PHYS 600: Homework 2

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September 22, 2023

Problem 1 Friedmann Equation II

We wish to derive

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{1}$$

from the equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{R_0^2 a^2} \tag{2}$$

$$0 = \dot{\rho} + 3H(\rho + P) \tag{3}$$

Begin by taking the time derivative on both sides of Eq. 2:

$$\begin{aligned}
2H\frac{a\ddot{a} - \dot{a}^2}{a^2} &= \frac{8\pi G}{3}\dot{\rho} - 2H\frac{\kappa}{R_0^2 a^2} && \left(H = \frac{\dot{a}}{a}\right) \\
2H\frac{\ddot{a}}{a} - 2H^3 &= -3H\frac{8\pi G}{3}(\rho + P) - 2H\frac{\kappa}{R_0^2 a^2} && \textit{(Eq. 3)} \\
\frac{\ddot{a}}{a} - H^2 &= -\frac{8\pi G}{3}\rho - \frac{4\pi G}{3}\rho - 3\frac{4\pi G}{3}P - \frac{\kappa}{R_0^2 a^2} \\
\frac{\ddot{a}}{a} - \cancel{H^2} &= -\cancel{H^2} - \frac{4\pi G}{3}\rho - 3\frac{4\pi G}{3}P && \textit{(Eq. 2)} \\
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P)
\end{aligned}$$

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Problem 2 Cosmological Dimming

We saw in class that the angular diameter distance goes as $(1+z)^{-1}$, which implies that angular size goes as $(1+z)$. Additionally, we note that the bolometric luminosity L scales as $(1+z)^{-2}$, where the two factors of $(1+z)$ are due to cosmological redshift and hubble drag, respectively.

By definition, the bolometric surface brightness of an object, I_e is given by

$$I_\mathrm{e} = \frac{L}{4\pi r^2},$$

where L is the intrinsic bolometric luminosity and $r$is the radius of the object. Using the scalings above, we then get for the observed surface brightness:

$$\begin{aligned}
I_\mathrm{o} &= \frac{L(1+z)^{-2}}{4\pi\left(r(1+z)\right)^2} \\
&= I_\mathrm{e}(1+z)^{-4}
\end{aligned}$$

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Problem 3 Magnitudes and K-corrections

- $$\begin{aligned}
m &= -2.5\log\left[\frac{f}{f_0}\right] \\
&= -2.5\log\left[\frac{f}{f(10\text{ pc})}\frac{f(10\text{ pc})}{f_0}\right] \\
&= -2.5\log\left[\frac{f(10\text{ pc})}{f_0}\right] - 2.5\log\left[\frac{f}{f(10\text{ pc})}\right] \\
&= M - 2.5\log\left[\frac{10\text{ pc}}{D_\mathrm{L}(z)}\right]^2 \\
&= M + 5\log\left[\frac{D_\mathrm{L}(z)}{10\text{ pc}}\right] \\
&= M + \mathrm{DM}(z)
\end{aligned}$$

- In general, flux S is related to bolometric luminosity L by

  $$S = \frac{L}{4\pi D_\mathrm{L}^2}$$

  For a flux in the frequency interval $(\nu, \nu + \mathrm{d}\nu)$, we must look not at the energy band being observed, but the emitted energy band, which is given by $\nu(1+z)$, hence the intrinsic luminosity band we are interested in is $L_{\nu(1+z)}$, and we must multiply by the ratio $L_{\nu(1+z)}/L_\nu$. Finally, we add a factor of $(1+z)$ to account for the fact that the photons have redshifted, so we need to convert back to the restframe. So we get:

  $$S_\nu = (1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_\mathrm{L}^2}$$

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Problem 4 A Static Universe

- Begin by differentiating Friedmann I with respect to time:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\dot{a}}{a}\right)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{8\pi G}{3}\left(\rho_{\mathrm{M}}+\rho_{\Lambda}\right)-\frac{k}{a^2}\right] \\
2\frac{\ddot{a}}{a}\frac{\dot{a}}{a} &= \frac{8\pi G}{3}\left(\dot{\rho_{\mathrm{M}}}+\dot{\rho_{\Lambda}}\right)-\left(-2k\frac{\dot{a}}{a^3}\right) \\
2\frac{\ddot{a}}{a}\frac{\dot{a}}{a} &= \frac{8\pi G}{3}\left[-3\frac{\dot{a}}{a}\left(\rho_{\mathrm{M}}+\cancelto{0}{P_{\mathrm{M}}}+\rho_{\Lambda}-\rho_{\Lambda}\right)\right]+2k\frac{\dot{a}}{a^3} \\
\implies \frac{\ddot{a}}{a} &= -4\pi G\rho_{\mathrm{M}}+\frac{k}{a^2}
\end{aligned}
$$

  We used the continuity equation with $P_{\mathrm{M}}=0$ and $P_{\Lambda}=-\rho_{\Lambda}$ in the third line.

- We wish the solve the following system of equations for Λ and k:

$$
\begin{aligned}
0 &= \frac{8\pi G}{3}\rho_{\mathrm{M}}+\frac{\Lambda}{3}-\frac{k}{a^2} \\
0 &= -4\pi G\rho_{\mathrm{M}}+\frac{k}{a^2}
\end{aligned}
$$

  Add them:

$$
\begin{aligned}
0 &= -\frac{4\pi G}{3}\rho_{\mathrm{M}}+\frac{\Lambda}{3} \\
\implies \Lambda &= 4\pi G\rho_{\mathrm{M}} \\
\implies k &= 4\pi G a^2\rho_{\mathrm{M}}
\end{aligned}
$$

  Since $k>0$, this is a closed universe.

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