear Power Spectrum and Structure Growth

- Plot the power spectrum from the simulation at the redshifts given (again on log scales). Compare this to the linear power spectrum.
- Now plot the ratio of the non-linear to linear power spectrum. Explain the shape of this ratio on large scales based on what you know about linear structure growth. At what k does linear structure growth start to break down (should be redshift dependent)?
- From this plot, extract the growth function as a function of redshift. How are you choosing to normalize this?
- Calculate the growth function for the cosmology of the simulation, using the method below. Compare this to your extracted growth function. How well do they agree?

Computing the Growth Function

Start by defining $g(a)$ as

$$D(a) \equiv g(a)a$$

where $D(a)$ is the linear growth function. Note that for a matter dominated Universe, $D(a) = a$, and so $g(a) = 1$. Define primes as derivatives with respect to $\ln a$. Then you can show (but you do not have to) that $g(a)$ satisfies the following equation (eg. Linder 2005)

$$g'' + \left(\frac{5}{2} - \frac{3}{2}w(a)\Omega_{DE}(a)\right) g' - \frac{3}{2}(1 - w(a))\Omega_{DE}(a)g = 0\,.$$

where $w(a)$ is the equation of state of dark energy, and $\Omega_{DE}(a)$ is the fraction of the total energy density in dark energy. Note that the quantity that appears in the equation is $\Omega_{DE}(a)$, i.e. the redshift dependent quantity.

This equation is quite general, and works for many different cosmologies with varying dark energy. Note that many of the other approximations you may find in the literature do not.

For the cosmology of the simulation, $w(a) = -1$, so the equation simplifies considerably. Finally, remember to multiply by a to get the growth rate.

In order to determine the growth rate for the previous problem, integrate this equation from some very early time (remember that you are integrating in $\ln a$); eg. $\ln a = -5.0$. At this time you may assume that the Universe is matter dominated, and so $g(a) = 1$, which you can use to set the initial conditions. Then integrate forward. When you get to $a = 1$, you should find that $g(a = 1) \approx 0.78$.

Halo Mass Function

The files you have include three different types of halo mass for each halo. You may use any of these, and if you are interested, you could explore the differences between them.

- The simulation we are using has a box size of $1h^{-1}$ Gpc, and has 1024^3 particles. What is the mass of each particle?
- Read in the halos, and measure the halo mass function at $z = 0$. Plot this on log-log scales. You will see a turnover at low masses. What do you think is causing this? How many particles are in halos at this mass?
- Using the linear power spectrum, calculate $\sigma(M)$, where $M = (4/3)\pi R^3 \bar{\rho}_m$. Plot this quantity.
- Using this quantity, compute the Press-Schechter mass function. Plot this on the same plot as the halo mass function. How well do they agree?
- Repeat this exercise at $z = 1$. The only change you need to make to compute the Press-Schechter mass function at $z = 1$ is to scale $\sigma(M)$ by $D(z)$.