8-09-2022

Limits:

cluster points

Definition:

Let $S \subset R$ be a set. A number $x \in R$ is called cluster point of S if for every $\varepsilon > 0$, the set $(x-\varepsilon, x+\varepsilon) \setminus \{x\}$. is not an empty set.

The neighbourhood of a point

I) The set $\{1/n \cdot n \in N\}$ has a unique Cluster point zero.

II) $(0, 1) \rightarrow$ for this open interval, the cluster points are all in the set of closed intervals. $[0, 1]$.

Proposition:- Let $S \subset R$, Then a $x \in R$ is a cluster point of S. iff $\exists$ a convergent sequence.

$\{x_n\}$ such that $x_n \neq x$ & $x_n \in S$ for all n, & $\lim_{n\to\infty} x_n = x$

Proof: Suppose x is a cluster point of S. For every $n \in \mathbb{N}$, pick x_n to be an arbitary point of the set $(x-\frac{1}{n}, x+\frac{1}{n}) \cap S \setminus \{x\}$, which is non-empty because x is a cluster point. Then x_n is within $\frac{1}{n}$ of x.

i.e. $|x - x_n| < \frac{1}{n}$

since $\{\frac{1}{n}\}$ converges to 0. Therefore $\{x_n\}$ converges to x.

⇒ On the other hand if we start with a sequence of numbers $\{x_n\}$ in S converging to x such that $x_n \neq x$ $\forall n \in \mathbb{N}$, then for every $\varepsilon > 0$ there exists an M such that

$$|x_M - x| < \varepsilon.$$

$$x_M \in (x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\}.$$

Limits of a function.

Definition: Let $f: S \to \mathbb{R}$ be a function & c a cluster point of $S \subset \mathbb{R}$. Suppose there exists an $L \in \mathbb{R}$ & for every $\varepsilon > 0$, $\exists$ $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ & $|x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

↪ $f(x)$ converges to L as x converges to c

$$\lim_{x \to c} f(x) = L$$

⇒ Let c be a cluster point of $S \subset \mathbb{R}$ & $f: S \to \mathbb{R}$ be a function such that $f(x)$ converges to x as x goes to c.

Then the limit of $f(x)$. x goes to c is unique.

⇒ Proof:

Let L_1 & L_2 be two numbers satisfying the definition

Take an $\varepsilon > 0$ & find a $\delta_1 > 0$ s.t. $|f(x) - L_1| < \varepsilon/2$. $\delta_2 > 0$ s.t.

$\delta = \max(\delta_1, \delta_2)$

suppose $x \in S$ $|x - c| < \delta$ & $x \neq c$.

As $\delta > 0$ & c is a cluster point such an x exists by the very definition of cluster points.

$\Rightarrow$ Then $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$.

$< |L_1 - f(x)| + |L_2 - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$.

$\Rightarrow$ $L_1 \to L_2$ L_1 converges to L_2.

Example

Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

Then for any $c \in \mathbb{R}$ $\lim_{x \to c} f(x) = \lim_{x \to c} x^2 = c^2$.

proof: Let $c \in \mathbb{R}$ be fixed & suppose $\varepsilon > 0$ is given. write

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|c|}\right\}$$ take $x \neq c$

such that $|x - c| < \delta$.

By reverse triangle inequality

$|x| - |c| \leq |x - c| < 1$.

adding $2c$ to both sides we have.

$\Rightarrow$ $|x| + |c| < 2c + 1$

$|f(x) - c^2| = |x^2 - c^2| = |(x+c)||(x-c)|$

$\leq (|x| + |c|)|x - c| < (2|c| + 1)|x - c|$

$= \frac{(2|c|+1)\,\varepsilon}{(2|c|+1)} = \varepsilon$.

$|f(x) - c^2| \Rightarrow |x^2 - c^2| < \varepsilon$.

Lemma:

Sequential Limits:

Lemma: Let $S \subset \mathbb{R}$, Let c be a cluster point of S, also let $f: S \to \mathbb{R}$ and the limit $L \in \mathbb{R}$. then $f(x) \to L$ as x goes to c. ($x \to c$) iff for every sequence $\{x_n\}$. Such that $x_n \in S \setminus \{c\}$ for $\forall n$, & Limit $x_n \to c$, we have $\{f(x_n)\}$. ~~they~~ all of them converge to L.

12-09-2022.

Sequential limits.

Lemma: Let $S \subset R$, Let c be a cluster point of S, Let $f: S \to R$ & let $L \in R$.

Then $f(x) \to L$ as $x \to c$.

Iff for every sequence $\{x_n\}$ of numbers such that $x_n \in S$ but $\{c\}$. $\forall n$, and such that

$\Rightarrow$ $\lim_{n \to \infty} x_n = c$. we have the sequence $\{f(x_n)\}$ converges to L.

$\Rightarrow$ proof:- suppose $f(x) \to L$ as $x \to c$ and $\{x_n\}$ is a sequence such that each of $x_n \in S \setminus \{c\}$ $\lim x_n = c$. let ε a real number is given $\hookrightarrow \varepsilon > 0$.

Find a $\delta > 0$ such that $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$. As $\{x_n\}$, converges to c, find a M such that for $n \geq M$, we have $|x_n - c| < \delta$.

Therefore for $n \geq M$ $|f(x_n) - L| < \varepsilon$.

Thus $\{f(x)\}$ converges to L.

Suppose it is not true that $f(x) \to L$ as $x \to c$. The negation of the definition is that $\exists$ & $\varepsilon > 0$ such that for every $\delta > 0$ $\exists$ $x \in S \setminus \{c\}$, where $|x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$

$\Rightarrow$ Let us use $1/n$ for δ to construct $\{x_n\}$ we have that $\exists$ $\varepsilon > 0$.

$\Rightarrow$ such that every n, $\exists$ $x_n \in S \setminus \{c\}$, where $|x_n - c| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$.

Proof: The sequence $\{x\}$ just constructed converges to c.

corollary 1:

Let $S \subset R$. and c be a cluster point of S. Suppose $f: S \to R$. and g $S \to R$ are functions such that $f(x) \leq g(x) \leq h(x)$ $\forall$ $x \in S$. $[\because h: S \to R]$.

$\Rightarrow$ suppose the limits of $f(x)$ & $h(x)$ both exists as $x \to c$, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$

then $\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$.

corollary-2.

Let $S \subseteq R$ and c be the cluster points and $f: S \to R$; $g: S \to R$ are functions such that the ~~limit~~ limits of then exists as $x \to c$;

Then

1) $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$.

2) $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

3) $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

$\lim_{x \to c} g(x) \neq 0$

~~corollary 3~~:

corollary 3:

$\lim_{x \to c} |f(x)| = |\lim_{x \to c} f(x)|$

=> Limits of restriction and one sided limits

-> sometimes we work with fun's defined on a subset.

Def: let $f: S \to R$ and $A \subseteq S$ define the funct. $f|_A : A \to R$ by $f|_A(x) = f(x)$ for $x \in A$

proposition:

Let $S \subseteq R$, $c \in R$ and $f: S \to R$ suppose $A \subseteq S$ is such that $\exists$ some $\alpha > 0$ such that $(A \setminus \{c\}) \cap (c - \alpha, c + \alpha) = (S \setminus \{c\}) \cap (c - \alpha, c + \alpha)$.

i) The point c is a cluster point of A iff c is a cluster point of S.

(ii) suppose c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $\Rightarrow$ $f|_A(x) \to L$ as $x \to c$.

proof:

First, let c be a ~~duster~~ cluster point of A

Since $A \subseteq S$, then if $(A \setminus \{c\}) \cap (c - \alpha, c + \alpha) \neq \{\phi\}$ $\forall \varepsilon > 0$.

then $(S \setminus \{c\}) \cap (c - \varepsilon, c + \varepsilon) \neq \{\phi\}$ $\forall \varepsilon > 0$.

Thus c is a cluster point of S.

=> secondly suppose c is a cluster point of S.

Then for $\varepsilon > 0$ such that $\varepsilon > \alpha$.

we get that. $(S \setminus \{c\}) \cap (c - \varepsilon, c + \varepsilon)$

$= (A \setminus \{c\}) \cap (c - \varepsilon, c + \varepsilon)$.

This is true for $\varepsilon < \alpha$ and hence $A - \{c\} \cap (c-\varepsilon, c+\varepsilon)$ $\neq \{\phi\}$.

$\downarrow$ c is also a cluster point of A.

$\Rightarrow$ Proof:

Now suppose c is a cluster point of s & $f(x) \to L$ as $x \to c$.

i.e for $\varepsilon > 0$ $\exists\, \delta > 0$ such that if $x \in S \setminus \{c\}$. and $|x-c| < \delta$, then $|f(x) - L| < \varepsilon$.

$\rightarrow$ Because $A \subseteq S$, if $x \in A \setminus \{c\}$, then $x \in S \setminus \{c\}$. and hence $f|_A(x) \to L$ as $x \to c$.

Finally suppose $f|_A(x) \to L$ as $x \to c$

for every $\varepsilon > 0$, $\exists\, \delta' > 0$ such that for $x \in A \setminus \{c\}$ & $|x-c| < \delta'$, $|f|_A(x) - L| < \varepsilon$.

As $|x-c| < \alpha$ then $x \in A \setminus \{c\}$ & $|x-c| < \delta'$

we have $|f(x) - L| = |f|_A(x) - L| < \varepsilon$.

$$\lim_{\substack{x \to c \\ x \in A}} f(x) = \lim_{x \to c} f|_A(x)$$

Definition: Let $f: S \to R$ be a function and c be the cluster ~~distribution~~ point of $S \cap (c, \infty)$ Then if the limits of restriction of f to $S \cap (c, \infty)$

define $\lim_{x \to c^+} f(x) = \lim_{x \to c} f|_{S \cap (c,\infty)}(x)$

$\Rightarrow$ Similarly $\lim_{x \to c^-} f(x) = \lim_{x \to c} f|_{S \cap (-\infty, c)}(x)$.

proposition:

let $S \subset R$ be such that c is a cluster point of both $S \cap (-\infty, c)$ $\wedge$ $S \cap (c, \infty)$.

let $f: S \to R$ be a function and $L \in R$.

Then c is a cluster point of s. and $\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$.

15-09-2022

1) show that $\{x_n\}$ defined by

$x_n = 1+\frac{1}{2} + \cdots \frac{1}{n}$ is divergent.

$x_n > 1+\frac{1}{2}+ \cdots \frac{1}{n}$.

$x_{2n} - x_n = \frac{1}{n+1} + \cdots \frac{1}{2n}$ $n \geq 1$.

$\geq \frac{1}{n+n} + \frac{1}{n+n} \cdots \frac{1}{2n}$.

$\frac{1}{2}$.

$x_n = 1+\frac{1}{2} + \cdots \frac{1}{n} - \ln(n)$ is convergent

$\ln n = \int_1^n \frac{1}{t} dt$ $\quad \alpha \; a \leq b$

$\frac{b-a}{b} \leq \int_a^b \frac{1}{t} dt \leq \frac{b-a}{a}$.

$1-\frac{a}{b} \leq \cancel{\log}\ln|b| - \cancel{\log}\ln|a| \leq \frac{b}{a} - 1$.

$\int f(t) dt = \sum_{i=1}^{n} f(t_i)\Delta_i \Leftrightarrow \Delta_i = (t_{i+1}-t_i)$.

$0.5 \leq \log|2| - \log|1| \leq$ 0.69

$\ln(n) = \int_1^n \frac{1}{t} dt = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{t} dt$.

$\ln n \leq \sum_{k=1}^{n-1} \frac{k+p-k}{k} = 1+\frac{1}{2}+\frac{1}{3} \cdots + \frac{1}{n-1}$

$x_n = 1+\frac{1}{2} + \cdots \frac{1}{n} - \ln(n) = \frac{1}{n} > 0$.

$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln(n) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t} dt$

$+ \int_n^{n+1} \frac{1}{t} dt \geq \frac{n+1-n}{n+1} = \frac{1}{n+1}$

$\frac{1}{n+1} - \int_n^{n+1} \frac{1}{t} dt \geq \frac{1}{n+1} - \left(\frac{1}{n+1}\right)$ or $> \frac{1}{n+1}$

thus $\frac{1}{n+1} - \int_n^{n+1} \frac{1}{t} dt \leq 0$.

2) Show that $\{x_n\}$ with

$x_n = \int_1^n \frac{\cos(t)}{t^2} dt$ is a cauchy sequence.

$\Rightarrow$

$\left|\int_1^n \frac{\cos t}{t^2} dt\right| \leq \int_1^n \frac{|\cos t|}{t^2} dt \leq \int_n^m \frac{1}{t^2} dt$

$= \frac{1}{n} - \frac{1}{m}$.

$$\cancel{x_{n+1}} - \cancel{x_n} = \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$$

$\cancel{|x_n|}$

Since $\underset{n\to\alpha}{Lt} \frac{1}{n} = 0$, then for any $\varepsilon > 0$, $\exists$ a $n_0 > 1$, such that for any $n \geq n_0$ we have $\frac{1}{n} < \varepsilon$.

which shows that the sequence is cauchy.

Discuss the convergence / divergence of

$$x_n = \frac{[\alpha] + [2\alpha] \dots + [n\alpha]}{n^2}$$

$[x] \to$ Means the largest integer $\leq x$.

Given

$$x_n = \frac{[\alpha] + [2\alpha] \dots + [n\alpha]}{n^2}.$$

$[x] \leq x < [x]+1$.

$x-1 < [x] \leq x$.

$$\frac{[\alpha] + [2\alpha] + \dots [n\alpha]}{n^2} > \frac{(\alpha-1) + (2\alpha-1) + \dots (n\alpha-1)}{n^2}$$

$$\frac{[\alpha]+[2\alpha] + \dots [n\alpha]}{n^2} \leq \frac{\alpha + 2\alpha + \dots n\alpha}{n^2}.$$

$$\Rightarrow \frac{(1+2+\dots+n)\alpha - n}{n^2} < \frac{(\ )}{n^2} \leq \frac{(1+2+\dots n)\alpha}{n^2}$$

$$\frac{\left[\frac{n(n+1)}{2}\alpha\right] - n}{n^2} < (\ ) \leq \frac{n(n+1)\alpha}{2 \cdot n^2}$$

$$\Rightarrow \left(\frac{(n+1)\alpha}{2n}\right) - \cancel{1} < (\ ) \leq \frac{n+1\ \alpha}{2n} < \frac{\alpha}{2}$$

$\Rightarrow$ $n \to \infty$ $\Rightarrow \frac{\alpha}{2} + \frac{\alpha}{2n} - \frac{1}{2} = \frac{\alpha}{2}$.

Thus $\underset{n\to\infty}{Lt}\, x_n = \frac{\alpha}{2}$.

$$x_n = \alpha^n - \beta^n / \alpha^n + \beta^n.$$

$|\alpha| \neq |\beta|$. $\Rightarrow \alpha \neq \beta$.

let $|\beta| > |\alpha|$

$$x_n = \frac{\left(\frac{\alpha}{\beta}\right)^n - 1}{\left(\frac{\alpha}{\beta}\right)^n + 1} = \frac{r^n - 1}{r^n + 1} \quad |r| < 1.$$

$\underset{n\to\infty}{Lt}\, r^n = 0$.

$\cancel{\frac{[(n+1)\alpha - 1)}{2n}}$

Cesaros Theorem:

Let ~~x[n]~~ $\{x_n\}$ converges to l such that

$$y_n = \frac{x_1 + x_2 \ldots x_n}{n}$$ also converges to l.

⇒ What is the converse?

$$y_n = \frac{x_1 + x_2 \ldots + x_n}{n}$$

$$y_n - l = \frac{x_1 + x_2 + \ldots x_n - nl}{n}$$

$$y_n - l = \frac{(x_1 - l) + (x_2 - l) \ldots + (x_n - l)}{n}$$

$$y_n - l \le \frac{|x_1 - l| + |x_2 - l| + \ldots |x_n - l|}{n} < \frac{n\varepsilon}{n}$$

$$y_n - l \le \varepsilon$$

thus y_n converges to l.

Formal proof:

Let $\varepsilon > 0$ since $\lim_{n \to \infty} x_n = l$, $\exists$ $N_0 > 1$ s.t for any $n \ge N_0$, we have $|x_n - l| < \varepsilon/2$

$$y_n - l = \frac{(x_1 - l) + (x_2 - l) + \ldots + (x_n - l)}{n}$$

$$= \frac{(x_1 - l) + (x_2 - l) \ldots (x_{N_0 - 1} - l)}{n} + \underbrace{\frac{(x_{N_0} - l) + \ldots (x_n - l)}{n}}_{\text{by definition it leads to } 0}$$

$$|y_n - l| \le \left| \frac{(x_1 - l) + (x_2 - l) + \ldots (x_{N_0 - 1} - l)}{n} \right| + \left| \frac{(x_{N_0} - l) + \ldots (x_n - l)}{n} \right|$$

$$|y_n - l| \le \frac{\varepsilon}{2} + \frac{(n - N_0)}{n} \cdot \frac{\varepsilon}{2}.$$

to be continued...

Given $x \ge 1$, such that

$$\lim_{n \to \infty} (2\sqrt[n]{x} - 1)^n = x^2$$

→ without loss of generalities we can write

$$0 < (\sqrt[n]{x} - 1)^2 = \sqrt[n]{x^2} - 2\sqrt[n]{x} + 1$$

$$2\sqrt[n]{x} - 1 < \sqrt[n]{x^2}$$

$$(2\sqrt[n]{x} - 1)^n < (\sqrt[n]{x^2})^n = x^2.$$

on the other hand.

$$(2\sqrt[n]{x} - 1)^n = x^2\left((2\sqrt[n]{x} - 1) / \sqrt[n]{x^2}\right)^n$$

$$= x^2\left(\frac{2}{\sqrt[n]{x}} - \frac{1}{\sqrt[n]{x^2}}\right)$$

$$= x^2\left(1 - \left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)^n.$$

Since, $\boxed{(1-h)^n \geq 1-nh.}$ for any $h \geq 0$. and $n \geq 1$.

$$\Rightarrow x^2 \left(1 \left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)^n \geq 1 - n\left(1 - \frac{1}{\sqrt[n]{x}}\right)^2$$

$$\Rightarrow x = \left(\sqrt[n]{x} - 1 + 1\right)^n \geq 1 + n\left(\sqrt[n]{x} - 1\right) > n\left(\sqrt[n]{x} - 1\right).$$

which implies $\left(\sqrt[n]{x} - 1\right)^2 < \frac{x^2}{n^2}$

$$\left(2\sqrt[n]{x} - 1\right)^n \geq x^2\left(1 - n\left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)$$

$$= x^2\left(1 - n\frac{\left(\sqrt[n]{x} - 1\right)^2}{\sqrt[n]{x^2}}\right)$$

$$\Rightarrow \left(2\sqrt[n]{x} - 1\right)^n > x^2\left(1 - \frac{x^2}{2\sqrt[n]{x^2}}\right)$$

$$x^2\left(1 - \frac{x^2}{n\sqrt[n]{x^2}}\right) < \left(2\sqrt[n]{x} - 1\right)^n < x^2$$

using squeeze theorem.

$$\lim_{n \to \infty} \left(2\sqrt[n]{x} - 1\right)^n = x^2.$$

26 - 09 - 2022.

Min-Max and intermediate value theorem.

$\Rightarrow$ Continuous function:

Definition: Let ~~$S \subset R$~~ $S \subset R$, $c \in S$ and Let $f: S \to R$ be a function. we say that f is continuous at c. if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that whatever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

When $f: S \to R$ is continuous, at all $c \in S$, then we say that $f(x)$ is a continuous function.

$\rightarrow$ proposition: Consider a function $f: S \to R$ defined on a set of real numbers $S \subset R$ and Let $c \in S$. Then

[~~No proofs were told~~].

1) If c is not a cluster point of S, then f is continuous at c.

2) If c is a cluster point, then f is c iff $\lim_{x \to c} f(x) = f(c)$

3) The function f is continuous at c if and only iff

using squeeze theorem.

$$\lim_{n \to \alpha} (2\sqrt[n]{x} - 1)^n = x^2.$$

26-09-2022.

Min-Max and intermediate value theorem:

⇒ **Continuous function:-**

Definition: Let ~~$S \subset R$~~ $S \subset R$, $c \in S$ and Let $f: S \to R$ be a function. we say that f is continuous at c, if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that whatever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

When $f: S \to R$ is continuous, at all $c \in S$, then we say that $f(x)$ is a continuous function.

→ proposition:- Consider a function $f: S \to R$ defined on a set of real numbers $S \subset R$ and Let $c \in S$.

Then

[No ~~proofs were told~~].

1) If c is not a cluster point of S, then f is continuous at c.

2) If c is a cluster point, then f is c iff
$$\lim_{x \to c} f(x) = f(c).$$

3) The function f is continuous at c if and only iff

for every sequence $\{x_n\}$, $x_n \in S$. and $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

proof: (i) Suppose c is not a cluster point of S.

Then there exists $\delta > 0$, such that $S \cap (c-\delta, c+\delta)$ $= \{c\}$ for any [$S \subset R$: $(x-\epsilon, x+\epsilon) \cap S \setminus \{x\}$ is not empty.] cluster point

$\epsilon > 0$, simply pick this given δ the only $x \in S$ such that $|x-c| < \delta$ is $x = c$.

$$|f(x) - f(c)| = 0 < \epsilon.$$

Example: $f : (0, \infty) \longrightarrow R$ defined by $f(x) = \frac{1}{x}$, Is it continuous?

proof: Fix $c \in (0, \infty)$. Let $\{x_n\}$ be a sequence in $(0, \infty)$ such that $\lim x_n = c$. Then

$$f(c) = \frac{1}{c} = \frac{1}{\lim_{n\to\infty} x_n} = \lim_{n\to\infty} f(x_n).$$

$$\lim_{x\to\infty} |f(x_n) - f(c)| < \epsilon.$$

proposition:

Let $f : R \to R$ be a polynomial

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots a_1 x + a_0.$$

for some constant $a_d, a_{d-1} \cdots$

proof: Let's fix some real number "c", $\{x_n\}$ such that $\lim_{n\to\infty} x_n = c$, Then $f(c) = a_d c^d + a_{d-1} c^{d-1} + a_{d-2} c^{d-2} \cdots a_1 c + a_0$

$$= a_d \left(\lim_{n\to\infty} c\right)^d + \cdots$$

$$= \lim_{n\to\infty} \left(a_d x^d + \cdots a_0\right) = \lim_{n\to\infty} \left(f(x_n)\right).$$

proposition: Let $f : S \to R$ and $g : S \to R$. f and g are continuous functions.

(i) The func $h(x) = f(x) + g(x)$ is continuous.

(ii) The function $h(x) = f(x) - g(x)$ is continuous

(iii) " " $h(x) = f(x) \cdot g(x)$ " "

(iv) " " $h(x) = f(x)/g(x)$ " ", if $g(x) \neq 0$

Example:

The functions $\sin x$ and $\cos x$ are continuous.

$$|\sin x - \sin c| = 2 \left|\sin\left(\frac{x+c}{2}\right)\right| \left|\cos\left(\frac{x+c}{2}\right)\right| \le 2 \left|\sin\left(\frac{x-c}{2}\right)\right|$$

$|\sin x| \leq |x|$.

Thus

$$2\left|\sin\left(\frac{x-c}{2}\right)\right| \leq 2\frac{|x-c|}{2} = |x-c|.$$

Composition of continuous functions:

Proposition: Let $A \subseteq B \subset R$ and a function $f: B \to R$.
$g: A \to B$.

be function's. If g is continuous at $C \subset A$.

If g is continuous at $c \in A$ and f is continuous at $g(c)$. Then ~~this~~ is $f\circ g: A \to R$. is continuous.

proof: Let $\{x_n\}$ be a sequence in A, such that $\lim\limits_{n\to\infty} x_n = c$. As g is continuous at c we have $\{g(x_n)\}$, converges to $g(c)$.

$\Rightarrow$ As f is continuous at $g(c)$ we have $\{f(g(x_n))\}$ converging to $f(g(c))$ $[f\circ g(c)]$.

Dis-continuous functions:

Let f takes $S \to R$ $[f: S \to R]$. be a function & $c \in S$.

Suppose $\exists \to \{x_n\}$, $x_n \in S$ and $\lim x_n = c$.

Such that $\{f(x_n)\}$ does not converge to $f(c)$.

Then ~~fnc~~ function f is discontinuous at c.

Example:

$\Rightarrow$ Greatest integer function [have some limitations].

$\Rightarrow$ Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

12-10-2022

Proposition:

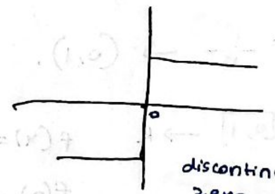
Let a function $f: S \to R$ be a function and $c \in S$.

Suppose $\exists \to$ a sequence $\{x_n\}$, where $x_n \in S$.

and $\lim x_n \to c$, such that the function ~~of~~ $\{f(x_n)\}$ does not converge to $f(c)$. Then f is discontinuous at c.

Example:

(i) $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x \geq 0 \end{cases}$



discontinuous at zero.

(ii) Dirichlet function.

Theorem: Min-Max ~~and~~ or extreme value theorem.

Lemma: A continuous function $f: [a,b] \to R$ is bounded.

Proof: Suppose f is not bounded. Then for each $n \in N$ There is an $x_n \in [a,b]$ such that $|f(x_n)| \geq n$.

$\Rightarrow$ The sequence $\{x_n\}$ is bounded as $a \leq x_n \leq b$ by Bolzano weierstress theorem.

There is a convergent subsequence $\{f(x_{n_i})\}$ is not bounded as $|f(x_{n_i})| \geq n_i \geq i$.

Thus f is not continuous at x. as $f(x) =$

$f\left(\lim_{i\to\infty} x_{ni}\right)$, $\lim_{i\to\infty} f(x_{ni})$ does not exist.

Minimum- Maximum Theorem

⇒ **Theorem:** A continuous function $f : [a,b] \to R$.

achieves both an absolute minimum & absolute maximum on these closed on bounded interval $[a,b]$.

Proof is provided in moodle notes.

Eg: $f(x) = x^2+1$ defined on the interval $[-1,2]$

$f(x) = \frac{1}{x} \to (0,1)$.

$f : [0,1] \to R$ $\quad f(x) = \frac{1}{x} \quad x>0$

$f(0) = 0$ [defined].

⇒ **Bolzano's Intermediate value Theorem:**

↓

Lemma: Let $f [a,b] \to R$ be a continuous function

⇒ suppose if $f(a)<0$ and if $f(b)>0$ then ∃ a number $c \in [a,b)$ is an $x_n \in [a,b]$.

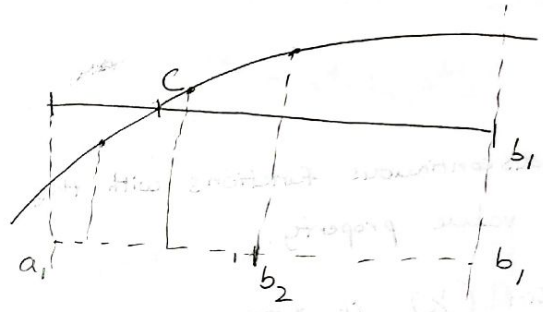
Such that $f(c) = 0$

Proof: Let's define a sequence $\{a_n\}$ & $\{b_n\}$

i) $a_1 = 0$ and $b_1 = b$.

(ii) If $f\left(a_n + b_n/2\right) \geq 0$ Let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$

(iii) If $f\left(\frac{a_n+b_2}{2}\right) < 0$. Let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$.



if $a_n<b_n$

$a_n < \frac{a_n+b_n}{2} < b_1$

~~$[a_{n+1} < b_n]$~~

$a_{n+1} < b_{n+1}$.

Then $a_n < b_n$

$a_n < b_n$

$\boxed{\begin{array}{l} a_{n+1} \geq a_n \\ b_{n+1} \leq b_n \end{array}}$ monotones

The sequence converges at c.

Let $c = \lim a_n$

$d = \lim b_n$ $\qquad a \leq c \leq d \leq b$.

we need to show $c=d$. Then, $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = 2^{1-n}(b-a)$

$\lim a_n = \lim b_n$.

$d - c = \lim_{n\to\infty} (b_n - a_n) = 0$ $\quad \boxed{d=c}$

Theorem: Let $f:[a,b]\to R$ be a continuous function

Suppose $y\in R$ such that if $f(a)<y<f(b)$ then there exists $\exists c\in(a,b)$ such that if $f(c)=y$

Examples:

There exists discontinuous functions with the intermediate value property.

$$f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$$ not const at $x=0$.

whenever $a<b$ and y is such that ~~f(a)<y<f(b)~~ $f(a)<y<f(b)$ and $f(a)>y>f(b)$. there exists a c such that $f(y)=c$.

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Proposition:

Let $f(x)$ be a polynomial of odd degree. Then f has a real root.

proof: suppose f is a polynomial of odd degree.

$f(x)=a_d x^d+a_{d-1}x^{d-1}+\dots+a_1x+a_0$, where $a_d\neq 0$.

$$g(x)=\frac{f(x)}{a_d}=x^d+b_{d-1}x^{d-1}+\dots+b_1x+b_0$$

with $b_k=\frac{a_k}{a_d}$

$$\left|\frac{b_{d-1}x^{d-1}+\dots+b_1n+b_0}{n^d}\right|=\frac{|b_{d-1}n^{d-1}+\dots+b_1n+b_0|}{n^d}$$

$$\leq\frac{|b_{d-1}|n^{d-1}+\dots+|b_1|n+|b_0|}{n^d}$$

$$\leq\frac{|b_{d-1}|n^{d-1}+\dots+|b_1|n^{d-1}+|b_0|n^{d-1}}{n^d}$$

$$=\frac{|b_{d-1}|+\dots+|b_1|+|b_0|}{n}$$

$$\lim_{n\to\infty}\frac{(\ )}{n}=\infty.$$

Therefore $\lim_{n\to\infty}\frac{b_{d-1}n^{d-1}+\dots+b_1n+b_0}{n^d}=0.$

$$\left|\frac{b_{d-1}M^{d-1} + \dots b_1 M + b_0}{M^d} - 0\right| < \varepsilon.$$

Therefore

$|b_{d-1} M^{d-1} + \dots b_0| < M^d$

$-(\_\_\_) < M^d$ $\left[\text{ie if } x < d \atop -x < d\right] x \text{ is +ve}$

$g(M) = M^d$ $g(M) > 0.$

⇒ Next consider $g(-n)$ for $n \in \mathbb{N}$.

By similar argument $\exists$ a $k \in \mathbb{N}_+$, such that

$b_{d-1}(-k)^{d-1} + \dots + b_1(-k) + b_0 < k^d$

$g(-k) < 0$ when d is odd. $[(-x)^d = -x^d]$.

$g(-k) < 0$

$g(M) > 0$ $[-k, M)$. $c \in (-k, M) \to g(c) = 0.$

⇓

The inequality is switched because $g(-k) = -g(k)$

Thus $\cancel{g(-m)} = \cancel{-g(M)}$. $g(m) = -g(-k)$

$g(M) = g(k).$

Uniform Continuity

Definition:- Let $S \subset \mathbb{R}$ and $f: S \to \mathbb{R}$. suppose for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that whenever $x, c \in S$ and $|x-c| \leq \delta$, then $|f(x) - f(c)| < \varepsilon$.

⇒ Then we say f is uniform continuous over S.

Example:-

The function $f: [0,1] \to \mathbb{R}$, define by $f(x) = x^2$

let $x \to c$.

⇒ $f(x) = x^2$ & $f(c) = c^2$. $|x-c| > \delta. > \varepsilon.$

$|x^2 - c^2| > 0$ we ~~defined~~ define a variable $\varepsilon > 0$

?

$|x^2 - c^2| = |x+c| \cdot |x-c|$

$\leq (|x| + |c|)\, |x-c|.$

$\leq 2\, |x-c|$

↳ max value of x & c is 1 & 1.

If $|x-c| < \delta = \varepsilon/2$ ⇒ $|f(x) - f(c)| < \varepsilon.$

$f: (0,1) \to \mathbb{R}$ $f(x) = 1/x$

⇒ $\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right|$ $|x-c| < \delta.$

⇒ $|x-c| < xc\varepsilon.$

for the given condition no single value of δ satisf which claims that the given function is not a uniform continuous function.

Theorem: Let $f:[a,b]\to R$ be a continuous func then f is uniformly continuous.

Continuous extension:

Lemma: Let $f: S\to R$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence Then $\{f(x_n)\}$ is also cauchy.

$|x_n - c| < \varepsilon.$

Proposition:

A function $f:(a,b)\to R$ is uniformly continuous iff the limits $L_a = \lim_{x\to b} f(x)$ exist & the function $f:[a,b]\to R$ defined by

$$F = \begin{cases} f(x) & \text{if } x\in(a,b) \\ L_a & x=a \\ L_b & x=b. \end{cases}$$

→ Lipschitz continuous functions:

Lipschitz continuous function. A function $f: S\to R$ is said to be L-continuous is $\exists$ a $k\in R$, Such that $|f(x)-f(y)|\le k|x-y|$ $\forall\ (x,y)\in S$.

A L.C function is uniformly continuous

proof: Let $f: S\to R$ and k be a constant real number, such that $|f(x)-f(y)|\le k|x-y|$ $\forall\ x,y\in S$.

→ Take $\delta = \varepsilon/k$ with $\varepsilon>0$. → $|x-y|<\delta$.

$$|f(x)-f(y)|\le k|x-y| < k\delta = \frac{k\,\varepsilon}{k} = \varepsilon.$$

Examples:

$\sin x$, $\cos x$.

→ Limits at infinity

Definition: we say ∞ is a cluster point of $S\subset R$. If for every $M\in R$ there exists an $x\in S$ such that $x\ge M$.

Prop. ⇒ The limit at $\infty, -\infty$ is unique if it exists

Let's prove infinity is a cluster point or not.

Let $\varepsilon>0$. Find $M>0$ so that $\frac{1}{M+1}<\varepsilon$.

If $x>M$: $\frac{1}{x+1}<\frac{1}{M+1}<\varepsilon$. It is also true for $\frac{1}{|x|+1}$.

Infinite Limit

Def: Let $f: S \to R$ and suppose S has ∞ as a cluster point. we say diverges to ∞ as x goes to ∞. if for every $N \in R$ there exists an $M \in R$ such that $f(x) > N$.

whenever $x \in S$ & $x \geq M$ we write

$$\lim_{x \to \infty} f(x) = \infty.$$

or we say that $f(x) \to \infty$ as $x \to \infty$.

Example: $\lim_{x \to \infty} \frac{1+x^2}{1+x}$

proof: for $x \geq 1$ we have.

$$\frac{1+x^2}{1+x} \geq \frac{x^2}{x+x} = \frac{x}{2}$$

Given $N \in R$, take $M = \max\{2N+1, 1\}$ if $x \geq M$,

then $x \geq 1$ & $\frac{x}{2} > N$. so

$$\frac{1+x^2}{1+x} \geq \frac{x}{2} > N.$$

Monotone functions

Definition: Let $S \subset R$ we say a function $f: S \to R$ is increasing is for $x, y \in S$:

$$x < y \to f(x) \leq f(y)$$

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The derivative.

Def: Let I be an interval, let $f: I \to R$ ~~out~~ at let $c \in I$, if the limit

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$ exist, then we say

$\hookrightarrow$ Difference quotient.

f is differenciable at $x = c$ $f'(c) = L$.

$\Rightarrow f(x) = x^2$ $L = \lim_{x \to c} x^2$

$$L = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = 2c.$$

$\Rightarrow$ **Proposition:** Let $f: I \to R$ be differentiable at $c \in I$. then It is continuous at c.

proof: $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ and we know that

$\lim_{x\to c} x-c = 0$.

furthermore, $f(x)-f(c)$ exists &

$$\lim_{x\to c} (f(x)-f(c)) = \left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right) \lim_{x\to c} (x-c)$$

$\lim_{x\to c} f(x) = f(c)$

Proposition:

Let I be an interval and $f: I\to R$ and $g: I\to R$. be differentiable at $x=c$. Let α be a real number. $\alpha \in R$.

i) Define $h: I\to R$ by $h(x) = \alpha f(x)$, Then h is differentiable at $x=c$

(ii) Define $h: I\to f(x)+g(x)$ Then h is differentiable at $x=c$. ⇒ Proof?

$$\lim_{x\to c} \frac{h x - h(c)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim \frac{g(x)-g(c)}{x-c}.$$

$$= f'(x) + g'(x)$$

⇒ $h(x) = f(x).g(x)$

⇒ $h'(x) = f(x) g'(x) + f'(x) g(x)$.

$$h'(x) = \lim_{x\to c} \frac{f(x).g(x) - f(c)g(c)}{x-c} = \lim_{x\to c} \frac{f(x).g(x)}{x-c} - \lim_{x\to c} \frac{f(c)g(c)}{x-c}$$

⇒ $\lim_{x\to c} f(x) g(x) \cdot \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$

$g(x) \; f'(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

⇒ $\lim_{x\to c} f(x) \left[\frac{gx - gc}{x-c}\right] +$

proof: $\frac{h(x)-h(c)}{x-c} = \frac{f(x).g(x) - f(c).g(c)}{x-c}$.

$$= \frac{f(x).g(x) - f(c)g(x) - f(c)g(c) + f(c)g(x)}{x-c}$$

$$= \frac{g(x)\left[f(x)-f(c)\right] + f(c)\left[g(x)-g(c)\right]}{x-c}$$

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{g(x)\left[f(x)-f(c)\right]}{x-c} + \lim_{x\to c} \frac{f(c)\left[g(x)-g(c)\right]}{x-c}$$

$$= g(x) f'(x) + f(x) g'(x).$$

Find differentiation of $\tan x$.

$$f'(x) = \lim_{x\to c} \frac{\tan(x) - \tan(c)}{x-c} = \frac{\tan(x-c) \times (1+\tan x \cdot \tan c)}{x-c}.$$

$f'(x) = \frac{f(x)-f(c)}{x-c}$ $\quad g'(x) = \frac{g(x)-g(c)}{x-c}$. $\quad f(x) g(x)$ $\quad f(c)g(c)$ $\quad f(c)\, g(x)$.

$$f'(x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan(x)}{h}$$

$$= \frac{(\tan(x+h) - \tan x)\,(1+\tan(x+h)(\tan x))}{h\,(1+\tan(x+h)\tan x)}.$$

$$= \lim_{h\to 0} \frac{\tan h}{h}\,(1+\tan x \cdot \tan(x+h))$$

$$= 1+\tan^2 x = \sec^2 x.$$

Chain Rule:

Proof: Let I_1, I_2 be intervals. Let g is a function which takes $I_1 \to I_2$ be differentiable at $c \in I_1$ and if $f: I_2 \to R$ be differentiable at $g(c)$. If $h: I_1 \to R$ is defined by

$$h(x) = (f \circ g(x)) = f(g(x)).$$

$\Rightarrow h$ is differentiable at $x = c$ &

$$h'(c) = f'(g(c))g'(c).$$

Proof: Let $d := g(c)$. Define $u: I_2 \to R$

$V: I_1 \to R$. by $u(y) : \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d \\ f'(d) & \text{if } y = d. \end{cases}$

$v(x) = \begin{cases} \frac{g(x)-g(c)}{x-c} & x \neq c \\ g'(c) & x = c. \end{cases}$

Because f is differentiable at $d = g(c)$, we find that u is continuous at d.

$\Rightarrow$ Similarly v is also continuous at $x = c$.

$\Rightarrow$ For any x & y, $f(y) - f(d)$ is equals to $u(y)(y-d)$ and $g(x) - g(c)$ is $v(x)(x-c)$

$$f(y) - f(d) = u(y)(y-d) \quad \& \quad g(x) - g(c) = v(x)(x-c).$$

$$\Rightarrow h(x) - h(c) = f(g(x)) - f(g(c)) = u(g(x))(g(x)-g(c))$$

$$= u(g(x))(v(x)(x-c)).$$

$$x \neq c \Rightarrow \frac{h(x)-h(c)}{x-c} = u(g(x))\, v(x)$$

$$= f'(g(c))\, g'(c).$$

Relative minima & maxima

Def: Let $S \subset R$ & $f: S \to R$. The function f is said to have a relative maxima at $c \in S$, if $\exists$ a $\delta > 0$ such that $\forall\, x \in S$, $|x-c| < \delta$, we have $f(x) \leq f(c)$.

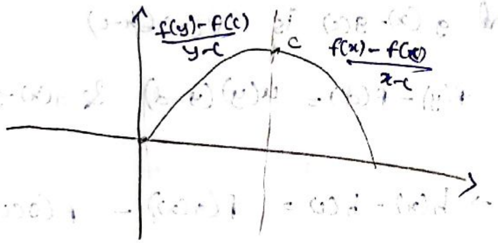
Lemma: Suppose $f:[a,b] \to R$ is differentiable at $c\,(a,b)$ and the function f has a relative minimum or maximum at c. Then $f'(c)=0$.

proof: maximum

c be a relative maximum of f. $|x-c|<f$.

we have $f(x)-f(c) \le 0$

$\to$ If $y<c$.



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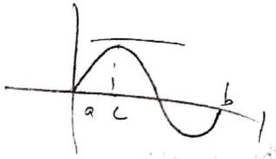
Mean Value theorem.

Rolle's theorem:



Suppose a function has the same value at both end points of an interval.

So min/max is/are in the interior of this interval

Rolli's theorem:

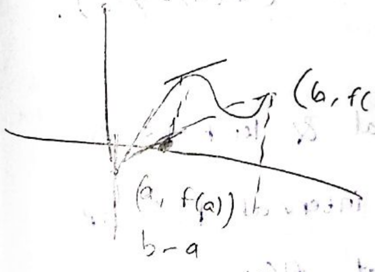
Let $f:[a,b] \to R$ be continuous function differentiable in (a,b) such that $f(a)=f(b)$

Then $\exists$ a 'c' $\in (a,b)$ such that $f'(c)=0$.

Mean Value theorem:

Theorem: Let $f:[a,b] \to R$ be a continuous & differentiable function in (a,b). then $\exists$ a point $c \in (a,b)$ s.t $f(b)-f(a) = f'(c)(b-a)$.

$\Rightarrow$ For a geometric interpolation of mean value theorem.



$\frac{f(b)-f(a)}{b-a}$ is the slope of the line joining $(a, f(a))$ with $(b, f(b))$.

proof: Define $g:[a,b] \to R$ by

$$g(x) = f(x)-f(b) - \frac{f(b)-f(a)}{b-a}(x-b)$$

$g(x)$ is differentiable and continuous (a,b)

Such that $g(a) = g(b) = 0$.

using Rolle's theorem

$$0 = g'(c) = f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(b) - f'(a) = f'(c)(b-a).$$

Cauchy's Mean value theorem :-

Let $f : [a,b] \to R$ & $\phi : [a,b] \to R$.

f, ϕ are continuous & differentiable;

Then $\exists$ $c\,(a,b)$ such that

$$(f(b)-f(a))\,\phi'(c) = f'(c)\,(\phi(b)-\phi(a)).$$

~~proof~~ Preposition:

Let I be the interval & let f be a function defined in that interval $f: I \to R$ is differentiable such that $f'(x) = 0.\ \forall x \in I$.

$\Rightarrow$ Then f is a constant.

Proof: $(x, y) \in J$ with $x<y$ Then f restricted to $[x,y]$ satisfies the hypothesis of mean value theorem:

$\Rightarrow$ $f(y) - f(x) = f'(c)(y-x)$, where $c \in (a,b)$.

$f(y) - f(x) = 0$ $[\because f'(c) = 0]$

$f(y) = f(x).\ \forall\ (x,y) \in I.$

$\Rightarrow$ Proposition:

Let I be an interval and f is defined in this interval & f is differentiable

$f: I \to R$

1) f is increasing iff $f'(x) \geq 0$ for all x in i. $\forall\ x \in I$.

2) similarly x is decreasing iff $f'(x) \leq 0$ $\forall x \in I$.

proof: Suppose f is increasing then $\forall\ x, c \in I$ with, $x \neq c$ we have ~~$[x>c]$~~

$$\lim_{x\to c} \frac{f(x) - f(c)}{x-c} \geq 0$$

$\boxed{f'(c) \geq 0}$ function is increasing.

take $[x, y] \in J$ $x < b$, and $c \in (x,y)$.

$$f(y) - f(x) = f'(c)(y-x).$$

$f(y) - f(x) \geq 0$ $\forall\ (x,y) \in I$ with $x<y$.

Continuity of derivatives; and intermediate

Value theorem:

Let $f:[a,b] \to R$ be differentiable.

suppose $y \in R$ such that $f'(a) \leq y \leq f'(b)$.

$\Rightarrow$ There exists some $c \in (a,b)$ such that

$f'(c) = y$.

Proposition:- ~~Let f : [a,b] → de~~

Let

$f: I \to R$ be a differentiable function

i) If $f'(x) > 0$ (strictly positive) $\forall x \in I$ then f is

strictly increasing.

we have $f(y)-f(x) = f'(c)(y-x)$. -(1)

we have (i) $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$, $x > c$.

from (1) we can write $\Rightarrow f'(x) > 0$,

$$\frac{f(y)-f(x)}{y-x} = f'(c).$$

we have $f'(c) > 0$,

Thus $\frac{f(y)-f(x)}{y-x} > 0$, thus $f(y) > f(x)$.

which proves that function is strictly increasing.