8-08-2022.

Real numbers & their properties

3.1 ~~Real numbers~~:-

The completeness axiom:-

Definition: An orderd field ϕ is said to be complete if every subset S of ϕ which is bounded above has the least upper bound.

$\Rightarrow$ Proposition: A non-empty subset S of an ordered field ϕ can have at most one least upper bound.

Proof:

Suppose λ & v are both least upper bounds of S. Then by the definition of least upper bound, we have $\lambda \leq v \leq \lambda$ thus $\lambda = v$.

Theorem 3.8

Let $x, y \in \mathbb{R}$ then

i) $\| x^2 \| = x \cdot x$.

ii) $\| x \cdot y \| \leq \| x \| \, \| y \|$

Proof :-

i) since $\|\underline{x}\| = (\underline{x}\cdot\underline{x})^{\frac{1}{2}}$ therefore $\|x\|^2 = \|\underline{x}\|\cdot\|\underline{x}\|$

(ii) For $l \in \mathbb{R}$ we have:-

$$0 \le \|\underline{x} - l\underline{y}\|^2 = (x - ly)(x - ly).$$

$$= \underline{x}(\underline{x} - l\underline{y}) + (-l\underline{y})(\underline{x} - l\underline{y}) = \underline{x}\,\underline{x} - l\,\underline{x}\,\underline{y}$$
$$- l\,\underline{x}\cdot\underline{y} + l^2\underline{y}$$

$$= \|x\|^2 - 2l(\underline{x}\cdot\underline{y}) + l^2\|\underline{y}\|^2.$$

Now put $l = \dfrac{\underline{x}\cdot\underline{y}}{\|\underline{y}\|^2}$ (certain real number).

$$\Rightarrow \quad 0 \le \|x\|^2 - 2\frac{(\underline{x}\cdot\underline{y})(\underline{x}\cdot\underline{y})}{\|\underline{y}\|^2} + \frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{y}\|^4}\|y\|^2.$$

$$\Rightarrow \quad 0 \le \|\underline{x}\|^2 - \frac{(\underline{x}\cdot y)^2}{\|\underline{y}\|^2} = \|\underline{x}\|^2\|\underline{y}\|^2 - \|\underline{x}\cdot\underline{y}\|^2.$$

$$\Rightarrow \quad 0 \le (\|\underline{x}\|\|\underline{y}\| + \|\underline{x}\cdot\underline{y})(\|\underline{x}\|\|\underline{y}\| - \|\underline{x}\cdot\underline{y})$$

which holds if

$$0 \le \|\underline{x}\|\|\underline{y}\| - \|\underline{x}\cdot\underline{y} \quad \text{i.e.} \quad \|\underline{x}\cdot\underline{y} \le \|\underline{x}\|\|\underline{y}\|.$$

Theorem : 3.9

Supposee $x, y, z \in \mathbb{R}^n$ then:-

a) $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ (Triangle inequality).

$\|x \cdot y\| \leq \|x\| \|y\|$ (causy sawn inequality). —(i)

$(\|\underline{x}+\underline{y}\|)^2 = \|x\|^2 + \|y\|^2 + 2\|\underline{x}\| \cdot \|\underline{y}\|$

Proof:

from (i),

$\leq \|\underline{x}\|^2 + 2\,\underline{x}\cdot\underline{y}\| + \|\underline{y}\|^2 = (\|\underline{x}\| + \|\underline{y}\|)^2.$

Thus $\|\underline{x}+\underline{y}\|^2 = (\|\underline{x}\| + \|\underline{y}\|)^2 \Rightarrow \|\underline{x}+\underline{y}\| \leq \|x\| + \|y\|$

$\Rightarrow (1+x)^n \geq (1+nx).$

if $n=1$

$(1+x) \geq (1+x).$

if $n=k$

$(1+x)^k \geq (1+kx).$

if $n=k+1$

$(1+x)^{k+1} \Rightarrow (1+x)^k (1+x) \geq (1+kx).(1+x)$

$(1+x)^{k+1} \geq (1+x+kx+kx^2)$

$(1+x)^{k+1} \geq (1+x(1+k)+kx^2)$

we know that

$$1+(k+1)x+kx^2 \geq 1+(k+1)x$$

Thus

$$(1+x)^{k+1} \geq 1+(k+1)x.$$

Bernoulli's inequality.

If $a_1, a_2 \ldots a_n$ are any real numbers, then

$$|a_1+a_2+\ldots a_n| \leq |a_1|+|a_2|+\ldots+|a_n|.$$

Tutorial 10-08-2022.

upper bound :
lower.

$S = \{ \ \}$.

$u \in \mathbb{R}$ $\quad s \leq u, \ \forall s \in S$.

$l \in \mathbb{R}$ $\quad l \leq s, \ \forall s \in S$.

$S = \{1, \ldots, 5\}$.

$u \geq 5$

largest lower bound

infimum

upper bound $(5, \infty)$

$(-\infty, 1]$ lower bound

lowest upper bound (supremum).

in set perspective Supremum is the largest element.

Note: Supremum may might not be in the set.

→ Let S be a set bounded above

An element s is said to be supremum of set S if it satisfies

→ (i) u must be an upperbound

⇒ $\forall u \in V, u \leq v$.

$V = \{$set of upper bounds of set $S\}$.

S

$(1, 3) = [3, a)$

(ii) $\forall u \in V, \lambda \geq u$.

[This can be proved using contradiction].

Finally supremum conditions:

1. $u \in$ upper bound

2. $u \le v \Rightarrow v \in$ Upper bound

↳ u is the least upper bound.

$$\boxed{f : D \to R.}$$

$$f(D) = \{ \underbrace{f(x) : x \in D}_{S} \}.$$

f is Bounded above if there exists an element B such that $f(x) \le B$, $\in R$, $\forall x \in D$.

↳ upper bound.

$B \le f(x)$

A function $f(x)$ is said to be bounded if there exists a lower bound & an upper bound.

$\Rightarrow$ $A = \{ \frac{(-1)^n}{n}, n \in N \}$.

supremum $= \frac{1}{2}$

infimum $= -1$

$S = \{ x \in R^2 : x^2 > 7 \}$.

Supremum $= \infty$, infimum $= -\infty$.

Reason:- There exists a number which always greater than the decided supremum. so ∞ is the supremum.

$C = \{ \frac{-1}{n} ; n \in N \}$.

$-1, 0$

show that

$$\sup \{ f(x) + g(x) \} \leq \sup \{ f(x) \} + \sup \{ g(x) \}$$

$f, g \quad A \to R$.

$f(x) \leq \sup \{ f(x) \} \ldots (i)$

$g(x) \leq \sup \{ g(x) \} \ldots (ii)$

$[f(x) + g(x)] \leq \sup (f(x)) + \sup (g(x))$

$\Downarrow$

$\Rightarrow \sup [f(x) + g(x)] \leq \sup (f(x)) + \sup (g(x))$.

we know that

$f(x) + g(x) \leq \sup (f(x) + g(x))$

The least upper bound is always less than or equal to any of the upper bound.

Cantors theorem:

$A = \{ \quad \}$.

cardinality of power set

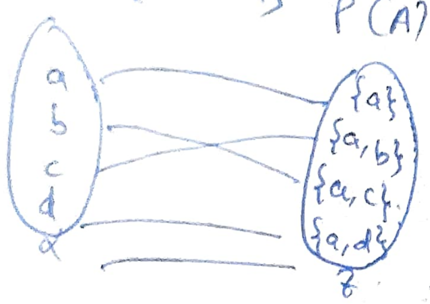
$|A| \lneq |P(A)|$.

L

$f: A \rightarrow P(A)$.

Surjection cant be defined.

$f: A \rightarrow P(A)$



$f(a) = \{a\}$

$f(b) = \{a, c\}$

$f(c) = \{a, b\}$

$f(d) = \{a, d\}$.

$X = \{ s : s \notin f(s) \}$. $\quad f(s) = s$.

a ~~dose~~ does. not belong to set X.

$b \in x$, $c \in X$

$f(\alpha) = z$.

$\alpha \in A$ [random]

~~$\alpha \in X$~~ ~~$[\alpha \in f(\alpha)]$~~

2 possibilities

$\alpha \in X$ (or) $\alpha \notin X$.

if $\alpha \in X$ [pre image of x is α].

$f(\alpha) = X$.

$\alpha \notin F(\alpha)$. [since α lies in X].

$\alpha \notin X$.

11-08-2022

Thursday.

Some Theorems:-

1) If z & a are elements of R with $\boxed{z + a = a}$

then $z = 0$.

Proof:- $z = z + 0 = z + (a + (-a))$

$= z + a + (-a)$

$= a + (-a) = 0$.

2) If u & b both are not equal to zero. & $\in R$.

with $u \cdot b = b$, then $u = 1$.

$u = u \cdot 1 = u\left(b \cdot \frac{1}{a}\right) = (u \cdot b)\frac{1}{b} = b \cdot \frac{1}{b} = 1$.

3) If a is a real number & non-zero

then $a \cdot 0 = 0$.

$\Rightarrow$ ~~$a \cdot 0$~~ $=$ $a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a(1+0) = a \cdot 1 = a$.

$a + a \cdot 0 = a$

$\Rightarrow$

$a \cdot 0 = 0.$

4) If $a \neq 0$ & $b \in \mathbb{R}$, such that $a \cdot b = 1$

then $b = 1/a.$

$\Rightarrow b \overset{b \cdot 1}{=} b\,(a \cdot 1/a) = (ba)\frac{1}{a} = \frac{1}{a}.$

$$\boxed{b = \frac{1}{a}.}$$

5) If $a \cdot b = 0$ then either $a = 0$ or $b = 0$.

[$a = a \cdot 1 = \cancel{a(2b-b)} = \cancel{2ab} - ab = 0.$

~~if a=0~~

$\cancel{b} = \cancel{b \cdot (1)} = \cancel{b(}$]

$b = b \cdot 1 = b\left(\frac{1}{a} \cdot a\right) = \frac{1}{a}(a \cdot b) = \frac{1}{a} = 0.$

since $\frac{1}{a}$ exists $a \neq 0.$

Rational & irrational numbers:

$\sqrt{2} \rightarrow$ This is ~~real par~~ not rational.

$\sqrt{2} = \frac{p}{q}$ $\Rightarrow p^2 = 2q^2$.

$\Rightarrow 2 = \left(\frac{p}{q}\right)^2$ $\hookrightarrow p^2$ is even.

~~$p = 2k$~~:

Theorems related to order properties:

$P \Rightarrow$ positive real numbers.

(a) If $a-b \in P$, then we write $a>b$ or $b<a$.

prove $\boxed{1 > 0.}$?

$1, 0 \in R$.

$\frac{1}{a} \cdot a > a-a$.

If $a \in \mathbb{R}$ such that $0 \leq a < \varepsilon \;\; \forall \; \varepsilon > 0$

then $a = 0$ & $\epsilon = a/2$.

$\Rightarrow$ 1) $ab > 0$

2) $a > 0$ & $b > 0$

3) $a < 0$ & $b < 0$.

$\Rightarrow$ $a > 0$ $\quad \frac{1}{a} > 0$,

$$b = 1 \cdot b = \left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a}(a \cdot b) > 0.$$

$\Rightarrow$ $\sqrt{ab} \leq \frac{a+b}{2}$.

$a > 0$, $b > 0$.

$\sqrt{a} > 0$, $\sqrt{b} > 0$

$(\sqrt{a} - \sqrt{b})^2 > 0$.

$a + b - 2\sqrt{ab} > 0$.

$\Rightarrow$ $a + b > 2\sqrt{ab}$

$\Rightarrow$ $\frac{a+b}{2} > \sqrt{ab}$.

⇒ Prove $(a_1 a_2 a_3 \ldots a_n)^{1/n} \le \frac{a_1+a_2+a_3 \ldots a_n}{n}$.

Bernoulli's Inequality:

$$(1+n)^n \ge 1+nx.$$

$n \in \mathbb{Z}$. $\cancel{A \in N}$

$x \in$ Positive real no.

$n=k \Rightarrow (1+x)^n \ge 1+kx$

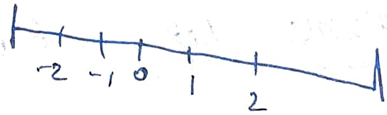
$n=k+1 \Rightarrow (1+x)^{k+1} = (1+x)^k(1+x)$

$\ge (1+kx)(1+x)$

$\ge 1+(k+1)x + k^2x$

$\ge 1+(k+1)x.$

Absolute Value:



$$|a| = \begin{cases} a & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -a & \text{if } a<0 \end{cases}$$

⇒ a) $|a \cdot b| = |a| \cdot |b|.$

b) If $c \ge 0$, then $|a| \le c$ if $c \le a \le c.$

Given $c > 0$

$$|a| \le c \cdot \begin{cases} a \le c & \text{if } a > 0 \\ a \le c & \text{if } a = 0 \\ a \ge -c & \text{if } a < 0 \end{cases}$$

$-c \le a \le c.$

$\Rightarrow$ ε. neighbourhood.

Definition: Let $a \in R$ & $\epsilon > 0$. Then

ε neighbourhood. if a is a set

$$V_\varepsilon(a) := \{ x \in R \mid |x-a| < \varepsilon \}.$$

$-\varepsilon < x-a < \varepsilon$

$\Rightarrow$ $a-\varepsilon < x < a+\varepsilon.$

Open set.

Theorem: Let $a \in R$, if x belongs to $V_\varepsilon(a)$ $\forall$ $\varepsilon > 0$, then $x = a$.

$|x-a| < \varepsilon \quad \forall \varepsilon$

$$\varepsilon_0 = \frac{|x-a|}{2}.$$

Completeness properties of R :

Supremum & infimum

a) The set (S) is set to be bounded above if there exists a number or "real number" such that $s \leq u$ $\forall$ $s \in S$ → upper bound.

Theorem: An upper bound u of a non-empty set $S \in R$ is the supremum iff for all $\varepsilon > 0$ there exists an $\varepsilon \in S$ such that $u - \varepsilon < s_{\varepsilon}$.

$f : D \to R$ $\quad f(x) \leq B$ $\forall$ $x \in D$.

$\sup f(D) \leq \sup g(D)$.

17-08-2022.

Tutorial - 2

$$\sqrt{2+\sqrt{2+\sqrt{2\cdots+\sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}}$$

$n=1$.

$$2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}.$$

$n=k$.

$$\sqrt{2+\sqrt{2+\cdots\sqrt{2}}} = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$$

$n=k+1$

$$2\cos\left(\frac{\pi}{2^{k+2}}\right) \Rightarrow 2\cos\frac{\pi}{(2^{k+1+1})} = 2\cos\left(\frac{\pi}{2^{k+1}\cdot 2}\right)$$

$\Rightarrow$ $P(k+1)$: $\sqrt{2+2\cos\frac{\pi}{2^{k+1}}} = 2\cos\frac{\pi}{2^{k+2}}$.

Q1 show that every integer greater than or equal to 2 can be factored into primes.

$n=2$

$2 \Rightarrow 2, 1$

$n=3$

$3 \Rightarrow 3, 1$

$n=4$

$4 \Rightarrow 2, 2.$

$n=5$

$\Rightarrow 5, 1$

$n=6$

$\Rightarrow 3, 2.$

$n=7$

$7, 1.$

$n=8 \Rightarrow 2\times2\times2.$

~~A~~ $n=9$

$\Rightarrow 3 \times 3$

$n=10$

5×2

Prime 2, 3, 5, 7, 11, 13, 17, 19, 23.

$n=k$

$k = \frac{k}{\text{prime}}$, Prime.

$\Rightarrow$ Step 1 $\Rightarrow$ $n=2$:

since 2 is prime it is factor for itself.

Step II Assume that $P(k)$ holds for all

$k \in \{2, 3, \ldots, k\}$.

Step III: $(k+1)$ prime... [same as base case].

$(k+1) = p \cdot q$. $p, q \leq k+1$. [composite].

$P(k) \Rightarrow P(k+1)$

For a bijection, $|S_1| = |S_2|$. cardinality.

$\Rightarrow$ A set S is called countable if $S \sim T$ for some $T \subseteq \mathbb{N}$.

$\Rightarrow$ If $T \subset \mathbb{N}$, S is countably finite.

$\Rightarrow$ If $T = \mathbb{N}$, S is countably infinite.

Countably finite → $\{1, 2, 3, 4\}$.

Countably Infinite → $\mathbb{Z}$.

Uncountably Finite → Does not exist.

Uncountably infinite → $\mathbb{R}$.

Let's assume.
⇒ There is a set "S" which contains all binary sequences
↳ countably infinite. [assumption].

$S = \quad s_1 = 0\ 0\ 0\ 0 \ldots$
$s_2 = 0\ 1\ 0\ 0 \ldots$
$s_3 = 0\ 1\ 1\ 0 \ldots$
$s_4 = 1\ 0\ 0\ 1 \ldots$

$s_n = 1\ 0\ 0\ 0 \ldots$ [assuming complements of diagonal elements]

We can always find another element s_n which does not exist in our set.

Here s_n is designed such that it does not exist in S

Thus we get a contradiction, thus our assumption that the set "S" as countably infinite is wrong.

Thus "S" is uncountably infinite.

set of all real numbers b/w 0 & 1

eg :- $S_1 =$ 0.3021 ...

0.4900 ...

0.0031 ...

010

18-08-2022.

Sequence and Series:

Archimedian property:

If $x \in \mathbb{R}$, then there exists a natural number n_x, such that $x < n_x$.

Corollaries:

1) $S = \{ 1/n : n \in N \}$. then infimum of $S = 0$.

2). If $t > 0$ $\exists$ $n_t \in N$ such that

$0 < \frac{1}{n_t} < t$.

3). If $y > 0$ $\exists$ $n_y \in N$ such that

$n_y - 1 \leq y \leq n_y$.

The existence of $\sqrt{2}$.

Proof: Let $S = \{ s \in \mathbb{R} \quad 0 \leq s, \; s^2 < 2 \}$.

Since $1 \in S$, it is not empty above.

S is also bounded by 2.

if $t > 2 \rightarrow t^2 > 4 \Rightarrow t \notin S$.

Then supremum property implies that S has a supremum in $\mathbb{R}$.

$\Rightarrow$ Let $x = \sup S$ Note $x > 1$

we will prove $x^2 = 2$. by ruling out $x^2 > 2$ & $x^2 < 2$.

$\Rightarrow$ Let us assume $x^2 < 2$.

consider $n \in \mathbb{N}$ such that $\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$

$\leq x^2 + \frac{2x+1}{n}$

$\frac{2x+1}{n} < 2 - x^2$ $[2 - x^2 > 0] \rightarrow (x + \frac{1}{n})^2 < x^2 + (2 - x^2)$

$\frac{2 - x^2}{2x+1} > 0$, we have some $n \in \mathbb{N}$. $= 2$.

$\frac{1}{n} < \frac{2-x^2}{2x+1}$

$\rightarrow x + \frac{1}{n} \in S \rightarrow$ contradiction $\rightarrow x^2 \not< 2$.

Supremum S.

Now we ~~need~~ ~~to~~ prove x^2

Then Supremum property implies that S has a Supremum in $\mathbb{R}$.

Let $x = \sup S$ Note $x > 1$.

Let $x^2 > 2$ & $m \in N$.

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

choose m, such that $\frac{2x}{m} < x^2 - 2$ $\{x^2 - 2 > 0\}$

$$\left(x - \frac{1}{m}\right)^2 > x^2 - (x^2 - 2) = 2.$$

$$\frac{1}{m} < \frac{x^2 - 2}{2x} \longrightarrow \left(x - \frac{1}{m}\right)^2 > 2.$$

$$\Downarrow$$

contradiction. $x^2 \not> 2$.

Thus

$\boxed{x^2 = 2}$ $[\because x^2 \not< 2$ & $x^2 \not> 2]$.