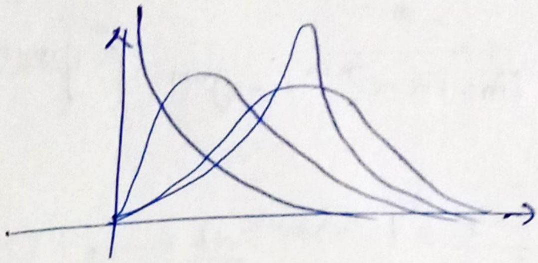
▸ Gamma Distribution

$X \sim \text{gamma}(\alpha, m)$ $[\text{mean} = m\alpha,]$ $[\text{var} = m\alpha^2]$

$Y \sim \text{gamma}(\alpha, n)$

can assume various shapes



$$f_x(x) = \frac{1}{\Gamma m \; \alpha^m} x^{m-1} \exp\left(\frac{-x}{\alpha}\right)$$

where $\Gamma m = \int_{-\infty}^{\infty} t^{mt} e^{-t} dt$

$f_z(z) = ?$ if $Z = \frac{X}{X+Y}$ $\quad 0 < z < 1$

$F_z(z) = P[Z < z]$ $\quad Y \in [0, \infty) \Rightarrow$ $\quad X = XZ + YZ$

$\cancel{x}$. $\quad X(1-Z) + YZ$

$X = \frac{YZ}{1-Z}$ or $Y = \frac{(1-Z)X}{Z}$

since $Y > 0$

$\frac{X(1-Z)}{Z} > 0$ $\quad Z \leq z \Rightarrow \frac{X}{Y} \leq \frac{z}{1-z}$

$$\Rightarrow F_z(z) = \int_0^{\infty} \int_0^{\frac{yz}{1-z}} f_{xy}(x, y)\, dx . dy$$

$$f_z(z) = \int_0^{\infty} \left[\frac{y(1-z) + yz}{(1-z)^2} . f_{xy}\left(\frac{yz}{1-z}, y\right) - 0 + 0 \right] dy$$

$$= \int_0^{\infty} \frac{y}{(1-z)^2} . f_{xy}\left(\frac{yz}{1-z}, y\right) dy$$

$\cancel{\text{let } \frac{yz}{1-z} = m \Rightarrow dm = \frac{y(1-z).1 + y.z}{(1-z)^2}}$

$\cancel{dm = \frac{y}{1-z}}$

$$f_Z(z) = \int_0^\infty \frac{1}{\Gamma(m)\,\Gamma n\,\alpha^{m+n}} \frac{y}{(1-z)^2} \left(\frac{yz}{1-z}\right)^{m-1} y^{n-1} \exp\left(\frac{-1}{\alpha}\left(\frac{yz}{1-z}+y\right)\right) \cdot dy$$

$$= \frac{z^{m-1}}{\Gamma m \cdot \Gamma n\,\alpha^{m+n}(1-z)^{m+1}} \int_0^\infty y^{m+n-1} \exp\left(\underbrace{\frac{-1}{\alpha}\left(\frac{y}{1-z}\right)}_{=t \text{ (say)}}\right) dy.$$

$$\Rightarrow \frac{+y}{\alpha(1-z)} = t \Rightarrow \frac{dt}{dy} = \frac{+1}{\alpha(1-z)} \qquad dy = +\alpha(1-z)\,dt$$

$$= \frac{z^{m-1}}{\Gamma m\,\Gamma n\,\alpha^{m+n}(1-z)^{m+1}} + \int_0^\infty [+t\alpha(1-z)]^{m+n-1} \exp(-t) . (+\alpha(1-z))\,dt$$

$$= \frac{z^{m-1}(1-z)^{m+n-1}\alpha^{m+n-1}}{\alpha^{-1}(1-z)^{-1}} \int_0^\infty (t)^{m+n-1} \exp(-t)\,dt$$

$$= \frac{z^{m-1}(1-z)^{n-1}}{\Gamma m * \Gamma n} \cdot \Gamma(m+n) \quad \leftarrow \beta \text{ distribution.}$$

$$\beta \text{ dis} = Z = \frac{X}{X+Y}$$

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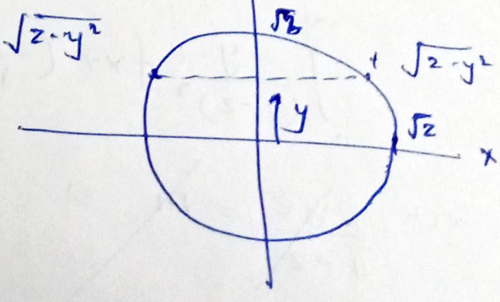
* <u>Sum of square to 2 R.V</u>

$Z = X^2 + Y^2$



$Z < z$

$X^2 + Y^2 < z$

$\Rightarrow \quad y \in [-\sqrt{z}, \sqrt{z}]$

$\quad x \in [-\sqrt{z-y^2}, +\sqrt{z-y^2}]$.

$F_Z(z) =$ PTO

$$F_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{+\sqrt{z-y^2}} f_{XY}(x, y)\, dx \cdot dy$$

$$f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \left[\frac{d}{dz} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{XY}(x,y)\, dx \right] dy$$

$$f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \left[\frac{1}{2\sqrt{z-y^2}} f_{XY}(\sqrt{z-y^2}, y) + \frac{1}{2\sqrt{z-y^2}} f_{XY}(-\sqrt{z-y^2}, y) + 0 \right] dy.$$

$$\cancel{f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} f_{XY}(\sqrt{z-y^2}, y)\, dy}$$

* if $X, Y \sim$ Gaussian $(0, \sigma^2)$ (independent).

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \sim Y$$

$$f_X(\sqrt{z-y^2}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(z-y^2)}{2\sigma^2}}$$

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$f_{XY}(-x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = f_{XY}(x,y)$$

$$\Rightarrow f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} \frac{1}{2\pi\sigma^2} e^{-\frac{(z-\cancel{y^2}+\cancel{y^2})}{2\sigma^2}} dy$$

$$= \cancel{\int_{-\sqrt{z}}^{\sqrt{z}} \left[\frac{+1}{2} \frac{(-2y)}{(z-y^2)^{3/2}} \right]_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\pi\sigma^2} e^{-\frac{z}{2\sigma^2}}}$$

$$= \cancel{2\sqrt{z}}$$

$$f_Z(z) = \frac{\exp\left(\frac{-z}{2\sigma^2}\right)}{2\pi\sigma^2} \cdot \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} dy. = \frac{\exp\left(\frac{-z}{2\sigma^2}\right)}{2\cancel{\pi}\sigma^2} \cdot \cancel{\left[\frac{\pi}{2} + \frac{\pi}{2}\right]}$$

exponential $\longrightarrow$ $\boxed{f_Z(z) = \exp\left(\frac{-z}{2\sigma^2}\right) \cdot \frac{1}{2\sigma^2}}$

__verification:__ $\mathbb{E}[Z] = \mathbb{E}[Y^2 + X^2] = \mathbb{E}[Y^2] + \mathbb{E}[Z^2] = 2\sigma^2$

Ex: $Y, X \sim N(0, \sigma^2)$

$Z = \sqrt{X^2 + Y^2}$

$Y \in [-z, z]$ $x = \sqrt{z^2 - y^2}$

$X \in [-\sqrt{z^2 - Y^2}, +\sqrt{z^2 - y^2}]$

$$F_Z(z) = \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{XY}(x, y)\, dx\, dy.$$

$$f_Z(z) = \int_{-z}^{z} \left[\frac{d}{dz} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{XY}(x, y)\, dx\right] dy$$

$$= \int_{-z}^{z} \left[\frac{\cancel{2} z}{\cancel{2}\sqrt{z^2-y^2}} f_{XY}(\sqrt{z^2-y^2}, y) + \frac{1}{\cancel{2}} \frac{\cancel{2} z}{\sqrt{z^2-y^2}} f_{XY}(-\sqrt{z^2-y^2}, y)\right] dy$$

$$= \int_{-z}^{z} \frac{z}{\sqrt{z^2-y^2}} \left[f_{XY}(\sqrt{z^2-y^2}, y) + f_{XY}(-\sqrt{z^2-y^2}, y)\right] dy$$

$$= \int_{-z}^{z} \frac{\cancel{2} z}{\sqrt{z^2-y^2}} \frac{1}{\cancel{2}\pi\sigma^2} \exp\left(-\frac{(z^2 - \cancel{y^2} + \cancel{y^2})}{2\sigma^2}\right) dy.$$

$$= \frac{z}{\pi\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right) \cdot \int_{-z}^{z} \frac{1}{\sqrt{z^2-y^2}}\, dy$$

$$= \frac{2z}{\pi\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right) \int_{0}^{z} \frac{dy}{\sqrt{z^2-y^2}}$$

$$\boxed{f_Z(z) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}} \leftarrow \text{Rayleigh Dist.}$$

$$E[Z] = \sigma\sqrt{\frac{\pi}{2}}$$

$$Var[Z] = \left(\frac{4-\pi}{2}\right) \cdot \sigma^2$$

▶ 2 functions of 2 R.V.

$z = g(X,Y)$ and $w = h(X,Y)$.

$F_{zw}(z,w) = P[\, g(X,Y) < z,\ h(X,Y) < w]$.

let $D_{zw} = \{(x,y) : g(x,y) < z,\ h(x,y) < w\}$.

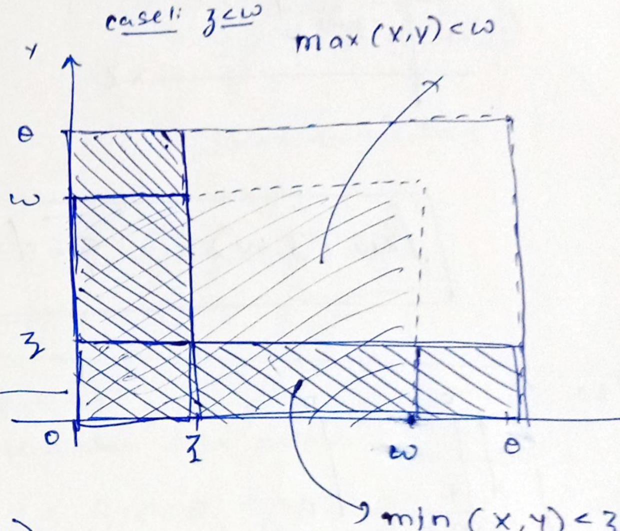
then.

$$F_{zw}(z,w) = \iint_{D_{zw}} f_{XY}(x,y)\, dx.dy.$$

Example: $\left.\begin{array}{l} Z = \min(X,Y) \\ W = \max(X,Y). \end{array}\right\}$ X, Y are iid over $[0, \theta]$

$D_{wz} = \{Z < z,\ W < w\}$

case 1: $z < w$



$F_{zw}(z,w) = F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z)$

if $z < w$

$F_{zw}(z,w) = F_{XY}(w,w)$

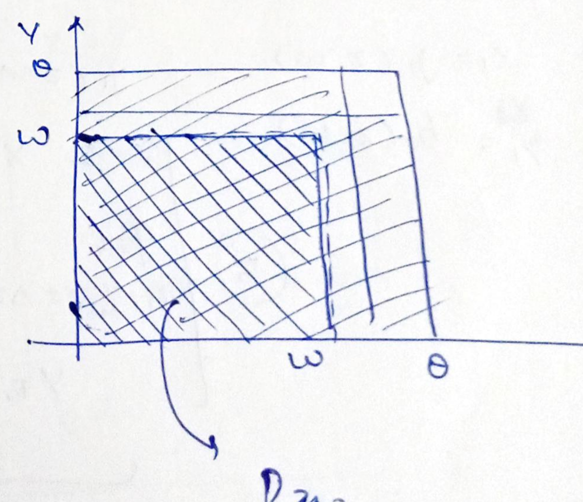
if $z \geq w$

D_{zw}

since $X, Y \sim U[0,\theta]$.

$f_{XY}(x,y) = \dfrac{x.y}{\theta^2}$ (since independent).

case 2: $z >= w$



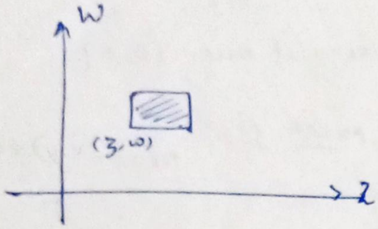
$$\boxed{\Rightarrow\ F_{zw}(z,w) = \begin{cases} \dfrac{zw}{\theta^2} + \dfrac{zw}{\theta^2} - \dfrac{z^2}{\theta^2} & z < w \\[2ex] \dfrac{w^2}{\theta^2} & z \geq w \end{cases}}$$

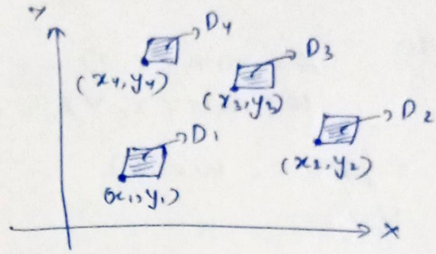
$\cancel{f_{z,w}(z,w) = d}$

* Joint Density of Z and W

$Z = g(X, Y)$
$W = h(X, Y)$

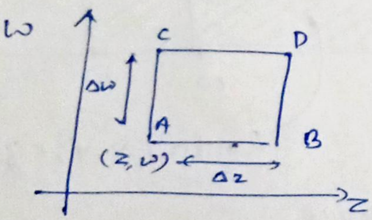
$$P[z < Z < z+\Delta z,\ w < W < w+\Delta w] = f_{ZW}(z, w)\cdot \Delta z \cdot \Delta w$$
$$= P(X, Y \in \cup D_i)$$
$$= \sum_i f_{XY}(x_i, y_i)\cdot \Delta_i$$

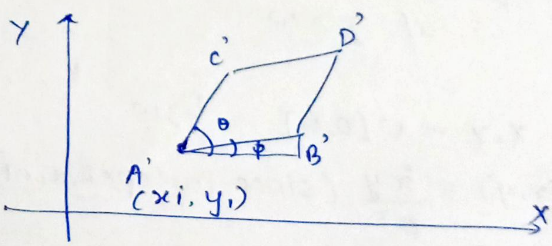




let area $(D_i) = \Delta_i$

$$\boxed{f_{ZW}(z, w) = \sum_i f_{XY}(x_i, y_i)\cdot \frac{\Delta_i}{\Delta z\, \Delta w}}$$





let $X_1 = g_1(z, w)$
$Y_1 = h_1(z, w)$

$$g_1(z+\Delta z, w) = g_1(z, w) + \frac{d}{dz} g_1(z, w)\cdot \Delta z$$

B'

$$X_{B'} = X_1 + \frac{d}{dz} g_1(z, w)\cdot \Delta z$$

$$h_1(z+\Delta z) = h_1(z, w) + \frac{d}{dz} h_1(z, w)\cdot \Delta z$$

$$Y_{B'} = Y_1 + \frac{d}{dz} h_1(z, w)\cdot \Delta z$$

1st order approximation.

$$\boxed{\begin{aligned} X_{B'} - X_1 &= A'B' \cos\phi \\ Y_{B'} - Y_1 &= A'B' \sin\phi \end{aligned}}$$

for ©.

$$x_0 = g_1(z, w+\Delta w) = \underbrace{x_1 + \frac{dg_1}{dw}\Delta w}_{\to A'C'\cos\theta}$$

$$y_0 = h_1(z, w+\Delta w) = \underbrace{y_1 + \frac{dh_1}{dw}\cdot\Delta w}_{\to A'C'\sin\theta}$$

$$\Delta i = A'B' \cdot A'C' \cdot \sin(\theta - \phi)$$

$$= A'B'\cos\phi \, A'C'\sin\theta - A'B'\cos\theta \, A'C'\sin\phi$$

$$\Delta i = \left(\frac{dg_1}{dz}\cdot\frac{dh_1}{dw} - \frac{dh_1}{dz}\cdot\frac{dg_1}{dw}\right)\Delta z \cdot \Delta w$$

$$\Rightarrow \frac{\Delta i}{\Delta z\,\Delta w} = \left(\frac{dg_1}{dz}\frac{dh_1}{dw} - \frac{dh_1}{dz}\cdot\frac{dg_1}{dw}\right)$$

$$\frac{\Delta i}{\Delta z\,\Delta w} = \begin{vmatrix} \frac{dg_1}{dz} & \frac{dh_1}{dz} \\ \frac{dg_1}{dw} & \frac{dh_1}{dw} \end{vmatrix} = |J(z,w)| = \frac{1}{|J(x_i, y_i)|}$$

Jacobian matrix.

$$\boxed{f_{w,z}(z,w) = \sum_i |J(z,w)| \cdot f_{xy}(x_i, y_i)}$$

Example: X, Y are iid $\sim N(0, \sigma^2)$

converting cartisian coordinates into polar.

$R = \sqrt{x^2+y^2}$ $\qquad X = R\cos\theta \leftarrow g_1$

$\theta = \tan^{-1}(y/x)$ $\qquad Y = R\sin\theta \leftarrow h_1$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(x^2+y^2)}{2\sigma^2}\right)$$

$$J(R,\theta) = \begin{vmatrix} \frac{dX}{dR} & \frac{dY}{dR} \\ \frac{dX}{d\theta} & \frac{dY}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -R\sin\theta & R\cos\theta \end{vmatrix}$$

$$J(R,\theta) = R\cos^2\theta + R\sin^2\theta = R$$

$$\cancel{f_{w,z}(z,w)} \; \boxed{f_{R,\theta}(r,\theta) = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-r^2}{2\sigma^2}\right)\cdot r}$$

$$\int_{r=0}^{\infty} f_{R\theta}(r,\theta)\, dr = \frac{1}{2\pi} \leftarrow \text{Marginal Dist of}$$

$$\int_{\theta=-\pi}^{\theta=\pi} f_{R,\theta}(r,\theta)\, d\theta = \frac{r}{\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2}\right) \leftarrow \text{marginal Dist of } R$$

$\Rightarrow$ R and θ are independent.

<u>Example:</u> X, Y iid $\sim Exp(\lambda)$.

$$\left.\begin{aligned} Z &= X+Y \\ W &= X-Y \end{aligned}\right\} \rightarrow \quad X = \underbrace{\frac{Z+W}{2}}_{g_1}, \quad Y = \underbrace{\frac{Z-W}{2}}_{h_1}$$

(single root).

$$J(z,w) = \begin{vmatrix} \frac{dg_1}{dz} & \frac{dh_1}{dz} \\ \frac{dg_1}{dw} & \frac{dh_1}{dw} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$f_{XY}(x,y) = \cancel{\lambda^2 e^{2\lambda}} \; \lambda^2 e^{-\lambda x} \cdot e^{-\lambda y} = \lambda^2 e^{-\lambda(x+y)}.$$

$$f_{Z,W}(z,w) = \lambda^2 e^{-\lambda(x+y)} \cdot \left|\left(-\frac{1}{2}\right)\right|$$

$$f_{Z,W}(z,w) = +\frac{\lambda^2}{2} e^{-\lambda(x+y)} = \frac{\lambda^2}{2} e^{-\lambda z}$$

$$\cancel{f_Z(z) = \underbrace{\left(\frac{\lambda^2 e^{-\lambda z}}{2}\right)}_{k} \cdot \int_{-\infty}^{\infty} dw = \infty \; k.}$$

$$\boxed{0 < |w| \le z < \infty}$$

$\Rightarrow$ $f_{Z,W}(z,w) = \frac{\lambda^2}{2} \exp(-\lambda z)$ is true for $0 < |w| \le z < \infty$

$f_{Z,W}(z,w) = 0$ otherwise

$$f_w(w) = \int_{|w|}^{\infty} f_{zw}(z, w) \cdot dz$$

$$= \cancel{\frac{\lambda^2}{2}} \frac{\lambda}{2} \left[\frac{e^{-\lambda z}}{\cancel{(-\lambda)}} \right]_{|w|}^{\infty} = \frac{1}{2} \left[0 - e^{-\lambda|w|} \right].$$

marginal dist. of w → $\boxed{f_w(w) = +\frac{\lambda}{2} e^{-\lambda|w|}}$

1). Markov: $P(X>a) \le \frac{E[X]}{a}, \; X \ge 0$

2) Chebyshev $P(|X-\mu| > \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$

3) Chernoff $P[X>a] \le e^{-sa} M_X(s)$

$$P[X>a] \le e^{-\phi_X(a)}$$

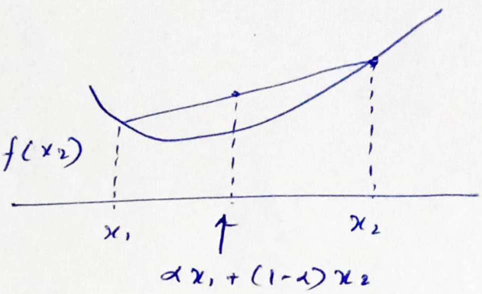
$$\phi_X(a) = \max_{s>0} \; sa - \ln(M_X(s))$$

Convex function:

if

$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2)$

then function $f(x)$ is convex



Now lets find $E[f(x)]$.

$$E[f(x)] = \sum_{i=1}^{N} \alpha_i f(x_i) \ge f\left(\sum_{i=1}^{N} \alpha_i x_i\right)$$

probabilities ($\Sigma = 1$)

$\Rightarrow$ $\boxed{E[f(x)] \ge f(E[X])}$ (Jensens Inequality)

The opposite result for

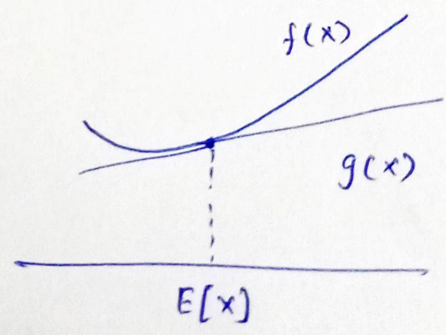
$$E[f(x)] \ge E[g(x)]$$

$\hookrightarrow a + bx$

$= a + bE[X]$

$= g(E[X])$

$= f(E[X])$

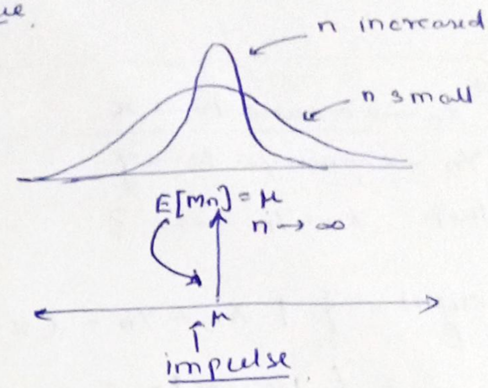


$X_1, X_2, \ldots + X_n \Rightarrow$ iid mean $= \mu$ and var $= \sigma^2$

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

M_n is R.V. for any n

but $M_n \to (\mathbb{E}[M_n] = \mu)$ as $n \to \infty$ } $\to$ Prove



* $E[M_n] = \mu$

* $\text{var}[M_n] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$$P[\,|M_n - \mu| > \epsilon\,] < \frac{\sigma^2}{n\epsilon^2} = 0 \quad \text{as } n \to \infty$$

* As $n \to \infty$, samples mean goes very close to mean.

(M_n lies close to μ with v. high prob. as $n \to \infty$)

[Weak Law of Large No.s]

Ex

$X_i \sim$ Bernoullie (0.5)

$X_n = \frac{1}{n} \sum X_i \to \tilde{X}_n$ (say) is Binomial $(n, 0.5)$

$$P = P[\,|X_n - 0.5| \le 0.1\,]$$
$$= P[\,0.4N < \tilde{X}_n < 0.6N\,]$$
$$= F_B(0.6N) - F_B(0.4N)$$

(F_B is cdf for Binomial).

(for $\epsilon = 0.1$, $p = 0.5$)

N	P
10	0.65
50	0.84
100	0.96
500	0.999
1000	1.000

$$\lim_{n\to\infty} P[\,|Y_n - \mu| > \epsilon\,] = 0$$

OR

$$\lim_{n\to\infty} P[\,|Y_n - \mu| \le \epsilon\,] = 1$$

Let

$X_n \rightarrow$ converges to $\rightarrow x$

$Y_n \rightarrow$ converges to $\rightarrow y$

then $X_n + Y_n \longrightarrow ?$

event $E = \{\, |X_n + Y_n - (x+y)| > \epsilon \,\}$

$= \{\, |X_n - x| > \frac{\epsilon}{2}$ OR $|Y_n - y| > \frac{\epsilon}{2} \,\}$

$\Rightarrow$ ~~$P(E)$~~

$X_n + Y_n \longrightarrow (x+y)$

---

if $\lim_{n\to\infty} E[(X_n - c)^2] = 0$

(~~then~~ X_n converge in mean square)

P.T. X_n converge in Prob.

$$P[\,|X_n - c|^2 > \varepsilon^2\,] \le \frac{E[(X_n - c)^2]}{\varepsilon^2} = 0 \quad \text{as } n \to \infty$$

$\Rightarrow$ X_n converge to c in Prob.

WLOLN

$X_1, X_2, \dots X_n$ are iid

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} \longrightarrow \mu \quad \left\{ \begin{matrix} \text{as} & n \to \infty \\ & M_n \to \mu \end{matrix} \right\}$$

OR

$$\lim_{n\to\infty} P(|M_n - \mu| > \varepsilon) = 0$$

$X_1, X_2, \dots X_n$ be iid

$$S_n = X_1 + X_2 + X_3 + \dots + X_n$$

$$E[S_n] = n.\mu$$

$$Var[S_n] = n.\sigma^2$$

Let $$Z_n = \frac{S_n - \mu n}{\sigma\sqrt{n}} = \frac{\tilde{X}_1 + \tilde{X}_2 + \dots + \tilde{X}_n}{\sigma\sqrt{n}}$$ where $\tilde{X}_i = X_i - \mu$

$\sum \tilde{X}_i = \sum X_i - \mu n$

Now finding MGF of Z_n.

$$M_{Z_n}(s) = E[e^{sZ_n}] = E\left[e^{s\frac{\sum\tilde{X}_i}{\sigma\sqrt{n}}}\right].$$

$$= E\left[\prod_{i=1}^{n} e^{\frac{s}{\sigma\sqrt{n}}\tilde{X}_i}\right]$$

$E[X^2] - E[X]^2 = 1$

$$= \prod_{i=1}^{n} E\left[e^{\frac{s}{\sigma\sqrt{n}}\tilde{X}_i}\right]$$

$$= E\left[e^{\frac{s}{\sigma\sqrt{n}}\tilde{X}_i}\right]^n$$

$$\boxed{M_{Z_n}(s) = [M_{\tilde{X}_i}(s')]^n} \qquad s' = \frac{s}{\sigma\sqrt{n}}$$

$$M_{Z_n}(s) = \Big[\underbrace{M_{\tilde{X}}(0)}_{1} + \underbrace{\frac{d}{ds'} M_{\tilde{X}}(s')\Big|_{s'=0}}_{E[\tilde{X}]=0}(s'-0) + \underbrace{\frac{1}{2!}\frac{d^2}{ds'^2} M_{\tilde{X}}(s')\Big|_{s'=0}}_{E[\tilde{X}]=\sigma^2}(s'-0)^2\Big]$$

$=$

$$M_{Z_n}(s) = \left[1 + 0 + \frac{1}{2}\sigma^2\frac{s^2}{\sigma^2 n}\right]^n$$

$$\lim_{n\to\infty} M_{Z_n}(s) = \lim_{n\to\infty}\left[1 + \frac{s^2/2}{n}\right]^n = e^{s^2/2}$$

$\hookrightarrow$ MGF(N(0,1))

Let $X_1, X_2, \dots X_n$ be a sequence of iid random variables with mean μ and variance σ^2 and

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} , \quad (S_n = \Sigma X_i).$$

Then, as $n \to \infty$

$$P[Z_n < c] = \phi(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} \exp\left(\frac{-x^2}{2}\right) dx$$

(Central Limit Theorem)

Now we find M_n in term of Z_n.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow \frac{S_n}{n} = \frac{Z_n \sigma\sqrt{n} + n\mu}{n} = M_n$$

$$\Rightarrow \boxed{M_n = Z_n\left(\frac{\sigma}{\sqrt{n}}\right) + \mu}$$

if $Z_n = N(0,1) \Rightarrow \boxed{M_n = N\left(\mu, \frac{\sigma^2}{n}\right)}$ } (valid for Large n only)

- **LOLN**: M_n get close to true mean μ as n grows large
- **CLT**: $M_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ as n grows Large

$$\boxed{\begin{aligned} S_n &= \sigma\sqrt{n} Z_n + \mu n \\ &\hookrightarrow N(n\mu, n\sigma^2) \end{aligned}}$$

Q. 100 item, $w_i \sim U[5, 50]$. $S_{100} = \sum_{i=1}^{100} w_i$ $P[S_{100} > 3000] =$

$S_{100} \sim N(n\mu, n\sigma^2)$. $\mu = \frac{55}{2}$ $\sigma^2 = \frac{(45)^2}{12}$

So $P[S_{100} > 3000] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{3000}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

We know $\lim_{n\to\infty}$ (p = moderate). $\text{Bino}(p, n) \sim N(np, npq)$.

we can prove it using CLT.

$\text{Bino}(p, n) = \sum_{i=1}^{n} \text{Bernoullie}(p)$; var = pq, mean = p

if $n \to \infty$ $\text{Bino}(p, n) = N(np, npq)$.

if we want to find $P[k \le S_n \le l]$.

Alternative method:

$k \le S_n \le l$

$\Rightarrow P\left[\frac{k-np}{\sqrt{npq}} \le \frac{S_n - np}{\sqrt{npq}} \le \frac{l-np}{\sqrt{npq}}\right]$

$\Rightarrow \left[\frac{k-np}{\sqrt{npq}} \le Z_n \sim N(0,1) \le \frac{l-np}{\sqrt{npq}}\right] = \phi\left(\frac{l-np}{\sqrt{npq}}\right) - \phi\left(\frac{k-np}{\sqrt{npq}}\right)$

Q. If X is the number scored in a throw of a fair die, then show that the chebyshef ineq $P[|X-\mu|>2.5]<0.47$ (μ: mean of X)

while the actual prob is 0

$\Rightarrow$ $\mu = \frac{6\times7}{2\times6} = 3.5$?

$\sigma^2 = E[x^2] - E[x]^2$

x^2 : 1 4 9 16 25 36.

$P[|X-\mu|>\varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$

$E[x^2] = \frac{n(n+1)(2n+1)}{6\times6}$

$= \frac{6\cdot7\cdot13}{6\cdot6} = \frac{13\cdot7}{6}$

$\frac{35\times4}{12\times25}$

$\boxed{\mu=3.5}$

$\sigma^2 = \frac{13\cdot7}{6} - \frac{7\cdot7}{4}$

$\leq \frac{7\times2}{6\times5}$

$X-\mu>2.5$

$= \frac{7}{2}\left[\frac{13\times2}{3\times2} - \frac{3\cdot7}{2\times3}\right] = \frac{(26-21)\times7}{2\times6}$

$\leq \frac{14}{30}$

$\mu - X$

or $\mu - X > 2.5$

$\sigma^2 = \frac{35}{12}$

$X>6$

$X<1$

Q How large a sample must be taken in order that Prob will be atleast 0.95 that $\bar{X}_n$ will be within 0.5 of μ (μ is unknown, $\sigma=1$)

$\bar{X}_n = \frac{X_1+X_2+\cdots X_n}{n}$

$E[\bar{X}_n] = \mu$

$P[(\bar{X}_n-\mu)<K] \geq 1-\frac{\sigma^2}{K^2} \rightarrow 0.95$

$P[(\bar{X}_n-\mu)<0.5] < 1-\frac{1}{(0.5)^2}$

$Var(\bar{X}_n) = \frac{\sum Var(X_i)}{n^2} = \frac{\sum 1}{n^2} = \frac{n}{n^2} = \frac{1}{n}$

13

Q
// $X_1, X_2 \ldots X_{20}$
be ind poisson r.v with mean 1. Use the markov ineq. to apply bound
on ~~prob of~~ $P\left(\sum_{i=1}^{20} X_i > 15\right)$ sin each X_i follows Poisson(mean = 1)

$$E\left[\sum_{i=1}^{20} X_i\right] = 20$$

$$P\left[\sum_{i=1}^{20} X_i > 15\right] < \frac{E[\sum X_i]}{15} = \frac{\cancel{20}^{4}}{\cancel{15}_{3}} = \frac{4}{3}$$

Q If 10 fair die are rolled, find prob that the sum ^approx is b/w 30 & 40

$\boxed{X = X_1 + \ldots X_{10}}$

X_i: No. opp. on j^{th} die

$$E[X_i] = \cancel{\frac{6\times7}{6\times2}} \cdot \cancel{\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{7}{6}} = \frac{7}{2}$$

$\boxed{E[X] = 35}$

$$Var(X_i) = \frac{35}{12}$$

$$\begin{array}{r} 35 \\ \times 5 \\ \hline 175 \end{array}$$

$$Var(X) = \sum Var(X_i) = \cancel{10}^{5} \cdot \frac{35}{\cancel{12}_{6}} = \frac{35\times5}{6} = \frac{175}{6}$$

$$\boxed{Var(X) = \frac{175}{6}} = \frac{29.1}{6} = \sigma^2$$

$$P\left[\,|\sum X_i - \mu| < 5\,\right] > 1 - \frac{\sigma^2}{\epsilon^2} = 1 - \frac{\left(\frac{175}{6}\right)^2}{25} \qquad 1 - \left(\frac{35}{6}\right)^2$$

$$\frac{36}{36} - \frac{35^2}{36}$$

$$30 < X < 40$$

$$\frac{30-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{40-\mu}{\sigma}$$

$$N \sim (0,1)$$

$$\frac{30-35}{\sqrt{\frac{175}{6}}} < N \sim (0,1) < \frac{40-35}{\sqrt{\frac{175}{6}}}$$

$$\frac{-5}{5.4006} < \qquad < \frac{5}{5.4006}$$

$$-0.9258 < \qquad < 0.9258$$

$$\begin{array}{r} 3159 \\ \times 2 \\ \hline 0.6318 \end{array}$$

$= 2 \times$ ~~3159~~ ~~0.3159~~ 0.3212

$\boxed{\text{Ans} = \cancel{0.6318}}$

0.6424

Q. If X is a random variable with $E[X] = 3$ and $E[X^2] = 13$
Then use WLOL to determine lower bound for

$P[-2 \le X \le 8]$ $\sigma^2 = 13 - 9 = 4$

$P[|X-3| \le 5]$

$P[|X-\mu| \le 5] \ge 1 - \frac{\sigma^2}{\epsilon^2} = \frac{25}{25} - \frac{4}{25} = \frac{21}{25}$

$-2 < X < 8$

$\frac{-2-3}{2} < \frac{X-\mu}{\sigma} < \frac{8-3}{2}$

$\frac{-5}{2} < \quad < \frac{5}{2}$

$-2.5 < \quad < 2.5$

$= 2 \times 0.4938$

$4938 \times 2 = 9876$

$= 0.9876$ (Ans)

---

1) WLOLN: $M_n \to \mu$ in Prob

2) CLT: $M_n \sim N(\mu, \sigma^2/n)$.

Convergence with Prob 1. (almost sure)

$Y_n \to c$ with Prob 1. if

$P[\lim_{n\to\infty} Y_n = c] = 1$

* Let $X_1, X_2, \ldots$ and $E[\sum_{i=1}^{\infty} |X_i|] < \infty$ (is finite)

then $\{X_n\} \to 0$ with Prob 1

(Valid only if X_i are all +ve quantities).
(or better consider $|X_i|$)

Example: Let $E[X_i] = 0 \; \forall i$ $\quad \left(M_n = \frac{1}{n} \sum X_i \right)$

then clearly $E[M_n] = 0 < \infty$

$\Rightarrow \left(\lim_{n \to \infty} M_n \right) \to 0$ with prob 1.

"We will Revisit this topic in next class"

(Dimitri Bertsekas : Intro to Prob)

$X = [X_1 \; X_2 \; X_3 \ldots X_N]^T$ (column vector)

R.V. vector

$f_X(x) = \prod_{i=1}^{N} f_{X_i}(x_i)$ if all X_i are independent.

(vector of x_i)

However, we can parametrize $f_X(x)$ using (μ, K)

mean vector — covariance matrix.

$$\mu = \left[E[X_1] \; E[X_2] \; \ldots \; E[X_n] \right]^T$$

$$\mu_j = \int_{-\infty}^{\infty} x_i f_{X_i}(x_i)\, dx_i$$

$$\boxed{f_{X_i} = \underbrace{\iiint \ldots \int}_{N-1} f_X(x) \underbrace{dx_1 \ldots dx_N}_{n-1}}$$

$$\boxed{K = E\left[(X - \mu)(X - \mu)^T \right]}$$ ← Covariance matrix. $(N \times N)$

$$K_{ij} = E\left[(X_i - \mu_i)(X_j - \mu_j) \right]$$

$$K_{ij} = \begin{cases} \operatorname{cov}(X_i, X_j), & i \neq j \\ \sigma_{X_i}^2, & i = j \end{cases}$$

$$K = \begin{bmatrix} \sigma_{X_1}^2 & cov(X_1, X_2) & \cdots & cov(X_1, X_n) \\ cov(X_2, X_1) & \sigma_{X_2}^2 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ & & & \sigma_{X_n}^2 \end{bmatrix}_{N \times N}$$

symmetric

$$K = \begin{bmatrix} E[X_1^2] - E[X_1]^2 & E[X_1X_2] - E[X_1]E[X_2] & \cdots \\ E[X_2X_1] - E[X_2]E[X_1] & E[X_2^2] - E[X_2]^2 & \\ \vdots & & \ddots \\ & & & E[X_n^2] - E[X_n]^2 \end{bmatrix}$$

$$K = \underbrace{E[XX^T]}_{R\,(say)} - \underbrace{E[X]E[X]^T}_{\mu \cdot \mu^T}$$

$$\Rightarrow \boxed{K = R - \mu\mu^T}$$

R: correlation matrix; μ: mean vector

Now Note:

* K is ~~diaognal~~ zero when X_i are ind.
* If X_i, X_j are ind. then $R_{ij} = E[X_iX_j] = E[X_i]E[X_j] = \mu_i\mu_j$ $\Rightarrow K = 0$
* K is positive definite matrix (diagonal values +ve) (det = +ve)

K is called positive definite iff $\forall z$, $z^TKz > 0$

Proof

$$z^TKz = z^T E[(X-\mu)(X-\mu)^T] z$$
$$= E[z^T(X-\mu)(X-\mu)^T z]$$
$$= E[Y^T \cdot Y]$$
$$= E[\|Y\|^2] \geq 0$$

Since k is positive definite, $\lambda_i \geq 0$ and $|k| \geq 0$

---

given a random variable vector X which is correlated.
can we convert X to Y (which is uncorrelated)

let $\boxed{Y = \phi X}$

we want to make Y (uncorrelated)
so that
k_Y is diagonal

we know $\phi_i^T \phi_i = 1$ or $\|\phi_i\| = 1$

We know $K_X \phi_i = \lambda_i \phi_i$

$$K_X \underbrace{[\phi_1 \cdots \phi_M]}_{\phi} = [\phi_1 \cdots \phi_M] \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_M \end{bmatrix}}_{\Lambda}.$$

$\boxed{K_Y = \phi^T K_X . \phi = \Lambda}$ as $\boxed{\begin{array}{l} \phi^T \phi = \phi \phi^T = I \\ \Rightarrow \phi^T = \phi^{-1} \end{array}}$

Now from this result
let $Y = \phi^T X.$

$\Rightarrow$ $E[Y] = \phi^T \mu = \mu_Y$

$K_Y = E[(Y - \mu_Y)(Y - \mu_Y)^T]$ let $\boxed{\mu_Y = 0}$

$K_Y = E[Y . Y^T]$

$= E[(\phi^T X)(\phi^T X)^T] = E[\phi^T X X^T \phi] = \phi^T E[X X^T] \phi$

$= \phi^T E[X X^T] \phi$

$\boxed{K_Y = \phi^T . K_X \phi}$ (verified). for $\boxed{\mu_X = \cancel{\mu_Y} = 0}$ $= 0$

but we can prove for any $\mu_X \neq 0$,

$\boxed{Y = \phi^T . X}$ will give uncorrelated R.Vs.

[Also called as whitening of noise
$X \rightarrow$ coloured noise (correlated) $\Longrightarrow$ $Y \rightarrow$ (white noise) (uncorrel.

Let $X_1, X_2 \ldots X_n$ are iid. and $S_n = \sum_{i=1}^{n} X_i$ and $M_n = \frac{S_n}{n}$

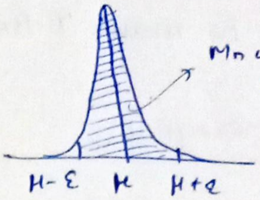
then $\lim_{n\to\infty} M_n = \mu$ with prob 1.

OR

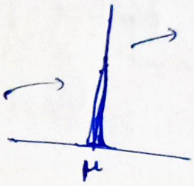
$$\boxed{P(\omega \in \Omega : \lim_{n\to\infty} M_n(\omega) = \mu) = 1}$$ (Almost sure conv.) (conv. with Prob 1)

strong Law of Large No

WLOLN: $\lim_{n\to\infty} P[\,|M_n - \mu| > \varepsilon\,] = 0$



$\forall\ \varepsilon > 0$



Let us define an event $\varepsilon = \{\omega \in \Omega : M_n \neq \mu\}$.

we have to prove $P[\varepsilon] = 0$

Let $\boxed{E[X_i] = \mu = 0}$

if $M_n \neq \mu = 0$

then $S_n \neq n\mu = 0$

this is same as $P[\,|S_n| > n\delta\,]$

$$P[\,|S_n| \geq n\delta\,] = P[\,|S_n|^4 \geq n^4 \delta^4\,] \leq \frac{E[S_n^4]}{n^4 \delta^4}$$ (using markovs ineq.)

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{ijkl} E[X_i X_j X_k X_l].$$

cases	Exp
$\{ i \neq j, k, l$	0
$i = j \neq k = l$; $i = k \neq j = l$; $i = l \neq k = j$	$E[X_i^2 X_k^2]\ 3.n(n-1)$
$\{ i = j = k = l$	$E[X_i^4].n$

Now also

$\left(x \cdot y \leq \frac{x^2+y^2}{2} \right)$

$$\Rightarrow \sum_{ijkl} E[X_i X_j X_k X_l] = n.E[X_i^4] + 3n(n-1)E[X_1^2 X_k^2]$$

$$\text{''} \leq n.E[X_i^4] + 3n(n-1)\frac{E[X_i^4 + X_k^4]}{2}$$

$$\text{''} \leq n E[X_i^4] + 3n(n-1)E[X_i^4].$$

$$\text{''} \leq (3n^2 - 2n)E[X_i^4]$$

$$\text{''} \leq 3n^2 E[X_i^4].$$

$\Rightarrow$ we get $P[|S_n| > n\delta] \leq \frac{3n^2.\ E[X_i^4]}{n^4 \delta^4}$

$$P[|S_n| > n\delta] \leq \frac{3E[X_i^4]}{n^2\delta^4}$$

$$\sum_{n=1}^{\infty} P[|S_n| > n\delta] \leq \sum_{n=1}^{\infty} \frac{c}{n^2} < \infty \text{ (is finite).}$$

$\Rightarrow$ the no. of times $|S_n| > n\delta$ is finite

$\Rightarrow$ ~~P(E)~~ $\boxed{P[\lim_{n\to\infty} |S_n| > n\delta| = 0 \quad \forall \delta > 0}$

$\Rightarrow$ $\boxed{P(\mathcal{E}) = 0}$

Covariance matrix:

$$X = [X_1 \; \dots \; X_n]^T . \quad \mu = [\mu_1 , \dots \; \mu_n]^T$$

$$K = E[\,(X-\mu)(X-\mu)^T]$$

$$K_{ij} = \begin{cases} E[\,(X_i - \mu_i)(X_j - \mu_j)] = \operatorname{cov}(X_i, X_j) & \text{if } i \neq j \\ \sigma_i^2 & \text{if } i = j \end{cases}$$

$X_i \sim$ i and $N(\mu_i, \sigma_i^2)$.

$$f_X(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i-\mu_i)^2}{2\sigma_i^2} \right)$$

$$f_X(x) = \frac{1}{(2\pi)^{n/2}(\sigma_1 . \sigma_2 \dots \sigma_n)} \cdot \exp\Big(-\frac{1}{2} \underbrace{\sum_{i=1}^{n} \frac{(x_i-\mu_i)^2}{\sigma_i^2}}\Big).$$

$$\downarrow$$

$$\Big\{ \underbrace{[(x_1-\mu_1) \dots (x_n-\mu_n)]}_{(X-\mu)^T} \underbrace{\begin{bmatrix} \frac{1}{\sigma_1^2} & \cdots & \\ \vdots & \frac{1}{\sigma_2^2} & \\ & & \ddots & \frac{1}{\sigma_n^2} \end{bmatrix}}_{K^{-1}} \underbrace{\begin{bmatrix} (x_1-\mu_1) \\ \vdots \\ (x_n-\mu_n) \end{bmatrix}}_{(X-\mu)} \Big\}$$

also. $(\sigma_1 \sigma_2 \dots \sigma_n) = \det(K)^{1/2}$

$$f_X(x) = \frac{1}{(2\pi)^{n/2} [\det(K)]^{1/2}} \cdot \exp.\left(-\frac{1}{2} (X-\mu)^T K^{-1} (X-\mu) \right) \; ; \; x \in \mathbb{R}^n$$

$$\downarrow$$

$N(\mu, K)$.

But here K is diagonal (X_i are ind).

what if K is not diagonal, (~~case~~ does this expression follows pdf)

Let $K \neq$ diaognal

but K is positive definite $\qquad \int_{\mathbb{R}^n} f_X(x)\, dx = 1.$

Let $(2\pi)^{n/2} (\det(K))^{1/2} = \alpha.$

we have to show $\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(X-\mu)^T K^{-1}(X-\mu)\right) dx = \alpha.$

Let $Z = X - \mu$ $\qquad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} Z^T . K^{-1} Z\right) dz = \alpha.$

Let $K = U \Sigma U^T$

$(UU^T = U^T U = I)$

and $C = \Sigma^{1/2} U^T$

then $\boxed{K^{-1} = U \Sigma^{-1} U^T}$

and $\boxed{C K^{-1} C^T = I}$

(can be verified).

Now let $Z = C^T Y.$

$\Rightarrow \quad Z^T K^{-1} Z$

$= Y^T C . K^{-1} C^T Y$

$= Y^T Y$

$= \|Y\|^2 = \Sigma Y_i^2$

Now also. since $Z = C^T . Y$

$dz = |\det(C)| . dy$

$\Rightarrow$ Int $\cancel{f_X(x)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \Sigma Y_i\right) dy \; |\det(C)|$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} y_1^2\right) . \exp\left(-\frac{1}{2} y_2^2\right) \ldots \exp\left(-\frac{1}{2} y_n^2\right) . dy_1 \ldots dy_n \,|\det(C$$

$$= \left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} y_i^2\right) dy_i\right)^n |\det(C)|$$

$$= (2\pi)^{n/2} |\det(C)|$$

but by our definition $\quad K = C . C^T \Rightarrow \; |K| = |C|^2$

$\det(C) = (\det(K))^{1/2}$

$\Rightarrow$ Int $= (2\pi)^{n/2} [\det(K)]^{1/2} = \alpha$

$$\Rightarrow \quad \boxed{\int_{-\infty}^{\infty} f_X(x)\, dx = 1}$$

We got

$$f_X(x) = \frac{1}{(2\pi)^{N/2}\,|\det(K)|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T K^{-1}(x-\mu)\right). \quad x \in \mathbb{R}^N.$$

Now lets visualize the case for $N=2$

$$x = [x_1\ x_2]^T \qquad K = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$
$$\mu = [\mu_1\ \mu_2]^T$$

$$\cancel{f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi}}$$

$$\det(K) = (\sigma_1\sigma_2)^2 - \rho^2(\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2)^2(1-\rho^2)$$

$$\cancel{(x-\mu)^T K^{-1}}$$

$$K^{-1} = \frac{1}{(1-\rho^2)}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}.$$

$$\Rightarrow (x-\mu)^T . K^{-1}(x-\mu) =$$

$$\frac{1}{(1-\rho^2)}[x_1-\mu_1 \quad x_2-\mu_2]\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)}\cdot\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right].$$

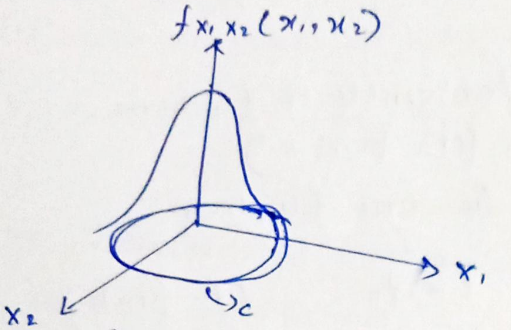
$$\boxed{f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right\}\right)}$$

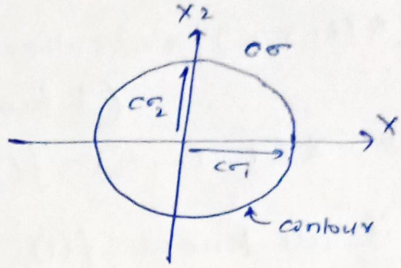
We can always find marginal pdf of X_1 or X_2

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\,dx_2 = N(\mu_1,\sigma_1)$$

Let $\rho=0$, $\sigma_1=\sigma_2$ and $\mu_1=\mu_2=0$.

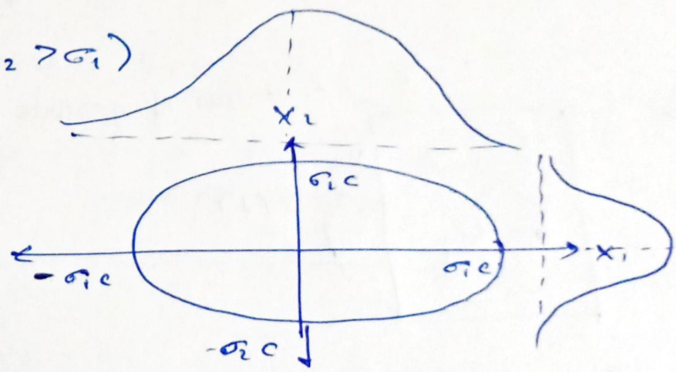
* when $\rho=0$ and $\sigma_1=\sigma_2$, we get perfectly symmetrical (circular countors).



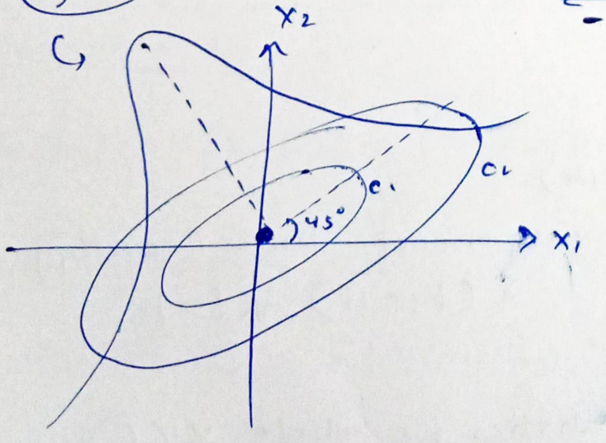


$\left\{ \text{assuming } f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\exp(c^2) \right\}$.

* If $\sigma_1 \neq \sigma_2$ (say $\sigma_2 > \sigma_1$)



* $\rho \neq 0$



$c_2 > c_1$

$\sigma_1 = \sigma_2 \Rightarrow$ angle $= 45^\circ$

RANDOM PROCESSES

R.V : outcomes $\longrightarrow$ ~~formula~~ value ($\mathbb{R}$)

R.P : experiments $\longrightarrow$ function (can be function of time (disc./cont)
, space (")
, angle (")
etc..))

eg:

$$X(t; r, \phi) = \underset{RV}{r} \cdot \cos(\omega t + \underset{RV}{\phi})$$

$$\Rightarrow \boxed{X(t) \rightarrow R.P}$$

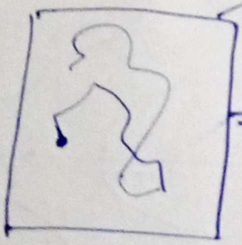
$X(t) = \{x(t; r, \theta)\}$ ← collection/ensemble of functions.
(1 function $f(t)$ $\forall$ r, θ)

Now let $X(t, r_0, \theta_0) = f(t)$ be one function

then if we know $f(t)$ for $t < t_0$ (t_0 finite)

* we can predict $f(t)$ $\forall$ t
* but we can not predict any other function of $X(t)$.

eg:



→ container of gaseous molecules.

$\Rightarrow X(t)$

$x(t; w, x)$

(t_0 finite)

If $X(t; w_i)$ observed for $t \leq t_0$, we cannot know (exactly) $X(t; w_i)$ $\forall$ $t > t_0$.

But under some conditions, we can describe statistics for whole $X(t)$.

$\omega \in \Omega$

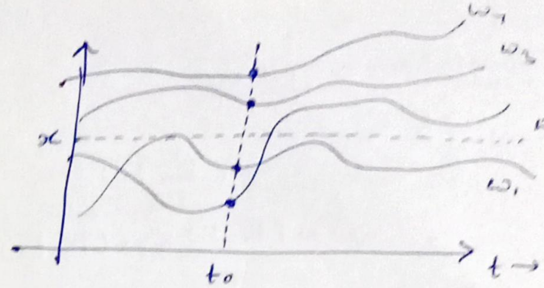
$X(t) \equiv X(t, \omega) \Rightarrow$ family / ensemble of functions

one for each $\omega \in \Omega$

fixed ω, t is variable $\Rightarrow$ $X(t) \rightarrow$ time function (sample function)

fixed t, ω is variable $\Rightarrow$ $X(t) \rightarrow$ R.V.

fixed t, fixed ω $\Rightarrow$ Real no. $X(t, \omega)$



Now let t be fixed:

1st order definition of $X(t)$

$$F_X(x; t) = \mathbb{P}[\underset{\text{R.V.}}{X(t)} \leq x]$$

ω_1 and ω_2 are $< x$

for $t = t_0 \Rightarrow \mathbb{P}[X(t_0) \leq x] = \frac{2}{4}$

* 1st order density function

$$f_X(x, t_0) = \frac{d}{dx} F_X(x, t_0)$$

* 2nd order distribution

$$F_X(x_1, x_2; t_1, t_2) = \mathbb{P}[X(t_1) \leq x_1, X(t_2) \leq x_2]$$

$$\cancel{f_X(x_1, x_2) =} \quad f_X(x_1, x_2; t_1, t_2) = \frac{d^2}{dx_1\, dx_2} F_X(x_1, x_2; t_1, t_2)$$

- Mean of $X(t_0)$

$$\eta_X(t_0) = E[X(t_0)] = \int_{-\infty}^{\infty} x \cdot f_X(x, t_0)\, dx$$

- Autocorrelation of $X(t)$

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

$$= \iint x_1 x_2 \cdot f_X(x_1, x_2; t_1, t_2)\, dx_1 \cdot dx_2$$

if $t_1 = t_2 \Rightarrow R_{XX}(t_1, t_1) = E[|X(t_1)|^2]$

Auto Covariance

$$C_{xx}(t_1, t_2) = E[\underbrace{(X(t_1) - \eta_x(t_1))}_{\tilde{X}(t_1)}\underbrace{(X(t_2) - \eta_x(t_2))}_{\tilde{X}(t_2)}] = R_{xx}(t_1, t_2) - \eta_x(t_1)\eta_x(t_2)$$

* for zero mean R.P.

$$R_{xx}(t_1, t_2) = C_{xx}(t_1, t_2)$$

Correlation-coefficient

$$\rho_{x}^{(t_1, t_2)} = \frac{C_{xx}(t_1, t_2)}{\sqrt{C_{xx}(t_1, t_1) \cdot C_{xx}(t_2, t_2)}} \quad \text{as} \quad C_{xx}(t_1, t_1) = var(X(t_1)).$$

Note if $X(t_1)$ and $X(t_2)$ are independent, then

$$R_{xx}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] = \eta_x(t_1) \cdot \eta_x(t_2).$$

and thus,

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - \eta_x(t_1) \cdot \eta_x(t_2) = 0$$

[If covariance is zero ⇒ R.V. are un-correlated]

Let $X(t) = \underset{\text{R.V.}}{r_0} \cos(\omega t + \underset{U[-\pi, \pi)}{\phi})$

$$R_{xx}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

$$= E[r^2 \cos(\omega t_1 + \phi) \cdot \cos(\omega t_2 + \phi)]$$

$$= E[r^2] \cdot E[\cos(\omega t_1 + \phi) \cdot \cos(\omega t_2 + \phi)]$$

$$= \frac{E[r^2]}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(\omega t_1 + \phi) \cdot \cos(\omega t_2 + \phi) \cdot d\phi$$