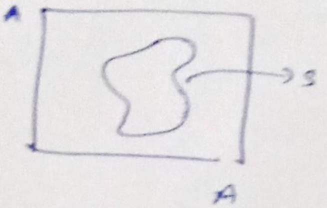
Ex:



find an exp. to find area of figure.

$$P(z \in S) = \iint_S \frac{1}{A^2}\, dx\, dy$$

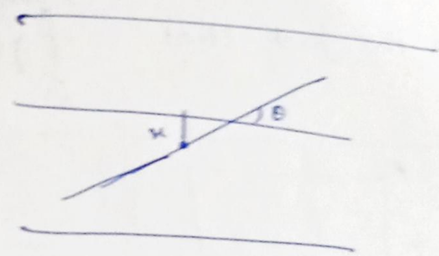
$$P(z \in S) = \frac{|S|}{A^2} \Rightarrow \boxed{|S| = P(z \in S) \cdot A^2}$$

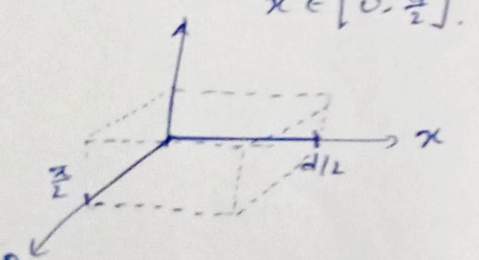
we find $P(z \in S)$ by large no. of trials.

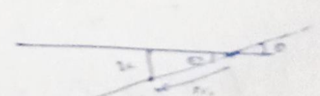
Ex: Buffon's Needle

we ke $\theta \in [0, \frac{\pi}{2}]$

$x \in [0, \frac{d}{2}]$.







$f_{X,\theta}(x, \theta) = \frac{4}{\pi d}$ (uniform distribution)

for intersection, $x < \frac{l}{2}\sin\theta$

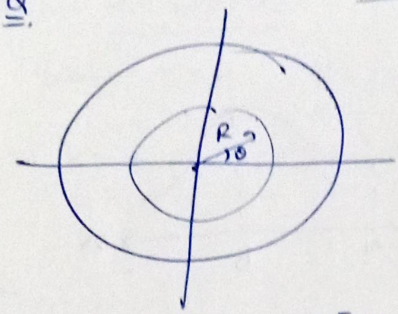
$$\left\{ \boxed{\theta > \sin^{-1}\left(\frac{x}{l/2}\right)} \quad \frac{x}{l/2} < \sin\theta \quad x < \frac{l}{2}\sin\theta \right\}.$$

$$P\left[x < \frac{l}{2}\sin\theta\right] = \int_0^{\pi/2}\int_0^{\frac{l}{2}\sin\theta} \frac{4}{\pi d} \cdot dx\, d\theta$$

$$= \int_0^{\pi/2} \frac{\overset{2}{\cancel{4}} \cdot \frac{l}{\cancel{2}}\sin\theta}{\pi d}\, d\theta$$

$$P[\text{intersection}] = \frac{2l}{\pi d}[1 - 0] = \frac{2l}{\pi d}$$

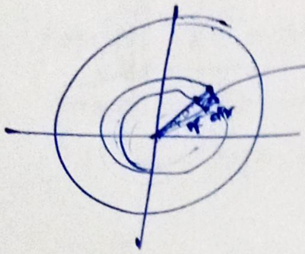
Q



$$F_R(r<R) = \frac{\pi r^2}{\pi R^2}$$

$$f_R(r) = \frac{2r}{R^2}$$

$$P[R \in [r, r+dr], \Theta \in [\phi, \phi + d\phi]] = \frac{r\, d\theta \cdot dr}{\pi A^2}$$



area = $(r\,d\theta)\cdot dr$

or more precisely. $\frac{d\theta}{2\pi}\left[\pi(r+dr)^2 - \pi r^2\right]$

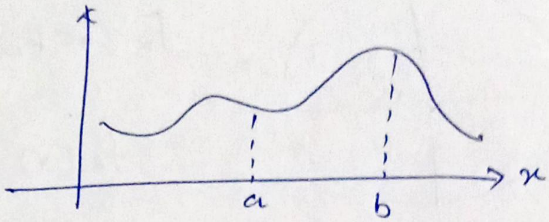
area = $\frac{d\theta}{2\pi}\left(\cancel{\pi r^2} + \pi(dr)^2 + 2\pi r\,dr - \cancel{\pi r^2}\right)$

area = $r\,d\theta\,dr$

$$\frac{r\,dr\,d\theta}{\pi A^2} = f_{R,\Theta}(r,\theta) \cdot dr \cdot d\theta$$

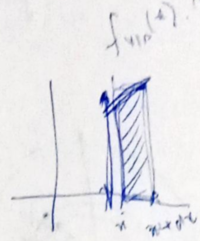
$$\Rightarrow \quad f_{R,\Theta}(r,\theta) = \frac{r}{\pi A^2} = \underbrace{\left(\frac{2r}{A^2}\right)}_{f_R(r)} \cdot \underbrace{\left(\frac{1}{2\pi}\right)}_{f_\Theta(\theta)}$$

Conditional Dist:



let $A = \{ X \in [a, b] \}$.

$$P[X \in B \mid A] = \frac{P[X \in A \cap B]}{P[X \in A]}$$



$$P[X \in B \mid A] = \frac{\int_{A \cap B} f_X(x)\,dx}{\int_A f_X(x)\,dx}$$

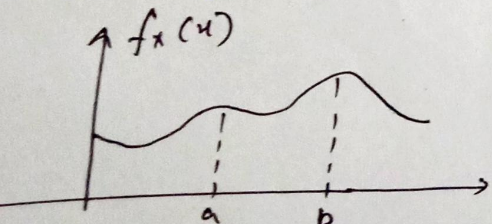
let $B = \{x, x + dx\}$ $\Rightarrow$ $P[X \in B \mid A] = f_{X|A}(x) \cdot dx$

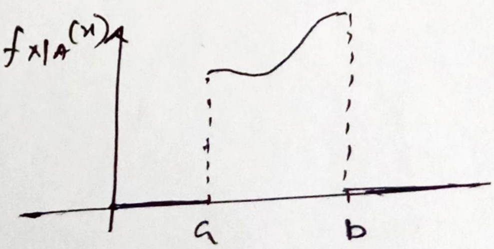
but as calculated prev. $P[X \in B \mid A] = \int_{A \cap B} f_X(x)\,dx \Big/ \int_A f_X(x)\,dx$

$$\Rightarrow \quad f_{X|A}(x)\,\cancel{dx} = \frac{f_X(x)\,.\cancel{dx}}{\int_A f_X(x)\,dx}$$

$$\Rightarrow \quad \boxed{f_{X|A}(x) = \frac{f_X(x)}{\int_a^b f_X(x)\,dx}}$$

graphically. $f_{X|A}(x)$ is just a scaled up version of f_X from a to b





where the scaling factor is $\left[\int_A f_X(x)\,dx \right]^{-1}$

Ex: $X \sim Exp(\lambda)$

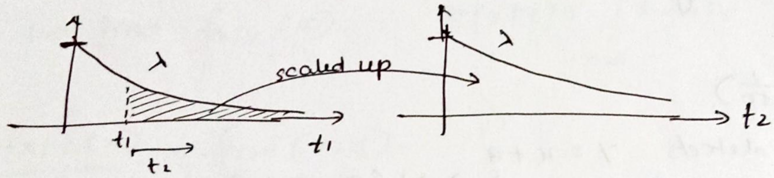
Given $X > t_1$, $X > t_1 + t_2 = ?$

$$P[X > t_1 + t_2 \mid X > t_1] = \frac{P[X > t_1 + t_2 \text{ and } X > t_1]}{P[X > t_1]}$$

$$= \frac{\exp(-\lambda(t_1 + t_2))}{\exp(-\lambda t_1)} = \exp(-\lambda t_2)$$

Result, $P[X > t_1 + t_2 \mid X > t_1]$ is independent of t_1

$\Rightarrow$ No matter how much we wait initially, the distribution is always exponential with same λ parameter.



Why this occurs with exponential R.V?

for $P[X > t_1]$ to cancel out, Numerator must be $P[X > t_1] \cdot P[X > t_2]$.

$\Rightarrow P[X > t_1 + t_2] = P[X > t_1] \cdot P[X > t_2]$.

Let $P[X > a]$ be a function $= S(a)$.

$\Rightarrow S(t_1 + t_2) = S(t_1) * S(t_2)$.

$\Rightarrow S(2) = S(1)^2$

$\Rightarrow S(1) = S(\tfrac{1}{2})^2$

$\Rightarrow S(3) = S(\tfrac{3}{2})^2$ or if $t_1 = t_2 = \frac{k}{2}$

$$S(k) = S\left(\frac{k}{2}\right)^2$$

or $S(\underbrace{1 + 1 + 1 + \dots 1}_{a \text{ times}}) = \underbrace{S(1) \cdot S(1) \dots S(1)}_{a \text{ times}}$

$S(a) = S(1)^a$ ~~Let~~ or $y = k^x$

$\log_e y = x \cdot \log_e k$

$\log_e y = k_1 \cdot x$

$$\boxed{e^{k_1 \cdot x} = y}$$

*. $f_{X,Y}(x,y)$.

given ~~Y~~ $Y=y$, find $f_{X|Y}(x) = ?$

$$P[X \in [x, x+dx] \mid Y \in [y, y+dy]] = \frac{P[X \in [x, x+dx], Y \in [y, y+dy]]}{P[Y \in [y, y+dy]]}.$$

~~= P[X∈[x, x+dx~~

$$\Rightarrow \quad f_{X|Y}(x).dx = \frac{f_{X,Y}(x,y).dx.dy}{f_Y(y).dy}$$

$$\Rightarrow \quad \boxed{f_{X|Y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

PDF of $X|Y$ when $Y=y$.

Ex: Speed $X \sim Exp(\frac{1}{50})$.

for $X=x$, Radar detects $Y = x + e$, $e \sim N(0, 0.1x)$.

$f_{X,Y}(x,y) = ?$

$$f_{X,Y}(x,y) = \frac{f_{X|Y}(x)}{f_Y(y)} = \frac{f_{Y|X}(y)}{f_X(x)}.$$

~~$f_{X|Y}(x) =$~~

$$f_{Y|X}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} + x$$

$$f_{Y|X}(y) = \frac{1}{\sqrt{2\pi}\, 0.1x} e^{-\frac{y^2}{0.02x^2}} + x$$

$$f_X(x) = \cancel{\exp(\tfrac{1}{50})}\ \frac{1}{50} e^{-\frac{x}{50}}$$

$$f_{X,Y}(x,y) = \frac{\frac{1}{\sqrt{2\pi}\, 0.1x} e^{-\frac{y^2}{0.02x^2}} + x}{\frac{1}{50} e^{-\frac{x}{50}}}$$

For $X = x$, $Y \sim N(x, 0.1x)$.

$\cancel{f_{Y|X}(x) =}$ $f_{Y|X}(y) = N(\underset{\text{mean}}{x}, \underset{\sigma}{0.1x})$

$$f_{Y|X}(y) = \frac{1}{\sqrt{2\pi}\, 0.1x} \exp\left(-\frac{(y-x)^2}{2(0.1x)^2}\right)$$

$$f_X(x) = \frac{1}{50} \exp\left(\frac{-1}{50} x\right).$$

$$f_{X,Y}(x,y) = \frac{f_{Y|X}(y)}{f_X(x)}$$

Now find $\mathbb{E}[X]$ given $Y = y$.

we find $f_{X|Y}(x)$. $\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x)\, dx$. → conditional expectation.

$$\frac{f_{X|Y}(x)}{f_Y(y)} = f_{X,Y}(x,y)$$

checking if this condition holds.

$$\int_{-\infty}^{\infty} \mathbb{E}[X|Y]\, f_Y(y)\, dy = \int_{-\infty}^{\infty} \cancel{\mathbb{E}[X]}\, x \cdot f_X(x)\, dx = \mathbb{E}[X].\ ?$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y)\, dx\, dy.$$

$$= \int_{-\infty}^{\infty} x \cdot \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\ \ dx$$

$$= \int_{-\infty}^{\infty} x\ f_X(x)\, dx$$

$$= \mathbb{E}[X].$$

Let $Y = \sum_{i=1}^{N} X_i$ $\quad$ X_i are iid or id

$y = x_1 + \ldots + y_n$

$E[Y] = \sum_n E[Y/N=n] \cdot P[N=n]$.

also we know that $E[Y|N=n] = n \cdot E[X_i]$

$E[Y] = E[X_i] \cdot \sum n \cdot P[N=n]$

$$\boxed{E[Y] = E[X_i]\, E[N]}$$ Walds equality

Finding variance of Y

$var[Y] = E[Y^2] - E[Y]^2$

finding $E[Y^2]$.

$E[Y^2 | N=n] = ?$ $\quad$ $Y = \sum_{n=1}^{N} X_i$

$\cancel{E(Y^2) = (}$

$E[Y^2 | N=n) = E[(X_1 + X_2 + \ldots X_n)^2 | N=n]$

$\cancel{= E[n^2 X_i^2 | N=n]}$.

$= \cancel{n^2 E[X_i^2 | N=n]}$

$E[Y^2 | N=n] = n\, E[X^2] - \underbrace{n(n-1)\, E[X]^2}$

(only if X_i are independent)

Now similarly.

$var(\cancel{Y}) \cancel{= \sum_n P[N=n] \cdot var(Y|N=n)}$

$\cancel{var(Y^2}$

$E[Y^2] = \sum_n P[N=n] \cdot E[Y^2 | N=n]$.

$E[Y^2] = \sum_n P[N=n] \cdot (n\, E[X^2] - n(n-1)\, E[X]^2)$.

Independence

X and Y are independent if.

$$\boxed{f_{X,Y}(x,y) = f_X(x).\ f_Y(y).}$$

Prove

$$P[X \in A,\ Y \in B] = P[X \in A].P[Y \in B].$$

$$LHS = \int_{X \in A}\int_{Y \in B} f_{X,Y}(x,y)$$

$$= \int_A\int_B f_X(x).\ f_Y(y)\,dx\,dy = \int_A f_X(x)\int_B f_Y(y)\,dy.\ dx$$

$$= \int_A f_X(x).\ P[Y \in B].\ dx$$

$$= P[Y \in B].\ \int_A f_X(x).\,dx$$

$$= P[Y \in B].\ P[X \in A].$$

Hence Proved.

Now let $A = [-\infty, x]$, $B = [-\infty, y]$.

then ~~$P[X \in A, Y \in B] =$~~

~~$f_{X,Y}(x,y) = F_{X,Y}(x,y).$~~

$$LHS = P[X \in A,\ Y \in B] = F_{X,Y}(x,y) = F_X(x).\ F_Y(y)$$

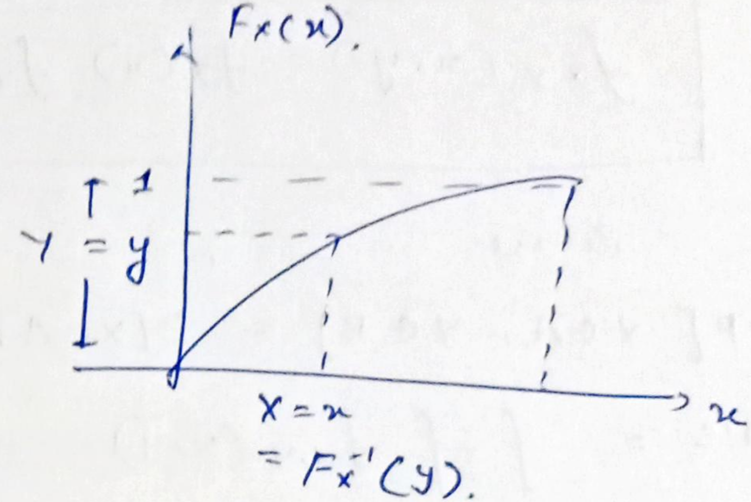
$$\boxed{F_{X,Y}(x,y) = F_X(x).\ F_Y(y)}$$

Suppose we are given $f_X(x)$ & $F_X(x)$ of a R.V. X.

we want to generate X s.t. it follows

Let us define $Y \sim U[0,1)$

$\Rightarrow$ $X = F^{-1}(Y)$.



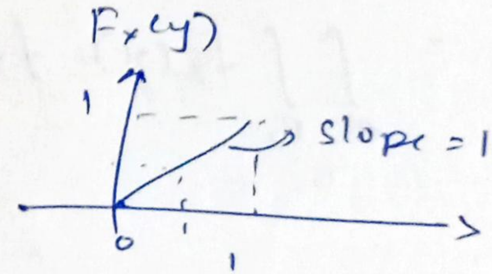
For x.

Let the event

$A := (X \leq x) \Rightarrow F_X[X] \leq F_X(x)$

$A \Rightarrow F_X(F_X^{-1}[Y]) \leq F_X(x)$

$\rightarrow A := (Y \leq F_X(x))$



$\Rightarrow$

$$P[X < x] = P[A] = \int_0^{F_X(x)} 1 \cdot dy = F_X(x) - 0 = F_X(x)$$

* For +ve R.V
for $X \geq 0$

$$E[X] = \int_0^\infty x \cdot f_X(x)\, dx$$

$$\int_0^\infty \mathbb{P}[X > x]\, dx = \int_0^\infty \left[\int_x^\infty f_X(y)\, dy \right] dx$$

$$= \int_0^\infty \int_x^\infty f_X(y)\, dy \cdot dx \quad \} \rightarrow \{(x,y) \mid y > x\}.$$

$$= \int_0^\infty f_X(y) \int_0^y dx \cdot dy$$

$$= \int_0^\infty f_X(y) \cdot y\, dy$$

$$= \mathbb{E}[X].$$

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > x]\, dx \quad \text{for } X \geq 0$$

$\hookrightarrow$ CDF

---

* COVARIANCE

$$\mathrm{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

$$= \mathbb{E}[X.Y] - \mathbb{E}[X] . \mathbb{E}[Y]$$

$$\mathbb{E}[(X - E(X))(Y - E(Y))] = E[XY - X E(Y) - Y E(X) + E(X) E(Y)]$$

$$= E[XY] + \cancel{E[X] E[Y]} - \cancel{E[Y]} E[X] - E[Y] E[X]$$

$$= E[XY] - E[X] E[Y].$$

Let $X = \sim U[-1, 1]$.

P_X

$Y = X^2$

$$E[X] = 0 \qquad E[Y] = E[X^2] = \sum x^2 P(x) = \cancel{\int}$$

$$f_X(x) = k$$

$$\int_{-1}^{1} k\, dx = 1 \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

$$E[X^2] = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\, dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{2}{6} = \frac{1}{3}.$$

$$E[X^3] = \int_{-1}^{1} x^3 \frac{1}{2}\, dx = 0. \qquad \cancel{\mathrm{cov}}\ E[XY] = 0, \quad E[Y] = \frac{1}{3} \quad E[X] = 0$$

* If X and Y are independent ...
* but uncorrelation (cov = 0) does not imply independence

Correlation Coefficient / Pearson correlation

$$\boxed{\rho_{XY} = \frac{cov[X,Y]}{\sqrt{var(X) \cdot var(Y)}}}$$

$|\rho_{XY}| \le 1.$

Properties of Covariance

1) $\boxed{cov(X,X) = var(X)}$

2) $cov(X, aY+b) = E[aXY + bX] - E[X](aE[Y]+b)$

$= a E[XY] + b\cancel{E[X]} - b\cancel{E[X]} - a E[X] \cdot E[Y]$

$= a(E[XY] - E[X]E[Y])$

$$\boxed{cov(X, aY+b) = a \cdot cov(X,Y)}$$

3) $cov(X, Y+Z) = E[XY + XZ] - E[X] \cdot (E[Y] + E[Z])$

$= E[XY] - E[X] \cdot E[Y] + E[XZ] - E[X] \cdot E[Z]$

$$\boxed{cov(X, Y+Z) = cov(X,Y) + cov(X,Z)}$$

4) $var(X+Y) = E[X^2 + Y^2 + 2XY] - (E[X] + E[Y])^2$

$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 + 2(E[XY] - E[X]E[Y])$

$$\boxed{var(X+Y) = var(X) + var(Y) + 2 \cdot cov(X,Y)}$$

by mathematical Induction, we can show.

$$\boxed{var\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} var[X_i] + \sum_{i \ne j} cov(X_i, X_j)}$$

5). $E[XY]^2 \le E[X^2] \cdot E[Y^2]$.

let us find $E[(X - tY)^2] \ge 0$

$\Rightarrow E[X^2] + t^2 E[Y^2] - 2t E[XY] \ge 0$

let $t = \frac{E[XY]}{E[Y^2]}$ $\Rightarrow$ $E[X^2] + \frac{E[XY]^2}{E[Y^2]} - 2\frac{E[XY]^2}{E[Y^2]} \ge 0$

$E[Y^2]$

Now $\quad cov(X,Y)^2 \le var(X).var(Y)$?

$E[XY]^2 + E(X)^2 E(Y)^2 - 2E[XY].E[X].E[Y] \le (E[X^2]-E(X)^2)$
$(E[Y^2]-E[Y]^2)$

$E[X^2]E[Y^2] + E(X)^2 E(Y)^2$
$- E(X)^2 E[Y^2]$
$- E[Y]^2 E[X^2].$

~~$E[XY]^2 - 2E[XY]E(X)E(Y)$~~ $\le$

$E[XY]^2 = cov(X,Y)^2 + 2E(XY)E(X).E(Y) - E(X)^2 E(Y)^2$
$\le E(X^2)E(Y^2)$

$Cov(XY)^2 \le 2[E(X^2)E(Y^2) - E(XY)E(X).E(Y)]$

$\Rightarrow \boxed{cov(XY)^2 \le var(X).var(Y)}$

since $\dfrac{cov(X,Y)^2}{var(X).var(Y)} \le 1 \Rightarrow \boxed{\rho^2 \le 1}$ or $\boxed{|\rho| \le 1}$

---

fair coin tossed n times.

$X :=$ no. of heads $\qquad Cov(X,Y) = ?$
$Y :=$ no of tails.

$X : Bino(n,p)$
$Y : Bino(n,q)$

$var(X) = npq = var(Y)$

$Cov(X,Y) = E[X.Y] - E[X].E[Y]. \quad | \quad E[X] = np$
$E[Y] = nq = n(1-p) = n - E[X].$

$X + Y = n$

$\rightarrow var(X+Y) = var(n) = 0 = var(X) + var(Y) + 2cov(X,Y)$

$cov(X,Y) = \dfrac{-var(X) - var(Y)}{2} = \dfrac{-npq \times 2}{2} = -npq$

$\rho = \dfrac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \dfrac{-npq}{npq} = -1$

Since correlation coefficient is -1

X, and Y are perfectly negatively correlated.

$$E[E[X|Y]] = E[X].$$

$$\mathrm{var}(X) = E[\mathrm{var}(X|Y)] + \mathrm{var}[E[X|Y]]$$

---

coin toss for n times

$X := \#$ heads.

$p = p(H) \sim U[0,1]$

$\cancel{E[X] = 1}$ $\quad E[X|P] = np$

$$E[E[X|P]] = E[n.P] = nE[P] = \frac{n}{2} = E[X]$$

$$\mathrm{var}(X|P) = nPq = n(P-P^2)$$

$$E[\mathrm{var}(X|P)] = nE[P.Q] = nE[P] - nE[P^2]$$

$$= n.\frac{1}{2} - n.\frac{1}{3} = \frac{n}{6} \quad \text{(i)}$$

$$E[X|P] = n.p.$$

$$\mathrm{var}(E[X|P]) = n^2 \mathrm{var}(P).$$

$$= n^2.\frac{1}{12} = \frac{n^2}{12} \quad \text{(ii)}$$

$$\Rightarrow \quad \mathrm{var}(X) = \frac{n}{6} + \frac{n^2}{12}$$

Now let $p(H) \sim \dfrac{\mu \exp(-\mu x)}{1 - \exp(-\mu)}$.

Now find $\mathrm{var}(X)$, $E[X]$. for $\bar{P} = E[P]$. fixed

find $\mathrm{var}(X)$, $E[X]$ for this.

part a

Let $\bar{p} = E[P]$.

$E[X] = \bar{p}N$

$Var(X) = \bar{p}(1-\bar{p}) \cdot N$

$$\bar{p} = \int_0^1 \frac{x \cdot \mu e^{-\mu x}}{1-e^{-\mu}}$$

$$= \frac{\mu}{1-e^{-\mu}} \int_0^1 x \cdot e^{-\mu x}$$

$$= \frac{\mu}{1-e^{-\mu}} \left[\frac{x \cdot e^{-\mu x}}{-\mu} - \int \frac{e^{-\mu x}}{-\mu} dx \right]$$

$$= \frac{\mu}{1-e^{-\mu}} \left[\frac{x \cdot [e^{-\mu x}]_0^1}{-\mu} + \frac{1}{\mu} \frac{[e^{-\mu x}]_0^1}{-\mu} \right]$$

$$= \frac{-\mu}{1-e^{-\mu}} \left[\frac{e^{-\mu}}{+\mu} + \frac{(e^{-\mu}-1)}{\mu^2} \right]$$

$$= \frac{1}{1-e^{-\mu}} \left[\frac{e^{-\mu}}{1} + \frac{e^{-\mu}}{\mu} - \frac{1}{\mu} \right]$$

$$\bar{p} = \frac{\mu e^{-\mu} + e^{-\mu} - 1}{\mu(1-e^{-\mu})}$$

part b

$E[X] = E(E[X|P])$.

$E[X|P] = nP$.

$E[E[X|P]] = E[nP] = nE[P]$

$= n(\bar{p})$.

▶ Moment generating Function (MGF)

$$M_X(s) = \mathbb{E}\left[e^{sX}\right]$$

* for discrete X:

$$M_X(s) = \sum_x e^{sx} P_X(X=x) = \sum_{k=-\infty}^{\infty} e^{sk} P_X(X=k)$$

* for continuous X:

$$M_X(s) = \int_{dom(X)} e^{sx} f_X(x)\, dx$$

$$e^{sX} = 1 + sX + \frac{(sX)^2}{2!} + \frac{(sX)^3}{3!} + \dots$$

$$\mathbb{E}[\cancel{s}e^{sX}] = 1 + s\,\mathbb{E}[X] + \frac{s^2}{2!}\mathbb{E}[X^2] + \frac{s^3}{3!}\mathbb{E}[X^3] + \dots$$

Now to find the 3rd moment of X's Expectation, we diff. the eqn 3 times wrts and put $s=0$.

$$\Rightarrow \boxed{\left.\frac{d^n M_X(s)}{ds^n}\right|_{s=0} = \mathbb{E}[X^n]}$$

Eg: $X \sim \exp(\mu)$

MGF = ?

$$M_X(s) = \mathbb{E}[e^{sX}] = \mu\int_0^\infty e^{sx} e^{-\mu x}\, dx$$

$$= \mu\int_0^\infty e^{x(s-\mu)}\, dx$$

$$= \mu\left[\frac{e^{x(s-\mu)}}{(s-\mu)}\right]_0^\infty = \mu\frac{(0-1)}{s-\mu} = \frac{\mu}{\mu - s}.$$

for $s-\mu<0$

$\boxed{s<\mu}$

$$\boxed{M_X(s) = \frac{\mu}{\mu - s}} \text{ with } \boxed{s<\mu}$$

$M_X(s) = \frac{\mu}{\mu - s}$

$\left.\begin{array}{l} \mathbb{E}[X] = \frac{1}{\mu} \\ \mathbb{E}[X^2] = \frac{2}{\mu^2} \end{array}\right\}$

~~$\mathbb{E}[X^3] \Rightarrow \mu(\mu-s)^{-1}$~~

$E[X] = -\mu(\mu - s)^{-2} = \frac{-\mu(-1)}{(\mu-s)^2} \Rightarrow \frac{1}{\mu}$

$E[X^2] = \frac{\mu(-2)(-1)}{(\mu-s)^3} = \frac{2\mu}{(\mu-s)^3} \Rightarrow \frac{2}{\mu^2}$

$E[X3] = \frac{2\mu(-3)(-1)}{(\mu-s)^4} \Rightarrow \frac{6\mu}{\mu^4} \Rightarrow \frac{6}{\mu 3}$

Let $X \sim$ Poisson (λ).

$f_X t \quad P_X(x = k) = e^{-\lambda} \frac{\lambda^k}{k!}$

$\Rightarrow \mathbb{E}[e^{sx}] = \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda} \frac{(e^s\lambda)^k}{k!} = e^{-\lambda} e^{e^s\lambda} = e^{\lambda(e^s -}$

$M_X(s) = e^{\lambda(e^s - 1)}$

$E[X] = e^{\lambda(e^s-1)} \cdot \lambda . e^s = \lambda . 1 = \lambda$

$E[X^2] = \frac{d}{ds}\left[\lambda . e^{\lambda e^s - \lambda + s}\right] = \lambda . e^{\lambda e^s - \lambda + s} . [\lambda e^s + 1]$

$E[X^2]. = \lambda e^{s} . [\lambda e^s + 1]$

$E[X^2] = \lambda . (\lambda + 1) = \lambda^2 + \lambda$

* what happens when $s \to \infty$ in $M_X(s)$.

$\Rightarrow$ Property: $M_X(0) = 1$

$\Rightarrow$ Property: $M_X(-\infty) = P_X(X = 0)$.

$\lim_{s \to -\infty} E[e^{sX}] = \lim_{s \to -\infty} \sum_{k=0}^{\infty} e^{sk} P_X(k) = \cancel{\lim_{s \to -\infty}} P_X(X=0)$.

at

$X_1 + X_2 + \dots + X_n = N$
$n X_1 = N$
$X_1 = \frac{N}{n}$

* $X \sim N(0,1)$

$\Rightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\Rightarrow \quad M_X(s) = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$= \frac{1}{e^{-s^2/2}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{s^2}{2} + sx} \, dx$$

$$= e^{s^2/2} \int \frac{1}{\sqrt{2\pi}} e^{\frac{(s-x)^2}{2}} \, dx$$

$$= e^{s^2/2} \left. \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{(x-s)^2}{2}} \, dx \right\} \sim N(s,1)$$

$$\boxed{M_X(s) = e^{s^2/2}}$$

* Now let $Y = \sigma X + \mu \Rightarrow Y \sim N(\mu, \sigma^2)$

~~$M_X[Y]$~~ $\quad M_Y(s) = \mathbb{E}[e^{sY}] = \mathbb{E}[e^{s\mu + s\sigma X}] = e^{s\mu} \, \mathbb{E}[e^{s\sigma X}]$

$\Rightarrow M_Y(s) = e^{s\mu} \, \mathbb{E}[e^{s\sigma X}]$

$M_Y(s) = e^{s\mu} M_X(s\sigma)$

$$M_Y(s) = e^{s\mu} e^{\frac{s^2\sigma^2}{2}} = e^{\frac{s^2\sigma^2}{2} + s\mu}$$

$$\boxed{M_Y(s) = e^{\frac{s^2\sigma^2}{2} + s\mu}}$$

(for general gaussian)

$$E[Y] = \left[e^{\frac{s^2\sigma^2}{2} + s\mu} \cdot \left(\frac{\cancel{2} s\sigma^2}{\cancel{2}} + \mu \right) \right]_{s=0} = \mu$$

$$E[Y^2] = (s\sigma^2 + \mu)^2 e^{\frac{s^2\sigma^2}{2} + s\mu} + e^{\frac{s^2\sigma^2}{2} + s\mu} (\sigma^2)$$

$$E[Y^2] = \mu^2 + \sigma^2$$

Property: if $M_X(s) = M_Y(s)$ then $F_X(s) = F_Y(s)$
OR $f_X(s) = f_Y(s)$

---

to obtain PDF/PMF from MGF, we write MGF in the form of polynomial and s^k coeff corresponds to $P_X(X=k)$
$\hookrightarrow$ check.

---

given $X \sim$ Geometric (P).

$$M_X(s) = \frac{\cancel{ps}\ pe^s}{1-(1-p)e^s} \ \text{(a)} = pe^s[1 + (1-p)e^s + (1-p)^2e^{2s} + \dots]$$

$$M_Y(s) \ \text{(b)} = [pe^s + p(1-p)e^{2s} + p(1-p)^2e^{3s} + \dots]$$

$\uparrow P(X=1)$ $\uparrow P(X=2)$ $\uparrow P(X=3) \dots$

in general: $P[X=k] =$ coeff of $e^k = p(1-p)^{k-1}$

---

let X and Y be independent

$\Rightarrow$ let $Z = X+Y$.

$$M_Z(s) = E[e^{s(X+Y)}] = E[e^{sX}.e^{sY}].$$

since X, Y are independent

e^{sX}, e^{sY} are independent $\Rightarrow E[e^{sX}.e^{sY}] = E[e^{sX}].E[e^{sY}]$

$$\boxed{M_Z(s) = M_X(s).M_Y(s).}$$

Now if X, and Y are iids, then

$$\boxed{M_Z(s) = (M_X(s))^2}$$

similarly, by mathematical induction, this result is valid for n iid/i random variables

$X \sim \text{Poisson}(\lambda_1)$

$Y \sim \text{Poiss}(\lambda')$

$* \quad Z = X + Y.$

$$M_X(s) = e^{\lambda_1(e^s - 1)}$$

$$M_Y(s) = e^{\lambda_2(e^s - 1)}$$

$$M_Z(s) = \left[e^{(e^s-1)}\right]^{(\lambda_1+\lambda_2)} \Rightarrow \text{Poisson}\,(\lambda_1+\lambda_2). = Z.$$

$$\boxed{\begin{array}{ccccc} Z & = & X & + & Y \\ \downarrow & & \downarrow & & \downarrow \\ Poi(\lambda_1+\lambda_2) & & Poi(\lambda_1) & & Poi(\lambda_2) \end{array}}$$

* Sum of independent Poison R.V. Is another poisson R.V. with parameters. sum of individual parameters

* <u>Similarly for Gaussian</u>

$$M_Z(s) = M_X(s) \cdot M_Y(s)$$

$$= e^{\frac{s^2\sigma_1^2}{2} + s\mu_1} \cdot e^{\frac{s^2\sigma_2^2}{2} + s\mu_2}$$

$$M_Z(s). = e^{\frac{s^2}{2}(\sigma_1^2+\sigma_2^2) + s(\mu_1+\mu_2)} \Rightarrow Z \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$$

This will also hold for N number of gauss. R.V.

if $Z = X_1 + X_2 + \dots X_N.$

$X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$.

$\Rightarrow Z \sim \text{Normal}\,(\Sigma \mu_i,\ \Sigma \sigma_i^2)$

---

* to find MGF of Binomial R.V. iids

use the fact that Bino. R.V = sum of infinite Bernoullie R.V

*** Let N also be a R.V

$Z = X_1 + X_2 + \dots X_N$

and all X_i are iids.

~~$MGF(Z) = M_Z(s) = [M_X(s)]^N$~~

$M_{Z|n}(s) = M_X(s)^n$

$M_Z(s) = \sum_{n=0}^{\infty} M_X(s)^n \cdot \mathbb{P}[N=n]$

$= \sum_{n=0}^{\infty} e^{\underline{n \log M_X(s)}} \underline{\mathbb{P}[N=n]}$

$M_X(s) = E(e^{sx})$
$= \sum_x e^{sx} \cdot p_X(x)$

$M_N(s)$

$$\boxed{M_Z(s) = M_N(\log M_X(s))}$$

$\left(\frac{\mu}{\mu - s}\right)^n$ $\frac{pe^s}{1-(1-p)e^s}$

Let $T_i \sim Expo(\mu)$
$N \sim Geo(p)$.

$T = \sum T_i$ T_i are iid.

$\Rightarrow M_T(s) = M_N(\log M_{T_i}(s))$

$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$ $M_{T_i}(s) = \frac{\mu}{\mu - s} = k$ (say)

$M_T(s) = \frac{p e^{\log k}}{1-(1-p)e^{\log k}} = \frac{pk}{1-(1-p)k} = \frac{\frac{p\mu}{\mu - s}}{\frac{\mu - s}{\mu - s} - \frac{(1-p)\mu}{\mu - s}}$

$M_T(s) = \frac{p\mu}{\cancel{\mu} - s - \cancel{\mu} + p\mu} = \frac{p\mu}{p\mu - s}$

$\boxed{M_T(s) = \frac{p\mu}{p\mu - s}} \Rightarrow$ ~~M_T is~~ $T \sim Expo(p\mu)$

* sum of Geometric no. of Exponential R.V.'s again exponential R
* sum of N exponential R.V. is not an expo. R.V.

Now we can find $Var(T) = \frac{1}{p^2\mu^2}$

Function of R.V

$Y = g(X)$.

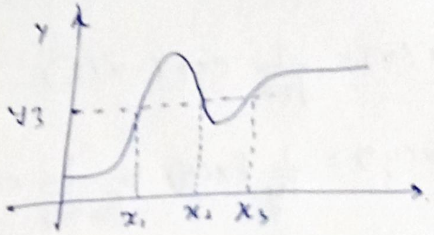
- $F_Y(y) = \mathbb{P}[Y \le y] = \mathbb{P}[X \in \mathbb{R}_y] \longrightarrow$ * $R_Y \ne \phi$ otherwise $f_Y(y) = 0$
  * R_1 should be union of countable set

where $\mathbb{R}_y$ is defined as

$Y \le y \Leftrightarrow g(X) < y$

$\{x : x \le g^{-1}(y)\} = \mathbb{R}_y$ (when invertible)

or better $\{x : g(x) \le y\} = R_y$

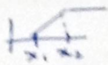


$Y < y_3 \Rightarrow X \le x_1$
or
$x_2 \le X \le x_3$

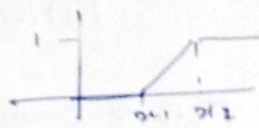
$F_Y(y_3) = P[X \le x_1] + P[x_2 < X \le x_3]$

$F_Y(y_3) = F_X(x_1) + F_X(x_3) - F_X(x_2)$

$R_{Y3} = \{ x \in \mathbb{R} \mid x \le x_1 \text{ or } x_2 \le x \le x_3 \}$



$$F_Y(y) = \sum_{x:\, g(x) < y} P_X(x).$$



*Let $Y = aX + b$ $\qquad Y \le y \Rightarrow X \le \frac{y-b}{a}$

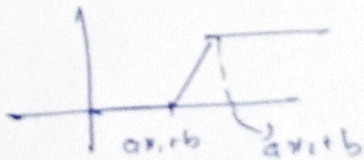
$\Rightarrow F_Y(y) = F_X\left(\frac{y-b}{a}\right)$

$X \sim U[x_1, x_2]$.

$Y = aX + b$

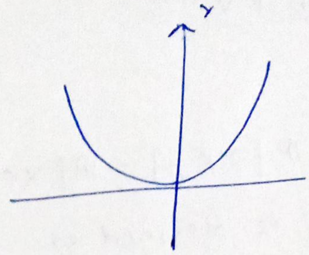
$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

$$= \frac{\frac{y-b}{a} - x_1}{x_2 - x_1} = \frac{y - (ax_1 + b)}{(ax_2 + b) - (ax_1 + b)}$$



(*).

$Y = X^2$



$F_Y(y) = P[X \in R_Y]$

$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$

$= \frac{1}{2}\frac{1}{\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}).$

thus, if $X \sim N(0,1)$

$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$

$f_Y(y) = \frac{1}{\sqrt{y}} f_X(y) = \frac{1}{\sqrt{2\pi y}} \exp(-y^2/2)$ for $y \geq 0$

[chi-square with degree of freedom 1].

* chi square with DOF K

$X(K)$

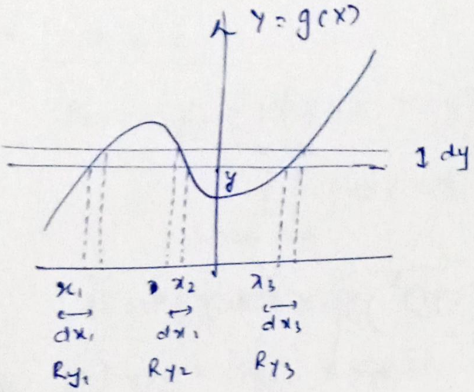
$$f_X(x) = \frac{1}{\Gamma(K/2)\, 2^{K/2}} x^{K/2-1} e^{-x/2}$$

$\Gamma(\cdot) \Rightarrow$ Gamma function

$\Gamma(m) = (m-1)!$ for $m =$ integer

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n+1) = n.\Gamma(n)$



$f_Y(y)\,dy = \mathbb{P}[Y \in [y, y+dy]]$

$= \mathbb{P}[X \in \cap (R_i)]$

$= P[X \in [x_1, x_1 + dx_1]]$

$+ P[X \in [x_2 - dx_2, x_2]]$

$+ P[X \in [x_3, x_3 + dx_3]]$.

$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$

$$\boxed{f_Y(y) = \sum_{i=1}^{3} \frac{f_X(x_i)}{|g'(x_i)|}} \qquad |g'(x_i)| \neq 0$$

* $Y = X^2 \Rightarrow$ roots, $x = \pm\sqrt{y}$

$\frac{dy}{dx} = g'(x) = 2x \Rightarrow |g'(x)| = 2|x|$

$\Rightarrow$

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

* $Y = aX + b$.

$g'(x) = a \rightarrow f_Y(y) = \dfrac{f_X\left(\frac{y-b}{a}\right)}{|a|}$

* $y = 1/x$

$g'(x) = \frac{-1}{x^2} \Rightarrow f_Y(y) = \dfrac{f_X\left(\frac{1}{y}\right)}{\cancel{x^2}} \left|-\frac{1}{y^2}\right|$.

$= f_Y(y) = \dfrac{f_X\left(\frac{1}{y}\right)}{y^2}$

⇒ <u>cauchy Dist:</u>

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} \qquad x \in (-\infty, \infty) \longrightarrow \text{cauchy}(\alpha)$$

⇒ if $f_Y(y) = \dfrac{\alpha/\pi}{\alpha^2 + 1/y^2} = \dfrac{1/\pi\alpha}{y^2 + 1/\alpha^2} = \text{cauchy}(1/\alpha)$

Let $y = e^x$

$$f_Y(y) = \frac{f_X(\ln y)}{e^{(\ln y)}} = \frac{f_X(\ln y)}{y}$$

$X \sim N(\mu, \sigma^2)$

$$\boxed{f_Y(y) = \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right)} \quad \text{for } y > 0$$

Log Normal distribution

* X, Y be R.V.

$$F_{XY}(x,y) = \mathbb{P}[X \le x, Y \le y]$$
$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

$$D := \{-\infty < Y < y,\ -\infty < X < x\}.$$

* $$\mathbb{P}[x_1 < X \le x_2,\ Y < y] = \int_{-\infty}^{y}\int_{x_1}^{x_2} f_{XY}(u,v)\,du\,dv = F_{XY}(x_2,y) - F_{XY}(x_1,y)$$

* $$\mathbb{P}[x_1 < X \le x_2,\ y_1 < Y \le y_2] = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$$
$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(u,v)\,du\,dv$$
$$= \int_{y_1}^{y_2}\left[\int_{-\infty}^{x_2}(\,)\,du - \int_{-\infty}^{x_1}(\,)\,du\right]dv$$
$$= \int_{-\infty}^{y_2}\left[\int_{-\infty}^{x_2}(\,)\,du - \int_{-\infty}^{x_1}(\,)\,du\right]dv - \int_{-\infty}^{y_1}\left[\int_{-\infty}^{x_2}(\,)\,du - \int_{-\infty}^{x_1}(\,)\,du\right]dv$$

$$\mathbb{P}[x_1 < X \le x_2,\ y_1 < Y \le y_2] = F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$

Let $\mathcal{E}$ be the event $X < Y$

$$P[\mathcal{E}] = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{v} f_{XY}(u,v)\,du.dv.$$

$$\mathcal{E} := D = \{(x,y) : x < y\}$$
$$D = \{-\infty < y < \infty,\ -\infty < x < y\}$$

Let $z = g(x,y)$

$F_Z(z) = \mathbb{P}[Z < z]$

$D_z = \{ (x,y) \mid g(x,y) < z \}$.

then $F_Z(z) = \iint\limits_{D_z} f_{x,y}(x,y)\,dx\,dy$

Leibnitz Integral Rule

If $F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx$

then $\frac{d}{d\theta} F(\theta) = b'(\theta)\, f(b(\theta), \theta) - a'(\theta)\, f(a(\theta), \theta) + \int_{a(\theta)}^{b(\theta)} \frac{d}{d\theta} f(x,\theta)\,dx$

If limits $a(\theta), b(\theta)$ are independent of θ

then $\frac{dF(\theta)}{d\theta} = \int_a^b \frac{d}{d\theta} f(x,\theta)\,dx$

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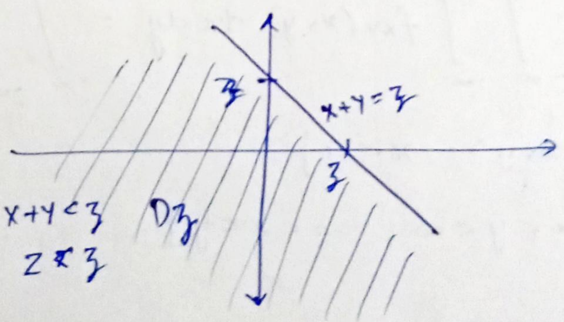
* ## Sum of 2 R.V.s

$Z = X + Y$

then $D_z := \{ Z < z \} = \{ (x,y) : x + y < z \}$.



$\mathbb{F}_Z(z) = \mathbb{P}[Z < z]$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(u,v)\,du,\,dv$

then $f_Z(z) = \frac{d}{dz} F_Z(z) = \int \left[\frac{d}{dz} \int_{-\infty} f_{XY}(u,v)\, du\right] dv$

$$= \int_{-\infty}^{\infty} \left[1 \cdot f_{XY}(z-y, y) - 0 + \int_{-\infty}^{z-y} \frac{d}{dz} \cancel{f(x,y)}\, dx \right] dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy$$

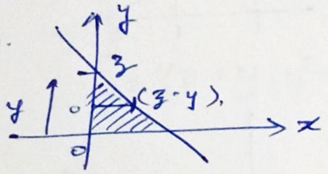
If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\, dy = [f_X * f_Y](z)$$

<u>Result</u>: distribution of sum of 2 independent R.V. is equal to convolution of their individual pdfs.

Now if we had Y and X ≥ 0,

then.



$D_z := \{ 0 < y < z,\ 0 < x < z - y \}$.

$D_z := \{ (x,y) : x + y \leq z,\ x \geq 0,\ y \geq 0 \}$.

$$F_Z(z) = \int_0^{z} \int_0^{z-y} f_{XY}(x,y)\, dx\, dy = \mathbb{P}[Z < z]$$

$$f_Z(z) = \frac{d}{dz}(F_Z(z)) = \cancel{1 \cdot \int_0^{z-y} f_{XY}(x,z)\, dx}$$

$$= 1 \cdot \int_0^{z-z} f_{XY}(x,z)\, dx - 0 + \int_0^{z} \frac{d}{dz} \int_0^{z-y} f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^{z} \left[1 \cdot f_{XY}(z-y, y) - 0 + \int_0^{z-y} \frac{d}{dz} \cancel{f_{XY}(x,y)}\, dx \right] dy$$

$$f_Z(z) = \int_0^{z} f_{XY}(z-y, y)\, dy$$

is the same expression in the prev. case with Limits refined to $[0, z]$

Eg: $X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$ } independent $\quad Z = X+Y \quad f_Z(z) = ?$

$$\cancel{f_Z(z) = \int_0^{\infty} f_X(x) \cdot f_Y(y)}$$

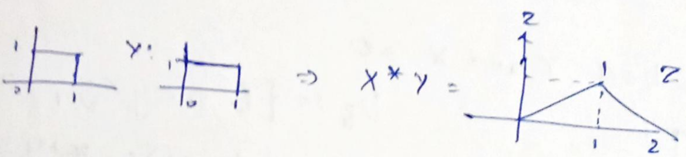
$$f_Z(z) = \int_0^{z} f_X(z-y)\, f_Y(y)\, dy$$

$$= \int_0^{z} \lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y}\, dy$$

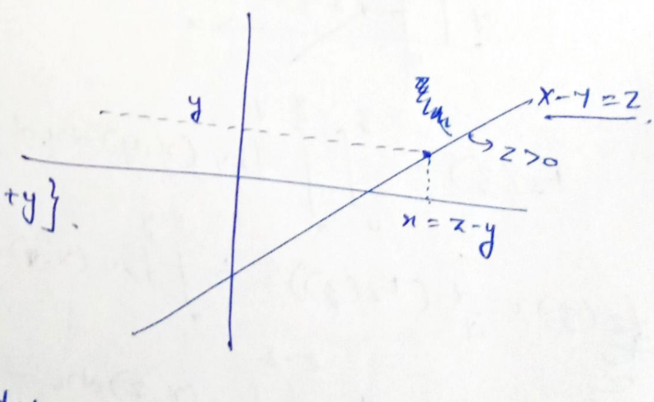
$$= \lambda^2 \int_0^{z} e^{-\lambda(z-\cancel{y}+\cancel{y})}\, dy = \lambda^2 \int_0^{z} e^{-\lambda z}\, dy$$

$$f_Z(z) = \lambda^2 e^{-\lambda z} \cdot z$$

Eg: if X: Y: $\Rightarrow$ $X * Y =$ $\quad Z = X+Y$



(*). Let $Z = X - Y$



$z = X - Y$

$$D_z := \{-\infty < y < \infty,\ -\infty < x < z+y\}.$$

$$\Rightarrow F_Z(z) = \mathbb{P}[Z < z]$$

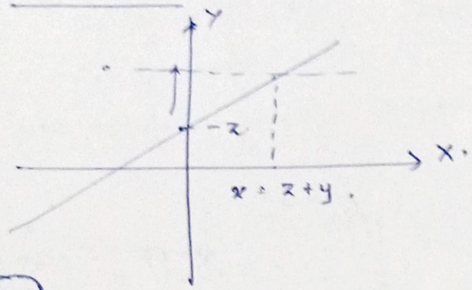
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z+y} f_{XY}(x,y)\, dx\, dy$$

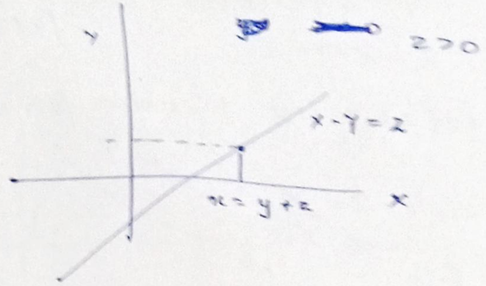
$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} \left[\frac{d}{dz} \int_{-\infty}^{z+y} f_{XY}(x,y)\, dx\right] dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} \left[1 \cdot f_{XY}(z+y, y) - 0 + 0\right] dy = \int_{-\infty}^{\infty} f_{XY}(z+y, y)\, dy$$

Now if we assume $X \geq 0$ and $Y > 0$.

then 2 cases:





#1 .

$D_z = \{-z < y < \infty,\ 0 < x < z + y\}$

$D_z = \{0 < y < \infty,\ 0 < x < z + y\}$.

$$F_Z(z) = \int_{-z}^{\infty}\int_{0}^{z+y} f_{XY}\, dx\, dy$$

$$f_Z(z) = -(-1) \cdot \int_{0}^{z-z} \cancel{f_{XY}(x,-z)}\, dx + \int_{-z}^{\infty} \frac{d}{dz}\int_{0}^{z+y} f_{XY}\, dx\, dy$$

$$= \int_{-z}^{\infty}\left[1 \cdot f_{XY}(z+y, y) - 0 + 0\right] dy$$

$$f_Z(z) = \int_{-z}^{\infty} f_{XY}(z+y, y)\, dy$$

#2

$D_z = \{0 < y < \infty,\ 0 < x < z + y\}$.

$$F_Z(z) = \int_{0}^{\infty}\int_{0}^{z+y} f_{XY}\, dx\, dy$$

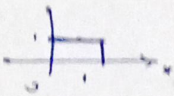
$$f_Z(z) = \int_{0}^{\infty} \frac{d}{dz}\left[\int_{0}^{z+y} f_{XY}(x, y)\, dx\right] dy$$

$$= \int_{0}^{\infty}\left[1 \cdot f_{XY}((z+y), y) - 0 + 0\right] dy$$

$$f_Z(z) = \int_{0}^{\infty} f_{XY}(z+y, y)\, dy$$

$$f_Z(z) = \begin{cases} \int_{-z}^{\infty} f_{XY}((z+y), y)\, dy & z < 0 \\ \int_{0}^{\infty} f_{XY}(z+y, y)\, dy & z > 0 \end{cases}$$

If $X \sim U[0,1]$, $Y \sim U[0,1]$ $\}$ ind. $\quad f_Y(y) = 1$, $f_X(x) = 1$

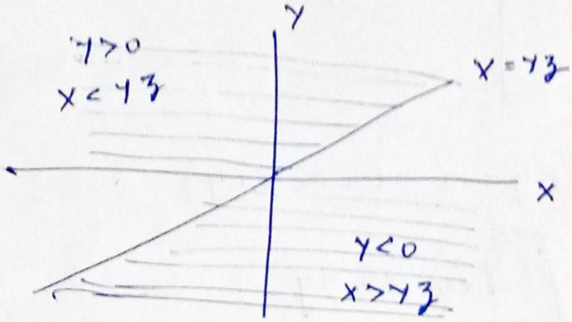


$\Rightarrow \quad Z = X - Y$

$f_Z(z)$ = use previous formula.

EX: Let $Z = \dfrac{X}{Y}$

$Z \leq z$



CDF

$$F_Z(z) = \int_0^{\infty}\int_{-\infty}^{yz} f_{XY}(x,y)\,dx\,dy$$

$$+ \int_{-\infty}^{0}\int_{yz}^{\infty} f_{XY}(x,y)\,dx\,.dy$$

PDF:

$$f_Z(z) = \int_0^{\infty}\left[y.\,f(yz,y) - 0 + \cancel{\int_{-\infty}^{yz}\frac{d}{dz} f_{XY}(x,y)\,dx}\right]dy$$

$$+ \int_{-\infty}^{0}\left[0 - y.\,f(yz,y) + 0\right]dy$$

$$= \int_0^{\infty} y.\,f(yz,y)\,dy + \int_{-\infty}^{0} -y\,f(yz,y)\,dy$$

$$= \int_{-\infty}^{\infty} |y|\,f(yz,y)\,dy$$