Sequence & Series:

A sequence is a function whose domain is the set (subset) of natural numbers.

$\Rightarrow$ If f is such a "sequence", Let $f(n) = x_n$, denotes the value of f at $n \in \mathbb{N}$. In this case we denote the sequence f by $(x)_{n=1}^{\infty}$ as simply (x_n).

An infinite sequence is an unending set (subset) of N.

Set of real numbers which are determined acc. to some rank. the value of f at $n \in N$.

In this case we denote

eg:- $\frac{n}{n+1}$ + $\{ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \} f(u_n)$.

Remarks:-

1) The order of the terms of a sequence is implied ~~the value of f~~ at

Bounded sequence : A sequence is (x_n) is said to be bounded if.

1) Bounded above if $\exists\ k \in R$ s.t $x_n \leq k\ \forall\ n \in N$.

2) Bounded below if $\exists\ t \in R$ s.t $x_n \geq t\ \forall\ n \in N$.

A sequence (x_n) is bounded, if $\exists\ M \geq 0$. s.t

$|x_n| \leq M.$

eg :-

$\left(\frac{1}{n}\right)$ $n \in N$ bounded

$(-1)^n$ bounded.

$\left(x + \frac{1}{x}\right)$ $x \in N$ bounded

lower bounded by 2.

upper bounded by ∞.

Convergent sequence :

A sequence (x_n) is said to be converge to a real number l, if given $\epsilon > 0$ there $\exists$ exists $N \in N$, related to ϵ such that $|x_n - l| < \epsilon$ $\forall$ $n \geq N$.

or $(\forall \epsilon > 0)$ $\exists N \Rightarrow |x_n - l| < \epsilon$.

$\text{Limit}_{n \to \infty} x_n = l$ $\quad x \to l$ as $x \to \infty$.

Diverging sequence :

A sequence (x_n) is said to diverge to ∞ denoted by $x_n \to \infty$ as $n \to \infty$ if for any particular real number k there $\exists$ a $N \in N$ such that $x_n > k$ $\forall$ $n \geq N$.

1) show that a sequence (x_n) Converges to zeros if the sequence $(|x_n|)$ converges to zeros.

proof: Lets assume (x_n) converges to zero.

$$|x_n - 0| < \varepsilon \quad \forall \quad n \geq N.$$

$$\forall \; n \geq N \longrightarrow \big|\,|x_n| - 0\big| = |x_n| < \varepsilon$$

$$\Rightarrow |x_n| \text{ converges to zero} \Leftarrow \big|\,|x_n| - 0\big| < \varepsilon$$

$$\boxed{\Rightarrow \text{Show that } \lim_{n\to\infty} 1/n = 0}$$

need to prove it using the similar procedure mentioned above.

using corollary (2) of Archimedian property $\{t > 0, \exists\, n_t \in \mathbb{N}$ Such that $0 < \frac{1}{n_t} < t\}$.

$N \in \mathbb{N} \Rightarrow 0 < \frac{1}{N} < \varepsilon$

$n > N$,

$(|x_n|)$ converges to zero.

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon,$$

⇒ Show that $\lim_{n \to \infty} \left(1 - \frac{1}{2n}\right) = 1$.

Proof:

Let $\varepsilon > 0$ $N \in \mathbb{N}$.

$$\left|\left(1 - \tfrac{1}{2n}\right) - 1\right| < \varepsilon \quad \forall\ n \geq N.$$

$$= \left|\left(1 - \frac{1}{2n}\right) - 1\right| = \left|\frac{1}{2n}\right| = \frac{1}{(1+1)^n}$$

$$= \frac{1}{2^n} = \frac{1}{(1+1)^n} \leq \frac{1}{n+1} < \frac{1}{n}$$

Theorem:

Let (s_n) & (t_n) be sequences of real no. & Let $s \in \mathbb{R}$, n_F for some $k > 0$ & $N_1 \in \mathbb{N}$,

$|s_n - s| \leq k\,|t_n|$ for all $n \geq N_1$ &

$\lim_{n \to \infty} t_n = 0$. then $\lim_{n \to \infty} s_n = s$.

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Theorem:

Let (a_n) and (t_n) be sequences of real numbers & let $s \in \mathbb{R}$ & for some positive real number k & some $N_1 \in \mathbb{N}$, are we have $|s_n - s| \le k|t_n|$ $\forall$ $n \ge N_1$

& if $\lim_{n\to\infty} t_n = 0$ & $\lim_{n\to\infty} s_n = s$.

Proof: let $\varepsilon > 0$ is given, since $t_n \to 0$ as $m \to \infty$, there exists an $N_n \in \mathbb{N}$ such that $|t_n| < \frac{\varepsilon}{k}$ for all $n \ge N_2$...

let $N = \max(N_1, N_2)$, then for all $n \ge N$, we have $|s_n - s| \le k|t_n| < k\frac{\varepsilon}{k} = \varepsilon$.

$\lim_{n\to\infty} |s_n - s| < \varepsilon \Rightarrow \lim_{n\to\infty} s_n = s$

$\Rightarrow$ Show that

$\lim_{n\to\infty} \sqrt[n]{n} = 1$. [theoretically we can say $(\infty)^{1/\infty} = \phi$].

Proof: $\sqrt[n]{n} \ge 1$. $\forall$ $n \in \mathbb{N}$, there is a non-negative real number a_n such that $\sqrt[n]{n} = 1 + a_n$.

$\Rightarrow \quad n = (1+a_n)^n$

$$= \sum_{k=1}^{n} \binom{n}{k} a_n^k = 1 + na_n + \frac{n(n-1)}{2} + \dots + a_n^n$$

$$n \geq 1 + \frac{n(n-1)}{2} a_n^2$$

& ~~adding~~ adding -1 on both sides

$$(n-1) \geq \frac{n(n-1)a_n^2}{2}$$

$$\Rightarrow \quad a_n^2 \leq \frac{2}{n} \quad \Rightarrow a_n \leq \sqrt{\frac{2}{n}}$$

$\forall \quad n \geq 2.$

$$\Rightarrow \quad \left| \sqrt[n]{n} - 1 \right| = |a_n| \Rightarrow a_n \leq \sqrt{\frac{2}{n}}$$

$$\Rightarrow \quad \lim_{n \to \infty} \sqrt{\frac{2}{n}} = 0.$$

$$\Rightarrow \quad \operatorname{Lt}_{n \to \infty} \left| \sqrt[n]{n} - 1 \right| = 0 \Rightarrow \lim_{n \to \infty} \sqrt[n]{n} = 1.$$

Theorem: Uniqueness of limit:

Let (s_n) be a sequence of real numbers.

If $\lim_{n\to\infty} s_n = l_1$ & $\lim_{n\to\infty} s_n = l_2$ then $l_1 = l_2$.

Proof: let $\varepsilon > 0$ be given, Then $\exists\ N_1, N_2 \in N$

s.t $|s_n - l| < \varepsilon/2 \ \forall\ n \geq N_1$ &

$|s_n - l_2| \leq \varepsilon/2 \ \forall\ n \geq N_2$

$\Rightarrow$ let $N = \max(N_1, N_2)$ then $\forall\ n \geq N$,

we have $|l_1 - l_2| = |l_1 - s_n + s_n - l_2|$

$\leq |s_n - l_1| + |s_n - l_2| < \varepsilon$.

Then $|l_1 - l_2| = \varepsilon$ & since $0 \leq |l_1 - l_2| \leq \varepsilon$

holds $\forall\ \varepsilon > 0$,

$l_1 = l_2$.

Proportion:

A sequence (x_n) converges to $l \in R$ iff

for each $\varepsilon > 0$ the set $\{n : x_n \notin (l-\varepsilon, l+\varepsilon)\}$

is finite

Theorem: Uniqueness of limit:

Let (s_n) be a sequence of real numbers. If $\lim_{n\to\infty} s_n = l_1$ & $\lim_{n\to\infty} s_n = l_2$ then $l_1 = l_2$.

Proof: let $\varepsilon > 0$ be given, Then $\exists$ $N_1, N_2 \in \mathbb{N}$ s.t $|s_n - l_1| < \varepsilon/2$ $\forall$ $n \geq N_1$ &

$|s_n - l_2| \leq \varepsilon/2$ $\forall$ $n \geq N_2$.

$\Rightarrow$ let $N = \max(N_1, N_2)$ then $\forall$ $n \geq N$, we have

$$|l_1 - l_2| = |l_1 - s_n + s_n - l_2|$$

$$\leq |s_n - l_1| + |s_n - l_2| < \varepsilon.$$

Then $|l_1 - l_2| = \varepsilon$ & since $0 \leq |l_1 - l_2| \leq \varepsilon$. holds $\forall$ $\varepsilon > 0$,

$l_1 = l_2$.

Proportion:

A sequence (x_n) converges to $l \in \mathbb{R}$ iff for each $\varepsilon > 0$ the set $\{n : x_n \notin (l-\varepsilon, l+\varepsilon)\}$ is finite

Theorem:

Every convergent sequence of real numbers is bounded.

Proof: Let (s_n) be a sequence of real numbers which converges to some real number s.

Then with $\varepsilon = 1$, $\forall$ an $N \in \mathbb{N}$ such that $|s_n - s| < 1$, $\forall\ n \geq N$.

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 1 + |s|,$$

$\forall\ n \geq N$.

$\Rightarrow$ Let $M = \max\{|s_1|, |s_2|, \ldots |s_N|, |s|+1\}$.

Then $|s_n| \leq M \quad \forall\ n \in \mathbb{N}$.

$\hookrightarrow (s_n)$ is bounded.

If a sequence is convergent then it is bounded but the converse is not true.

Theorem: (squeeze theorem).

suppose (s_n), (t_n) & (u_n) are sequences such that

$$s \leq t_n \leq u_n \quad \forall\ n \in \mathbb{N}.$$

$\Rightarrow$ If $\lim\limits_{n\to\infty} s_n = l = \lim\limits_{n\to\infty} u_n$, then $\lim\limits_{n\to\infty} t_n = l$.

Proof: Let $\varepsilon > 0$ be given, then $\exists\ N_1, N_2 \in \mathbb{N}$

s.t. $|s_n - l| < \varepsilon \quad \forall\ n \geq N_1$ &

$|u_n - l| < \varepsilon \quad \forall\ n \geq N_2$

i.e. ~~$l - \varepsilon < s_n$~~

$$l - \varepsilon < s_n < l + \varepsilon \quad \forall\ n \geq N_1$$

& $l - \varepsilon < u_n < l + \varepsilon \quad \forall\ n \geq N_2$

$N = \max(N_1, N_2)$. $\quad l - \varepsilon < s_n \leq t_n \leq u_n \leq l + \varepsilon$

$\Rightarrow \quad l - \varepsilon \leq t_n \leq l + \varepsilon. \quad \forall\ n \geq N.$

$\Downarrow$

$|t_n - l| \leq \varepsilon \quad \forall\ n \geq N.$

$\lim\limits_{n\to\infty}(t_n) = l \quad \Rightarrow \quad \lim\limits_{n\to\infty} t_n = l.$

Exercise: S.T $\lim\limits_{n\to\infty} \left| \dfrac{\cos(n\pi/2)}{n^2} \right| = 0.$

$\left| \dfrac{\cos(n\pi/2)}{} \right.$

$\forall\ x$ with $|x|<1$, $\lim_{n\to\infty} n x^n = 0$.

$\Rightarrow$ ~~$\lim_{n\to\infty} nx^n$~~

$|x| < 1 \Rightarrow \frac{1}{|x|} > 1$.

$\Rightarrow \frac{1}{|x|} = 1 + a$.

$\Rightarrow \frac{1}{|x|^n} = (1+a)^n = \sum_{k=1}^{n} \binom{n}{k} a^k$

$= 1 + a + \frac{n(n-1)a^2}{2} + \ldots > \frac{n(n-1)a^2}{2}$

$\frac{1}{|x|^n} \geq \frac{n(n+1)a^2}{2}$.

$\Rightarrow \frac{-2}{(n-1)a^2} \leq n x^n \leq \frac{2}{(n-1)a^2}$

now, $n \to \infty$ both limits tends to 0.

$\lim_{n\to 0} n x^n = 0$.

$\Rightarrow$ Algebra of limit sequences:

1) Let 2 sequences s_n & t_n, be the sequences of real no.'s converging to s & t $\in \mathbb{R}$.

1) $\lim_{n\to\infty} (S_n + t_n) = S+t$

(2) $\lim_{n\to\infty} (S_n t_n) = S \cdot t$

(3) $\lim_{n\to\infty} S_n/t_n = S/t$ if $t \neq 0$.

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Let S_n & t_n be sequences of real no.s which converges to s & t respectively.

Thm

$$\lim_{n\to\infty} (S_n + t_n) = S+t$$

Proof:

Let $\varepsilon > 0$ be given, then $\exists (N_1, N_2) \in \mathbb{N}$, such that $|S_n - s| < \varepsilon/2$ & $|t_n - s| < \varepsilon/2$.

$\lim |S_n - t_n - s - t_n| = \lim_{n\to\infty} |(S_n - s) - (t_n - t)|$

$\leq \lim_{n\to\infty} |S_n - s| + |t_n - t| < \varepsilon$.

$\lim_{n\to\infty} |S_n - S_n - (S+t)| < \varepsilon \rightarrow S_n + t_n$ converges to $S \cdot t$.

ii) $\lim_{n\to\infty} (s_n t_n) = s \cdot t$.

Let $\varepsilon > 0$ be given, Now.

$|s_n t_n - (st)| = |s_n t_n - s t_n + s t_n - st|$.

$= |(s_n - s) t_n + (t_n - t) s|$.

$\leq |s_n - s| |t_n| + t_n - t|s|$.

Since t_n is convergent, it is bounded.

There exist a $k \in \mathbb{N}$ such that $t_n \leq k$ for all $n \in \mathbb{N}$.

Thus $|s_n t_n - st| \leq |s_n - s| k + |t_n - t| s$.

Let $M = \max\{k, |s|\}$, then $|s_n t_n - st| \leq M (|s_n - s| + |t_n - t|)$.

Since $s_n \to s$ & $t_n \to s$ & $n \to \infty$. then there exist N_1 & N_2 in $\mathbb{N}$ such that $|s_n - s| < \frac{\varepsilon}{M}$ for all $n \geq N_1$ & $|t_n - t| < \frac{\varepsilon}{M}$ for all $n \geq N_2$.

$\Rightarrow$ Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have.

$|s_n t_n - st| \leq M (|s_n - s| + |t_n - t|)$

$< M |s_n - s| + M |t_n - t|$..

Therefore

$\rightarrow$ show

to s,

to

Monotonic

Let $(s_n$

that

(a) Increas

(b) strictl

(c) Decrea

(d) stric

(e) Mono

(f) stric

or de

$\Rightarrow \left\{\frac{n+1}{n}\right.$

s_{n+1}

$$\epsilon M\left(\frac{\epsilon}{M}\right) + M\left(\frac{\epsilon}{M}\right) = \epsilon.$$

therefore $\lim_{n\to\infty} s_n t_n = st$.

) show that if the sequence (s_n) converges to s, then the sequence (s_n^2) converges to s^2.

Monotonic Sequences:

Let (s_n) be a sequence of real numbers we say that (s_n) is

(a) Increasing if for each $n \in N$ $s_n \leq s_{n+1}$.

(b) strictly increasing if for each $n \in N$ $s_n < s_{n+1}$.

(c) Decreasing if for each $n \in N$. $s_n \geq s_{n+1}$.

(d) strictly Decreasing if for each $n \in N$ $s_n > s_{n+1}$

(e) Monotone if (s_n) is increasing or decreasing.

(f) strictly Monotone if (s_n) is strictly increasing or decreasing.

$$\Rightarrow \left\{\frac{n+1}{n}\right\} = s_n$$

$$s_{n+1} = \frac{n+2}{n+1} \Rightarrow \frac{s_{n+1}}{s_n} = \frac{n+2/n+1}{n+1/n} = \frac{n(n+2)}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2} < \frac{(n+1)^2}{(n+1)^2} = 1.$$

$$\epsilon M\left(\frac{\epsilon}{M}\right) + M\left(\frac{\epsilon}{M}\right) = \epsilon.$$

Therefore $\lim_{n\to\infty} s_n t_n = st$.

-) Show that if the sequence (s_n) converges to s, then the sequence (s_n^2) converges to s^2?

Monotonic Sequences:-

Let (s_n) be a sequence of real numbers we say that (s_n) is

(a) Increasing if for each $n \in \mathbb{N}$ $s_n \leq s_{n+1}$.

(b) Strictly increasing if for each $n \in \mathbb{N}$ $s_n < s_{n+1}$.

(c) Decreasing if for each $n \in \mathbb{N}$. $s_n \geq s_{n+1}$.

(d) Strictly Decreasing if for each $n \in \mathbb{N}$ $s_n > s_{n+1}$

(e) Monotone if (s_n) is increasing or decreasing.

(f) Strictly Monotone if (s_n) is strictly increasing or decreasing.

=> $\left\{\frac{n+1}{n}\right\} = s_n$

$$s_{n+1} = \frac{n+2}{n+1} \Rightarrow \frac{s_{n+1}}{s_n} = \frac{n+2/n+1}{n+1/n} = \frac{n(n+2)}{(n+1)^2}.$$

$$= \frac{n^2+2n}{(n+1)^2} < \frac{(n+1)^2}{(n+1)^2} = 1.$$

$S_{n+1} < 1.$

Thus it is strictly decreasing.

Theorem: Let a sequence (S_n) be a bounded sequence.

1) If (S_n) is monotonically ↑, then it converges to its supremum.

2) If S_n is monotonically decreasing then it converges to its infimum.

Proof: Let $S_1 = \sup S_n$ & $S_2 = \inf S_n$ & take $\varepsilon > 0$.

(i). $\sup S_n = S_1$ there exists S_{n_0} such that $S_1 - \varepsilon < S_{n_0}$.

Since S_n is increasing then $S_1 - \varepsilon < S_{n_0} < S_n < S_1 < S_1 + \varepsilon$.

$\Rightarrow$ $S_1 - \varepsilon < S_n < S_1 + \varepsilon$ for all $n > n_0$.

$\Rightarrow$ $\lim_{n \to \infty} S_n = S_1$.

(ii) Since $\inf s_n = s_2$ there exists s_{n_1} such that

$s_{n_1} < s_2 + \varepsilon$.

Since s_n is decreasing then: $s_2 - \varepsilon < s_n < s_{n_1} < s_2$

$< s_2 + \varepsilon$

$\Rightarrow$ $s_2 - \varepsilon < s_n < s_1 + \varepsilon$ for all $n > n_1$.

$\Rightarrow$ $|s_n - s_2| < \varepsilon$ for all $n > n_1$

$\Rightarrow$ $\lim\limits_{n\to\infty} s_n = s_2$

Theorem:

A monotonic sequence converges if and only if it is bounded

proof: Find $\Rightarrow$ Every convergent sequence is bounded (proved before).

$\Rightarrow$ To prove the converse let (s_n) be a bounded increasing sequence & let $S = \{s_n | n \in \mathbb{N}\}$

Since S is bounded above it has a supremum.

$\sup S = s$ say. we claim that $\lim\limits_{n\to\infty} S = s$.

let $\varepsilon > 0$ be given, by the characterization of supremum. there exists $s_N \in S$ such that

$s - \varepsilon < s_N < s_n < s + \varepsilon$ for all $n \geq N$. Thus

$|s_n - s| < \varepsilon$. for all $n \geq N$.

Subsequence:

Let (s_n) be a sequence of real numbers & let $(n_k)_k \in \mathbb{N}$ be a sequence of natural numbers such that $n_1 < n_2 < \dots$ Then the sequence (s_{n_k}) is called a subsequence of s_n. That is a subsequence (s_{n_k}) of sequence (s_n) is strictly increasing function $\phi : k \mapsto s_{n_k}$.

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eg:

$(s_n) = (1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 9, \frac{1}{9})$ (sequence)

$\underbrace{(1, 2, 3, 9 \dots) \ \& \ (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{9} \dots)}_{\text{(subsequence)}}$

Theorem:

Let s_n be a sequence which converges to s then any subsequence of (s_n) converges to s.

Proof: Let (s_{m_n}) be a subsequence of (s_n) & Let $\varepsilon > 0$ be given.

Then there is a natural number $N \in \mathbb{N}$

such that $|s_n - s| < \varepsilon \quad \forall \; n \geq N.$

Thus when $k \geq N$, we have $n_k \geq k \geq N$,

so that $|s_{n_k} - s| < \varepsilon \quad \forall \; k \geq N.$

$\Rightarrow$ Thus, $\lim_{n \to \infty} s_{n_k} = s$.

Bolzano weierstrass theorem:

Every bounded infinite sequence (s_n) of real numbers has a convergent subsequence.

Cauchy Sequence:-

Statement: A sequence (s_n) ~~of real numbers~~ is a cauchy sequence if given any $\varepsilon > 0$, there exists natural number such that

$$|s_n - s_m| < \varepsilon \quad \forall \; n, m \geq N.$$

(or), $(\forall \; \varepsilon > 0) \; (\exists \; N \in \mathbb{N}) \; (\forall \; (n, m) \geq N)$
$(n \geq M) \wedge (m \geq N) \Rightarrow |s_n - s_m| < \varepsilon.$

or: (s_n) is a cauchy sequence if

$$\lim_{n \to \infty} |s_n - s_m| < \varepsilon$$

Example:

$s_n = \frac{n+1}{n}$ is a cauchy sequence.

proof: $\forall (n, m) \in N$ $|s_n - s_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right|$
$n < m$.

$= \left|\frac{n-m}{mn}\right| < \left|\frac{n+m}{nm}\right|$

$= \left|\frac{1}{n} + \frac{1}{m}\right| < \left|\frac{2}{n}\right|.$

$\lim_{n \to \infty} |s_n - s_m| < \varepsilon.$

Theorem: Every converging sequence is a cauchy sequence.

proof: Assume (s_n) converges to s. Then given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t $|s_n - s| < \varepsilon/2$.

$|s_n - s_m| = |(s_n - s) + (s - s_m)|$

$\leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow |s_n - s_m| < \varepsilon \; \forall (n, m) \geq N \in \mathbb{N}.$

Theorem:

Every cauchy sequence is bounded.

proof:

Let $\varepsilon = 1$, then there exist $N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $n, m \geq N$.

choose $a k \geq N$ & we would observe that

$$|s_n| = |s_n - s_k + s_k| < |s_n - s_k| + |s_k|$$

$$< 1 + |s_k|.$$

Let $M = \max(|s_1|, |s_2|, \dots |s_k| + 1)$

Then $s_n < M$ for all $n \geq N$ and therefore (s_n) is bounded.

Theorem:

Every cauchy sequence of real numbers converges.

Proof:

Let (s_n) be a cauchy, sequence then it is bounded. By, B.W theorem (s_n) has a sub sequence (s_{n_k}) that converges to $l \in \mathbb{R}$.

let $\varepsilon > 0$ given Then $\exists (N_1, N_2) \in \mathbb{N}$ s.t. $|s_n - s_m| < \varepsilon/2 \;\forall\; (n, m) > N$.

$\Rightarrow |s_{n_k} - l| < \varepsilon/2 \quad \forall \quad k \geq N_2.$

Take $N = \max(N_1, N_2)$. Then $\forall \; n \geq N$,

$$|s_n - l| = |s_n - s_{n_k} + s_{n_k} - l|$$

$$\leq |s_n - s_{n_k}| + |s_{n_k} - l|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\Rightarrow |s_n - l| < \varepsilon.$

$\Rightarrow \lim\limits_{n \to \infty} s_n = l.$

Example:- $\left(\frac{(-1)^n}{n}\right)$ converges.

$\rightarrow \left(\frac{(-1)^n}{n}\right)$ $\quad n = 1, 2, 3, 4$

$\rightarrow -1, 1, -1, 1 \ldots n \in N$

let $\varepsilon > 0$ & $s_n = \frac{(-1)^n}{n}$.

for all $m, n \in N$, $m, n > 2/\varepsilon$, with $m \geq n$ $|s_n - s_m| =$

$$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| < \cancel{\frac{1}{m}} \; \frac{1}{n} + \frac{1}{m} < \frac{2}{n}.$$

So there is an $N \in N$ such that $\frac{2}{N} < \varepsilon$. Thus for all $n \geq N$,

we have $|s_n - s_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{2}{N} < \varepsilon.$

Thus $\frac{(-1)^n}{n}$ is a cauchy sequence & so it converges.

c) $S_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n}$. $\boxed{n > m}$

$$|S_n - S_m| = \left|1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} - \left(1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{m}\right)\right|$$

$$= \left|\frac{1}{m+1} + \frac{1}{m+2} \cdots + \frac{1}{n}\right|$$

$$= \left\{\frac{1}{m+1} + \frac{1}{m+2} \cdots + \frac{1}{n}\right.$$

$$\geq \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots \frac{1}{n}}_{n-m \text{ times}} = \frac{n-m}{n}.$$

if $n = 2m$ we get

$$|S_n - S_m| = \left|\left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{m}\right)\right| \geq \frac{n-m}{n}$$

$$= \frac{m}{n} = \frac{1}{2}.$$

$|S_n - S_m| \geq \frac{1}{2}$ thus (S_n) is not a cauchy sequence. so it diverges.