$E_p = A^2 T_b$

$E_q = A^2 T_b$

$$\boxed{P_b = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N}}\right) = Q\left(\sqrt{\frac{1.217 E_b}{N}}\right) \quad \text{for } \delta_f = \frac{1.43}{2T_b}}$$

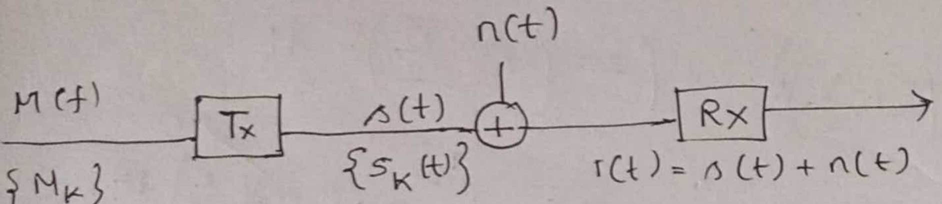
BER Efficient

$$\boxed{P_b = Q\left(\sqrt{\frac{E_b}{N}}\right) \quad \text{for } \delta_f = \frac{1}{2T_b}}$$

B.W Efficient.

M-Ary scheme

Assumption: $n(t)$ is AWGN



$S_k(t)$ can be written in terms of Basis f^n (orthogonal)

M: Max. no. elements in basis.

$$s_K(t) = \sum_{j=1}^{N} s_{Kj}\, \phi_j(t) \qquad N \leq M$$

where $s_{Kj} = \int_0^{T_s} s_K(t)\phi_j(t)\,dt$ is projection of $s_K(t)$ on j^{th} Basis f^n.

$r(t) = \underbrace{s(t)}_{\cancel{\text{random}}} + \underbrace{n(t)}_{\text{random}}$ → It would require infinite basis f^n to write down completely.
But we will use basis similar to $s(t)$

$$= \sum_{j=1}^{N} s_{Kj}\, \phi_j(t) + \sum_{j=1}^{\infty} n_j\, \phi_j(t)$$

Noise projected on basis of $\underbrace{\phi_j(t)}_{\text{signal space}}$ is important

$$= \sum_{j=1}^{N} (s_{Kj} + n_j)\, \phi_j(t) + \underbrace{\sum_{j=N+1}^{\infty} n_j\, \phi_j(t)}_{\text{Irrelevant noise}}$$

where $n_j = \int_0^{T_s} n(t)\, \phi_j(t)\, dt$ projection of noise on j^{th} basis f^n.

$$= \sum_{j=1}^{N} s_{Kj}\,\phi_j(t) + \sum_{j=1}^{N} n_j\,\phi_j(t) + \sum_{j=N+1}^{\infty} n_j\,\phi_j(t)$$

$$= \underbrace{s_K(t) + n'(t)}_{q(t)} + \underbrace{n''(t)}$$

$$= q(t) + n''(t)$$

$q = [q_1 \ldots\ldots q_N]$

$s_K = [s_{K_1} \ldots\ldots s_{K_N}]$ Each of them are Independent Gaussian r.v.

$\rightarrow q = s_K + n$

$q(t) = \sum_{j=1}^{N} q_j \phi_j(t)$ $n' = [n_1 \ldots\ldots n_N]$

$\rightarrow$ Design of noise receiver depends on noise statistics. So we need to find distribution of noise vector n.

$\rightarrow n = [n_1 \ldots\ldots, n_N]$; $n_j \in \mathcal{N}\left(\underset{\text{Mean}}{0}, \underset{\text{Covariance Matrix}}{\frac{NI}{2}}\right)$

$$n_j = \int_0^{T_s} n(t)\,\phi_j(t)\,dt$$

$\mathbb{E}[n_j] = 0$ as $\mathbb{E}[n(t)] = 0 \rightarrow \because$ AWGN

$$\mathbb{E}[n_j n_K] = \mathbb{E}\left[\int_0^{T_s} n(\tau)\,\phi_j(\tau)\,d\tau \int_0^{T_s} n(\beta)\,\phi_K(\beta)\,d\beta\right]$$

$$= \mathbb{E}\left[\int_0^{T_s}\int_0^{T_s} n(\tau)\,n(\beta)\,\phi_j(\tau)\,\phi_K(\beta)\,d\tau\,d\beta\right]$$

$$= \int_0^{T_s}\int_0^{T_s} \mathbb{E}[n(\tau)\,n(\beta)]\,\phi_j(\tau)\,\phi_K(\beta)\,d\tau\,d\beta$$

$$= \int_0^{T_s}\int_0^{T_s} \underbrace{R_n(\tau-\beta)}_{\text{correlation of AWGN.}}\,\phi_j(\tau)\,\phi_K(\beta)\,d\tau\,d\beta$$

N: PSD of AWGN

$$R_n(\tau-\beta) = \frac{N}{2}\,\delta(\tau-\beta)$$

$$= \frac{N}{2}\underbrace{\int_0^{T_s} \phi_j(\tau)\,\phi_K(\tau)\,d\tau}$$

when $j = K \rightarrow 1$

$j \neq K \rightarrow 0$

$$= \begin{cases} N/2, & \text{if } j = K \\ 0, & \text{else} \end{cases}$$

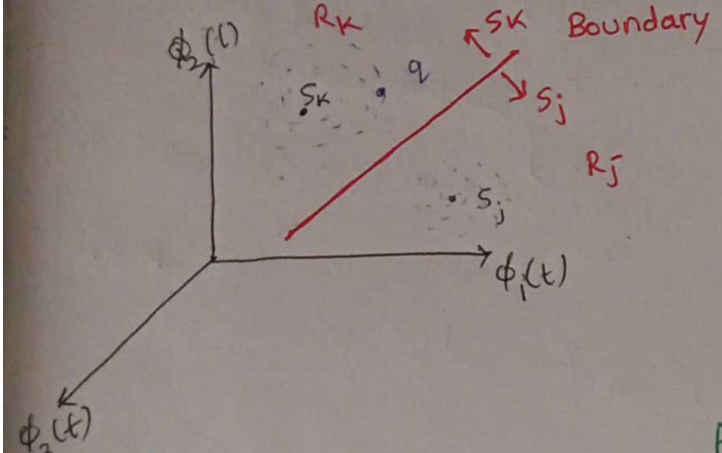
Covariance Matriax ~~with~~ diagonal matrix.

$n = [n_1 \;.\;.\;.\;.\;.\; n_N]$

$$P_n(n) = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi \frac{N}{2}}} \exp\left(-\frac{n_j^2}{N}\right)$$

$$\boxed{P_n(n) = \frac{1}{(\pi \mathcal{N})^{N/2}} \exp\left(-\frac{\|n\|^2}{\mathcal{N}}\right)}$$

Distribution of Multivariate r.v. with mean = 0, covariance matrix = diagonal



$q = s_K + n$

We need to find optimal Boundary. This will have some BER.

$$P_e(s_K) = \int_{R_j} p(q/s_K)\, dq$$

As $\mathcal{N} \uparrow$, dispersion $\uparrow$, $P_e \uparrow$.

$q = m_K + n$ Probability of detection

$$\underbrace{P(c \mid q = q)}_{\text{P of correct decision for some } q = q} = \underbrace{P(m_K \mid q = q)}_{\text{P of detecting } m_K \text{ when } q \text{ was received}}$$

We need this to be maximum.

$$P(c) = \int \underbrace{P(c \mid q = q)}_{\text{We need to maximise this}} \underbrace{P(q)}\, dq = \int P(m_K \mid q)\, p(q)\, dq.$$

Probability of observation Not in our control.

$$\max_K P(c) = \max_K \underbrace{P(m_K \mid q)}$$

$\hookrightarrow$ Aposteriori probability

$p(m_K)$

$\hookrightarrow$ Prior probability.

M values for each possible sym

$$\boxed{\hat{m} = m_K \quad \text{if } P(m_K \mid q) \geq P(m_j \mid q)} \quad j = 1, ..M \quad j \neq K.$$

MAP Detector : Maximum a posterior detec

$$P(m_K \mid q) = \frac{P(q \mid m_K)\, p(m_K)}{p(q)}$$

$p(q) \longrightarrow$ constant for All M values

if $P(m_K) = \frac{1}{M}$ then, $p(m_K)$ would be constant also.

$$\max_{K} P(m_K|q) = \max_{K} p(q|m_K)$$

Likelihood f^n of m_K

Maximum Likelihood Detector

MAP Decoder under equi-probable ~~p~~ give ML Detector

$$q = s_K + n \Rightarrow n = q - s_K$$

$$\max_{K} P(m_K) P_q(\tilde{q}|m_K) = P(m_K) P_n(\tilde{q} - s_K)$$

$$= \underline{\frac{P(m_K)}{(\pi N)^{N/2}}} \exp\left(-\frac{\|\tilde{q} - s_K\|^2}{N}\right)$$

Law of Transformation.

Multiply with $\frac{N}{2}$

$$\frac{N}{2} \log\left(\frac{P(m_K)}{(\pi N)^{N/2}}\right) - \frac{\|\tilde{q} - s_K\|^2}{2}$$

$$\hat{m} = \arg\max_{m_K} P(q \mid m_K)\, P(m_K)$$

$q = s_K + \underline{n}$ — s_K: symbol selected for message bit

$n'' = \underline{n} + n'$ — n'': Total noise; $\underline{n}$: Noise in dimension/space of message vector; n': Noise not in space of message.

$$\equiv \max_{m_K} \frac{N}{2} \ln P(m_K) - \frac{1}{2} \| q - s_K \|^2 \equiv \underbrace{\frac{N}{2} \ln P(m_K) - \frac{1}{2} \| s_K \|^2}_{b_K} + \underbrace{\langle q, s_K \rangle}_{\langle q, s_K \rangle}$$

We need a filter to find inner product of q i.e. corrupted s_K and s_K received bits.

Let $h(t)$ is filter used. and $q(t)$ is given in input

$$\int_{-\infty}^{\infty} q(\tau)\, h(T_M - \tau)\, dt$$

To detect s_K, it needs to be a mathed filter i.e. $h(t) = s_K(T_M - t)$

$$\langle q, s_K \rangle = \int q(\tau)\, s_K(\tau)\, d\tau$$

$q(t)$: Recieved signal in signal space.

$r(t)$: Received signal. $r(t) = q(t) + \underbrace{n''(t)}_{\text{noise not in signal space or othogoral}}$

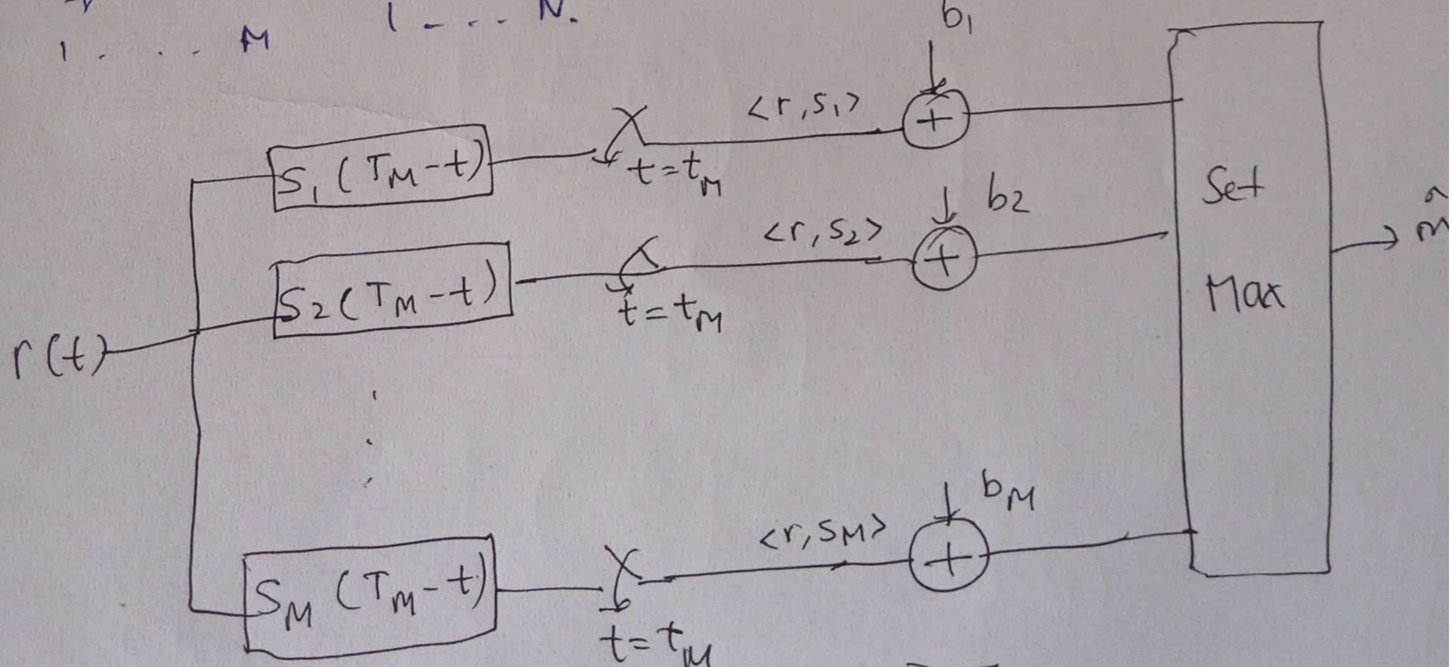
$$\therefore \int n''(t)\, s_i(t)\, dt = 0$$

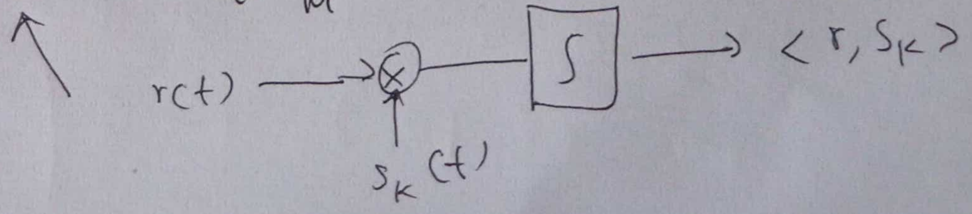
$$\therefore \langle q, s_K \rangle \int q(\tau)\, s_K(\tau)\, d\tau = \int r(\tau)\, s_K(\tau)\, d\tau$$

$$= \langle r, s_K \rangle$$

$$s_K = \sum_{j=1}^{N} s_{Kj}\, \phi_j(t)$$

$s_K \to 1 \ldots M$, $s_{Kj} \to 1 \ldots N$.

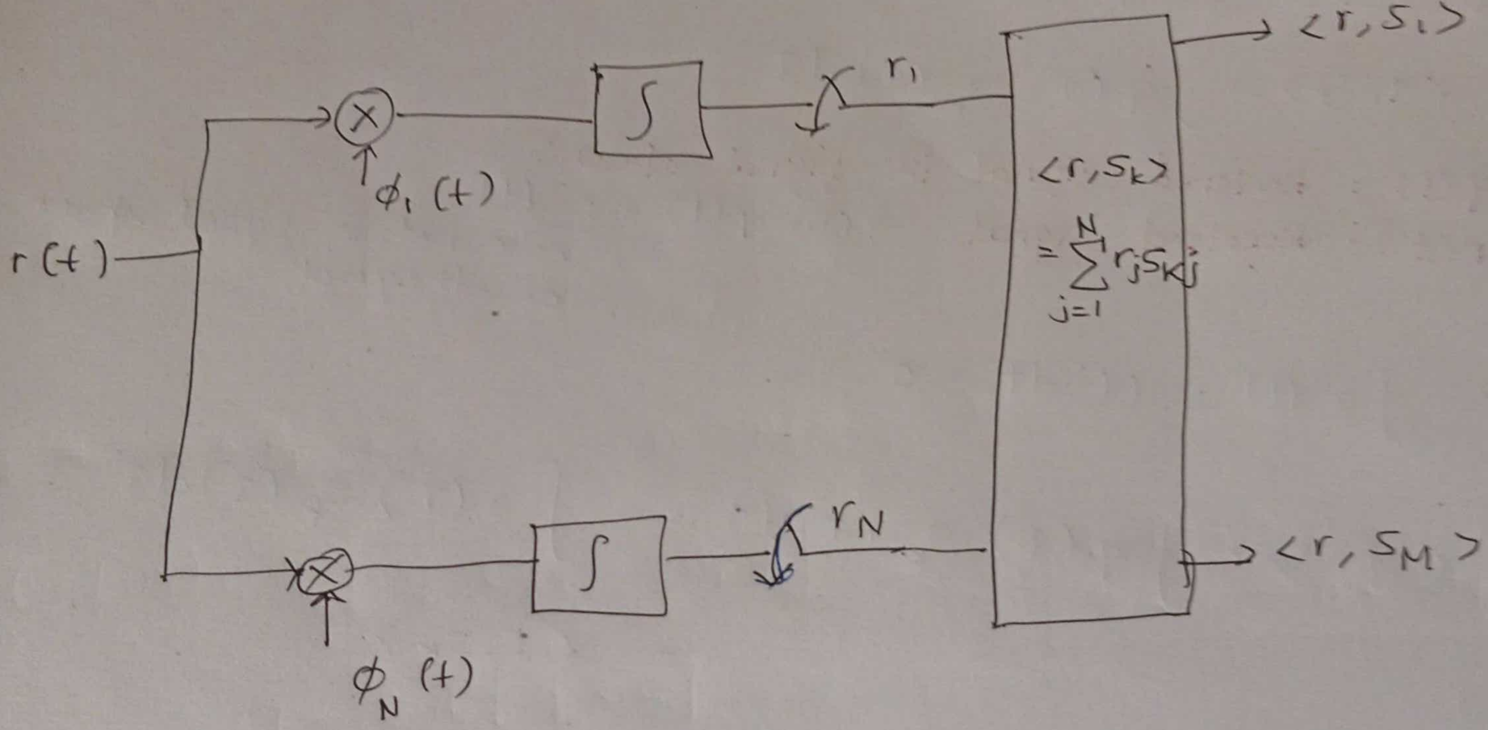




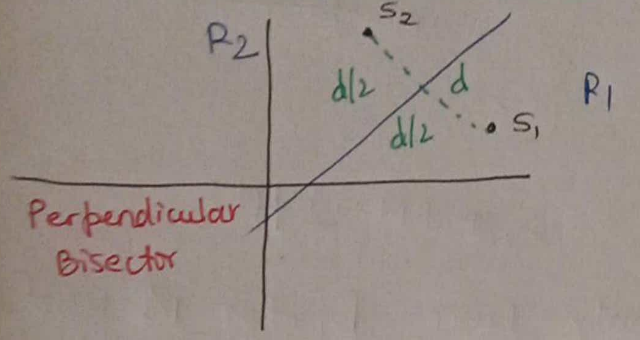
$$\langle q, s_k \rangle = \int_0^{T_M} q(\tau)\, s_k(\tau)\, d\tau = \int r(\tau)\, s_k(\tau)\, d\tau = \langle r, s_k \rangle$$

$$r(t) = \sum r_j\, \phi_j(t)$$

$$r_j = \int r(t)\, \phi_j(t)\, dt$$



Bit Error Rate



signal space with two symbols

We decide boundary and region to

The criterion we follow for it is:

$$\frac{N}{2}\ln P(M_1) - \underbrace{\|q - s_1\|^2}_{d_1^2} \;\overset{m_1}{\underset{m_2}{\gtrless}}\; \frac{N}{2}\ln P(M_2) - \underbrace{\|q - s_2\|^2}_{d_2^2}$$

$$d_2^2 - d_1^2 > \frac{N}{2}\ln\left(\frac{P(M_2)}{P(M_1)}\right) = C$$

$d = \|s_1 - s_2\|$

CASE 1: Equi probable

$C = 0$, if $P(M_1) = P(M_2)$

$$d_2^2 - d_1^2 > 0$$

$$d_1 \overset{m_1}{\underset{m_2}{\gtrless}} d_2$$

$$d_2 - d_1 > 0 \quad (\text{as } d_1 + d_2 > 0)$$

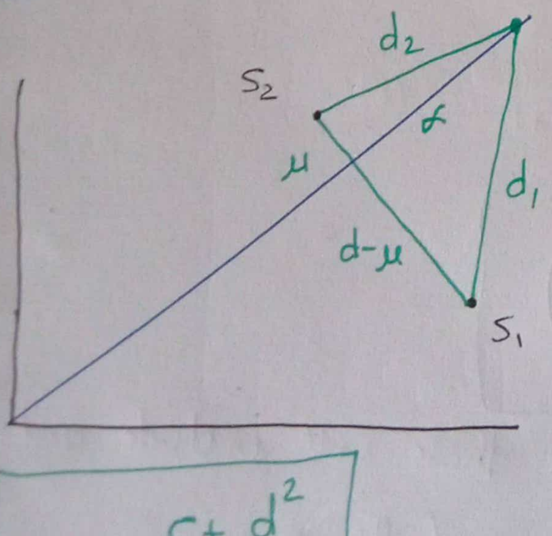
$$d_2 > d_1 \rightarrow \text{For } m_1.$$

CASE 2: M_1 is more favourable $\rightarrow$ C is negative R_2 gets higher

$\rightarrow$ μ is smaller $\rightarrow$

We will shift ⊥ bisector towards S_2

$$\boxed{C = \frac{N}{2}\ln\frac{P(M_2)}{P(M_1)}}$$



$$d_2^2 - d_1^2 = C$$

$$\begin{cases} \alpha^2 + \mu^2 = d_2^2 \\ \alpha^2 + (d-\mu)^2 = d_1^2 \end{cases}$$

$$\alpha^2 + d^2 + \mu^2 - 2d\mu = d_1^2$$

$$\boxed{\mu = \frac{C + d^2}{2d}}$$

M-ary PAM : Pulse

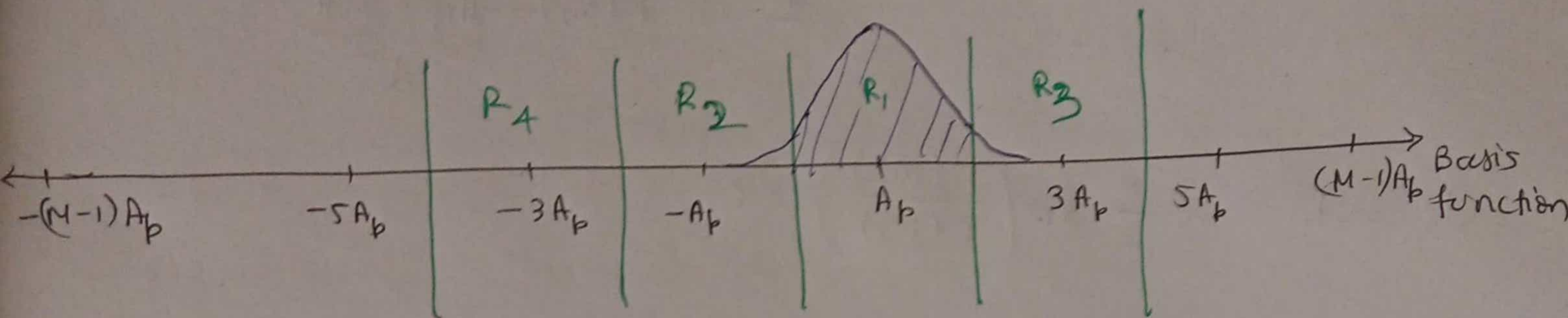
↳ only 1 basis f^n required.

~~Also for M-ary FSK~~

Constellation Points

$\pm p(t), \pm 3p(t), \pm 5p(t), \ldots\ldots, \pm(M-1)p(t)$

Only noise component along the only basis function is used,



$q = \pm K A_p + n \qquad n \sim N(0, \sigma_n^2)$

$$\boxed{\begin{array}{l} q = A_p + n \\ q < 0 \rightarrow n < -A_p \end{array}}$$

$$P(c|m_1) = P[q \in [0, 2A_p]]$$

$$= 1 - 2P[q<0]$$

$$= 1 - 2P[n \leq -A_p] \longrightarrow$$ Then q would be less than 0

$$= 1 - 2\left(1 - Q\left(\frac{-A_p - 0}{\sigma_n}\right)\right)$$

$$= 1 - 2Q\left(\frac{A_p}{\sigma_n}\right)$$

$$\boxed{P(c|m_i) = 1 - 2Q\left(\frac{A_p}{\sigma_n}\right) \quad i = 1 \ldots (M-2)}$$

We not including extreme regions, as those will include one region

$M_i = M-1, M \qquad P(c|m_i) = 1 - Q\left(\frac{A_p}{\sigma_n}\right)$

Error Probability

$$P_e(m_i) = 2Q\left(\frac{A_p}{\sigma_n}\right) \qquad i = 1, \ldots, M-2$$

$$P_e(m_i) = Q\left(\frac{A_p}{\sigma_n}\right) \qquad i = M-1, M$$

Probability of error in message

$$P_{eM} = \sum_{j=1}^{M} P_e(M_i)\, P(M_i)$$

$$= \frac{1}{M}\sum_{j=1}^{M} P_e(M_i)$$

$$= \frac{1}{M}\left((M-2)\,2\,Q\left(\frac{A_p}{\sigma_n}\right) + 2Q\left(\frac{A_p}{\sigma_n}\right)\right)$$

$$= \frac{2(M-1)}{M}\, Q\left(\frac{A_p}{\sigma_n}\right)$$

Relation of E_p with A_p will help us find above probability in term of E_p. we find E_p and divide it by M. So give equal energy.

$$\pm A_p, \pm 3A_p, \pm 5A_p, \pm \ldots \pm (M-1)A_p.$$

$$E_{avg} = 2 \times \frac{A_p^2}{2} \times \left(1 + 9 + \ldots + (M-1)^2\right)$$

$$E_{avg} = \frac{M^2-1}{3} E_p \qquad \left(E_p = \frac{A_p^2}{\cancel{\;}}\right)$$

$$E_b = \frac{E_{avg}}{\log_2 M} = \frac{M^2-1}{3\log_2 M} \times \frac{A_p^2}{2}$$

$$\sigma_n = \sqrt{\frac{N}{2}}$$

$$\boxed{P_{em} = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M}{(M^2-1)}\frac{E_b}{N}}\right)}$$

PAR Analysis of 16-QAM

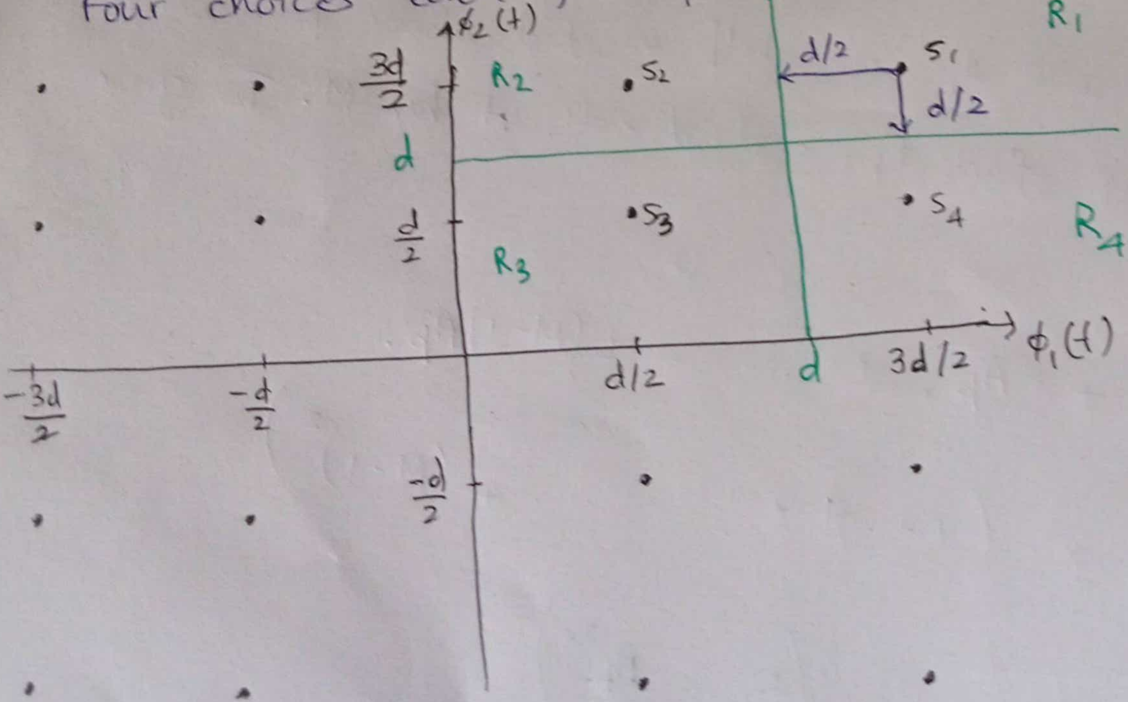
Receiver Structure

PSK.

16 - QAM

$$s_i(t) = a_i\,\phi_1(t) + b_i\,\phi_2(t)$$

Four choices each for a_i and b_i



$$q = s_i + n$$

$$= \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$\rightarrow$ $P(c|m_1) = P[q \in R_1]$

$$= P\left[n_1 > \frac{-d}{2},\ n_2 > \frac{-d}{2}\right]$$

$\rightarrow$ $P(c|m_3) = P[q \in R_3]$

$$= P\left[\frac{-d}{2} < n_1 < \frac{d}{2},\ \frac{-d}{2} < n_2 < \frac{d}{2}\right]$$

$\rightarrow$ $P(c|m_2) = P[q \in R_4]$

$= P(c|m_4)$

$$= P\left[\frac{-d}{2} < n_1 < \frac{d}{2},\ \frac{-d}{2} < n_2\right]$$

These will apply equally to all four quadrants.