PHYS 600: Homework 3

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October 8, 2023

Problem 1 Reviewing the Background

Density Parameters In general, ρ_{m} scales as $(1+z)^3$ and ρ_Λ is constant. We can now evaluate the desired quantities:

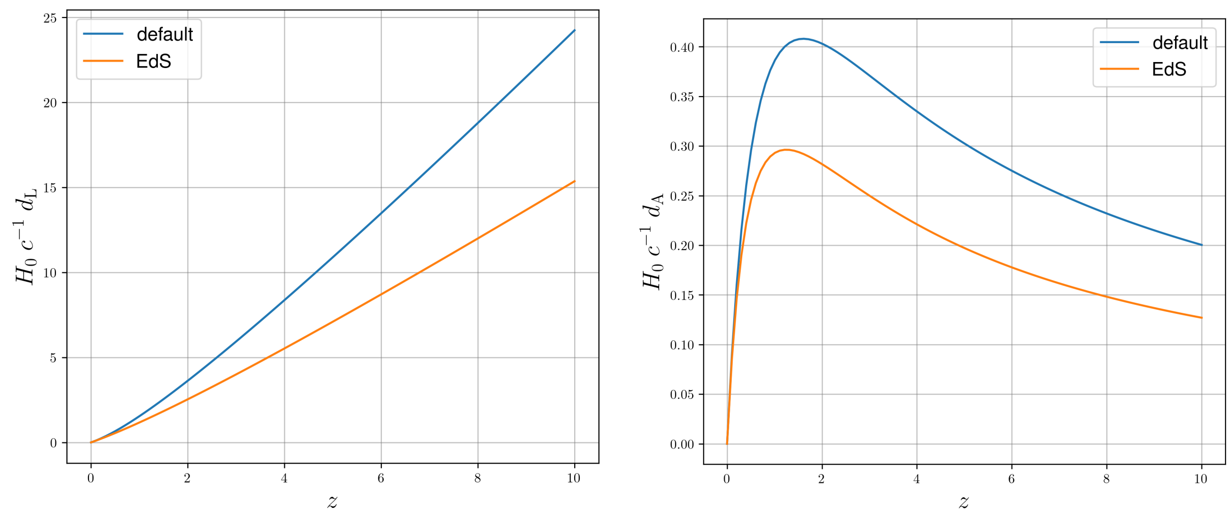
$$\begin{aligned}
\Omega_{\mathrm{m}}(z=0.5) &= \frac{\rho_{\mathrm{m}}(z)}{\rho_{\mathrm{c}}(z)} \\
&= \Omega_{\mathrm{m},0}(1+z)^3 \frac{H_0^2}{H^2(z)} \\
&= \frac{\Omega_{\mathrm{m},0}(1+z)^3}{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\mathrm{k},0}(1+z)^2 + \Omega_\Lambda} \\
&= \frac{0.3\cdot(1+0.5)^3}{0.3\cdot(1+0.5)^3 + 0 + 0 + 0.7} = \boxed{0.591} \\
\Omega_\Lambda(z=0.5) &= \frac{\rho_\Lambda}{\rho_{\mathrm{c}}(z)} \\
&= \Omega_\Lambda \frac{H_0^2}{H^2(z)} \\
&= \frac{\Omega_\Lambda}{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\mathrm{k},0}(1+z)^2 + \Omega_\Lambda} \\
&= \frac{0.7}{0.3\cdot(1+0.5)^3 + 0 + 0 + 0.7} = \boxed{0.409}
\end{aligned}$$

We have approximated $\rho_{\mathrm{r}} = 0$ at all times extending back to at least $z = 0.5$, and assumed a flat universe with $\Omega_{\mathrm{k}}(z) = 0$. These results are confirmed with `Astropy` (see appendix).

Luminosity and Angular Diameter Distances The luminosity distance from us as a particular redshift, d_{L} is given by

$$d_{\mathrm{L}} = (1+z)d_{\mathrm{M}}; \qquad d_{\mathrm{M}} = \int_0^z \frac{\mathrm{d}z'}{H(z')}$$

$$\implies d_{\mathrm{L}}(z) = (1+z)\int_0^z \frac{\mathrm{d}z'}{H_0\left[\Omega_{\mathrm{m},0}(1+z')^3 + \Omega_{\mathrm{r},0}(1+z')^4 + \Omega_{\mathrm{k}}(1+z')^2 + \Omega_\Lambda\right]^{1/2}}$$



(a) (b)

Figure 1: The luminosity distance and angular diameter distance as a function of redshift for two different cosmologies.

$$\approx (1+z)\int_0^z \frac{\mathrm{d}z'}{H_0\left[\Omega_{\mathrm{m},0}(1+z')^3+\Omega_\Lambda\right]^{1/2}}$$

The angular diameter distance is given by

$$\begin{aligned} d_\mathrm{A} &= \frac{d_\mathrm{M}}{(1+z)} \\ &\approx (1+z)^{-1}\int_0^z \frac{\mathrm{d}z'}{H_0\left[\Omega_{\mathrm{m},0}(1+z')^3+\Omega_\Lambda\right]^{1/2}} \end{aligned}$$

We have approximated $\rho_\mathrm{r}=0$ at all times extending back to at least $z=10$, and assumed a flat universe with $\Omega_\mathrm{k}(z)=0$.

To plot these, we use the `numpy.trapz` method for numerical integration, which uses a trapezoidal approximation.

To make the quantities dimensionless, we will compute $H_0 d_\mathrm{L}$ and $H_0 d_\mathrm{A}$. See plots in Fig. 1. Code can be found in the appendix.

Note in Fig. 1b, that there is a critical point at $z\approx 0.7$. This corresponds to the epoch when the Universe goes from matter dominated to Λ dominated.

Looking Back In general, we can find the lookback time from the comoving distance (we will explicitly include c to make getting units of time easier):

$$\mathrm{d}d_\mathrm{M} = \frac{c}{H(z')}\,\mathrm{d}z$$

Now, in each differential interval $\mathrm{d}z$, the distance light must travel is $(1+z)^{-1}\,\mathrm{d}d_{\mathrm{M}}$. We get:

$$t_{\mathrm{L}} = \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')\left[\Omega_{\mathrm{m},0}(1+z')^3+\Omega_\Lambda\right]^{1/2}}.$$

We use a bisection method to invert this function and solve for z numerically. See the code in the appendix. Our results is $\boxed{z = 1.86}$. This is verified using Astropy.

A Λ-dominated Universe We will use the first Friedmann equation to solve for $a(t)$:

$$\begin{aligned}\left(\frac{\dot a}{a}\right)^2 &= \frac{\Lambda}{3}\\ \implies \frac{\mathrm{d}a}{\mathrm{d}t} &= \sqrt{\frac{\Lambda}{3}}a\\ \implies a(t) &= Ae^{\sqrt{\Lambda/3}\;t}\end{aligned}$$

We throw away the negative square root, since we observe $\dot a$ to be positive. To find the age of the Universe, we look for t such that $a(t) = 0$. But there is no such finite value in this model (i.e., it requires $t = -\infty$). This implies an infinitely old Universe.

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Problem 2 Massive Neutrinos

- In general, the energy density of a species in an equilibrium gas is given by (e.g., 3.9 in Baumann):

$$\rho = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\; f(p,T)E(p)$$

Where $E(p) = \sqrt{m^2+p^2}$ and $f(p,T)$ is the distribution function over phase space. Using the Fermi-Dirac distribution (neutrinos are Fermions), and setting the chemical potential to zero for very early times, we get

$$\rho_\nu = \frac{g_\nu}{2\pi^2}T_\nu^4\int_0^\infty \mathrm{d}\xi\frac{\xi^2\sqrt{\xi^2+m_\nu^2/T_\nu^2}}{\exp\left[\sqrt{\xi^2+m_\nu^2/T_\nu^2}\right]+1}$$

with $\xi = p/T$. For relativistic neutrinos, we have $m_\nu \ll T\nu$, and we will use $g = 2$ for the spin degrees of freedom for a single neutrino species (e.g. `https://physics.stackexchange.com/questions/335061/degrees-of-freedom-of-neutrinos`). We get

$$\rho_\nu = \frac{T_\nu^4}{2\pi^2}\int_0^\infty \mathrm{d}\xi\frac{\xi^2\sqrt{\xi^2+m_\nu^2/T_\nu^2}}{e^\xi+1}$$

- In general, we have the expansion $\sqrt{1+\epsilon^2} = 1 + \frac{\epsilon^2}{2} - \cdots$. We manipulate as follows:

$$\begin{aligned}
\xi^2\sqrt{\xi^2 + m_\nu^2/T_\nu^2} &= \xi^3\sqrt{1 + \frac{m_\nu^2}{\xi^2 T_\nu^2}} \\
&= \xi^3\left[1 + \frac{m_\nu^2}{2\xi^2 T_\nu^2} + \cdots\right] \\
\implies \rho_\nu \approx \frac{T_\nu^4}{2\pi^2}\int_0^\infty \mathrm{d}\xi \frac{\xi^3\left[1 + \frac{m_\nu^2}{2\xi^2 T_\nu^2}\right]}{e^\xi + 1} \\
&= \frac{T_\nu^4}{2\pi^2}\int_0^\infty \mathrm{d}\xi \frac{\xi^3}{e^\xi + 1} + \frac{T_\nu^4}{2\pi^2}\frac{m_\nu^2}{2T_\nu^2}\int_0^\infty \mathrm{d}\xi \frac{\xi}{e^\xi+1} \\
&= \underbrace{\frac{T_\nu^4}{2\pi^2}\frac{7\pi^4}{120}}_{\equiv \rho_{\nu,0}} + \frac{T_\nu^4}{2\pi^2}\frac{m_\nu^2}{2T_\nu^2}\frac{\pi^2}{12} \\
&= \rho_{\nu,0}\left[1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right]
\end{aligned}$$

I used Wolfram Alpha to evaluate the integrals.

- Let's suppose that in order the be detectable, we require $\rho_{\nu,\mathrm{CMB}} \gtrsim 10\rho_{\nu,0}$. We will also assume that the relativistic approximation used above is still valid at the epoch of the CMB. We get

$$\begin{aligned}
1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2} &\gtrsim 10 \\
m_\nu^2 &\gtrsim \frac{63\pi^2}{5}T_\nu^2 \\
m_\nu &\gtrsim \sqrt{\frac{63}{5}}\pi T_\nu
\end{aligned}$$

In class, we found that after neutrino decoupling, we have $T_\nu = (4/11)^{1/3}T_\gamma$:

$$\begin{aligned}
m_\nu &\gtrsim \sqrt{\frac{63}{5}}\pi\left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}(1+z) \\
&\approx \sqrt{\frac{63}{5}}\pi\left(\frac{4}{11}\right)^{1/3} 0.235\ \mathrm{meV} \times 1000 \\
&\approx \boxed{1.87\ \mathrm{eV}}
\end{aligned}$$

- When a particle goes non-relativistic, its thermal energy becomes much smaller than its rest mass energy. We look for the redshift at which $m_\nu \sim T_\nu$:

$$T_\nu \sim m_\nu$$

$$
\begin{aligned}
T_{\nu,0}(1+z) &\sim m_\nu \\
\left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}(1+z) &\sim m_\nu \\
\implies z_{\mathrm{non-rel}} &\sim \left(\frac{11}{4}\right)^{1/3} \frac{m_\nu}{0.235\ \mathrm{meV}} - 1
\end{aligned}
$$

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Problem 3 Measuring the Expansion History with Standard Candles

- In order to expand χ to third order in z, we expand the integrand to second order. Using Mathematica (see code in the appendix), we get:

$$
\begin{aligned}
\mathrm{d}\chi = & \frac{1}{H_0 \left[\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\Lambda,0} + (1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0})(1+z)^2\right]^{1/2}} \\
\approx & \frac{1}{H_0}\left[1 + \frac{-2 + 2\Omega_{\Lambda,0} - \Omega_{\mathrm{m},0}}{2} z + \frac{8 - 20\Omega_{\Lambda,0} + 12\Omega_{\Lambda,0}^2 + 4\Omega_{\mathrm{m},0} + 3\Omega_{\mathrm{m},0}^2 - 12\Omega_{\Lambda,0}\Omega_{\mathrm{m},0}}{8} z^2\right]
\end{aligned}
$$

- Now we go back to Python to find the z value at which our third order approximation of χ diverges from the exact integral by 10%. We again implement a bisection method. See the Python code in the appendix.

  We find that this occurs at $\boxed{z = 1.204}$.

- We can use the same procedure to find where the first order approximation $\chi = \frac{z}{H_0}$ (which does not involved the cosmological density parametersdd) differs by the exact integral by at least 10%. See the Python code in the appendix. We find that this occurs at $z = 0.386$.

  Our statement can be explained using two approaches:

  Intuitively On an intuitive level, at low redshifts (the local universe), the Hubble parameter is roughly equal to what it is at present day. We are hardly probing the epochs at which it would be different, so we should only expect to find a constant relationship between recession velocity and distance. Since the cosmological parameters are related to H's *evolution*, we do not probe them in the local universe.

  Mathematically Using the expansion we just derived, we find that only for z not close to 0 χ significantly deviate from the first order expansion, which is sensitive only to H_0. Cosmological density parameters are only probed for higher z.

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A Python code

B Mathematica code

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Problem 3

Expand the integrand for χ to second order:

In[ ]:=
```
dχ[z_] := c / (h0 Sqrt[Ωm (1 + z)^3 + Ωlam + (1 - Ωm - Ωlam) (1 + z)^2])
```

$$d\chi[z\_] := \frac{c}{h0\sqrt{\Omega m\,(1+z)^3+\Omega lam+(1-\Omega m-\Omega lam)\,(1+z)^2}}$$

In[ ]:=
```
dχSeries = Series[dχ[z], {z, 0, 2}]
```

Out[ ]=

$$\frac{c}{h0}+\frac{c\,(-2+2\,\Omega lam-\Omega m)\,z}{2\,h0}+\frac{c\,\left(8-20\,\Omega lam+12\,\Omega lam^2+4\,\Omega m-12\,\Omega lam\,\Omega m+3\,\Omega m^2\right)\,z^2}{8\,h0}+O[z]^3$$