Infinite series:

$\Rightarrow$ ~~S_1~~ ~~$\sum_{n=}^{\infty} a_n$~~

Partial sum $S_n = \sum_{i=1}^{n} a_i$.

If the sequence of partial sum converges to S, the series is Convergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$\Rightarrow$ $\frac{1}{2} + 1/4 + 1/8 + \ldots$

$S_1 = 1/2$, $S_2 = 1/2 + 1/4 = 3/4$ ... $S_n = 1/2 + 1/4 + 1/8 \ldots$

$= 2^n - 1 / 2^n$

$\lim_{n \to \infty} S_n = 1$

$\Rightarrow t_n = \frac{1}{n} - \frac{1}{n+1}$

$\sum_{i=1}^{n} t_i$ $\quad S_n = 1 - \frac{1}{n+1}$

$(t_1 - t_2) + (t_2 - t_3)$

$\hookrightarrow$ telescopic series.

$t_n = \frac{2}{4n^2 - 1}$

$\Rightarrow \frac{1}{2n-1} - \frac{1}{2n+1}$

$S_n = 1 - \frac{1}{2n+1}$

---

a, r

$S_n = \frac{a(1-r^n)}{1-r}$.

$\Rightarrow S_n = a + ar + ar^2 + \dots + ar^{n-1} \cdot (i)$

Let us multiply this equation by r

$rS_n = ar + ar^2 + ar^3 \dots + ar^n \dots (ii)$

$\Rightarrow (ii) - (i)$

$S_n(r-1) = a - ar^n$.

$S_n = \frac{a-ar^n}{r-1}$; if $r \neq 1$.

if $|r| > 1$ $r^n \to \infty$ then $\lim_{n\to\infty} \frac{a-ar^n}{1-r} = \infty$.

$\to$ if $|r| < 1$, $r^n \to 0$ as $n \to \infty$ that is $\lim_{n\to\infty} \frac{a-ar^n}{1-r} = \frac{a}{1-r}$

$\Rightarrow$ Thus series converges to sum $\frac{a}{1-r}$.

$S_n = 0.\overline{08}$.

$S_n = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} \dots = \sum_{n=0}^{\infty} \frac{8}{10^2}\left(\frac{1}{10^2}\right)^n$.

$a = \frac{8}{10^2}$ & $r = \frac{1}{10^2}$.

Thus sum is given by

$$S_n = 0.\overline{08} = \frac{a}{1-r} = \frac{a/10^2}{1-1/10^2} = \frac{8}{99}$$

<u>5-09-2022</u>

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

$|x+y| \leq |x|$

$|x+y| \leq |y|$.

$|x+y| \geq \max(|x|, |y|)$.

⇒ consider case (i)

$|x+y| \leq |x|$

Since $|x+y| \leq |x|$, we get

$|x+y| + |x| \, |x+y| \leq |x| + |x||x+y|$.

So, $|x+y| (1+|x|) \leq |x| (1+|x+y|)$.

implying $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{\ldots}$ ⇒

$$\Rightarrow \frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Case 3:-

$$|x+y| \geq \max(|x|, |y|).$$

Triangle inequalities

$$|x+y| \leq |x| + |y|.$$

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|x+y|}$$

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Holder inequalities:-

Prove that for $n \in N$ & $x_j \in R^+$

for $1 \leq j \leq n$, one has that

$$\sqrt[n]{\prod_{j=1}^{n} x_j} \leq \frac{1}{n} \sum_{j=1}^{n} x_j$$

~~Youn~~ Young's inequality

for $p \in (1, \alpha)$, we have $xy \le \frac{x^p}{p} + \frac{y^q}{q}$

for $x, y \in R^+$ & $\frac{1}{p} + \frac{1}{q} = 1$

$$q = \frac{p}{p-1}$$

Induction $n = 1$. its true.

$n = 2$, true.

Let us consider this inequality to be true for n.

Young's inequalities for $p \in (1, \alpha)$, we have $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y \in R^+$.

$$\sqrt[n+1]{\prod_{j=1}^{n+1} x_j} \le \underbrace{\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)^{\frac{n}{n+1}}}_{x} \cdot \underbrace{\left(x_{n+1}\right)^{1/n+1}}_{y}$$

$$\le \frac{1}{n+1}\sum_{j=1}^{n} x_j + x_n + 1.$$

$$\frac{}{n+1} = \frac{1}{n+1}\sum_{j=1}^{n+1} x_j$$

l_p norm ~~area~~ $|x|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$

For $p \in (1, \infty)$ & $x = (x_1, x_2, x_3 \ldots x_n) \in \mathbb{R}^n$

show that $\sum_{j=1}^{n} |x_j y_j| \leq |x|_p |y|_q$

we think of loss of generality $x \neq 0, y \neq 0$.

$$\frac{|x_j|}{|x|_p} \cdot \frac{|y_j|}{|y|_p} \leq \frac{1}{p} \frac{|x_j|^p}{|x|_p^p} + \frac{1}{q} \frac{|y_j|^q}{|y_j|_q^q}$$

$\forall \quad 1 \leq j \leq n \Rightarrow x \cdot y \leq \frac{x^p}{p} + \frac{y^q}{q}$.

$$\frac{\sum |x_j y_j|}{|x|_p |y|_p} \leq \sum \sum_{j=1}^{n} \frac{|x_j|^p}{|x_p|^p} \cdot \frac{1}{p} + \frac{\sum^y |y_j|^q}{|y|_q^q \, q}$$

$$= \sum_{j=1}^{n} |x_j y_j| \leq |x|_p |y|_p.$$

$$\leq \frac{1}{p} + \frac{1}{q} = 1.$$

Convergence.

$$x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} - \frac{n^2}{\sqrt{n^6+n}}$$

we have $\sqrt{n^6+i} > \sqrt{n^6}$ $\forall\ i \in \{0,1,2 \ldots n\}$.

$$\Rightarrow x_n < \frac{n^2}{\sqrt{n^6}} + \frac{n^2}{\sqrt{n^6}} + \ldots \text{ n times.}$$

$$\because \frac{n}{\sqrt{n^6+i}} < \frac{n}{\sqrt{n^6}}$$

Thus $\frac{n^2 n}{n^3} = 1 \Rightarrow x_n < 1$ -- (1).

Now, $n^6+i \le n^6+n \ \forall\ \{i \in n\}$.

$$\Rightarrow x_n > \frac{n^2}{\sqrt{n^6+n}} + \ldots \text{ n times.}$$

$$\Rightarrow x_n > \frac{n+}{\sqrt{\frac{n^6+n}{n^6}}} \Rightarrow x_n > \frac{1}{\sqrt{1+\frac{1}{n^5}}} \quad (2).$$

By ① & ②

$$\frac{1}{\sqrt{1+\frac{1}{n^5}}} < x_n < 1 \quad (3)$$

we have

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{1-\frac{1}{n^5}}}\right) = 1,$$

& $\lim_{n\to\infty} 1 = 1.$

By squeeze theorem, we get

$\lim_{n\to\infty} x_n = 1$, $\qquad \frac{1}{p}+\frac{1}{q}=1.$

⇒ Let $\{x_n\}$ defined by

$x_1 = 1$ & $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$

S.T $\{x_n\}$ is convergent & find its limit.

Let us first show by induction

$x_n \geq 0$. & $1 \leq x_n^2 \leq 2$.

$$x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + \frac{4}{x_n^2} + 4\right) \geq \frac{1}{4}\left(x^2 + \frac{4}{x_n^2}\right) + 1$$

$$(2 - x_n)^2 \geq 0.$$

$$4 - 4x_n^2 + x_n^4 \geq 0.$$

$$\Rightarrow \frac{x_n^4 + 4}{4x_n^2} \leq 1. \Rightarrow \cancel{1}$$

$$\boxed{x_{n+1}^2 \leq 1 + 1 = 2}$$

$$\Rightarrow 1 \leq x_{n+1}^2 \leq 2.$$