COMMUNICATION THEORY (3-1-0).

Office hours: Thur., 10 to 11   Tutorial

Prerequisite: Signal and Systems, Probability and Random Process.

Reference Book:

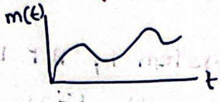
* ① Modem Digital and Analog Comm. system by B. P. Lathi
② An intro to dig. & analog comm. by S. Hyatkins.
③ Intro to comm. system by U Madow.

Evaluation	Grading: Absolute	
End Sem – 40%.	$A \geq 70$	30
Mid Sem – 20%.	$A^{-1} \geq 63$	15
Quiz (2 of 3) – 20%.	$B \geq 57$	15
Surprise Test. – 10%.	$B^{-} \geq 50$	8
Assignment(6) – 10%.	$C \geq 45$	8
	$C^{-} \geq 40$	76%
	$D \geq 35$	

Basics

Source of Information → $m(t)$ → Trans → $s(t)$ → Channel → $y(t)$ → Reciever → Destination

Analog signal - Defined at every point in time.



Channel:
- → Wireless
- → Co-axial
- → Optical Fibre Cable
- → Satellite Link.

voice needs to be converted to electrical by transducer before transmission.
Each channel has diffrent properties and prerequisites.

★ Signal might not have quality to transform in a specific channel.
So, a transmitter is used. This makes signal suitable for transmission

★ Now, we need to undo the effect of transmitter.
The device used is Receiver.
This will be done for a specific channel. We assume a very simple channel model for this and focus on Tx and Rx in this course

Blocks in Transmitter (Tx)

→ Modulator – For both Digital and Analog Communication

It convert the BaseBand signal. to passband signal

It convert the BaseBand signal st. it becomes suitable for given type of channel.
Transform signal st. it becomes suitable for given type of channel.
To do this, the signal is superimposed on a carrier with high frequency.

base band
0 → kHz
signal in original form

★ Base band Signal

→ A / D Converter

→ Encoder

Blocks in Receiver (Rx)

02 Aug 2022

→ Demodulator

→ D/A Convertor

→ Decoder.

Channel Impairments on signal

Due to some channel impairments, the signal sent by the transmitter and signal received by Receiver are different.

Some common Channel Impairments

① **Attenuation:**

Wireless P_s: Power of signal sent

P_R: Power of signal received

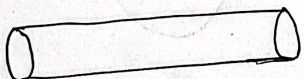
Antenna 1

Antenna 2

$$P_R \propto \frac{P_S}{d^{\alpha}}$$

'α' depends on environment

Wired



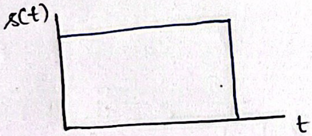
Co-axial Cable.

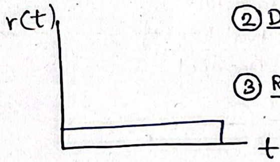
Common reasons for losses

① Diameter Length

② Dielectric of material

③ Radiation Losses





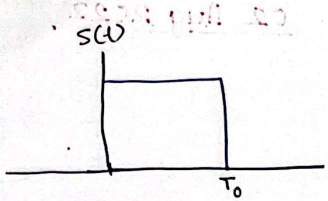
② **Distortion**

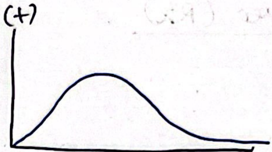
Wired Cut off freq. of LPF will be determined by co-axial cable.

width, T_0

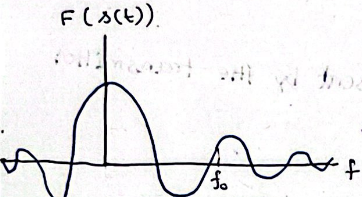
freq, f_0

Square pulse tends to get elongated.

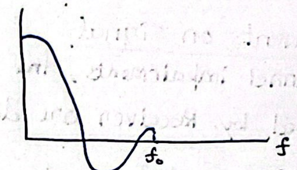




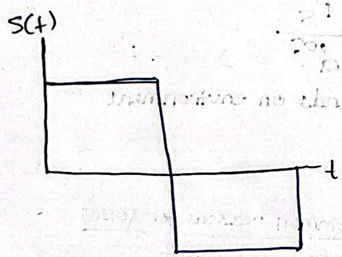
Low Pass representation of Pulse.

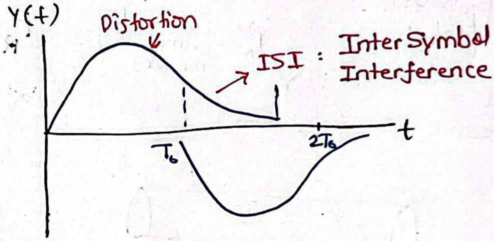


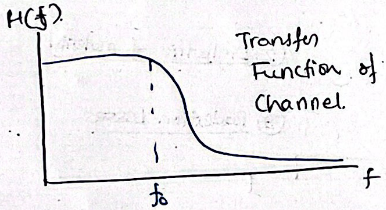
Freq. Response of s(t)



Freq response of s(t) after Distortion

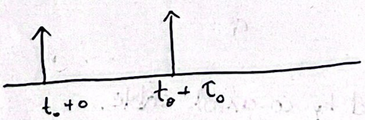




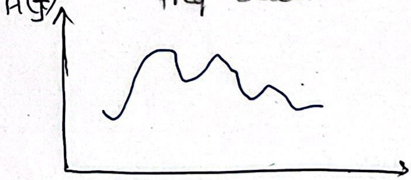


Transfer Function of Channel.

Wireless



Freq Selective

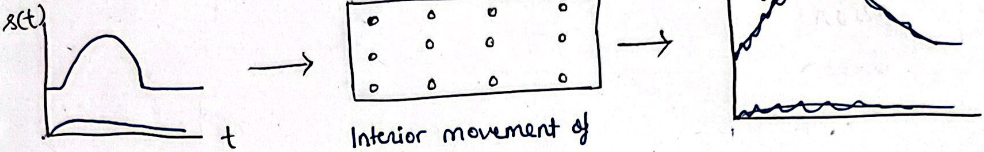


③ Noise

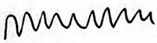
Not part of original signal but still adding to it.

Wired

Thermal Noise



Interior movement of electrons.



If Amplitude of signal is large it will have large Signal to Noise Ratio.

$$SNR = \frac{P_s}{N_0}$$

P_s : Power of signal

N_0 : Power of Noise.

For low amplitude Signal, it will be just noise.

④ Delay

Modulator

Need

① For transforming signal to make it suitable for a given channel.

RA design

② Diameter of Receiver Antenna

$$\boxed{l_A \propto \lambda_{wave}}$$

l_A : Length / Dimension of Antenna

λ_{wave} : Wavelength of wave

Eg. 3000 Hz $\lambda = \frac{c}{f} = 10^5 = 100$ km ; size of antenna $= 0.1\lambda = 10$ km

Eg. 30 MHz $\lambda = \frac{c}{f} = 10$m ; size of antenna $= 0.1\lambda = 1$m

So, to keep the size of antenna small, we need to increase the frequency of signal. This is another advantage of modulation.

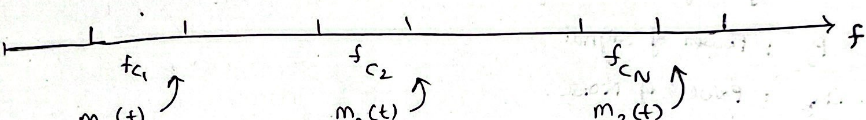
③ Multiplexing FDM Freq. Div. Multiplexing.

Transmitting multiple signal at diff freq.

MW

Base Band → Pass Band.

put message into Diff. carrier freq ~~f~~



We are changing one feature of ~~using~~ carrier ~~freq.~~ using $m(t)$ modulated signal

f_c decide location in spectrum

$m(t)$ — A or D.

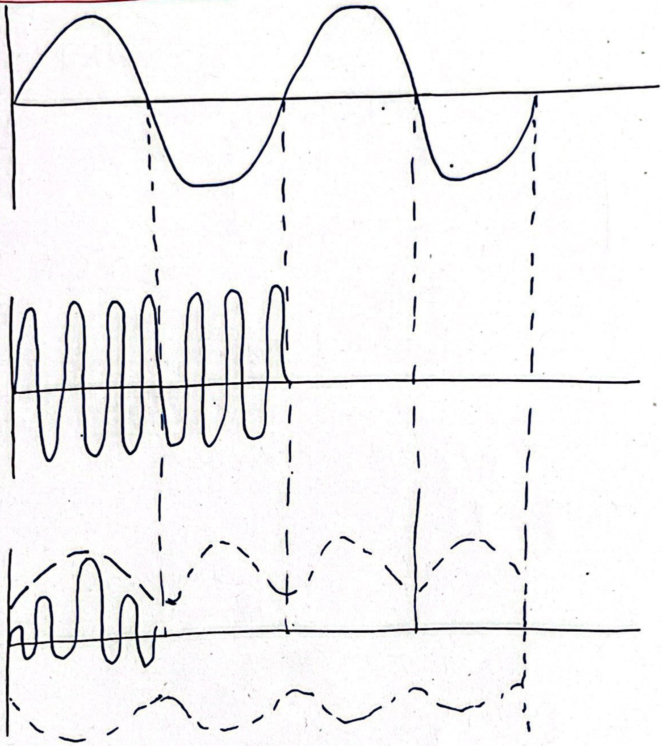
$c(t) = \underline{A} \cos(2\pi \underline{f_c} t + \underline{\theta})$

AM	FM	PM
ASK	FSK	PSK

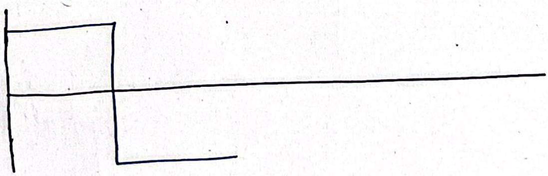
Angle Modulation

continuous wave Modulation.

Amplitude Modulation



Amplitude Shift Keying : Digital Counterpart.



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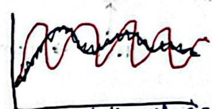
$S \rightarrow Tx \rightarrow Ch \rightarrow Rx \rightarrow D$ (SNRI at input of Rx, SNR_o at output of Rx)

- Tx (Both): Modulator, A/D (D), Encoder (D)
- Ch: Attenuation, Distortion, Noise
- Rx: Demodulator, D/A Converter, Demodulator.

Performance of Demodulator: BER/SER - For Digital

$SNR = \frac{P_s}{N_o}$ - For Analog. - How strong is un signal wrt. noise

We want high SNR output.



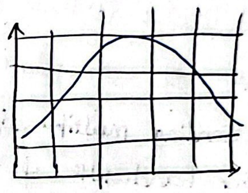
Power ↑ : Better representation of signal

$\frac{SNR_o}{SNR_I}$ = Noise figure of receiver.

Converting Analog to Digital Signal

1. sampling:

$T_s < \frac{1}{2B}$ Nyquist Theorem.



2. Quantisation: Approximating signal value to nearest level.

Quantisation Error: We show statistical description of error.

n level we can encode using $\log_2 n$ bits.

Symbol (Transmission) rate : $R_s = \frac{1}{T_s}$ Symbols sent over channel per unit time.

$$\boxed{R_b = \frac{\log_2 k}{T_s}}$$

Bits ~~per~~ of sample transmitted per unit time.

More R_b, better capacity of channel.

But R_b value is limited ∴ There is capacity of channels.

Encoders

① Source Coding: Coding signal with lesser bits.

② Error Correction Codes

$S_1 \rightarrow 0$, $\quad S_1 \rightarrow 0\ 0\ 0$

$\hat{S}_1 \rightarrow 0 \quad \hat{S}_1 \rightarrow 1\ 0\ 0$

We introduce redundancy to decrease probability of Error.

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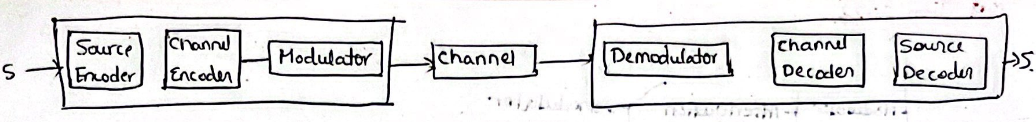


AWGN : Added to White Gaussian Noise

$Y(t) \longrightarrow \oplus \longrightarrow r(t) = y(t) + n(t)$

$N(t) \rightarrow$ Complex Gaussian $\underbrace{\text{White}}_{\text{Uncorrelated.}}$ Noise $\rightarrow CN(0, \sigma_n^2)$

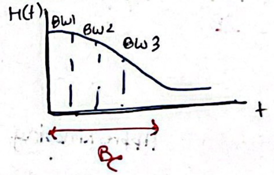
Pr

<u>Resources of Signal:</u> (1) Power, P_s (2) Bandwidth, $B. < B_c$, channel bandwidth
$\boxed{B_M > B}$ $B_M \rightarrow$ Bandwidth of modulated signal

Transfer Function of communication channel. decides bandwidth of signal.

Multiplexing: Sending multiple signal over channel simultaneously

If we reduce bandwidth of modulated. signal.



<u>~~Construction~~ Constraint</u> complexity of T_x / R_x.

<u>Representing a signal using vector</u>

Shannon Channel Chappacity Theorem

$C = B_c \log_2 (1 + SNR)$ bit/sec

You can transfer C bit/sec, while keeping BER as small as possible

~~B is premodulated sig~~ → B_c is channel Bandwith.

Thus, we cannot have channel with capacity > C with smaller BER.

Digital Signal is sent using square pulses. Duration of these pulse and carrier freq. defines bandwidth of signal.

Energy

We deal with finite duration pulses / decreasing

Energy dissipated by a 1 Ω resistor at time t is $x^2(t)$.

$x(t) \longrightarrow [t_1 \;\; t_2]$

$$E_x = \int_{t_1}^{t_2} |x^2(t)|\, dt \quad < \infty$$

Power

If signal is increasing with time, we define power / infinite duration

So, when E is ∞, we define power.

$$P_s = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x^2(t)|\, dt$$

If signal is complex, we take absolute value of signal.

Energy signal : If energy is finite. $P = 0$

Power Signal : If energy is infinite or zero, $P =$ finite.

All periodic signal are power signal

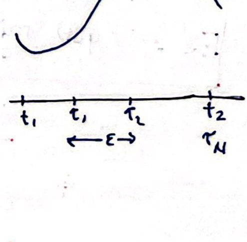
$x(t) \sim N(0, \sigma^2)$ Its power is variance.

$P_x = E[|x(t)|^2] = \sigma^2$

∴ Power can be characterised by variance and mean

Representing a signal using vector

$g(t) = [g(\tau_1) \; \ldots \; g(\tau_N)]$ $\quad \lim_{N \to \infty} g \longrightarrow g(t)$



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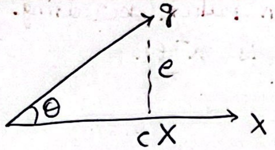


1. Scalar Product $\langle g, g \rangle = \|g\|^2 = (\text{norm})^2$
2. Scalar g and x : $\langle g, x \rangle = \|x\| \, \|g\| \cos\theta$ Inner Product.
3. Component of g along x: $c\|X\| = \|g\| \cos\theta$ Projection of g along X.

- $g = cX + e$

$$c = \frac{\|x\| \, \|g\| \cos\theta}{\|x\|^2} = \frac{\langle g, x \rangle}{\langle x, x \rangle}$$

4. If g and x are ortho $c = 0$ or $\theta = \frac{\pi}{2}$. $\langle g, x \rangle = 0$

$$\langle g, x \rangle = \sum g_i x_i$$

$$\langle g(t), x(t) \rangle = \sum_{i=0}^{\infty} g(\tau_i)\, x(\tau_i) = \int_{t_1}^{t_2} g(\tau)\, x(\tau)\, d\tau$$

$$\|x(t)\|^2 = \langle x(t), x(t) \rangle = \int_{t_1}^{t_2} |x^2(t)|dt = E_x$$ Energy is norm squared

If g(t) and x(t) are ortho $\boxed{\int_{t_1}^{t_2} g(t)\, x^*(t)\, dt = 0}$ No component of g(t) falls on x(t).

★ $g(t) \simeq c\,x(t)$ to reduce to the energy of error, g(t) is approximated by a factor of c

$g(t) = cX(t) + \underbrace{\cancel{e(t)}}_{E_e}$ Approximating g(t) with x(t) linearly so that error is minimised. Resolved by component orthogonal to X.

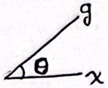
$$\boxed{E_e = \cancel{g(t)} \min_{c} \int_{t_1}^{t_2} [g(t) - c\,x(t)]^2\, dt}$$ Error wrt. c.

To minimise error signal, optimum 'c' can be found by differentiating.

$$\frac{dE_e}{dc} = 0 \Rightarrow \boxed{c = \frac{\int_{t_1}^{t_2} g(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt} = \frac{\langle g(t), x(t) \rangle}{E_x}}$$ factor of 2 ?

★ $\langle x(t), x(t) \rangle = \int_{t_1}^{t_2} \underbrace{x(t)\, x^*(t)}_{|x(t)|^2}\, dt = E_x$

Correlation of two signals



$\cos\theta$ captures correlation.

It denotes amount of component of g along x.

It captures dependency and not amount of dependency.

$\rightarrow \langle g, x \rangle = \|x\|\, \|g\| \cos\theta$

$\rightarrow g = cx$ if $\theta = 0°$ $\because$ Then $e = 0$ So, g should be along x to represent it completely in terms of x without any error.

$\rightarrow c\|x\| = \|g\| \cos\theta$

$\rightarrow \cos\theta = \dfrac{\langle g, x \rangle}{\|g\|\, \|x\|} = \rho$

$$\Rightarrow \boxed{\rho_{gx} = \frac{\int_{t_1}^{t_2} g(t)\, x^*(t)\, dt}{\underbrace{\sqrt{E_g E_x}}_{}}}$$

$\sqrt{E_g E_x} \rightarrow$ Normalising Factor

Correlation Function

It captures dependency of $g(t)$ on delayed $x(t)$. i.e. $x(t+\tau)$

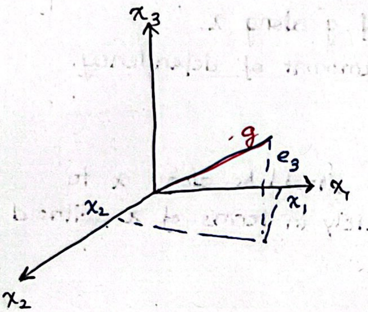
$$\psi_{x,g}(\tau) = \int_{t_1}^{t_2} g(t)\, x^*(t+\tau)\, dt$$

The use of this function: Received signal is a delayed ~~of~~ version of transmitted signal. So, when we receive signal, $g(t)$, we correlate it with transmitted signal, $x(t)$ and take the value of delay which gives max. value of this function. $\because$ then error would be least.

So, we can get most close signal.

Auto - correlation Function:

$$\psi_{x,x}(\tau) = \int_{t_1}^{t_2} x(t)\, x^*(t+\tau)\, dt$$



$$g = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$\langle x_1\ x_2 \rangle = 0 \quad \langle x_3, x_1 \rangle = 0 \quad \langle x_3, x_2 \rangle = 0$

Any 3D vector can be represented as linear combination of 3 orthogonal vectors.

$$c_i = \frac{\langle g, x_i \rangle}{\langle x_i ; x_i \rangle}$$

$\{x_1, x_2, x_3\} \rightarrow$ Complete basis set.

Generalising to n dimensions

$$g(t) = \sum_{i=1}^{n} c_i\, x_i(t) + e_n$$

There would be some unresolved error for $x_{n+1}(t)$, $x_{n+2}(t)$ ..... components.

Basis set : $\{x_i(t)\}_{i=1}^{N}$

Generalising to infinite dimensions

$$g(t) = \sum_{i=-\infty}^{\infty} c_i\, x_i(t)$$

Decomposing signal to infinite sum of orthogonal basis functions.

Basis Set : $\{x_i(t)\}_{i=1}^{\infty}$

Generalised version of Fourier Series.

i.e. $\|e_n\|^2 \longrightarrow 0$ as $N \rightarrow \infty$

$$c_i = \frac{\langle g(t), x_i(t) \rangle}{\langle x_i(t), x_i(t) \rangle}$$

Fourier Series.

Complex Exponentials as Basis Function

$$x_n(t) = \exp(j\, 2\pi n f_0 t) \qquad f_0 = \frac{1}{T_0}$$

$n = -\infty$ to ∞.

To show the x_n form basis set, we need to show any pair of n, m, st. $n \neq m$ are orthogonal.

$$\rightarrow \int_0^{T_0} x_n(t)\, x_m^*(t)\, dt$$

$$\rightarrow \int_0^{T_0} \exp(j\, 2\pi (n-m) f_0 t)\, dt = \begin{cases} 0, & m \neq n \\ T_0, & m = n \end{cases}$$

sum of sine and cos in exponential would be zero over a period.
And if $n = m$, $n - m = 0$, so $\int_0^{T_0} 1\, dt = T_0$.

$$\rightarrow g(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j\, 2\pi n f_0 t), \quad f_0 = \frac{1}{T_0}$$

IFT of $G(f)$

Expressing signal as linear combination of basis vector.

$$\text{where,} \quad c_n = \frac{1}{T_0} \int_0^{T_0} g(t) \exp(-j\, 2\pi n f_0 t)\, dt = G(f)$$

FT of $g(t)$

$$T_0 = \int_0^{T_0} \exp(j\, 2\pi n f_0 t) \exp(-j\, 2\pi n f_0 t)\, dt$$

$$g(t) \xleftrightarrow{FT} g(f)$$

$$\bigstar\ E_g = \sum_{n=-\infty}^{\infty} \|c_n\|^2 \quad \text{Parseval's Theorem.}$$

$$\|g(t)\|^2 = E_g = \sum_{n=-\infty}^{\infty} \|c_n \exp(j\, 2\pi n f_0 t)\|^2 = \sum_{n=-\infty}^{\infty} \|c_n\|^2 \cancel{\|\exp(j 2\pi n f_0 t)\|^2} \quad (\text{as } \|re^{i\theta}\| = r)$$

So, x_n are special basis function leading to Fourier Series

Addition of Orthogonal signals

$X = X_1 + X_2$

$\|X\|^2 = \|X_1\|^2 + \|X_2\|^2$

$$\boxed{\begin{aligned} X(t) &= X_1(t) + X_2(t) \\ E_x &= E_{x_1} + E_{x_2} \end{aligned}}$$

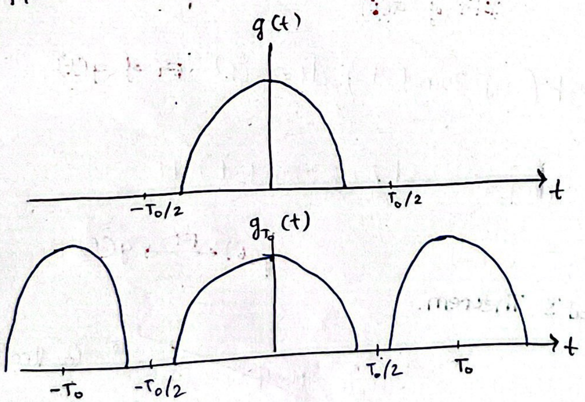
why ?

The Fourier Series is not applicable for aperiodic signals

Fourier series → Periodic → Finite Duration

Fourier Transform → Aperiodic → Infinite Duration.

How to extend it for longer duration or aperiodic signal ?



$$\underbrace{\lim_{T_0 \to \infty} g_{T_0}(t)}_{\text{Periodic Signal.}} = g(t)$$

Fourier Transform / Fourier Spectrum

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j 2\pi n f_0 t) \; ; \quad f_0 = \frac{1}{T_0}$$

$$\text{where, } D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) \exp(-j 2\pi n f_0 t)\, dt$$

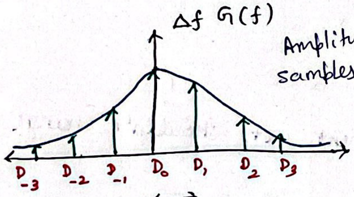
As $g_{T_0}(t)$ and $g(t)$ are same in interval $\left(\frac{-T_0}{2}, \frac{T_0}{2}\right)$.

$$D_n = \frac{1}{T_0} \underbrace{\int_{-\infty}^{\infty} g(t) \exp(-j 2\pi f t)\, dt}_{G(f)} \Big|_{f = n f_0}$$

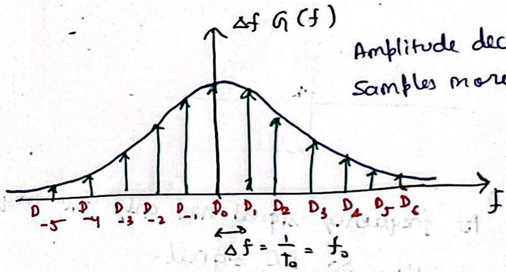
$$= \frac{1}{T_0} G(n f_0)$$

Let $\Delta f = f_0$ = spacing between frequency components

$$= \Delta f \; G(n \Delta f)$$



Smaller T_0



Larger T_0

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{T_0 \to \infty} \sum_{n=-\infty}^{\infty} \Delta f \; G(n \Delta f)\, e^{j 2\pi n \Delta f t}$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j 2\pi f t}\, df \qquad G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j 2\pi f t}\, dt$$

$|G(f)|$ $|g(t)|\, e^{-j2\pi f t}$

Properties of Fourier Transform

① g(t) is real. $G(-f) = G^*(f)$

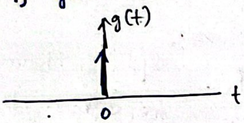
Magnitude $\rightarrow |G(-f)| = |G^*(f)| = |G(f)|$ Even Symmetric

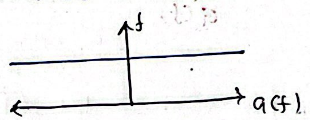
Phase $\rightarrow \theta(G(-f)) = -\theta(g(f))$ odd Symmetric

② Linearity:

$\alpha_1 g_1(t) + \alpha_2 g_2(t) = \alpha_1 G_1(f) + \alpha_2 G_2(f)$

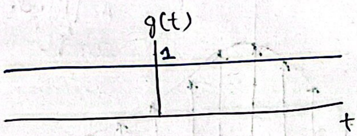
③ If $g(t) = \delta(t) \longleftrightarrow G(f) = 1 \quad \forall f$

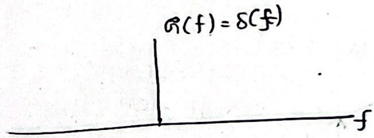




All frequencies are present in equal magnitude.

④

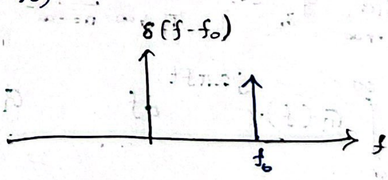




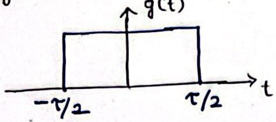
As frequency signal has only one freq. component, g(t) shouldn't change at all. So, DC signal.

⑤ $g(t)\, e^{+j2\pi f_0 t} \xrightarrow{FT} \cancel{G(f)}\ G(f - f_0)$

$e^{j2\pi f_0 t} \xrightarrow{FT} \delta(f - f_0)$



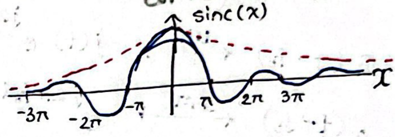
⑥ $g(t)$ is rectangular signal



$$G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t}\, dt = \tau \frac{\sin(\pi f \tau)}{\pi f \tau} = \tau \, \mathrm{sinc}(\pi f \tau)$$

$$= \left[\frac{e^{-j2\pi f t}}{j2\pi f} \right]_{-\tau/2}^{\tau/2}$$

Bandwidth
Range of spectrum where 90-95% of signal energy is concentrated



Sinc Function $\quad \mathrm{sinc}(x) = \dfrac{\sin x}{x}$

1). Even function : As it is product of two odd functions

2). zero at $x = \pm k\pi$, $k > 0$

3). $\mathrm{sinc}(0) = 1 = \lim_{x \to 0} \dfrac{\sin x}{x}$

4). Amplitude $\propto \dfrac{1}{x}$

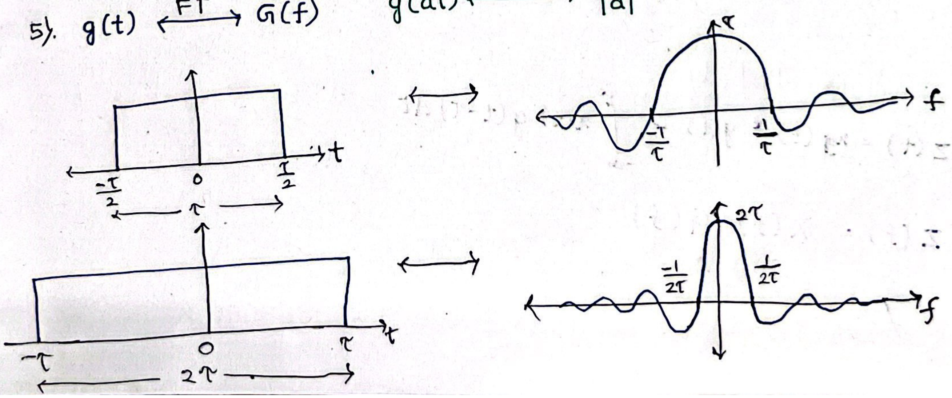
So, we need to amplitude $\propto \left(\dfrac{1}{x}\right)^n$
$n > 1$
higher order.

Expansion in time
$\downarrow\uparrow$
compression in freq domain.
spectral Exp, comp.

All freq. component gets diff phase shift

Time Scaling

5). $g(t) \xleftrightarrow{FT} G(f) \qquad g(at) \longleftrightarrow \dfrac{1}{|a|} G(f/a)$



6). $g(t-t_0) \longleftrightarrow G(f)e^{-j2\pi f t_0}$ $\quad g(t) \longleftrightarrow G(f)$ Time Shifting

No change in amplitude of ~~phase~~ frequency spectrum

Amplitude spectrum remains unchanged.

Phase spectrum changes by $-2\pi f t_0$.

-

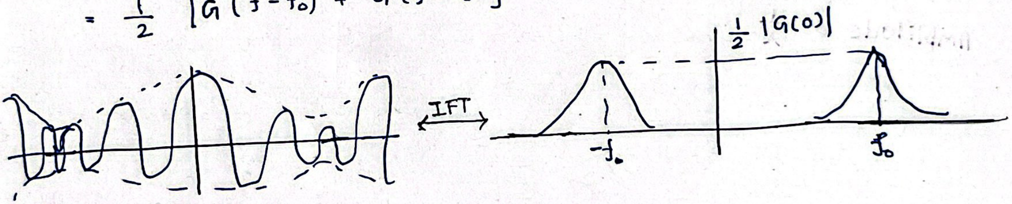
7). $G(f-f_0) \xleftrightarrow{IFT} g(t)e^{j2\pi f_0 t}$ Frequency shifting Modulation Property

$\delta(f-f_0) \xleftrightarrow{IFT} e^{j2\pi f_0 t}$

Q. Find Fourier Transform of $g(t)\cos(2\pi f_0 t)$

Ans. $\frac{g(t)}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) = \frac{1}{2}\left[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}\right]$

$= \frac{1}{2}\left[G(f-f_0) + G(f+f_0)\right]$



Modulation is changing frequency using carrier.

8). Convolution Property

(a). $z(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(t)\, g(t-\tau)\, d\tau$

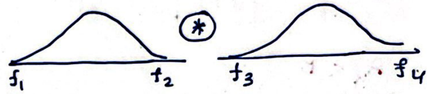
$Z(f) = X(f) \cdot G(f)$

(b). $Z(f) = X(f) * G(f)$

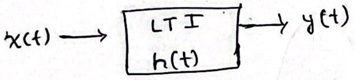
$z(t) = x(t)\, g(t)$ → Bandwidth, B_y

Bandwidth, B_z Bandwidth, B_x

$$\boxed{B_z = B_x + B_y}$$



Span of Convolution



$y(t) = x(t) * h(t)$ → Impulse Response

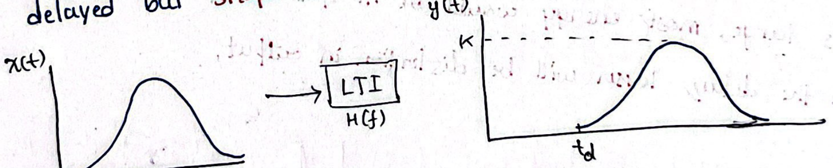
$Y(f) = X(f)\; H(f)$ → Transfer Function.

FT pair are unique. So, any LTI System can be described by $h(t)$ or $H(f)$ completely.

$H(f) = |H(f)|\, e^{-j2\pi f}$

Distortionless System/ Filter (DF)

Output doesn't distort the shape of input signal. It can get attenuated or delayed but shape shouldn't change.

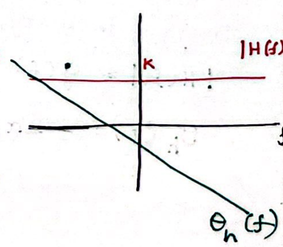


$y(t) = K\, x(t - t_d)$

$Y(f) = K\, X(f)\, e^{-j2\pi f t_d}$

For LTI System to be a DF

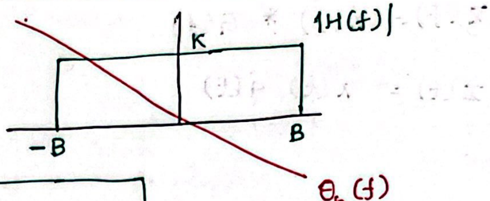
$$\boxed{H(f) = K\, e^{-j2\pi f t_d} \qquad h(t) = K\, \delta(t - t_d)}$$



Different frequency should have different phase shifts.

$x(t) \longrightarrow \boxed{LPF} \longrightarrow y(t)$

$B_x < B$



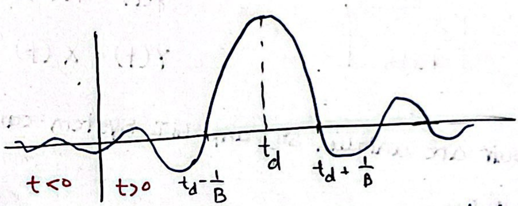
$$\boxed{H(f) = \begin{cases} K e^{-j2\pi f t_d} & , \forall f \in [-B, B] \\ 0 & , \text{otherwise.} \end{cases}}$$

slope = $2\pi f t_d$

K: Factor of attenuation of input

We want LPF system to be distortionless. So, it should provide constant attenuation in Bandwidth of signal B_x. which is limited by B. So, we can draw spectrum and write H(f) as above.

Corresponding h(t) would be:



which makes system non-causal.

It depends on signal, t<0,

So, we ignore part of signal for t<0.

$\hat{h}(t) = h(t)\, u(t)$

If t_d is large, most energy would be in positive part.

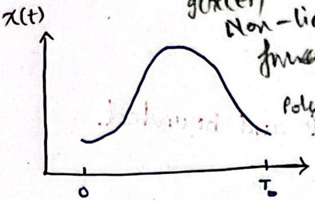
So, higher the delay, lesser will be distortion in output.

Magnitude of Transfer freq should be constant with freq. of i/p signal involved with phase $\propto$ frequency.

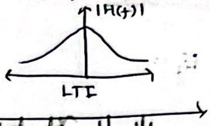
$|H(f)| = g(f)$

$\theta_h(f) = -2\pi f t_d$

If magnitude is not constant ? → Linear Distortion

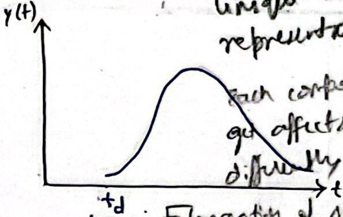


g(x(t)) Non-linear function



Polynomial fn of i/p

$\theta_h(f) = -2\pi f t_d$



unique representation

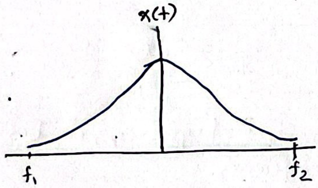
Each component gets affects differently

Elongation of signal

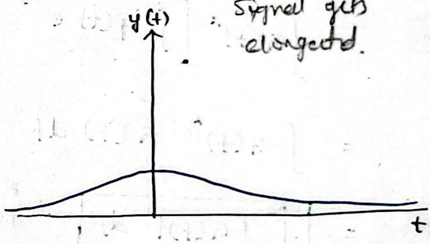
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_n t}$$

$$y(t) = \sum c_n' e^{j2\pi f_n t}$$

Non - Linear Distortion

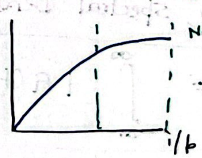


System
$y(t) = g(x(t))$



Signal gets elongated.

Eg. Amplifier : Not linear above cut off voltage. o/p



$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \cdots$$

2B   3B   B: Bandwidth.

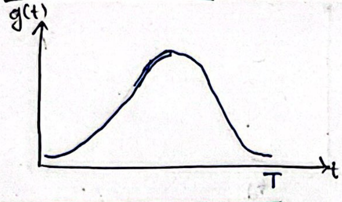
↳ Non - Linear Polynomial function of input.

Energy/power spectral density

For M.G. signal.

Energy Spectral Density

Finite bounded.



$E_g =$

g(t) should be finite and bounded.

$$\boxed{E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt} = \int_{-\infty}^{\infty} g(t)\, g^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft} \, df \right]^* dt$$

$$= \int_{-\infty}^{\infty} g(t) \left[\int G(f)^* \, e^{-2j\pi ft} \, df \right] dt$$

$$= \int G(f)^* \left[\int g(t)\, e^{-j2\pi ft} \, dt \right] df \quad \text{(Replacing Integrals)}$$

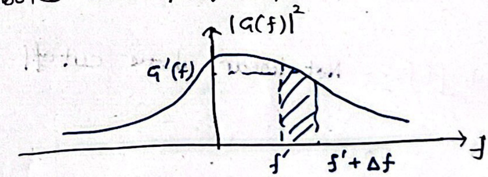
$$= \int G(f)^* \, G(f)\, df$$

$$= \boxed{\int_{-\infty}^{\infty} |G(f)|^2 \, df}$$

Energy Spectral Density

Area concentrated around frequency. i.e. Energy Distributed over frequency.

$$\boxed{E_g = \int_{-\infty}^{\infty} |G(f)|^2 \, df}$$



Energy of shaded frequency range: $|G'(f)|^2 \Delta f$

Statistical Matrix

Energy Spectrum Density : For Random Signal For random signal

$$|G(f)|^2$$

$t+\tau$?

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$ Auto - Correlation function.

$$E_g = \int_{-\infty}^{\infty} \psi_g(f)\, df = \int_{-\infty}^{\infty} g^2(t)\, dt$$

$$E_g = R_g(0) = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

$\updownarrow$ FT

$$\psi_g(f) = \int_{-\infty}^{\infty} R_g(\tau)\, e^{-2\pi j f\tau}\, d\tau$$

↳ Fourier Transfer of $R_g(\tau)$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt\right] e^{-2\pi j f\tau}\, d\tau = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt\right] e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g^*(t-\tau)\, e^{-j2\pi f\tau}\, d\tau\right] g(t)\, dt$$

$$= \int_{-\infty}^{\infty} G^*(f)\, e^{j2\pi f t}\, g(t)\, dt$$

$$R_g(0) = E_g$$

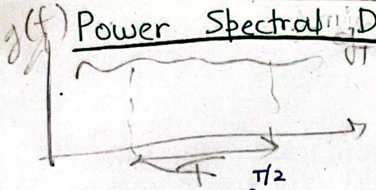
$$= G^*(f)\, G(-f) = |G(f)|^2 \quad (FT)$$

ESD $\psi_g(f) \xleftrightarrow{FT} R_g(\tau)$ Auto Corr.

$$R_g(\tau) = \int |G(f)|^2\, e^{j2\pi f\tau}\, df$$

$$|R_g(\tau)| \leq R_g(0) = \int |G(f)|^2\, df = E_g$$

Power Spectral Density



$$\int_{-\infty}^{\infty} |G_T(f)|^2 \, df$$ Infinite

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g_T(t)|^2 \, dt = \lim_{T\to\infty} \frac{E_{g_T}}{T}$$

Power of interval

T

$T \to \infty \to$ Power

constant or converging to constant

Both should converge at same rate

$\therefore$ E_g should incr. with time.

$\therefore$ spectral density ~~incre.~~ should incr.

Montone convergence theorm $\to$ Exchange limit & $\int$.

Fraction of power conc around freq f.

$$P_g = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} \, df$$

$|G_T(f)|^2 \to$ FT of truncated version of signal

PSD $\phi_g(t) = \lim_{T\to\infty} \dfrac{|G_T(f)|^2}{T}$ $\to$ Power Spectral Density

$\Updownarrow$ FT

Auto $\Psi_g(\tau)$

$$\boxed{P_g = \int_{-\infty}^{\infty} \phi_g(f) \, df = \Psi_g(0)}$$

Finite Duration pulse

$g(t) \longrightarrow$ [LTI $H(f)$] $\longrightarrow y(t)$

$\downarrow$ ESD $= |G(f)|^2$

$\downarrow$ ESD $= |Y(f)|^2 = |G(f)H(f)|^2$

$= |G(f)|^2 \, |H(f)|^2$

LTI system Doesn't change spectral density of i/p either energy power

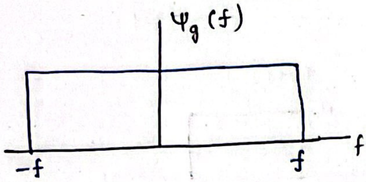
$$\boxed{\Psi_y(f) = \Psi_g(f) \, |H(f)|^2}$$

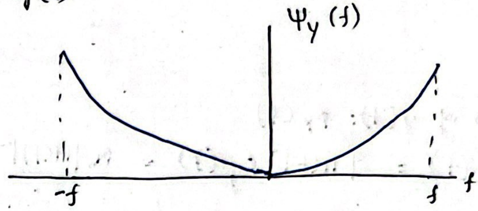
Valuable for random signal. Auto corr $\to$ Energy or power.

For Differentiator system

$$g(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

Differentiator





$$y(t) = \frac{d\,g(t)}{dt} = h(t) * g(t)$$

$$Y(f) = H(f)\, G(f)$$

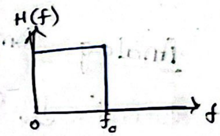
$H(f) = j2\pi f \rightarrow$ Fourier Transform of Differentiator.

$$\boxed{\Psi_Y(f) = |j2\pi f|^2\, \Psi_g(f)}$$

0 mean noise.

For ideal low pass Filter

We have an ideal low pass filter with transfer function:



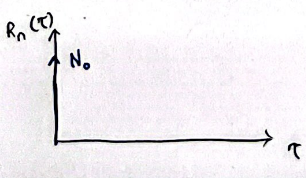
Let we give a random signal, i.e, 0 mean noise as input.

$$R_n(\tau) = E[g(t)\, g(t+\tau)] = \begin{cases} 0, & \tau \neq 0 \\ N_0, & \tau = 0 \end{cases}$$ Variance as mean is 0.

As $g(t)$ is uncorrellated, $E[g(t)\, g(t+\tau)] = \cancel{E[g(t)]}^{0}\, E[g(t+\tau)] = 0 \quad \forall\, \tau \neq 0$

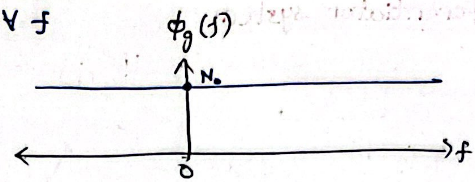
$\because$ 0 mean noise

For, $\tau = 0$,



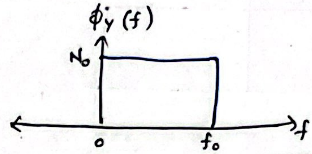
$$R_n(\tau) = E[g(t)\, g(t+\tau)] = N_0\, \delta(\tau)$$

ESD of $g(t)$: $\phi_g(f) = N_0 \;\; \forall f$



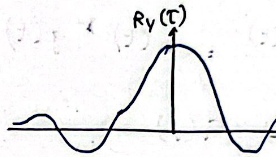
ESD of $y(t)$: $\phi_y(f)$

$\phi_y(f) = |H(f)|^2 . \phi_g(f) = N_0 |H(f)|^2$



Auto-corr.

$R_y(\tau)$: Inverse FT of $\phi_y(f)$: Sinc.



uncorrelated + correlated. And filter.

<u>Conclusion:</u>

Given uncorrelated function, the output becomes correlated regardless of type of filter used.

<u>Analog Communication</u>

Carrier; $A \cos(2\pi f_c t + \theta)$ Message: $m(t)$

$A(t) = g_1(m(t)) \rightarrow$ AM

$f(t) = g_2(m(t)) \rightarrow$ FM

$\theta(t) = g_3(m(t)) \rightarrow$ PM

FM, PM: Angle Modulation

AM, FM, PM: Carrier Modulation

<u>Base band Communication</u>

When message is transmitted as it is, i.e., without modulation

Amplitude Modulation: SC: Double Side Band Modulation SC-DSB

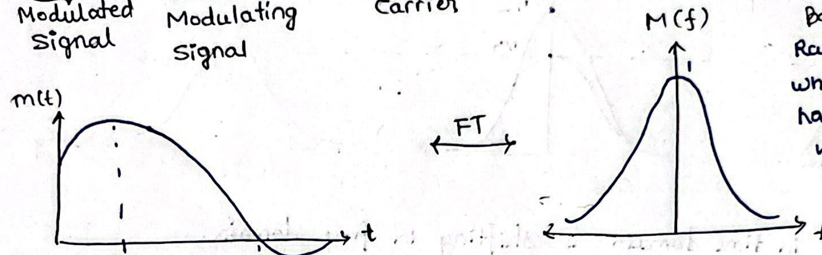
Varying amplitude of carrier signal instantaneously w.r.t. message signal

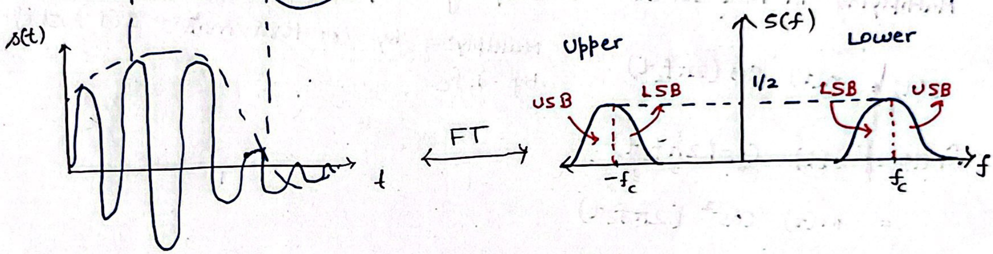
Suppressed carrier

Shifts signal by $\pm f_c$ frequency.

$$\underbrace{s(t)}_{\text{Modulated Signal}} = \underbrace{m(t)}_{\text{Modulating Signal}} \underbrace{\cos(2\pi f_c t)}_{\text{Carrier}}$$

Bandwidth
Range of freq. where signal has significant value.





$$S(f) = \frac{1}{2}\left[M(f_c + f) + M(-f_c + f) \right]$$

$$E_m = \int_{-\infty}^{\infty} |M(f)|^2 \, df$$

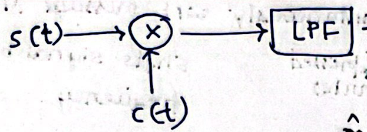
$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 \, df = \int_{-\infty}^{0} |S(f)|^2 \, df + \int_{0}^{\infty} |S(f)|^2 \, df$$

$$= \frac{1}{4} E_m + \frac{1}{4} E_m = \frac{1}{2} E_m \qquad \boxed{E_s = \frac{1}{2} E_m} \qquad \boxed{P_s = \frac{1}{2} P_m}$$

We are multiplying, but still energy of signal is decreasing as energy involves squaring for input which ~~this~~ lies between (0,1) so squaring causes decrease in energy.

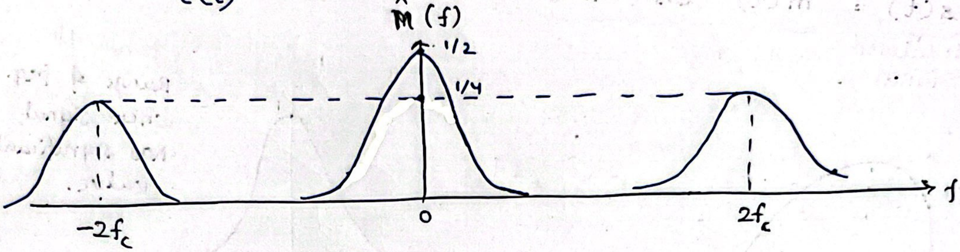
$$\boxed{B_s = 2 B_m}$$

Amplitude Demodulation : Synchronised / Coherent Demodulation



$\hat{m}(t) = \frac{m(t)}{2}$

This again shifts signal by $\pm f_c$.



Multiplying in time domain is shifting in freq. domain.

$s(t) = m(t) \cos(2\pi f_c t)$ ∵ Multiplying by cos term makes $S(f)$ shift by $\pm f_c$.

$\hat{m}(t) = s(t) \cos(2\pi f_c t)$

$= m(t) \cos^2(2\pi f_c t)$

$m(t) = \frac{m(t)}{2} + \frac{m(t)}{2} \cos(4\pi f_c t)$

$= \frac{m(t)}{2}$ (After LPF)

FT $\updownarrow$

$\hat{M}(f) = \frac{1}{2} M(f) + \frac{1}{4} \left[M(f - 2f_c) + M(f + 2f_c) \right]$: Passed through LPF to get $\frac{m(t)}{2}$.

Carrier way, perfectly synchronised by frequency and phase
We need to have the same carrier wave for the demodulation that was used in modulation. Hence the name.

Why is it called SC-DSB?
Because only frequency around carrier frequency is present but carrier is absent.

Scanned with CamScanner

What if carrier at Receiver is not synchronised ?

$$\boxed{\Delta f << f_c}$$

$$s(t) \cos(2\pi(f_c + \Delta f)t + \theta)$$

$$\hat{m}(t) = m(t) \cos(2\pi f_c t) \cos(2\pi(f_c + \Delta f)t + \theta)$$

$$\frac{m(t)}{2} \cos(2\pi \Delta f t + \theta) + \underbrace{\frac{m(t)}{2} \cos(2\pi(2f_c + \Delta f)t + \theta)}_{0\text{, when passed through LPF}}$$

CASE 1: $\Delta f = 0$ Constant phase shift..

$$\hat{m}(t) = \frac{m(t)}{2} \cos(\theta).$$

So, constant attenuation in modulated signal is observed when there is a constant phase shift

CASE 2: $\theta = 0$

$$\hat{m}(t) = \frac{m(t)}{2} \cos(2\pi \Delta f t)$$

If $\Delta f = 1$ Hz, then we are changing amplitude of $m(t)$ by $\cos(2\pi t)$

i.e. freq. of 1 Hz. i.e. every second.

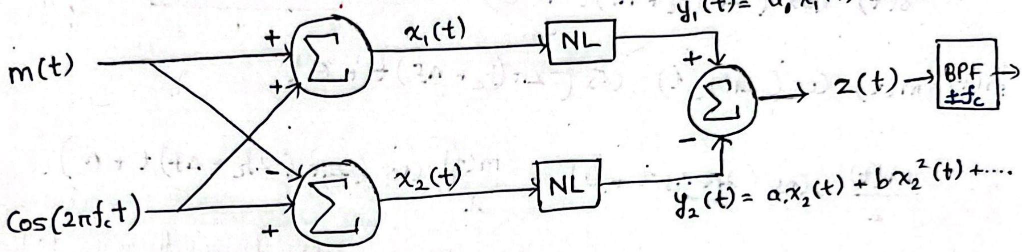
$\therefore$ Increase and decrease in amplitude with a time period of $\frac{1}{\Delta f}$.

Modulator

(a) Non- Linear Modulator

Types of modulator which gives product of two signals.

Let say we have a Non-Linear System.



$$z(t) = y_1(t) - y_2(t)$$

$$= a x_1(t) + b x_1^2(t) - a x_2(t) - b x_2^2(t)$$

$$= 2a\, m(t) + 4b\, m(t) \cos(2\pi f_c t)$$

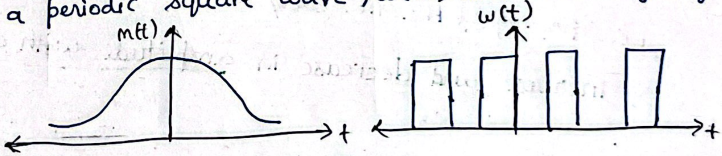
We need to pass it through a BPF, to get $4b\, m(t) \cos(2\pi f_c t)$

So, we can use a quadratic Non-Linear system as then, there would not be any higher terms.

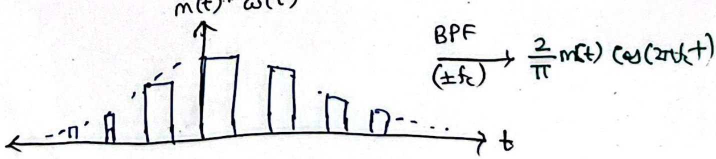
Switching

Let we multiply by a periodic square wave, $w(t)$, and the message signal.

$m(t) * w(t)$



switching on and off action



$$w(t) = \sum_{n=-\infty}^{\infty} C_n \cos(2\pi n f_c t)$$

(Fourier Transform)
(Sum of Cosines)

$\rightarrow m(t)\, w(t) = \sum_{n=-\infty}^{\infty} m(t)\, C_n \cos(2\pi n f_c t)$

Its Freq. spectrum will included scaled $M(f)$, $M(f \pm f_c)$, $M(f \pm 2f_c)$, ...

As we are interested in only $M(f \pm f_c)$ part of frequency component,

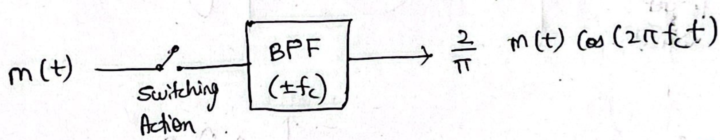
we can pass it through BPF.

$\rightarrow g(t) = m(t)\, w(t) = m(t) \left[\frac{1}{2} + \frac{2}{\pi} \left\{ \cos(2\pi f_c t) - \frac{1}{3}\cos(3\pi f_c t) + \frac{1}{5}\cos(5\pi f_c t) \right\} \right]$

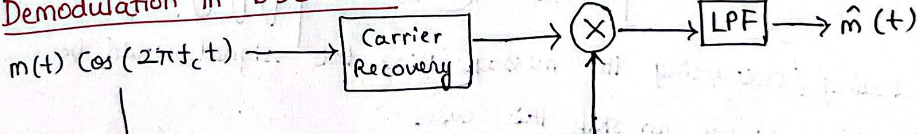
$$= \underbrace{\frac{1}{2} m(t)}_{\downarrow} + \underbrace{\frac{2}{\pi} m(t) \cos(2\pi f_c t)} - \underbrace{\frac{2m(t)}{3\pi} \cos(6\pi f_c t)} + \cdots$$

$$G(f) = \frac{1}{2} M(f) + \frac{1}{\pi} M(f \pm f_c) - \frac{1}{3\pi} M(f \pm 3f_c) \cdots$$

$\rightarrow$ We can pass this through BPF centered around $\pm f_c$ to get the modulated signal.

$\rightarrow$ 

Demodulation in DSB-SC



Carrier Recovery has higly complex circuit making Receiver costly.

Recovery circuit recover the exact carrier frequency used in modulation.

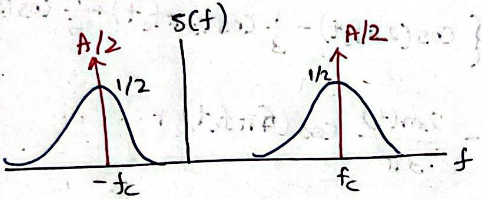
So, we use AM-Modulation ....

Amplitude Modulation DSB – FC (Full Carrier)

As we need carrier signal for demodulation, we send carrier signal along with modulated signal also.

$$s(t) = \overbrace{A \cos(2\pi f_c t)}^{\text{carrier}} + \overbrace{m(t) \cos(2\pi f_c t)}^{\text{DSB}} = [A + m(t)] \cos(2\pi f_c t)$$

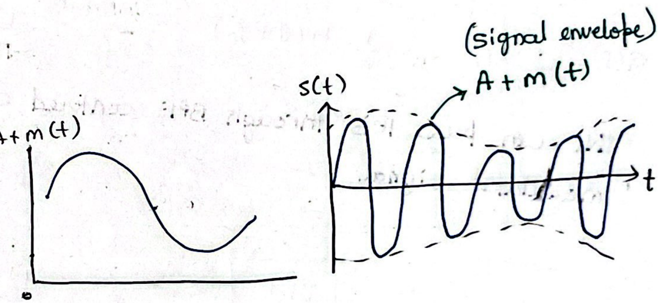
$$S(f) = \frac{A}{2}\left[\delta(f+f_c) + \delta(f-f_c)\right] + \frac{1}{2}\left[M(f+f_c) + M(f-f_c)\right]$$



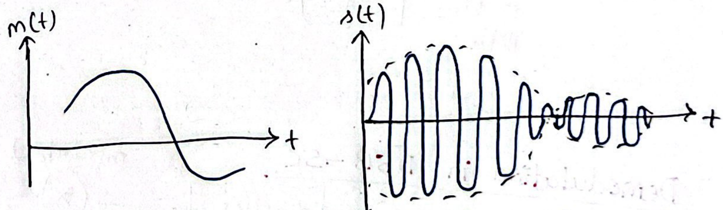
$$P_s = \frac{P_m}{2} + \underbrace{\frac{A^2}{2}}_{\text{Additional Power}}$$

CASE 1: $A + m(t) \geq 0$

We can signal through Envelope Detector to get back the message signal.



CASE 2: $A + m(t) < 0$



Detecting / recovering the message using this signal would be difficulting. So, we can skip this case

$A \pm m_p > 0 \quad 1 \pm \mu > 0 \quad m_p$: Peak-to peak amplitude of modulated signal

$$|m(t)| \leq m_p$$

$$\mu = \frac{m_p}{A}$$

μ: Modulation Index : $0 < \mu < 1$

Tone Modulation

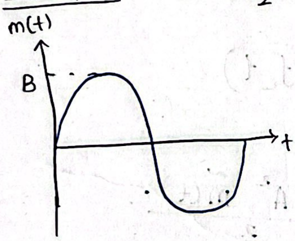
Tone Modulation.

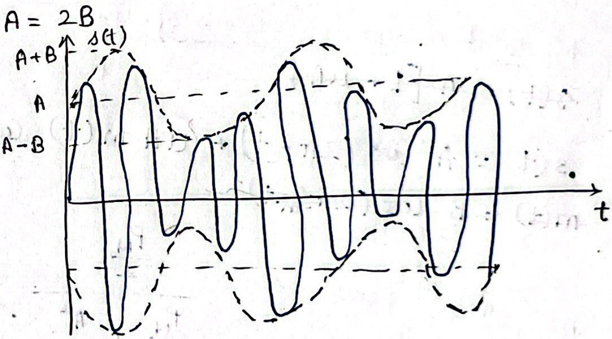
When the message signal consisting of only one frequency is used in modulation, it

$$m(t) = B \cos(2\pi f_m t)$$

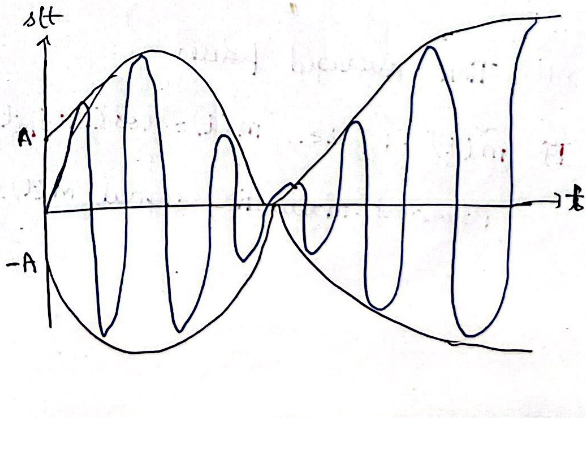
$$\mu = \frac{B}{A}$$

Example 1 $\quad \mu = \frac{1}{2} \quad$ i.e. $A = 2B$



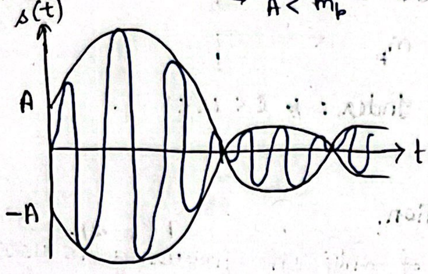


CASE2 $\quad \mu = 1 \quad A = B$



CASE3 $\mu > 1$ Over Modulation

$\rightarrow A < m_p$



If modulation index, $0 < \mu, 1$, then, we can easily recover the message signal using envelope detector.

What if $m(t) > 0$?

Then there is no need to add another DC component.

Modulated signal in terms of μ

$\cancel{s(t) = A[1 + \mu A]}$

$$s(t) = \underbrace{A \cos(2\pi f_c t)}_{\text{undesired Power}} + \overbrace{\mu A\, m(t) \cos(2\pi f_c t)}^{\text{Desired Power}}$$

$\cancel{m(t) = B \cos(2\pi f_m t)}$

$$\eta = \frac{\text{desired power}}{\text{total power}} = \frac{\frac{P_M}{2}}{\frac{P_M}{2} + \frac{A^2}{2}} = \frac{\frac{\mu^2 A^2 \tilde{m}(t)}{2}}{\frac{\mu^2 A^2 \tilde{m}(t)}{2} + \frac{A^2}{2}}$$

$\tilde{m}(t)$: Time - averaged power

$\cancel{\text{If } \tilde{m}(t) = 1, \text{ i.e., } m(t) = \cos(2\pi f_m t) \quad \eta = \frac{\mu^2}{\mu^2 + 1}}$

P_M : Power of modulating signal, $m(t)$.

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Eg. $m(t) = m_p \cos(2\pi f_m t) = \mu A \cos(2\pi f_m t) \rightarrow P_M = \frac{(\mu A)^2}{2}$

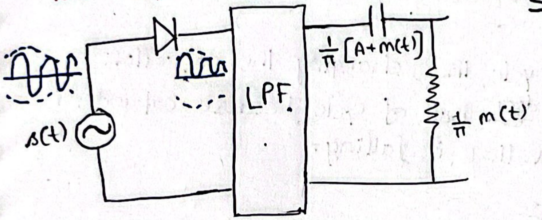
$$\boxed{\eta = \frac{\mu^2}{\mu^2+2}}$$

Max. value at $\mu = 1$: $\boxed{\eta = \frac{1}{3}}$ Max. efficiency for tone modulation

Reducing hardware complexity at cost of power efficiency.

Rectifier Demodulator : Rectifier + LPF

AM helps to get demodulated signal easily at the cost of Power-efficiency.



Switching at regular intervals.

$s(t)\, \underbrace{w(t)}_{\text{+ve cycle: 1} \quad \text{-ve cycle: 0}} = \text{AM} \times \text{Switch}$

The output of low pass filter have a DC component 'A' and an attenuation of $\frac{1}{\pi}$. So, to block DC part a suitable capacitance can be used. So, output becomes $\frac{1}{\pi}m(t)$.

$s(t)\, w(t)$

$$= \underbrace{[A + m(t)] \cos(2\pi f_c t)}\left[\frac{1}{2} + \frac{2}{\pi}\left[\underbrace{\cos(2\pi f_c t)} - \frac{1}{3}\cos(6\pi f_c t) + \frac{1}{5}\cos(10\pi f_c t) \cdots\right]\right]$$

$$= \underbrace{\frac{1}{\pi}[A + m(t)]}_{\text{shift by } \pm f_c} + \text{High } \cancel{\text{frequency}} \text{ term} \xrightarrow{\text{LPF}}$$

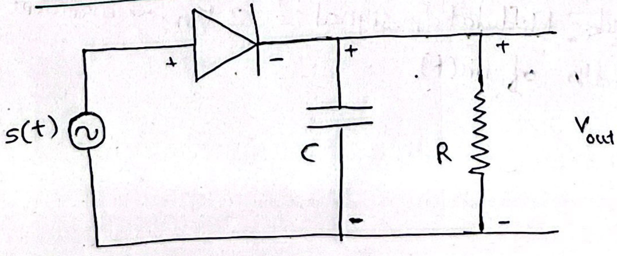
Cut off frequency of LPF should be freq of modulating signal, m(t).

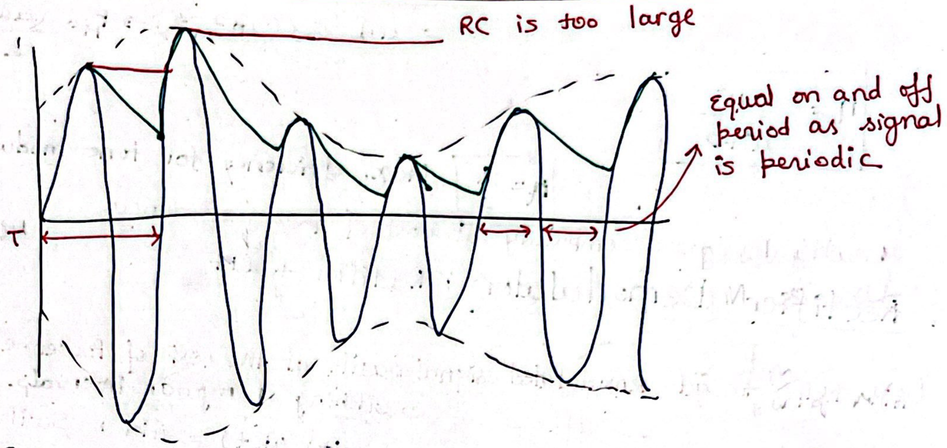
$$\frac{2}{\pi}[A + m(t)] \cos^2(2\pi f_c t) = \frac{2}{\pi}[A + m(t)]\left[\frac{1 + \cos(4\pi f_c t)}{2}\right]$$

$$= \frac{1}{\pi}[A + m(t)] + \text{Higher frequency terms}$$

Advantage: No synchronisation of carrier wave required

Envelope Demodulation





Working

· Diode ~~is~~ conducts in ~~+ve~~ incr half of cycle, thus, charging the capacitor.
Diode will be in reverse bias in ~~-ve~~ decr half of cycle, because capacitor is now charged and other node's voltage is falling.

Suitable values for R and C

→ We need to have large values for RC st. that there is a very large discharge time st. there is no drop in voltage across capacitor. This makes sure capacitor remains in reverse bias during decreasing half

→ Also, we cannot make it too large as then it would be ~~u~~able to detect the forthcoming waveform.
So, its discharging time should be less than one period of modulating signal, m(t).

→ Combining these two, we can write

$$\frac{1}{2\pi f_c} << RC << \frac{1}{2\pi f_m}$$

Bandwidth of Amplitude Modulated signal = $2B_m$ → Inefficient
where, B_m is bandwidth of m(t).

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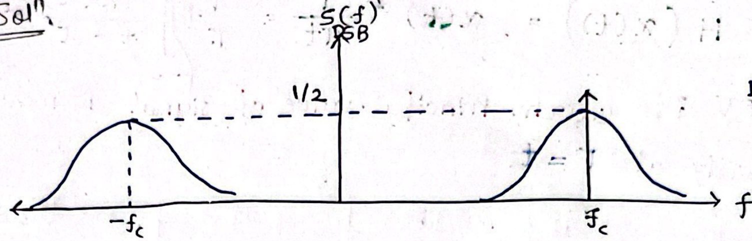
Single Side Band - Suppressed carrier (SSB - SC)

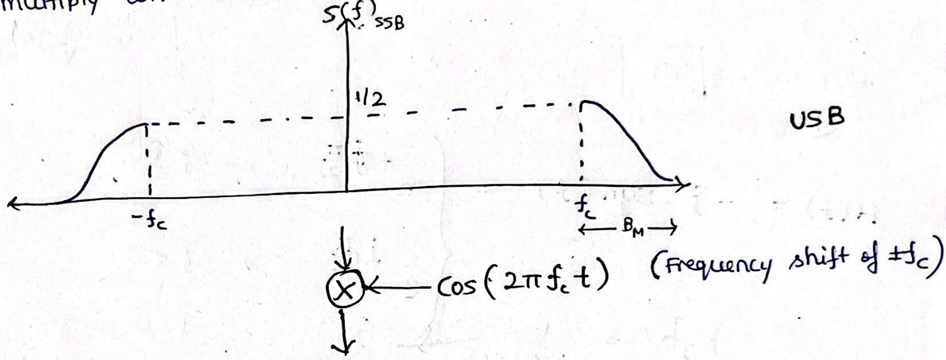
Q4 Soln.



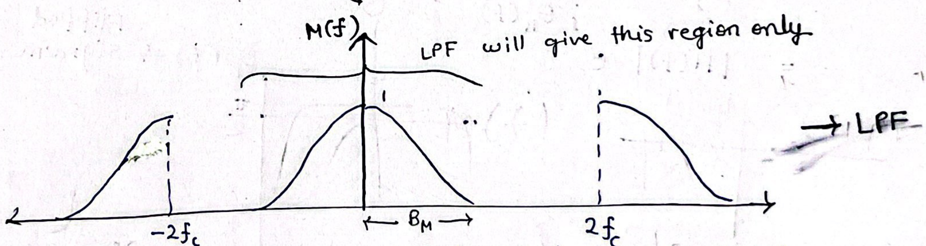
DSB-SC/AM

$B = 2B_M$

If somehow we can get the USB (Upper side Band), then, we can multiply with carrier wave to get back original signal.



USB



We can pass this through a LPF to get back the signal.

$B_S = 2B_M$ $\qquad P_S = \frac{P_M}{4}$

Hilbert Transform

$$x_h(t) = H(x(t)) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

where PV is Cauchy Principal value of signal to remove singularity at $\tau = t$.

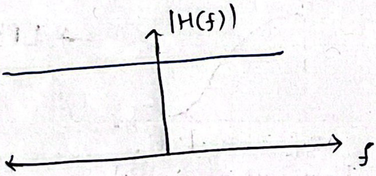
$$\frac{1}{\pi} PV \left[\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \right] = \lim_{\epsilon \to 0} \int_{-\infty}^{t-\epsilon} \ldots d\tau + \int_{t+\epsilon}^{\infty} \ldots d\tau.$$

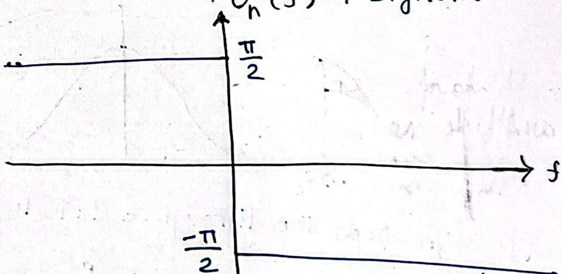
$$x(t) \longrightarrow \underset{H(f)}{\boxed{HT}} \longrightarrow x_h(t)$$

$$H(f) = -j \, \text{sgn}(f) = \begin{cases} e^{-j\frac{\pi}{2}} &, f > 0 \\ e^{j\frac{\pi}{2}} &, f < 0 \\ 0 &, f = 0 \end{cases}$$

$$= |H(f)| \, e^{j\theta_h(f)}$$

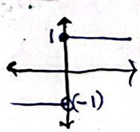
$\theta_h(f) \rightarrow$ Flipped signum



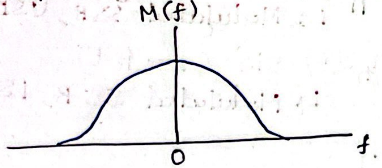


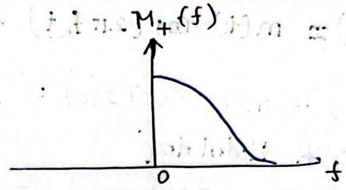
So, H(f) provides only phase shift, and no change in magnitude.
Also, there is no phase change for DC values.

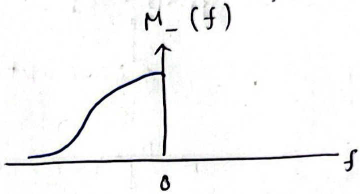
$$X_h(f) = X(f) \, FT\left(\frac{1}{\pi t}\right) = -j \, X(f) \underbrace{\text{sgn}(f)}$$



Represent a lobe







$$j X_h(f) = X(f)\, sgn(f)$$

$$M_+(f) = M(f)\, u(f)$$

$$= M(f)\ \frac{1}{2}\left[1 + sgn(f)\right]$$

$$= \frac{1}{2}\left[M(f) + j M_h(f)\right]$$

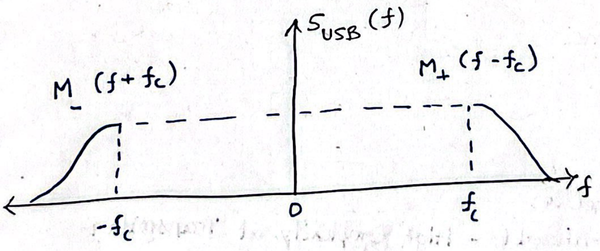
$$\frac{1}{2} j\, M_h(f) = \frac{1}{2} j\,(-j\, sgn(f))$$
$$= \frac{1}{2}\, sgn(f)$$

$$M_-(f) = M(f)\, u(-f)$$

$$= M(f)\ \frac{1}{2}\left[1 - sgn(f)\right]$$

$$= \frac{1}{2}\left[M(f) - j M_h(f)\right]$$

$$-\frac{1}{2} j\, M_h(f) = \frac{-1}{2} j\,(-j\, sgn(f))$$
$$= \frac{-1}{2}\, sgn(f)$$



$\therefore$ Sideband are suppressed in Hilbert transform.

SSB with USB

$$S_{USB}(f) = M_+(f - f_c) + M_-(f + f_c)$$

$$= \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right] - \frac{1}{2j}\left[M_h(f - f_c) - M_h(f + f_c)\right]$$

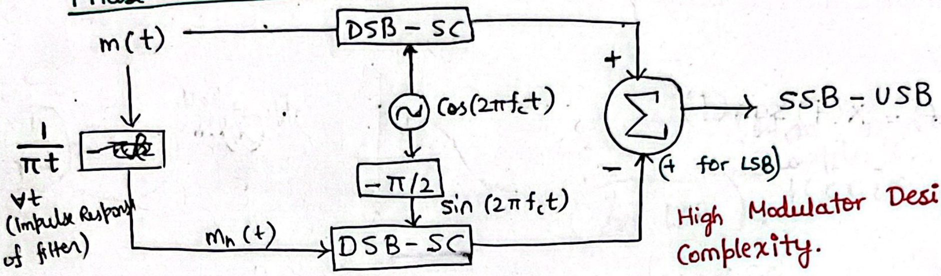
$$S_{USB}(t) = m(t)\cos(2\pi f_c t) - m_h(t)\sin(2\pi f_c t)$$

↳ Modulated SSB, USB

$$S_{LSB}(t) = m(t)\cos(2\pi f_c t) + m_h(t)\sin(2\pi f_c t)$$

↳ Modulated SSB, LSB

<u>Phase Shift Modulator</u>

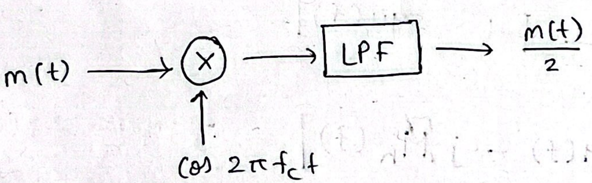


High Modulator Design Complexity.

Demodulation

$$s(t) * \cos(2\pi f_c t)$$

$$= \frac{1}{2}m(t) + \cancel{\frac{1}{2}\cos(4\pi f_c t) - \frac{m_h(t)}{2}\sin(4\pi f_c t)} \rightarrow \text{LPF}$$



<u>Advantage</u>

→ In SSB, Bandwidth is same

<u>Disadvantage</u>

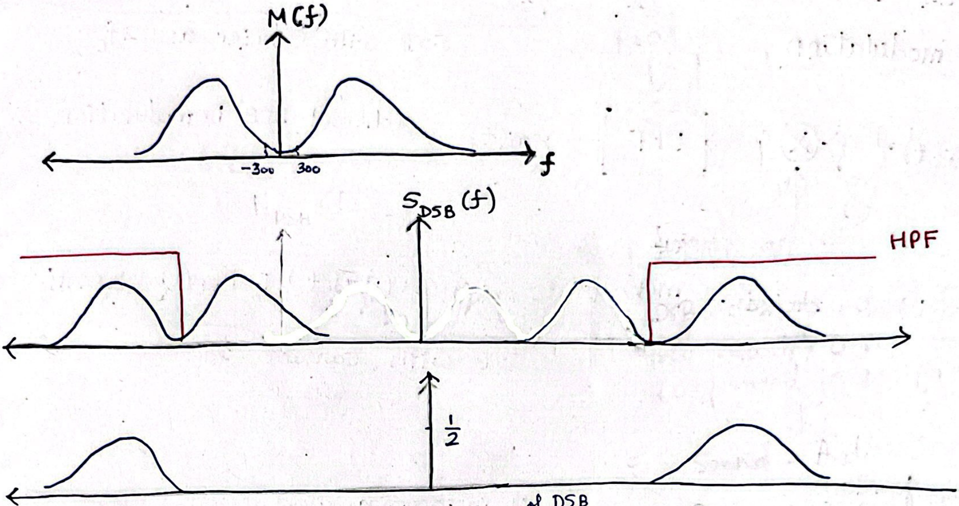
→ Too much complexity at receiver

→ Hibert Transform is not realisable - High Complexity at Transmitter

→ Coherent Demodulation as we need to have same carrier frequency

Hilbert Transform is not causal ∵ $-\infty < t < \infty$, and hence not realisable.

Frequency Selective Modulator



→ We use an HPF to allow only USB of DSB to pass through. Even though the filter is real, it gives good results if lower frequency components aren't significant

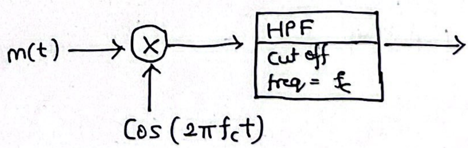
→ This method is particularly good for signals whose DSB is close to 0 at origin i.e. Frequency components nearby 0 has to have smaller components

Eg. Audio Signal : Sound have freq. from 300 Hz − 34 kHz.

$B_S = B_M$ Simplicity ↑ Efficient ?

$P_S = \dfrac{P_M}{4}$

Block Diagram



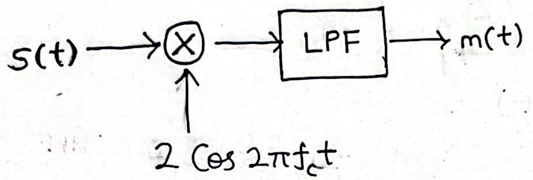
(By LPF) ≈ 0

$$s(t) = m(t)\cos(2\pi f_c t) \mp m_h(t)\sin(2\pi f_c t)$$

$$s(t)\cos(2\pi f_c t) = \frac{m(t)}{2} + \underbrace{\cancel{\frac{1}{2}m(t)\cos(4\pi f_c t) \mp \frac{m_h(t)}{2}\sin(4\pi f_c t)}}_{\text{SSB with Carrier as } 2f_c}$$

Demodulation

Coherent DSB Demodulation



$$s(t)\cos 2\pi f_c t = \frac{m(t)}{2} + \underbrace{\frac{1}{2}m(t)\cos(4\pi f_c t) \mp \frac{m_H(t)}{2}\sin(4\pi f_c t)}_{\text{SSB with carrier } 2f_c}$$

Disadvantage

→ Doppler's Effect   Frequency shift in incoming signal

$$f_D = \mp \frac{v\lambda_c}{c}\cos\theta$$

$$f_R = f_c \pm f_D$$

λ_c : Carrier Wavelength

f_c : Carrier Frequency f_R : Received Frequency

f_D : Doppler's Frequency change

θ : Angle of Arrival

SSB with Carrier

$$s(t) = (A + m(t)) \cos 2\pi f_c t + m_h(t) \sin(2\pi f_c t)$$

Can we detect this using Envelope Detector?
Yes, if we can write this in terms of $m(t)$.

$$s(t) = \left[[A + m(t)]^2 + [m_h(t)]^2 \right]^{1/2} \cos(2\pi f_c t + \theta)$$

$a \cos x + b \sin x = R \cos(x - \alpha)$, $R = \sqrt{a^2 + b^2}$

$$= A \left[1 + \underbrace{\frac{m^2(t)}{A^2}}_{} + \frac{2m(t)}{A} + \underbrace{\frac{m_h^2(t)}{A^2}}_{} \right]^{1/2} \cos(2\pi f_c t + \theta)$$

Square of fractions are ignored assuming $\boxed{A >> m_p}$

$$\approx A \left[1 + \frac{2m(t)}{A} \right]^{1/2} \cos(2\pi f_c t + \theta)$$

Now we have Amplitude (Envelope) in term of $m(t)$

$$\approx [A + m(t)] \cos(2\pi f_c t + \theta) \quad \text{(First Order Expansion)}$$

where, m_p : Peak-to-Peak amplitude of modulating signal.
$A >> m_{H_p}$ also, as H.T. doesn't change amplitude.

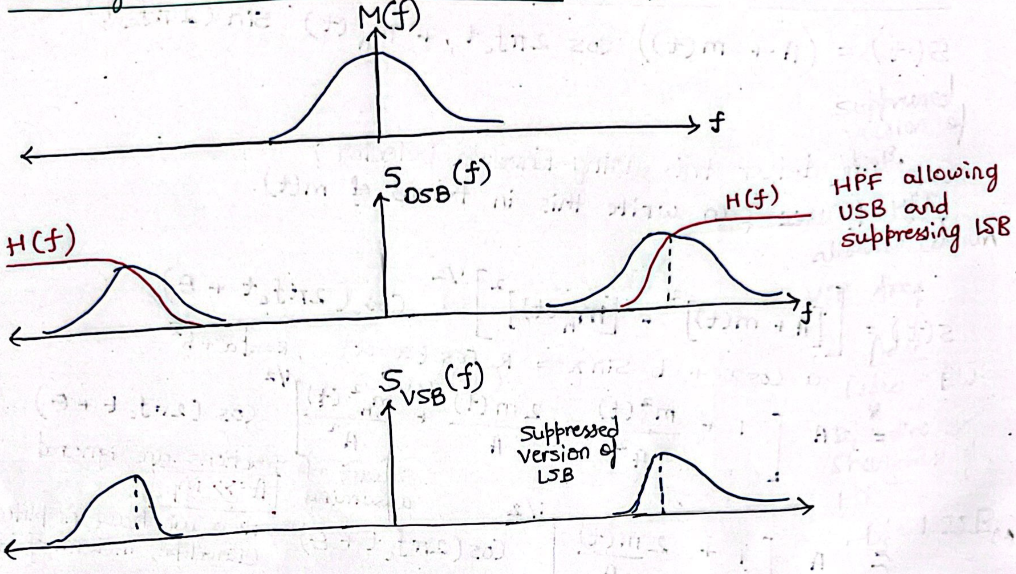
As $A + m(t) > 0$ always as $A >> m_p$, we can use envelope detector for demodulation

Disadvantage Highly power inefficient as $A >> m_p$.

DSB → Simple Structure, High B.W. = $2B_M$

SSB → High Complexity, Low B.W. = B_M

Vestigial Sideband Modulation (VSB)



Advantage

→ Reduce Bandwidth: Depending on sharpness of HPF.

$B_S = 1.25\, B_M$

Block Diagram

$m(t) \longrightarrow \boxed{DSB} \longrightarrow \boxed{HPF, H(f)} \longrightarrow S_{VSB}(f)$

$$S_{VSB}(f) = \frac{1}{2}\left[M(f+f_c) + M(f-f_c)\right] H(f)$$

Coherent Demodulation

Shift by $\pm f_c$.

$S_{VSB}(t) \longrightarrow \otimes \xrightarrow{X(t)} G(t) \longrightarrow m(t)$

$\uparrow \cos(2\pi f_c t)$

Increase in Bandwidth

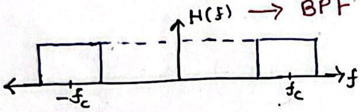
Decrease in Modulation, Demodulation structure complexity

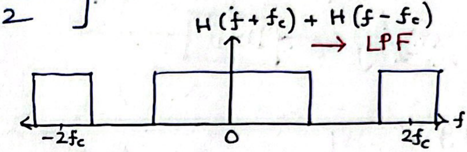
$$X(f) = \frac{1}{2}\left[S_{VSB}(f+f_c) + S_{VSB}(f-f_c) \right]$$

(Shift of S_{VSB} by $\pm f_c$)

$$= \frac{1}{4}\left[\{M(f) + M(f-2f_c)\} H(f-f_c) + \{M(f) + M(f+2f_c)\} H(f+f_c) \right]$$

$$X(f)G(f) = \frac{M(f)}{2}\left[\frac{H(f+f_c)}{2} + \frac{H(f-f_c)}{2} \right] G(f)$$





$$G(f) = \frac{1}{H(f+f_c) + H(f-f_c)}$$

$\forall\ |f| < f_m$

Quadrature Amplitude Modulation

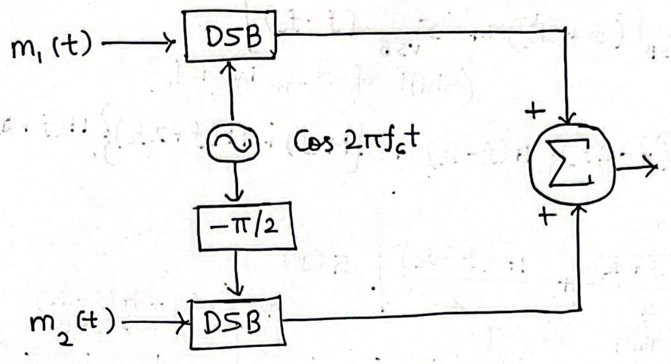
Here, we try to transmit 2 message signal over 2 bandwidth. This would help us save Bandwidth. $\because$ Bandwidth required per signal would be B_M.

But, we can do this only if we can retrieve the signal back.

$$S_{SSB}(t) = m(t)\cos 2\pi f_c t + m_h(t)\sin(2\pi f_c t)$$

$$S_{DSB}(t) = m(t)\cos(2\pi f_c t)$$

$$\underbrace{S_{QAM}(f)}_{2B_M} = \underbrace{\underbrace{m_1(t)\cos(2\pi f_c t)}_{\text{In Phase Component}}}_{2B_M} + \underbrace{\overbrace{m_2(t)\sin(2\pi f_c t)}^{\text{Quadrature Phase Component.}}}_{2B_M}$$



Demodulation

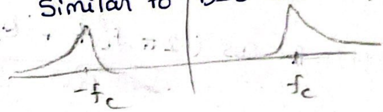
$$S_{QAM}(t) \times \sin(2\pi f_c t) = \frac{m_1(t)}{2} \sin(4\pi f_c t)$$

X cos

Sin

LPF

Demod

	P_S	B_S	Complexity.
DSB - SC	$\frac{P_M}{2}$	$2B_M$	Carrier Recovery required
AM/ DSB - FC (Full Carrier)	$\frac{A^2}{2}+\frac{P_M}{2}$ $(A+m_p>0)$	$\approx 2B_M$	Reduced complexity via use of envelope detector.
SSB	$\frac{P_M}{4}$	B_M	Modulation requires HT which increases complexity.
VSB (Vegistial Side Band)	$\frac{P_M}{2}>P_S>\frac{P_M}{4}$	$2B_M>B_S>B_M$	Similar to DSB-SC 

PLL : Phase Lock Loop Recovers carrier generating FM signal using VCO.

VCO : Voltage Controller Oscillator Voltage change wrt input voltage.

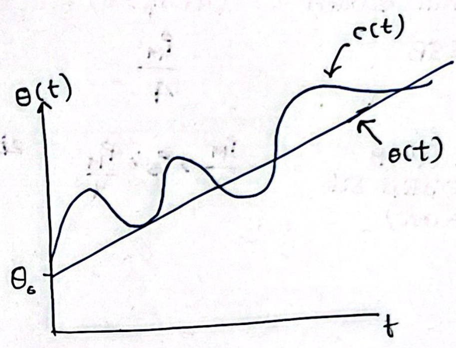
Frequency Modulation

Changing instantaneous frequency of carrier linearly with modulating signal, m(t).

$$\boxed{f_i(t) = f_c + \frac{K_f}{2\pi} m(t)} \quad \text{— (5)}$$

$-m_p < m(t) < m_p$

$$= f_c \pm \frac{K_f}{2\pi} m_p = f_c'$$



$$c(t) = A \cos(\underbrace{2\pi f_c' t + \theta_0}_{\theta(t)})$$

$$c(t) = A \cos(\theta(t))$$

where, $\theta(t) = 2\pi f_c' t + \theta_0$

$$\boxed{f_i(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt}} \qquad \boxed{\theta(t) = 2\pi \int_{-\infty}^{t} f_i(t)\, dt}$$

Int. Freq is a derivative of Int. angle ALWAYS.

FM ⇒ int. frequency ∝ m(t) Modulating signal

Phase Modulation Changing inst. phase of carrier linearly with ~~time~~ m(t).

$$\boxed{\phi_i(t) = \phi_0 + K_p m(t)} \quad \text{— (3)}$$

$$\theta(t) = 2\pi f_c t + \cancel{\theta_0}^{0} + K_p m(t).$$

$$\boxed{S_{PM}(t) = A \cos(2\pi f_c t + K_p m(t))} \quad \text{PM - SM — (1)}$$

$$f_i(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt} = \boxed{f_c + \frac{K_p}{2\pi} \dot{m}(t) = f_i(t)} \quad \text{— (2)}$$

PM ⇒ int. freq ∝ $\dot{m}(t)$ Derivative of modulating signal.

Frequency Modulation (FM)

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$$

$$\theta(t) = 2\pi \int_{-\infty}^{t} f_i(t)\,dt = 2\pi f_c t + K_f \int_{-\infty}^{t} m(\tau)\,d\tau$$

$$\boxed{S_{FM}(t) = A \cos\left(2\pi f_c t + K_f \int_{-\infty}^{t} m(\tau)\,d\tau\right)} \quad — (4)$$

$$\boxed{\text{FM : Int } \underset{(\text{Phase})}{\overset{\text{Angle}}{\cancel{\text{Freq}}}} \propto \int m(\tau)\,d\tau} \quad — (5)$$

Phase and Frequency modulation are equivalent and interchangeable

$$m(t) \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{\dot{m}(t)} \boxed{FM} \longrightarrow S_{PM}(t)$$

In Eqn (1) Replacing $m(t) \to \int m(\tau)d\tau$ will give Eq 3 of PM.

~~$S_{FM}(t) = A \cos(2\pi f_c t + K_f \int_{-\infty}^{t} m'(\tau)\,d\tau$~~

A signal that is a FM wave wrt $m(t)$ is also PM wrt $\int m(\tau)d\tau$

$$m(t) \longrightarrow \boxed{\int(\,).dt} \xrightarrow{\int_{-\infty}^{t} m(\tau)d\tau} \boxed{PM} \longrightarrow S_{FM}(t)$$

In Eqn (4), Replacing $m(\tau) \Leftrightarrow \dot{m}(t)$

A signal that is a PM wave wrt. $m(t)$ is also FM wrt $\dot{m}(t)$

$$\text{FM: } \theta(t) \propto \int_{-\infty}^{t} m(\tau)\,d\tau = u(t) \circledast m(t)$$

Transfer Function of LTI Systems

$$\text{PM: } \theta(t) \propto m(t) = \delta(t) \circledast m(t)$$

$m(t) \longrightarrow \boxed{h(t)} \longrightarrow \theta(t) = m(t) \circledast h(t)$

FM: $K_f \, u(t)$

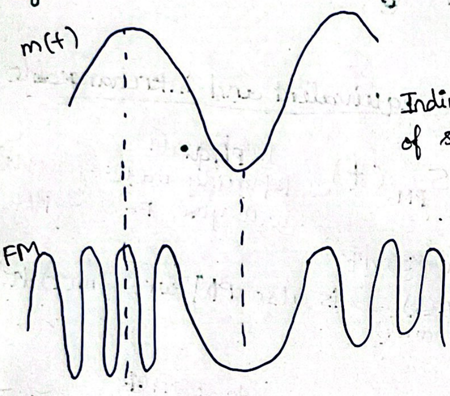
PM: $K_p \, \delta(t)$

* We cannot ~~differentiate~~ distinguish between FM and PM, generally
* But sometimes we can

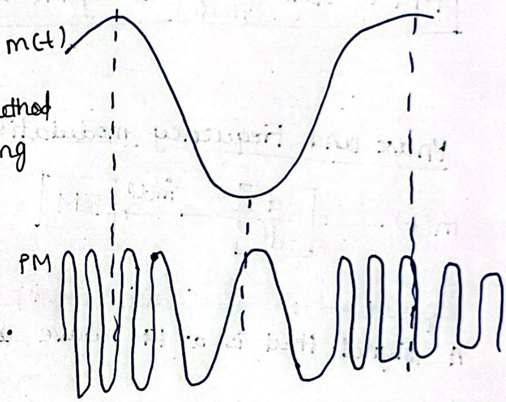
FM: $f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$

PM: $f_i(t) = f_c + \frac{K_p}{2\pi} \dot{m}(t)$

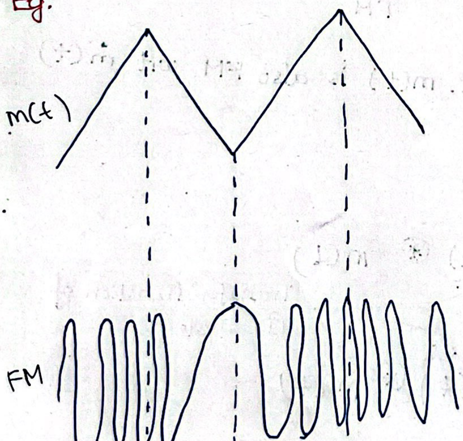
* Eg.

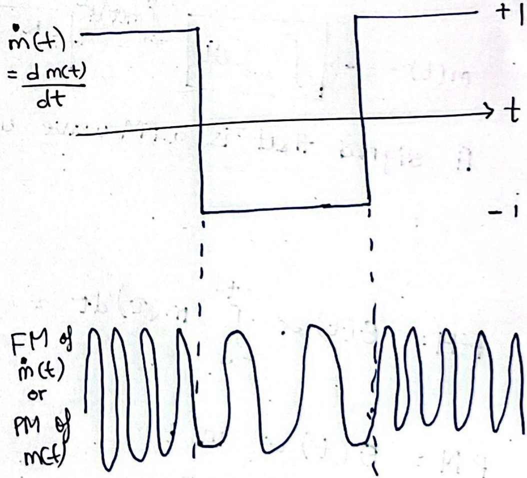


Indirect Method of sketching



* Eg.





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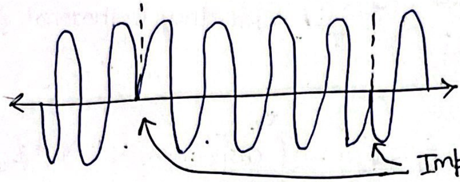
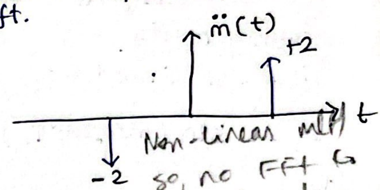
PM of $\dot{m}(t)$ At transitions we can observe impulses and hence can observe 180° phase shift.





Non-Linear m(t) so, no FFT to determine freq components to find BW.

* <u>How?</u>

FM / PM : $\cancel{P_S = \frac{P_M}{2}}$?

$$\boxed{P_{FM} = P_{PM} = \frac{A^2}{2}}$$

$$S_{FM}(t) = A \cos(2\pi f_c t) + K_f \int_{-\infty}^{t} m(\tau)\, d\tau$$

In angle modulated waves, instantaneous frequency and phase change with time but amplitude A remains constant.

* $S_{PM}(t) = A \cos(2\pi f_c t + K_p m(t))$

$$= \begin{cases} A \cos(2\pi f_c t + K_p), & \text{when } \dot{m}(t) = 1 \\ A \cos(2\pi f_c t - K_p), & \text{when } \dot{m}(t) = (-1) \end{cases}$$

Direct method of sketching Used when there are discontinuities ~~of~~ or impulses.

Tutorial 22 Feb. 2023

Orthogonality

$s_1 = \sin(n\omega_0 t + \phi_1)$ $\quad n \neq m \;,\; \omega_0 = \frac{2\pi}{T}$ $\quad$ Are they Orthogonal ?

$s_2 = \sin(m\omega_0 t + \phi_2)$

$$\int_{t_1}^{t_2} s_1(t)\, s_2(t)\, dt = \frac{1}{2} \int_{t_1}^{t_2} \cos\left((n-m)\omega_0 t + \phi_1 - \phi_2\right) - \int_{t_1}^{t_2} \cos\left[\left((n+m)\,\omega_0 t + \phi_1 + \phi_2\right]\right.$$

$t_1 = 0, \quad t_2 = T, \quad \omega_0 = \frac{2\pi}{T}$

$$\int_{t_1}^{t_2} s_1(t)\, s_2(t)\, dt = \frac{1}{2}\left[\frac{\sin[(n-m)\,\omega_0 t + \phi_1 - \phi_2]\Big|_0^T}{(n-m)\,\omega_0} - \frac{\sin[(n+m)\omega_0 t + \phi_1 + \phi_2]\Big|_0^T}{(n+m)\,\omega_0}\right]$$

$= 0$ $\quad$ $t \to 0$ to T. $\quad \omega_0 t \to 0$ to 2π $\quad$ as $\int_0^{2\pi} \sin(at+b) = 0$

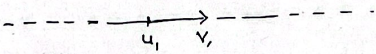
Gram Schmidt's Orthogonalisation

$V = \{v_1, v_2, \ldots\ldots, v_n\} \xrightarrow{GSO} \{u_1, u_2, \ldots, u_n\}$

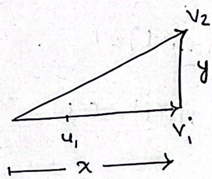
Linearly Independent Basis $\quad$ Orthogonal Basis.

$u_1 = \frac{\vec{v_1}}{\|v_1\|}$

1-D



2-D



$v_2 = c u_1 + y$

$y = v_2 - x$

$= v_2 -$ proj. of v_2 on v_1

$= (v_2 \cdot v_1) \cdot v_1$

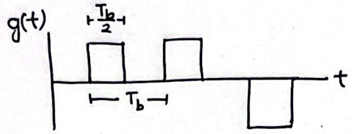
$u_2 = \frac{y}{\|y\|}$

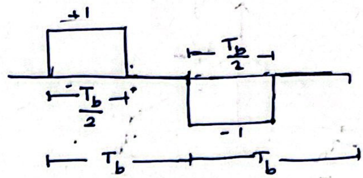
3-D

$x_3 - x$

Pulse Train, $g(t)$. Bit Time $= \frac{T_b}{2}$, Bit changes every T_b sec.

$$\left.\begin{array}{l} 1 \rightarrow +1 \\ 0 \rightarrow -1 \end{array}\right\}$$ These two symbols are equally likely and occur randomly. Find PSD





Sol^n.

$$R_g(\tau) \xleftrightarrow{FT} PSD$$

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int g(t)\, g(t-\tau)\, dt$$ Auto-correlation for random signal

$T = N\, T_b$ where, N is number of pulses.

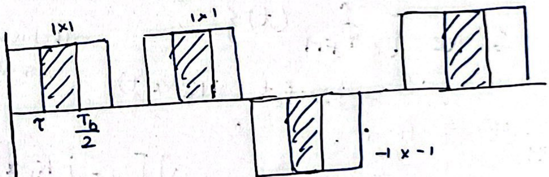
$T \to \infty$ as $N \to \infty$

$$R_g(\tau) = \lim_{N\to\infty} \frac{1}{N T_b} \int g(t)\, g(t-\tau)\, dt$$

CASE 1 $\tau < \frac{T_b}{2}$

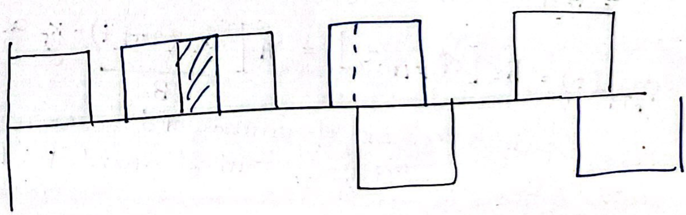
CASE 2 $\tau > \frac{T_b}{2}$

CASE 1



Same symbols are overlapping.

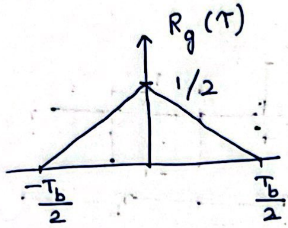
CASE 2



Next Symbol Overlap.

Hence, sum would average out to 0.

$$\frac{1}{T_b}\left(\frac{T_b}{2}-\tau\right) = \frac{1}{2} - \frac{\tau}{T_b} = \underbrace{\frac{1}{2}\left(1-\frac{2\tau}{T_b}\right)}_{\text{Triangular Pulse}}$$



Bandwidth of angle modulated signals

Angle modulated waves are non-linear* of modulating signal. Hence, we cannot use FFT to determine frequency components to find bandwidth.

$a(t)$ is an integral of $m(t)$. If $M(f)$ is band limited to B, $A(f)$ is also band limited to B. $\because$ Integration is a linear operator. $m(t) \to \boxed{\int} \to a(t)$

$$\boxed{A(f) = \frac{1}{2j\pi f} M(f)}$$

spectrum of $a^2(t)$ is $A(f) \circledast A(f)$. and is Band-Limited to $2B$.

Similarly, spectrum of $a^n(t)$ is Band Limited to nB.

$$\phi_{FM}(t) = A \cos\left(2\pi f_c t + K_f\, a(t)\right), \quad \text{where } a(t) = \int_{-\infty}^{t} m(\tau)\,d\tau$$

$$= Re\left\{ \hat{\phi}_{FM}(t) \right\}$$ (Representing in terms of complex exponentials)

where, $\hat{\phi}_{FM}(t) = A\, e^{j2\pi f_c t}\, e^{jK_f a(t)}$

$$= A\left[\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right]\left[1 + jK_f a(t) - \frac{(K_f a(t))^2}{2!} - j\frac{(K(f) a(t))^3}{3!} + \dots\right]$$

$$\phi_{FM}(t) = Re\left[\phi_{FM}(t)\right] = A\left[\underbrace{\cos(2\pi f_c t)}_{B} - \underbrace{K_f\, a(t)\sin(2\pi f_c t)}_{2B} \underbrace{- \frac{(K_f a(t))^2 \cos(2\pi f_c t)}{2!} + \dots}_{4B}\right]$$

Hence, spectrum consists of unmodulated carrier spectra plus spectra of $[a(t)]^i$ centered around f_c that are DSB with $a(t), a^2(t), \dots, a^n(t), \dots$ as modulating signal.

K_f, m_p (p-p amplitude) decides what frequency components would be present in FM signal.

Although BW of FM wave is theoretically infinite, for practical signals with bounded $|a(t)|$, $|K_f a(t)|$ will remain finite. Because,

$$\frac{[K_f \, a(t)]^n}{n!} \simeq 0 \quad \text{for large } n.$$

It has a bandwidth of $2nB$.

Hence, most of modulate signal power resides in a finite BW.

CASE1: $|K_f m_p| << 1$

$$\phi_{FM}(t) \simeq A \cos(2\pi f_c t) - A \underbrace{K_f}_{K_p \text{ for PM}} a(t) \sin(2\pi f_c t)$$ DSB-SC

$B_{FM} = 2B_M$ Narrow Band FM Modulation (NBFM)

$B_{PM} = 2B_M$ Narrow Band PM Modulation (NBPM)

As the two terms in FM are out of phase, they cancel each other to give constant amplitude.

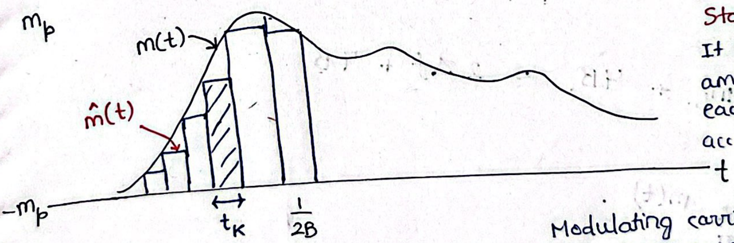
$$\phi_{PM}(t) \approx A \cos(2\pi f_c t) - A K_p m(t) \sin(2\pi f_c t)$$

CASE2: $|K_f m_p| >> 1$ Wide Band Frequency Modulation (WBFM)

Approximating signal $m(t)$ with staircase signal $\hat{m}(t)$

Staircase Approximation

It has pulses of constant amplitude, with width of each pulse as $\frac{1}{2B}$ i.e. acc. to Nyquist interval for sampling.



Modulating carrier by approximated staircase signal as modulating signal.

$$f_i(t_K) = f_c + \frac{K_f}{2\pi} \hat{m}(t_K)$$

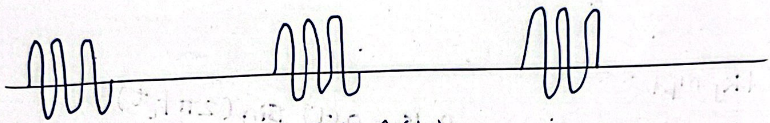
The FM Spectrum of $\hat{m}(t)$ consists of sum of Fourier Transforms of all the ~~rectangular pulse~~, sinusoidal pulses corresponding to all the cells.

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & , \text{ for } |t| < \tau \\ 0 & ; \text{ otherwise} \end{cases} \longleftrightarrow \tau \operatorname{sinc}\left(\frac{f\tau}{2}\right)$$

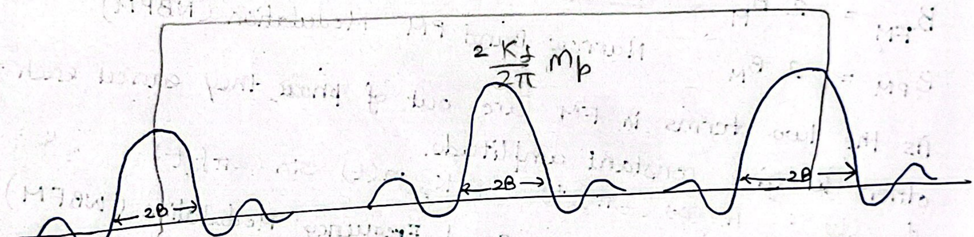
$$\Pi(t\, 2B) \cos\left(2\pi f_c t + K_f \hat{m}(t)\right) \longleftrightarrow \frac{1}{2} \operatorname{sinc}\left(\frac{2\pi f \pm \left[2\pi f_c + K_f \hat{m}(t_k)\right]}{4B}\right)$$

$2\pi f_c + K_f \hat{m}(t_k) \rightarrow$ Center Frequency

$$\boxed{\tau = \frac{1}{2B}}$$



$f_c - \frac{K_f}{2\pi} m_p$ $\quad$ $f_c + \frac{K_f}{2\pi} \hat{m}(t_k)$ $\quad$ $f_c + \frac{K_f}{2\pi} m_p$



$f_c - \frac{K_f}{2\pi} m_p$ $\quad$ $f_c + \frac{K_f}{2\pi} \hat{m}(t_k)$ $\quad$ $f_c + \frac{K_f}{2\pi} m_p$

$$B_{\hat{\phi}} = \frac{2K_f}{2\pi} m_p + 4B = 2\Delta f + 4B$$

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$$

$$\Delta f = \frac{K_f}{2\pi} \frac{[m_p - (-m_p)]}{2} = \text{Frequency Deviation} = \frac{K_f m_p}{2\pi}$$

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As we did stair case approximation, we overestimated bandwith. $\because$ as rectangular pulse have more abrupt changes, it has more frequency components. Hence, overestimation.

$$B_\phi \in [\,2\Delta f \text{ to } 2\Delta f + 4B\,]$$

<u>CASE I</u>: $\Delta f \approx 0 \Rightarrow B_{\hat{\phi}} = 4B$

$K_f\, m_p << 1$ NBFM $\phi_{NBFM} = 2B$

$$\hat{B}_\phi \approx 2\Delta f + 2B$$
$$= 2\beta B + 2B = 2B(\beta + 1)$$

Modulation Index, $\boxed{\beta = \frac{\Delta f}{B}}$

Carson's Rule

Carson's Bandwidth

$2\Delta f$ to

$B = (2\Delta f + 2B)$ M

$= 2B(1+\beta)$

$\Delta f = \frac{K_f}{2\pi} m_p$

Bandwidth analysis for Tone modulation.

For tone modulation, we can still do frequency domain analysis, find spectrum and do bandwidth analysis.

$\rightarrow m(t) = \alpha \cos(2\pi f_m t)$

$\rightarrow a(t) = \alpha \int_{-\infty}^{t} m(\tau) d\tau = \frac{\alpha}{2\pi f_m} \sin(2\pi f_m t)$, $a(-\infty) = 0$

$\rightarrow \phi_{FM}(t) = A \cos\left(2\pi f_c t + \frac{\alpha k_f}{2\pi f_m} \sin(2\pi f_m t)\right)$

$\rightarrow \phi_{FM}(t) = Re\left\{ \underbrace{A e^{j\left(2\pi f_c t + \underbrace{\frac{\alpha k_f}{2\pi f_m}}_{\beta} \sin(2\pi f_m t)\right)}}_{\hat{\phi}_{FM}(t)} \right\}$

$\boxed{\frac{\Delta f}{f_m} = \frac{\alpha k_f}{2\pi f_m} = \beta = \text{Deviation}}$ For Tone Modulation

$\rightarrow \hat{\phi}_{FM}(t) = A e^{j2\pi f_c t} e^{j\beta \sin(2\pi f_m t)}$ (Periodic Complex signal)

$\rightarrow e^{j\beta(\sin(2\pi f_m t))} = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi n f_m t}$ (FS)

correlation of and conjg of

$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt$

$\rightarrow$ We can do substitution,

$2\pi f_m t = x \qquad dt = \frac{dx}{2\pi f_m} \qquad x = \pm 2\pi f_m \frac{T_0}{2} = \pm \pi$

$D_n = \frac{1}{2\pi f_m T_0} \int_{-\pi}^{\pi} e^{j[\{\beta \sin(x)\} - (nx)]} dx = J_n(\beta)$

n^{th} order Bessel function of first ~~order~~ kind.

$$\rightarrow e^{j\beta \sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j 2\pi n f_m t}$$

$\rightarrow k_f\, a(t)$

$\beta = k_f$

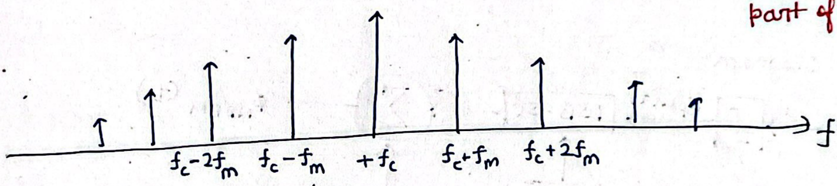
$$\rightarrow \hat{\phi}_{FM}(t) = A\, e^{j2\pi f_c t}\, e^{j\beta \sin(2\pi f_m t)}$$

$$\hat{\phi}_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{j(2\pi f_c t + 2n\pi f_m t)}$$

$$\rightarrow \phi_{FM}(t) = \sum_{n=-\infty}^{\infty} A\, J_n(\beta) \cos(2\pi f_c t + 2\pi n f_m t)$$ (Real part of $\hat{\phi}_{FM}(t)$)

→ It would be similar to show for any periodic signal, in general.

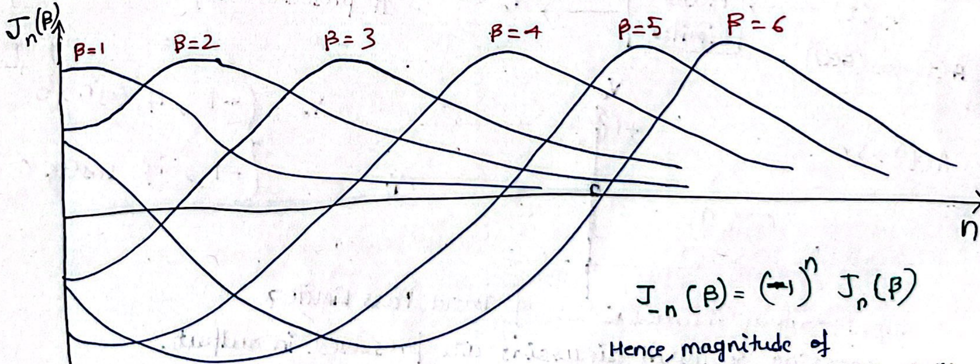
This show +ve part of spectrum



Tone Modulated Waveform

$\rightarrow n > \beta + 1$; $J_n(\beta) \approx 0$ $\quad \therefore$ No. of significant sideband impulses $= \beta + 1$. +ve freq components

Bandwidth : $(\beta+1) f_m + (\beta+1) f_m = 2 f_m (\beta+1)$.



Variation of $J_n(\beta)$ with β.

$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

Hence, magnitude of

LSB at $\omega = \omega_c - n\omega_n$ =

USB at $\omega = \omega_c + n\omega_n$

Indirect Method — NBFM Generation

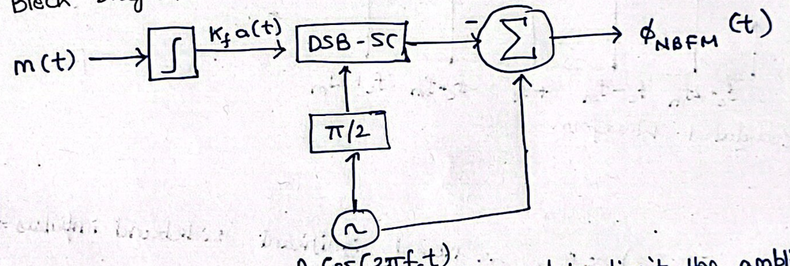
→ $\phi_{NBFM}(t) = \underbrace{A\cos(2\pi f_c t)}_{\text{Carrier}} - \underbrace{A K_f\, a(t) \sin(2\pi f_c t)}_{\text{DSB-SC}}$

→ It has Amplitude Variations Because we did first order approximation of Taylor's Series.

→ For $K_f m_p \ll 1$, NBFM ⇒ DSB-SC + Carrier + Amplitude Variation Removal

Freq deviation Conversion → FM signal for given Δf

→ Block Diagram



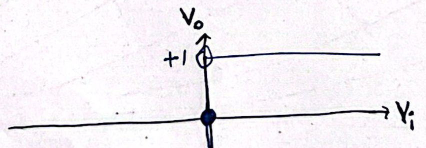
→ Hard Limiter: A non-linear device designed to limit the amplitude of a band pass signal.

$v_i(t)$ → Hard Limiter → $v_o(t)$ → Angle modulated square wave.

$A(t)\cos(\theta(t))$

It preserves angle modulation info.

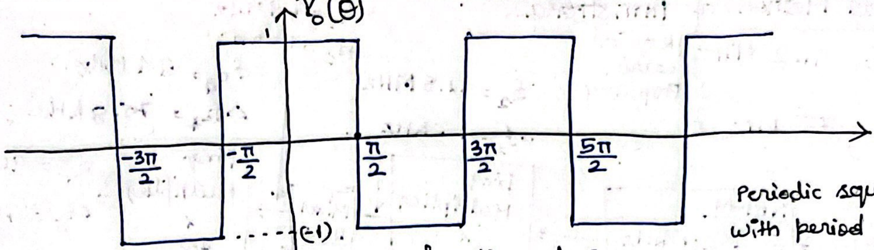
$A(t) \gg 0$



$$V_o = \begin{cases} +1, & \text{if } \cos\theta > 0 \\ -1, & \text{if } \cos\theta \leq 0 \end{cases}$$

Input-output characteristics of Band Pass Limiter

Zero crossing of input sinusoids are preserved in output.



Periodic square wave with period 2π.

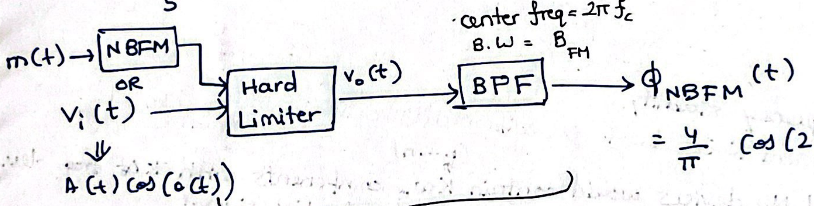
Hard limiter output as a function of θ.

$$\theta(t) = 2\pi f_c(t) + k_f \int_{-\infty}^{t} m(\tau)\, d\tau$$

$$V_o(\theta) = \frac{4}{\pi}\left[\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \dots\right]$$

$$V_o(\theta(t)) = \frac{4}{\pi}\Big[\overbrace{\cos(2\pi f_c t + k_f a(t))}^{\text{original FM Wave}} - \underbrace{\frac{1}{3}\cos(2\pi\, 3f_c t + 3k_f \Delta f)}_{\text{Frequency multiplied FM Wave}}$$

$$+ \frac{1}{5}\cos(2\pi\, 5 f_c t + 5k_f a(t))$$



Band Pass Limiter

Remove amplitude variation

For $K_f m_p \ll 1$

Indirect Method in FM wave Armstrong

$y(t) = a_2 x^2(t)$ Non-Linear device

$$= a_2 \cos^2(2\pi f_c t + k_f a(t))$$

$$= \frac{a_2}{2} + \frac{a_2}{2}\cos(2\pi\, 2f_c t + 2k_f a(t))$$

Passing output through BPF centered at $2f_c$ would recover FM with twice original instantaneous freq.

$2f_c,\ 2\Delta f$

NBFM generated above is converted to WBFM using frequency multipliers. They are realised by a non-linear device followed by a bandpass filter. ~~y(t)~~ It can increase both carrier frequency and frequency deviation by an integer 'K'.

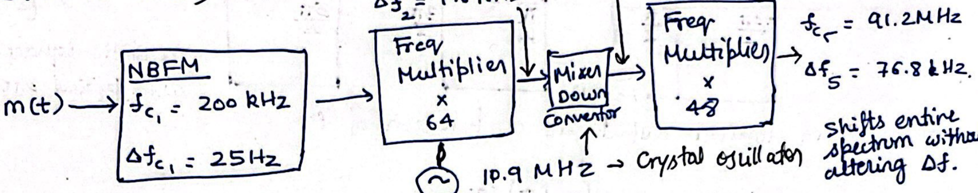
~~Indirect Method of Armstrong~~

$f_c = 91.2$ MHz
$\Delta f = 75$ kHz
} Required carrier Frequency

$f_{c_3} = 1.9$ MHz
$\Delta f_3 = 1.6$ kHz

$f_{c_2} = 12.8$ MHz
$\Delta f_2 = 1.6$ kHz.

$f_{c_4} = 91.4$ MHz
$\Delta f_{c_4} = 76.8$ kHz.



10.9 MHz → Crystal oscillator

Shifts entire spectrum without altering Δf.

$m(t) = 50$ Hz to 20 kHz

$$\frac{\Delta f}{\Delta f_{c_1}} = \frac{75 \text{ kHz}}{\Delta f_{c_1}} = 3000 \approx 3072$$

64 → Six Doubler
48 → Four Double + One Tribler

$$\beta = \frac{\Delta f_{c_1}}{f_{min}} = \frac{25}{50} = 0.5$$

Advantage: Frequency stability.

Disadvantage: Inherent Noise due to excessive multiplication and distortion at lower modulating freq.

In general, output of NL devices would contain $K\omega_c$ $(1,\dots,n)$ components with $K\Delta f$ freq dev.

So, BPF centered at $K\omega_c$ would recover that specific FM signal.

Generally, we require to increase Δf. Thus, carrier freq also incr. This may not be needed. Hence, frequency mixing is applied to shift down carrier frequency to desired value.

$\boxed{200 \text{ kHz}}$ → Easy to construct stable crystal oscillators. And also
→ Balanced Modulators

$\boxed{25 \text{ Hz}}$ → $\beta = \frac{\Delta f}{f_m} << 1$ → $\Delta f = 25$ Hz give $\beta = 0.5$ for worst possible case of $f_m = 50$ as Baseband spectrum required for high fidelity purposes ranges from 50 Hz to 15 kHz.

ASSIGNMENT - 1

Q1 Soln.

a). $\text{sinc}^3 t$

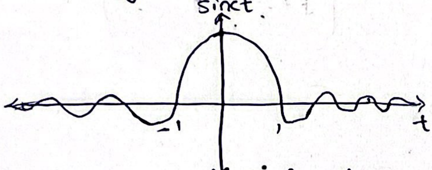
Soln.

$\rightarrow FT(\sin^3 t) = FT(\text{sinc}^2 T) * FT(\text{sinc}\, t)$

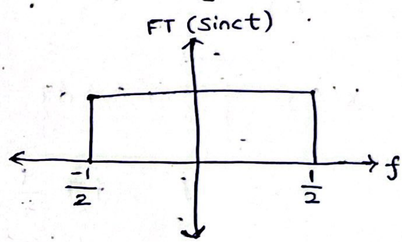
$= [FT(\text{sinc}\, t) * FT(\text{sinc}\, t)] * FT(\text{sinc}\, t)$

Using Convolution Property, $FT(x(t) \cdot y(t)) = FT(x(t)) * FT(y(t))$

$\rightarrow$ Finding FT (sinct t)

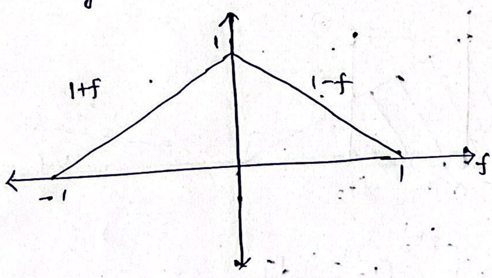


$\xrightarrow{FT}$

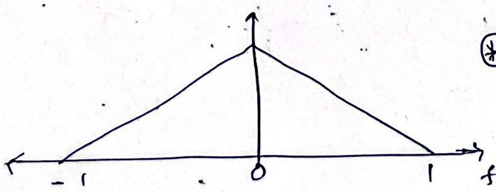


Duality Principle for sinc.

$\rightarrow$ Finding $FT(\text{sinc}\, t) * FT(\text{sinc}\, t)$



$\rightarrow FT(\text{sinc}^3 t)$



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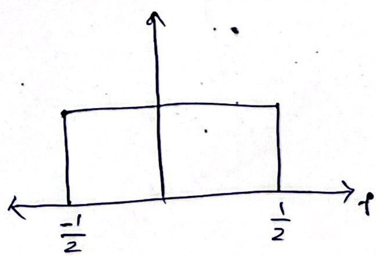
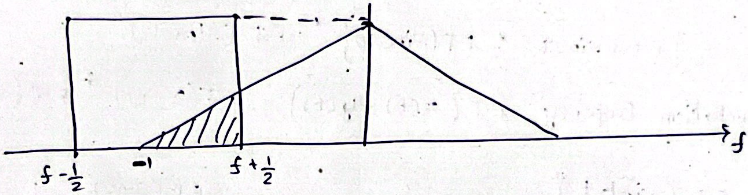


→ Let $X(F) = FT(\text{sinc}^3 t)$

→ <u>CASE 1</u> :

$f < \frac{-3}{2}$: No Overlap → $X(f) = 0$

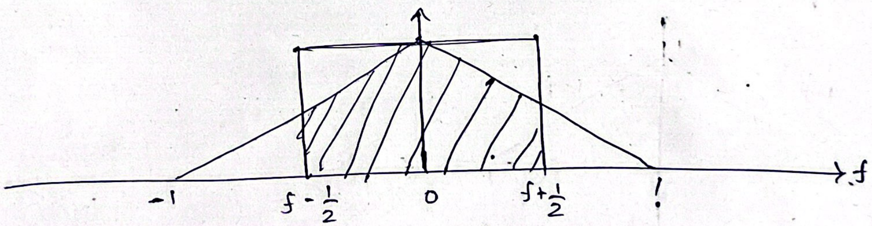
→ <u>CASE 2 :</u> $\frac{-3}{2} \leq f < \frac{-1}{2}$



$$X(f) = \int_{-1}^{f+\frac{1}{2}} (1+\tau)\, d\tau = f + \frac{3}{2} + \frac{1}{2}\left(f+\frac{3}{2}\right)\left(f-\frac{1}{2}\right)$$

$$= \frac{f^2}{2} + f + \frac{9}{8}$$

→ <u>CASE 3:</u> $\frac{-1}{2} \leq f \leq \frac{1}{2}$

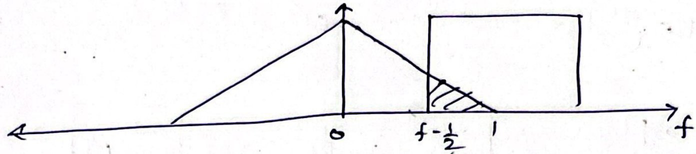


$$X(f) = \int_{f-\frac{1}{2}}^{0} (1+\tau)\, d\tau + \int_{0}^{f+\frac{1}{2}} (1-\tau)\, d\tau$$

$$= \frac{1}{2} - \cancel{f} - \frac{f^2}{2} - \frac{1}{8} + \cancel{\frac{f}{2}} + \frac{1}{2} + \cancel{f} - \frac{1}{8} - \cancel{\frac{f}{2}} - \frac{f^2}{2}$$

$$= \frac{3}{4} - f^2$$

CASE IV $\quad f - \frac{1}{2} < 1 \quad$ and $\quad f < \frac{3}{2}$



$$X(f) = \int_{f-\frac{1}{2}}^{1} (1-\tau)\, d\tau = \frac{3}{2} - f + \frac{1}{2}\left[f^2 - f - \frac{3}{4} \right]$$

$$= \frac{f^2}{2} - \frac{3f}{2} + \frac{9}{8}$$

CASE 5 $\quad f \geq \frac{3}{2} \quad X(f) = 0 \quad$ No overlap.

$$\rightarrow X(f) = FT(\sin^3 t) = \begin{cases} \frac{f^2}{2} + f + \frac{9}{8} &, -\frac{3}{2} \leq f < \frac{-1}{2} \\ \frac{3}{4} - f^2 &, \frac{-1}{2} \leq f < \frac{1}{2} \\ \frac{f^2}{2} - \frac{3f}{2} + \frac{9}{8} &, \frac{1}{2} \leq f \leq \frac{3}{2} \\ 0 &, \text{otherwise.} \end{cases}$$

b). $t * \text{sinc}t \quad$ Assuming it is convolution.

$\rightarrow$ FT $(t * \text{sinc}t)$ = FT (t) . FT $(\text{sinc}t)$

Acc. to convolution property: FT $(x(t) * y(t))$ = FT $(x(t))$. FT $(y(t))$:

$$\rightarrow FT(t) = \int t\, e^{-j\omega t}\, dt = \frac{-1}{\omega^2}$$

$$\rightarrow FT(\text{sinc}t) = \int \text{sinc}t\, e^{-j\omega t}\, dt = \text{rect}(\omega/2\pi)$$

$$\rightarrow FT(t * \text{sinc}t) = \frac{-j}{\omega^2} \times \text{rect}(\omega/2\pi)$$

$$= \frac{-[illegible]}{\omega^2}\, \text{rect}(\omega/2\pi)$$

c). $t e^{-\alpha t} \cos \beta t$

$\rightarrow$ ~~FT~~ $FT(e^{-\alpha t}) = \frac{1}{\alpha + j\omega} \quad (\alpha > 0)$

$\rightarrow FT(e^{-\alpha t} \cos \beta t) = \frac{1}{2}\left[\frac{1}{\alpha + j(\omega - \beta)} + \frac{1}{\alpha + j(\omega + \beta)}\right]$

$$= \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \beta^2}$$

$\rightarrow FT(t e^{-\alpha t} \cos \beta t) = j\left[\frac{j(\alpha + j\omega)^2 + j\beta^2 - 2j(\alpha + j\omega)^2}{[(\alpha + j\omega)^2 + \beta^2]^2}\right]$

$$= \frac{-(\alpha + j\omega)^2 - \beta^2}{[(\alpha + j\omega)^2 + \beta^2]^2}$$

Q2 Sol^n.

$\rightarrow u_p(t) = \text{sinc}(2t) \cos(100\pi t)$

$v_p(t) = \text{sinc}(t) \sin\left(101\pi t + \frac{\pi}{4}\right)$

$\rightarrow$ Finding $u(t)$ and $v(t)$, $f_c = 50$ Hz.

$v_p(t) = \text{Re}\left\{\text{sinc}(t)\left(-j e^{j\left(101\pi t + \frac{\pi}{4}\right)}\right)\right\}$

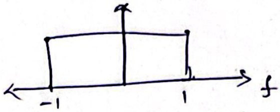
$= \text{Re}\left[\text{sinc}(t)\, e^{j\left(\pi t - \frac{\pi}{4}\right)} e^{j(100\pi t)}\right]$

$\therefore v(t) = \text{sinc}(t)\, e^{j\left(\pi t - \frac{\pi}{4}\right)}$

→ Finding Bandwidth of $u_p(t)$ and $v_p(t)$.

- BW of $u_p(t)$ = BW of $u(t)$ = 2

$\text{sinc}(t) \xrightarrow{FT}$

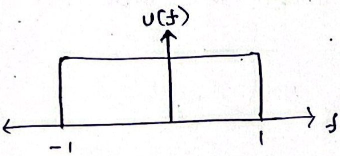


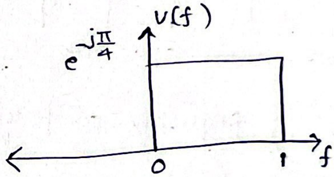
- BW of $v_p(t)$ = BW of $v(t)$ = 1

Because there would be no change in BW while multiplying by $e^{j\omega_0 t}$, it would only shift the spectrum.

→ $\langle u_p, v_p \rangle$ = $Re\left\{ \int u(f) \cdot v^*(f)\, df \right\} = Re\left(\frac{1}{2} e^{j\frac{\pi}{4}}\right)$

$= \frac{1}{2\sqrt{2}}$





→ $y_p(t) = u_p * v_p$

$Y_p(f) = U_p(f)\, V_p(f)$ (convolution Property)

$= \langle u_p, v_p \rangle \cdot U(f) \cdot V(f)$

$= \frac{1}{4\sqrt{2}} V(f)$

$y_p(t) = \text{Inv FT}(V(f))$

$= \frac{1}{4\sqrt{2}} \text{sinc}(t) \sin\left(101\pi t + \frac{\pi}{4}\right)$

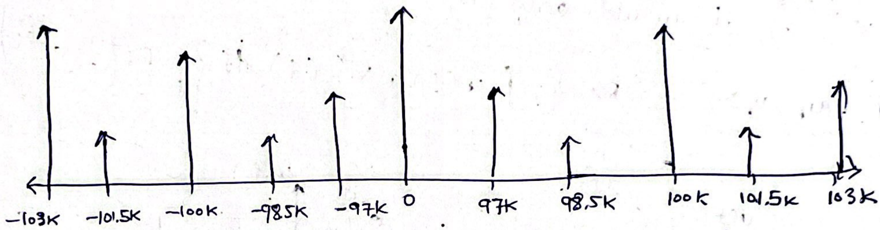
Q3 Sol^n

$\rightarrow u(t) = [20 + 2\cos 3000\pi t + 10\cos 6000\pi t]\cos(2\pi f_c t)$

$f_c = 10^5$ Hz.

$\rightarrow$ Spectrum of $u(t)$

$u(t) = 20\cos(2\pi(100k)t) + \cos((101.5)2\pi t) + \cos(2\pi(98.5k t))$
$+ 5\cos(2\pi(103k t)) + 5\cos(2\pi(97k)t)$.



$\rightarrow$ Power in frequency components

$P_{100K} = \frac{A^2}{2} = 200$ W

$P_{97K} = \frac{A^2}{2} = 12.5$ W

$P_{98K} = \frac{A^2}{2} = 0.5$ W

$P_{101K} = \frac{A^2}{2} = 0.5$ W

$P_{103K} = \frac{A^2}{2} = 12.5$ W

$\rightarrow$ Modulation Index

$m(t) = 2\cos(3000\pi t) + 10\cos(6000\pi t)$

$$\mu_1 = \frac{m_{p_1}}{A} = 0.1 \qquad \mu_2 = \frac{m_{p_2}}{A} = 0.5$$

$$\mu = \sqrt{\mu_1^2 + \mu_2^2} = \sqrt{0.26} \approx 0.5$$

→ Total Power and Power in Sidebands

Power in Carrier $= \frac{A^2}{2} = 200\ W$

Power in Side-Band $= 200 \times \frac{\mu^2}{2} = \frac{200 \times 0.26}{2} = 26\ W$

Total Power $= 226\ W =$ Power in Carrier and Side-Band.

Q5 Sol^n:

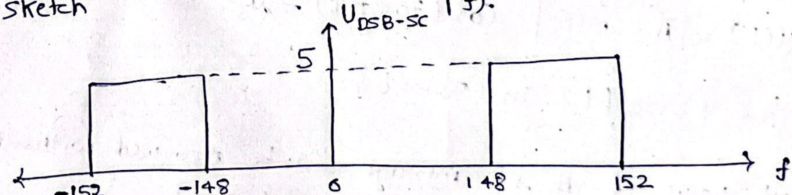
→ $M(f) = I_{[-2,2]}(f)$.

→ $m(t) = 4\ \text{sinc}(4t)$: (Inverse FFT)

→ $u_{DSB-SC}(t) = 10\ m(t) \cos(300\pi t)$

→ $U_{DSB-SC}(f) = 5\ [M(f-150) + M(f+150)]$

→ sketch



→ Bandwidth $= f_H - f_L = 4\ Hz$

→ Power $= 0$, As message and thus DSB are of finite energy

→ Energy $= u^2_{DSB}(t)$

$= 100\, m^2(t) \cos^2(300\pi t)$

$= 50\, m^2(t) + 50\, m^2(t) \cos(600\pi t)$

$= 50\, E_m$

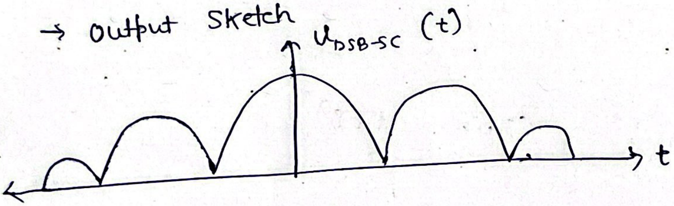
$= 50 \int_{-2}^{2} 1^2 df = 200$ (Parseveral's Relation)

→ <u>Output when passed through Envelope detector</u>

Let $y(t)$ be output of envelope.

$y(t) = 40\,|\operatorname{sinc}(4t)|$

→ Output sketch



→ Relation with message signal.

u_{DSB-SC} is modulus of ~~DSB-SC~~ message signal.

So, part of information in message signal is lost.

→ $u_A\; M(t) = [A + m(t)] \cos(300\pi t)$ $= \cancel{u_s(t) \sin(300\pi t)}$

$= (A + 4 \operatorname{sinc} 4t) \cos(300\pi t)$

→ Smallest value of A s.t message can be recovered without distortion from AM signal by envelope detection.

$A > 4\,|\min_t \operatorname{sinc} 4t)$

≥ 0.88 $\quad (\because |\min_t \sin(4t)| \simeq 0.22)$.

$\rightarrow$ $u_p(t) = u_c(t) \cos(300\pi t) - u_s(t) \sin(300\pi t)$

$u(t) = 20 \operatorname{sinc} 2t \; e^{j2\pi t}$ (Inverse FT)

$u_c(t) = 20 \operatorname{sinc}(2t) \cos(2\pi t)$

$= 20 \operatorname{sinc}(4t)$

$= 5\, m(t)$

Similarly, $u_s(t) = 5\, \tilde{m}(t)$.

Q6. Soln

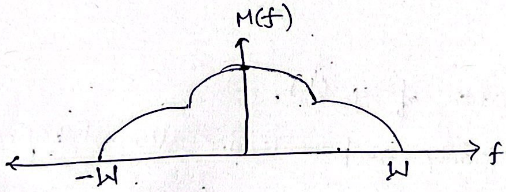
$\rightarrow$ $m(t) \rightarrow \oplus \xrightarrow{x(t)} \boxed{\text{Square Law Device}} \xrightarrow{y_1(t) = x^2(t)} \boxed{\text{Bandpass Filter}} \xrightarrow{y_2(t)} \otimes \xrightarrow{y_3(t)} \boxed{\text{LPF}} \rightarrow y_4(t)$

$\cos(2\pi f_0 t)$ into $\oplus$; $\cos(2\pi f_0 t)$ into $\otimes$

BPF = BW = $2W$

LPF = BW = W

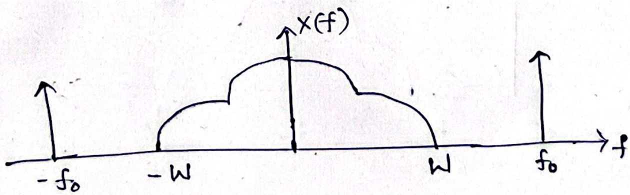
$\rightarrow$



$\rightarrow$ Spectra and Bandwidth of $X(f)$

$x(t) = m(t) + \cos(2\pi f_0 t)$

$X(f) = M(f) + \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$



Bandwidth $= f_0$.

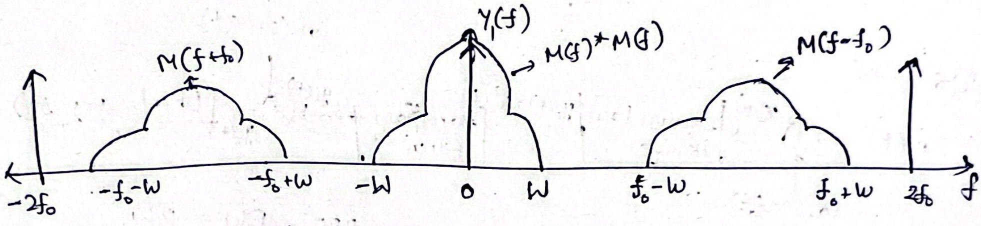
→ **Spectra and Bandwidth of $y_1(t) = x^2(t)$**

$$y_1(t) = [m(t) + \cos(2\pi f_0 t)]^2$$

$$= m^2(t) + \frac{1}{2} + \frac{\cos(4\pi f_0 t)}{2} + 2m(t)\cos(2\pi f_0 t)$$

$$Y_1(f) = M(f) \circledast M(f) + \frac{1}{2}\delta(f) + \frac{1}{4}[\delta(f-2f_0) + \delta(f+2f_0)]$$

$$+ \frac{\cancel{2}[M(f - f_0) + M(f+f_0)]}{\cancel{2}}$$

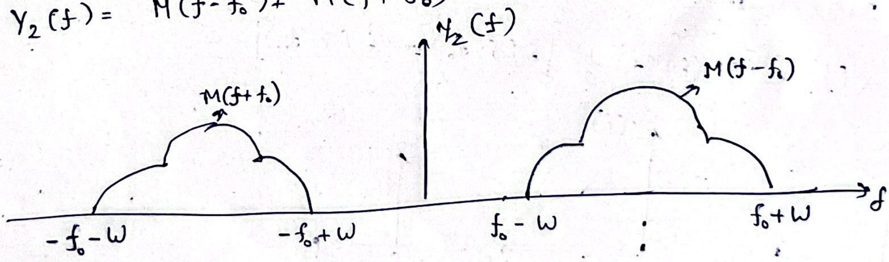


Bandwidth = $2f_0$

→ **Spectra and Bandwidth of $y_2(t)$.**

As it is output of BPF, only components in allowed region, i.e. ($f_0 - W$ to $f_0 + W$) and ($-f_0 - W$ to $-f_0 + W$)

$$Y_2(f) = M(f - f_0) + M(f + f_0)$$



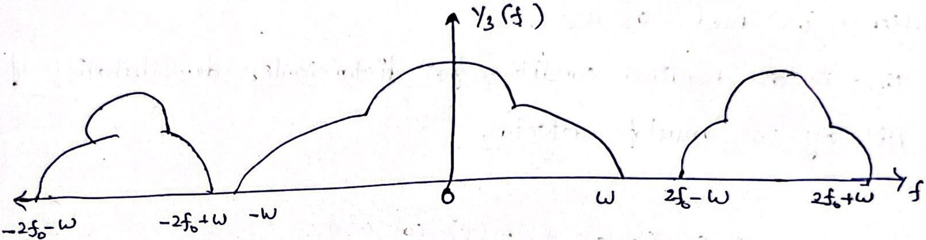
Bandwidth = $2W$

→ Spectra and Bandwidth of $y_3(t)$.

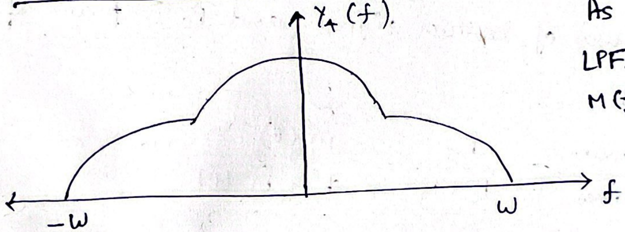
$$y_3(t) = 2\,m(t)\cos^2(2\pi f_0 t)$$

$$= m(t) + m(t)\cos(4\pi f_0 t)$$

$$Y_3(f) = M(f) + \frac{1}{2}\left[M(f-2f_0) + M(f+2f_0)\right]$$



→ Spectra and Bandwidth of $y_4(t)$



As it is output of LPF. with BW W, only $M(f)$ would be left.

Bandwidth $= W$

→ Significance of Modulation Index

① Modulation Index is defined as ; $\mu = \frac{m_p}{A}$.

For envelope detection, to be distortionless, the condition is $A \geq m_p$.

Hence, we need $0 \leq \mu \leq 1$.

This is the required condition for distortionless demodulation of AM by an envelope detector.

②

$$\phi_{AM}(t) = A\cos(\omega_c t) + m(t)\cos\omega_c t$$

→ $P_c = \frac{A^2}{2}$ and $P_s = \frac{1}{2}\tilde{m}^2(t)$

where, P_c is Power of carrier signal and P_s is power of sideband signal.

→ $\eta = \frac{\text{useful power}}{\text{total power}} = \frac{P_s}{P_c + P_s} = \frac{\tilde{m}^2(t)}{A^2 + \tilde{m}^2(t)}$ 100 %.

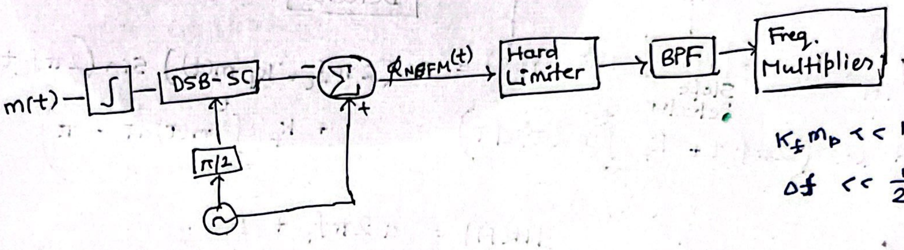
→ For tone modulation, $m(t) = \mu A \cos\omega_m t$

$$\tilde{m}^2(t) = \frac{(\mu A)^2}{2}$$

→ Hence, $\eta = \frac{\mu^2}{2+\mu^2}$ 100%.

→ η increases monotonically with μ, and η_{max} occurs at $\mu = 1$.

∴ $\eta_{max} = 33\%$. for tone modulation under best condition $(\mu = 1)$



$K_f m_p << 1$

$\Delta f << \frac{1}{2\pi}$

Width of Square wave depends on instantaneous frequency

Direct Method

$$m(t) \longrightarrow \boxed{H(f)} \longrightarrow \phi_{FM}(t)$$

↑
VCO
↑

$$\boxed{\begin{aligned} f_0 &= \frac{1}{2\pi\sqrt{LC}} \\ C &= C_0 - K\, m(t) \end{aligned}}$$

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t).$$

$$f_i(t) = \frac{1}{2\pi\sqrt{LC}} \left[1 - \frac{K}{C_0} m(t)\right]^{-1/2}.$$

$$\simeq f_0 \left[1 + \frac{K}{2C_0} m(t)\right] \quad \text{if } \frac{K}{C_0} m(t) << 1$$

$$\simeq f_0 + \frac{f_0 K}{2C_0} m(t).$$

In Voltage-controlled oscillator (VCO), frequency is controlled by external voltage

FM Demodulation

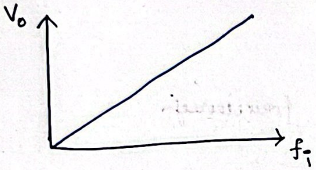
$$\phi_{FM}(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t) \longrightarrow \boxed{\text{Envelope Detector}} \longrightarrow A\left(2\pi f_c + K_f\, m(t)\right)$$

Differentiator

Slope Detector

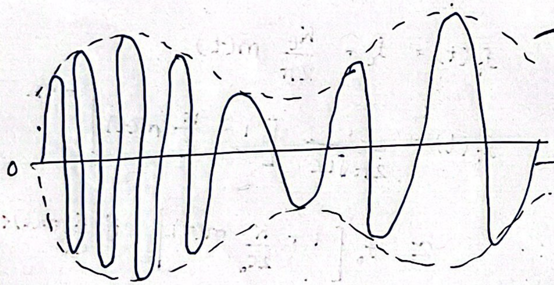
$$= A\cos\left(2\pi f_c t + K_f \int_{-\infty}^{t} m(\tau)\,d\tau\right)$$

$$y(t) = A\left(2\pi f_c + K_f\, m(t)\right)\sin\left(2\pi f_c t + K_f \int_{-\infty}^{t} m(\tau)\,d\tau - \pi\right)$$



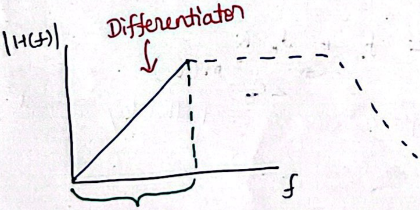
$$|H(f)| = a\,2\pi f_i + b$$

$$H(f) = j\,2\pi f_i$$



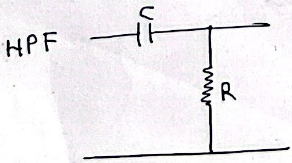
→ Demodulated / recovered signal → Envelope.

→ Rectified FM signal



Band pass filter (BPF) used as differentior in highlighted region

HPF



if $f \ll \dfrac{1}{2\pi RC}$

$$H(f) = \frac{j2\pi f RC}{1 + j2\pi f RC} \simeq j2\pi f RC$$

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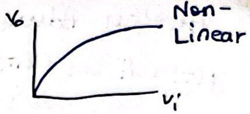


Features of FM

$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + \ldots$$

$$x(t) = A \cos(2\pi f_c t + K_f a(t))$$

$$y(t) = c_0 + c_1 \cos(2\pi f_c t + K_f a(t)) + c_2 \cos(4\pi f_c t + 2K_f a(t)) + \ldots$$

$$\rightarrow \boxed{BPF} \rightarrow c_1 \cos(2\pi f_c t + K_f a(t))$$

FM signal is ~~now~~ immune to non-linearity

$$x(t) = m(t)\cos(2\pi f_c t) \qquad y(t) = a_1 x(t) + a_3 x^3(t)$$

$$y(t) = a_1 m(t)\cos(2\pi f_c t) + a_3\left[(m(t))^3 \cos^3(2\pi f_c t)\right]$$

$$= a_1 m(t)\cos(2\pi f_c t) + a_3 [m(t)]^3 \left[\frac{3}{4}\cos(2\pi f_c t) + \frac{1}{3}\cos(6\pi f_c t)\right]$$

$$x = m(t)\left[a_1 \cos(2\pi f_c t) + a_3 [m(t)]^2 \frac{3}{4}\cos(2\pi f_c t)\right] + a_3 [m(t)]^3 \frac{1}{3}\cos(6\pi f_c t)$$

$$= \cos(2\pi f_c t)\left[a_1 m(t) + a_3 m^3(t)\right]$$

Not immune to non-linearity

FM signal handles non-linearity in a better fashion as compared to AM signal.

FM > AM Bandwidth $\rightarrow$ Disadvantage of FM.

Another Advantage: Handles ~~linearity~~ Interference better

$$y(t) = A\cos(2\pi f_c t) + I\cos(2\pi(f_c+f)t)$$

Interference of carrier signal.

This is received at receiver.

$$= \underbrace{[A + I\cos(2\pi f t)]}_{a}\cos(2\pi f_c t) - \underbrace{I\sin(2\pi f t)}_{b}\sin(2\pi f_c t)$$

$$= \underbrace{R(t)}_{\sqrt{a^2+b^2}}\cos(2\pi f_c t - \underbrace{\psi(t)}_{\tan^{-1}\left(\frac{b}{a}\right)})$$

$$\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\frac{I\sin(2\pi f t)}{A + I\cos(2\pi f t)}$$

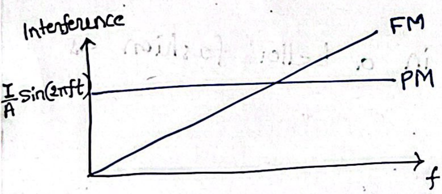
Approximation.

① $A \gg I$, $\frac{I}{A} \ll 1$

$$\psi(t) \simeq \frac{I}{A}\sin(2\pi f t)$$

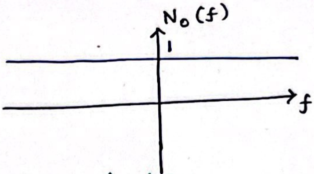
$R(t)\cos(2\pi f_c t - \psi(t)) \longrightarrow$ $\boxed{\text{PM Demodulation}}$ $\longrightarrow z(t) = \psi(t) = \frac{I}{A}\sin(2\pi f t)$

$\boxed{\text{FM Demodulation}}$ $\longrightarrow z(t) = \frac{d\psi(t)}{dt} = \frac{I}{A}(2\pi f)\cos(2\pi f t)$



$A \cos(2\pi f_c t) + n(t)$

Interfering Noise

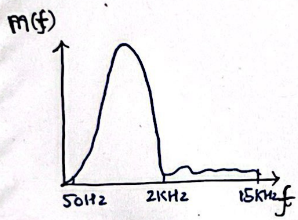


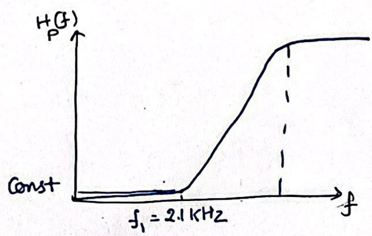
Noise Power Spectrum.

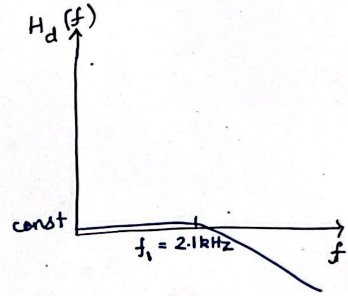
⊛ FM signal handles interference in a better fashion

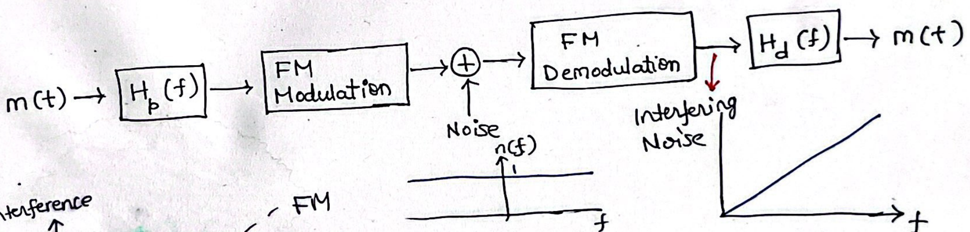
⊛ For smaller freq. separation, FM is better

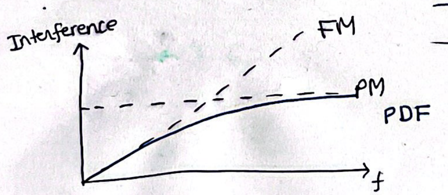
For larger freq. separation, PM is better





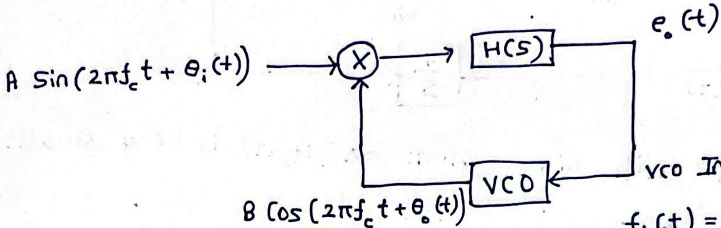






Phase Lock Loop (PLL)

can be used as demodulator of FM signal and also for carrier recovery



VCO Instantaneous Freq

$$f_i(t) = f_c + c\,e_o(t)$$

$$f_i(t) = f_c + \frac{1}{2\pi}\dot{\theta}_o(t)$$

$$\boxed{\dot{\theta}_o(t) = \underbrace{2\pi c}_{c'}\, e_o(t)} \quad —①$$

$$\boxed{e_o(t) = \frac{1}{c}\dot{\theta}_o(t) \qquad \because \dot{\theta}_e(t) \approx 0}$$

→ PLL is an circuit with VCO that constantly adjusts to match the frequency of an input signal.

→ PLL is in lock with input signal $A\sin[\omega_c t + \theta_i(t)]$ and output error $e_o(t)$.

→ $AB\,\sin(2\pi f_c t + \theta_i(t))\cos(2\pi f_c t + \theta_o(t))$

$$= \frac{AB}{2}\left[\sin(\theta_i(t) - \theta_o(t)) + \cancel{\sin(4\pi f_c t + \theta_i(t) + \theta_o(t))}\right]$$

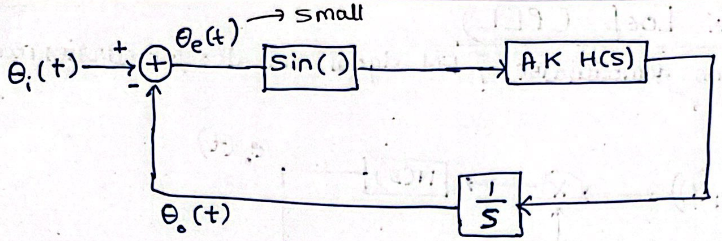
suppressed by LPF.

Loop Filter is designed st. it suppresses the higher frequency.

→ $e_o(t) = \frac{AB}{2}\sin(\theta_e(t)) \circledast h(t)$, where —②

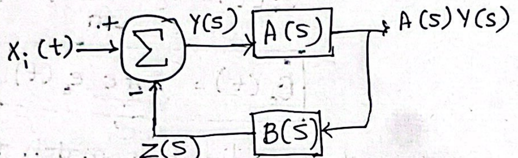
$$\theta_e(t) = \theta_i(t) - \theta_o(t)$$

→ $$\boxed{\dot{\theta}_o(t) = \underbrace{\frac{ABc'}{2}}_{K}\sin(\theta_e(t)) * h(t)}$$ From eqn ① and ②



We can just skip sin( ) block as $\theta_e(t)$ is very small. So,

$\sin(\theta_e(t)) \approx \theta_e(t)$



$Z(s) = A(s)\,B(s)\,Y(s)$

$Y(s) = A(s)\,B(s)\,Y(s) + X(s) = \dfrac{X(s)}{1 + A(s)\,B(s)}$

$$\boxed{\frac{Z(s)}{Y(s)} = \frac{A(s)\;B(s)}{1 + A(s)\,B(s)}}$$

$$\theta_i(t) \longrightarrow \boxed{\frac{AK\,H(s)}{s + AK\,H(s)}} \longrightarrow \theta_o(t)$$

$$\theta_e(t) = \theta_i(t) - \theta_o(t) = \left[1 - \frac{\theta_o(t)}{\theta_i(t)}\right] \theta_i(t)$$

$$\Theta_e(s) = \Theta_i(s) - \Theta_o(s) = \left[1 - \frac{\Theta_o(s)}{\Theta_i(s)}\right] \Theta_i(s)$$

$$\boxed{\Theta_e(s) = \frac{s}{s + AK\,H(s)}\;\Theta_i(s)}$$

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$A \sin(2\pi f_o t + \psi)$

$$A \sin(2\pi f_c t + \underbrace{2\pi(f_o - f_c)t + \psi}_{\theta_i(t)})$$

$$\therefore \theta_i(s) = \frac{2\pi(f_o - f_c)}{s^2} + \frac{\psi}{s}$$

Specific Examples of $H(s)$

(*) $H(s) = 1$

$$\lim_{s\to 0} s\,\Theta_e(s) = \frac{s^2}{s + AK}\left[\frac{2\pi(f_o - f_c)}{s^2} + \frac{\psi}{s}\right] = \frac{2\pi(f_o - f_c)}{AK}$$

(Final Value Theorem)

$$= \lim_{t\to\infty} \theta_e(t)$$

$$\boxed{\Theta_e(s) = \frac{s}{s + AK\,H(s)}\,\dot{\theta}_i(s)}$$

At steady state we get constant phase difference which is predetermined and hence can be adjusted.

(*) $H(s) = \dfrac{s+a}{s}$

$$\lim_{s\to 0} s\,\Theta_e(s) = \frac{s^3}{s^2 + AK(s+a)}\left[\frac{2\pi(f_o - f_c)}{s^2} + \frac{\psi}{s}\right] = 0 = \lim_{t\to\infty} \theta_e(t)$$

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PLL as FM Demodulation

$$\sin\big(\underbrace{2\pi f_c t + K_f\, a(t) + \tfrac{\pi}{2}}_{\theta_i(t)}\big) \longrightarrow \boxed{PLL} \longrightarrow e_o(t) = \frac{1}{c'}\,\dot{\theta}_o(t)$$

$$\theta_e(t) = \theta_i(t) - \theta_o(t)$$

$$e_o(t) = \frac{1}{c'}\,\frac{d}{dt}\left[\theta_i(t) - \theta_e(t)\right] = \frac{K_f}{c'}\,m(t)$$

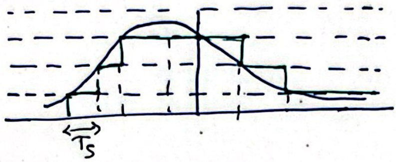
When PLL is locked, i.e., $\theta_e(t) \approx 0$

$$\sin\left(2\pi f_c t + K_p\, m(t) + \tfrac{\pi}{2}\right) \longrightarrow \boxed{PLL} \xrightarrow[\text{at large } t]{K_p\,\dot{m}(t)} \boxed{\int} \longrightarrow m(t)$$

Analog to Digital Converter

20 March 2023

We need represent analog signal using discretised amplitude and time.

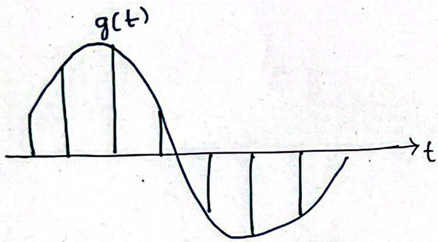


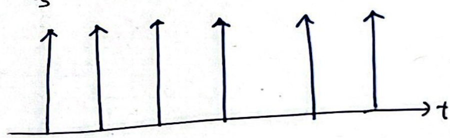
$$g(t) \longrightarrow \boxed{\text{Sampler}} \longrightarrow \boxed{\text{Quantiser}} \longrightarrow g[n] \Longrightarrow \hat{g}(t) = g(nT_s)$$

$$T_s \le \frac{1}{2B}$$

We need to digitise signal st. we can recover back original signal reliably.

→ Nyquist Sampling Rate





$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \delta_{T_s}(t)$$

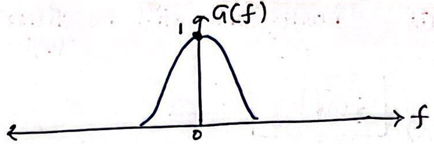
$$\bar{g}(t) = g(t)\,\delta_{T_s}(t)$$

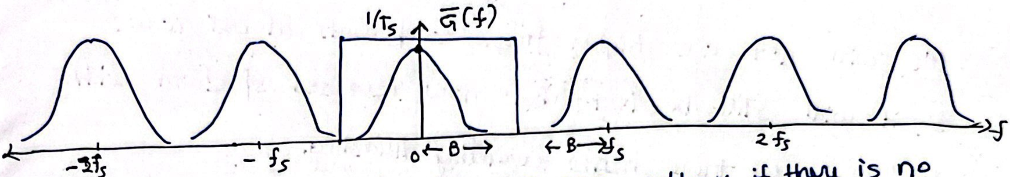
$$\bar{g}(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s)$$

$$\bar{G}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(f - nf_s)$$

To retrive signal back, we need to recover $G(f)$ from $\bar{G}(f)$

acc. to Uniqueness of FT, recovering $G(F) \sim$ recovering $g(t)$.





We can recover signal when we can suppress if there is no overlap. Then, we can design required filter accordingly.

$$\boxed{\frac{1}{T_s} = f_s \geq 2B}$$

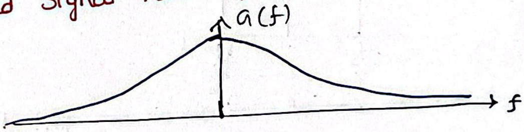
$f_s = 2B =$ Nyquist Sampling Rate

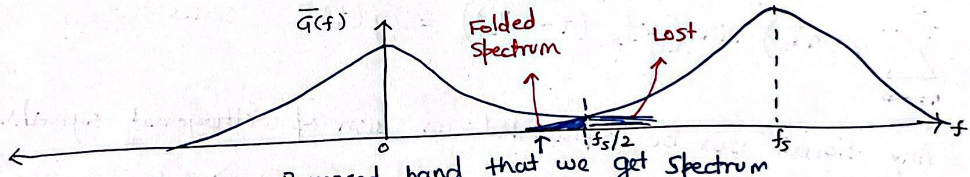
Shorter signal has larger B.W. Signal with larger B.W are longer duration signal.

Time limited signal has infinite BW.

Band limited signal has infinite time.

Time limited
Unlimite B.W.





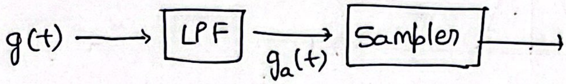
① Folded spectrum: Reversed band that we get spectrum

This interferes and thus distorts the signal. Hence, causing Aliasing Effect.

② Lost

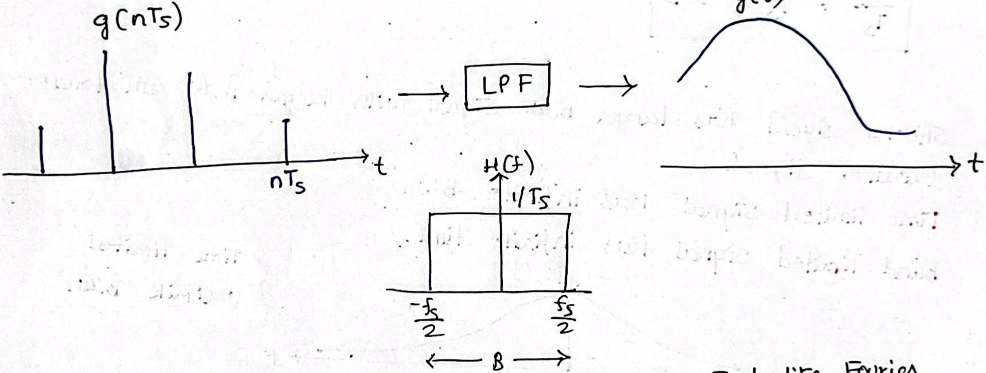
When we pass G(f) through LPF, we will loose higher freq. components.

Avoiding Aliasing.... ☺ Lost will still be ~~there~~ lost though! ☹



We will not use higher freq. components at all now. So, it will still be lost. But now, reversed spectrum will also not be there hence avoiding Aliasing.

Recovering Signal



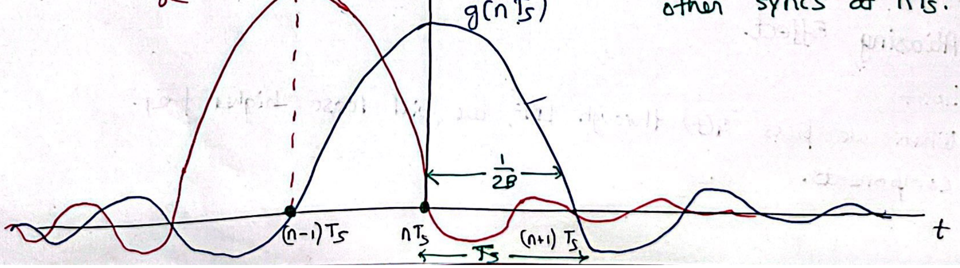
$$\sum_{n=-\infty}^{\infty} g(nT_s)\,\text{sinc}\left(\pi f_s (t-nT_s)\right) = g(t)$$

Just like Fourier Series

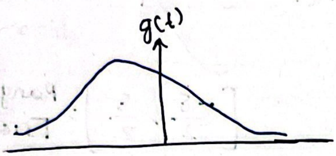
Any signal can be represented as sum of orthogonal signals.

We observe that theses sync are orthogonal signals.

No interventions from other sync's at nT_s.

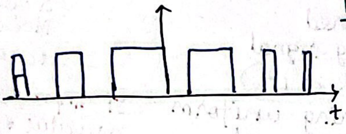


Pulse Modulations...



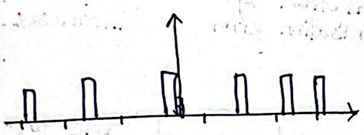
PAM
Pulse Amplitude Modulation

Fixed in time.
$\therefore$ Fixed B.W.



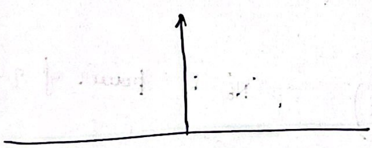
PWM
Pulse Width Modulation

Width is small → High B.W.



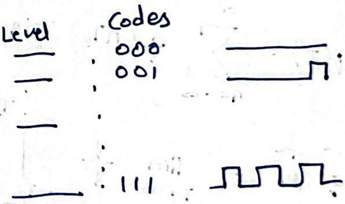
PPM
Pulse Position Modulation

Changing position of pulse (Delay) in an interval acc. to inst. amplitude



PCM
Pulse Code Modulation

$g(t) \longrightarrow \boxed{A/D} \longrightarrow$ L-ary signal.

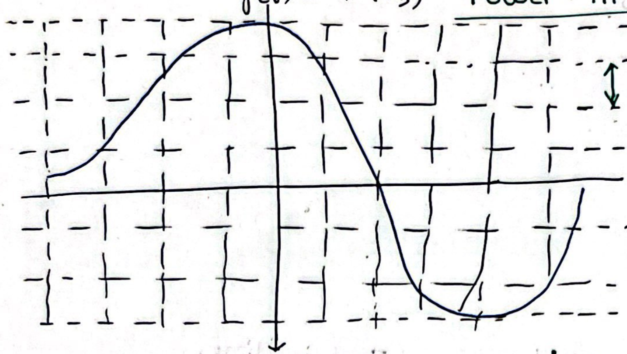


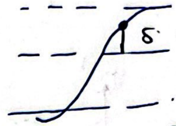
Discretised signal with L levels for amplitude,
$\therefore$ No. of Bits $= \log_2 L = M$

$$\hat{m}(nT_s) = \underbrace{m(nT_s)}_{\text{Message}} + \underbrace{q(nT_s)}_{\text{Noise}}$$

If SNR is high, it is good.

Power in Noise





$\left[\frac{-\delta}{2}, \frac{\delta}{2}\right]$: Range of Error

$\hat{m}(nT_s) \rightarrow \boxed{LPF} \rightarrow \hat{m}(t) = \underbrace{m(t)}_{S_o} + \underbrace{q(t)}_{N_o}$

$= \underbrace{m(nT_s)}_{\text{Digital Signal}} + \underbrace{q(nT_s)}_{\text{Error introduced during quantisation}}$

$\hat{m}(t)$: Recovered Analog Signal

N_o: Quantisation Noise Signal Power

$$\boxed{SNR = \frac{\widetilde{m(t)^2}}{\widetilde{q(t)^2}} = \frac{3L^2 \widetilde{m^2(t)}}{m_p^2}}$$

SNR $\propto L^2$

Assuming uniform distribution of quantisation error

It improves as value of L increases.

We are trying to find N_o.

$q(nT_s) \longrightarrow \boxed{LPF} \longrightarrow q(t) \rightarrow$ weighted sum of sinc fn.

Interpolation

'N_o' is power of $q(t)$.

$$q(t) = \sum_{n=-\infty}^{\infty} q(nT_s) \operatorname{sinc}(2Bt - n\pi)$$

$$\widetilde{q^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} q^2(t)\, dt$$

We know sinc functions are orthogonal. So,

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi Bt - n\pi) \operatorname{sinc}(\pi Bt - k\pi)\, dt = \begin{cases} 0, & n \neq k \\ \frac{1}{2B}, & n = k. \end{cases}$$

$$\widetilde{q^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left(\sum q(nT_s)\, \text{sinc}\,(2Bt - n\pi)\right)^2$$

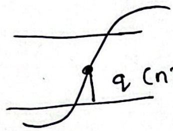
$$= \lim_{T\to\infty} \frac{1}{\underbrace{T\,2B}_{\text{Total no. of samples} \,=\, T(2B)}} \sum_{n=-\infty}^{\infty} q^2(nT_s)$$

$$\boxed{\widetilde{q^2(t)} = \widetilde{q^2(nT_s)}}$$

Finding average noise per observation

We need to know the distribution along with range to find average.

So, assuming uniform distribution



Error

$q(nT_s) \in \left[\frac{-\delta}{2}, \frac{\delta}{2}\right]$

$$\widetilde{q(nT_s)^2} = \int_{-\delta/2}^{\delta/2} q^2 \, f_q(q)\, dq$$

$$= \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} q^2 \, dq$$

$$= \frac{1}{\delta} \left[\frac{q^3}{3}\right]_{-\delta/2}^{\delta/2} = \frac{1}{\delta}\left[\frac{\delta^3}{24} + \frac{\delta^3}{24}\right] = \frac{\delta^2}{12} = \frac{m_p^2}{3L^2}$$

$$\boxed{\widetilde{q^2(nT_s)} = \frac{\delta^2}{12} = \frac{m_p^2}{3L^2}}$$

Minimum Channel B.W required. $\boxed{B_T = nB}$

Bandwidth SNR Tradeoff

$$\delta = \frac{2m_p}{2^n} \qquad q^2 = \frac{\delta^2}{12}$$

$SNR \propto L^2 \propto 2^{2n}$ (Large SNR)

But, $B_T = nB$ (Large Bandwidth)

Larger SNR requires larger B.W.

Uniform Quantiser

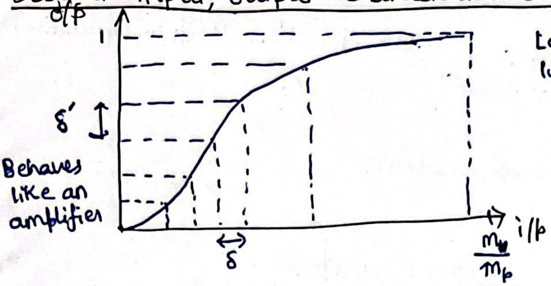
$$N_o = \frac{\delta^2}{12} = \frac{m_p^2}{3L^2}$$

$$\frac{S_o}{N_o} = SNR = \frac{3L^2}{m_p^2}\widetilde{m(t)^2} = \frac{\widetilde{m^2(t)}}{N_o}$$

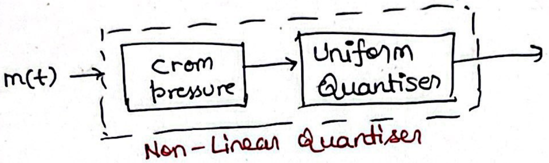
Small Amplitude of m(t) ⇒ Small N_o ⇒ step size δ small

Large " " " ⇒ Large N_o ⇒ " " " large

Desired input, output characteristics of quantiser



Looks like log func.

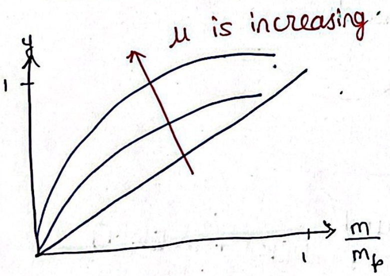


$\boxed{\text{Expander}}$

To negate effect ~~from~~ of crom pressure

μ-Law

$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \mu\frac{m}{m_p}\right) \quad , \mu = 255 \text{ standard value}$$



A-Law

$$y = \begin{cases} \dfrac{A}{1+\ln A} \cdot \dfrac{m}{m_p} & \forall\ 0 < \dfrac{m}{m_p} < \dfrac{1}{\pi} \\[2ex] \dfrac{1}{1+\ln A}\left(1 + \ln\left(\dfrac{m}{m_p}\right)\right) & \forall\ \dfrac{1}{\pi} < \dfrac{m}{m_p} < 1 \end{cases}$$

* Same characteristics as μ-Law.

SNR for uniform Quantiser

$$\frac{S_o}{N_o} = \frac{3L^2}{m_p^2}\,\widetilde{m^2(t)} = \underbrace{\frac{3L^2}{[\ln(1+\mu)]^2}}$$ SNR for Non-Uniform Quantiser

Lets say $c = \begin{cases} \dfrac{3\,\widetilde{m^2(t)}}{m_p^2} & \rightarrow \text{uniforn} \\ \dfrac{3}{[\ln(1+\mu)]^2} & \rightarrow \text{Non-uniform} \end{cases}$

Constant

$$SNR = cL^2 = c\,2^{2n}$$

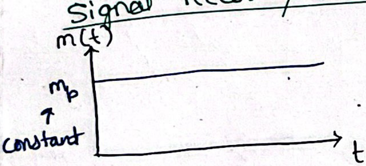
$$SNR_{dB} = 10 \log_{10}(SNR)$$

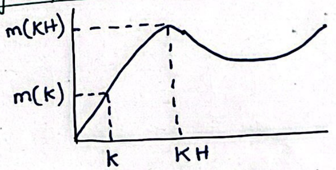
$$= 10 \log_{10} c + 20n \log_{10} 2$$

$$\boxed{SNR_{dB} = \alpha + 6n}$$

Now, $\boxed{B_T = nB \qquad SNR_{dB} = \alpha + 6n}$ For PCM

Signal Recovery using different approach





If inputs at time instants are correlated

$$d[K] = m[K] - m[K-1] \Rightarrow m[K] = d[K] + m[K-1]$$

$\rightarrow$ Differential PCM $\rightarrow$ dPCM.

$\boxed{m_d << m_p}$

$\underbrace{\delta' = \frac{2m_d}{L}}$

Low Power Noise

$\delta = \frac{2m_p}{L}$

① $\frac{m_p}{m_d} \rightarrow \delta\downarrow \rightarrow N_o\downarrow \rightarrow SNR\uparrow$ Adv

② $nB\downarrow$ = Transition B.W $\downarrow$

Scanned with CamScanner

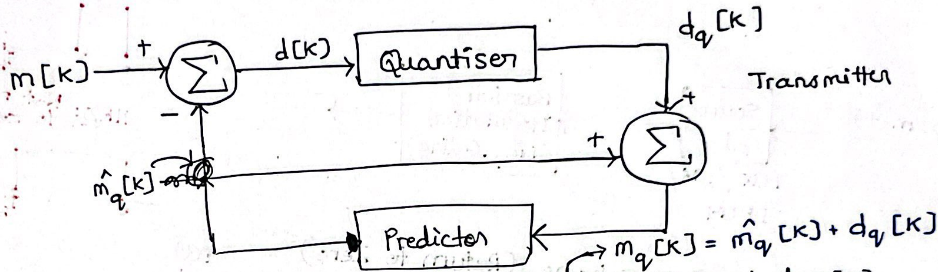
We can reduce noise power even further using prediction approach using predictor

$d[K] = m[K] - \underbrace{\hat{m}_q[K]}$ Predicted signal

Predictor is used on quantised signal, not on actual signal.

Hence, Reduced $\delta' \Rightarrow$ Same n, SNR ↑, BT same

Same SNR, reduce n, $B_T \downarrow$



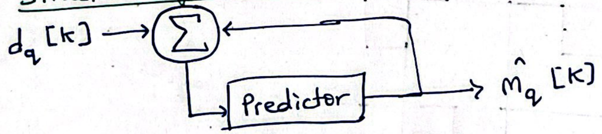
$$d[K] = m[K] - \hat{m}_q[K]$$

$$d_q[K] = d[K] + q[K]$$

$$m_q[K] = \hat{m}_q[K] + d_q[K] = m[K] - d[K] + d_q[K]$$

$$\boxed{m_q[K] = m[K] + q[K]}$$

Structure of Receiver



Gain in B.W after using PCM

$$\frac{SNR_{DPCM}}{SNR_{PCM}} = \frac{N_{o\,PCM}}{N_{o\,DPCM}} = \frac{m_p^2}{m_d^2}$$

$$SNR^{dB}_{dPCM} = 10 \log_{10}\left(\frac{m_p}{m_d}\right)^2 + \underbrace{(\alpha + 6n)\ dB}_{SNR^{dB}_{PCM}}$$

Delta Modulation (Self Read)

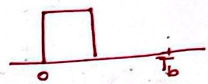
25 March 2023

$B_{PCM} = \frac{R_b}{2} = \frac{n2B}{2} = nB$ $\qquad SNR_{PCM, dB} = \alpha + 6n$

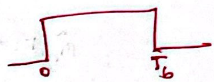
$SNR_{DPCM} = G\ SNR_{PCC}$ $\qquad SNR_{DPSM, dB} = 20 \log_{10}\left(\frac{m_p}{m_d}\right) + \alpha + 6n.$

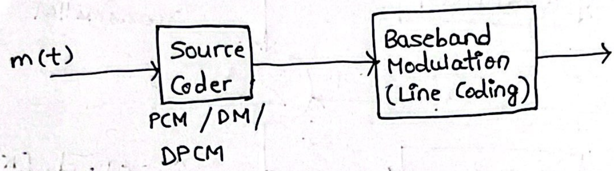
$m_d \Rightarrow d[K] = m[K] - \hat{m}_q[K]$

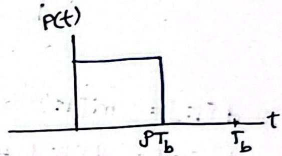
RZ : Return to Zero



NRZ: Not return to zero.



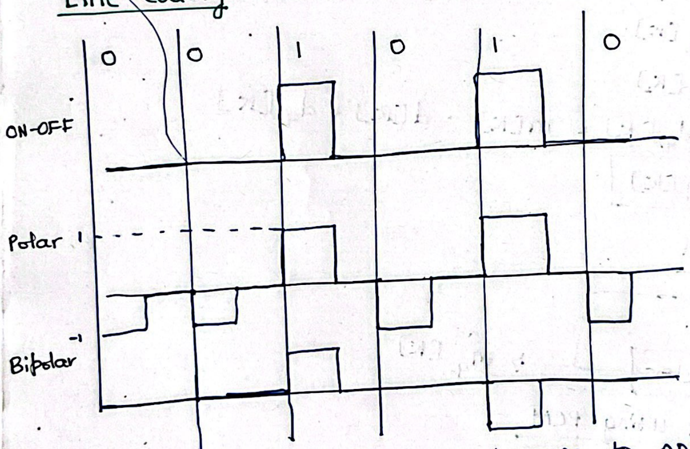




↳ Fractional

~~Pulse Coded signal~~

Line Coding : RZ Scheme (Return to Zero)



nature of freq matching error

⊛ Polar ~~needs~~ mo scheme is more immune to noise and distance b/n + and - is more than +, 0.

Hence, polar scheme is more immune to noise and more power efficient compared to ON-OFF.

⊛ We can count the number of ones.

Hence, Bipolar has error detection capabilities.

25 March

$$SNR_{ON-OFF} = \frac{A'^2}{2N_0} = K \qquad SNR_{Polar} = \frac{A^2}{N_0} = K$$

$A' > A$ i.e. $A' = \sqrt{2}\,A$

'K' is suppose the desired SNR. So, to equate the SNR for both schemes, ON-OFF will require amplitude and hence more power

ISI control
PSD of pulsing
more power
PLL
clk/carrier recovery

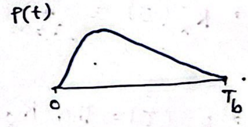
Favourable requirements for Line Encoding.

(i) Power efficient
(ii) BW efficient
(iii) Favourable PSD (Power Spectral Density) of pulses to avoid ISI (Inter-Symbolic Interference)
(iv) Error ~~computation~~ correct / detection capability
(v) Favourable for Clock recovery

Calculating Power Spectral Density (PSD) of Pulses

Pulse Coded Signal

Deterministic component of a Line Coding Scheme
PSD/FT



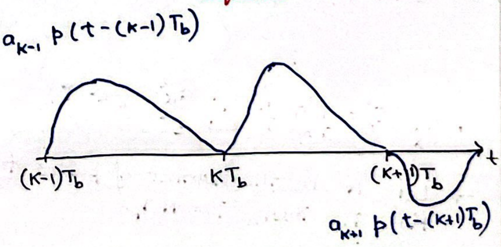
ON-OFF $a_K = \begin{cases} 1, & \text{for '1'} \\ 0, & \text{for '0'} \end{cases}$

Polar $a_K = \begin{cases} +1, & \text{for '1'} \\ -1, & \text{for '0'} \end{cases}$

Bipolar

$Y(t)$: Line Coding Signal

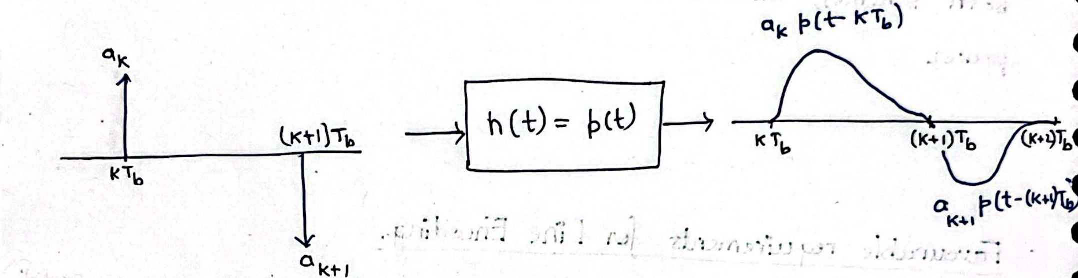
$$Y(t) = \sum_{K=-\infty}^{\infty} \underbrace{a_K}_{\text{Random sequence}} \underbrace{p(t - KT_b)}_{\text{Deterministic}}$$



Representing $Y(t)$ as an output of a filter will help us calculate its PSD directly acc. to PSD of input. using formula:

$$S_Y(f) = |P(f)|^2 \, S_X(f)$$

This helps for decoupling dependency of $Y(t)$ from a_K and $p(t-KT_b)$



$x(t)$: impulse at multiples of T_b.

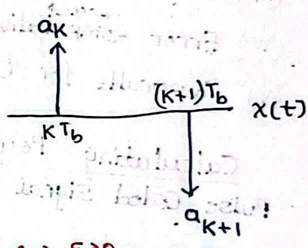
$$Y(t) = \sum_K a_K \, p(t - KT_b)$$

Finding PSD of input.

$\rightarrow S_X(f) \overset{\wedge}{=} FT\{R_X(\tau)\}$

$\rightarrow R_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t-\tau)\, dt$

$x(t)$: original signal.



$\rightarrow R_X(\tau) = \lim_{\varepsilon\to 0} R_{\hat{X}}(\tau)$.

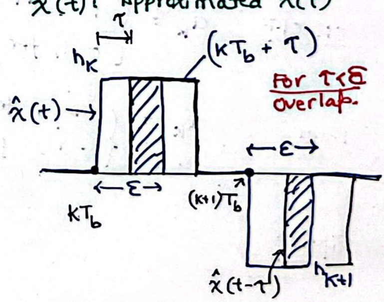
We will calculate above integration with $x(t) = \hat{x}(t)$ using cases below:

$\hat{x}(t) \xrightarrow{\varepsilon \to 0} x(t)$

$\hat{x}(t)$: Approximated $x(t)$

CASE I

For $\tau < \varepsilon \Rightarrow \lim_{T\to\infty} \frac{1}{T} \sum_K h_K^2 (\varepsilon - \tau)$



$\rightarrow$ Impulses can be seen as limiting case for rectangular pulses.

$\rightarrow R_{\hat{X}}(\tau) = \lim_{T\to\infty} \frac{1}{T} \sum \frac{a_K^2}{\varepsilon^2} (\varepsilon - \tau)$

$h_K \varepsilon = a_K$ as $\varepsilon \to 0$

$\rightarrow$ Let N be the number of pulses transmitted in time T. Then, $T = NT_b$.

$h_K = \frac{a_K}{\varepsilon}$ as $\varepsilon \to 0$

$\rightarrow R_{\hat{x}}(\tau) = \underbrace{\left(\lim_{N\to\infty} \frac{1}{N} \sum a_k^2\right)}_{R_0} \frac{1}{T_b\,\varepsilon}\left(1 - \frac{\tau}{\varepsilon}\right)$ $\quad R_0 = \widetilde{a_k}$

= Time average of a_k.

$\rightarrow R_{\hat{x}}(\tau) = \frac{R_0}{T_b\,\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)$ for $|\tau| < \varepsilon$

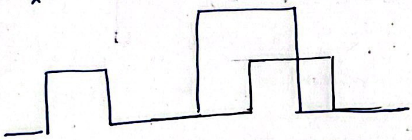
CASE 2

$\rightarrow R_{\hat{x}}(\tau) = 0$ for $\varepsilon < \tau \le T_b - \varepsilon$

As there would be no overlap.

$\rightarrow R_{\hat{x}}(\tau)$ for $T_b - \varepsilon < \tau < T_b + \varepsilon$ CASE 3

sampl avg $\neq$ exact avg time



$$R_{\hat{x}}(\tau) = \lim_{T\to\infty} \frac{1}{T} \sum h_k\, h_{k+1}\,(\varepsilon - \tau)$$

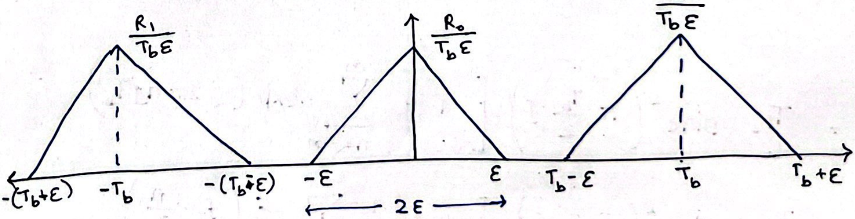
$$= \frac{R_1}{T_b\,\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)$$

where, $R_1 = \lim_{N\to\infty} \frac{1}{N} \sum a_k a_{k+1} = \widetilde{a_k a_{k+1}}$

$\rightarrow$ For $nT_b - \varepsilon < \tau < nT_b + \varepsilon$

$$R_{\hat{x}}(\tau) = \frac{R_n}{T_b\,\varepsilon}\left(1 - \frac{|\tau|}{\varepsilon}\right)$$

$\rightarrow$



$\rightarrow R_X(\tau) = \lim_{\epsilon \to 0} R_{\hat{X}}(\tau)$

$$R_X(\tau) = \frac{1}{T_b} \sum_n R_n \, \delta(t - nT_b)$$

$$S_X(f) = \frac{1}{T_b} \sum_n R_n \exp(-j 2\pi f n T_b)$$

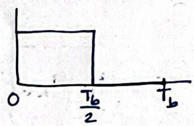
$$S_Y(f) = \frac{|P(f)|^2}{T_b} \sum_{n=-\infty}^{\infty} R_n \exp(-j 2\pi f n T_b)$$

$$= \frac{|P(f)|^2}{T_b} \left[\sum_{n=1}^{\infty} 2 R_n \cos(2\pi f n T_b) + R_0 \right]$$

$\rightarrow R_0 = \widetilde{a_K^2} = 1^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{2} = \frac{1}{2}$

$R_n = \widetilde{a_K \, a_{K+n}} = 1^2 \cdot \frac{1}{4} + 0^2 \, \frac{3}{4} = \frac{1}{4}$

$\rightarrow$ For $p(t) =$



$$P(f) = \frac{T_b}{2} \, \text{sinc}\left(\pi \frac{T_b}{2} f\right)$$

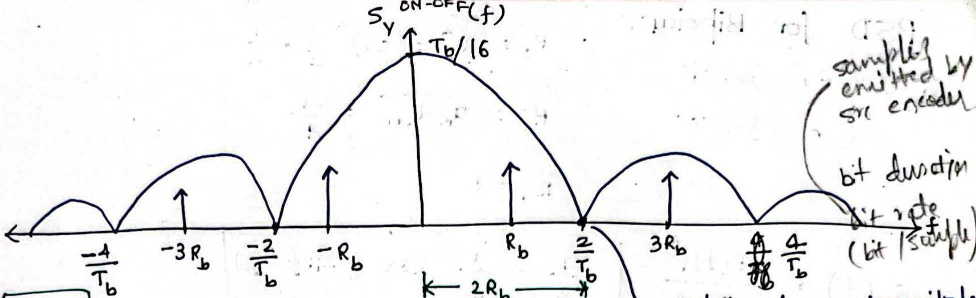
$$|P(f)|^2 = \frac{T_b^2}{4} \, \text{sinc}^2\left(\pi \frac{T_b}{2} f\right)$$

$$\rightarrow S_Y^{\text{ON-OFF}}(f) = \frac{T_b^2}{4T_b} \, \text{sinc}^2\left(\pi \frac{T_b}{2} f\right) \left[\frac{1}{2} + \sum_{n \neq 0} \frac{1}{4} \exp(j 2\pi f n T_b) \right]$$

$$= \frac{T_b}{16} \, \text{sinc}^2\left(\pi \frac{T_b}{2} f\right) \left[1 + \sum_{n=-\infty}^{\infty} \exp(j 2\pi n f T_b) \right]$$

$$= \frac{T_b}{16} \, \text{sinc}^2\left(\frac{\pi}{2} T_b f\right) \left[1 + \frac{1}{T_b} \sum \delta\left(f - \frac{n}{T_b}\right) \right]$$

Fourier series represent of $\sum \delta\left(f - \frac{n}{T_b}\right)$ is $\sum \exp(j 2\pi n f T_b)$.



sampling emitted by src encoder

bit duration

bit rate (bit/sample)

$\boxed{R_b = \frac{1}{T_b}}$

So, $S_y(f)$ consists of both continuous (sync) and discrete (impulse) part

Null-to-Null BW

Ideally, BW should be infinite, but we look at null to null BW. We consider BW to ~~consi~~ have main lobe in sinc function.

$B_T = \frac{B_W}{2}$ (min BW require

channel BW.

$BW_{ON-OFF} = 2R_b = 4B_T$ if pulse width is $T_b/2$.

$BW_{ON-OFF} = R_b = 2B_T$ if pulse width is T_b if For full T_b

where, $B_T = \frac{R_b}{2}$ B_T : Minimum channel B.W.

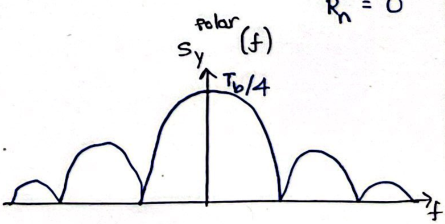
We need to increase pulse width to make it more closer to B_T.

More power eff. $\because$ sinc amplitude is less

PSD for Polar

Polar $\Rightarrow a_k = \begin{cases} +1, & \text{for '1'} \\ -1, & \text{for '0'} \end{cases} \Rightarrow R_n = \overline{a_k a_{k+n}} \Rightarrow \begin{matrix} R_0 = 1 \\ R_n = 0 \end{matrix}$

$$S_y^{polar}(f) = \frac{T_b}{4} \operatorname{sinc}^2\left(\pi \frac{T_b}{2} f\right)$$



BW is same as ON-OFF

Scanned with CamScanner

PSD for Bipolar.

$$a_k = \begin{cases} \pm 1 & \text{for '1'} \\ 0 & \text{for '0'} \end{cases}$$

$$R_0 = \widetilde{a_k^2} = (\pm 1)^2 \frac{1}{2} = \frac{1}{2}$$

$$R_1 = \widetilde{a_k a_{k+1}} = \frac{-1}{4}$$

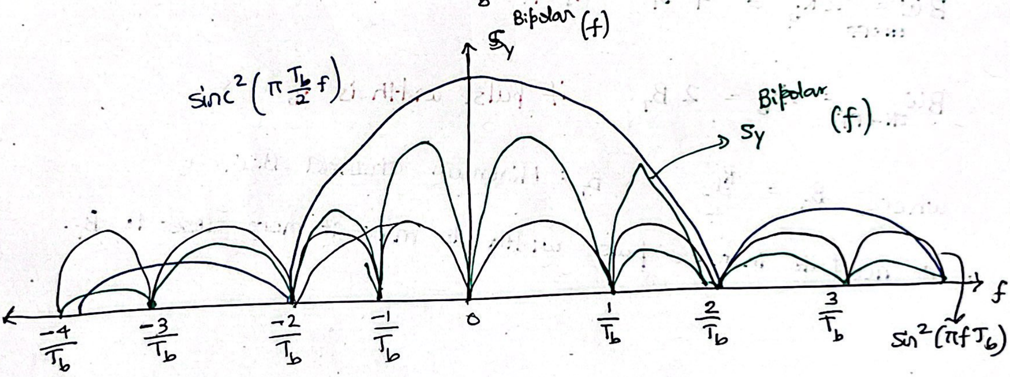
$$R_n = 0$$

$$S_y(f) = \frac{|P(f)|^2}{T_b} \left[\frac{1}{2} - \frac{1}{2} \cos(2\pi f T_b) \right]$$

$$= \frac{|P(f)|^2}{2T_b} \left[1 - \cos(2\pi f T_b) \right]$$

$$= \frac{T_b}{4} \underbrace{\text{sinc}^2\left(\pi \frac{T_b}{2} f\right)}_{\hookrightarrow \pm \frac{n2}{T_b}} \underbrace{\sin^2(\pi f T_b)}_{\hookrightarrow \frac{\pm n}{T_b}}$$

Values at which these terms become zero.



$$BW = R_b$$

ASSIGNMENT – 2

Q1. To show periodic rectangular waveform with period T_p can be used as carrier to generate DSB-SC signal.

Sol^n:

→ $s(t)$ is a periodic signal with period T_p.. Let $f_p = \frac{1}{T_p}$

$$s(t) = \sum_n x(t - nT_p)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_p t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_p t)$$

(Fourier Series Representation)

→ As $s(t)$ is an odd signal, $a_n = 0 \quad \forall n \geq 1$.

→ $s(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_p t)$

→ $a_0 = \frac{1}{T_p} \int_{\langle T_p \rangle} s(t)\, dt = \frac{1}{T_p}\left[\int_0^{T_p/2} 1\, dt + \int_{T_p/2}^{T_p} (-1)\, dt\right] = 0$

→ $b_n = \frac{2}{T_p}\left[\int_0^{T_p} s(t) \sin(2\pi n f_p t)\, dt\right]$

$$= \frac{2}{T_p}\left[\int_0^{T_p/2} \sin(2\pi n f_p t)\, dt + \int_{T_p/2}^{T_p} -\sin(2\pi n f_p t) dt\right]$$

$$= \frac{2}{T_p}\left[\frac{\cos(2\pi n f_p t)\Big|_{T_p/2}^{0}}{2\pi n f_p} + \frac{\cos(2\pi n f_p t)\Big|_{T_p/2}^{T_p}}{2\pi n f_p}\right]$$

$$= \frac{2}{T_p}\left[\frac{1 - \cos(\pi n f_p T_p) + \cos(2\pi n f_p T_p) - \cos(\pi n f_p T_p)}{2\pi n f_p}\right]$$

$$= \frac{2}{T_p}\left[\frac{2 - 2\cos(n\pi)}{2\pi n f_p}\right] = \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

$$= \begin{cases} 0 & , \text{if 'n' is odd} \\ 4/n\pi & , \text{if 'n' is even} \end{cases}$$

$$\rightarrow s(t) = \frac{4}{\pi} \sum_{n\,\in\,odd} \frac{\sin(2\pi n f_p t)}{n}$$

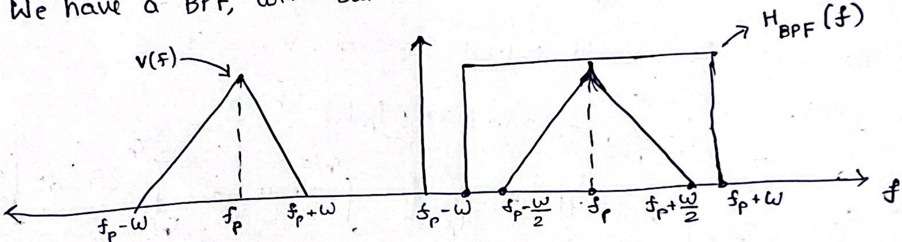
$$\rightarrow v(t) = m(t)\, s(t)$$

$$= m(t) \cdot \frac{4}{\pi} \sum_{n\,\in\,odd} \frac{\sin(2\pi n f_p t)}{n}$$

$$\rightarrow V(f) = \frac{4}{\pi}\left[\frac{M(f-f_p) - M(f+f_p)}{2j} + \frac{M(f-3f_p) - M(f+3f_p)}{2j} + \dots\right]$$

(Fourier Transform)

→ We have a BPF, with bandwidth 2W, tuned to f_p.



→ Only a specific component of V(f) will be filtered out.

$$\rightarrow V(f) = \frac{M(f-f_p) - M(f+f_p)}{2j} \quad (\text{assuming } W \ll f_p)$$

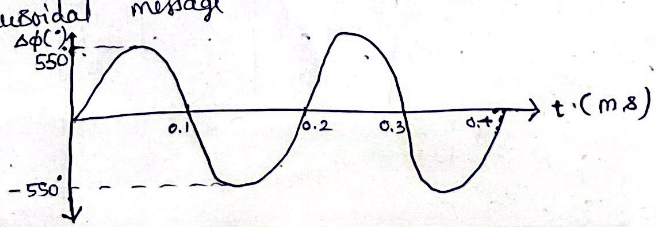
$$\rightarrow v(t) = m(t) \sin(2\pi f_p t)$$

$$\rightarrow u(t) = m(t) \sin(2\pi f_p t)$$

→ As, $v(t) = u(t)$, same DSB-SC signals are produced.

Hence, proved.

Q2. Phase deviation of Band pass FM signal modulated by a sinusoidal message



Sol^n

→ Amplitude of phase deviation, $\Delta\phi \approx 550^\circ \approx 3\pi$

→ Time period of phase deviation, $\Delta\phi \approx 0.2ms = T_p$. $\quad f_p = \frac{1}{T_p} = 5kHz$

→ We can write phase of frequency modulated signal as follows:

$$\phi_{FM}(t) = 2\pi f_p t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$$

→ Hence, the phase difference would be:

$$\Delta\phi_{FM}(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\,d\tau \quad — ①$$

$$= \frac{2\pi k_f (A_m \sin\omega_m t)}{\omega_m}$$ (Assuming $m(t) = A_m \cos\omega_m t$ as it is sinusoidal)

$$= \frac{A_m K_f \sin\omega_m t}{f_m}$$

$$= \alpha \sin\omega_m t$$ where, $\alpha = \frac{A_m K_f}{f_m}$

→ Also, $f_p = f_m = 5\ kHz$.

(i) Modulation Index

It is equal to amplitude of phase deviation $= 3\pi$

(Estimating to integer multiple of π)

(ii) Message Bandwidth

$f_m = 5\ kHz$

(iii). Bandwidth of FM signal

$$B_{FM} = 2(BW)(\beta+1)$$ (Carson's formula)

$$= 2 \times 5 \cdot (3\pi + 1)\ kHz$$

$$\approx 104\ kHz.$$

Q3. $M(f) = \begin{cases} j2\pi f & |f| < 1 \\ 0 & \text{else} \end{cases}$

$u(t) = A \cos(2\pi f_c t + \phi(t))$

K_f of FM modulator is 1.

Soln.

(a) Finding $\phi(t)$

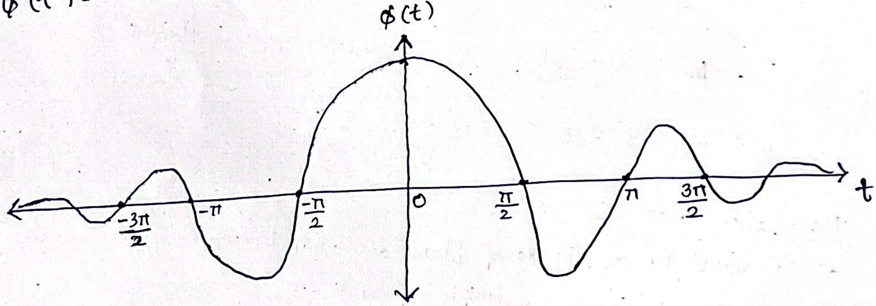
$\rightarrow \phi(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau$

$\rightarrow \Phi(f) = 2\pi K_f \times \frac{1}{j2\pi f} \times j2\pi f = 2\pi \;\forall\; |f| < 1.$

(Fourier Transform)

$\rightarrow \phi(t) = 4\pi \operatorname{sinc}(2t)$ (Inverse Fourier Transform)

$\rightarrow$



(b) Magnitude of instantaneous frequency deviation from carrier at $t = \frac{1}{4}$.

$\rightarrow \Delta f_{FM}(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = \frac{1}{2\pi} \times 4\pi \times \frac{d(\operatorname{sinc}(2t))}{dt}$

$= 2 \frac{d}{dt}\left(\frac{\sin 2\pi t}{2\pi t}\right)$

$= \frac{2}{2\pi}\left(\frac{2\pi \cos(2\pi t)\, t - \sin(2\pi t)}{t^2}\right)$

$$\rightarrow \Delta f_{FM}\left(t=\frac{1}{4}\right) = \frac{1}{\pi}\left(\frac{2\pi \cos\left(\frac{\pi}{2}\right) \times \frac{1}{4} - \sin\left(2\pi \times \frac{1}{4}\right)}{\left(\frac{1}{4}\right)^2}\right)$$

$$= \frac{1}{\pi}\left[-16\right]$$

$$\therefore \left|\Delta f_{FM}\left(t=\frac{1}{4}\right)\right| = \frac{16}{\pi} \approx 5.09$$

(c) Estimating B.W. of $u(t)$

$\rightarrow$ Assuming $\max(\Delta f_{FM}(t)) = \left|\Delta f_{FM}\left(t=\frac{1}{4}\right)\right|$

$$\rightarrow \beta = \frac{\max(\Delta f_{FM}(t))}{f_m} = 5.09$$

(Carlson's Method)

$$\rightarrow \text{B.W.} = 2B(\beta+1)$$
$$= 2 \times 1 (5.09+1)$$
$$= 12.18 \text{ Hz}$$

Q4. $p(t) = I_{[-\frac{1}{2},\frac{1}{2}]}(t) \rightarrow$ rectangular pulse of unit duration

$$m(t) = \sum_{n=-\infty}^{\infty} (-1)^n p(t-n).$$

$$u(t) = 20 \cos(2\pi f_c t + \phi(t))$$

$$\phi(t) = 20\pi \int_{-\infty}^{t} m(\tau)\, d\tau + a \qquad \text{s.t. } \phi(0) = 0$$

Sol^n.

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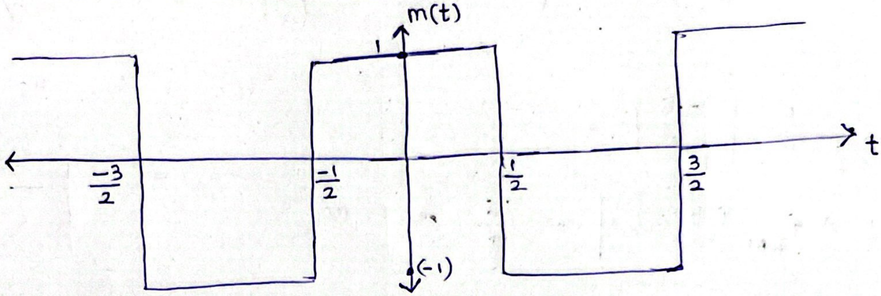
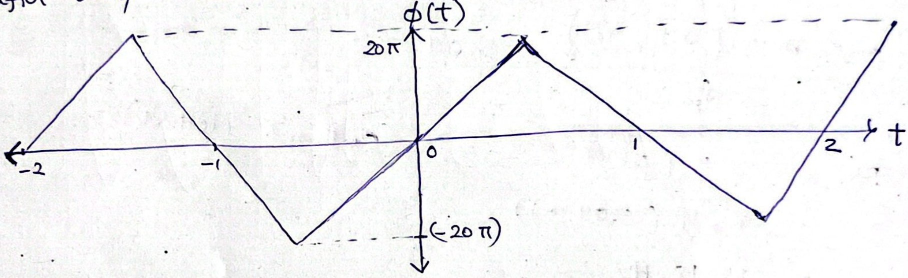
(a). Sketch of m(t) and $\phi(t)$



As $m(t) = \sum_{n=-\infty}^{\infty} (-1)^n \, p(t-n)$, we will observe +1, -1, changes after every 1 second interval as shown.



$$\phi(t) = 20\pi \int_{-\infty}^{t} m(\tau)\, d\tau + a \quad \text{st. } \phi(0) = 0.$$

We need to integrate m(t) which is rectangular wave. Hence, a ramp $\phi(t)$ is to be formed as shown.

(b). B.W of m(t) ~~and $\phi(t)$~~ , $W \approx 2$.

$\rightarrow$ $f(t) = \frac{1}{2\pi} \frac{d\phi}{dt} \Rightarrow -10 \leq f(t) \leq 10$

$\therefore \Delta f_{max} = 10$ Hz

$\rightarrow$ $BW_{u(t)} = 2 \left[BW_{m(t)} + \Delta f_{max} \right]$ (Carson's formula)

$= 2 \left[2 + 10 \right] = 24$ Hz.

$\rightarrow$ Hence, BW of u(t) is 24 Hz

(c) 4

→ Narrow ideal BPF placed at $f_c + \alpha$.

→ As $m(t)$, $\phi(t)$ and complex envelope $e^{j\phi(t)}$ are all periodic, we can find F.S coefficients of $e^{j\phi(t)}$ as kf_m, where f_m is fundamental frequency $= \frac{1}{2}$.

→ Hence, only integral multiples of $f_m = \frac{1}{2}$ would get non-zero component.

→ Hence, only $\alpha = 0.5$ and $\alpha = 1$ would give non-zero component output at BPF where $\alpha = 0.75$ would give zero component.

Q5.

(a) Sketch of $\Psi_{FM}(t)$ and $\Psi_{PM}(t)$, of modulating signal $m(t)$, a periodic sawtooth signal.

$\omega_c = 2\pi 10^6$, $K_f = 2000\pi$, $K_p = \frac{\pi}{2}$

→ $f_i = f_c + \frac{K_f}{2\pi} m(t) = 10^6 + 1000\, m(t)$

Hence, $1001\ kHz \geq f_i \geq 999\ kHz$. $(\because -1 \leq m(t) \leq 1.)$

→ For sketching PM signal:

→ $\phi_{PM}(t) = \cos\left[2\pi f_c t + K_p m(t)\right]$

$= \cos\left[2\pi(10^6 + 500)t\right]$ $(\because m(t) = 2000t)$

→ At discontinuity, we need to use indirect method.

At discontinuity, carrier frequency is not constant throughout.

Phase discontinuity $= K_p \times 2 = \pi$.

→ So, $K_p < \pi$ is kept to avoid difficulty in modulation.

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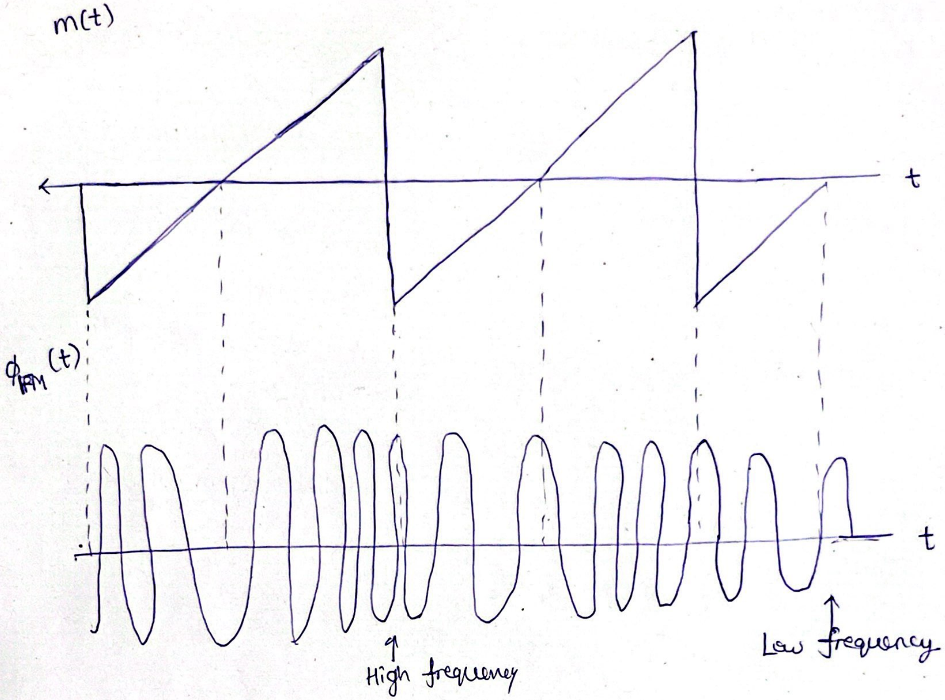
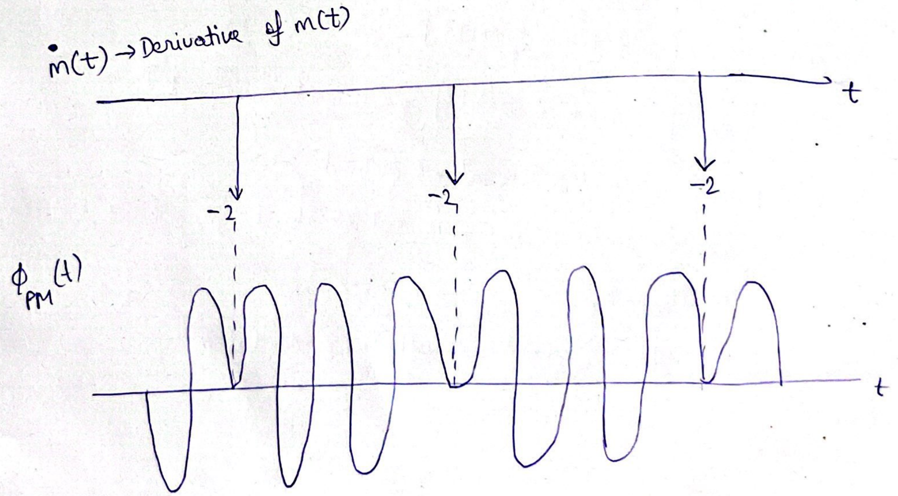

m(t)
t
φFM(t)
t
High frequency
Low frequency


ṁ(t) → Derivative of m(t)
t
-2
-2
-2
φPM(t)
t

(b). B.W of $\psi_{FM}(t)$ and $\psi_{PM}(t)$

→ Assuming B.W of modulating signal till the fifth harmonic of m(t)

$$B.W_{m(t)} = 5 f_m = 5 \times 10^3 \text{ Hz} = 5 \text{ kHz}.$$

→ $$\Delta f_{FM} = \frac{K_f}{2\pi} m_p = 1000 = 1 \text{kHz}$$

→ $$B.W_{FM} = 2(\Delta f_{FM} + BW_{m(t)})$$

$$BW_{FM} = 2(1+5) = 12 \text{ kHz}$$

→ ~~$B.W_{PM} =$~~

→ $$\Delta f_{PM} = \frac{K_p}{2\pi} \dot{m}_p = \frac{\pi}{2\pi \times 2} \times 2 \times 10^3 = 0.5 \text{ kHz}$$

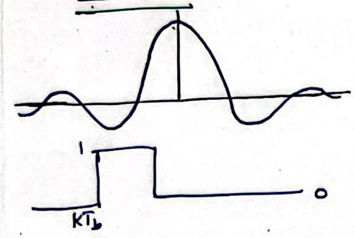
→ $$B.W_{PM} = 2(\Delta f_{PM} + BW_{m(t)})$$

$$B.W_{PM} = 2(0.5+5) = 11 \text{ kHz}$$

$$B.W_{PM} = 11 \text{ kHz}$$

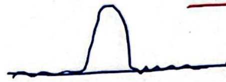
ISI

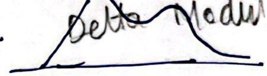
30 March 2023 Mid Sem to Delta Modul.

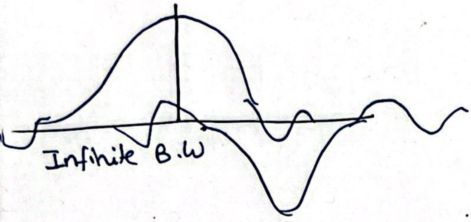


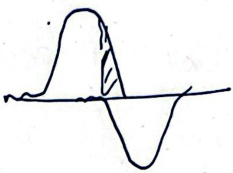
→ channel $B.W = 2R_b$

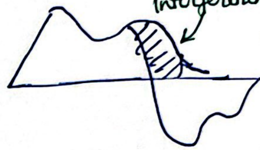
B.W limited Channel.



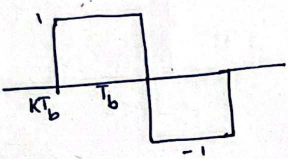








ISI: Inter Symbolic Interference



Bandwidth limited pulse → Pulse overlapping in time → ISI before even folding to channel

Time limited pulse → Infinite B.W → Output of B.W limited Channel causing plus spreading in time → ISI at output.

Avoiding ISI

→ $$\boxed{P(t) = \begin{cases} 1 & \text{at } t = 0 \\ 0 & \text{at } t = \pm nT_b \end{cases}}$$

Infinite time duration pulse

B.W limited pulse.

$P(t)$ can take any value at 't' other than $\pm nT_b \cup \{0\}$.

→ $$Y(t) = \sum a_K \, p(t - KT_b)$$

$$= a_l \, P(t - lT_b) + \sum_{K \neq l} a_K \, P(t - KT_b)$$

→ $$Y(lT_b) = a_l$$

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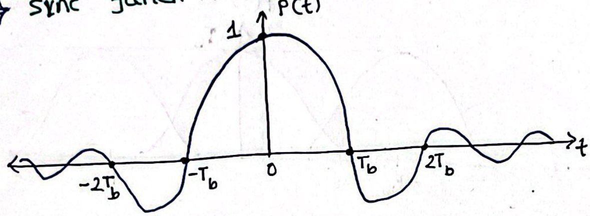
→ $B_T = \frac{R_b}{2}$

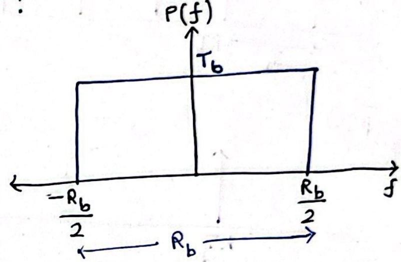
Min Channel B.W to transmit bits sampled at rate R_b.

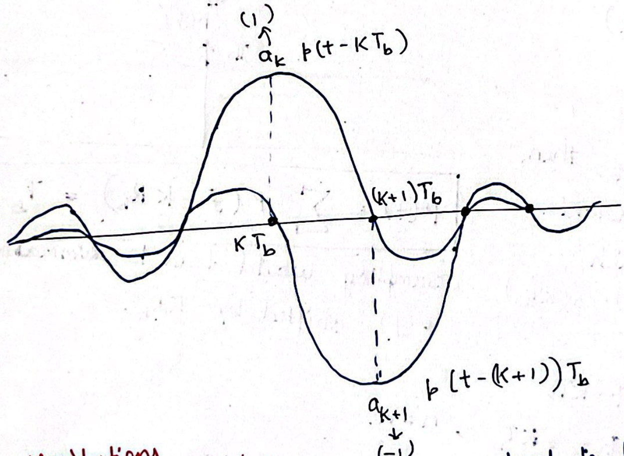
→ So, $\boxed{BW \text{ of } p(t) \leq B(t).}$

As now pulse is associated with each bit now. p(t) should also have the same constraint of B.W so that it is transmitted effectively.

→ sinc function have both the property !



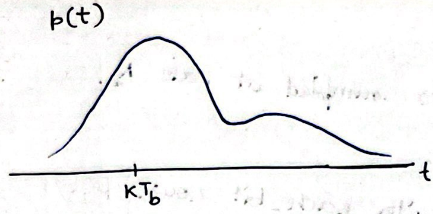


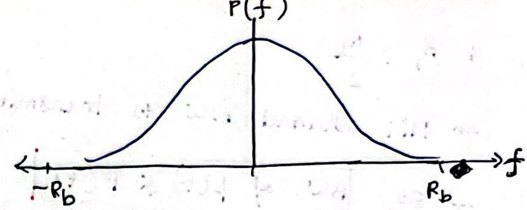


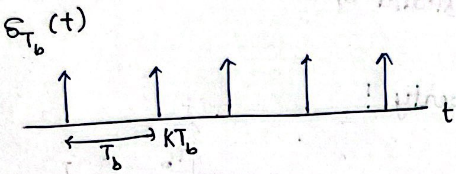
Limitations

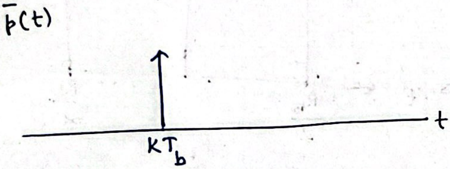
① Little bit error in sampling can lead to high levels of ISI.

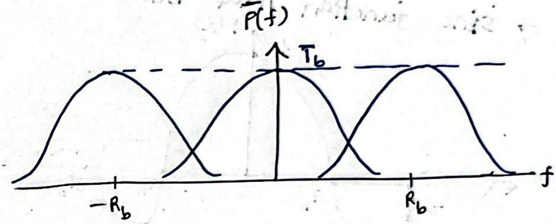
$\because \ \text{sinc}\,x = \frac{\sin x}{x}$ $\sum \frac{1}{x}$ → large value

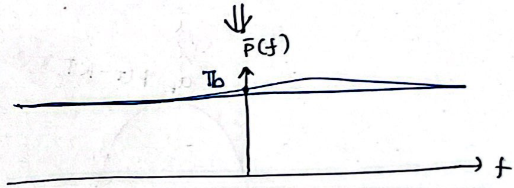












$\bar{p}(t) = p(t) \times \delta_{T_b}(t)$

If we choose $\bar{P}(f) = T_b$, then we would get a constant amplitude of T_b which would lead to $\bar{p}(t) = \delta(KT_b)$ which we want.

$$\boxed{\bar{P}(f) = \sum P(f - KR_b) = T_b}$$

Sampling would lead to summation of $P(f)$ shifted by KR_b.

$$\bar{p}(t) = \begin{cases} 1, & \text{for } t = KT_b \\ 0, & \text{for } t = \pm KT_b, \; n \neq K. \end{cases}$$

Over any band (frequency) only two pulse are overlapping.

$$T_b = P(f + 0.5R_b) + P(f - 0.5R_b)$$

1) $p(t) = 1 \quad t = 0$

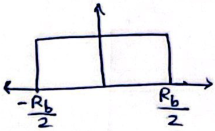
$\quad\quad 0 \quad t = \pm n T_b$

$m(t) = \sum a_K p(t - K t_b)$

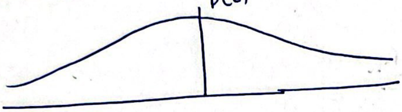
2) $B \leq \frac{R_b}{2}$

$p(t) = R_b \operatorname{sinc}(\pi R_b t)$

$P(f) = \Pi\left(\frac{f}{R_b}\right)$



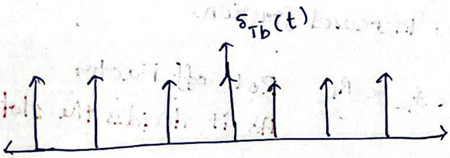
We will relax the second criteria.

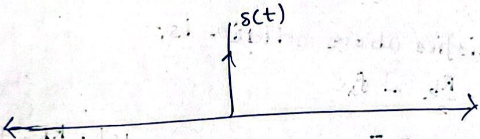


$\overline{p(t)} = p(t)\, \delta_{T_b}(t) = \delta(t)$

$R_b \sum_i P(f - K R_b) = 1$

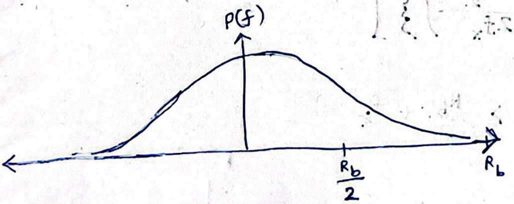
$$\boxed{\sum P(f - K R_b) = T_b}$$





$P\left(f + \frac{R_b}{2}\right) + P\left(f - \frac{R_b}{2}\right) = T_b$

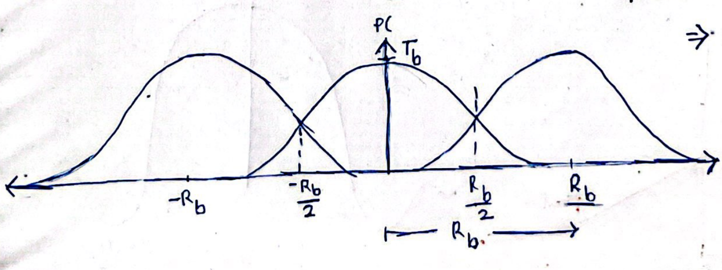
$\left|P\left(f + \frac{R_b}{2}\right)\right| + \left|P\left(-f + \frac{R_b}{2}\right)\right| = T_b$

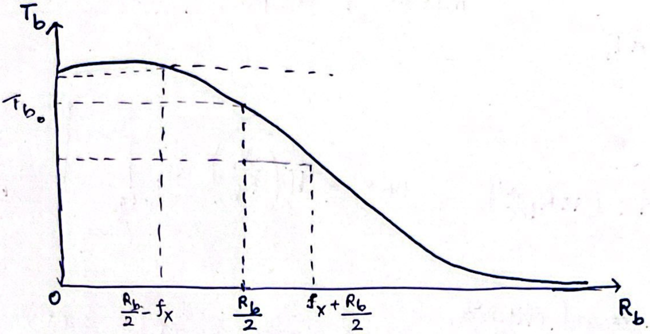


$p(t) \Rightarrow$ Real

$P(f) \Rightarrow P(f) = P^*(-f)$

$\Rightarrow |P(f)| = |P(-f)|$





$$\left|P\left(\frac{R_b}{2}\right)\right| = \frac{1}{2}|P(0)|$$

Let say sinc = 0 at $B.W = f_x + \frac{R_b}{2}$

B.W increased from $\frac{R_b}{2}$ to $f_x + \frac{R_b}{2}$.

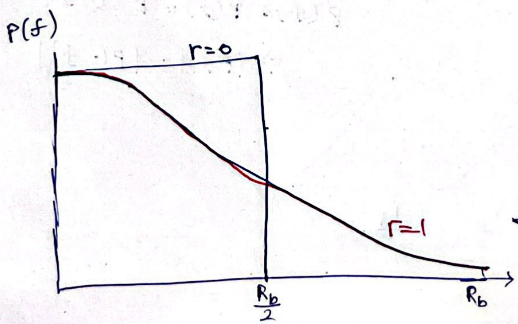
$B.W. = \frac{R_b}{2}(1+r)$ where r : Increased fraction.

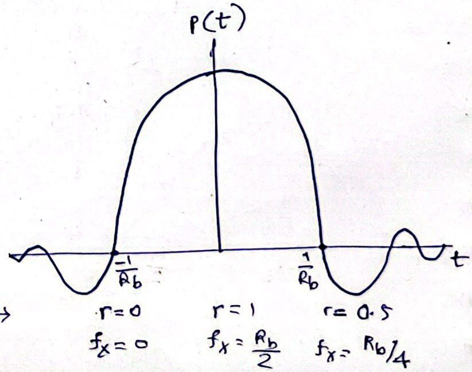
$\boxed{r = 2 f_x T_b}$ $0 < r < 1$, $0 < f_x < \frac{R_b}{2}$ Roll off Factor

As it decides the slope

The family of pulses which satisfies above criteria is:

$$P(f) = \begin{cases} 1 & \text{for } |f| \leq \frac{R_b}{2} - f_x \\ \frac{1}{2}\left[1 - \sin\left\{\pi\left(\frac{f - \frac{R_b}{2}}{2f_x}\right)\right\}\right] & \text{for } \frac{R_b}{2} - f_x \leq |f| \leq \frac{R_b}{2} + f_x \\ 0 & \text{for } |f| \geq \frac{R_b}{2} + f_x \end{cases}$$





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For $r = 1$

$$P(f) = \frac{1}{2}\left[1 - \sin\left(\pi f R_b - \frac{\pi}{2}\right)\right] \qquad 0 < |f| \leq R_b$$

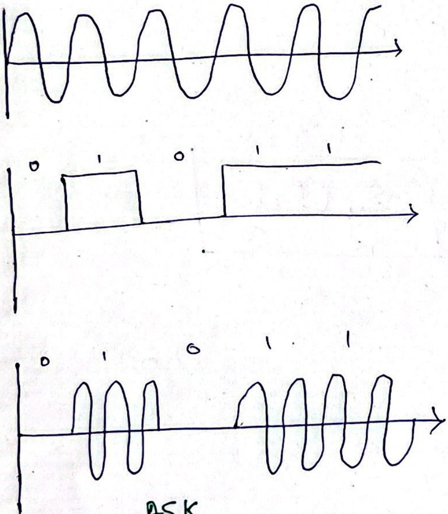
$$= \frac{1}{2}\left[1 + \cos\left(\frac{\pi f}{R_b}\right)\right] \Pi\left(\frac{f}{R_b}\right)$$

$$= \cos^2\left(\frac{\pi f}{2R_b}\right) \Pi\left(\frac{f}{R_b}\right)$$

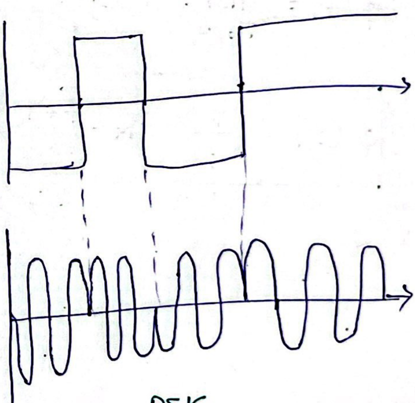
$$\boxed{p(t) = \frac{R_b \cos(\pi t R_b)}{1 - \pi R_b^2 f^2} \; \frac{\sin(\pi t R_b)}{\pi t R_b}}$$ Raised Cosine Pulse.

Baseband Modulation

$$m(t) = \sum a_K \, p(t - K T_b)$$



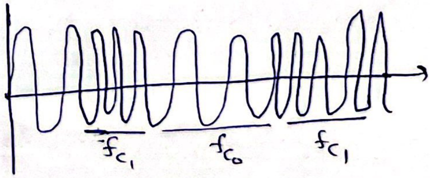
ASK



PSK

$$a_{K_{PSK/ON\text{-}OFF}} = \begin{cases} 1 & \text{for '1'} \\ 0 & \text{for '0'} \end{cases}$$

$$\phi_{ASK}(t) = \begin{cases} \cos(2\pi f_c t) & \text{for '1'} \\ 0 & \text{for '0'} \end{cases}$$



FSK

$$\phi_{PSK} = \begin{cases} \cos(2\pi f_c t) & \text{for '1'} \quad 0 \leq t \leq T_b \\ -\cos(2\pi f_c t) & \text{for '0'} \end{cases}$$

$$\phi_{FSK} = \begin{cases} \cos(2\pi f_{c_1} t) & \text{for '1'} \quad 0 \leq t \leq T_b \\ \cos(2\pi f_{c_0} \cdot t) & \text{for '0'} \end{cases}$$

$$m(t) = \sum_i a_K \, p(t - KT_b) \qquad \begin{cases} \text{ON-OFF} & - \text{ASK} \\ \text{Polar} & - \text{PSK} \end{cases}$$

$$\phi(t) = m(t) \cos(2\pi f_c t)$$

$$\psi(f) = \frac{M(f + f_c) + M(f - f_c)}{2}$$

$$\phi(f) = \lim_{T \to \infty} \frac{|\psi_T(f)|^2}{T}$$

$$\boxed{\phi(f) = \frac{1}{4} S_M(f + f_c) + \frac{1}{4} S_M(f - f_c)}$$

ASSIGNMENT-3

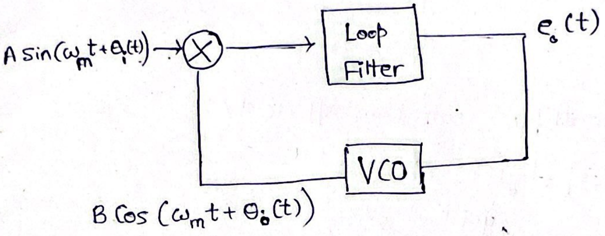
Q1. PLL Realisation and response plot

(i). Mixer as phase detector

Sol^n

→ Let message signal be $A \sin[\omega_m t + \theta_i(t)]$,

→ First, lets draw PLL Block Diagram.



→ We can obtain VCO output as above as VOC is an electric oscillator whose oscillation frequency is controlled by a voltage input.

$$\omega_i(t) = \omega_m + c\, e_o(t)$$

$$\omega_i(t) = \omega_m + \dot{\theta}_o(t) \qquad \left(\because e_o(t) = \frac{1}{c}\dot{\theta}_o(t) \text{ as } \theta_e(t) \approx 0\right)$$

$$\dot{\theta}_o(t) = c' e_o(t)$$

→ Hence, output of the multiplier can be written as:

$$AB \sin(\omega_m t + \theta_i(t)) \cos(\omega_m t + \theta_o(t))$$

$$= \frac{AB}{2} \sin(\theta_i(t) - \theta_o(t)) + \underbrace{\sin(2\omega_m t + \theta_i(t) + \theta_o(t))}_{\text{Suppressed by LPF.}}$$

LPF is designed st. it suppresses the higher frequency component.

→ So, the final output of multiplier after passing through LPF is :

$$\frac{AB}{2} \sin[\theta_i(t) - \theta_o(t)]$$

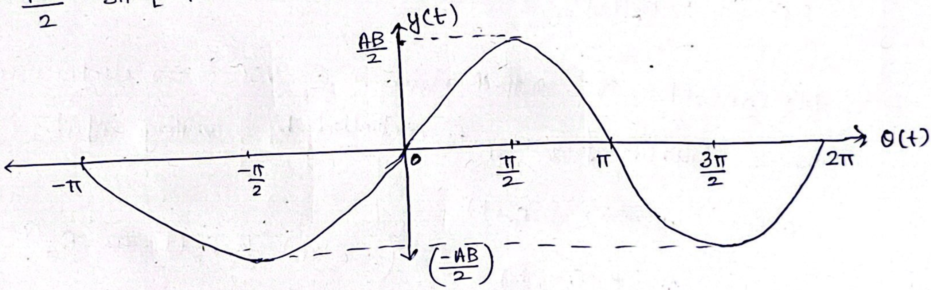
→ Out of Loop Filter with transfer function $H(s)$ and impulse response $h(t)$.

$$e_o(t) = h(t) * \frac{AB}{2} \sin[\theta_i(t) - \theta_o(t)]$$

→ Response Plot

As seen above, the multiplier output is effectively :

$$\frac{AB}{2} \sin[\theta_i(t) - \theta_o(t)] = y(t)$$



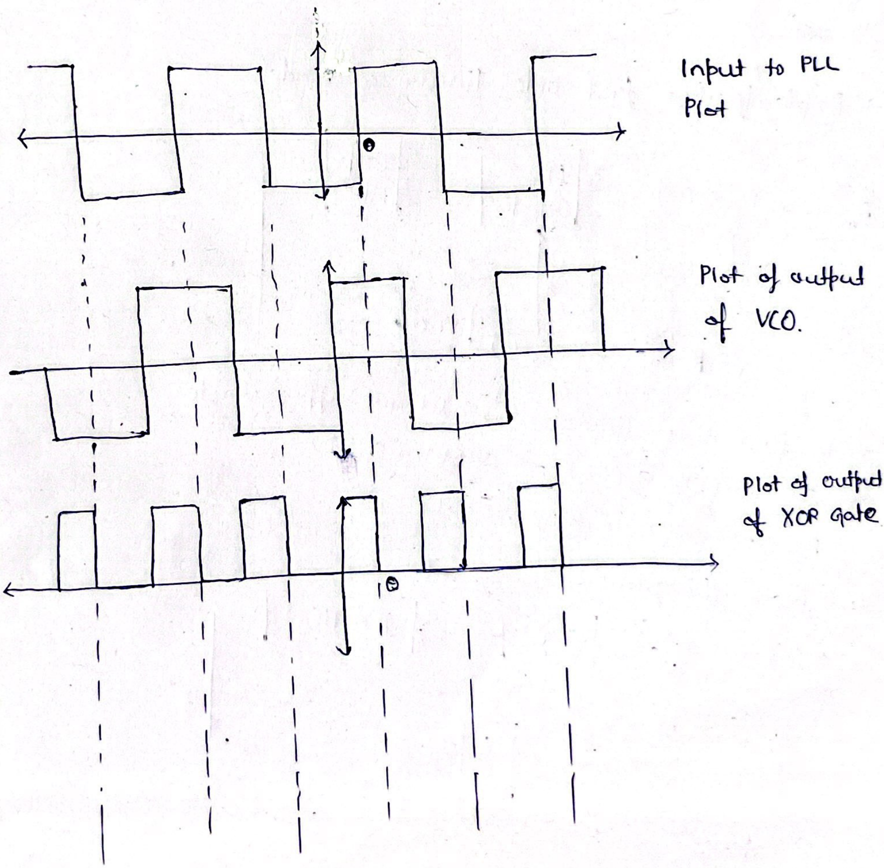
where, $\theta(t) = \theta_i(t) - \theta_o(t)$

(b). XOR Gate as Phase Detector

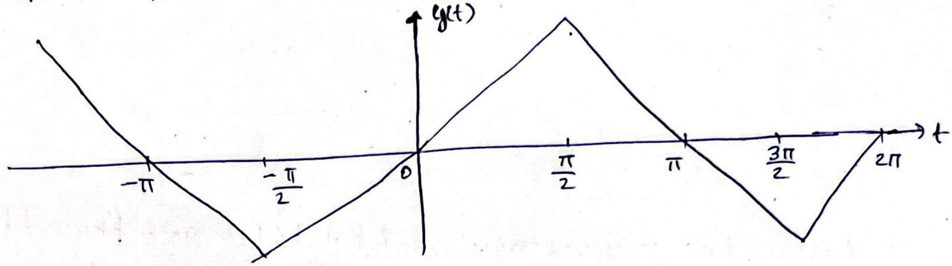
→ Truth Table for XOR Gate

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0

→ Let us take a square wave as input to PLL with phase difference ϕ. And output of VCO again a square wave with period of π. Then, we can plot the following:



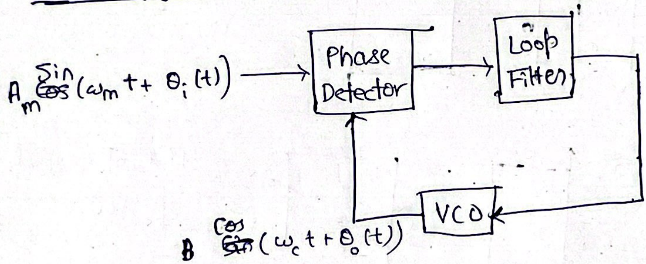
→ We can also plot the average value of output shifted by appropriately so that it passes through origin;



i.e. The average value of output is linear in nature

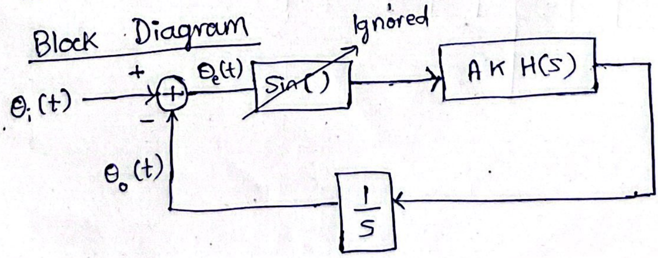
<u>Q2. Solⁿ.</u>

→ <u>Explaining PLL first-order linearized model</u>



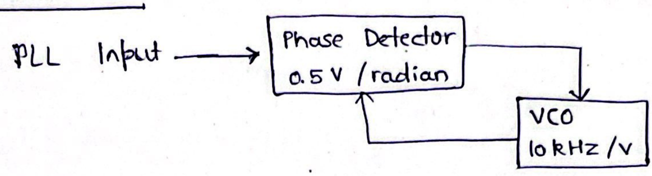
→ In first order linearized model we approximate
$\sin(\theta_i(t) - \theta_o(t)) \approx \theta_i(t) - \theta_o(t)$
As the angle is very small

→ <u>Block Diagram</u>



Linearized first-order model

→ Numerical



→ Input jumps by e = 2.72 rad at t = 0

→ $\theta_i(t) = e \xrightarrow[\text{Transform}]{\text{Laplace}} \Theta_i(s) = \frac{e}{s}$.

→ $\Theta_e(s) = \frac{s}{s+K} \cdot \frac{e}{s} \xrightarrow[\text{Transform.}]{\text{Inverse Laplace}} \theta_e(t) = e^{1-Kt}$

→ Difference between PLL input phase and VCO output phase shrink to 1 rad. say at $t = t_0$.

So, $\theta_e(t_0) = 1 = e^{1-Kt_0} \rightarrow t_0 = \frac{1}{K}$

→ Hence, $\boxed{t_0 = \frac{1}{5\,kHz} = 0.2\,ms}$ (as $K = A_c A_v = 5\,kHz$)

→ Limiting value of phase error if frequency jumps by 1kHz just after t=0.

→ $\theta_i(s) = \frac{2\pi\,\Delta f}{s^2}$

→ $\Theta_e(s) = H_e(s) \cdot \Theta_i(s)$

$= \frac{2\pi\,\Delta f}{s^2} \times \frac{s}{s+K} = \frac{2\pi\,\Delta f}{s(s+K)}$

$$\rightarrow \lim_{s\to 0} s\,\Theta_e(s) = \frac{2\pi\Delta f}{K} \qquad \text{(Final Value Theorem)}$$

$$= \frac{2\pi}{5} \text{ rad.}$$

$\rightarrow$ Hence, $\boxed{\lim_{t\to\infty} \theta_e(t) = \frac{2\pi}{5} \text{ rad.}}$

AM signal with modulation index = 1. Demodulation: $\begin{cases}\text{Coherent Demodulation}\\ \text{Envelope Detector}\end{cases}$

Line coding

ASK $\phi(t) = m(t)\cos(2\pi f_c t)$ $\qquad m(t) = \sum a_K\, p(t - KT_b)$ $\qquad$ ON-OFF

PSK $\phi(t) = A\cos(2\pi f_c t + \theta_K)$ $\qquad \theta_K = \begin{cases}\pi, & \text{for '0'}\\ 0, & \text{for '1'}\end{cases}$ $\qquad$ POLAR

$$= \pm A\cos(2\pi f_c t) = m(t)\cos(2\pi f_c t)$$

FSK $\phi(t) = \begin{cases} A\cos(2\pi f_{c_1} t) & \text{for '1'}\\ A\cos(2\pi f_{c_2} t) & \text{for '0'.}\end{cases}$

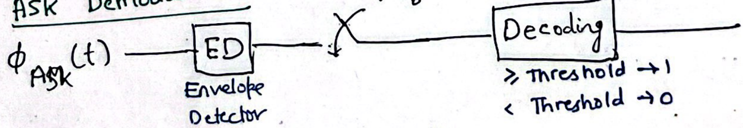
FSK as a sum of two ASK.

$$\sum \underbrace{a_K}_{\text{ON-OFF}}\, p(t - KT_b)\cos(2\pi f_{c_1} t) + \sum \underbrace{(1 - a_K)}_{\text{ON-OFF}}\, p(t - KT_b)\cos(2\pi f_{c_2} t)$$

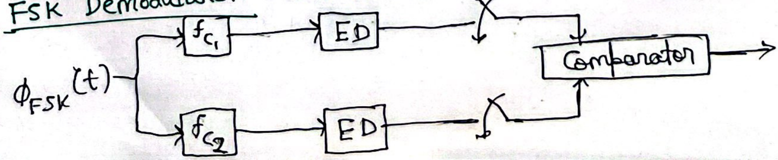
Spectrum of FSK

$$\phi(f) = \frac{1}{4}\left[S_M(f + f_c) + S_M(f - f_c)\right]$$

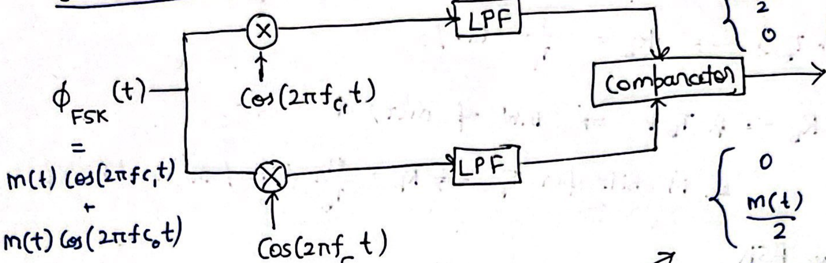
ASK Demodulation



FSK Demodulation $\qquad$ Non-Coherent



Coherent Demodulation of FSK



$\phi_{FSK}(t) = m(t)\cos(2\pi f_{c_1} t) + m(t)\cos(2\pi f_{c_0} t)$

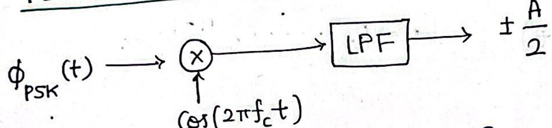
$\begin{cases} \frac{m(t)}{2} & \text{for '1'} \\ 0 & \text{for '0'} \end{cases}$

$\begin{cases} 0 & \text{for '1'} \\ \frac{m(t)}{2} & \text{for '0'} \end{cases}$

$$\frac{m(t)}{2} + \cancel{\frac{m(t)}{2}\cos(4\pi f_{c_1} t)} + \cancel{\frac{m(t)}{2}\cos\Big(2\pi\underbrace{(f_{c_1} - f_{c_0})}_{\text{Well separated frequency}}t\Big)} + \cancel{\frac{m(t)}{2}\cos\big(2\pi(f_{c_1}+f_{c_0})t\big)}$$

Filtered by LPF

PSK Demodulation

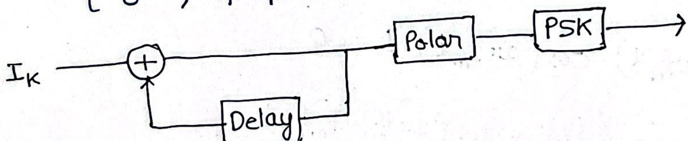


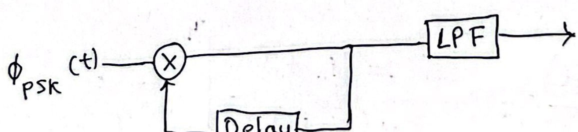
$\phi_{PSK}(t) = A\cos(2\pi f_c t + \theta_K)$ where, $\theta_K = \begin{cases} 0 & \text{for '1'} \\ \pi & \text{for '0'} \end{cases}$

$\phi_{PSK}(t)\, A\cos(2\pi f_c t) = \pm A\cos(2\pi f_c t)\cos(2\pi f_c t)$

Differential PSK — dPSK

$\theta_K = \begin{cases} \pi, & \text{if previous} \neq \text{cur} \\ 0, & \text{if previous} = \text{cur} \end{cases}$





$-\frac{A^2}{2} \Rightarrow 1$

$\frac{A^2}{2} \Rightarrow 0$

$I_K \rightarrow$	1	0	1	1	0	0	0	1	
$Q_K \rightarrow$	1	1	0	1	1	1	1	0	

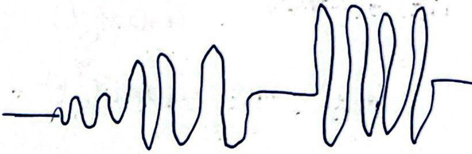
1 Bit per T_b s $\rightarrow$ $R_b = \frac{1}{T_b}$ bit/s

Increase R_b $\rightarrow$ 1). $T_b \downarrow$ $\Rightarrow$ B.W of $m(t)$ $\uparrow$

2). M bits per T_b $\Rightarrow$ $R_b = \frac{M}{T_b}$ bits/s. M-ary

M-ary ASK

$$\phi_{ASK}(t) = a_K \cos(2\pi f_c t) \quad 0 \leq t \leq T_b$$



$$a_K = \{0, A, 2A, \ldots\ldots, (N-1)A\}$$

M-ary FSK

$$\phi_{FSK}(t) = A \cos(2\pi f_n t) \qquad n = 0, 1, \ldots\ldots, n-1$$

$$f_n = f_0 + (n-1)\,\delta_F$$

$$B.W = 2\Delta f + 2\underbrace{B}_{\rightarrow \frac{1}{T_b}} \qquad \text{Carson's Rule}$$

$$\Delta f = \frac{f_N - f_0}{2} = \frac{(N-1)}{2}\delta_F$$

$\rightarrow$ Two signals are different seen if they are orthogonal.
So, we need to find Δf minimum posible for signal to be orthogonal.

$$\int_0^{T_b} A^2 \cos(2\pi f_n t)\cos(2\pi f_m t)\,dt = 0$$

$$\int_0^{T_b} \frac{A^2}{2} \cos(2\pi(f_m + f_n)t) + \cos(2\pi(f_n - f_m)t)$$

$$\frac{A^2}{2}\left[\frac{+\sin(2\pi(f_m+f_n)t)}{2\pi(f_m+f_n)} + \frac{\sin(2\pi(f_n-f_m)t)}{2\pi(f_n-f_m)}\right]_0^{T_b} = 0$$

$$\cancel{\frac{\sin(2\pi(f_n+f_m)T_b)}{2\pi\underbrace{(f_n+f_m)}_{\text{Large Freq}}}}^{\approx 0} + \frac{\sin(2\pi(f_n-f_m)T_b)}{2\pi(f_n-f_m)} = 0$$

$f_n - f_m = (n-m)\,\delta_f$ $\boxed{\delta_f = \frac{1}{2T_b}}$ optimal Choice

$\cancel{\pi = T_b\, 2\pi (f_n - f_m)}$

$$B.W = 2\Delta f + 2B = 2\,\frac{(N-1)}{2}\,\delta_f + \frac{2}{T_b}$$

$$\boxed{B.W = \frac{(N-1)}{2T_b} + \frac{2}{T_b}}$$

Scanned with CamScanner

1). optimal Design of Receiver to min BER

Bit Error Rate (BER): A measure of performance.

$\pm p(t) + n(t)$

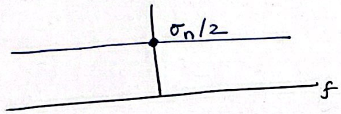
Assumed properties of $n(t)$

① Additive ② Gaussian $(0, \sigma_n^2)$ ③ $n(t_1)$ & $n(t_2)$ are independent.

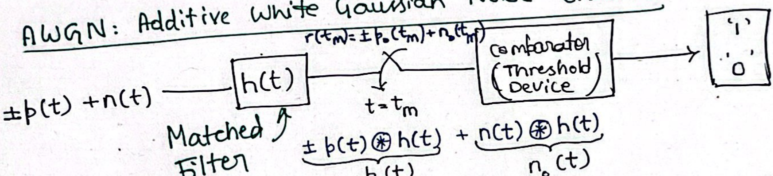
$R_n(\tau) = \sigma_n^2 \, \delta(\tau) \Big|_{\tau=0} = E[n^2(t)]$ → Whiteness

$R_n(\tau) = E[n(t+\tau)\, n(t)]$

PSD = FT$\{R_n(\tau)\}$



AWGN: Additive White Gaussian Noise Channel



$\underbrace{\pm p(t) \circledast h(t)}_{p_o(t)} + \underbrace{n(t) \circledast h(t)}_{n_o(t)}$

$n_o(t_m) = \int n(\tau)\, h(\tau - T_m)\, d\tau$

$n_o(t_m) \rightarrow N(0, \sigma_n^2)$

$n(\tau) \rightarrow$ AWGN

$E[n_o(t_m)] = 0$

$E[(n_o(t_m))^2] = \sigma_n^2$

$E(x^2) - \cancel{(E(x))^2} = Var(x)$

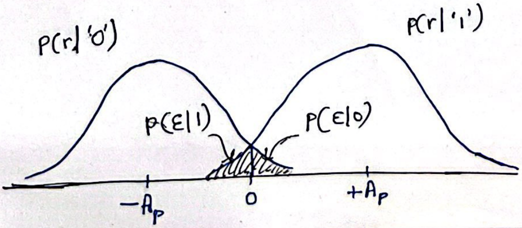
As convolution is weighted sum, $n_o(t_m)$ is again Gaussian.

$A_p = P_o(t_m)$

$r(t_m) = \pm A_p + n_o(t_m)$

$n_o(t_m) \rightarrow N\left(0, \frac{\sigma_n^2}{2}\right)$

$r(t_m) \rightarrow N(\pm A_p, \sigma_n^2)$



$\rightarrow r(t_m) > 0 \quad \Rightarrow$ bit '1'

$< 0 \quad \Rightarrow$ bit '0'

$\rightarrow P(\varepsilon) = P(\varepsilon | 1) P(1) + P(\varepsilon | 0) P(0)$ —— ①

$\rightarrow P(\varepsilon | '0') = P(r(t_m) > 0 \,|\, '0') = \int_0^\infty P(r|'0')\, dr$

$$\frac{1}{\sqrt{2\pi \sigma_n^2}} \int_0^\infty \exp\left(-\frac{(r-(-A_p))^2}{2\sigma_n^2}\right) dr \qquad \frac{r+A_p}{\sigma_n} = z$$

$$\frac{\cancel{\sigma_n}}{\sqrt{2\pi}\,\cancel{\sigma_n}} \int_{A_p/\sigma_n}^{\infty} \exp(-z^2/2)\, dz = Q\left(\frac{A_p}{\sigma_n}\right) \qquad \text{where, } Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz$$

$\rightarrow P(\varepsilon | '1') = Q\left(\frac{A_p}{\sigma_n}\right)$ Similarly as above.

As $P(\varepsilon|1) = P(\varepsilon|0) = P$ (say)

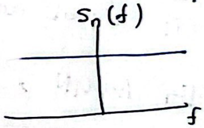
From Eqⁿ ① So, $P(\varepsilon) = P(P(1) + P(0)) = P$

$$\rightarrow \boxed{P(\varepsilon) = Q\left(\frac{A_p}{\sigma_n}\right)}$$

$\rightarrow P(\varepsilon) = Q(\rho) \qquad \rho = \frac{A_p}{\sigma_n} = \frac{p_o(t_m)}{\sqrt{E[(n_o(t_m))^2]}}$

$\rightarrow p_o(t) = p(t) \circledast h(t) \qquad n_o(t) = n(t) \circledast h(t)$

$n(t) \longrightarrow \boxed{h(t)} \xrightarrow{n_o(t)} \quad S_{n_o}(f) = S_n(f)\, |H(f)|^2$



$\rightarrow \rho^2 = \text{SNR at } t = t_m$ To maximise SNR, ρ^2 needs to be maximum.

$$\max_{H(f)} \rho^2 = \frac{[p_o(t_m)]^2}{E[n_o(t_m)]^2}$$

$$\rightarrow \sigma_n^2 = E[n_o^2(t)] = R_{n_o}(0) = \int_{-\infty}^{\infty} S_{n_o}(f) \underbrace{e^{j2\pi f \tau}}_{1 \text{ as } \tau = 0} df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 S_n(f)\, df$$

$$\rightarrow p_o(t_m) = p(t) \circledast h(t) = F^{-1}\{P(f) H(f)\}$$

$$= \int_{-\infty}^{\infty} H(f) P(f)\, e^{+j2\pi f t_m} df$$

→ Using Cauchy - Schwarz Inequality

$$\left|\int X(f)\,Y(f)\,df\right|^2 \leq \int |X(f)|^2 df \;*\; \int |Y(f)|^2 df$$

If $X(f) = c\,Y^*(f)$, then Equality will hold.

→
$$P_o(t_m) = \frac{\left|\int_{-\infty}^{\infty} H(f)\sqrt{S_n(f)}\,\frac{P(f)}{\sqrt{S_n(f)}}\,e^{j2\pi f t_m}\,df\right|^2}{\int |H(f)|^2\,S_n(f)\,df}$$

$$\leq \frac{\int_{-\infty}^{\infty} |H(f)|^2\,S_n(f)\,df \;*\; \int_{-\infty}^{\infty} \frac{|P(f)|^2}{S_n(f)}\,d(f)}{\int |H(f)|^2\,S_n(f)\,df} = \int \frac{|P(f)|^2}{S_n(f)}\,df$$

→ For equality to hold, we will get max. value

$$\boxed{H(f) = c\,\frac{P^*(f)}{|S_n(f)|}\,e^{-j2\pi f t_m} = c\,\frac{P(f)}{|S_n(f)|}\,e^{-j2\pi f t_m}}$$

As signals are conjugate symmetric.

special case: AWGN Channel.

→ For AWGN, $S_n(f) = \frac{N}{2}$

$$\boxed{\rho^2_{max} = \frac{2}{N}\int_{-\infty}^{\infty}|P(f)|^2 df}$$

$$\boxed{\rho^2_{max} = \frac{2E_p}{N}}$$

$$\boxed{\begin{aligned} H(f) &= c'\,P^*(f)\,e^{-j2\pi f t_m} \\ h(t) &= c'\,P(t_m - t) \end{aligned}}$$

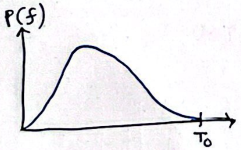
$c' = \frac{c}{S_n(f)}$

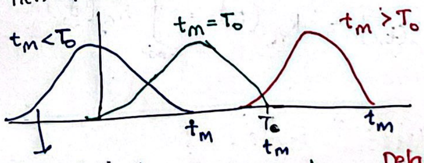
as $P^*(f) = P(-f)$

Constant c' doesn't affect SNR. As both input and noise are scaled equally

$h(t) = p(t_m - t)$

→



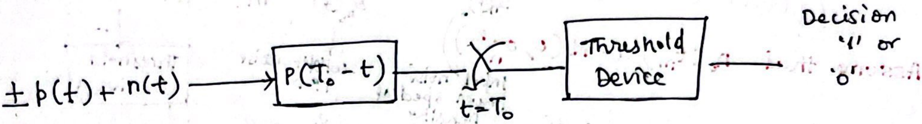


Non-Causal

Maximum of maximum options

Delayed result as $t_m - T_0$ needs to be waited for.

$$\rho = \frac{P_o(t_m)^2}{E[n_o(t_m)^2]} = \frac{2}{N}\int_{-\infty}^{\infty} p(\tau)\, h(\tau - t_m)\, d\tau$$



$$P_o(t) = \int p(\tau)\, p(\tau - T_o + t)\, d\tau$$

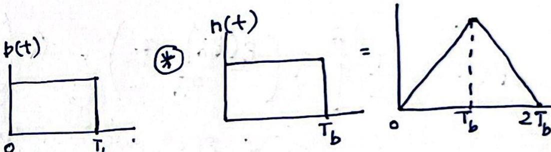
$$P_o(T_o) = \int p^2(\tau)\, d\tau \quad (\text{for } t = T_o)$$

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N}}\right)}$$

i.e. Probability of energy is dependent only on E_p. which is the energy that we put in pulses. It is independent of shape of $p(t)$.

Let noise, $n(t)$ is absent....

output = $p(t) * p(T_b - t)$ =



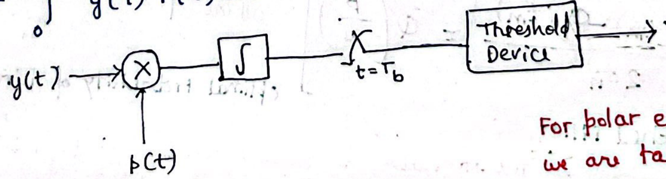
So, $T_m = T_b$ attains maximum convolution.

Alternate representation of Matched Filter

$z(t) = y(t) * h(t) = y(t) * p(T_b - t)$ = Filter output.

$$z(t) = \int_{-\infty}^{\infty} y(\tau)\, p(\tau - T_b + t)\, d\tau$$

$$z(T_b) = \int_0^{T_b} y(\tau)\, p(\tau)\, d\tau \qquad \text{Co-relation of } y(t) \text{ and } p(t).$$

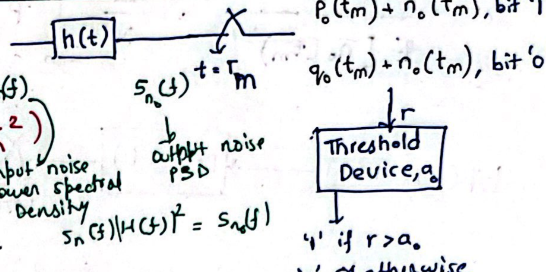


Correlator Receiver

For polar encoding ONLY, as we are taking $\pm p(t)$.

Filter for different Pulse Encoding for '1' and '0'

$$y(t) = \begin{cases} p(t) + n(t), & \text{for '1'} \\ q(t) + n(t), & \text{for '0'} \end{cases}$$



Assume that $n_o(t_m) \sim N(0, \sigma_n^2)$

input noise power spectral Density

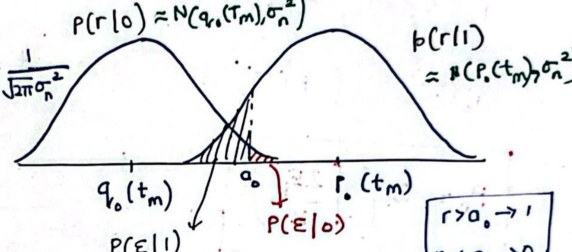
$S_n(f)|H(f)|^2 = S_{n_o}(f)$

Calculation of Probability of Error

$\rightarrow P(\varepsilon) = P(1)\, P(\varepsilon|1) + P(0)\, P(\varepsilon|0)$

$$\rightarrow P(\varepsilon|0) = \int_{a_o}^{\infty} \exp\left(-\frac{(r - (q(t_m))^2}{2\sigma_n^2}\right) dr \frac{1}{\sqrt{2\pi\sigma_n^2}}$$

$$= Q\left(\frac{a_o - q_o(t_m)}{\sigma_n}\right) = P(r > a_o | '0')$$



$r > a_o \rightarrow 1$
$r < a_o \rightarrow 0$

$$\rightarrow P(\varepsilon|1) = Q\left(\frac{p_o(t_m) - a_o}{\sigma_n}\right)$$

$$= 1 - Q\left(\frac{a_o - p_o(t_m)}{\sigma_n}\right) = Q\left(\frac{p_o(t_m) - a_o}{\sigma_n}\right) \quad (\text{as } 1 - Q(x) = Q(-x))$$

$$\rightarrow P(\varepsilon) = \frac{1}{2}\left[Q\left(\frac{p_o(t_m) - a_o}{\sigma_n}\right) + Q\left(\frac{a_o - q_o(t_m)}{\sigma_n}\right)\right]$$

$\rightarrow$ To find minimum of $P(\varepsilon)$, we can find derivative.

$$\frac{dP_e}{da_o} = 0 \Rightarrow \boxed{a_o = \frac{p_o(t_m) + q_o(t_m)}{2}}$$ optimal Threshold value

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt \qquad Q'(x) = -e^{-x^2/2}$$ Newton–Leibnitz Rule

At a_o = Average value of $p_o(t_m)$ and $q_o(t_m)$, $\boxed{P_e = P(\varepsilon|0) = P(\varepsilon|1)}$

$$\boxed{P(\varepsilon) = Q\left(\frac{p_o(t_m) - q_o(t_m)}{2\sigma_n}\right)} = Q\left(\frac{\beta}{2}\right) \quad \text{where } \beta = \frac{p_o(t_m) - q_o(t_m)}{\sigma_n}$$

optimal Probability of Error

Obtaining Matched Filter

$\max_{H(f)} \beta^2$

Q() is monotonically decreasing function, so, to minimise $Q\left(\frac{\beta}{2}\right)$ we increase β. which depends on $p_o(t_m)$ and $q_o(t_m)$ which depends on $H(f)$.

$$\beta^2 = \frac{|p_o(t) - q_o(t)|^2}{\sigma_n^2} = \frac{|(p(t) - q(t)) * h(t)|^2}{\sigma_n^2}$$

$$= \frac{\left| \int_{-\infty}^{\infty} P(f) - Q(f) \; H(f) \, e^{j 2\pi f t_m} \, df \right|^2}{\int_{-\infty}^{\infty} S_n(f) \, |H(f)|^2 \, df}$$

Quiz 2

As we want to maximise $H(f)$ value, we needed to convert time domain expression to frequency domain. So, we took FT.

last sut.

We can use Cauchy Inequality

$$\boxed{\beta^2 = \int \frac{|P(f) - Q(f)|^2}{S_n(f)} \, df}$$

$$\boxed{H(f) = \frac{P(-f) - Q(-f)}{S_n(f)} \, e^{-j 2\pi f T_b}}$$

Noise Distribution → Gaussian
PSD → Constant
} AWGN

For AWGN : $S_n(f) = \frac{N}{2}$

$$\beta^2 = \frac{2}{N} \int_{-\infty}^{\infty} |P(f) - Q(f)|^2 \, df = \frac{2}{N} \int_0^{T_b} [p(t) - q(t)]^2 \, dt$$

$$= \frac{2}{N} \left[\int_0^{T_b} p^2(t) \, dt + \int_0^{T_b} q^2(t) \, dt - 2 \underbrace{\int p(t) \, q(t) \, dt}_{E_{pq}} \right]$$

$$\frac{2}{N} \left(E_p + E_q - 2 E_{pq} \right) = \beta^2_{max}$$

$$\boxed{P_e = Q\left(\frac{\beta_{max}}{2}\right) = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right)}$$

$P(\varepsilon)$ is now not independent of shape of pulses as E_{pq} would depend on shape of the two pulses.

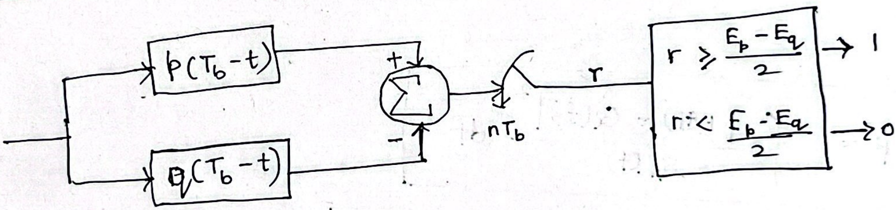
$$H(f) = P(-f)\, e^{-j2\pi f T_b} - Q(-f)\, e^{-j2\pi f T_b}$$

($S_n(f)$ was ignored as it is just a constant)

$$\boxed{h(t) = p(T_b - t) - q(T_b - t)}$$

$$\boxed{a_0 = \frac{p_0(T_b) + q_0(T_b)}{2} = \frac{1}{2}(E_p - E_q)}$$

Block Diagram for complete receiver



Polar

'1' $\rightarrow p(t)$

'0' $\rightarrow q(t) = -p(t)$

$$E_p = \int_0^{T_b} p^2(t)\,dt \qquad E_q = E_p$$

$$E_{pq} = \int_0^{T_b} p(t)\,q(t)\,dt = -E_p$$

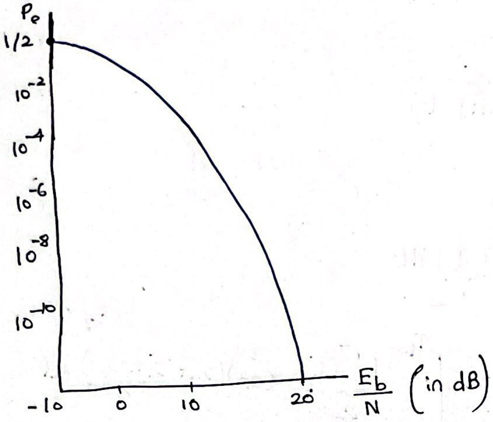
$$P_e = Q\left(\sqrt{\frac{3E_p + E_p}{2N}}\right) = Q\left(\sqrt{\frac{2E_p}{N}}\right) = Q\left(\sqrt{\frac{2E_b}{N}}\right)$$

$$h(t) = p(T_b - t)$$

$$a_0 = 0$$

E_b = Energy transmitted per bit $= \frac{E_p}{2} + \frac{E_q}{2} = \frac{E_p + E_q}{2}$

$E_b = E_p$ for polar



For ON-OFF

'1': $p(t)$ $\quad E_p \quad E_{pq} = 0$

'0': $q(t) = 0 \quad E_q = 0$

$P_e = P_b$ Probability of Error is probabability of error in bit

$a_0 = \frac{E_p}{2} \qquad h(t) = p(T_b - t)$

$$P_b = Q\left(\sqrt{\frac{E_p}{2N}}\right) = Q\left(\sqrt{\frac{E_b}{N}}\right)$$

Less power efficient than Polar as amplitude is less in Q( ) which is monotonically decreasing

Doubling E_b results in 3dB increase in E_b.

For Orthogonal $p(t)$ and $q(t)$

'1' $p(t) \rightarrow E_p \qquad E_{pq} = \int_0^{T_b} p(t)\, q(t)\, dt = 0$

'0' $q(t) \rightarrow E_q = E_p$

$$P_b = Q\left(\sqrt{\frac{E_p}{N}}\right) = Q\left(\sqrt{\frac{E_b}{N}}\right)$$

NOTE All the pulse we are transmitting ie. $p(t)$ and $q(t)$ ~~and~~ are Baseband signals. Which can also be pulses to avoid ISI.

Scanned with CamScanner

But in many cases we modulate these Baseband signals.

BPSK - Polar

ASK - ON-OFF

FSK -

BPSK

$$p(t) = \pm\sqrt{2}\, p'(t) \cos(2\pi f_c t)$$

$$\boxed{h(t) = p(T_b - t)}$$

$$E_p = 2\int_0^{T_b} p'^2(t) \cos^2(2\pi f_c t)\, dt$$

$$= 2\int_0^{T_b} \frac{p'^2(t)}{2}\, dt + \cancel{2\int_0^{T_b} \frac{p'^2(t)}{2} \cos(2\pi\, 2 f_c t)\, dt} \to 0$$

$$\boxed{E_p = E_p'}$$

$$\boxed{E_q = E_q'}$$

$$\boxed{BER = P_b = Q\left(\sqrt{\frac{E_b}{N}}\right)}$$

ASK

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BFSK

Normalisation Factor: $\sqrt{2}$

'1' $p(t) = \sqrt{2}A\cos\left(2\pi\left(f_c + \frac{\delta f}{2}\right)t\right)$

'0' $q(t) = \sqrt{2}A\cos\left(2\pi\left(f_c - \frac{\delta f}{2}\right)t\right)$

$\delta f = \frac{1}{2T_b}$ → $p(t)$ and $q(t)$ are orthogonal → $\int_0^{T_b} p(t)\,q(t)\,dt = 0 = E_{pq}$

Bandwidth optimal choice

BER Optimal Choice

For negatively correlated signal, E_{pq} would be negative so, it would make argument for Q() f^n greater hence Q() ↓.

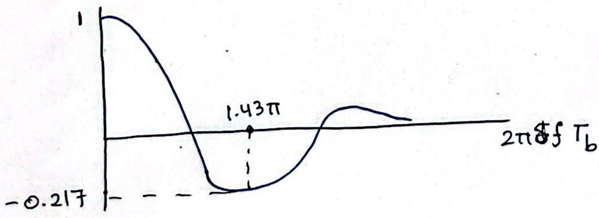
So, BER would reduce

$$E_{pq} = 2A^2 \int_0^{T_b} \cos\left(2\pi\left(f_c + \frac{\delta f}{2}\right)t\right) \cos\left(2\pi\left(f_c - \frac{\delta f}{2}\right)t\right) dt$$

$$= \frac{2A^2}{2}\int_0^{T_b} \cos(2\pi\,\delta f\,t)\,dt + \frac{2A^2}{2}\cancel{\int_0^{T_b} \cos(2\pi 2f_c t)\,dt}^{\,\to 0}$$

$$= A^2 T_b \frac{\sin(2\pi\,\delta f\,T_b)}{2\pi\,\delta f\,T_b} = \boxed{A^2 T_b\,\mathrm{Sinc}(2\pi\,\delta f\,T_b) = E_{pq}}$$

$$\delta f = \frac{1.43\pi}{2\pi T_b} \Rightarrow \boxed{\delta f = \frac{1.43}{2T_b}}$$



$$\boxed{E_{pq} = -0.217\,A^2 T_b}$$

$E_p = A^2 T_b$

$E_q = A^2 T_b$

$$\boxed{P_b = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right) = Q\left(\sqrt{\frac{1.217 E_b}{N}}\right) \quad \text{for } \delta_f = \frac{1.43}{2T_b}}$$

BER Efficient

$$\boxed{P_b = Q\left(\sqrt{\frac{E_b}{N}}\right) \quad \text{for } \delta_f = \frac{1}{2T_b}}$$

B.W Efficient.