3

Analog Signal Transmission and Reception

A large number of information sources are analog sources. Analog sources can be modulated and transmitted directly or can be converted to digital data and transmitted using digital modulation techniques. The notion of analog to digital conversion will be examined in detail in Chapter 6.

Speech, image, and video are examples of analog sources of information. Each of these sources is characterized by its bandwidth, dynamic range, and the nature of the signal. For instance, in case of audio, and black and white video, the signal has just one component measuring the pressure or intensity, but in case of color video, the signal has four components measuring red, green, and blue color components, and the intensity.

In spite of the general trend toward digital transmission of analog signals, there is still today a significant amount of analog signal transmission, especially in audio and video broadcast. In this chapter, we treat the transmission of analog signals by carrier modulation. The treatment of the performance of these systems in the presence of noise is being deferred to Chapter 5. We consider the transmission of an analog signal by impressing it on either the amplitude, the phase, or the frequency of a sinusoidal carrier. Methods for demodulation of the carrier-modulated signal to recover the analog information signal are also described.

3.1 INTRODUCTION TO MODULATION

The analog signal to be transmitted is denoted by $m(t)$, which is assumed to be a lowpass signal of bandwidth W, in other words, $M(f) \equiv 0$, for $|f| > W$. We also assume that the signal is a power-type signal (see Section 2.3) whose power is denoted by P_m, where

$$P_m = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |m(t)|^2\, dt$$

The message signal $m(t)$ is transmitted through the communication channel by impressing it on a carrier signal of the form

$$c(t) = A_c \cos(2\pi f_c t + \phi_c) \tag{3.1.1}$$

where A_c is the carrier amplitude, f_c is the carrier frequency, and ϕ_c is the carrier phase. We say that the message signal $m(t)$ modulates the carrier signal $c(t)$ in either amplitude, frequency, or phase, if after modulation, the amplitude, frequency, or phase of the signal become functions of the message signal. In effect, modulation converts the message signal $m(t)$ from lowpass to bandpass, in the neighborhood of the center frequency f_c.

Modulation of the carrier $c(t)$ by the message signal $m(t)$ is performed in order to achieve one or more of the following objectives: (1) The lowpass signal is translated in frequency to the passband of the channel so that the spectrum of the transmitted bandpass signal will match the passband characteristics of the channel; (2) to accommodate for simultaneous transmission of signals from several message sources, by means of frequency-division multiplexing (see Section 3.2.6); and (3) to expand the bandwidth of the transmitted signal in order to increase its noise immunity in transmission over a noisy channel, as we will see in our discussion of angle-modulation noise performance in Chapter 5. We will see that objectives (1) and (2) are met by all of the modulation methods described next. Objective (3) is met by employing angle modulation to spread the signal $m(t)$ over a larger bandwidth.

In the following sections of this chapter we consider the transmission and reception of analog signals by carrier-amplitude modulation (AM), carrier-frequency modulation (FM) and carrier-phase modulation (PM). Comparisons will be made among these modulation methods on the basis of their bandwidth requirements and their implementation complexity. Their performance in the presence of additive noise disturbances and their power efficiency, will be treated in Chapter 5.

3.2 AMPLITUDE MODULATION (AM)

In amplitude modulation, the message signal $m(t)$ is impressed on the amplitude of the carrier signal $c(t)$. There are several different ways of amplitude modulating the carrier signal by $m(t)$, each of which results in different spectral characteristics for the transmitted signal. Next, we describe these methods, which are called (1) double-sideband, suppressed carrier AM, (2) conventional double-sideband AM, (3) single-sideband AM, and (4) vestigial-sideband AM.

3.2.1 Double-Sideband Suppressed Carrier AM

A double-sideband, suppressed carrier (DSB-SC) AM signal is obtained by multiplying the message signal $m(t)$ with the carrier signal $c(t)$. Thus, we have the amplitude modulated signal

$$
\begin{aligned}
u(t) &= m(t)c(t) \\
&= A_c m(t)\cos(2\pi f_c t + \phi_c)
\end{aligned}
$$

Bandwidth Requirements. The spectrum of the modulated signal can be obtained by taking the Fourier transform of $u(t)$.

$$
\begin{aligned}
U(f) &= \mathcal{F}[m(t)] \star \mathcal{F}[A_c\cos(2\pi f_c t + \phi_c)] \\
&= M(f) \star \frac{A_c}{2}\left[e^{j\phi_c}\delta(f - f_c) + e^{-j\phi_c}\delta(f + f_c)\right] \\
&= \frac{A_c}{2}\left[M(f - f_c)e^{j\phi_c} + M(f + f_c)e^{-j\phi_c}\right]
\end{aligned}
$$

Figure 3.1 illustrates the magnitude and phase spectra for $M(f)$ and $U(f)$.

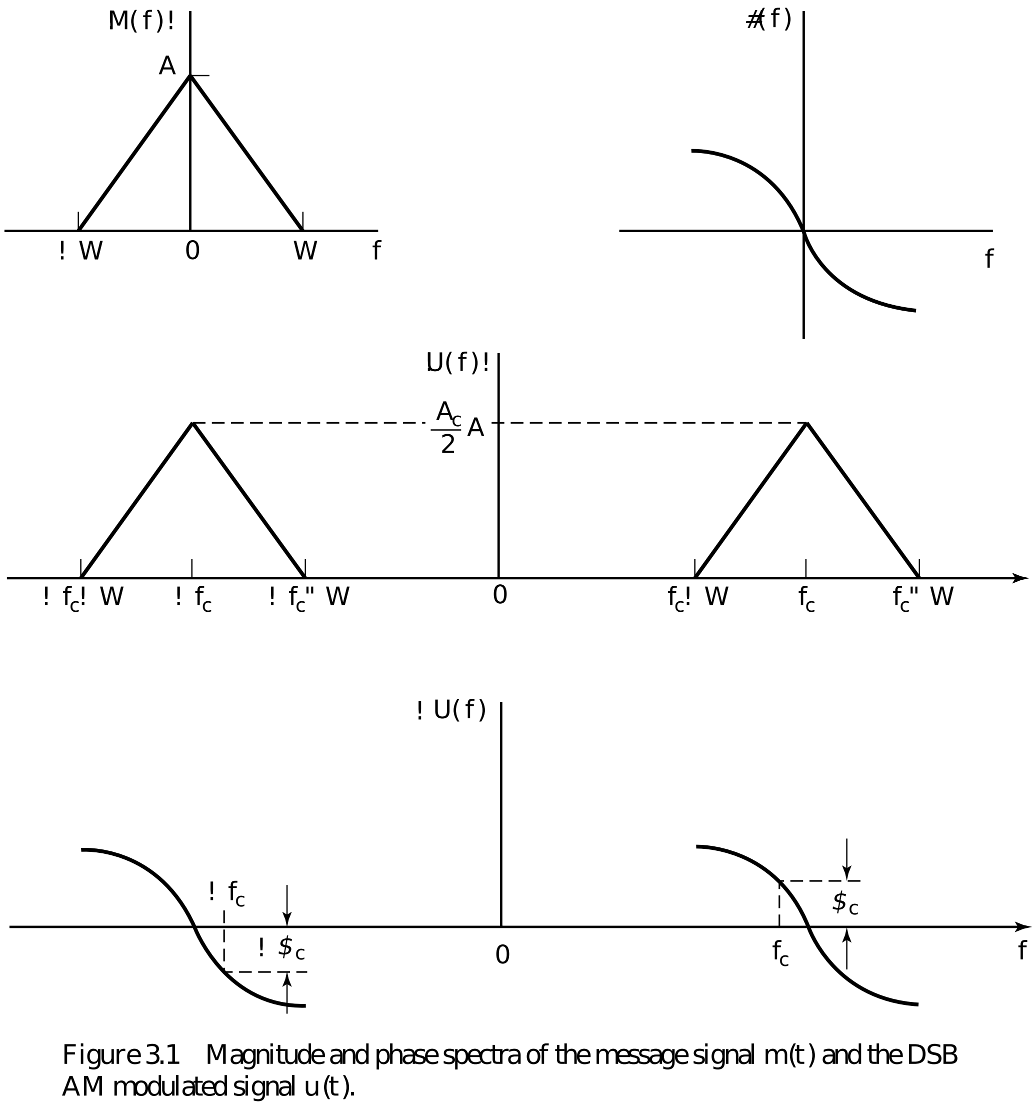

Figure 3.1 Magnitude and phase spectra of the message signal $m(t)$ and the DSB AM modulated signal $u(t)$.

We observe that the magnitude of the spectrum of the message signal $m(t)$ has been translated or shifted in frequency by an amount f_c. The phase of the message signal has been translated in frequency and offset by the carrier phase ϕ_c. Furthermore, the bandwidth occupancy of the amplitude-modulated signal is $2W$, whereas the bandwidth of the message signal $m(t)$ is W. Therefore, the channel bandwidth required to transmit the modulated signal $u(t)$ is $B_c = 2W$.

The frequency content of the modulated signal $u(t)$ in the frequency band $|f| > f_c$ is called the upper sideband of $U(f)$, and the frequency content in the frequency band $|f| < f_c$ is called the lower sideband of $U(f)$. It is important to note that either one of the sidebands of $U(f)$ contains all the frequencies that are in $M(f)$. That is, the frequency content of $U(f)$ for $f > f_c$ corresponds to the frequency content of $M(f)$ for $f > 0$, and the frequency content of $U(f)$ for $f < -f_c$ corresponds to the frequency content of $M(f)$ for $f < 0$. Hence, the upper sideband of $U(f)$ contains all the frequencies in $M(f)$. A similar statement applies to the lower sideband of $U(f)$. Therefore, the lower sideband of $U(f)$ contains all the frequency content of the message signal $M(f)$. Since $U(f)$ contains both the upper and the lower sidebands, it is called a double-sideband (DSB) AM signal.

The other characteristic of the modulated signal $u(t)$ is that it does not contain a carrier component. That is, all the transmitted power is contained in the modulating (message) signal $m(t)$. This is evident from observing the spectrum of $U(f)$. We note that, as long as $m(t)$ does not have any DC component, there is no impulse in $U(f)$ at $f = f_c$, which would be the case if a carrier component was contained in the modulated signal $u(t)$. For this reason, $u(t)$ is called a suppressed-carrier signal. Therefore, $u(t)$ is a DSB-SC AM signal.

Example 3.2.1

Suppose that the modulating signal $m(t)$ is a sinusoid of the form

$$m(t) = a \cos 2\pi f_m t \qquad f_m \ll f_c$$

Determine the DSB-SC AM signal and its upper and lower sidebands.

Solution The DSB-SC AM is expressed in the time domain as

$$\begin{aligned} u(t) = m(t)c(t) &= A_c a \cos 2\pi f_m t \cos(2\pi f_c t + \phi_c) \\ &= \frac{A_c a}{2} \cos[2\pi(f_c - f_m)t + \phi_c] + \frac{A_c a}{2} \cos[2\pi(f_c + f_m)t + \phi_c] \end{aligned}$$

In the frequency domain, the modulated signal has the form

$$\begin{aligned} U(f) = {} & \frac{A_c a}{4} \left[e^{j\phi_c} \delta(f - f_c + f_m) + e^{-j\phi_c} \delta(f + f_c - f_m) \right] \\ & + \frac{A_c a}{4} \left[e^{j\phi_c} \delta(f - f_c - f_m) + e^{-j\phi_c} \delta(f + f_c + f_m) \right] \end{aligned}$$

This spectrum is shown in Figure 3.2(a).

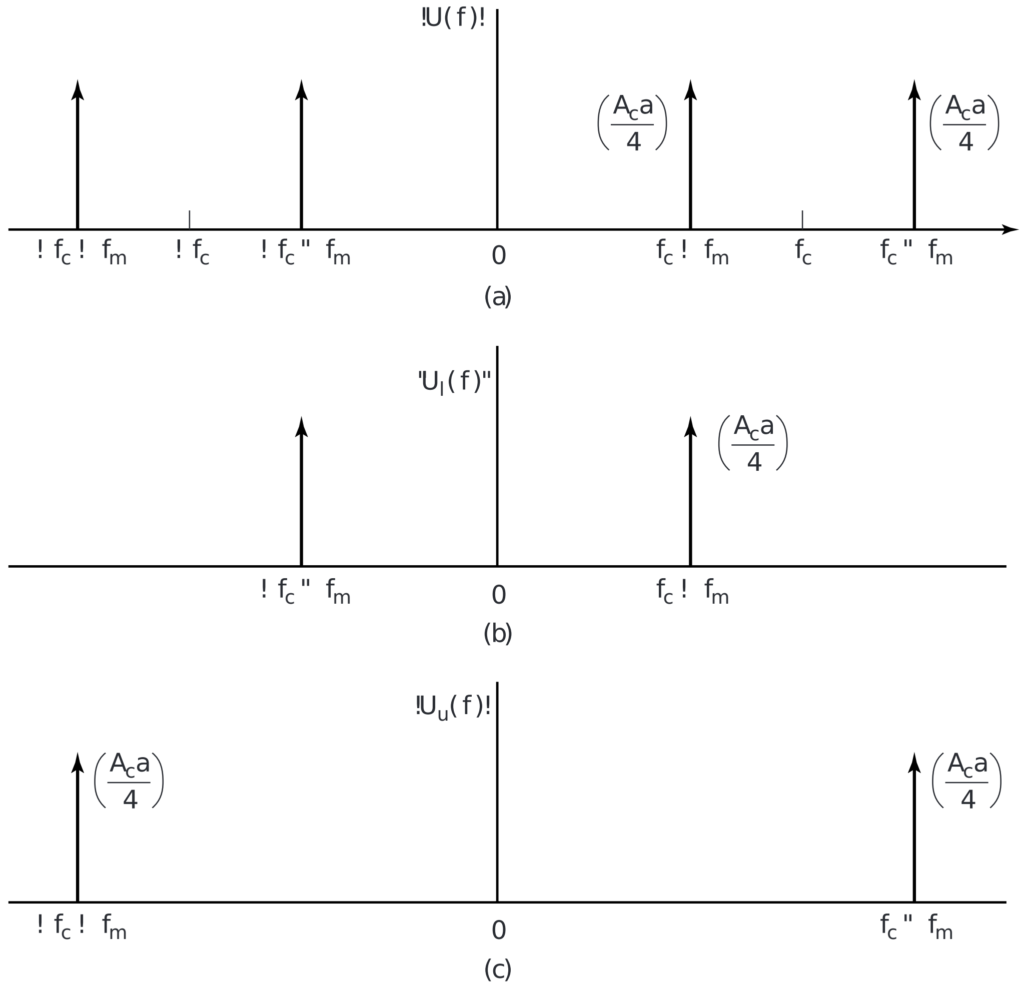

Figure 3.2 (a) The (magnitude) spectrum of a DSB-SC AM signal for a sinusoidal message signal and (b) its lower and (c) upper sidebands.

The lower sideband of $u(t)$ is the signal

$$u_l(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t + \phi_c]$$

and its spectrum is illustrated in Figure 3.2(b). Finally, the upper sideband of $u(t)$ is the signal

$$u_u(t) = \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t + \phi_c]$$

and its spectrum is illustrated in Figure 3.2(c).

Power Content of DSB-SC Signals. In order to compute the power content of the DSB-SC signal, without loss of generality we can assume that the phase of the signal is set to zero. This is because the power in a signal is independent of the phase of the signal. The time-average autocorrelation function of the signal $u(t)$ is given by

$$
\begin{aligned}
R_u(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u(t)u(t-\tau)\,dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_c^2 m(t)m(t-\tau)\cos(2\pi f_c t)\cos(2\pi f_c(t-\tau))\,dt \\
&= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} m(t)m(t-\tau)[\cos(4\pi f_c t - 2\pi f_c \tau) + \cos(2\pi f_c \tau)]\,dt \\
&= \frac{A_c^2}{2} R_m(\tau)\cos(2\pi f_c \tau) \qquad (3.2.1)
\end{aligned}
$$

where we have used the fact that

$$\lim_{T\to\infty} \int_{-T/2}^{T/2} m(t)m(t-\tau)\cos(4\pi f_c t - 2\pi f_c \tau)\,dt = 0$$

This is because

$$
\begin{aligned}
&\int_{-\infty}^{\infty} m(t)m(t-\tau)\cos(4\pi f_c t - 2\pi f_c \tau)\,dt \\
&\overset{(a)}{=} \int_{-\infty}^{\infty} \mathcal{F}[m(t-\tau)]\{\mathcal{F}[m(t)\cos(4\pi f_c t - 2\pi f_c \tau)]\}^* df \\
&= \int_{-\infty}^{\infty} e^{-j2\pi f\tau} M(f) \left[\frac{M(f-2f_c)e^{-j2\pi f_c \tau}}{2} + \frac{M(f+2f_c)e^{j2\pi f_c \tau}}{2}\right]^* df \\
&\overset{(b)}{=} 0
\end{aligned}
$$

where in (a) we have used Parseval's relation [see Equation (2.2.11)] and in (b) we have used the fact that $M(f)$ is limited to the frequency band $[-W, W]$ and $W \ll f_c$, hence, there is no frequency overlap between $M(f)$ and $M(f \pm 2f_c)$.

By taking the Fourier transform of both sides of Equation (3.2.1), we can obtain the power-spectral density of the modulated signal as

$$
\begin{aligned}
S_u(f) &= \mathcal{F}\left[\frac{A_c^2}{2} R_m(\tau)\cos(2\pi f_c \tau)\right] \\
&= \frac{A_c^2}{4}[S_m(f-f_c) + S_m(f+f_c)] \qquad (3.2.2)
\end{aligned}
$$

This shows that the power-spectral density of the DSB-SC signal is the power-spectral density of the message shifted upward and downward by f_c and scaled by $A_c^2/4$. To obtain the total power in the modulated signal, we can either substitute $\tau = 0$ in the time-average autocorrelation function as given in Equation (3.2.1), or we can integrate the power-spectral density of the modulated signal given in Equation (3.2.2) over all frequencies. Using the first approach from Equation (3.2.1), we have

$$\begin{aligned} P_u &= \frac{A_c^2}{2} R_m(\tau)\cos(2\pi f_c\tau)|_{\tau=0} \\ &= \frac{A_c^2}{2} R_m(0) \\ &= \frac{A_c^2}{2} P_m \end{aligned} \tag{3.2.3}$$

where $P_m = R_m(0)$ is the power in the message signal.

Example 3.2.2

In Example 3.2.1, determine the power-spectral density of the modulated signal, the power in the modulated signal, and the power in each of the sidebands.

Solution The message signal is $m(t) = a\cos 2\pi f_m t$, its power-spectral density is given by

$$S_m(f) = \frac{a^2}{4}\delta(f - f_m) + \frac{a^2}{4}\delta(f + f_m) \tag{3.2.4}$$

Substituting in Equation (3.2.2) we obtain the power-spectral density of the modulated signal as

$$S_u(f) = \frac{A_c^2 a^2}{16}[\delta(f - f_m - f_c) + \delta(f + f_m - f_c) + \delta(f - f_m + f_c) + \delta(f - f_m - f_c)]$$

The total power in the modulated signal is the integral of $S_u(f)$ and is given by

$$P_u = \int_{-\infty}^{\infty} S_u(f)\,df = \frac{A_c^2 a^2}{4}$$

Because of symmetry of sidebands the powers the upper and lower sidebands, P_{us} and P_{ls}, are equal and given by

$$P_{us} = P_{ls} = \frac{A_c^2 a^2}{8}$$

It can also be observed from the power-spectral density of the DSB-SC signal [see Equation (3.2.2)] that the bandwidth of the modulated signal is $2W$, twice the bandwidth of the message signal, and that there exists no impulse at the carrier frequency $\pm f_c$ in the power-spectral density. Therefore, the modulated signal is suppressed carrier (SC).

Demodulation of DSB-SC AM Signals. In the absence of noise, and with the assumption of an ideal channel, the received signal is equal to the modulated signal; i.e.,

$$\begin{aligned} r(t) &= u(t) \\ &= A_c m(t)\cos(2\pi f_c t + \phi_c) \end{aligned} \tag{3.2.5}$$

Suppose we demodulate the received signal by first multiplying $r(t)$ by a locally generated sinusoid $\cos(2\pi f_c t + \phi)$, where ϕ is the phase of the sinusoid, and then passing the product signal through an ideal lowpass filter having a bandwidth W. The multiplication of $r(t)$ with $\cos(2\pi f_c t + \phi)$ yields

$$\begin{aligned} r(t)\cos(2\pi f_c t + \phi) &= A_c m(t)\cos(2\pi f_c t + \phi_c)\cos(2\pi f_c t + \phi) \\ &= \tfrac{1}{2} A_c m(t)\cos(\phi_c - \phi) + \tfrac{1}{2} A_c m(t)\cos(4\pi f_c t + \phi + \phi_c) \end{aligned}$$

The lowpass filter rejects the double frequency components and passes only the lowpass components. Hence, its output is

$$y_\ell(t) = \tfrac{1}{2} A_c m(t)\cos(\phi_c - \phi) \tag{3.2.6}$$

Note that $m(t)$ is multiplied by $\cos(\phi_c - \phi)$. Thus, the desired signal is scaled in amplitude by a factor that depends on the phase difference between the phase ϕ_c of the carrier in the received signal and the phase ϕ of the locally generated sinusoid. When $\phi_c \neq \phi$, the amplitude of the desired signal is reduced by the factor $\cos(\phi_c - \phi)$. If $\phi_c - \phi = 45^\circ$, the amplitude of the desired signal is reduced by $\sqrt{2}$ and the signal power is reduced by a factor of two. If $\phi_c - \phi = 90^\circ$, the desired signal component vanishes.

The above discussion demonstrates the need for a phase-coherent or synchronous demodulator for recovering the message signal $m(t)$ from the received signal. That is, the phase ϕ of the locally generated sinusoid should ideally be equal to the phase ϕ_c of the received carrier signal.

A sinusoid that is phase-locked to the phase of the received carrier can be generated at the receiver in one of two ways. One method is to add a carrier component into the transmitted signal as illustrated in Figure 3.3.

We call such a carrier component "a pilot tone." Its amplitude A_p and, hence, its power $A_p^2/2$ is selected to be significantly smaller than that of the modulated signal $u(t)$. Thus the transmitted signal is double-sideband, but it is no longer a suppressed carrier signal. At the receiver, a narrowband filter tuned to frequency f_c is used to filter out the pilot signal component and its output is used to multiply the received signal as shown in Figure 3.4.

The reader may show that the presence of the pilot signal results in a dc component in the demodulated signal, which must be subtracted out in order to recover $m(t)$.

The addition of a pilot tone to the transmitted signal has the disadvantage of requiring that a certain portion of the transmitted signal power must be allocated to the transmission of the pilot. As an alternative, we may generate a phase-locked sinusoidal

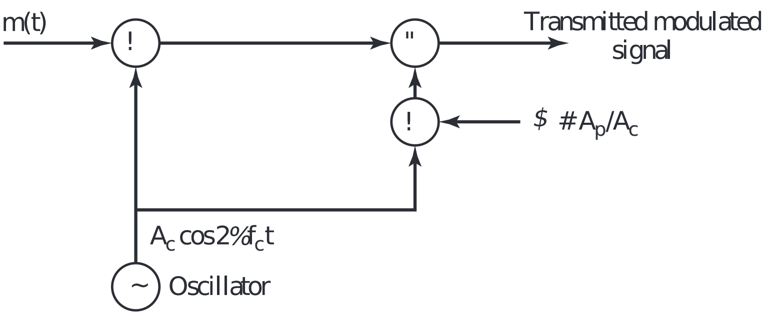

Figure 3.3 Addition of a pilot tone to a DSB AM signal.

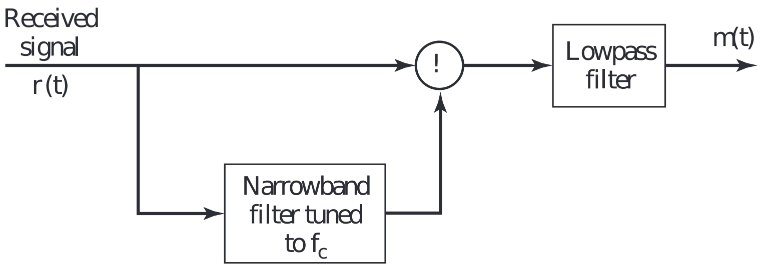

Figure 3.4 Use of a pilot tone to demodulate a DSB AM signal.

carrier from the received signal $r(t)$ without the need of a pilot signal. This can be accomplished by use of a phase-locked loop (PLL) as described in Section 5.2.

3.2.2 Conventional Amplitude Modulation

A conventional AM signal consists of a large carrier component in addition to the double-sideband AM modulated signal. The transmitted signal is expressed mathematically as

$$u(t) = A_c[1 + m(t)]\cos(2\pi f_c t + \phi_c) \tag{3.2.7}$$

where the message waveform is constrained to satisfy the condition that $|m(t)| \le 1$. We observe that $A_c m(t)\cos(2\pi f_c t + \phi_c)$ is a double-sideband AM signal and $A_c \cos(2\pi f_c t + \phi_c)$ is the carrier component. Figure 3.5 illustrates an AM signal in the time domain.

As long as $|m(t)| \le 1$, the amplitude $A_c[1 + m(t)]$ is always positive. This is the desired condition for conventional DSB AM that makes it easy to demodulate, as described next. On the other hand, if $m(t) < -1$ for some t, the AM signal is said to be overmodulated and its demodulation is rendered more complex. In practice, $m(t)$ is scaled so that its magnitude is always less than unity.

It is sometimes convenient to express $m(t)$ as

$$m(t) = am_n(t)$$

where $m_n(t)$ is normalized such that its minimum value is -1. This can be done, for

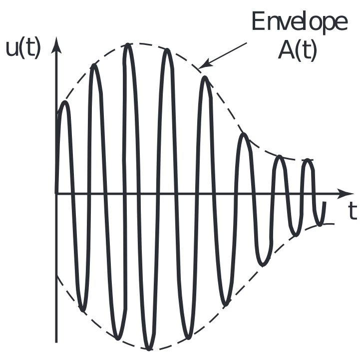

Figure 3.5 A conventional AM signal in the time domain.

example, by defining

$$m_n(t) = \frac{m(t)}{\max|m(t)|}$$

The scale factor a is called the modulation index. Then the modulated signal can be expressed as

$$u(t) = A_c\,[1 + a m_n(t)] \cos 2\pi f_c t \tag{3.2.8}$$

Let us consider the spectral characteristics of the transmitted signal $u(t)$.

Bandwidth of Conventional AM Signal. If $m(t)$ is a deterministic signal with Fourier transform (spectrum) $M(f)$, the spectrum of the amplitude-modulated signal $u(t)$ is

$$\begin{aligned}
U(f) &= \mathcal{F}[a m_n(t)] \star \mathcal{F}[A_c \cos(2\pi f_c t + \varphi_c)] + \mathcal{F}[A_c \cos(2\pi f_c t + \varphi_c)] \\
&= a M_n(f) \star \frac{A_c}{2}\left[e^{j\varphi_c}\delta(f - f_c) + e^{-j\varphi_c}\delta(f + f_c)\right] \\
&\quad + \frac{A_c}{2}\left[e^{j\varphi_c}\delta(f - f_c) + e^{-j\varphi_c}\delta(f + f_c)\right]
\end{aligned}$$

$$\begin{aligned}
U(f) = \frac{A_c}{2}\Big[&e^{j\varphi_c} a M_n(f - f_c) + e^{j\varphi_c}\delta(f - f_c) \\
&+ e^{-j\varphi_c} a M_n(f + f_c) + e^{-j\varphi_c}\delta(f + f_c)\Big]
\end{aligned}$$

Obviously, the spectrum of a conventional AM signal occupies a bandwidth twice the bandwidth of the message signal.

Example 3.2.3

Suppose that the modulating signal $m_n(t)$ is a sinusoid of the form

$$m_n(t) = \cos 2\pi f_m t \qquad f_m \ll f_c$$

Determine the DSB AM signal, its upper and lower sidebands, and its spectrum, assuming a modulation index of a.

Solution From Equation (3.2.8), the DSB AM signal is expressed as

$$\begin{aligned}
u(t) &= A_c[1 + a\cos 2\pi f_m t]\cos(2\pi f_c t + \varphi_c) \\
&= A_c \cos(2\pi f_c t + \varphi_c) + \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t + \varphi_c] \\
&\quad + \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t + \varphi_c]
\end{aligned}$$

The lower sideband component is

$$u_\ell(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t + \varphi_c]$$

while the upper sideband component is

$$u_u(t) = \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t + \varphi_c]$$

The spectrum of the DSB AM signal $u(t)$ is

$$U(f) = \frac{A_c}{2}\left[e^{j\varphi_c}\delta(f - f_c) + e^{-j\varphi_c}\delta(f + f_c)\right]$$
$$+ \frac{A_c a}{4}\left[e^{j\varphi_c}\delta(f - f_c + f_m) + e^{-j\varphi_c}\delta(f + f_c - f_m)\right]$$
$$+ \frac{A_c a}{4}\left[e^{j\varphi_c}\delta(f - f_c - f_m) + e^{-j\varphi_c}\delta(f + f_c + f_m)\right]$$

The spectrum $|U(f)|$ is shown in Figure 3.6. It is interesting to note that the power of the carrier component, which is $A_c^2/2$, exceeds the total power $(A_c^2 a^2/2)$ of the two sidebands.

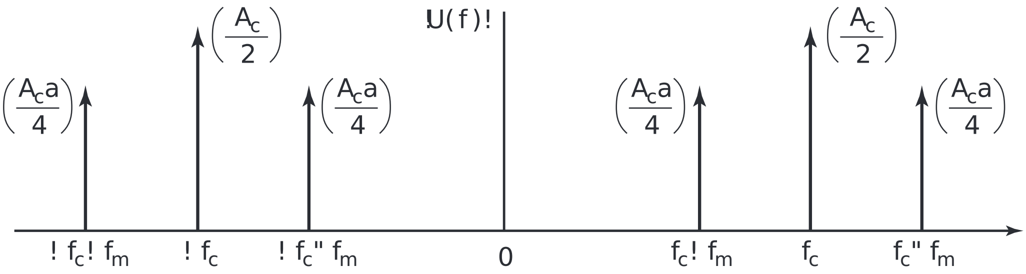

Figure 3.6 Spectrum of a DSB AM signal in Example 3.2.3.

Power for Conventional AM Signal. Conventional AM signal is similar to DSB when $m(t)$ is substituted with $1 + am_n(t)$. As we have already seen in the DSB-SC case, the power in the modulated signal is [see Equation (3.2.3)]

$$P_u = \frac{A_c^2}{2} P_m$$

where P_m denotes the power in the message signal. For the conventional AM

$$P_m = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} (1 + am_n(t))^2\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left(1 + a^2 m_n^2(t)\right) dt$$

where we have assumed that the average of $m_n(t)$ is zero. This is a reasonable assumption for many signals including audio signals. Therefore, for conventional AM

$$P_m = 1 + a^2 P_{m_n}$$

and, hence,

$$P_u = \frac{A_c^2}{2} + \frac{A_c^2}{2} a^2 P_{m_n}$$

The first component in the the preceding relation is due to the existence of the carrier and this component does not carry any information. The second component is the information carrying component. Note that the second component is usually much smaller than the first component ($|m_n(t)| < 1$, and for signals with large dynamic range, $P_{m_n} \ll 1$). This shows that the conventional AM systems are far less power efficient compared with DSB-SC systems. The advantage of conventional AM is that it is easily demodulated.

Demodulation of Conventional DSB AM Signals. The major advantage of conventional AM signal transmission is the ease with which the signal can be demodulated. There is no need for a synchronous demodulator. Since the message signal $m(t)$ satisfies the condition $|m(t)| < 1$, the envelope (amplitude) $1 + m(t) > 0$. If we rectify the received signal, we eliminate the negative values without affecting the message signal as shown in Figure 3.7. The rectified signal is equal to $u(t)$ when $u(t) > 0$ and zero when $u(t) < 0$. The message signal is recovered by passing the rectified signal through a lowpass filter whose bandwidth matches that of the message signal. The combination of the rectifier and the lowpass filter is called an envelope detector.

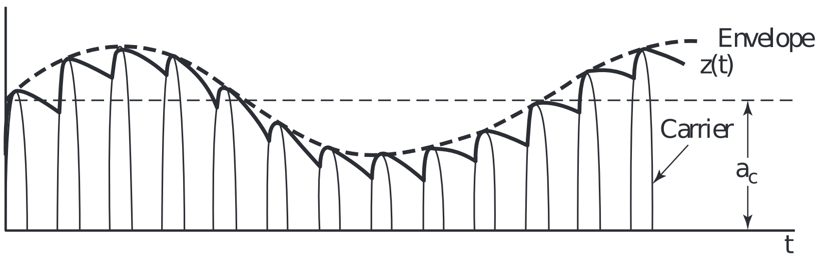

Figure 3.7 Envelope detection of conventional AM signal.

Ideally, the output of the envelope detector is of the form

$$d(t) = g_1 + g_2 m(t) \tag{3.2.9}$$

where g_1 represents a dc component and g_2 is a gain factor due to the signal demodulator. The dc component can be eliminated by passing $d(t)$ through a transformer, whose output is $g_2 m(t)$.

The simplicity of the demodulator has made conventional DSB AM a practical choice for AM radio broadcasting. Since there are literally billions of radio receivers, an inexpensive implementation of the demodulator is extremely important. The power inefficiency of conventional AM is justified by the fact that there are few broadcast transmitters relative to the number of receivers. Consequently, it is cost effective to construct powerful transmitters and sacrifice power efficiency in order to simplify the signal demodulation at the receivers.

3.2.3 Single-Sideband AM

In Section 3.2.1 we showed that a DSB-SC AM signal required a channel bandwidth of $B_c = 2W$ Hz for transmission, where W is the bandwidth of the baseband signal. However, the two sidebands are redundant. Next, we demonstrate that the transmission of either sideband is sufficient to reconstruct the message signal $m(t)$ at the receiver. Thus, we reduce the bandwidth of the transmitted to that of the baseband signal.

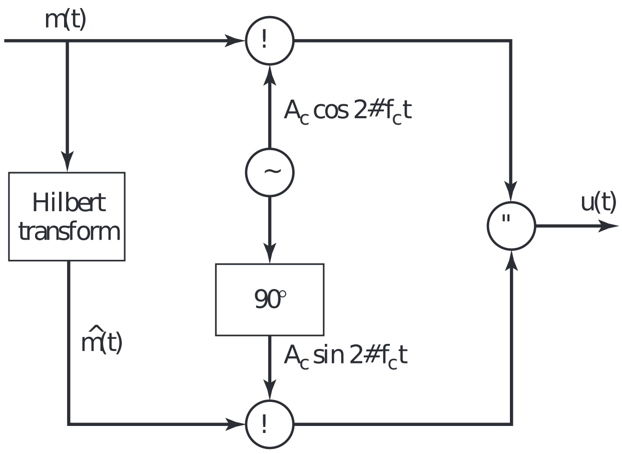

Figure 3.8 Generation of a single-sideband AM signal.

First, we demonstrate that a single-sideband (SSB) AM signal is represented mathematically as

$$u(t) = A_c m(t) \cos 2\pi f_c t \mp A_c \hat{m}(t) \sin 2\pi f_c t \tag{3.2.10}$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$ that was introduced in Section 2.5, and the plus-or-minus sign determines which sideband we obtain. We recall that the Hilbert transform may be viewed as a linear filter with impulse response $h(t) = 1/\pi t$ and frequency response

$$H(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \\ 0, & f = 0 \end{cases} \tag{3.2.11}$$

Therefore, the SSB AM signal $u(t)$ may be generated by using the system configuration shown in Figure 3.8.

The method shown in Figure 3.8 for generating a SSB AM signal is one that employs a Hilbert transform filter. Another method, illustrated in Figure 3.9, generates a DSB-SC AM signal and then employs a filter which selects either the upper sideband or the lower sideband of the double-sideband AM signal.

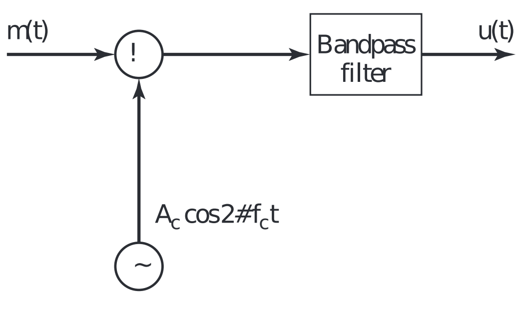

Figure 3.9 Generation of a single-sideband AM signal by filtering one of the sidebands of a DSB-SC AM signal.

Spectral Characteristics of the Single Sideband Signal. Let $m(t)$ be a signal with Fourier transform (spectrum) $M(f)$. An upper single-sideband amplitude-modulated signal (USSB AM) is obtained by eliminating the lower sideband of a DSB amplitude-modulated signal. Suppose we eliminate the lower sideband of the DSB AM signal, $u_{DSB}(t) = 2A_c m(t)\cos 2\pi f_c t$, by passing it through a highpass filter whose transfer function is given by

$$H(f) = \begin{cases} 1, & |f| > f_c \\ 0, & \text{otherwise} \end{cases}$$

Obviously $H(f)$ can be written as

$$H(f) = u_{-1}(f - f_c) + u_{-1}(-f - f_c)$$

where $u_{-1}(\cdot)$ represents the unit step function. Therefore the spectrum of the USSB AM signal is given by

$$U_u(f) = A_c M(f - f_c)u_{-1}(f - f_c) + A_c M(f + f_c)u_{-1}(-f - f_c)$$

or equivalently

$$U_u(f) = A_c M(f)u_{-1}(f)\big|_{f=f-f_c} + A_c M(f)u_{-1}(-f)\big|_{f=f+f_c} \tag{3.2.12}$$

Taking inverse Fourier transform of both sides of Equation (3.2.12) and using the modulation property of the Fourier transform, we obtain

$$u_u(t) = A_c m(t) \star \mathscr{F}^{-1}[u_{-1}(f)]\, e^{j2\pi f_c t} + A_c m(t) \star \mathscr{F}^{-1}[u_{-1}(-f)]\, e^{-j2\pi f_c t} \tag{3.2.13}$$

By noting that

$$\begin{aligned} \mathscr{F}\left[\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right] &= u_{-1}(f) \\ \mathscr{F}\left[\frac{1}{2}\delta(t) - \frac{j}{2\pi t}\right] &= u_{-1}(-f) \end{aligned} \tag{3.2.14}$$

and substituting Equation (3.2.14) in Equation (3.2.13), we obtain

$$\begin{aligned} u_u(t) &= A_c m(t) \star \left[\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right] e^{j2\pi f_c t} + A_c m(t) \star \left[\frac{1}{2}\delta(t) - \frac{j}{2\pi t}\right] e^{-j2\pi f_c t} \\ &= \frac{A_c}{2}[m(t) + j\hat{m}(t)]e^{j2\pi f_c t} + \frac{A_c}{2}[m(t) - j\hat{m}(t)]e^{-j2\pi f_c t} \end{aligned} \tag{3.2.15}$$

where we have used the identities

$$\begin{aligned} m(t) \star \delta(t) &= m(t) \\ m(t) \star \frac{1}{\pi t} &= \hat{m}(t) \end{aligned}$$

Using Euler's relations in Equation (3.2.15), we obtain

$$u_u(t) = A_c m(t)\cos 2\pi f_c t - A_c \hat{m}(t)\sin 2\pi f_c t \tag{3.2.16}$$

which is the time-domain representation of an USSB AM signal. The expression for the LSSB AM signal can be derived by noting that

$$u_u(t) + u_\ell(t) = u_{DSB}(t)$$

or

$$A_c m(t) \cos 2\pi f_c t - A_c \hat{m}(t) \sin 2\pi f_c t + u_\ell(t) = 2A_c m(t) \cos 2\pi f_c t$$

and, therefore,

$$u_\ell(t) = A_c m(t) \cos 2\pi f_c t + A_c \hat{m}(t) \sin 2\pi f_c t \tag{3.2.17}$$

Thus, the time domain representation of a SSB AM signal can in general be expressed as

$$u_{SSB}(t) = A_c m(t) \cos 2\pi f_c t \mp A_c \hat{m}(t) \sin 2\pi f_c t \tag{3.2.18}$$

where the minus sign corresponds to the USSB AM signal and the plus sign corresponds to the LSSB AM signal.

Example 3.2.4

Suppose that the modulating signal is a sinusoid of the form

$$m(t) = \cos 2\pi f_m t, \quad f_m \ll f_c$$

Determine the two possible SSB AM signals

Solution The Hilbert transform of $m(t)$ is

$$\hat{m}(t) = \sin 2\pi f_m t \tag{3.2.19}$$

Hence,

$$u(t) = A_c \cos 2\pi f_m t \cos 2\pi f_c t \mp A_c \sin 2\pi f_m t \sin 2\pi f_c t \tag{3.2.20}$$

If we take the upper $(-)$ sign we obtain the upper sideband signal

$$u_u(t) = A_c \cos 2\pi (f_c + f_m)t$$

On the other hand, if we take the lower $(+)$ sign in Equation (3.2.20) we obtain the lower sideband signal

$$u_\ell(t) = A_c \cos 2\pi (f_c - f_m)t$$

The spectra of $u_u(t)$ and $u_\ell(t)$ were previously given in Figure 3.2.

Demodulation of SSB AM Signals. To recover the message signal $m(t)$ in the received SSB AM signal, we require a phase coherent or synchronous demodulator, as was the case for DSB-SC AM signals. Thus, for the USSB signal as given in Equation (3.2.18), we have

$$\begin{aligned} r(t) \cos 2\pi f_c t &= u(t) \cos(2\pi f_c t + \phi) \\ &= \tfrac{1}{2} A_c m(t) \cos \phi + \tfrac{1}{2} A_c \hat{m}(t) \sin \phi + \text{double frequency terms} \end{aligned} \tag{3.2.21}$$

By passing the product signal in Equation (3.2.21) through an ideal lowpass filter, the double-frequency components are eliminated, leaving us with

$$y_\ell(t) = \tfrac{1}{2}A_c m(t)\cos\phi + \tfrac{1}{2}A_c \hat{m}(t)\sin\phi \tag{3.2.22}$$

Note that the effect of the phase offset is not only to reduce the amplitude of the desired signal $m(t)$ by $\cos\phi$, but it also results in an undesirable sideband signal due to the presence of $\hat{m}(t)$ in $y_\ell(t)$. The latter component was not present in a DSB-SC signal and, hence, it was not a factor. However, it is an important element that contributes to the distortion of the demodulated SSB signal.

The transmission of a pilot tone at the carrier frequency is a very effective method for providing a phase-coherent reference signal for performing synchronous demodulation at the receiver. Thus, the undesirable sideband signal component is eliminated. However, this means that a portion of the transmitted power must be allocated to the transmission of the carrier.

The spectral efficiency of SSB AM makes this modulation method very attractive for use in voice communications over telephone channels (wire lines and cables). In this application, a pilot tone is transmitted for synchronous demodulation and shared among several channels.

The filter method shown in Figure 3.9 for selecting one of the two signal sidebands for transmission is particularly difficult to implement when the message signal $m(t)$ has a large power concentrated in the vicinity of $f = 0$. In such a case, the sideband filter must have an extremely sharp cutoff in the vicinity of the carrier in order to reject the second sideband. Such filter characteristics are very difficult to implement in practice.

3.2.4 Vestigial-Sideband AM

The stringent frequency-response requirements on the sideband filter in a SSB AM system can be relaxed by allowing a part, called a vestige, of the unwanted sideband to appear at the output of the modulator. Thus, we simplify the design of the sideband filter at the cost of a modest increase in the channel bandwidth required to transmit the signal. The resulting signal is called vestigial-sideband (VSB) AM.

To generate a VSB AM signal we begin by generating a DSB-SC AM signal and passing it through a sideband filter with frequency response $H(f)$ as shown in Figure 3.10. In the time domain the VSB signal may be expressed as

$$u(t) = [A_c m(t)\cos 2\pi f_c t] \star h(t) \tag{3.2.23}$$

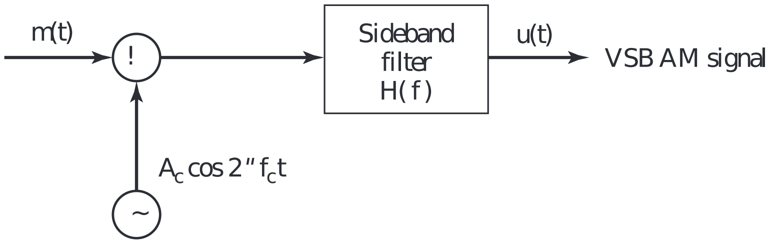

Figure 3.10 Generation of VSB AM signal.

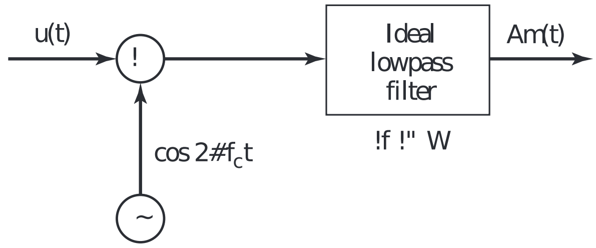

Figure 3.11 Demodulation of VSB signal.

where $h(t)$ is the impulse response of the VSB filter. In the frequency domain, the corresponding expression is

$$U(f) = \frac{A_c}{2}[M(f - f_c) + M(f + f_c)]H(f) \tag{3.2.24}$$

To determine the frequency-response characteristics of the filter, let us consider the demodulation of the VSB signal $u(t)$. We multiply $u(t)$ by the carrier component $\cos 2\pi f_c t$ and pass the result through an ideal lowpass filter, as shown in Figure 3.11. Thus, the product signal is

$$v(t) = u(t) \cos 2\pi f_c t$$

or, equivalently,

$$V(f) = \tfrac{1}{2}[U(f - f_c) + U(f + f_c)] \tag{3.2.25}$$

If we substitute for $U(f)$ from Equation (3.2.24) into Equation (3.2.25), we obtain

$$\begin{aligned} V(f) = {} & \frac{A_c}{4}[M(f - 2f_c) + M(f)]H(f - f_c) \\ & + \frac{A_c}{4}[M(f) + M(f + 2f_c)]H(f + f_c) \end{aligned} \tag{3.2.26}$$

The lowpass filter rejects the double-frequency terms and passes only the components in the frequency range $|f| \leq W$. Hence, the signal spectrum at the output of the ideal lowpass filter is

$$V_\ell(f) = \frac{A_c}{4} M(f)[H(f - f_c) + H(f + f_c)] \tag{3.2.27}$$

We require that the message signal at the output of the lowpass filter be undistorted. Hence, the VSB filter characteristic must satisfy the condition

$$H(f - f_c) + H(f + f_c) = \text{constant}, \quad |f| \leq W \tag{3.2.28}$$

This condition is satisfied by a filter that has the frequency-response characteristic shown in Figure 3.12. We note that $H(f)$ selects the upper sideband and a vestige of the lower sideband. It has odd symmetry about the carrier frequency f_c, in the frequency range $f_c - f_a < f < f_c + f_a$, where f_a is a conveniently selected frequency that is

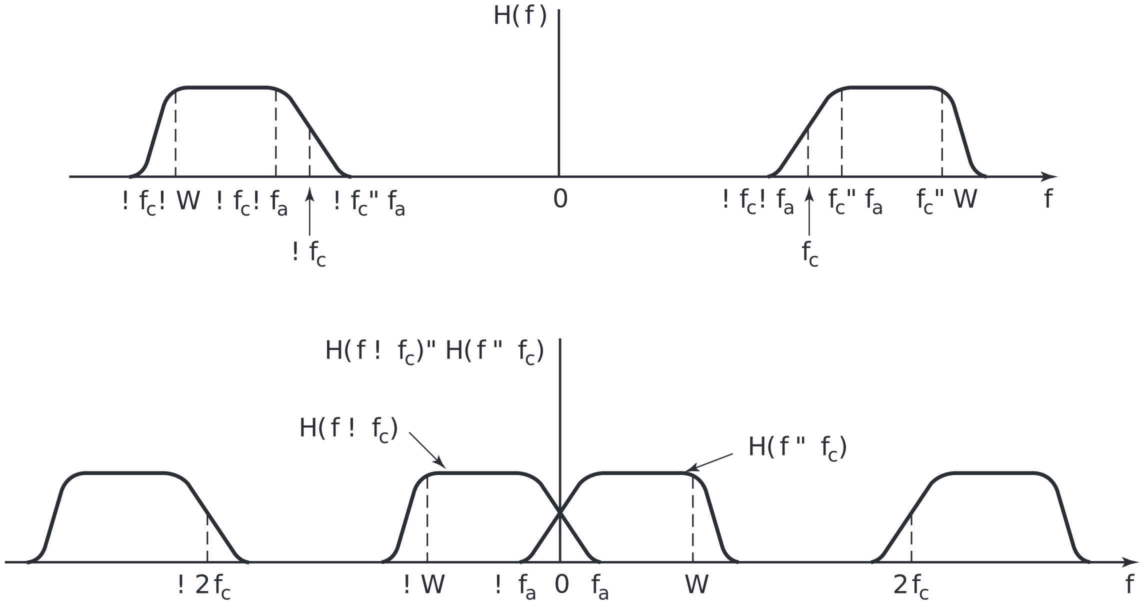

Figure 3.12 VSB filter characteristics.

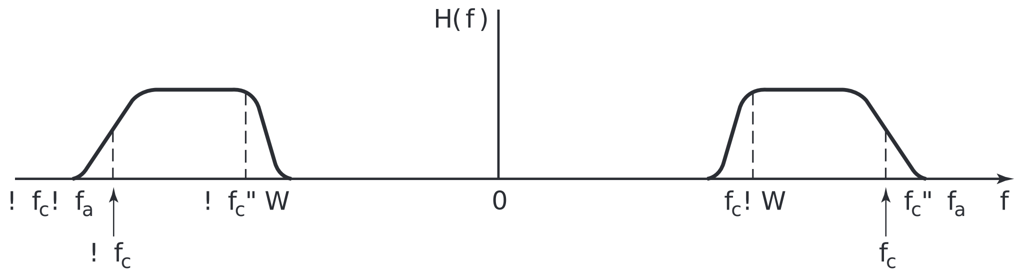

Figure 3.13 Frequency response of VSB filter for selecting the lower sideband of the message signals.

some small fraction of W; i.e., $f_a \ll W$. Thus, we obtain an undistorted version of the transmitted signal. Figure 3.13 illustrates the frequency response of a VSB filter that selects the lower sideband and a vestige of the upper sideband.

In practice, the VSB filter is designed to have some specified phase characteristic. To avoid distortion of the message signal, the VSB filter should be designed to have linear phase over its passband $f_c - f_a \le |f| \le f_c + W$.

Example 3.2.5

Suppose that the message signal is given as

$$m(t) = 10 + 4\cos 2\pi t + 8\cos 4\pi t + 10\cos 20\pi t$$

Specify the frequency-response characteristic of a VSB filter that passes the upper sideband and the first frequency component of the lower sideband.

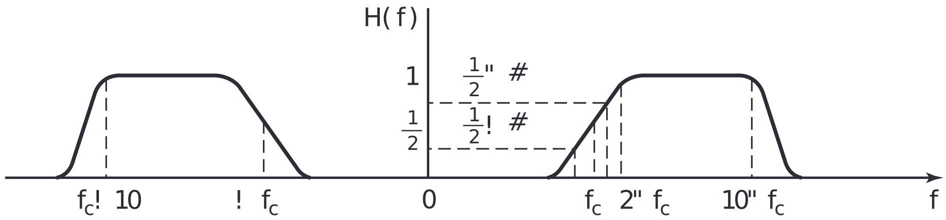

Figure 3.14 Frequency response characteristics of VSB filter in Example 3.2.5.

Solution The spectrum of the DSB-SC AM signal $u(t) = m(t)\cos 2\pi f_c t$ is

$$U(f) = 5[\delta(f - f_c) + \delta(f + f_c)] + 2[\delta(f - f_c - 1) + \delta(f + f_c + 1)]$$
$$+ 4[\delta(f - f_c - 2) + \delta(f + f_c + 2)] + 5[\delta(f - f_c - 10) + \delta(f + f_c + 10)]$$

The VSB filter can be designed to have unity gain in the range $2 \le |f - f_c| \le 10$, a gain of $1/2$ at $f = f_c$, a gain of $1/2 + \alpha$ at $f = f_c + 1$, and a gain of $1/2 - \alpha$ at $f = f_c - 1$, where α is some conveniently selected parameter that satisfies the condition $0 < \alpha < 1/2$. Figure 3.14 illustrates the frequency-response characteristic of the VSB filter.

3.2.5 Implementation of AM Modulators and Demodulators

There are several different methods for generating AM modulated signals. We shall describe the methods most commonly used in practice. Since the process of modulation involves the generation of new frequency components, modulators are generally characterized as nonlinear and, or, time-variant systems.

Power-Law Modulation. Let us consider the use of a nonlinear device such as a P-N diode which has a voltage-current characteristic as shown in Figure 3.15. Suppose that the voltage input to such a device is the sum of the message signal $m(t)$ and the carrier $A_c \cos 2\pi f_c t$, as illustrated in Figure 3.16. The nonlinearity will generate a product of the message $m(t)$ with the carrier, plus additional terms. The desired modulated signal can be filtered out by passing the output of the nonlinear device through a bandpass filter.

To elaborate on this method, suppose that the nonlinear device has an input–output (square-law) characteristic of the form

$$v_0(t) = a_1 v_i(t) + a_2 v_i^2(t) \tag{3.2.29}$$

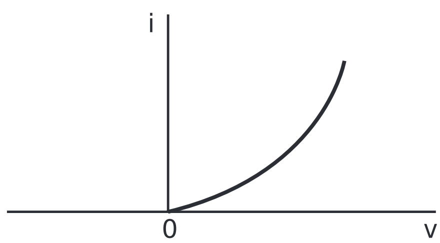

Figure 3.15 Voltage-current characteristic of P-N diode.

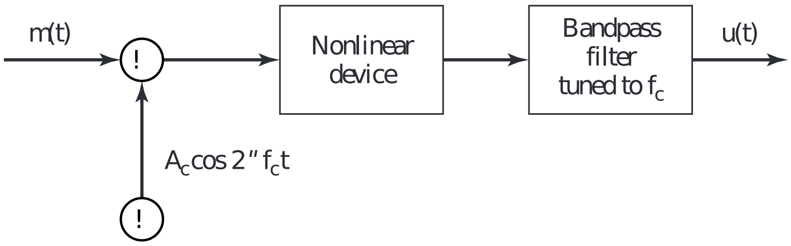

Figure 3.16 Block diagram of power-law AM modulator.

where $v_i(t)$ is the input signal, $v_0(t)$ is the output signal, and the parameters (a_1, a_2) are constants. Then, if the input to the nonlinear device is

$$v_i(t) = m(t) + A_c \cos 2\pi f_c t, \tag{3.2.30}$$

its output is

$$\begin{aligned} v_o(t) &= a_1[m(t) + A_c \cos 2\pi f_c t] + a_2[m(t) + A_c \cos 2\pi f_c t]^2 \\ &= a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2 2\pi f_c t + A_c a_1 \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos 2\pi f_c t \end{aligned} \tag{3.2.31}$$

The output of the bandpass filter with bandwidth $2W$ centered at $f = f_c$ yields

$$u(t) = A_c a_1 \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos 2\pi f_c t \tag{3.2.32}$$

where $2a_2|m(t)|/a_1 < 1$ by design. Thus, the signal generated by this method is a conventional DSB AM signal.

Switching Modulator. Another method for generating an AM modulated signal is by means of a switching modulator. Such a modulator can be implemented by the system illustrated in Figure 3.17(a). The sum of the message signal and the carrier; i.e., $v_i(t)$ given by Equation (3.2.30), are applied to a diode that has the input–output voltage characteristic shown in Figure 3.17(b), where $A_c \gg m(t)$. The output across the load resistor is simply

$$v_0(t) = \begin{cases} v_i(t), & c(t) > 0 \\ 0, & c(t) < 0 \end{cases} \tag{3.2.33}$$

This switching operation may be viewed mathematically as a multiplication of the input $v_i(t)$ with the switching function $s(t)$; i.e.,

$$v_0(t) = [m(t) + A_c \cos 2\pi f_c t] s(t) \tag{3.2.34}$$

where $s(t)$ is shown in Figure 3.17(c).

Since $s(t)$ is a periodic function, it is represented in the Fourier series as

$$s(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos[2\pi f_c t(2n-1)] \tag{3.2.35}$$

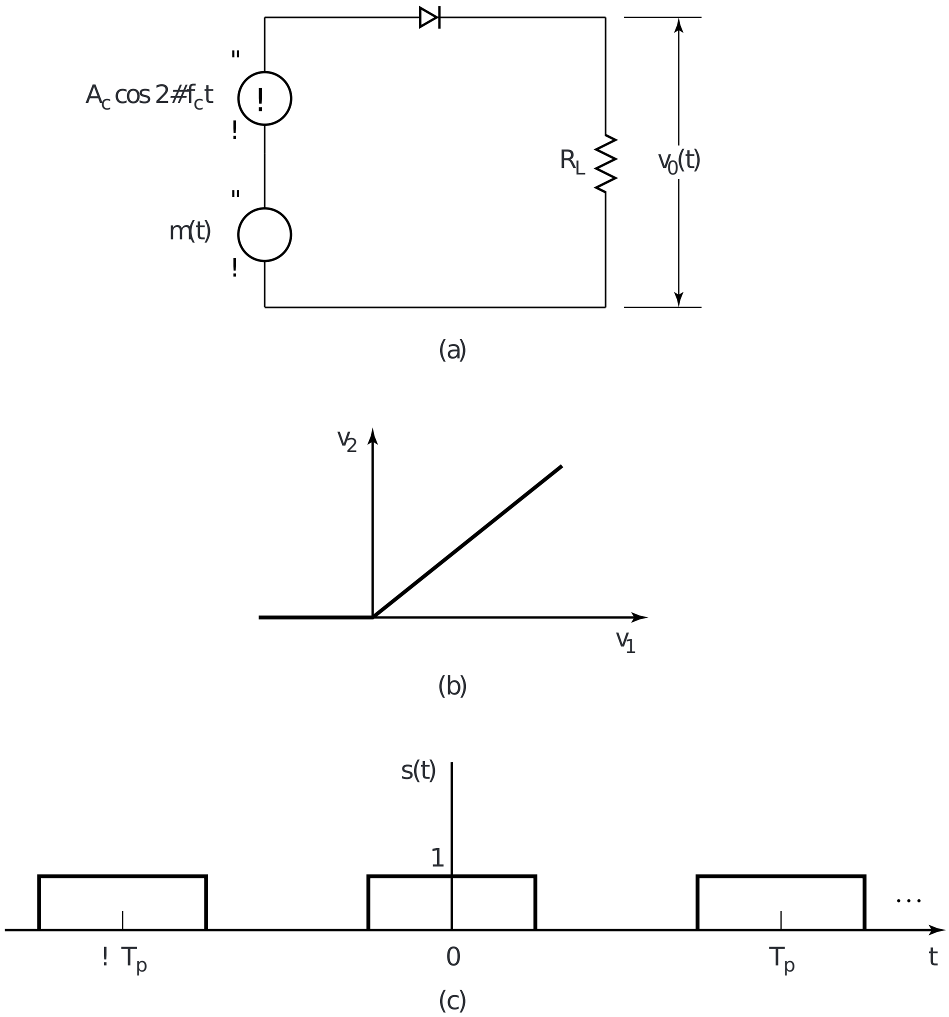

Figure 3.17 Switching modulator and periodic switching signal.

Hence,

$$v_0(t) = [m(t) + A_c \cos 2\pi f_c t]s(t)$$
$$= \frac{A_c}{2}\left(1 + \frac{4}{\pi A_c} m(t)\right) \cos 2\pi f_c t + \text{other terms} \tag{3.2.36}$$

The desired AM modulated signal is obtained by passing $v_0(t)$ through a bandpass filter with center frequency $f = f_c$ and bandwidth $2W$. At its output, we have the desired conventional DSB AM signal

$$u(t) = \frac{A_c}{2}\left(1 + \frac{4}{\pi A_c} m(t)\right) \cos 2\pi f_c t \tag{3.2.37}$$

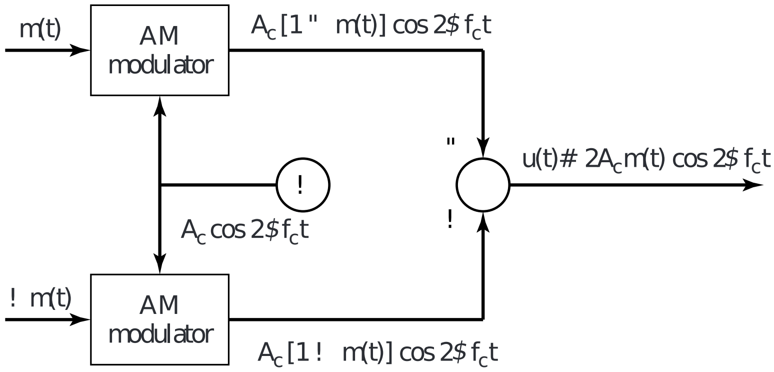

Figure 3.18 Block diagram of a balanced modulator.

Balanced Modulator. A relatively simple method to generate a DSB-SC AM signal is to use two conventional AM modulators arranged in the configuration illustrated in Figure 3.18. For example, we may use two square-law AM modulators as described above. Care must be taken to select modulators with approximately identical characteristics so that the carrier component cancels out at the summing junction.

Ring Modulator. Another type of modulator for generating a DSB-SC AM signal is the ring modulator illustrated in Figure 3.19. The switching of the diodes is controlled by a square wave of frequency f_c, denoted as $c(t)$, which is applied to the center taps of the two transformers. When $c(t) > 0$, the top and bottom diodes conduct, while the two diodes in the crossarms are off. In this case, the message signal $m(t)$ is multiplied by $+1$. When $c(t) < 0$, the diodes in the crossarms of the ring conduct, while the other two are switched off. In this case, the message signal $m(t)$ is multiplied by -1. Consequently, the operation of the ring modulator may be described mathematically as

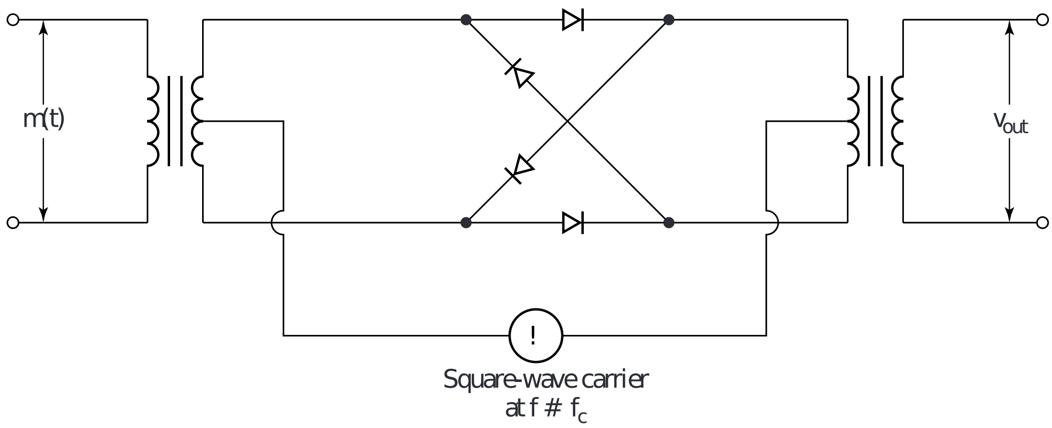

Figure 3.19 Ring modulator for generating DSB-SC AM signal.

a multiplier of $m(t)$ by the square-wave carrier $c(t)$; i.e.,

$$v_0(t) = m(t)c(t) \tag{3.2.38}$$

as shown in Figure 3.19.

Since $c(t)$ is a periodic function, it is represented by the Fourier series

$$c(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos[2\pi f_c(2n-1)t] \tag{3.2.39}$$

Hence, the desired DSB-SC AM signal $u(t)$ is obtained by passing $v_0(t)$ through a bandpass filter with center frequency f_c and bandwidth $2W$.

From the discussion above, we observe that the balanced modulator and the ring modulator systems, in effect, multiply the message signal $m(t)$ with the carrier to produce a DSB-SC AM signal. The multiplication of $m(t)$ with $A_c \cos w_c t$ is called a mixing operation. Hence, a mixer is basically a balanced modulator.

The method shown in Figure 3.8 for generating a SSB signal requires two mixers; i.e., two balanced modulators, in addition to the Hilbert transformer. On the other hand, the filter method illustrated in Figure 3.9 for generating a SSB signal requires a single balanced modulator and a sideband filter.

Let us now consider the demodulation of AM signals. We begin with a description of the envelope detector.

Envelope Detector. As previously indicated, conventional DSB AM signals are easily demodulated by means of an envelope detector. A circuit diagram for an envelope detector is shown in Figure 3.20. It consists of a diode and an RC circuit, which is basically a simple lowpass filter.

During the positive half-cycle of the input signal, the diode is conducting and the capacitor charges up to the peak value of the input signal. When the input falls below the voltage on the capacitor, the diode becomes reverse-biased and the input becomes disconnected from the output. During this period, the capacitor discharges slowly through the load resistor R. On the next cycle of the carrier, the diode conducts again when the input signal exceeds the voltage across the capacitor. The capacitor charges up again to the peak value of the input signal and the process is repeated again.

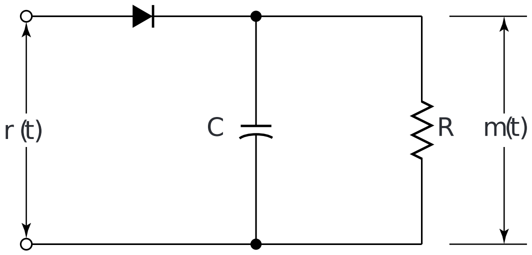

Figure 3.20 An envelope detector.

The time constant RC must be selected so as to follow the variations in the envelope of the carrier-modulated signal. In effect,

$$\frac{1}{f_c} \ll RC \ll \frac{1}{W}$$

In such a case, the capacitor discharges slowly through the resistor and, thus, the output of the envelope detector closely follows the message signal.

Demodulation of DSB-SC AM Signals. As previously indicated, the demodulation of a DSB-SC AM signal requires a synchronous demodulator. That is, the demodulator must use a coherent phase reference, which is usually generated by means of a phase-locked loop (PLL) (see Section 5.2), to demodulate the received signal.

The general configuration is shown in Figure 3.21. A PLL is used to generate a phase-coherent carrier signal that is mixed with the received signal in a balanced modulator. The output of the balanced modulator is passed through a lowpass filter of bandwidth W that passes the desired signal and rejects all signal and noise components above W Hz. The characteristics and operation of the PLL are described in Section 5.2.

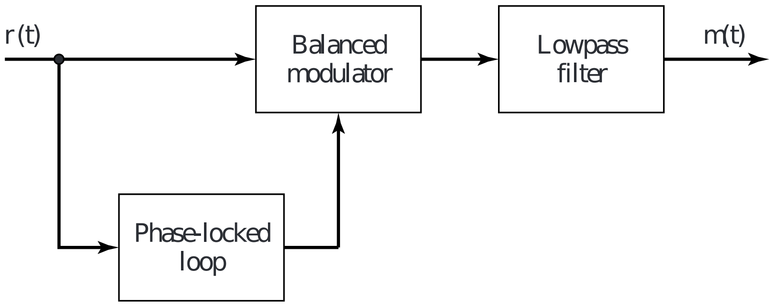

Figure 3.21 Demodulator for DSB-SC AM signal.

Demodulation of SSB Signals. The demodulation of SSB AM signals also requires the use of a phase coherent reference. In the case of signals such as speech, that have relatively little or no power content at dc, it is straightforward to generate the SSB signal, as shown in Figure 3.9, and then to insert a small carrier component that is transmitted along with the message. In such a case we may use the configuration shown in Figure 3.22 to demodulate the SSB signal. We observe that a balanced modulator

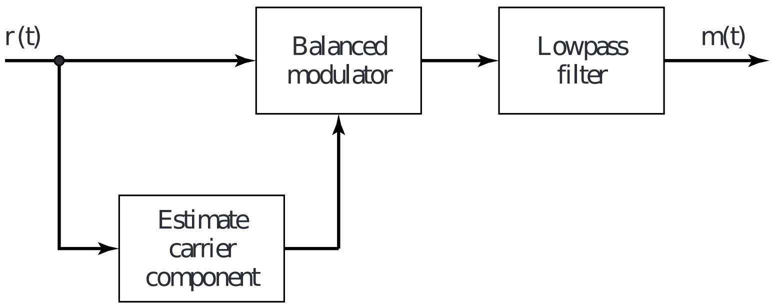

Figure 3.22 Demodulation of SSB AM signal with a carrier component.

is used for the purpose of frequency conversion of the bandpass signal to lowpass or baseband.

Demodulation of VSB Signals. In VSB a carrier component is generally transmitted along with the message sidebands. The existence of the carrier component makes it possible to extract a phase-coherent reference for demodulation in a balanced modulator, as shown in Figure 3.22.

In some applications such as TV broadcast, a large carrier component is transmitted along with the message in the VSB signal. In such a case, it is possible to recover the message by passing the received VSB signal through an envelope detector.

3.2.6 Signal Multiplexing

We have seen that amplitude modulation of a sinusoidal carrier by a message signal $m(t)$ translates the message signal in frequency by an amount equal to the carrier frequency f_c. If we have two or more message signals to transmit simultaneously over the communications channel, it is possible to have each message signal modulate a carrier of a different frequency, where the minimum separation between two adjacent carriers is either $2W$ (for DSB AM) or W (for SSB AM), where W is the bandwidth of each of the message signals. Thus, the various message signals occupy separate frequency bands of the channel and do not interfere with one another in transmission over the channel.

The process of combining a number of separate message signals into a composite signal for transmission over a common channel is called multiplexing. There are two commonly used methods for signal multiplexing: (1) time-division multiplexing and (2) frequency-division multiplexing. Time-division multiplexing is usually used in the transmission of digital information and will be described in Chapter 6. Frequency-division multiplexing (FDM) may be used with either analog or digital signal transmission.

In FDM, the message signals are separated in frequency as described above. A typical configuration of an FDM system is shown in Figure 3.23. This figure illustrates the frequency-division multiplexing of K message signals at the transmitter and their demodulation at the receiver. The lowpass filters at the transmitter are used to ensure that the bandwidth of the message signals is limited to W Hz. Each signal modulates a separate carrier; hence, K modulators are required. Then, the signals from the K modulators are summed and transmitted over the channel. For SSB and VSB modulation, the modulator outputs are filtered prior to summing the modulated signals.

At the receiver of an FDM system, the signals are usually separated by passing through a parallel band of bandpass filters, where each filter is tuned to one of the carrier frequencies and has a bandwidth that is sufficiently wide to pass the desired signal. The output of each bandpass filter is demodulated and each demodulated signal is fed to a lowpass filter that passes the baseband message signal and eliminates the double frequency components.

FDM is widely used in radio and telephone communications. For example, in telephone communications, each voice-message signal occupies a nominal bandwidth of 3 kHz. The message signal is single-sideband modulated for bandwidth efficient

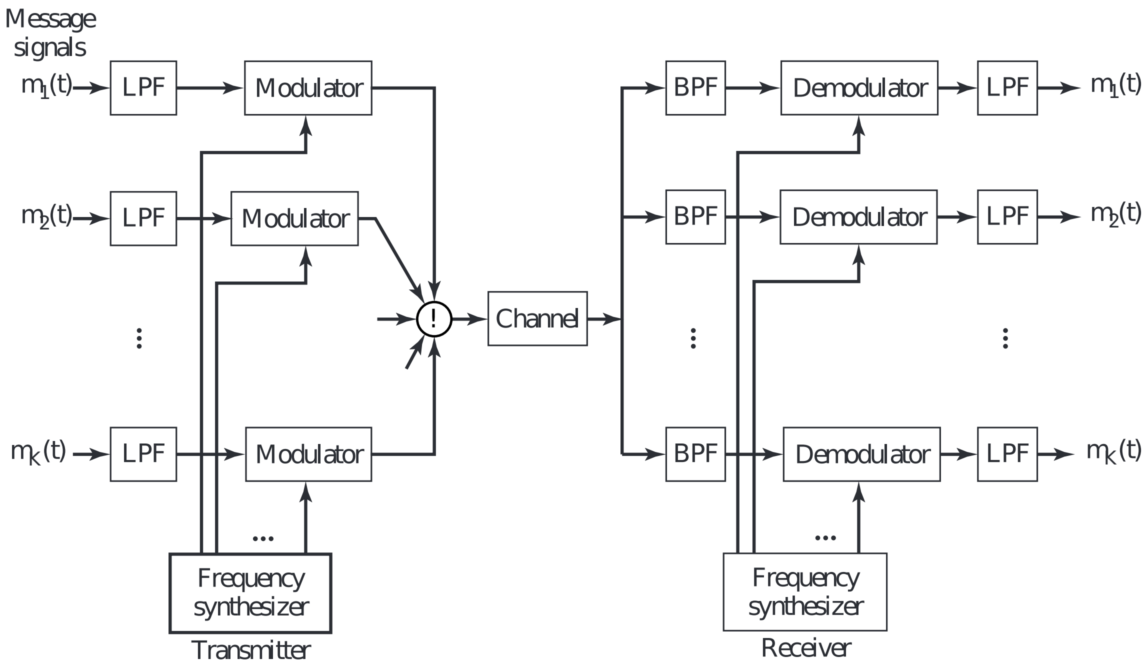

Figure 3.23 Frequency-division multiplexing of multiple signals.

transmission. In the first level of multiplexing, 12 signals are stacked in frequency, with a frequency separation of 4 kHz between adjacent carriers. Thus, a composite 48-kHz channel, called a group channel, is used to transmit the 12 voice-band signals simultaneously. In the next level of FDM, a number of group channels (typically five or six) are stacked together in frequency to form a supergroup channel, and the composite signal is transmitted over the channel. Higher-order multiplexing is obtained by combining several supergroup channels. Thus, an FDM hierarchy is employed in telephone communication systems.

Quadrature-Carrier Multiplexing. A totally different type of multiplexing allows us to transmit two message signals on the same carrier frequency, using two quadrature carriers $A_c \cos 2\pi f_c t$ and $A_c \sin 2\pi f_c t$. To elaborate, suppose that $m_1(t)$ and $m_2(t)$ are two separate message signals to be transmitted over the channel. The signal $m_1(t)$ amplitude modulates the carrier $A_c \cos 2\pi f_c t$ and the signal $m_2(t)$ amplitude modulates the quadrature carrier $A_c \sin 2\pi f_c t$. The two signals are added and transmitted over the channel. Hence, the transmitted signal is

$$u(t) = A_c m_1(t) \cos 2\pi f_c t + A_c m_2(t) \sin 2\pi f_c t \tag{3.2.40}$$

Therefore, each message signal is transmitted by DSB-SC AM. This type of signal multiplexing is called quadrature-carrier multiplexing.

Figure 3.24 illustrates the modulation and demodulation of the quadrature-carrier multiplexed signals. As shown, a synchronous demodulator is required at the receiver to separate and recover the quadrature-carrier modulated signals.

Quadrature-carrier multiplexing results in a bandwidth-efficient communication system that is comparable in bandwidth efficiency to SSB AM.

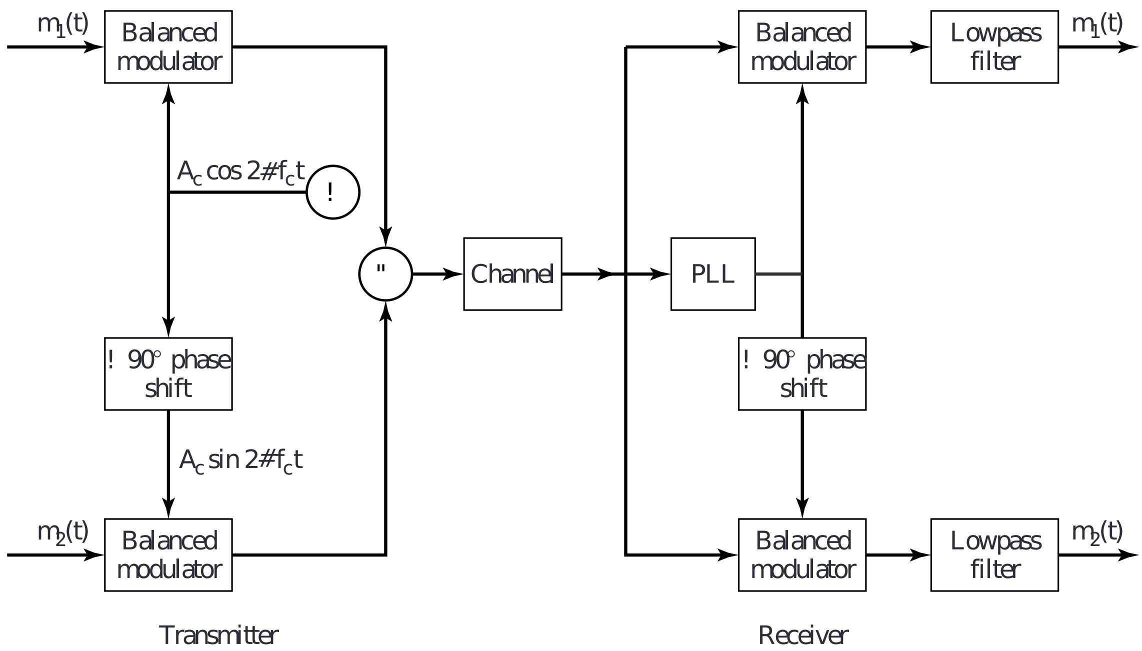

Figure 3.24 Quadrature-carrier multiplexing.

3.3 ANGLE MODULATION

In the previous section we considered amplitude modulation of the carrier as a means for transmitting the message signal. Amplitude-modulation methods are also called linear-modulation methods, although conventional AM is not linear in the strict sense.

Another class of modulation methods are frequency and phase modulation which are described in this section. In frequency-modulation (FM) systems, the frequency of the carrier f_c is changed by the message signal and in phase-modulation (PM) systems the phase of the carrier is changed according to the variations in the message signal. Frequency and phase modulation are obviously quite nonlinear, and very often they are jointly referred to as angle-modulation methods. As our analysis in the following sections will show, angle modulation, due to its inherent nonlinearity, is more complex to implement, and much more difficult to analyze. In many cases only an approximate analysis can be done. Another property of angle modulation is its bandwidth-expansion property. Frequency and phase-modulation systems generally expand the bandwidth such that the effective bandwidth of the modulated signal is usually many times the bandwidth of the message signal.[†] With a higher implementation complexity and a higher bandwidth occupancy, one would naturally raise a question as to the usefulness of these systems. As our analysis in Chapter 5 will show, the major benefit of these systems is their high degree of noise immunity. In fact these systems trade-off bandwidth for high noise immunity. That is the reason that FM systems are widely used in high-fidelity music broadcasting and point-to-point communication systems where the transmitter power is quite limited.

[†]Strictly speaking, the bandwidth of the modulated signal, as it will be shown later, is infinite. That is why we talk about the effective bandwidth.

3.3.1 Representation of FM and PM Signals

An angle-modulated signal in general can be written as

$$u(t) = A_c \cos(\theta(t))$$

$\theta(t)$ is the phase of the signal, and its instantaneous frequency $f_i(t)$ is given by

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) \tag{3.3.1}$$

Since $u(t)$ is a bandpass signal, it can be represented as

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) \tag{3.3.2}$$

and, therefore,

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \tag{3.3.3}$$

If $m(t)$ is the message signal, then in a PM system we have

$$\phi(t) = k_p m(t) \tag{3.3.4}$$

and in an FM system we have

$$f_i(t) - f_c = k_f m(t) = \frac{1}{2\pi}\frac{d}{dt}\phi(t) \tag{3.3.5}$$

where k_p and k_f are phase and frequency deviation constants. From the above relationships we have

$$\phi(t) = \begin{cases} k_p m(t), & \text{PM} \\ 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau, & \text{FM} \end{cases} \tag{3.3.6}$$

Equation (3.3.6) shows the close and interesting relation between FM and PM systems. This close relationship makes it possible to analyze these systems in parallel and only emphasize their main differences. The first interesting result observed from Equation (3.3.6) is that if we phase modulate the carrier with the integral of a message, it is equivalent to frequency modulation of the carrier with the original message. On the other hand, in Equation (3.3.6) the relation can be expressed as

$$\frac{d}{dt}\phi(t) = \begin{cases} k_p \frac{d}{dt} m(t), & \text{PM} \\ 2\pi m(t), & \text{FM} \end{cases} \tag{3.3.7}$$

which shows that if we frequency modulate the carrier with the derivative of a message, the result is equivalent to phase modulation of the carrier with the message itself. Figure 3.25 shows the above relation between FM and PM. Figure 3.26 illustrates a square-wave signal and its integral, a sawtooth signal, and their corresponding FM and PM signals.

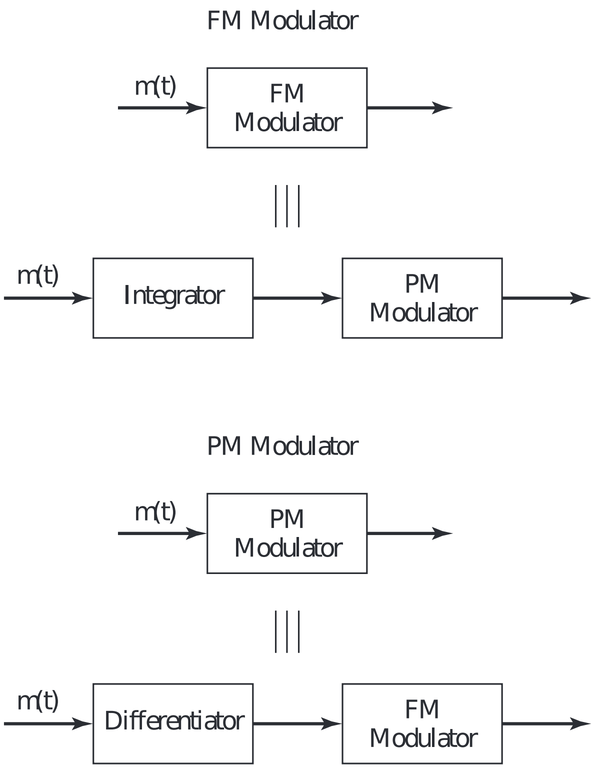

Figure 3.25 A comparison of FM and PM modulators.

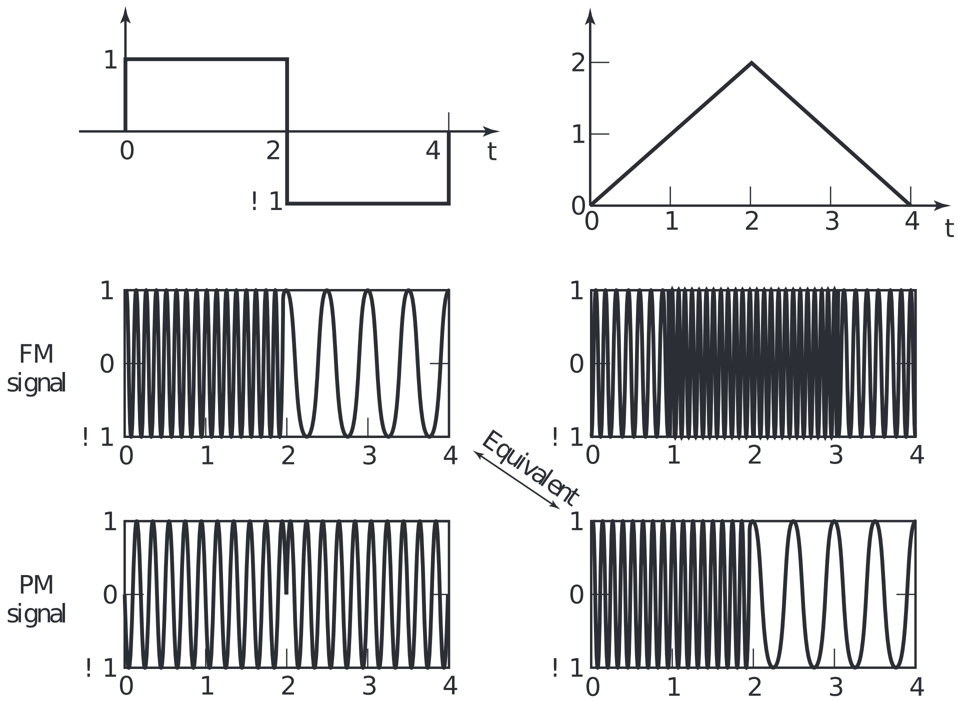

Figure 3.26 FM and PM of square and sawtooth waves.

The demodulation of an FM signal involves finding the instantaneous frequency of the modulated signal and then subtracting the carrier frequency from it. In the demodulation of PM, the demodulation process is done by finding the phase of the signal and then recovering $m(t)$. The maximum phase deviation in a PM system is given by

$$\Delta\phi_{max} = k_p \max[|m(t)|] \tag{3.3.8}$$

and the maximum frequency-deviation in an FM system is given by

$$\Delta f_{max} = k_f \max[|m(t)|] \tag{3.3.9}$$

Example 3.3.1

The message signal

$$m(t) = a\cos(2\pi f_m t)$$

is used to either frequency modulate or phase modulate the carrier $A_c \cos(2\pi f_c t)$. Find the modulated signal in each case.

Solution In PM we have

$$\phi(t) = k_p m(t) = k_p a \cos(2\pi f_m t) \tag{3.3.10}$$

and in FM we have

$$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau = \frac{k_f a}{f_m} \sin(2\pi f_m t) \tag{3.3.11}$$

Therefore, the modulated signals will be

$$u(t) = \begin{cases} A_c \cos(2\pi f_c t + k_p a \cos(2\pi f_m t)), & \text{PM} \\ A_c \cos\left(2\pi f_c t + \frac{k_f a}{f_m} \sin(2\pi f_m t)\right), & \text{FM} \end{cases} \tag{3.3.12}$$

By defining

$$\beta_p = k_p a \tag{3.3.13}$$

$$\beta_f = \frac{k_f a}{f_m} \tag{3.3.14}$$

we have

$$u(t) = \begin{cases} A_c \cos(2\pi f_c t + \beta_p \cos(2\pi f_m t)), & \text{PM} \\ A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t)), & \text{FM} \end{cases} \tag{3.3.15}$$

The parameters β_p and β_f are called the modulation indices of the PM and FM systems respectively.

We can extend the definition of the modulation index for a general nonsinusoidal signal $m(t)$ as

$$\beta_p = k_p \max[|m(t)|] \tag{3.3.16}$$

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} \tag{3.3.17}$$

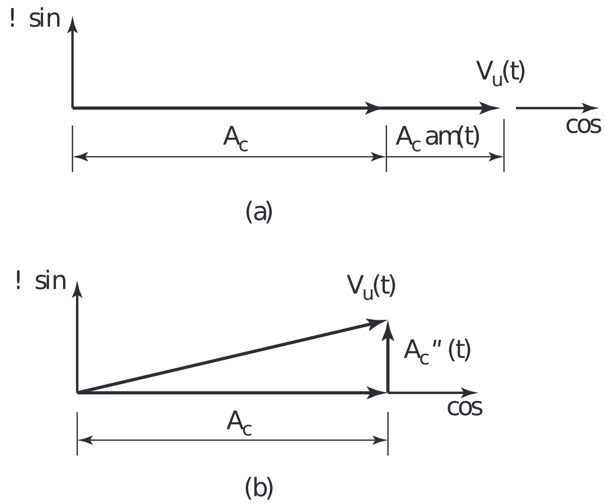

Figure 3.27 Phasor diagram for the conventional AM (a) and narrowband angle modulation (b).

where W denotes the bandwidth of the message signal $m(t)$. In terms of the maximum phase and frequency deviation $\Delta\phi_{max}$ and Δf_{max}, we have

$$\beta_p = \Delta\phi_{max} \tag{3.3.18}$$

$$\beta_f = \frac{\Delta f_{max}}{W} \tag{3.3.19}$$

Narrowband Angle Modulation.[†] If in an angle-modulation system the deviation constants k_p and k_f and the message signal $m(t)$ are such that for all t we have $\phi(t) \ll 1$, then we can use a simple approximation to expand $u(t)$ as

$$\begin{aligned} u(t) &= A_c \cos 2\pi f_c t \cos\phi(t) - A_c \sin 2\pi f_c t \sin\phi(t) \\ &\approx A_c \cos 2\pi f_c t - A_c\phi(t) \sin 2\pi f_c t \end{aligned} \tag{3.3.20}$$

This last equation shows that in this case the modulated signal is very similar to a conventional AM signal. The only difference is that the message signal $m(t)$ is modulated on a sine carrier rather than a cosine carrier. The bandwidth of this signal is similar to the bandwidth of a conventional AM signal, which is twice the bandwidth of the message signal. Of course this bandwidth is only an approximation to the real bandwidth of the FM signal. A phasor diagram for this signal and the comparable conventional AM signal are given in Figure 3.27. Note that compared to conventional AM, the narrowband angle-modulation scheme has far less amplitude variations. Of course, the angle-modulation system has constant amplitude and, hence, there should be no amplitude variations in the phasor-diagram representation of the system. The slight variations here are due to the first-order approximation that we have used for the expansions of $\sin(\phi(t))$ and $\cos(\phi(t))$. As we will see in Chapter 5, the narrowband angle-modulation method does not provide any better noise immunity compared to a conventional AM system. Therefore, narrowband angle modulation is seldom used in practice for communication purposes. However, these systems can be used as an intermediate stage for generation of wideband angle-modulated signals as we will discuss in Section 3.3.3.

[†]Also known as low-index angle modulation.

3.3.2 Spectral Characteristics of Angle-Modulated Signals

Due to the inherent nonlinearity of angle-modulation systems the precise characterization of their spectral properties, even for simple message signals, is mathematically intractable. Therefore, the derivation of the spectral characteristics of these signals usually involves the study of very simple modulating signals and certain approximations. Then the results are generalized to the more complicated messages. We will study the spectral characteristics of an angle-modulated signal in three cases: when the modulating signal is a sinusoidal signal, when the modulating signal is a general periodic signal, and when the modulating signal is a general nonperiodic signal.

Angle Modulation by a Sinusoidal Signal. Let us begin with the case where the message signal is a sinusoidal signal. As we have seen, in this case for both FM and PM, we have

$$u(t) = A_c \cos(2\pi f_c t + \beta \sin 2\pi f_m t) \tag{3.3.21}$$

where β is the modulation index that can be either β_p or β_f. Therefore, the modulated signal can be written as

$$u(t) = \mathrm{Re}\left(A_c e^{j2\pi f_c t} e^{j\beta \sin 2\pi f_m t}\right) \tag{3.3.22}$$

Since $\sin 2\pi f_m t$ is periodic with period $T_m = \frac{1}{f_m}$, the same is true for the complex exponential signal

$$e^{j\beta \sin 2\pi f_m t}$$

Therefore, it can be expanded in a Fourier series representation. The Fourier series coefficients are obtained from the integral

$$\begin{aligned} c_n &= f_m \int_0^{\frac{1}{f_m}} e^{j\beta \sin 2\pi f_m t} e^{-jn2\pi f_m t}\, dt \\ &\overset{u=2\pi f_m t}{=} \frac{1}{2\pi} \int_0^{2\pi} e^{j(\beta \sin u - nu)}\, du \end{aligned} \tag{3.3.23}$$

This latter integral is a well-known integral known as the Bessel function of the first kind of order n and is denoted by $J_n(\beta)$. Therefore, we have the Fourier series for the complex exponential as

$$e^{j\beta \sin 2\pi f_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} \tag{3.3.24}$$

By substituting Equation (3.3.24) in Equation (3.3.22), we obtain

$$\begin{aligned} u(t) &= \mathrm{Re}\left(A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} e^{j2\pi f_c t}\right) \\ &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi (f_c + n f_m) t) \end{aligned} \tag{3.3.25}$$

Equation (3.3.25) shows that even in this very simple case, where the modulating signal is a sinusoid of frequency f_m, the angle-modulated signal contains all frequencies of the form $f_c + nf_m$ for $n = 0, \pm 1, \pm 2, \ldots$. Therefore, the actual bandwidth of the modulated signal is infinite. However, the amplitude of the sinusoidal components of frequencies $f_c \pm nf_m$ for large n is very small. Hence, we can define a finite effective bandwidth for the modulated signal. A series expansion for the Bessel function is given by

$$J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{n+2k}}{k!(k+n)!} \tag{3.3.26}$$

The above expansion shows that for small β, we can use the approximation

$$J_n(\beta) \approx \frac{\beta^n}{2^n n!} \tag{3.3.27}$$

Thus for a small modulation index β, only the first sideband corresponding to $n = 1$ is of importance. Also, using the above expansion, it is easy to verify the following symmetry properties of the Bessel function.

$$J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases} \tag{3.3.28}$$

Plots of $J_n(\beta)$ for various values of n are given in Figure 3.28, and a table of the values of the Bessel function is given in Table 3.1.

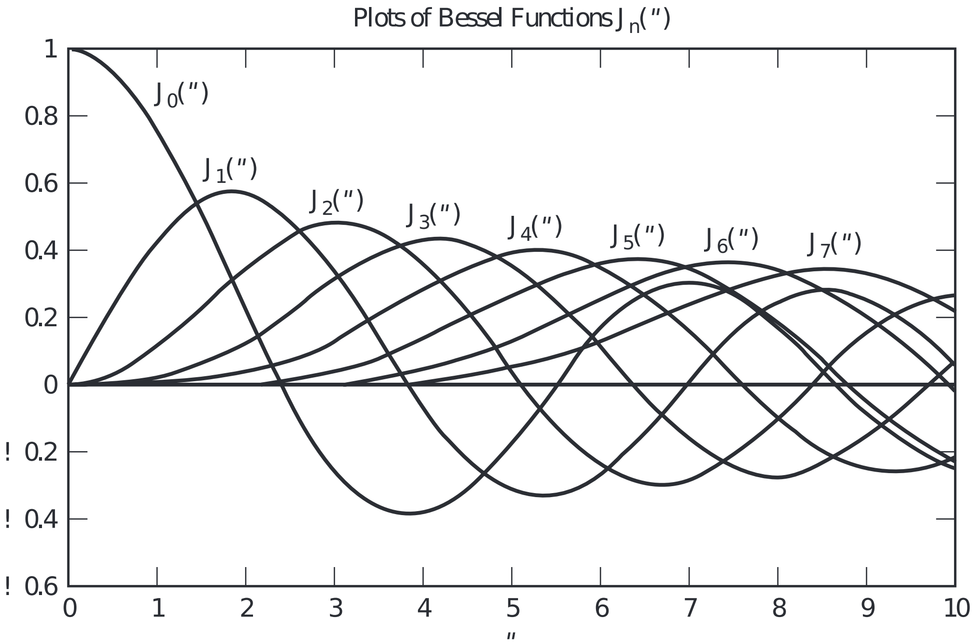

Figure 3.28 Bessel functions for various values of n.

TABLE 3.1 TABLE OF BESSEL FUNCTION VALUES

n	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 8$	$\beta = 10$
0	0.997	0.990	0.938	0.765	0.224	−0.178	0.172	−0.246
1	0.050	0.100	0.242	0.440	0.577	−0.328	0.235	0.043
2	0.001	0.005	0.031	0.115	0.353	0.047	−0.113	0.255
3				0.020	0.129	0.365	−0.291	0.058
4				0.002	0.034	0.391	−0.105	−0.220
5					0.007	0.261	0.186	−0.234
6					0.001	0.131	0.338	−0.014
7						0.053	0.321	0.217
8						0.018	0.223	0.318
9						0.006	0.126	0.292
10						0.001	0.061	0.207
11							0.026	0.123
12							0.010	0.063
13							0.003	0.029
14							0.001	0.012
15								0.004
16								0.001

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Example 3.3.2

Let the carrier be given by $c(t) = 10\cos(2\pi f_c t)$ and let the message signal be $\cos(20\pi t)$. Further assume that the message is used to frequency modulate the carrier with $k_f = 50$. Find the expression for the modulated signal and determine how many harmonics should be selected to contain 99% of the modulated signal power.

Solution The power content of the carrier signal is given by

$$P_c = \frac{A_c^2}{2} = \frac{100}{2} = 50 \tag{3.3.29}$$

The modulated signal is represented by

$$\begin{aligned} u(t) &= 10\cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} \cos(20\pi\tau)\, d\tau\right) \\ &= 10\cos\left(2\pi f_c t + \frac{50}{10}\sin(20\pi t)\right) \\ &= 10\cos(2\pi f_c t + 5\sin(20\pi t)) \end{aligned} \tag{3.3.30}$$

The modulation index is given by

$$\beta = k_f \frac{\max[|m(t)|]}{f_m} = 5 \tag{3.3.31}$$

and, therefore, the FM-modulated signal is

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$
$$= \sum_{n=-\infty}^{\infty} 10 J_n(5)\cos(2\pi(f_c + 10n)t) \tag{3.3.32}$$

It is seen that the frequency content of the modulated signal is concentrated at frequencies of the form $f_c + 10n$ for various n. To make sure that at least 99% of the total power is within the effective bandwidth, we have to choose k large enough such that

$$\sum_{n=-k}^{n=k} \frac{100 J_n^2(5)}{2} \geq 0.99 \times 50 \tag{3.3.33}$$

This is a nonlinear equation and its solution (for k) can be found by trial and error and by using tables of the Bessel functions. Of course, in finding the solution to this equation we have to employ the symmetry properties of the Bessel function given in Equation (3.3.28). Using these properties we have

$$50\left[J_0^2(5) + 2\sum_{n=1}^{k} J_n^2(5)\right] \geq 49.5 \tag{3.3.34}$$

Starting with small values of k and increasing it, we see that the smallest value of k for which the left-hand side exceeds the right-hand side is $k = 6$. Therefore, taking frequencies $f_c \pm 10k$ for $0 \leq k \leq 6$ guarantees that 99% of the power of the modulated signal has been included and only one per cent has been left out. This means that, if the modulated signal is passed through an ideal bandpass filter centered at f_c with a bandwidth of at least 120 Hz, only 1% of the signal power will be eliminated. This gives us a practical way to define the effective bandwidth of the angle-modulated signal as being 120 Hz. Figure 3.29 shows the frequencies present in the effective bandwidth of the modulated signal.

In general the effective bandwidth of an angle-modulated signal, which contains at least 98% of the signal power, is given by the relation

$$B_c = 2(\beta + 1) f_m \tag{3.3.35}$$

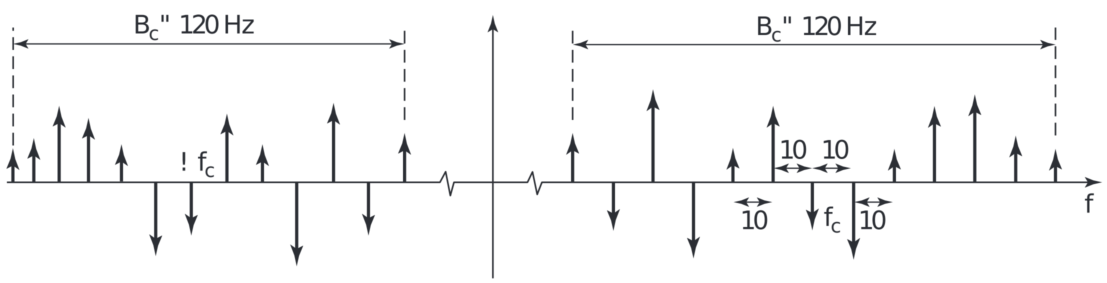


Figure 3.29 The harmonics present inside the effective bandwidth of Example 3.3.2.

where β is the modulation index and f_m is the frequency of the sinusoidal message signal. It is instructive to study the effect of the amplitude and frequency of the sinusoidal message signal on the bandwidth and the number of harmonics in the modulated signal. Let the message signal be given by

$$m(t) = a\cos(2\pi f_m t) \tag{3.3.36}$$

The bandwidth† of the modulated signal is given by

$$B_c = 2(\beta + 1) f_m = \begin{cases} 2(k_p a + 1) f_m, & \text{PM} \\ 2\left(\frac{k_f a}{f_m} + 1\right) f_m, & \text{FM} \end{cases} \tag{3.3.37}$$

or

$$B_c = \begin{cases} 2(k_p a + 1) f_m, & \text{PM} \\ 2(k_f a + f_m), & \text{FM} \end{cases} \tag{3.3.38}$$

Equation (3.3.38) shows that increasing a, the amplitude of the modulating signal, in PM and FM has almost the same effect on increasing the bandwidth B_c. On the other hand, increasing f_m, the frequency of the message signal, has a more profound effect in increasing the bandwidth of a PM signal as compared to an FM signal. In both PM and FM the bandwidth B_c increases by increasing f_m, but in PM this increase is a proportional increase and in FM this is only an additive increase, which in most cases of interest, (for large β) is not substantial. Now if we look at the number of harmonics in the bandwidth (including the carrier) and denote it with M_c, we have

$$M_c = 2\lfloor\beta\rfloor + 3 = \begin{cases} 2\lfloor k_p a\rfloor + 3, & \text{PM} \\ 2\left\lfloor\frac{k_f a}{f_m}\right\rfloor + 3, & \text{FM} \end{cases} \tag{3.3.39}$$

Increasing the amplitude a increases the number of harmonics in the bandwidth of the modulated signal in both cases. However, increasing f_m, has no effect on the number of harmonics in the bandwidth of the PM signal and decreases the number of harmonics in the FM signal almost linearly. This explains the relative insensitivity of the bandwidth of the FM signal to the message frequency. On the one hand, increasing f_m decreases the number of harmonics in the bandwidth and, at the same time, it increases the spacing between the harmonics. The net effect is a slight increase in the bandwidth. In PM, however, the number of harmonics remains constant and only the spacing between them increases. Therefore, the net effect is a linear increase in bandwidth. Figure 3.30 shows the effect of increasing the frequency of the message in both FM and PM.

Angle Modulation by a Periodic Message Signal. To generalize the preceding results, we now consider angle modulation by an arbitrary periodic message signal $m(t)$. Let us consider a PM-modulated signal where

$$u(t) = A_c\cos(2\pi f_c t + \beta m(t)) \tag{3.3.40}$$

†From now on, by bandwidth we mean effective bandwidth unless otherwise stated.

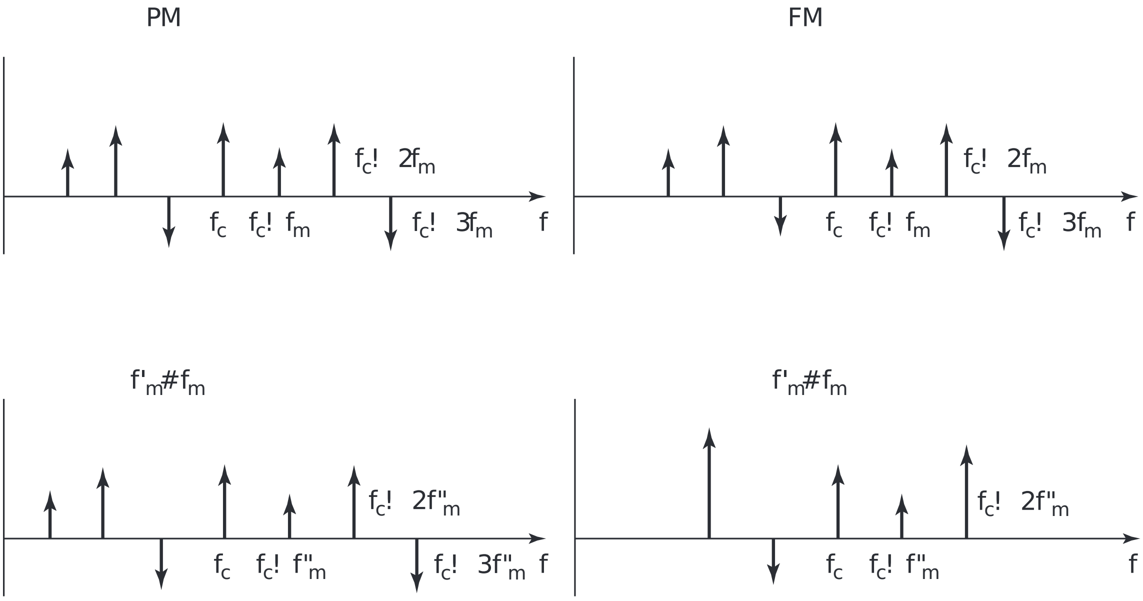


Figure 3.30 The effect of increasing bandwidth of the message in FM and PM.

We can write this as

$$u(t) = A_c \operatorname{Re}\left[e^{j2\pi f_c t} e^{j\beta m(t)}\right] \tag{3.3.41}$$

We are assuming that $m(t)$ is periodic with period $T_m = \frac{1}{f_m}$. Therefore, $e^{j\beta m(t)}$ will be a periodic signal with the same period, and we can find its Fourier series expansion as

$$e^{j\beta m(t)} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t} \tag{3.3.42}$$

where

$$\begin{aligned} c_n &= \frac{1}{T_m} \int_0^{T_m} e^{j\beta m(t)} e^{-j2\pi n f_m t}\, dt \\ &\overset{u=2\pi f_m t}{=} \frac{1}{2\pi} \int_0^{2\pi} e^{j\left[\beta m\left(\frac{u}{2\pi f_m}\right) - nu\right]}\, du \end{aligned} \tag{3.3.43}$$

and

$$\begin{aligned} u(t) &= A_c \operatorname{Re}\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_c t} e^{j2\pi n f_m t}\right] \\ &= A_c \sum_{n=-\infty}^{\infty} |c_n| \cos(2\pi (f_c + n f_m)t + \angle c_n) \end{aligned} \tag{3.3.44}$$

It is seen again that the modulated signal contains all frequencies of the form $f_c + n f_m$.

The detailed treatment of the spectral characteristics of an angle-modulated signal for a general nonperiodic deterministic message signal $m(t)$ is quite involved due to the nonlinear nature of the modulation process. However, there exists an approximate relation for the effective bandwidth of the modulated signal, known as the Carson's rule, and given by

$$B_c = 2(\beta + 1)W \tag{3.3.45}$$

where β is the modulation index defined as

$$\beta = \begin{cases} k_p \max[|m(t)|], & \text{PM} \\ \frac{k_f \max[|m(t)|]}{W}, & \text{FM} \end{cases} \tag{3.3.46}$$

and W is the bandwidth of the message signal $m(t)$. Since in wideband FM the value of β is usually around 5 or more, it is seen that the bandwidth of an angle-modulated signal is much greater than the bandwidth of various amplitude-modulation schemes, which is either W (in SSB) or $2W$ (in DSB or conventional AM).

3.3.3 Implementation of Angle Modulators and Demodulators

Any modulation and demodulation process involves the generation of new frequencies that were not present in the input signal. This is true for both amplitude and angle-modulation systems. This means that, if we interpret the modulator as a system with the message signal $m(t)$ as the input and with the modulated signal $u(t)$ as the output, this system has frequencies in its output that were not present in the input. Therefore, a modulator (and demodulator) can not be modeled as a linear time-invariant system because a linear time-invariant system can not produce any frequency components in the output that are not present in the input signal.

Angle modulators are, in general, time-varying and nonlinear systems. One method for generating an FM signal directly is to design an oscillator whose frequency changes with the input voltage. When the input voltage is zero, the oscillator generates a sinusoid with frequency f_c, and when the input voltage changes, this frequency changes accordingly. There are two approaches to designing such an oscillator, usually called a VCO or voltage-controlled oscillator. One approach is to use a varactor diode. A varactor diode is a capacitor whose capacitance changes with the applied voltage. Therefore, if this capacitor is used in the tuned circuit of the oscillator and the message signal is applied to it, the frequency of the tuned circuit, and the oscillator, will change in accordance with the message signal. Let us assume that the inductance of the inductor in the tuned circuit of Figure 3.31 is L_0 and the capacitance of the varactor diode is given by

$$C(t) = C_0 + k_0 m(t) \tag{3.3.47}$$

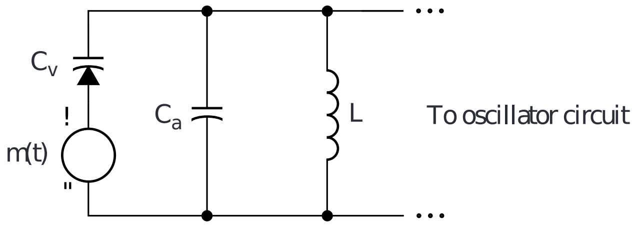


Figure 3.31 Varactor diode implementation of an angle modulator.

When $m(t) = 0$, the frequency of the tuned circuit is given by $f_c = \frac{1}{2\pi\sqrt{L_0 C_0}}$. In general, for nonzero $m(t)$, we have

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi\sqrt{L_0(C_0 + k_0 m(t))}} \\ &= \frac{1}{2\pi\sqrt{L_0 C_0}} \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}} \\ &= f_c \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}} \end{aligned} \tag{3.3.48}$$

Assuming that

$$\epsilon = \frac{k_0}{C_0} m(t) \ll 1$$

and using the approximations

$$\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2} \tag{3.3.49}$$

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon, \tag{3.3.50}$$

we obtain

$$f_i(t) \approx f_c \left[1 - \frac{k_0}{2C_0} m(t) \right] \tag{3.3.51}$$

which is the relation for a frequency-modulated signal.

A second approach for generating an FM signal is by use of a reactance tube. In the reactance-tube implementation, an inductor whose inductance varies with the applied voltage is employed and the analysis is very similar to the analysis presented for the varactor diode. It should be noted that although we described these methods for generation of FM signals, due to the close relation between FM and PM signals, basically the same methods can be applied for generation of PM signals (see Figure 3.25).

Another approach for generating an angle-modulated signal is to first generate a narrowband angle-modulated signal, and then change it to a wideband signal. This method is usually known as the indirect method for generation of FM and PM signals. Due to the similarity of conventional AM signals, generation of narrowband

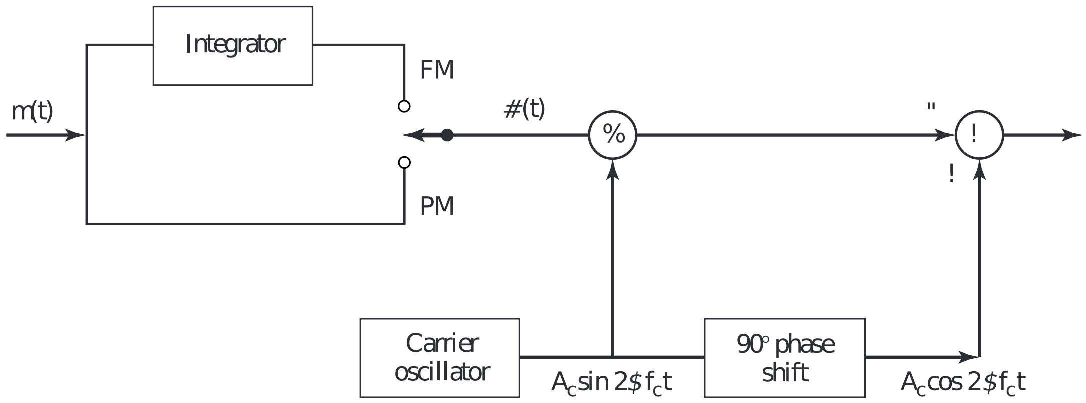


Figure 3.32 Generation of narrowband angle-modulated signal.

angle-modulated signals is straightforward. In fact any modulator for conventional AM generation can be easily modified to generate a narrowband angle-modulated signal. Figure 3.32 shows the block diagram of a narrowband angle modulator. The next step is to use the narrowband angle-modulated signal to generate a wideband angle-modulated signal. Figure 3.33 shows the block diagram of a system that generates wideband angle-modulated signals from narrowband angle-modulated signals. The first stage of such a system is, of course, a narrowband angle-modulator such as the one shown in Figure 3.32. The narrowband angle-modulated signal enters a frequency multiplier that multiplies the instantaneous frequency of the input by some constant n. This is usually done by applying the input signal to a nonlinear element and then passing its output through a bandpass filter tuned to the desired central frequency. If the narrowband-modulated signal is represented by

$$u_n(t) = A_c \cos(2\pi f_c t + \phi(t)) \tag{3.3.52}$$

the output of the frequency multiplier (output of the bandpass filter) is given by

$$y(t) = A_c \cos(2\pi n f_c t + n\phi(t)) \tag{3.3.53}$$

In general, this is, of course, a wideband angle-modulated signal. However, there is no guarantee that the carrier frequency of this signal, nf_c, will be the desired carrier frequency. The last stage of the modulator performs an up or down conversion to shift

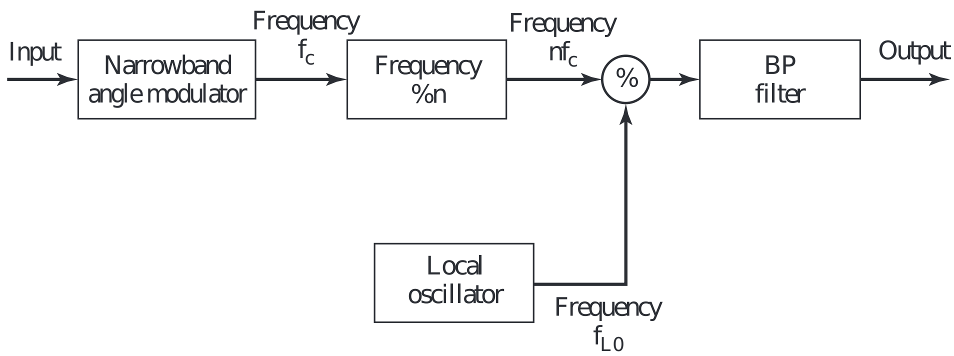


Figure 3.33 Indirect generation of angle-modulated signals.

the modulated signal to the desired center frequency. This stage consists of a mixer and a bandpass filter. If the frequency of the local oscillator of the mixer is f_{L0} and we are using a down converter, the final wideband angle-modulated signal is given by

$$u(t) = A_c \cos(2\pi(nf_c - f_{L0})t + n\phi(t)) \tag{3.3.54}$$

Since we can freely choose n and f_{L0}, we can generate any modulation index at any desired carrier frequency by this method.

FM demodulators are implemented by generating an AM signal whose amplitude is proportional to the instantaneous frequency of the FM signal, and then using an AM demodulator to recover the message signal. To implement the first step; i.e., transforming the FM signal into an AM signal, it is enough to pass the FM signal through an LTI system whose frequency response is approximately a straight line in the frequency band of the FM signal. If the frequency response of such a system is given by

$$|H(f)| = V_0 + k(f - f_c) \quad \text{for } |f - f_c| < \frac{B_c}{2} \tag{3.3.55}$$

and if the input to the system is

$$u(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau\right), \tag{3.3.56}$$

then, the output will be the signal

$$v_o(t) = A_c(V_0 + kk_f m(t)) \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau\right) \tag{3.3.57}$$

The next step is to demodulate this signal to obtain $A_c(V_0 + kk_f m(t))$, from which the message $m(t)$ can be recovered. Figure 3.34 shows a block diagram of these two steps.

There exist many circuits that can be used to implement the first stage of an FM demodulator; i.e., FM to AM conversion. One such candidate is a simple differentiator with

$$|H(f)| = 2\pi f \tag{3.3.58}$$

Another candidate is the rising half of the frequency characteristics of a tuned circuit as shown in Figure 3.35. Such a circuit can be easily implemented, but usually the linear region of the frequency characteristic may not be wide enough. To obtain a linear characteristic over a wider range of frequencies, usually two circuits tuned at two frequencies, f_1 and f_2, are connected in a configuration which is known as a balanced discriminator. A balanced discriminator with the corresponding frequency characteristics is shown in Figure 3.36.

The FM demodulation methods described here that transform the FM signal into an AM signal have a bandwidth equal to the channel bandwidth B_c occupied by the FM signal. Consequently, the noise that is passed by the demodulator is the noise contained within B_c.

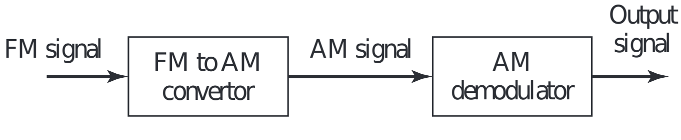


Figure 3.34 A general FM demodulator.

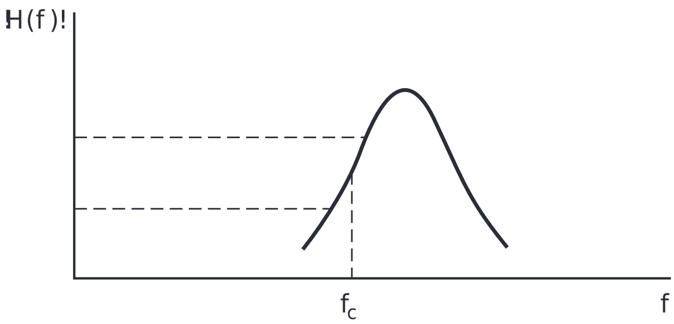


Figure 3.35 A tuned circuit used in an FM demodulator.

A totally different approach to FM signal demodulation is to use feedback in the FM demodulator to narrow the bandwidth of the FM detector and, as will be seen in Chapter 5, to reduce the noise power at the output of the demodulator. Figure 3.37 illustrates a system in which the FM discrimination is placed in the feedback branch of a feedback system that employs a voltage-controlled oscillator (VCO) path. The bandwidth of the discriminator and the subsequent lowpass filter is designed to match the bandwidth of the message signal $m(t)$. The output of the lowpass filter is the desired message signal. This type of FM demodulator is called an FM demodulator with feedback (FMFB). An alternative to FMFB demodulator is the use of a phase-locked loop (PLL), as shown in Figure 3.38. The input to the PLL is the angle-modulated signal (we neglect the presence of noise in this discussion)

$$u(t) = A_c \cos[2\pi f_c t + \phi(t)] \tag{3.3.59}$$

where, for FM,

$$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau \tag{3.3.60}$$

The VCO generates a sinusoid of a fixed frequency, in this case the carrier frequency f_c, in the absence of an input control voltage.

Now, suppose that the control voltage to the VCO is the output of the loop filter, denoted as $v(t)$. Then, the instantaneous frequency of the VCO is

$$f_v(t) = f_c + k_v v(t) \tag{3.3.61}$$

where k_v is a deviation constant with units of Hz/volt. Consequently, the VCO output may be expressed as

$$y_v(t) = A_v \sin[2\pi f_c t + \phi_v(t)] \tag{3.3.62}$$

where

$$\phi_v(t) = 2\pi k_v \int_{0}^{t} v(\tau)\, d\tau \tag{3.3.63}$$

The phase comparator is basically a multiplier and filter that rejects the signal component centered at $2 f_c$. Hence, its output may be expressed as

$$e(t) = \tfrac{1}{2} A_v A_c \sin[\phi(t) - \phi_v(t)] \tag{3.3.64}$$

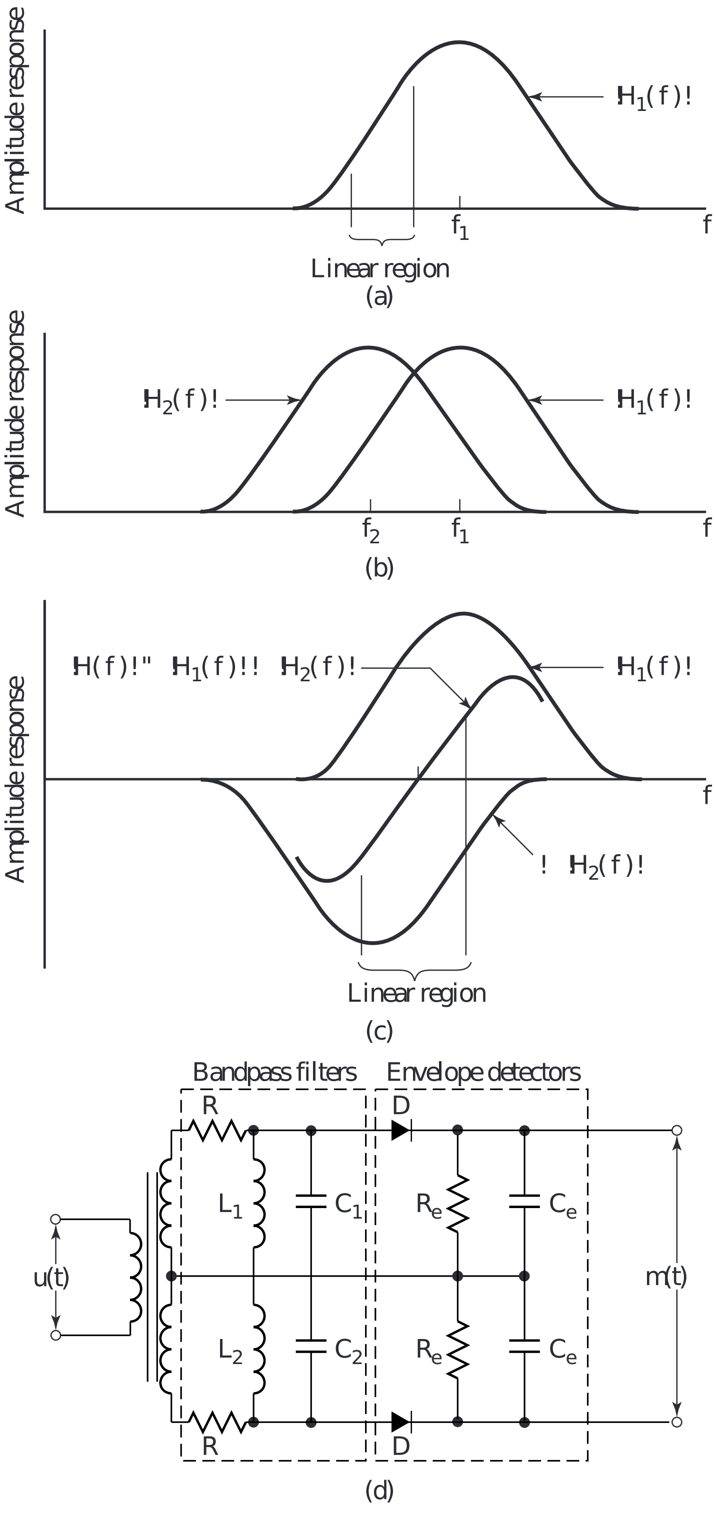


Figure 3.36 A balanced discriminator and the corresponding frequency response.

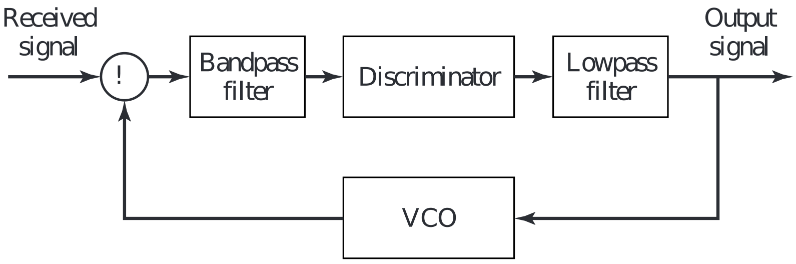


Figure 3.37 Block diagram of FMFB demodulator.

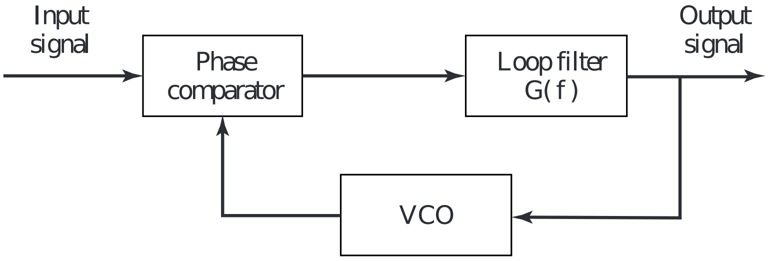


Figure 3.38 Block diagram of PLL-FM demodulator.

where the difference, $\phi(t) - \phi_v(t) \equiv \phi_e(t)$, constitutes the phase error. The signal $e(t)$ is the input to the loop filter.

Let us assume that the PLL is in lock, so that the phase error is small. Then,

$$\sin[\phi(t) - \phi_v(t)] \approx \phi(t) - \phi_v(t) = \phi_e(t) \tag{3.3.65}$$

Under this condition, we may deal with the linearized model of the PLL, shown in Figure 3.39. We may express the phase error as

$$\phi_e(t) = \phi(t) - 2\pi k_v \int_0^t v(\tau)\, d\tau \tag{3.3.66}$$

or, equivalently, either as

$$\frac{d}{dt}\phi_e(t) + 2\pi k_v v(t) = \frac{d}{dt}\phi(t) \tag{3.3.67}$$

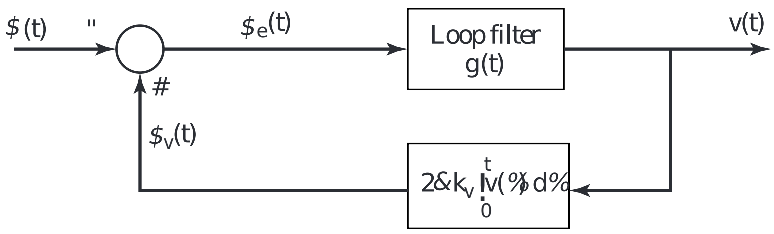


Figure 3.39 Linearized PLL.

or as

$$\frac{d}{dt}\phi_e(t) + 2\pi k_v \int_0^\infty \phi_e(\tau) g(t-\tau)\, d\tau = \frac{d}{dt}\phi(t) \tag{3.3.68}$$

The Fourier transform of the integro-differential equation in Equation (3.3.68) is

$$(j2\pi f)\Phi_e(f) + 2\pi k_v \Phi_e(f) G(f) = (j2\pi f)\Phi(f) \tag{3.3.69}$$

and, hence,

$$\Phi_e(f) = \frac{1}{1 + \left(\frac{k_v}{jf}\right) G(f)} \Phi(f) \tag{3.3.70}$$

The corresponding equation for the control voltage to the VCO is

$$\begin{aligned} V(f) &= \Phi_e(f) G(f) \\ &= \frac{G(f)}{1 + \left(\frac{k_v}{jf}\right) G(f)} \Phi(f) \end{aligned} \tag{3.3.71}$$

Now, suppose that we design $G(f)$ such that

$$\left| k_v \frac{G(f)}{jf} \right| \gg 1 \tag{3.3.72}$$

in the frequency band $|f| < W$ of the message signal. Then from Equation (3.3.71), we have

$$V(f) = \frac{j2\pi f}{2\pi k_v} \Phi(f) \tag{3.3.73}$$

or, equivalently,

$$\begin{aligned} v(t) &= \frac{1}{2\pi k_v} \frac{d}{dt}\phi(t) \\ &= \frac{k_f}{k_v} m(t) \end{aligned} \tag{3.3.74}$$

Since the control voltage of the VCO is proportional to the message signal, $v(t)$ is the demodulated signal.

We observe that the output of the loop filter with frequency response $G(f)$ is the desired message signal. Hence, the bandwidth of $G(f)$ should be the same as the bandwidth W of the message signal. Consequently, the noise at the output of the loop filter is also limited to the bandwidth W. On the other hand, the output from the VCO is a wideband FM signal with an instantaneous frequency that follows the instantaneous frequency of the received FM signal.

The major benefit of using feedback in FM signal demodulation is to reduce the threshold effect that occurs when the input signal-to-noise-ratio to the FM demodulator drops below a critical value. The threshold effect is treated in Chapter 5.

3.4 RADIO AND TELEVISION BROADCASTING

Radio and television broadcasting is the most familiar form of communication via analog signal transmission. Next, we describe three types of broadcasting, namely, AM radio, FM radio, and television.

3.4.1 AM Radio Broadcasting

Commercial AM radio broadcasting utilizes the frequency band 535–1605 kHz for transmission of voice and music. The carrier frequency allocations range from 540–1600 kHz with 10-kHz spacing.

Radio stations employ conventional AM for signal transmission. The baseband message signal $m(t)$ is limited to a bandwidth of approximately 5 kHz. Since there are billions of receivers and relatively few radio transmitters, the use of conventional AM for broadcast is justified from an economic standpoint. The major objective is to reduce the cost of implementing the receiver.

The receiver most commonly used in AM radio broadcast is the so called superheterodyne receiver shown in Figure 3.40. It consists of a radio frequency (RF) tuned amplifier, a mixer, a local oscillator, an intermediate frequency (IF) amplifier, an envelope detector, an audio frequency amplifier, and a loudspeaker. Tuning for the desired radio frequency is provided by a variable capacitor, which simultaneously tunes the RF amplifier and the frequency of the local oscillator.

In the superheterodyne receiver, every AM radio signal is converted to a common IF frequency of $f_{IF} = 455$ kHz. This conversion allows the use of a single tuned IF amplifier for signals from any radio station in the frequency band. The IF amplifier is designed to have a bandwidth of 10 kHz, which matches the bandwidth of the transmitted signal.

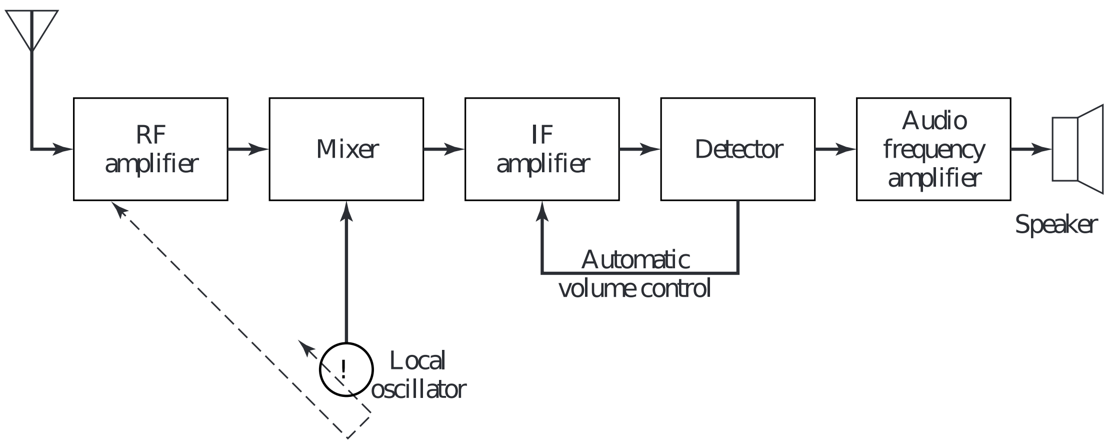


Figure 3.40 Superheterodyne AM receiver.

The frequency conversion to IF is performed by the combination of the RF amplifier and the mixer. The frequency of the local oscillator is

$$f_{LO} = f_c + f_{IF}$$

where f_c is the carrier frequency of the desired AM radio signal. The tuning range of the local oscillator is 955–2055 kHz. By tuning the RF amplifier to the frequency f_c and mixing its output with the local oscillator frequency $f_{LO} = f_c + f_{IF}$, we obtain two signal components, one centered at the difference frequency f_{IF} and the second centered at the sum frequency $2f_c + f_{IF}$. Only the first component is passed by the IF amplifier.

At the input to the RF amplifier we have signals picked up by the antenna from all radio stations. By limiting the bandwidth of the RF amplifier to the range $B_c < B_{RF} < 2f_{IF}$ where B_c is the bandwidth of the AM radio signal (10 kHz), we can reject the radio signal transmitted at the so-called image frequency, $f_c' = f_{LO} + f_{IF}$. Note that when we mix the local oscillator output, $\cos 2\pi f_{LO}t$, with the received signals

$$\begin{aligned} r_1(t) &= A_c[1 + m_1(t)] \cos 2\pi f_c t \\ r_2(t) &= A_c[1 + m_2(t)] \cos 2\pi f_c' t \end{aligned} \tag{3.4.1}$$

where $f_c = f_{LO} - f_{IF}$ and $f_c' = f_{LO} + f_{IF}$, the mixer output consists of the two signals

$$\begin{aligned} y_1(t) &= A_c[1 + m_1(t)] \cos 2\pi f_{IF} t + \text{double frequency term} \\ y_2(t) &= A_c[1 + m_2(t)] \cos 2\pi f_{IF} t + \text{double frequency term} \end{aligned} \tag{3.4.2}$$

where $m_1(t)$ represents the desired signal and $m_2(t)$ is the signal transmitted by the radio station transmitting at the carrier frequency $f_c' = f_{LO} + f_{IF}$. In order to prevent the signal $r_2(t)$ from interfering with the demodulation of the desired signal $r_1(t)$, the RF amplifier bandwidth is designed to be sufficiently narrow so that the image frequency signal is rejected. Hence, $B_{RF} < 2f_{IF}$ is the upper limit on the bandwidth of the RF amplifier. In spite of this constraint, the bandwidth of the RF amplifier is still considerably wider than the bandwidth of the IF amplifier. Thus, the IF amplifier, with its narrow bandwidth, provides signal rejection from adjacent channels and the RF amplifier provides signal rejection from image channels. Figure 3.41 illustrates the bandwidths of the RF and IF amplifiers and the requirement for rejecting the image frequency signal.

The output of the IF amplifier is passed through an envelope detector which produces the desired audio-band message signal $m(t)$. Finally, the output of the envelope detector is amplified and the amplified signal drives a loudspeaker. Automatic volume control (AVC) is provided by a feedback control loop which adjusts the gain of the IF amplifier based on the power level of the signal at the envelope detector.

3.4.2 FM Radio Broadcasting

Commercial FM radio broadcasting utilizes the frequency band 88–108 MHz for transmission of voice and music signals. The carrier frequencies are separated by 200 kHz and the peak-frequency deviation is fixed at 75 kHz. Pre-emphasis is generally used,

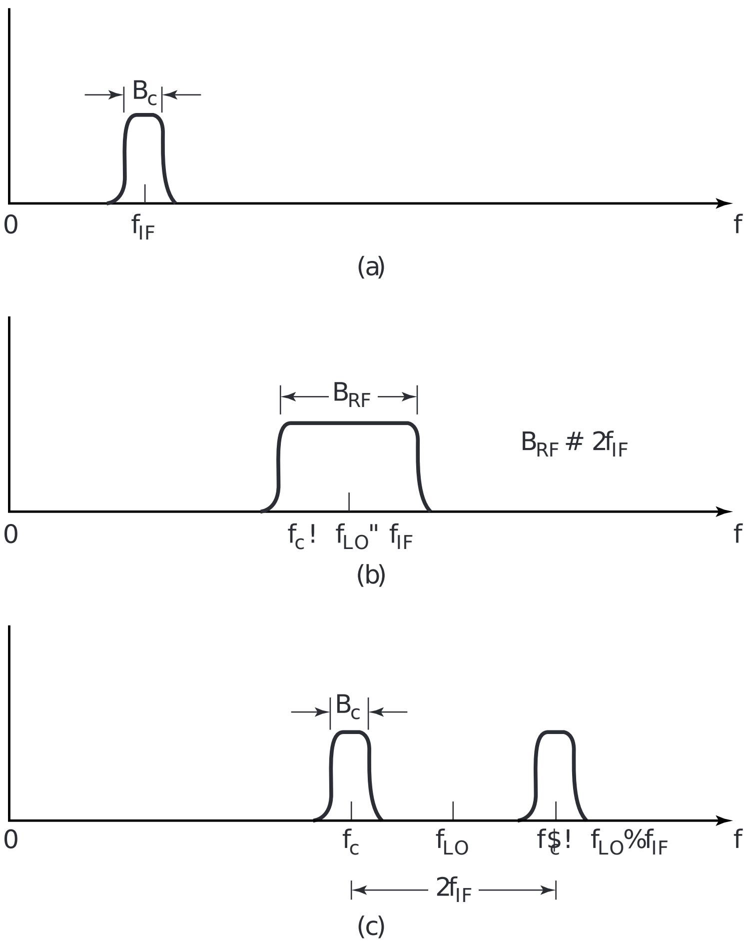


Figure 3.41 Frequency response characteristics of IF and RF amplifiers.

as described in Chapter 5, to improve the demodulator performance in the presence of noise in the received signal.

The receiver most commonly used in FM radio broadcast is a superheterodyne type. The block diagram of such a receiver is shown in Figure 3.42. As in AM radio reception, common tuning between the RF amplifier and the local oscillator allows the mixer to bring all FM radio signals to a common IF bandwidth of 200 kHz, centered at $f_{IF} = 10.7$ MHz. Since the message signal $m(t)$ is embedded in the frequency of the carrier, any amplitude variations in the received signal are a result of additive noise and interference. The amplitude limiter removes any amplitude variations in the received signal at the output of the IF amplifier by band-limiting the signal. A bandpass filter centered at $f_{IF} = 10.7$ MHz with a bandwidth of 200 kHz is included in the limiter to remove higher order frequency components introduced by the nonlinearity inherent in the hard limiter.

A balanced frequency discriminator is used for frequency demodulation. The resulting message signal is then passed to the audio frequency amplifier which performs the functions of de-emphasis and amplification. The output of the audio amplifier is further filtered by a lowpass filter to remove out-of-band noise and its output is used to drive a loudspeaker.

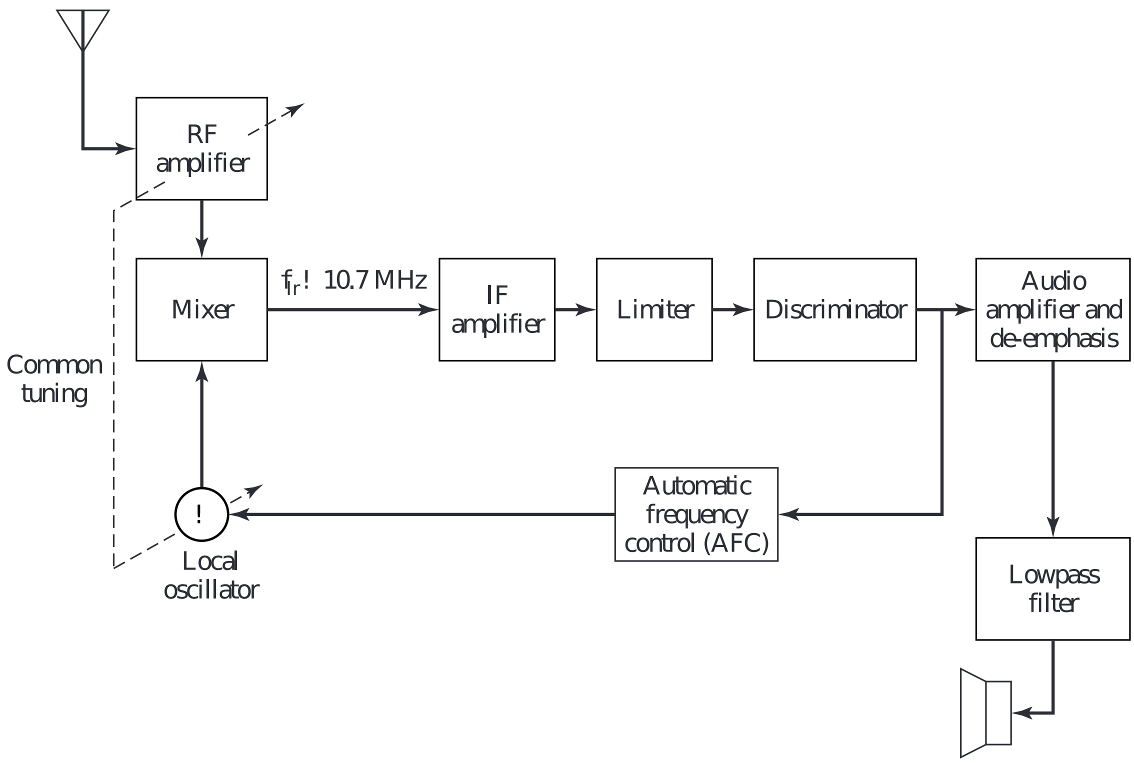


Figure 3.42 Block diagram of a superheterodyne FM radio receiver.

FM Stereo Broadcasting. Many FM radio stations transmit music programs in stereo by using the outputs of two microphones placed in two different parts of the stage. Figure 3.43 shows a block diagram of an FM stereo transmitter. The signals from the left and right microphones, $m_l(t)$ and $m_r(t)$, are added and subtracted as shown. The sum signal $m_l(t) + m_r(t)$ is left as is and occupies the frequency band 0–15 kHz. The difference signal $m_l(t) - m_r(t)$ is used to AM modulate (DSB-SC) a 38-kHz carrier that is generated from a 19-kHz oscillator. A pilot tone at the frequency of 19 kHz is added to the signal for the purpose of demodulating the DSB-SC AM signal. The reason for placing the pilot tone at 19 kHz instead of 38 kHz is that the pilot is more easily separated from the composite signal at the receiver. The combined signal is used to frequency modulate a carrier.

By configuring the baseband signal as an FDM signal, a monophonic FM receiver can recover the sum signal $m_l(t) + m_r(t)$ by use of a conventional FM demodulator. Hence, FM stereo broadcasting is compatible with conventional FM. The second requirement is that the resulting FM signal does not exceed the allocated 200-kHz bandwidth.

The FM demodulator for FM stereo is basically the same as a conventional FM demodulator down to the limiter/discriminator. Thus, the received signal is converted to baseband. Following the discriminator, the baseband message signal is separated into the two signals $m_l(t) + m_r(t)$ and $m_l(t) - m_r(t)$ and passed through de-emphasis filters, as shown in Figure 3.44. The difference signal is obtained from the DSB-SC signal by means of a synchronous demodulator using the pilot tone. By taking the sum

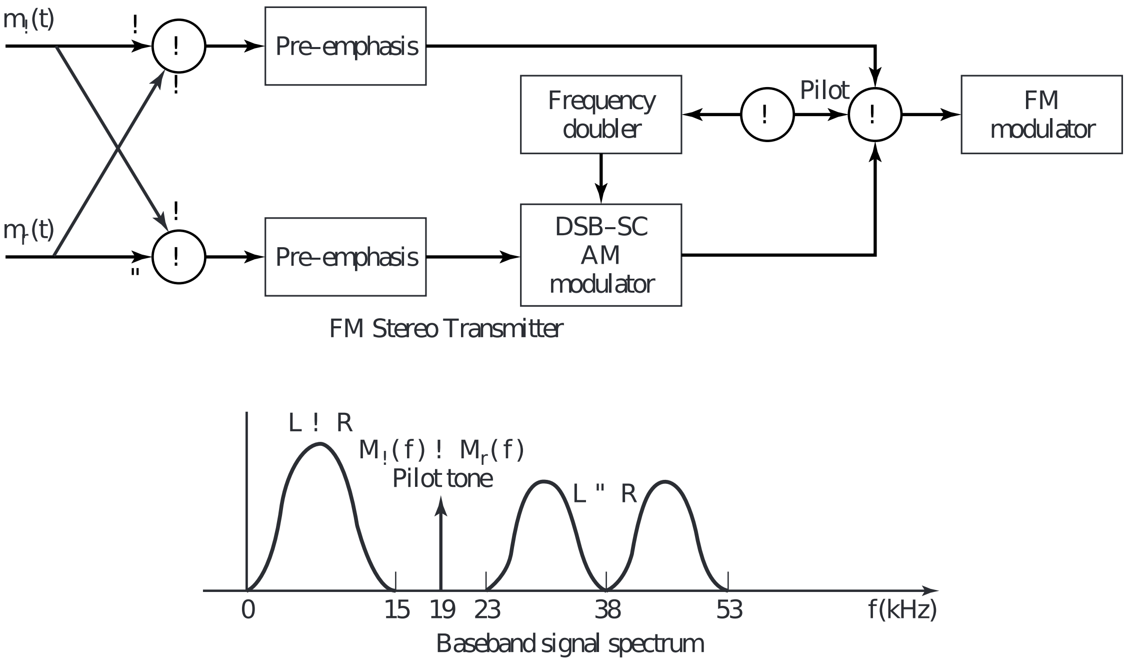


Figure 3.43 FM stereo transmitter and signal spacing.

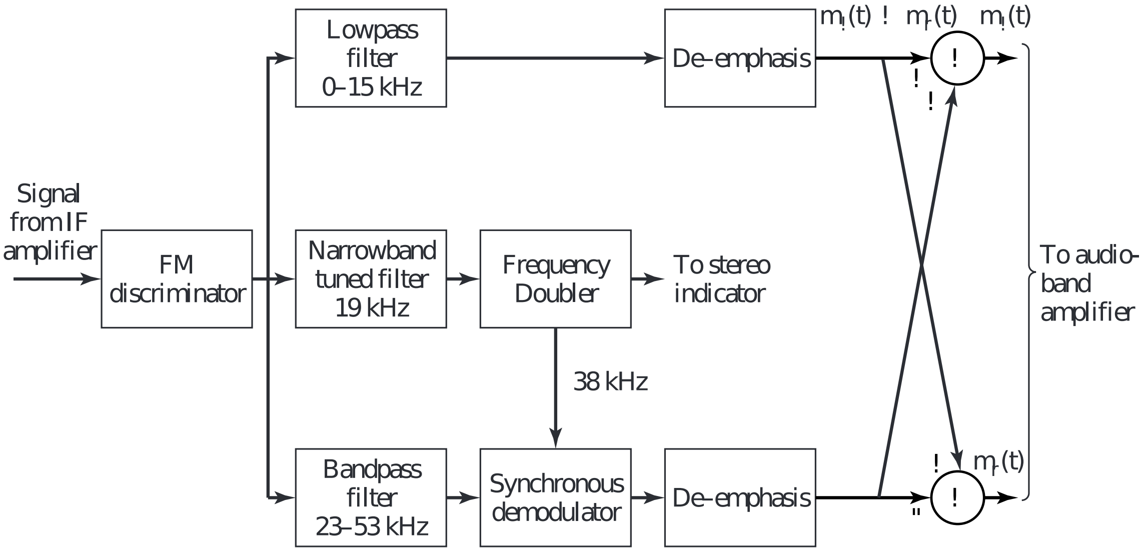


Figure 3.44 FM stereo receiver.

and difference of the two composite signals, we recover the two signals $m_l(t)$ and $m_r(t)$. These audio signals are amplified by audio-band amplifiers and the two outputs drive dual loudspeakers. As indicated above, an FM receiver that is not configured to receive the FM stereo sees only the baseband signal $m_l(t) + m_r(t)$ in the frequency range 0–15 kHz. Thus, it produces a monophonic output signal which consists of the sum of the signals at the two microphones.

3.4.3 Television Broadcasting

Commercial TV broadcasting began as black-and-white picture transmission in London in 1936 by the British Broadcasting Corporation (BBC). Color television was demonstrated a few years later, but the move of commercial TV stations to color TV signal transmission was slow in developing. To a large extent, this was due to the high cost of color TV receivers. With the development of the transistor and microelectronics components, the cost of color TV receivers decreased significantly, so that by the middle 1960s color TV broadcasting was widely used by the industry.

The frequencies allocated for TV broadcasting fall in the VHF and UHF frequency bands. Table 5.2 lists the TV channel allocations in the United States. We observe that the channel bandwidth allocated for transmission of TV signals is 6 MHz.

In contrast to radio broadcasting, standards for television signal transmission vary from country to country. The U.S. standard, which we describe below, was set by the National Television Systems Committee (NTSC).

Black-and-White TV Signals. The first step in TV signal transmission is to convert a visual image into an electrical signal. The two-dimensional image or picture is converted to a one-dimensional electrical signal by sequentially scanning the image and producing an electrical signal that is proportional to the brightness level of the image. The scanning is performed in a TV camera, which optically focuses the image on a photo cathode tube that consists of a photosensitive surface.

The scanning of the image is performed by an electron beam that produces an output current or voltage which is proportional to the brightness of the image. The resulting electrical signal is called a video signal.

The scanning of the electron beam is controlled by two voltages applied across the horizontal and vertical deflection plates. These two voltages are shown in Figure 3.45.

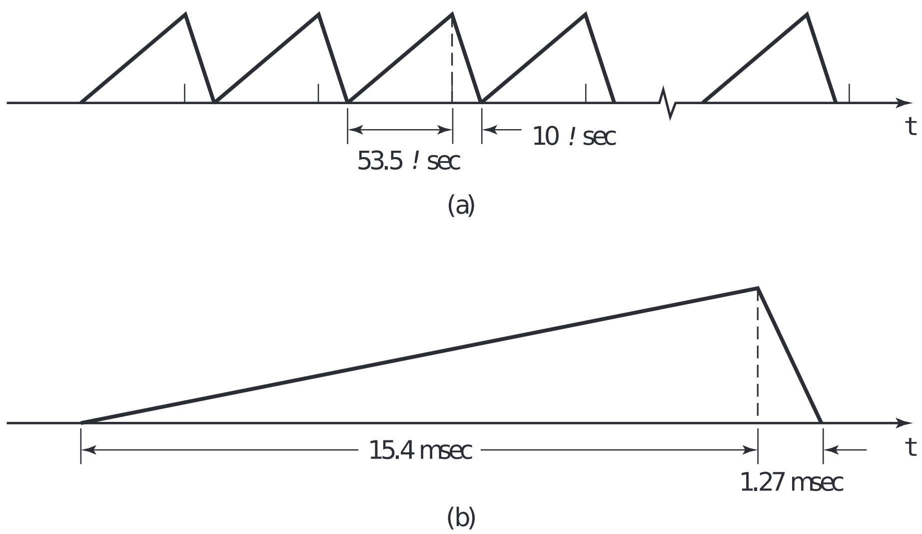


Figure 3.45 Signal waveforms applied to horizontal (a) and vertical (b) deflection plates.

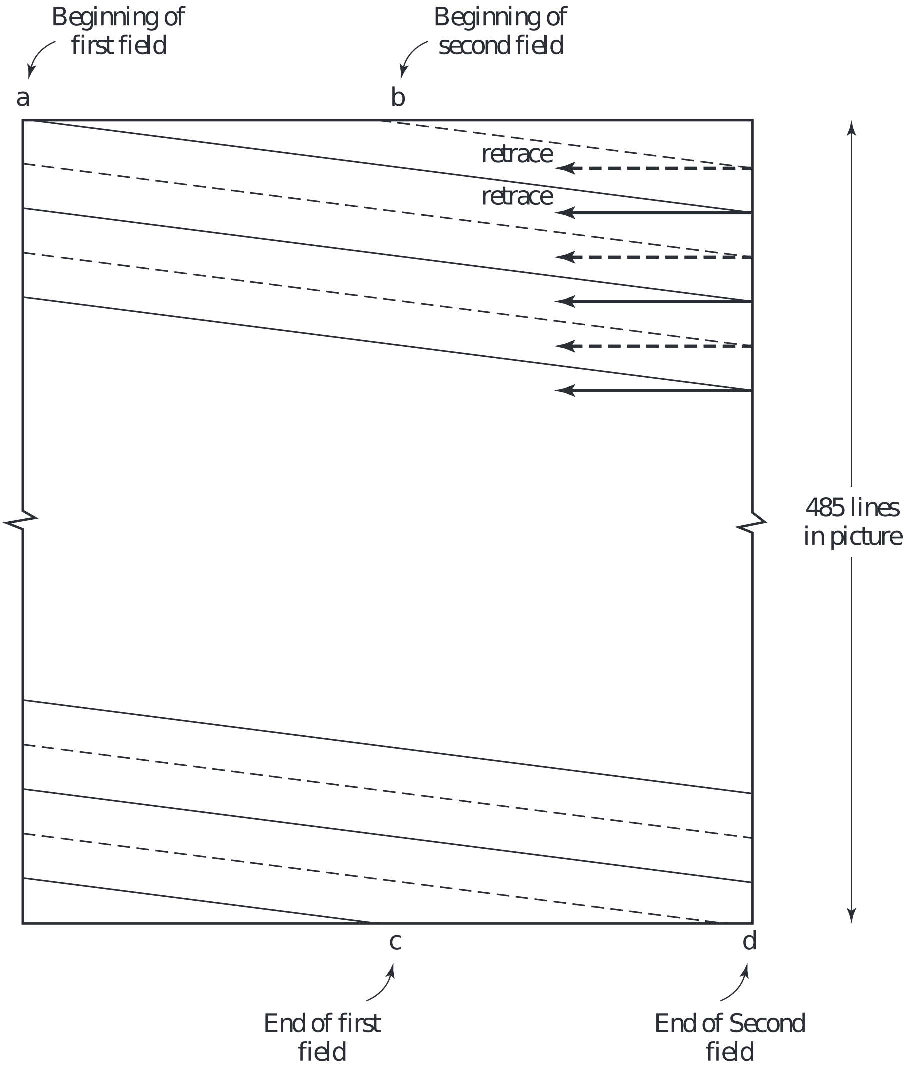


Figure 3.46 Interlaced scanning pattern.

In this scanning method, the image is divided into 525 lines that define a frame, as illustrated in Figure 3.46. The resulting signal is transmitted in 1/30 of a second. The number of lines determines the picture resolution and, in combination with the rate of transmission, determine the channel bandwidth required for transmission of the image.

The time interval of 1/30 second to transmit a complete image is generally not fast enough to avoid flicker that is annoying to the eyes of the average viewer. To overcome flicker, the scanning of the image is performed in an interlaced pattern as shown in Figure 3.46. The interlaced pattern consists of two fields, each consisting of 262.5 lines. Each field is transmitted in 1/60 of a second, which exceeds the flicker rate that is observed by the average eye. The first field begins at point “a” and terminates at point “c.” The second field begins at point “b” and terminates at point “d.”

A horizontal line is scanned in 53.5 μsec as indicated by the sawtooth signal waveform applied to the horizontal deflection plates. The beam has 10 μsec to move to

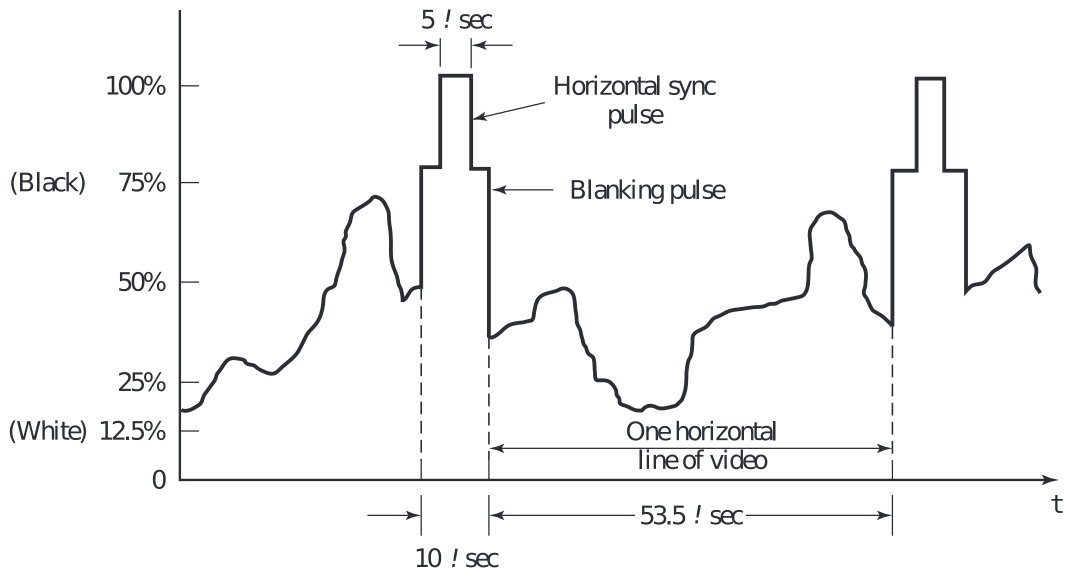


Figure 3.47 Typical video signal for one horizontal sweep.

the next line. During this interval, a blanking pulse is inserted to avoid the appearance of retrace lines across the TV receiver. A 5 μsec pulse is added to the blanking pulse to provide synchronization for the horizontal sweep circuit at the receiver. A typical video signal is shown in Figure 3.47.

After the transmission of one interlaced field, the vertical sawtooth signal waveform applied to the vertical deflection plates is reset to zero. The retrace interval of 1.27 msec, corresponding to 20 line scans, allows the beam to move from the bottom to the top of the picture. A vertical blanking pulse is inserted during the interval to avoid the appearance of retrace lines at the TV receiver. When we allow for the vertical retrace (twice per frame), the actual number of horizontal lines in the image is 485.

The bandwidth of the video signal can be estimated by viewing the image as a rectangular array of 485 rows by (485)(4/3) columns, where 4/3 is the aspect ratio (the ratio of the width to height of the image). Thus, we have 313,633 picture elements (pixels) per frame, which are transmitted in 1/30 of a second. This is equivalent to a sampling rate of 10.5 MHz, which is sufficient to represent a signal as large as 5.25 MHz. However, the light intensity of adjacent pixels in an image is highly correlated. Hence, the bandwidth of the video signal is less than 5.25 MHz. In commercial TV broadcasting, the bandwidth of the video signal is limited to $W = 4.2$ MHz.

Since the allocated channel bandwidth for commercial TV is 6 MHz, it is clear that DSB transmission is not possible. The large low-frequency content of the video signal also rules out SSB as a practical modulation method. Hence, VSB is the only viable alternative. By transmitting a large carrier component, the received VSB signal can be simply demodulated by means of an envelope detector. This type of detection significantly simplifies the implementation of the receiver.

The range of frequencies occupied by the transmitted video signal is shown in Figure 3.48. We note that the full upper sideband (4.2 MHz) of the video signal is

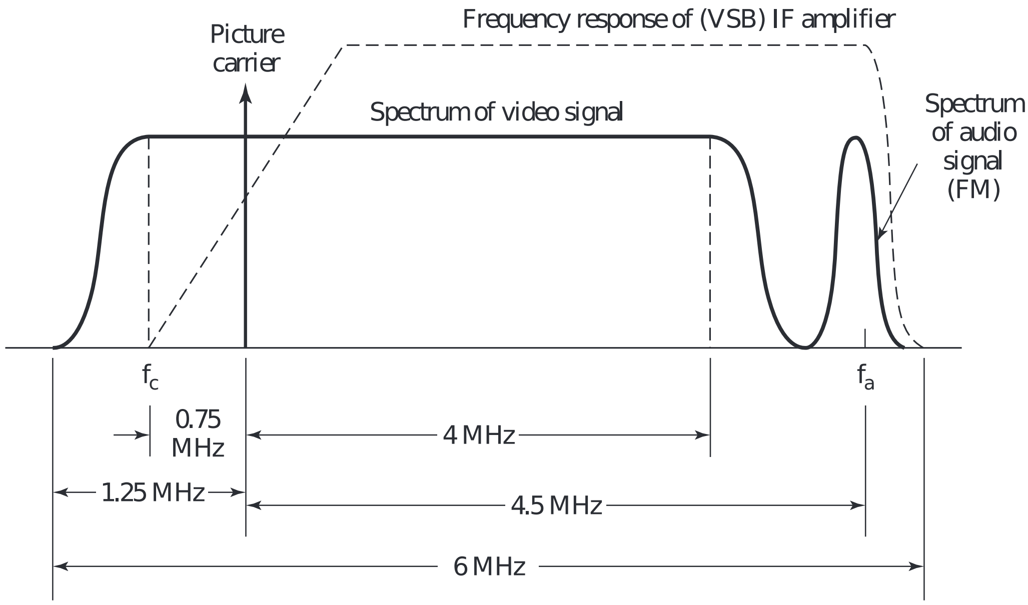


Figure 3.48 Spectral characteristics of black-and-white television signal.

transmitted along with a portion (1.25 MHz) of the lower sideband. Unlike the conventional VSB spectral shaping described in Section 5.2.4, the lower sideband signal in the frequency range f_c and $f_c - 0.75$ MHz is transmitted without attenuation. The frequencies in the range $f_c - 1.25$ to $f_c - 0.75$ MHz are attenuated as shown in Figure 3.48 and all frequency components below $f_c - 1.25$ MHz are blocked. VSB spectral shaping is performed at the IF amplifier of the receiver.

In addition to the video signal, the audio portion of the TV signal is transmitted by frequency modulating a carrier at $f_c + 4.5$ MHz. The audio-signal bandwidth is limited to $W = 10$ kHz. The frequency deviation in the FM modulated signal is selected as 25 kHz and the FM signal bandwidth is 70 kHz. Hence, the total channel bandwidth required to transmit the video and audio signals is 5.785 MHz.

Figure 3.49 shows a block diagram of a black-and-white TV transmitter. The corresponding receiver is shown in Figure 3.50. It is a heterodyne receiver. We note that there are two separate tuners, one for the UHF band and one for the VHF band. The TV signals in the UHF band are brought down to the VHF band by a UHF mixer. This frequency conversion makes it possible to use a common RF amplifier for the two frequency bands. Then, the video signal selected by the tuner is translated to a common IF frequency band of 41–47 MHz. The IF amplifier also provides the VSB shaping required prior to signal detection. The output of the IF amplifier is envelope detected to produce the baseband signal.

The audio portion of the signal centered at 4.5 MHz is filtered out by means of an IF filter amplifier and passed to the FM demodulator. The demodulated audio band signal is then amplified by an audio amplifier and its output drives the speaker.

The video component of the baseband signal is passed through a video amplifier which passes frequency components in the range 0–4.2 MHz. Its output is passed to the

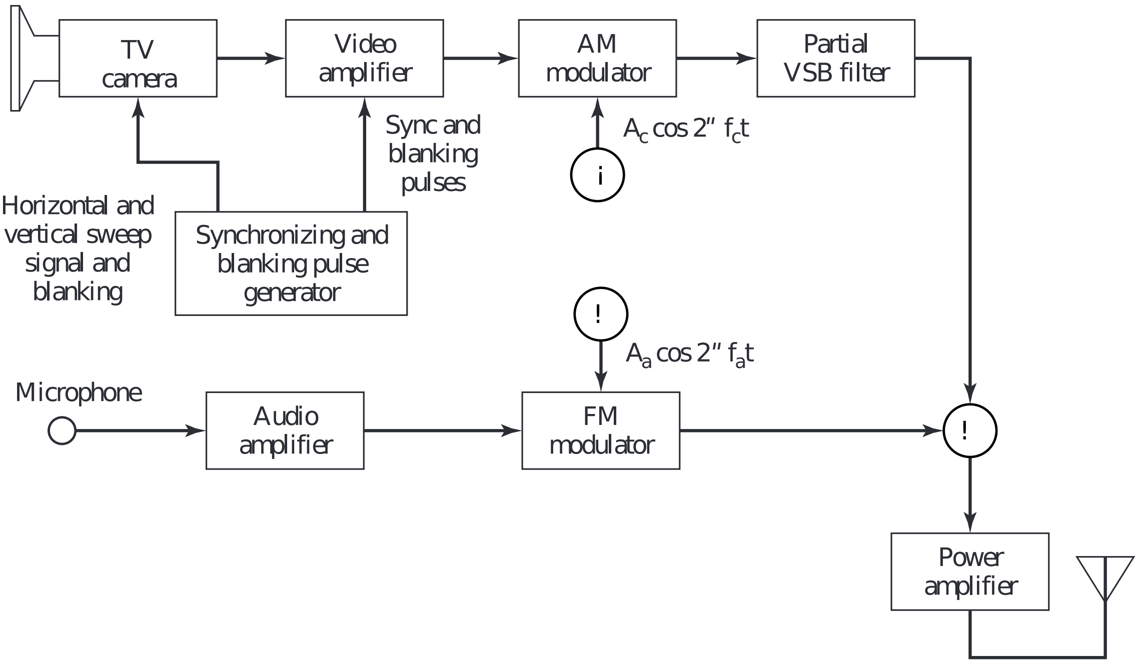


Figure 3.49 Block diagram of a black-and-white TV transmitter.

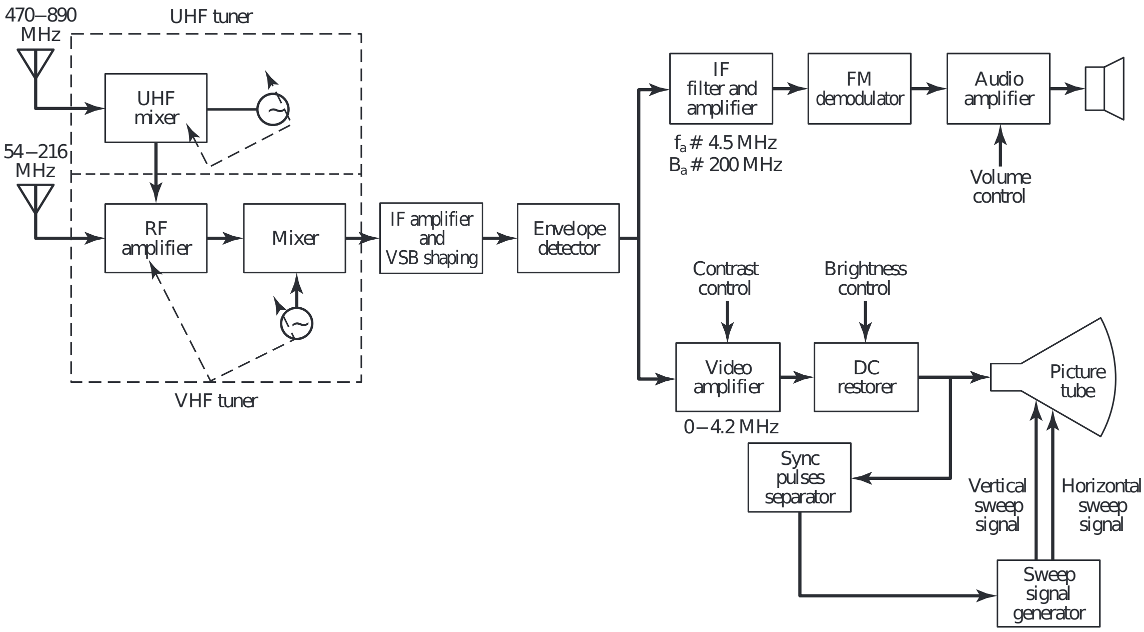


Figure 3.50 Block diagram of a black-and-white TV receiver.

DC restorer that clamps the blanking pulses and sets the correct dc level. The dc-restored video signal is then fed to the picture tube. The synchronizing pulses contained in the received video signal are separated and applied to the horizontal and vertical sweep generators.

Compatible Color Television. The transmission of color information contained in an image can be accomplished by decomposing the colors of pixels into primary colors and transmitting the electrical signals corresponding to these colors. In general, all natural colors are well approximated by appropriate mixing of three primary colors: blue, green, and red. Consequently, if we employ three cameras, one with a blue filter, one with a green filter, and one with a red filter, and transmit the electrical signals $m_b(t)$, $m_g(t)$, and $m_r(t)$, generated by the three color cameras that view the color image, the received signals can be combined to produce a replica of the original color image.

Such a transmission scheme has two major disadvantages. First, it requires three times the channel bandwidth of black-and-white television. Second, the transmitted color TV signal cannot be received by a black-and-white (monochrome) TV receiver.

The NTSC standard adopted in 1953 in the United States avoids these two problems by transmitting a mixture of the three primary-color signals. Specifically, the three signals transmitted in the standard color TV system are the following three linearly independent combinations:

$$\begin{aligned} m_L(t) &= 0.11 m_b(t) + 0.59 m_g(t) + 0.30 m_r(t) \\ m_I(t) &= -0.32 m_b(t) - 0.28 m_g(t) + 0.60 m_r(t) \\ m_Q(t) &= 0.31 m_b(t) - 0.52 m_g(t) + 0.21 m_r(t) \end{aligned} \tag{3.4.3}$$

The transformation matrix

$$M = \begin{bmatrix} 0.11 & 0.59 & 0.30 \\ -0.32 & -0.28 & 0.60 \\ 0.31 & -0.52 & 0.21 \end{bmatrix} \tag{3.4.4}$$

that is used to construct the new transmitted signals $m_L(t)$, $m_I(t)$, and $m_Q(t)$ is nonsingular and is inverted at the receiver to recover the primary-color signals $m_b(t)$, $m_g(t)$ and $m_r(t)$ from $m_L(t)$, $m_I(t)$ and $m_Q(t)$.

The signal $m_L(t)$ is called the *luminance signal*. It is assigned a bandwidth of 4.2 MHz and transmitted via VSB AM as in monochrome TV transmission. When this signal is received by a monochrome receiver, the result is a conventional black-and-white version of the color image. Thus, compatibility with monochrome TV broadcasting is achieved by transmitting $m_L(t)$. There remains the problem of transmitting the additional color information that can be used by a color TV receiver to reconstruct the color image. It is remarkable that the two composite color signals $m_I(t)$ and $m_Q(t)$ can be transmitted in the same bandwidth as $m_L(t)$, without interfering with $m_L(t)$.

The signals $m_I(t)$ and $m_Q(t)$ are called *chrominance signals* and are related to hue and saturation of colors. It has been determined experimentally, through subjective tests, that human vision cannot discriminate changes in $m_I(t)$ and $m_Q(t)$ over short time intervals and, hence, over small areas of the image. This implies that the high frequency content in the signals $m_I(t)$ and $m_Q(t)$ can be eliminated without significantly compromising the quality of the reconstructed image. The end result is that $m_I(t)$ is limited in bandwidth to 1.6 MHz and $m_Q(t)$ is limited to 0.6 MHz prior to transmission. These two signals are quadrature-carrier multiplexed on a subcarrier frequency

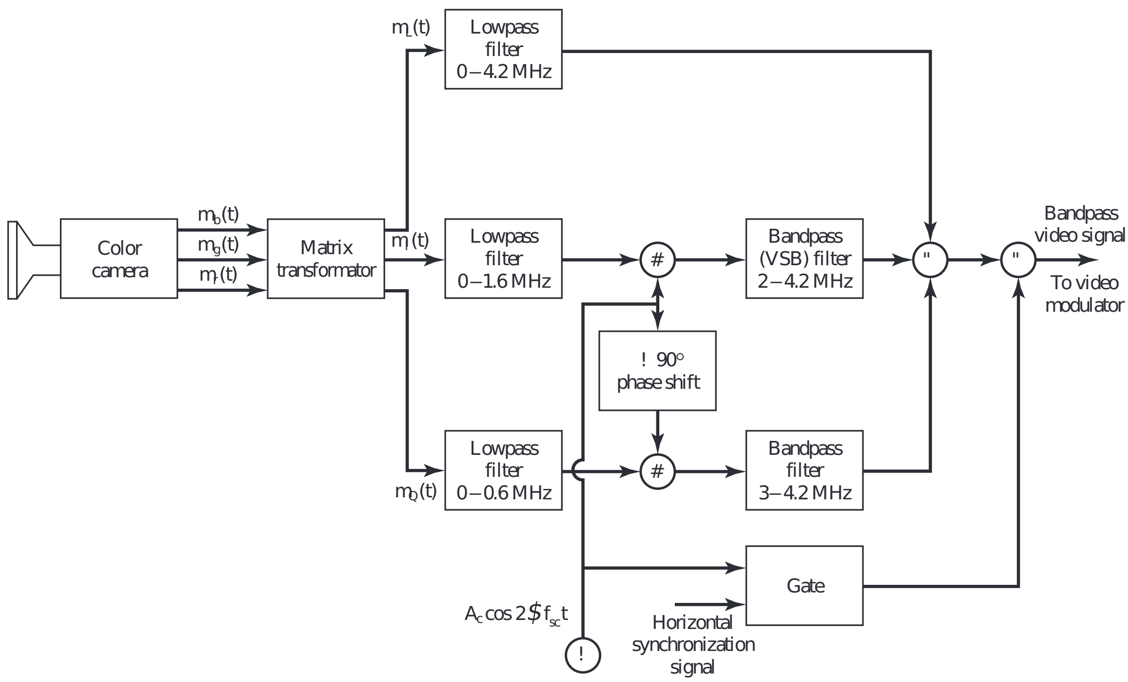


Figure 3.51 Transmission of primary-color signals and multiplexing of chrominance and luminance signals.

$f_{sc} = f_c + 3.579545$ MHz, as illustrated in Figure 3.51. The signal $m_I(t)$ is passed through a VSB filter that removes a part of the upper sideband, above 4.2 MHz. The signal $m_Q(t)$ is transmitted by DSB-SC amplitude modulation. Therefore, the composite video signal may be expressed as

$$m(t) = m_L(t) + m_Q(t) \sin 2\pi f_{sc} t + m_I(t) \cos 2\pi f_{sc} t + \hat{m}_I^{!}(t) \sin 2\pi f_{sc} t \qquad (3.4.5)$$

The last two terms in (3.4.5) involving $m_I(t)$ and $\hat{m}_I^{!}(t)$, constitute the VSB AM signal for the chrominance $m_I(t)$. The composite signal $m(t)$ is transmitted by VSB plus carrier in a 6 MHz bandwidth, as shown in Figure 3.52.

The spectrum of the luminance signal $m_L(t)$ has periodic gaps between harmonics of the horizontal sweep frequency f_h, which in color TV is 4.5 MHz/286. The subcarrier frequency $f_{sc} = 3.579545$ for transmission of the chrominance signals was chosen because it corresponds to one of these gaps in the spectrum of $m_L(t)$. Specifically, it falls between the 227 and 228 harmonics of f_h. Thus, the chrominance signals are interlaced in the frequency domain with the luminance signal as illustrated in Figure 3.53. As a consequence, the effect of the chrominance signal on the luminance signal $m_L(t)$ is not perceived by the human eye, due to the persistence of human vision. Therefore, the chrominance signals do not interfere with the demodulation of the luminance signal in both a monochrome TV receiver and a color TV receiver.

Horizontal and vertical synchronization pulses are added to $m(t)$ at the transmitter. In addition, eight cycles of the color subcarrier $A_c \cos 2\pi f_{sc} t$, called a "color burst," are superimposed on the trailing edge of the blanking pulses, as shown in Figure 3.54, for the purpose of providing a signal for subcarrier phase synchronization at the receiver.

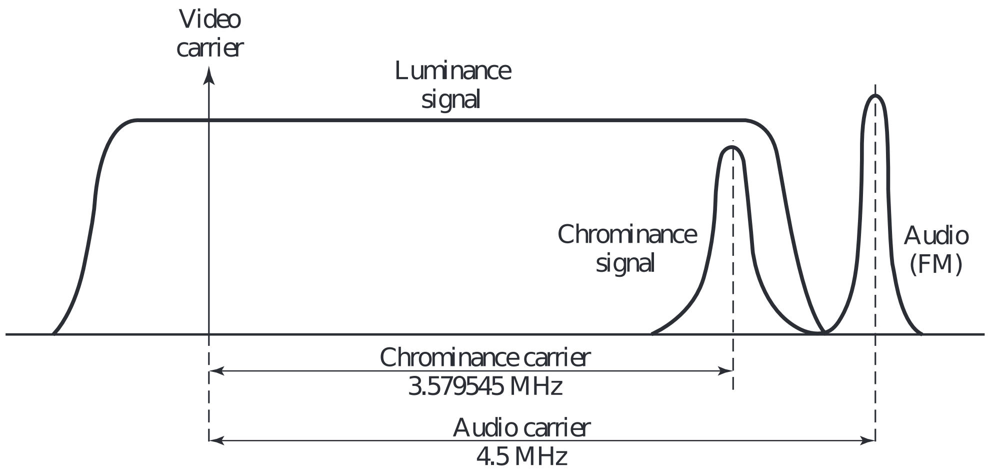


Figure 3.52 Spectral characteristics of color TV signal.

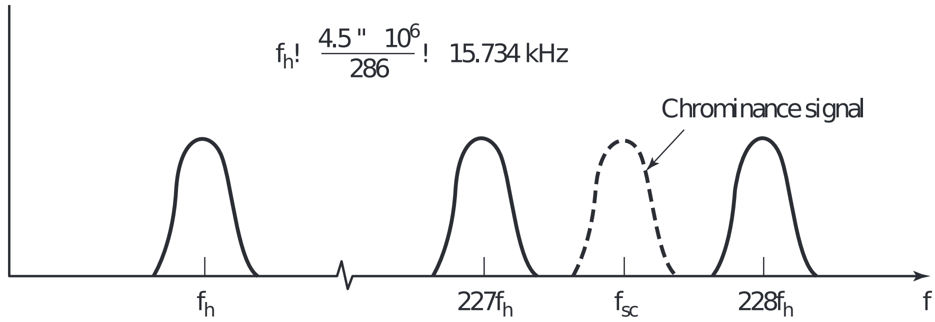


Figure 3.53 Interlacing of chrominance signals with luminance signals.

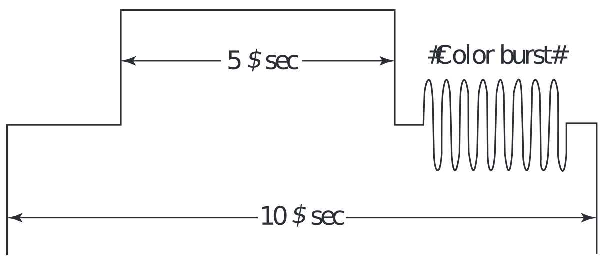


Figure 3.54 Blanking pulse with color subcarrier.

The front end of the color TV receiver is basically the same as that of a monochrome receiver, down to the envelope detector which converts the 6 MHz VSB signal to baseband. The remaining demultiplexing operations in the color TV receiver are shown in Figure 3.55. We note that a lowpass filter with bandwidth 4.2 MHz is used to recover the luminance signal $m_L(t)$. The chrominance signals are stripped off by bandpass filtering and demodulated by the quadrature-carrier demodulator using the output of a VCO that is phase locked to the received color-carrier frequency burst transmitted in each horizontal sweep. The demodulated chrominance signals are lowpass filtered and, along with the luminance signal, are passed to the "inverse matrix" converter that

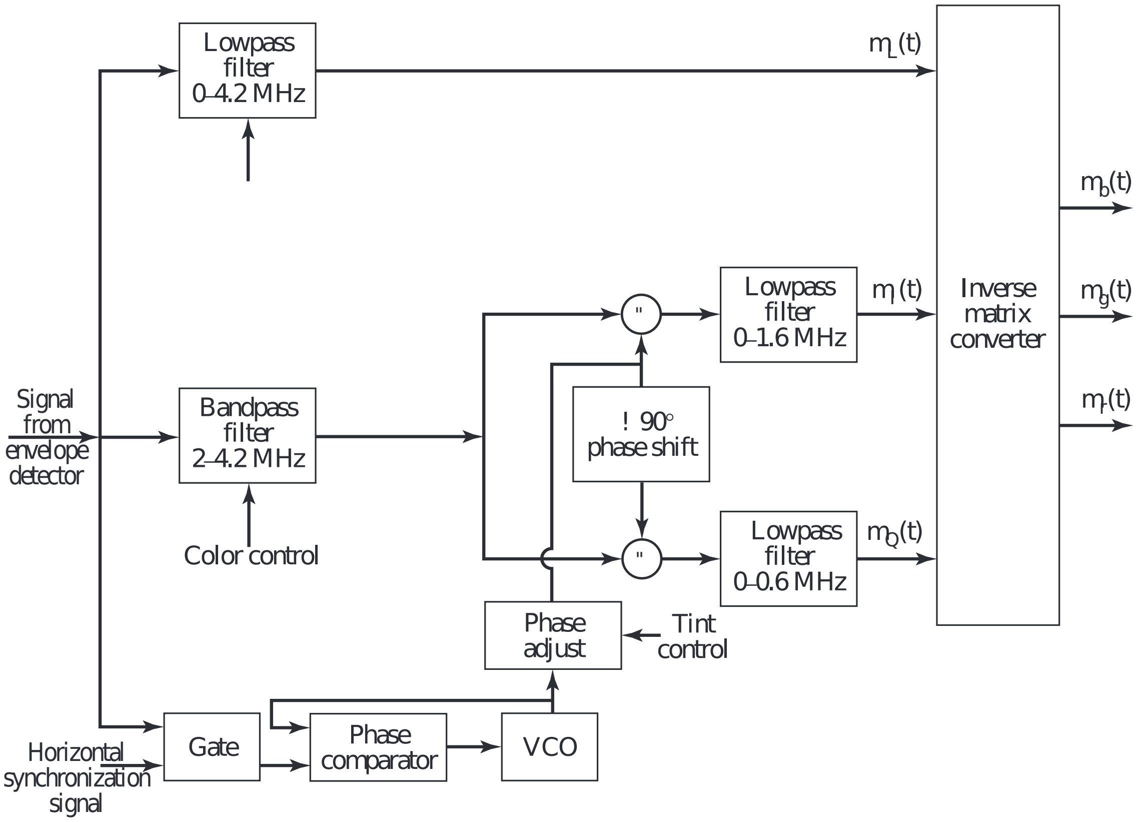


Figure 3.55 Demultiplexing and demodulation of luminance and chrominance signals in a color TV receiver.

reconstructs the three color signals $m_b(t)$, $m_g(t)$ and $m_r(t)$; i.e.,

$$\begin{bmatrix} m_b(t) \\ m_g(t) \\ m_r(t) \end{bmatrix} = \begin{bmatrix} 1.00 & -1.10 & 1.70 \\ 1.00 & -0.28 & -0.64 \\ 1.00 & -0.96 & 0.62 \end{bmatrix} \begin{bmatrix} m_L(t) \\ m_I(t) \\ m_Q(t) \end{bmatrix} \tag{3.4.6}$$

The resulting color signals control the three electron guns that strike corresponding blue, green, and red picture elements in a color picture tube. Although color picture tubes are constructed in many different ways, the color mask tube is commonly used in practice. The face of the picture tube contains a matrix of dots of phosphor of the three primary colors with three such dots in each group. Behind each dot color group there is a mask with holes, one hole for each group. The three electron guns are aligned so that each gun can excite one of the three types of color dots. Thus, the three types of color dots are excited simultaneously in different intensities to generate color images.

3.5 MOBILE RADIO SYSTEMS

The demand to provide telephone service for people traveling in automobiles, buses, trains, and airplanes has been steadily increasing over the past three to four decades. To meet this demand, radio transmission systems have been developed that link the mobile

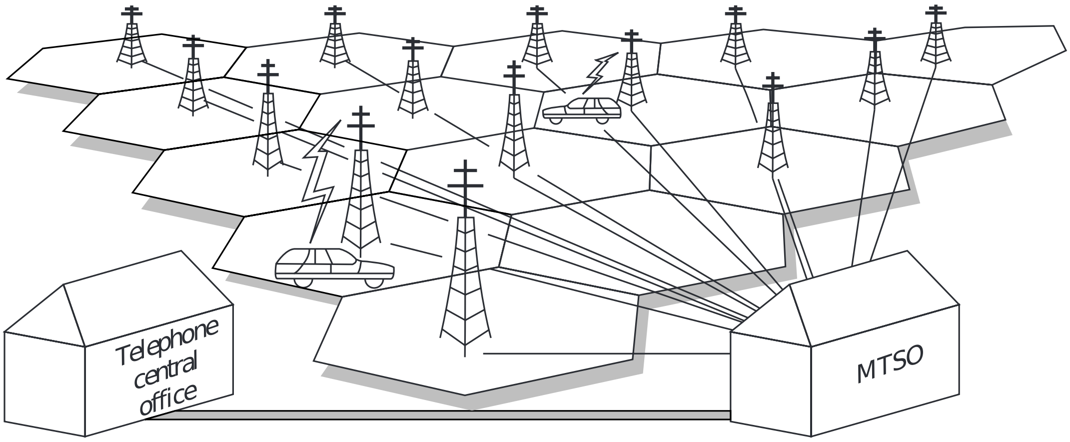


Figure 3.56 Mobile radio system.

telephone user to the terrestrial telephone network. Today, radio-based systems make it possible for people to communicate via telephone while traveling in airplanes and motor vehicles. In this section, we will briefly describe the analog cellular telephone system that provides telephone service to people with hand-held portable telephones and automobile telephones. Digital cellular systems are studied in Chapter 10.

A major problem with the establishment of any radio communication system is the availability of a portion of the radio spectrum. In the case of radio telephone service, the Federal Communications Commission (FCC) in the United States has assigned parts of the UHF frequency band in the range 806–890 MHz for this use. Similar frequency assignments in the UHF band have been made in Europe and Japan.

The cellular radio concept was adopted as a method for efficient utilization of the available frequency spectrum, especially in highly populated metropolitan areas where the demand for mobile telephone services is the greatest. A geographic area is subdivided into cells, each of which contains a base station, as illustrated in Figure 3.56. Each base station is connected via telephone lines to a mobile telephone switching office (MTSO), which in turn is connected via telephone lines to a telephone central office (CO) of the terrestrial telephone network.

A mobile user communicates via radio with the base station within the cell. The base station routes the call through the MTSO to another base station if the called party is located in another cell or to the central office of the terrestrial telephone network if the called party is not a mobile. Each mobile telephone is identified by its telephone number and the telephone serial number assigned by the manufacturer. These numbers are automatically transmitted to the MTSO during the initialization of the call for purposes of authentication and billing.

A mobile user initiates a telephone call in the usual manner by keying in the desired telephone number and pressing the “send” button. The MTSO checks the authentication of the mobile user and assigns an available frequency channel for radio transmission of the voice signal from the mobile to the base station. The frequency assignment is sent to the mobile telephone via a supervisory control channel. A second frequency is assigned for radio transmission from the base station to the mobile user. The simultaneous transmission between the two parties is called full-duplex operation. The MTSO interfaces with the central office of the telephone network to complete the connection to the called party. All telephone communications between the MTSO and the telephone network are by means of wideband trunk lines that carry speech signals from many users. Upon completion of the telephone call, when the two parties hang up, the radio channel becomes available for another user.

During the phone call, the MTSO monitors the signal strength of the radio transmission from the mobile user to the base station and, if the signal strength drops below a preset threshold, the MTSO views this as an indication that the mobile user is moving out of the initial cell into a neighboring cell. By communicating with the base stations of neighboring cells, the MTSO finds a neighboring cell that receives a stronger signal and automatically switches, or hands-off, the mobile user to the base station of the adjacent cell. The switching is performed in a fraction of a second and is generally transparent to the two parties. When a mobile user is outside of the assigned service area, the mobile telephone may be placed in a "roam" mode, which allows the mobile user to initiate and receive telephone calls.

In analog transmission of voice-band audio signals via radio, between the base station and the mobile user, the 3-kHz wide audio signal is transmitted via FM using a channel bandwidth of 30 kHz. This represents a bandwidth expansion of approximately a factor of 10. Such a large bandwidth expansion is necessary to obtain a sufficiently large signal-to-noise ratio (SNR) at the output of the FM demodulator. However, the use of FM is highly wasteful of the radio frequency spectrum. The new generation of cellular telephone systems discussed in Chapter 10 use digital transmission of digitized compressed speech (at bit rates of about 10,000 bps) based on LPC encoding and vector quantization of the speech-model parameters as described in Chapter 6. With digital transmission, the cellular telephone system can accommodate a four-fold to tenth-fold increase in the number of simultaneous users with the same available channel bandwidth.

The cellular radio telephone system is designed such that the transmitter powers of the base station and the mobile users are sufficiently small, so that signals do not propagate beyond immediately adjacent cells. This allows for frequencies to be reused in other cells outside of the immediately adjacent cells. Consequently, by making the cells smaller and reducing the radiated power, it is possible to increase frequency reuse and, thus, to increase the bandwidth efficiency and the number of mobile users. Current cellular systems employ cells with a radius in the range of 5–18 km. The base station normally transmits at a power level of 35 W or less and the mobile users transmit at a power level of 3 W or less, approximately. Digital transmission systems are capable of communicating reliably at lower power levels (see Chapter 10).

The cellular radio concept is being extended to different types of personal communication services using low-power, hand-held radio transmitter and receiver. These emerging communication services are made possible by rapid advances in the fabrication of small and powerful integrated circuits that consume very little power and are relatively inexpensive. As a consequence, we will continue to experience exciting new developments in the telecommunications industry, well into the twenty-first century.

3.6 FURTHER READING

Analog communication systems are treated in numerous books on basic communication theory, including Sakrison (1968), Shanmugam (1979), Carlson (1986), Stremler (1990), Ziemer and Tranter (1990), Couch (1993), Gibson (1993), and Haykin (2000). Implementation of analog communications systems are dealt with in depth in Clarke and Hess (1971).

PROBLEMS

3.1 The message signal $m(t) = 2\cos 400t + 4\sin(500t + \frac{\pi}{3})$ modulates the carrier signal $c(t) = A\cos(8000\pi t)$, using DSB amplitude modulation. Find the time domain and frequency domain representation of the modulated signal and plot the spectrum (Fourier transform) of the modulated signal. What is the power content of the modulated signal?

3.2 In a DSB system the carrier is $c(t) = A\cos 2\pi f_c t$ and the message signal is given by $m(t) = \mathrm{sinc}(t) + \mathrm{sinc}^2(t)$. Find the frequency domain representation and the bandwidth of the modulated signal.

3.3 The two signals (a) and (b) shown in Figure P-3.3 DSB modulate a carrier signal $c(t) = A\cos 2\pi f_0 t$. Precisely plot the resulting modulated signals as a function of time and discuss their differences and similarities.

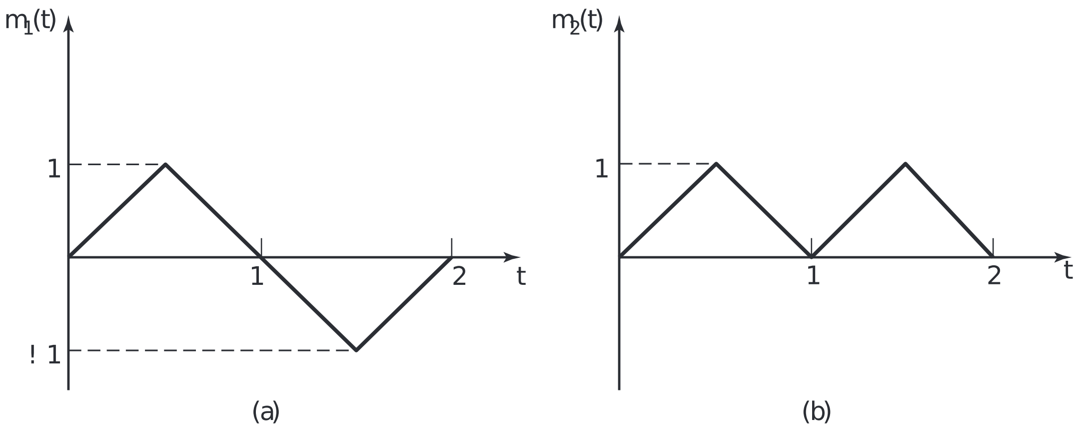


Figure P-3.3

3.4 Suppose the signal $x(t) = m(t) + \cos 2\pi f_c t$ is applied to a nonlinear system whose output is $y(t) = x(t) + \frac{1}{2}x^2(t)$. Determine and sketch the spectrum of $y(t)$ when $M(f)$ is as shown in Figure P-3.4 and $W \ll f_c$.

3.5 The modulating signal

$$m(t) = 2\cos 4000\pi t + 5\cos 6000\pi t$$

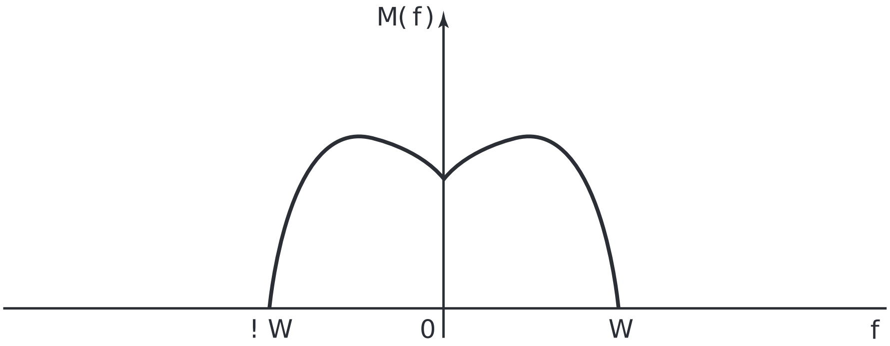


Figure P-3.4

is multiplied by the carrier

$$c(t) = 100 \cos 2\pi f_c t$$

where $f_c = 50$ kHz. Determine and sketch the power-spectral density of the DSB signal.

3.6 A DSB-modulated signal $u(t) = Am(t) \cos 2\pi f_c t$ is mixed (multiplied) with a local carrier $x_L(t) = \cos(2\pi f_c t + \theta)$ and the output is passed through a LPF with a bandwidth equal to the bandwidth of the message $m(t)$. Denoting the power of the signal at the output of the lowpass filter by P_{out} and the power of the modulated signal by P_U, plot $\frac{P_{out}}{P_U}$ as a function of θ for $0 \leq \theta \leq \pi$.

3.7 An AM signal has the form

$$u(t) = [20 + 2 \cos 3000\pi t + 10 \cos 6000\pi t] \cos 2\pi f_c t$$

where $f_c = 10^5$ Hz.

1. Sketch the (voltage) spectrum of $u(t)$.
2. Determine the power in each of the frequency components.
3. Determine the modulation index.
4. Determine the power in the sidebands, the total power, and the ratio of the sidebands power to the total power.

3.8 A message signal $m(t) = \cos 2000\pi t + 2 \cos 4000\pi t$ modulates the carrier $c(t) = 100 \cos 2\pi f_c t$ where $f_c = 1$ MHz to produce the DSB signal $m(t)c(t)$.

1. Determine the expression for the upper sideband (USB) signal.
2. Determine and sketch the spectrum of the USB signal.

3.9 A DSB-SC signal is generated by multiplying the message signal $m(t)$ with the periodic rectangular waveform shown in Figure P-3.9 and filtering the product with a bandpass filter tuned to the reciprocal of the period T_p, with bandwidth $2W$, where W is the bandwidth of the message signal. Demonstrate that the output

$u(t)$ of the BPF is the desired DSB-SC AM signal

$$u(t) = m(t) \sin 2\pi f_c t$$

where $f_c = 1/T_p$.

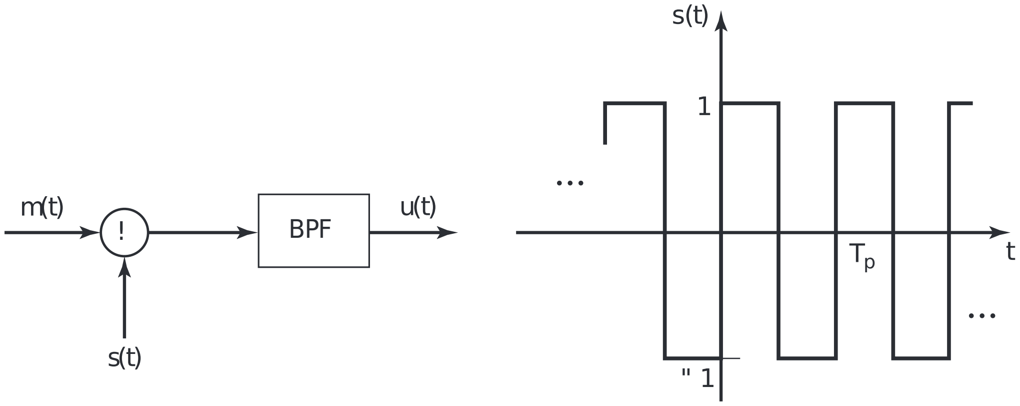


Figure P-3.9

3.10 Show that in generating a DSB-SC signal as in Problem P-3.9, it is not necessary that the periodic signal be rectangular. This means that any periodic signal with period T_p can substitute for the rectangular signal in Figure P-3.9.

3.11 The message signal $m(t)$ has a Fourier transform shown in Figure P-3.11(a). This signal is applied to the system shown in Figure P-3.11(b) to generate the signal $y(t)$.

1. Plot $Y(f)$, the Fourier transform of $y(t)$.
2. Show that if $y(t)$ is transmitted, the receiver can pass it through a replica of the system shown in Figure P-3.11(b) to obtain $m(t)$ back. This means that this system can be used as a simple scrambler to enhance communication privacy.

3.12 Show that in a DSB-modulated signal, the envelope of the resulting bandpass signal is proportional to the absolute value of the message signal. This means that an envelope detector can be employed as a DSB demodulator if we know that the message signal is always positive.

3.13 An AM signal is generated by modulating the carrier $f_c = 800$ kHz by the signal

$$m(t) = \sin 2000\pi t + 5 \cos 4000\pi t t$$

The AM signal

$$u(t) = 100[1 + m(t)] \cos 2\pi f_c t$$

is fed to a $50\ \Omega$ load.

1. Determine and sketch the spectrum of the AM signal.
2. Determine the average power in the carrier and in the sidebands.

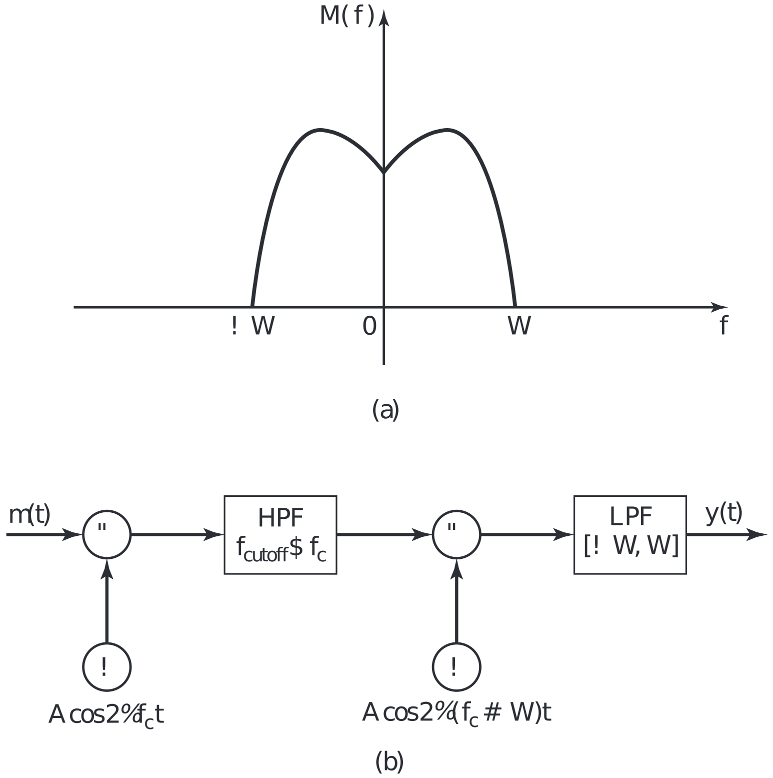


Figure P-3.11

3. What is the modulation index?
4. What is the peak power delivered to the load?

3.14 The output signal from an AM modulator is

$$u(t) = 5\cos 1800\pi t + 20\cos 2000\pi t + 5\cos 2200\pi t$$

1. Determine the modulating signal $m(t)$ and the carrier $c(t)$.
2. Determine the modulation index.
3. Determine the ratio of the power in the sidebands to the power in the carrier.

3.15 A DSB-SC AM signal is modulated by the signal

$$m(t) = 2\cos 2000\pi t + \cos 6000\pi t$$

The modulated signal is

$$u(t) = 100m(t)\cos 2\pi f_c t$$

where $f_c = 1$ MHz.

1. Determine and sketch the spectrum of the AM signal.
2. Determine the average power in the frequency components.

3.16 A SSB AM signal is generated by modulating an 800-kHz carrier by the signal $m(t) = \cos 2000\pi t + 2 \sin 2000\pi t$. The amplitude of the carrier is $A_c = 100$.

1. Determine the signal $\hat{m}(t)$.
2. Determine the (time domain) expression for the lower sideband of the SSB AM signal.
3. Determine the magnitude spectrum of the lower sideband SSB signal.

3.17 Weaver's SSB modulator is illustrated in Figure P-3.17. By taking the input signal as $m(t) = \cos 2\pi f_m t$, where $f_m < W$, demonstrate that by proper choice of f_1 and f_2 the output is a SSB signal.

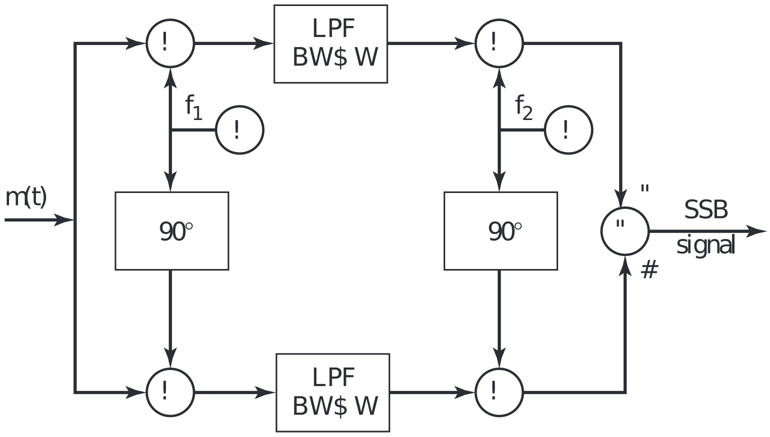


Figure P-3.17

3.18 The message signal $m(t)$ whose spectrum is shown in Figure P-3.18 is passed through the system shown in the same figure.

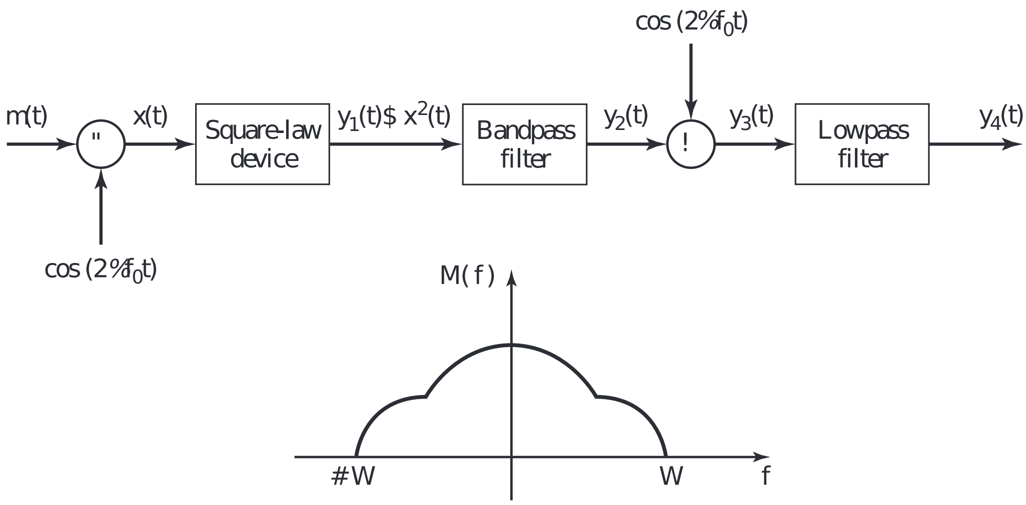


Figure P-3.18

The bandpass filter has a bandwidth of $2W$ centered at f_0 and the lowpass filter has a bandwidth of W. Plot the spectra of the signals $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$, and $y_4(t)$. What are the bandwidths of these signals?

3.19 The system shown in Figure P-3.19 is used to generate an AM signal. The modulating signal $m(t)$ has zero mean and its maximum (absolute) value is $A_m = \max |m(t)|$. The nonlinear device has a input–output characteristic

$$y(t) = ax(t) + bx^2(t)$$

1. Express $y(t)$ in terms of the modulating signal $m(t)$ and the carrier $c(t) = \cos 2\pi f_c t$.
2. What is the modulation index?
3. Specify the filter characteristics that yield an AM signal at its output.

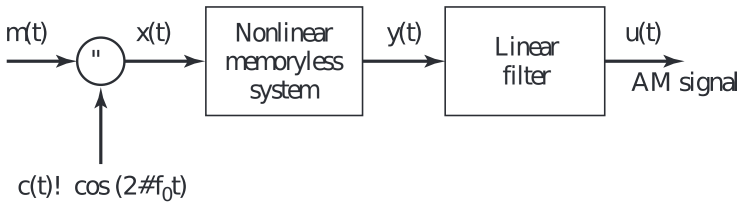


Figure P-3.19

3.20 The signal $m(t)$ whose Fourier transform $M(f)$ is shown in Figure P-3.20 is to be transmitted from point A to point B. It is known that the signal is normalized, meaning that $-1 \leq m(t) \leq 1$.

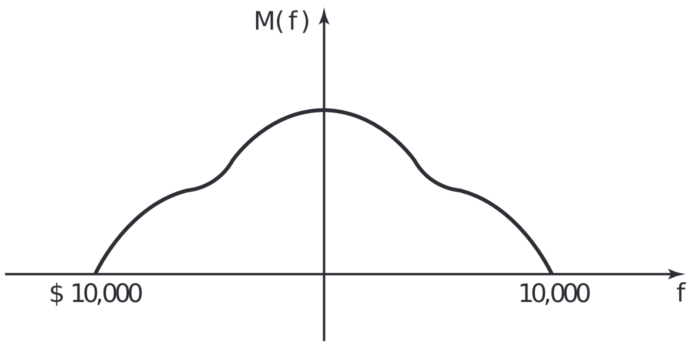


Figure P-3.20

1. If USSB is employed, what is the bandwidth of the modulated signal?
2. If DSB is employed, what is the bandwidth of the modulated signal?
3. If an AM modulation scheme with $a = 0.8$ is used, what is the bandwidth of the modulated signal?

4. If an FM signal with $k_f = 60$ kHz is used, what is the bandwidth of the modulated signal?

3.21 A vestigial sideband modulation system is shown in Figure P-3.21. The bandwidth of the message signal $m(t)$ is W and the transfer function of the bandpass filter is shown in the figure.

1. Determine $h_l(t)$ the lowpass equivalent of $h(t)$, where $h(t)$ represents the impulse response of the bandpass filter.
2. Derive an expression for the modulated signal $u(t)$.

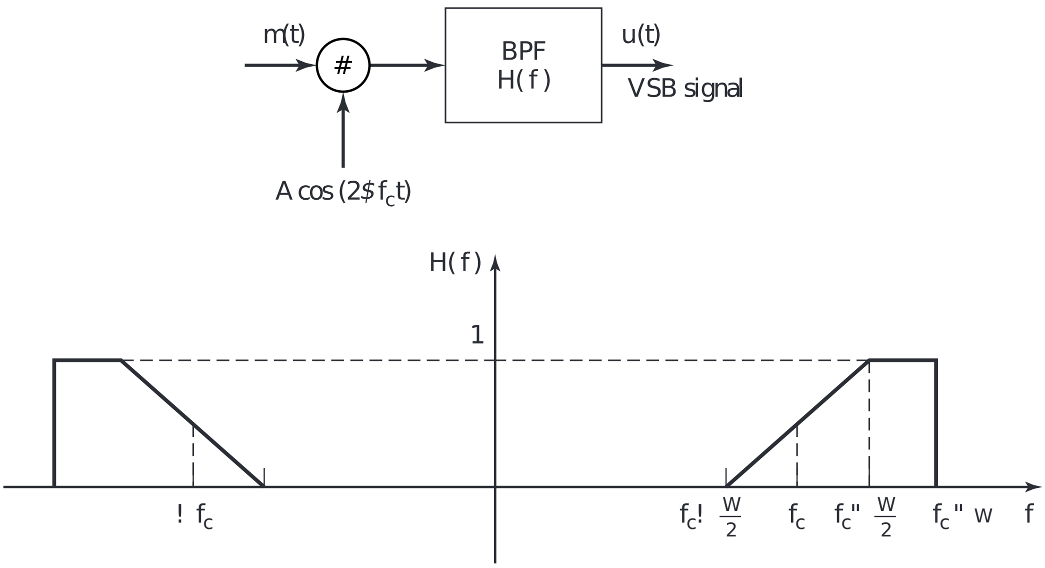


Figure P-3.21

3.22 Find expressions for the in-phase and quadrature components, $x_c(t)$ and $x_s(t)$, and envelope and phase, $V(t)$ and ! (t), for DSB, SSB, Conventional AM, USSB, LSSB, FM, and PM.

3.23 The normalized signal $m_n(t)$ has a bandwidth of 10,000 Hz and its power content is 0.5 W. The carrier $A \cos 2\pi f_0 t$ has a power content of 200 W.

1. If $m_n(t)$ modulates the carrier using SSB amplitude modulation, what will be the bandwidth and the power content of the modulated signal?
2. If the modulation scheme is DSB-SC, what will be the answer to part 1?
3. If the modulation scheme is AM with modulation index of 0.6, what will be the answer to part 1?
4. If modulation is FM with $k_f = 50{,}000$, what will be the answer to part 1?

3.24 The message signal $m(t) = 10\,\mathrm{sinc}(400t)$ frequency modulates the carrier $c(t) = 100 \cos 2\pi f_c t$. The modulation index is 6.

1. Write an expression for the modulated signal $u(t)$?
2. What is the maximum frequency deviation of the modulated signal?
3. What is the power content of the modulated signal?
4. Find the bandwidth of the modulated signal.

3.25 Signal $m(t)$ is shown in Figure P-3.25. This signal is used once to frequency modulate a carrier and once to phase modulate the same carrier.

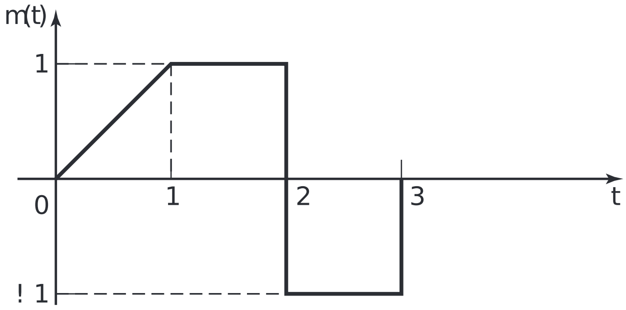


Figure P-3.25

1. Find a relation between k_p and k_f such that the maximum phase of the modulated signals in both cases are equal.
2. If $k_p = f_d = 1$, what is the maximum instantaneous frequency in each case?

3.26 An angle modulated signal has the form

$$u(t) = 100 \cos[2\pi f_c t + 4 \sin 2000\pi t]$$

where $f_c = 10$ MHz.

1. Determine the average transmitted power.
2. Determine the peak-phase deviation.
3. Determine the peak-frequency deviation.
4. Is this an FM or a PM signal? Explain.

3.27 Find the smallest value of the modulation index in an FM system that guarantees that all the modulated signal power is contained in the sidebands and no power is transmitted at the carrier frequency.

3.28 Wideband FM can be generated by first generating a narrowband FM signal and then using frequency multiplication to spread the signal bandwidth. Figure P-3.28 illustrates such a scheme, which is called an Armstrong-type FM modulator. The narrowband FM signal has a maximum angular deviation of 0.10 radians in order to keep distortion under control.

1. If the message signal has a bandwidth of 15 kHz and the output frequency from the oscillator is 100 kHz, determine the frequency multiplication that is

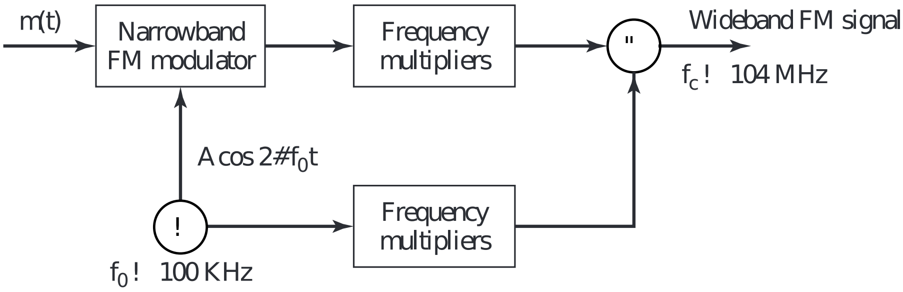


Figure P-3.28 Armstrong-type FM Modulator.

necessary to generate an FM signal at a carrier frequency of $f_c = 104$ MHz and a frequency deviation of $\Delta f = 75$ kHz.

2. If the carrier frequency for the wideband FM signal is to be within ± 2 Hz, determine the maximum allowable drift of the 100 kHz oscillator.

3.29 Determine the amplitude and phase of various frequency components of a PM signal with $k_p = 1$ and $m(t)$, a periodic signal given by

$$m(t) = \begin{cases} 1, & 0 \le t \le \frac{T_m}{2} \\ -1, & \frac{T_m}{2} \le t \le T_m \end{cases} \tag{3.5.1}$$

in one period.

3.30 An FM signal is given as

$$u(t) = 100 \cos\left[2\pi f_c t + 100 \int_{-\infty}^{t} m(\tau)\, d\tau\right]$$

where $m(t)$ is shown in Figure P-3.30.

1. Sketch the instantaneous frequency as a function of time.
2. Determine the peak-frequency deviation.

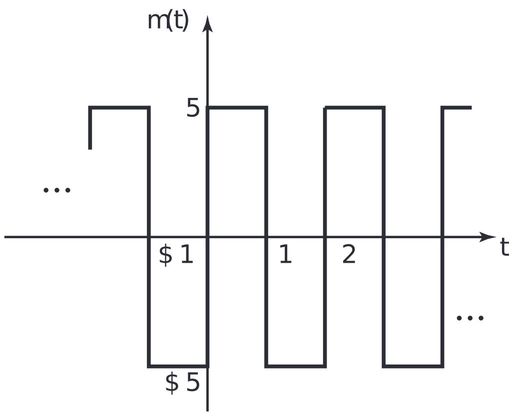


Figure P-3.30

3.31 The carrier $c(t) = 100\cos 2\pi f_c t$ is frequency modulated by the signal $m(t) = 5\cos 20000\pi t$, where $f_c = 10^8$ Hz. The peak frequency deviation is 20 kHz.

1. Determine the amplitude and frequency of all signal components that have a power level of at least 10% of the power of the unmodulated carrier component.
2. From Carson's rule, determine the approximate bandwidth of the FM signal.

3.32 The carrier $c(t) = A\cos 2\pi 10^6 t$ is angle modulated (PM or FM) by the sinusoid signal $m(t) = 2\cos 2000\pi t$. The deviation constants are $k_p = 1.5$ rad/V and $k_f = 3000$ Hz/V.

1. Determine β_f and β_p.
2. Determine the bandwidth in each case using Carson's rule.
3. Plot the spectrum of the modulated signal in each case (plot only those frequency components that lie within the bandwidth derived in part 2.)
4. If the amplitude of $m(t)$ is decreased by a factor of two, how would your answers to parts 1–3 change?
5. If the frequency of $m(t)$ is increased by a factor of two, how would your answers to parts 1–3 change?

3.33 The carrier $c(t) = 100\cos 2\pi f_c t$ is phase modulated by the signal $m(t) = 5\cos 2000\pi t$. The PM signal has a peak-phase deviation of $\pi/2$. The carrier frequency is $f_c = 10^8$ Hz.

1. Determine the magnitude spectrum of the sinusoidal components and sketch the results.
2. Using Carson's rule, determine the approximate bandwidth of the PM signal and compare the results with the analytical result in part 1.

3.34 An angle-modulated signal has the form

$$u(t) = 100\cos[2\pi f_c t + 4\sin 2\pi f_m t]$$

where $f_c = 10$ MHz and $f_m = 1000$ Hz.

1. Assuming that this is an FM signal, determine the modulation index and the transmitted signal bandwidth.
2. Repeat part 1 if f_m is doubled.
3. Assuming that this is a PM signal determine the modulation index and the transmitted signal bandwidth.
4. Repeat part 3 if f_m is doubled.

3.35 It is easy to demonstrate that amplitude modulation satisfies the superposition principle, whereas angle modulation does not. To be specific, let $m_1(t)$ and $m_2(t)$ be two message signals, and let $u_1(t)$ and $u_2(t)$ be the corresponding modulated versions.

1. Show that when the combined message signal $m_1(t) + m_2(t)$ DSB modulates a carrier $A_c \cos 2\pi f_c t$, the result is the sum of the two DSB amplitude-modulated signals $u_1(t) + u_2(t)$.
2. Show that if $m_1(t) + m_2(t)$ frequency modulates a carrier, the modulated signal is not equal to $u_1(t) + u_2(t)$.

3.36 An FM discriminator is shown in Figure P-3.36. The envelope detector is assumed to be ideal and has an infinite input impedance. Select the values for L and C if the discriminator is to be used to demodulate an FM signal with a carrier $f_c = 80$ MHz and a peak-frequency deviation of 6 MHz.

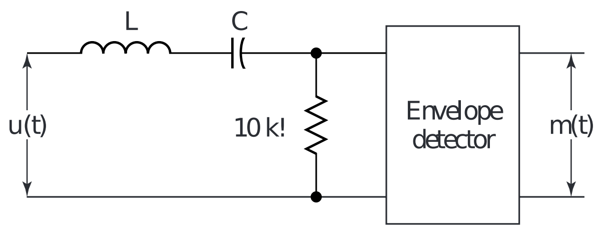


Figure P-3.36

3.37 An angle-modulated signal is given as

$$u(t) = 100 \cos [2000\pi t + \phi(t)]$$

where (a) $\phi(t) = 5 \sin 20\pi t$ and (b) $\phi(t) = 5 \cos 20\pi t$. Determine and sketch the amplitude and phase spectra for (a) and (b), and compare the results.

3.38 The message signal $m(t)$ into an FM modulator with peak-frequency deviation $f_d = 25$ Hz/V is shown in Figure P-3.38. Plot the frequency deviation in Hz and the phase deviation in radians.

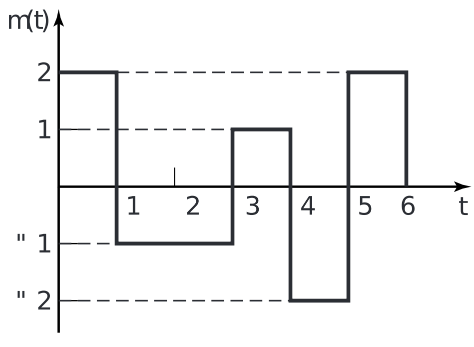


Figure P-3.38

3.39 A message signal $m(t)$ has a bandwidth of 10 kHz and a peak magnitude $|m(t)|$ of 1 V. Estimate the bandwidth of the signal $u(t)$ obtained when $m(t)$ frequency modulates a carrier with a peak frequency deviation of (a) $f_d = 10$ Hz/V, (b) 100 Hz/V, and (c) 1000 Hz/V.

3.40 The modulating signal into an FM modulator is

$$m(t) = 10 \cos 16\pi t$$

The output of the FM modulator is

$$u(t) = 10 \cos\left(400\pi t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau\right)$$

where $k_f = 10$. If the output of the FM modulator is passed through an ideal BPF centered at $f_c = 2000$ with a bandwidth of 62 Hz, determine the power of the frequency components at the output of the filter (see Figure P-3.40). What percentage of the transmitter power appears at the output of the BPF?

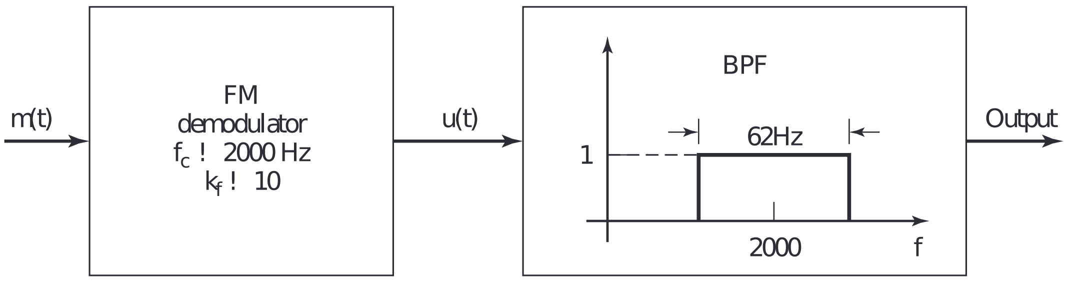


Figure P-3.40

3.41 The message signal $m_1(t)$ is shown in Figure P-3.41.

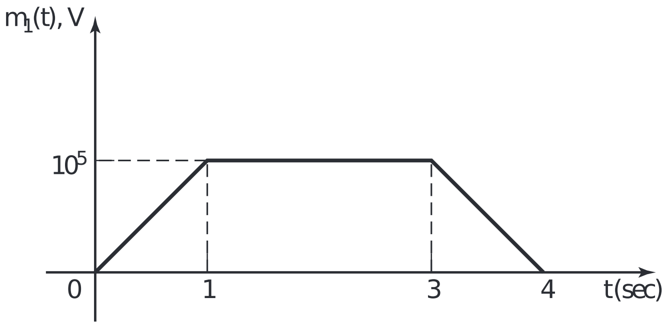


Figure P-3.41

and the message signal $m_2(t) = \text{sinc}(2 \times 10^4 t)$, in volts again.

1. If $m_1(t)$ is frequency modulated on a carrier with frequency 10^6 Hz with a frequency deviation constant (k_f) equal to 5 Hz/V, what is the maximum instantaneous frequency of the modulated signal?
2. If $m_1(t)$ is phase modulated with phase-deviation constant $k_p = 3$ radians/V, what is the maximum instantaneous frequency of the modulated signal? What is the minimum instantaneous frequency of the modulated signal?

3. If $m_2(t)$ is frequency modulated with $k_f = 10^3$ Hz/V, what is the maximum instantaneous frequency of the modulated signal? What is the bandwidth of the modulated signal?

3.42 We wish to transmit 60 voice-band signals by SSB (upper sideband) modulation and frequency-division multiplexing (FDM). Each of the 60 signals has a spectrum as shown in Figure P-3.42. Note that the voice-band signal is band limited to 3 kHz. If each signal is frequency translated separately, we require a frequency synthesizer that produces 60 carrier frequencies to perform the frequency-division multiplexing. On the other band, if we subdivide the channels into L groups of K subchannels each, such that $LK = 60$, we may reduce the number of frequencies from the synthesizer to $L + K$.

1. Illustrate the spectrum of the SSB signals in a group of K subchannels. Assume that a 1 kHz guard band separates the signals in adjacent frequency subchannels and that the carrier frequencies are $f_{c_1} = 10$ kHz, $f_{c_2} = 14$ kHz, . . . , etc.
2. Sketch L and K such that $LK = 60$ and $L + K$ is a minimum.
3. Determine the frequencies of the carriers if the 60 FDM signals occupy the frequency band 300 kHz to 540 kHz, and each group of K signals occupies the band 10 kHz to $(10 + 4K)$ kHz.

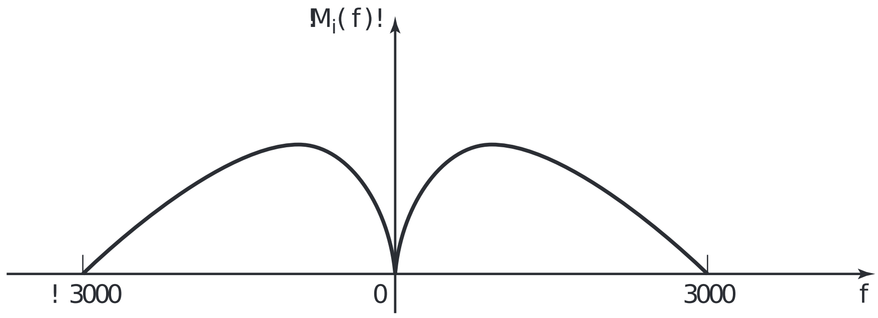


Figure P-3.42

3.43 A superheterodyne FM receiver operates in the frequency range of 88–108 MHz. The IF and local oscillator frequencies are chosen such that $f_{IF} < f_{LO}$. We require that the image frequency f_c' fall outside of the 88–108 MHz region. Determine the minimum required f_{IF} and the range of variations in f_{LO}?

Chapter 3

Problem 3.1

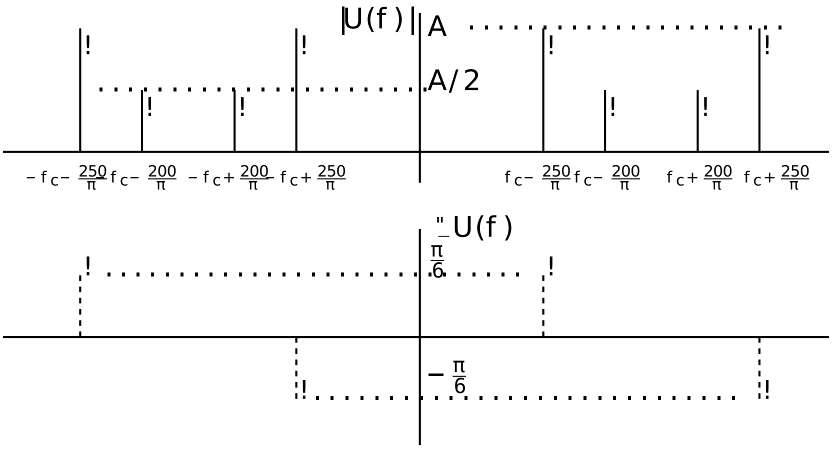
The modulated signal is

$$
\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t) \\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t + \frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t) \\
&= A\cos(2\pi(4\times 10^3 + \frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3 - \frac{200}{\pi})t) \\
&\quad +2A\sin(2\pi(4\times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3})
\end{aligned}
$$

Taking the Fourier transform of the previous relation, we obtain

$$
\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi}) + \delta(f+\frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right] \\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3) + \delta(f+4\times 10^3)] \\
&= \frac{A}{2}\left[\delta(f-4\times 10^3 - \frac{200}{\pi}) + \delta(f-4\times 10^3 + \frac{200}{\pi})\right. \\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3 + \frac{250}{\pi}) \\
&\quad +\delta(f+4\times 10^3 - \frac{200}{\pi}) + \delta(f+4\times 10^3 + \frac{200}{\pi}) \\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3 + \frac{250}{\pi})\right]
\end{aligned}
$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



To find the power content of the modulated signal we write $u^2(t)$ as

$$
\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times 10^3 + \frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times 10^3 - \frac{200}{\pi})t) \\
&\quad +4A^2\sin^2(2\pi(4\times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) + 4A^2\sin^2(2\pi(4\times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3}) \\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}
$$

Hence,

$$
P = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)dt = \frac{A^2}{2} + \frac{A^2}{2} + \frac{4A^2}{2} + \frac{4A^2}{2} = 5A^2
$$

Problem 3.2

$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t))\cos(2\pi f_c t)$$

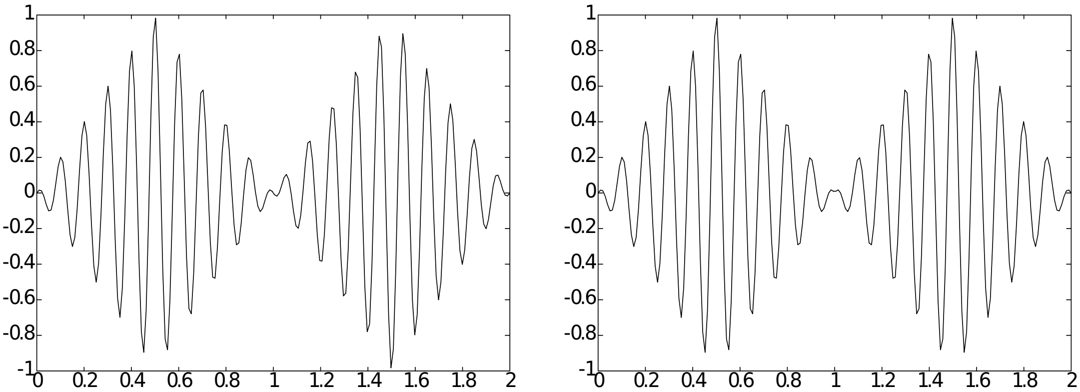
Taking the Fourier transform of both sides, we obtain

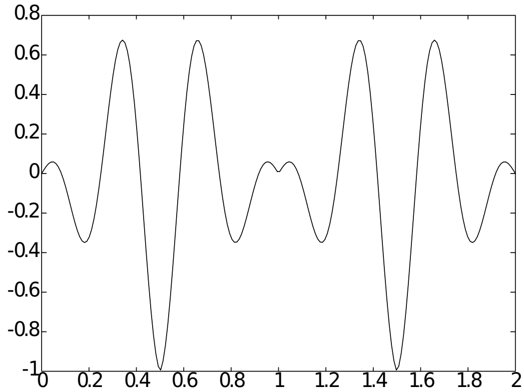
$$\begin{aligned} U(f) &= \frac{A}{2}[\Pi(f) + \Lambda(f)] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2}[\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)] \end{aligned}$$

$\Pi(f - f_c) \neq 0$ for $|f - f_c| < \frac{1}{2}$, whereas $\Lambda(f - f_c) \neq 0$ for $|f - f_c| < 1$. Hence, the bandwidth of the bandpass filter is 2.

Problem 3.3

The following figure shows the modulated signals for $A = 1$ and $f_0 = 10$. As it is observed both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ (right plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.





Problem 3.4

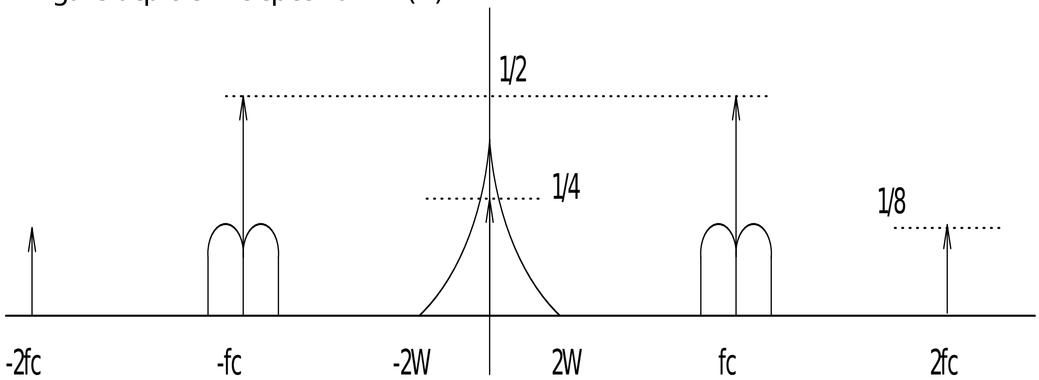
$$y(t) = x(t) + \frac{1}{2}x^2(t)$$

$$
\begin{aligned}
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}
$$

Taking the Fourier transform of the previous, we obtain

$$
\begin{aligned}
Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}(M(f - f_c) + M(f + f_c)) \\
&\quad + \frac{1}{4}\delta(f) + \frac{1}{2}(\delta(f - f_c) + \delta(f + f_c)) + \frac{1}{8}(\delta(f - 2f_c) + \delta(f + 2f_c))
\end{aligned}
$$

The next figure depicts the spectrum $Y(f)$



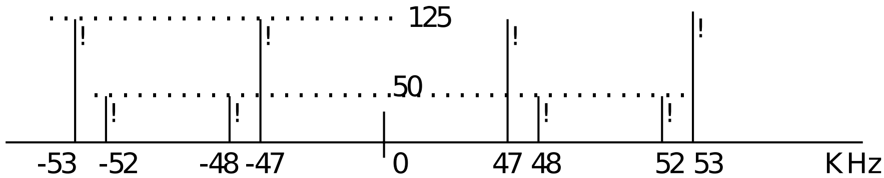
Problem 3.5

$$
\begin{aligned}
u(t) &= m(t) \cdot c(t) \\
&= 100(2\cos(2\pi 2000t) + 5\cos(2\pi 3000t))\cos(2\pi f_c t)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U(f) &= \frac{100}{2}\left[\delta(f - 2000) + \delta(f + 2000) + \frac{5}{2}(\delta(f - 3000) + \delta(f + 3000))\right] \\
&\quad \star [\delta(f - 50000) + \delta(f + 50000)] \\
&= 50\left[\delta(f - 52000) + \delta(f - 48000) + \frac{5}{2}\delta(f - 53000) + \frac{5}{2}\delta(f - 47000)\right. \\
&\quad \left. + \delta(f + 52000) + \delta(f + 48000) + \frac{5}{2}\delta(f + 53000) + \frac{5}{2}\delta(f + 47000)\right]
\end{aligned}
$$

A plot of the spectrum of the modulated signal is given in the next figure



Problem 3.6

The mixed signal $y(t)$ is given by

$$
\begin{aligned}
y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\
&= \frac{A}{2}m(t)\left[\cos(2\pi 2f_c t + \theta) + \cos(\theta)\right]
\end{aligned}
$$

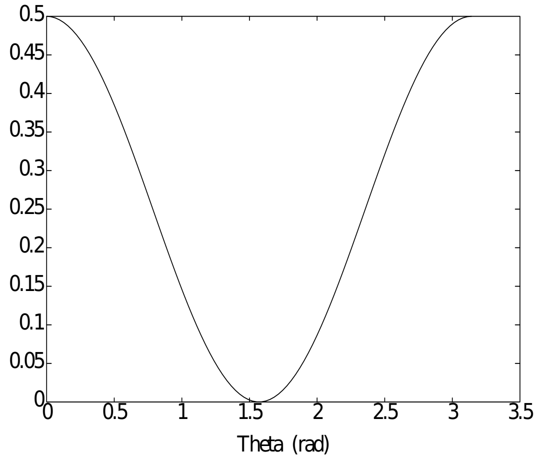
The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t) \cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{out} = P_M \frac{A^2}{4} \cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2} P_M$. Hence,

$$\frac{P_{out}}{P_U} = \frac{1}{2}\cos^2(\theta)$$

A plot of $\frac{P_{out}}{P_U}$ for $0 \leq \theta \leq \pi$ is given in the next figure.

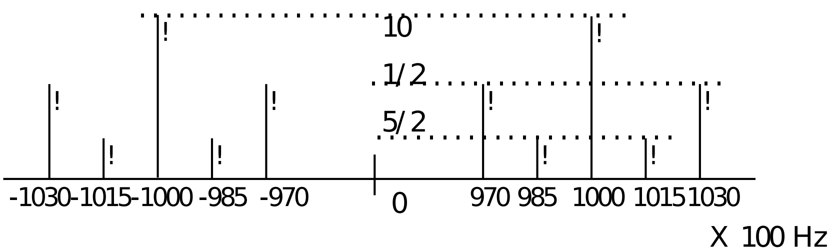


Problem 3.7

1) The spectrum of $u(t)$ is

$$\begin{aligned}
U(f) &= \frac{20}{2}[\delta(f - f_c) + \delta(f + f_c)] \\
&+ \frac{2}{4}[\delta(f - f_c - 1500) + \delta(f - f_c + 1500) \\
&+ \delta(f + f_c - 1500) + \delta(f + f_c + 1500)] \\
&+ \frac{10}{4}[\delta(f - f_c - 3000) + \delta(f - f_c + 3000) \\
&+ \delta(f + f_c - 3000) + \delta(f + f_c + 3000)]
\end{aligned}$$

The next figure depicts the spectrum of $u(t)$.



2) The square of the modulated signal is

$$\begin{aligned}
u^2(t) &= 400\cos^2(2\pi f_c t) + \cos^2(2\pi(f_c - 1500)t) + \cos^2(2\pi(f_c + 1500)t) \\
&+ 25\cos^2(2\pi(f_c - 3000)t) + 25\cos^2(2\pi(f_c + 3000)t) \\
&+ \text{terms that are multiples of cosines}
\end{aligned}$$

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \rightarrow \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t) \\ &= \cos^2(2\pi 1500t) + \frac{1}{10}\cos(2\pi 1500t) - \frac{1}{2} \end{aligned}$$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. Hence, the modulation index is

$$\alpha = -\frac{201}{400}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &= 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226}$$

Problem 3.8

1)

$$\begin{aligned} u(t) &= m(t)c(t) \\ &= 100(\cos(2\pi 1000t) + 2\cos(2\pi 2000t))\cos(2\pi f_c t) \\ &= 100\cos(2\pi 1000t)\cos(2\pi f_c t) + 200\cos(2\pi 2000t)\cos(2\pi f_c t) \\ &= \frac{100}{2}[\cos(2\pi(f_c + 1000)t) + \cos(2\pi(f_c - 1000)t)] \\ &\quad \frac{200}{2}[\cos(2\pi(f_c + 2000)t) + \cos(2\pi(f_c - 2000)t)] \end{aligned}$$

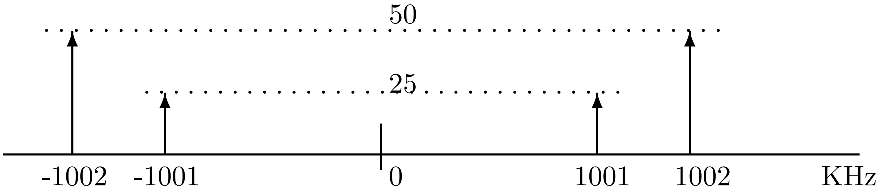
Thus, the upper sideband (USB) signal is

$$u_u(t) = 50\cos(2\pi(f_c + 1000)t) + 100\cos(2\pi(f_c + 2000)t)$$

2) Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U_u(f) &= 25\left(\delta(f-(f_c+1000))+\delta(f+(f_c+1000))\right) \\ &\quad +50\left(\delta(f-(f_c+2000))+\delta(f+(f_c+2000))\right) \end{aligned}$$

A plot of $U_u(f)$ is given in the next figure.



Problem 3.9
If we let

$$x(t) = -\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)$$

then using the results of Problem 2.23, we obtain

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t-nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where

$$\begin{aligned} X(\frac{n}{T_p}) &= \mathcal{F}\left[-\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)\right]\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(f\frac{T_p}{2})\left(e^{-j2\pi f\frac{T_p}{4}} - e^{j2\pi f\frac{T_p}{4}}\right)\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2}) \end{aligned}$$

Hence, the Fourier transform of $v(t)$ is

$$V(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2})M(f-\frac{n}{T_p})$$

The bandpass filter will cut-off all the frequencies except the ones centered at $\frac{1}{T_p}$, that is for $n=\pm 1$. Thus, the output spectrum is

$$\begin{aligned} U(f) &= \mathrm{sinc}(\frac{1}{2})(-j)M(f-\frac{1}{T_p}) + \mathrm{sinc}(\frac{1}{2})jM(f+\frac{1}{T_p}) \\ &= -\frac{2}{\pi}jM(f-\frac{1}{T_p}) + \frac{2}{\pi}jM(f+\frac{1}{T_p}) \\ &= \frac{4}{\pi}M(f)\star\left[\frac{1}{2j}\delta(f-\frac{1}{T_p}) - \frac{1}{2j}\delta(f+\frac{1}{T_p})\right] \end{aligned}$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t) = \frac{4}{\pi}m(t)\sin(2\pi\frac{1}{T_p}t)$$

which has the form of a DSB-SC AM signal, with $c(t) = \frac{4}{\pi}\sin(2\pi\frac{1}{T_p}t)$ being the carrier signal.

Problem 3.10
Assume that $s(t)$ is a periodic signal with period T_p, i.e. $s(t) = \sum_n x(t - nT_p)$. Then

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t - nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where $X(\frac{n}{T_p}) = \mathcal{F}[x(t)]|_{f=\frac{n}{T_p}}$. The Fourier transform of $v(t)$ is

$$\begin{aligned} V(f) &= \frac{1}{T_p}\mathcal{F}\left[\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t}\right] \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})M(f - \frac{n}{T_p}) \end{aligned}$$

The bandpass filter will cut-off all the frequency components except the ones centered at $f_c = \pm\frac{1}{T_p}$. Hence, the spectrum at the output of the BPF is

$$U(f) = \frac{1}{T_p}X(\frac{1}{T_p})M(f - \frac{1}{T_p}) + \frac{1}{T_p}X(-\frac{1}{T_p})M(f + \frac{1}{T_p})$$

In the time domain the output of the BPF is given by

$$\begin{aligned} u(t) &= \frac{1}{T_p}X(\frac{1}{T_p})m(t)e^{j2\pi\frac{1}{T_p}t} + \frac{1}{T_p}X^*(\frac{1}{T_p})m(t)e^{-j2\pi\frac{1}{T_p}t} \\ &= \frac{1}{T_p}m(t)\left[X(\frac{1}{T_p})e^{j2\pi\frac{1}{T_p}t} + X^*(\frac{1}{T_p})e^{-j2\pi\frac{1}{T_p}t}\right] \\ &= \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))\mathrm{m(t)}\cos(2\pi\frac{1}{\mathrm{T_p}}\mathrm{t}) \end{aligned}$$

As it is observed $u(t)$ has the form a modulated DSB-SC signal. The amplitude of the modulating signal is $A_c = \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))$ and the carrier frequency $f_c = \frac{1}{T_p}$.

Problem 3.11
1) The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]$$

The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)]$$

Multiplying the output of the HPF with $A\cos(2\pi(f_c+W)t)$ results in the signal $z(t)$ with spectrum

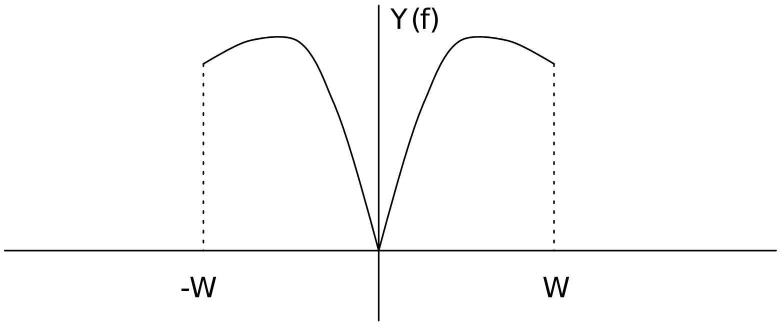
$$\begin{aligned} Z(f) &= \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)] \\ &\star\frac{A}{2}[\delta(f - (f_c + W)) + \delta(f + f_c + W)] \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{4}(M(f+f_c-f_c-W)u_{-1}(-f+f_c+W-f_c) \\
&\quad +M(f+f_c-f_c+W)u_{-1}(f+f_c+W-f_c) \\
&\quad +M(f-2f_c-W)u_{-1}(f-2f_c-W) \\
&\quad +M(f+2f_c+W)u_{-1}(-f-2f_c-W)) \\
&= \frac{A^2}{4}(M(f-W)u_{-1}(-f+W)+M(f+W)u_{-1}(f+W) \\
&\quad +M(f-2f_c-W)u_{-1}(f-2f_c-W)+M(f+2f_c+W)u_{-1}(-f-2f_c-W))
\end{aligned}
$$

The LPF will cut-off the double frequency components, leaving the spectrum

$$
Y(f)=\frac{A^2}{4}[M(f-W)u_{-1}(-f+W)+M(f+W)u_{-1}(f+W)]
$$

The next figure depicts $Y(f)$ for $M(f)$ as shown in Fig. P-5.12.



2) As it is observed from the spectrum $Y(f)$, the system shifts the positive frequency components to the negative frequency axis and the negative frequency components to the positive frequency axis. If we transmit the signal $y(t)$ through the system, then we will get a scaled version of the original spectrum $M(f)$.

Problem 3.12

The modulated signal can be written as

$$
\begin{aligned}
u(t) &= m(t)\cos(2\pi f_c t+\phi) \\
&= m(t)\cos(2\pi f_c t)\cos(\phi)-m(t)\sin(2\pi f_c t)\sin(\phi) \\
&= u_c(t)\cos(2\pi f_c t)-u_s(t)\sin(2\pi f_c t)
\end{aligned}
$$

where we identify $u_c(t)=m(t)\cos(\phi)$ as the in-phase component and $u_s(t)=m(t)\sin(\phi)$ as the quadrature component. The envelope of the bandpass signal is

$$
\begin{aligned}
V_u(t) &= \sqrt{u_c^2(t)+u_s^2(t)}=\sqrt{m^2(t)\cos^2(\phi)+m^2(t)\sin^2(\phi)} \\
&= \sqrt{m^2(t)}=|m(t)|
\end{aligned}
$$

Hence, the envelope is proportional to the absolute value of the message signal.

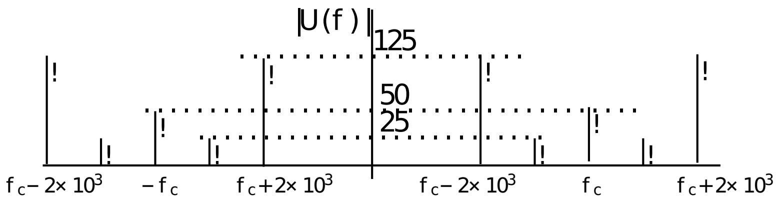
Problem 3.13

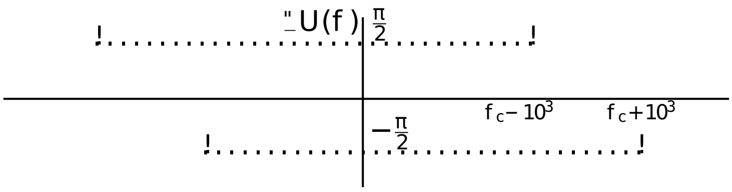
1) The modulated signal is

$$
\begin{aligned}
u(t) &= 100[1+m(t)]\cos(2\pi 8\times 10^5 t) \\
&= 100\cos(2\pi 8\times 10^5 t)+100\sin(2\pi 10^3 t)\cos(2\pi 8\times 10^5 t) \\
&\quad +500\cos(2\pi 2\times 10^3 t)\cos(2\pi 8\times 10^5 t) \\
&= 100\cos(2\pi 8\times 10^5 t)+50[\sin(2\pi(10^3+8\times 10^5)t)-\sin(2\pi(8\times 10^5-10^3)t)] \\
&\quad +250[\cos(2\pi(2\times 10^3+8\times 10^5)t)+\cos(2\pi(8\times 10^5-2\times 10^3)t)]
\end{aligned}
$$

Taking the Fourier transform of the previous expression, we obtain

$$
\begin{aligned}
U(f) &= 50[\delta(f-8\times10^5)+\delta(f+8\times10^5)] \\
&\quad +25\left(\frac{1}{j}\delta(f-8\times10^5-10^3)-\frac{1}{j}\delta(f+8\times10^5+10^3)\right) \\
&\quad -25\left(\frac{1}{j}\delta(f-8\times10^5+10^3)-\frac{1}{j}\delta(f+8\times10^5-10^3)\right) \\
&\quad +125\left(\delta(f-8\times10^5-2\times10^3)+\delta(f+8\times10^5+2\times10^3)\right) \\
&\quad +125\left(\delta(f-8\times10^5-2\times10^3)+\delta(f+8\times10^5+2\times10^3)\right) \\
&= 50[\delta(f-8\times10^5)+\delta(f+8\times10^5)] \\
&\quad +25\left(\delta(f-8\times10^5-10^3)e^{-j\frac{\pi}{2}}+\delta(f+8\times10^5+10^3)e^{j\frac{\pi}{2}}\right) \\
&\quad +25\left(\delta(f-8\times10^5+10^3)e^{j\frac{\pi}{2}}+\delta(f+8\times10^5-10^3)e^{-j\frac{\pi}{2}}\right) \\
&\quad +125\left(\delta(f-8\times10^5-2\times10^3)+\delta(f+8\times10^5+2\times10^3)\right) \\
&\quad +125\left(\delta(f-8\times10^5-2\times10^3)+\delta(f+8\times10^5+2\times10^3)\right)
\end{aligned}
$$





2) The average power in the carrier is

$$P_{carrier}=\frac{A_c^2}{2}=\frac{100^2}{2}=5000$$

The power in the sidebands is

$$P_{sidebands}=\frac{50^2}{2}+\frac{50^2}{2}+\frac{250^2}{2}+\frac{250^2}{2}=65000$$

3) The message signal can be written as

$$
\begin{aligned}
m(t) &= \sin(2\pi10^3t)+5\cos(2\pi2\times10^3t) \\
&= -10\sin(2\pi10^3t)+\sin(2\pi10^3t)+5
\end{aligned}
$$

As it is seen the minimum value of $m(t)$ is -6 and is achieved for $\sin(2\pi10^3t)=-1$ or $t=\frac{3}{4\times10^3}+\frac{1}{10^3}k$, with $k\in Z$. Hence, the modulation index is $\alpha=6$.

4) The power delivered to the load is

$$P_{load}=\frac{|u(t)|^2}{50}=\frac{100^2(1+m(t))^2\cos^2(2\pi f_ct)}{50}$$

The maximum absolute value of $1+m(t)$ is 6.025 and is achieved for $\sin(2\pi 10^3 t) = \frac{1}{20}$ or $t = \frac{\arcsin(\frac{1}{20})}{2\pi 10^3} + \frac{k}{10^3}$. Since $2\times 10^3 \ll f_c$ the peak power delivered to the load is approximately equal to

$$\max(P_{\text{load}}) = \frac{(100\times 6.025)^2}{50} = 72.6012$$

Problem 3.14

1)

$$\begin{aligned} u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\ &= 20(1+\frac{1}{2}\cos(200\pi t))\cos(2000\pi t) \end{aligned}$$

The modulating signal is $m(t) = \cos(2\pi 100t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000t)$.

2) Since $-1 \le \cos(2\pi 100t) \le 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

Problem 3.15

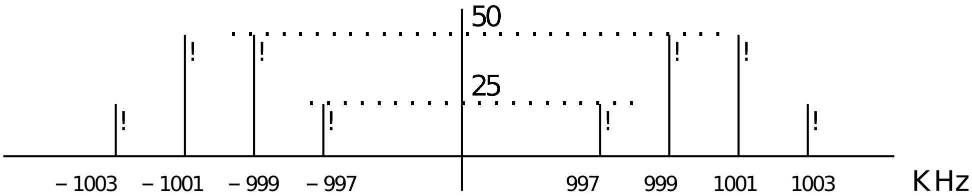
1) The modulated signal is written as

$$\begin{aligned} u(t) &= 100(2\cos(2\pi 10^3 t) + \cos(2\pi 3\times 10^3 t))\cos(2\pi f_c t) \\ &= 200\cos(2\pi 10^3 t)\cos(2\pi f_c t) + 100\cos(2\pi 3\times 10^3 t)\cos(2\pi f_c t) \\ &= 100\left(\cos(2\pi(f_c+10^3)t) + \cos(2\pi(f_c-10^3)t)\right) \\ &\quad +50\left(\cos(2\pi(f_c+3\times 10^3)t) + \cos(2\pi(f_c-3\times 10^3)t)\right) \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50\left(\delta(f-(f_c+10^3)) + \delta(f+f_c+10^3)\right. \\ &+ \left.\delta(f-(f_c-10^3)) + \delta(f+f_c-10^3)\right) \\ &+ 25\left(\delta(f-(f_c+3\times 10^3)) + \delta(f+f_c+3\times 10^3)\right. \\ &+ \left.\delta(f-(f_c-3\times 10^3)) + \delta(f+f_c-3\times 10^3)\right) \end{aligned}$$

The spectrum of the signal is depicted in the next figure



2) The average power in the frequencies f_c+1000 and f_c-1000 is

$$P_{f_c+1000} = P_{f_c-1000} = \frac{100^2}{2} = 5000$$

The average power in the frequencies f_c+3000 and f_c-3000 is

$$P_{f_c+3000} = P_{f_c-3000} = \frac{50^2}{2} = 1250$$

Problem 3.16

1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of $\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned} u_l(t) &= 100\,[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)]\cos(2\pi f_c t) \\ &+ 100\,[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)]\sin(2\pi f_c t) \\ &= 100\,[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)] \\ &+ 200\,[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)] \\ &= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t) \end{aligned}$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned} U_l(f) &= 50\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\ &+ 100j\,(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)) \\ &= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000) \end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned} |U_l(f)| &= \sqrt{50^2 + 100^2}\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\ &= 10\sqrt{125}\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \end{aligned}$$

Problem 3.17

The input to the upper LPF is

$$\begin{aligned} u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\ &= \frac{1}{2}[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)] \end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned} u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\ &= \frac{1}{2}[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)] \end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned} y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\ y_l(t) &= \sin(2\pi(f_1 - f_m)t) \end{aligned}$$

The output of the Weaver's modulator is

$$u(t) = \cos(2\pi(f_1 - f_m)t)\cos(2\pi f_2 t) - \sin(2\pi(f_1 - f_m)t)\sin(2\pi f_2 t)$$

which has the form of a SSB signal since $\sin(2\pi(f_1 - f_m)t)$ is the Hilbert transform of $\cos(2\pi(f_1 - f_m)t)$. If we write $u(t)$ as

$$u(t) = \cos(2\pi(f_1 + f_2 - f_m)t)$$

then with $f_1+f_2-f_m = f_c+f_m$ we obtain an USSB signal centered at f_c, whereas with $f_1+f_2-f_m = f_c - f_m$ we obtain the LSSB signal. In both cases the choice of f_c and f_1 uniquely determine f_2.

Problem 3.18

The signal $x(t)$ is $m(t) + \cos(2\pi f_0 t)$. The spectrum of this signal is $X(f) = M(f) + \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$ and its bandwidth equals to $W_x = f_0$. The signal $y_1(t)$ after the Square Law Device is

$$\begin{aligned}
y_1(t) &= x^2(t) = (m(t) + \cos(2\pi f_0 t))^2 \\
&= m^2(t) + \cos^2(2\pi f_0 t) + 2m(t)\cos(2\pi f_0 t) \\
&= m^2(t) + \frac{1}{2} + \frac{1}{2}\cos(2\pi 2f_0 t) + 2m(t)\cos(2\pi f_0 t)
\end{aligned}$$

The spectrum of this signal is given by

$$Y_1(f) = M(f) \star M(f) + \frac{1}{2}\delta(f) + \frac{1}{4}(\delta(f - 2f_0) + \delta(f + 2f_0)) + M(f - f_0) + M(f + f_0)$$

and its bandwidth is $W_1 = 2f_0$. The bandpass filter will cut-off the low-frequency components $M(f) \star M(f) + \frac{1}{2}\delta(f)$ and the terms with the double frequency components $\frac{1}{4}(\delta(f - 2f_0) + \delta(f + 2f_0))$. Thus the spectrum $Y_2(f)$ is given by

$$Y_2(f) = M(f - f_0) + M(f + f_0)$$

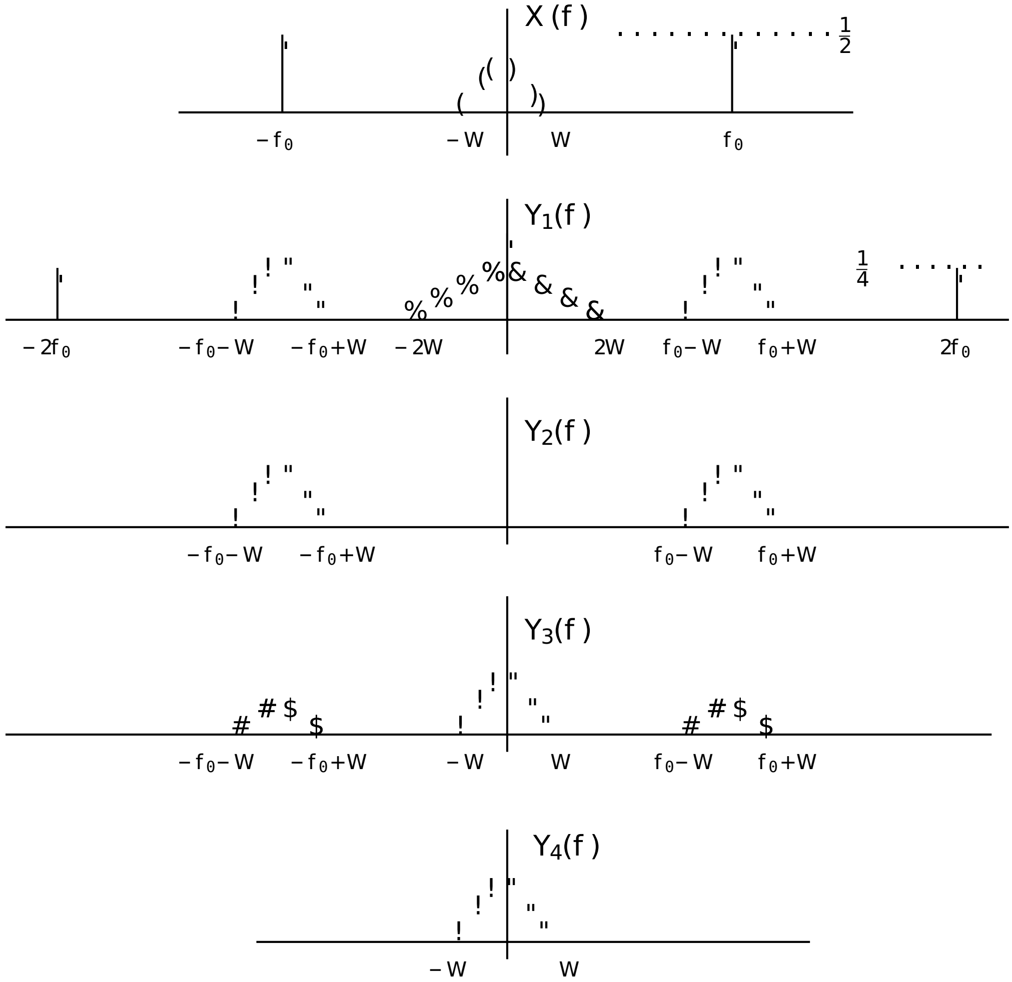
and the bandwidth of $y_2(t)$ is $W_2 = 2W$. The signal $y_3(t)$ is

$$y_3(t) = 2m(t)\cos^2(2\pi f_0 t) = m(t) + m(t)\cos(2\pi f_0 t)$$

with spectrum

$$Y_3(t) = M(f) + \frac{1}{2}(M(f - f_0) + M(f + f_0))$$

and bandwidth $W_3 = f_0 + W$. The lowpass filter will eliminate the spectral components $\frac{1}{2}(M(f - f_0) + M(f + f_0))$, so that $y_4(t) = m(t)$ with spectrum $Y_4 = M(f)$ and bandwidth $W_4 = W$. The next figure depicts the spectra of the signals $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$.



Problem 3.19

1)

$$\begin{aligned} y(t) &= ax(t) + bx^2(t) \\ &= a(m(t) + \cos(2\pi f_0 t)) + b(m(t) + \cos(2\pi f_0 t))^2 \\ &= am(t) + bm^2(t) + a\cos(2\pi f_0 t) \\ &\quad + b\cos^2(2\pi f_0 t) + 2bm(t)\cos(2\pi f_0 t) \end{aligned}$$

2) The filter should reject the low frequency components, the terms of double frequency and pass only the signal with spectrum centered at f_0. Thus the filter should be a BPF with center frequency f_0 and bandwidth W such that $f_0 - W_M > f_0 - \frac{W}{2} > 2W_M$ where W_M is the bandwidth of the message signal $m(t)$.

3) The AM output signal can be written as

$$u(t) = a(1 + \frac{2b}{a} m(t)) \cos(2\pi f_0 t)$$

Since $A_m = \max[|m(t)|]$ we conclude that the modulation index is

$$\alpha = \frac{2bA_m}{a}$$

Problem 3.20

1) When USSB is employed the bandwidth of the modulated signal is the same with the bandwidth of the message signal. Hence,

$$W_{USSB} = W = 10^4 \text{ Hz}$$

2) When DSB is used, then the bandwidth of the transmitted signal is twice the bandwidth of the message signal. Thus,

$$W_{DSB} = 2W = 2 \times 10^4 \text{ Hz}$$

3) If conventional AM is employed, then

$$W_{AM} = 2W = 2 \times 10^4 \text{ Hz}$$

4) Using Carson's rule, the effective bandwidth of the FM modulated signal is

$$B_c = (2\beta + 1)W = 2\left(\frac{k_f \max[|m(t)|]}{W} + 1\right)W = 2(k_f + W) = 140000 \text{ Hz}$$

Problem 3.21

1) The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u_{-1}(f + f_c)H(f + f_c) = 2\begin{cases} \frac{1}{W}f + \frac{1}{2} & |f| \leq \frac{W}{2} \\ 1 & \frac{W}{2} < f \leq W \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
h_l(t) &= \mathcal{F}^{-1}[H_l(f)] = \int_{-\frac{W}{2}}^{W} H_l(f) e^{j2\pi f t} df \\
&= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} \left(\frac{1}{W}f + \frac{1}{2}\right) e^{j2\pi f t} df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi f t} df \\
&= \frac{2}{W}\left(\frac{1}{j2\pi t} f e^{j2\pi f t} + \frac{1}{4\pi^2 t^2} e^{j2\pi f t}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t} e^{j2\pi f t}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t} e^{j2\pi f t}\Bigg|_{\frac{W}{2}}^{W} \\
&= \frac{1}{j\pi t} e^{j2\pi W t} + \frac{j}{\pi^2 t^2 W} \sin(\pi W t) \\
&= \frac{j}{\pi t}\left[\text{sinc}(Wt) - e^{j2\pi W t}\right]
\end{aligned}$$

2) An expression for the modulated signal is obtained as follows

$$\begin{aligned}
u(t) &= \text{Re}[(m(t) \star h_l(t)) e^{j2\pi f_c t}] \\
&= \text{Re}\left[\left(m(t) \star \frac{j}{\pi t}(\text{sinc}(Wt) - e^{j2\pi W t})\right) e^{j2\pi f_c t}\right] \\
&= \text{Re}\left[\left(m(t) \star \left(\frac{j}{\pi t}\text{sinc}(Wt)\right)\right) e^{j2\pi f_c t} + \left(m(t) \star \frac{1}{j\pi t} e^{j2\pi W t}\right) e^{j2\pi f_c t}\right]
\end{aligned}$$

Note that

$$\mathcal{F}\left[m(t) \star \frac{1}{j\pi t} e^{j2\pi W t}\right] = -M(f)\text{sgn}(f - W) = M(f)$$

since $\text{sgn}(f - W) = -1$ for $f < W$. Thus,

$$\begin{aligned} u(t) &= \text{Re}\left[(m(t) \star (\frac{j}{\pi t}\text{sinc}(Wt)))e^{j2\pi f_c t} + m(t)e^{j2\pi f_c t}\right] \\ &= m(t)\cos(2\pi f_c t) - m(t) \star (\frac{1}{\pi t}\text{sinc}(Wt))\sin(2\pi f_c t) \end{aligned}$$

Problem 3.22

a) A DSB modulated signal is written as

$$\begin{aligned} u(t) &= Am(t)\cos(2\pi f_0 t + \phi) \\ &= Am(t)\cos(\phi)\cos(2\pi f_0 t) - Am(t)\sin(\phi)\sin(2\pi f_0 t) \end{aligned}$$

Hence,

$$\begin{aligned} x_c(t) &= Am(t)\cos(\phi) \\ x_s(t) &= Am(t)\sin(\phi) \\ V(t) &= \sqrt{A^2 m^2(t)(\cos^2(\phi) + \sin^2(\phi))} = |Am(t)| \\ \Theta(t) &= \arctan\left(\frac{Am(t)\cos(\phi)}{Am(t)\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi \end{aligned}$$

b) A SSB signal has the form

$$u_{SSB}(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Thus, for the USSB signal (minus sign)

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= A\hat{m}(t) \\ V(t) &= \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)} \\ \Theta(t) &= \arctan\left(\frac{\hat{m}(t)}{m(t)}\right) \end{aligned}$$

For the LSSB signal (plus sign)

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= -A\hat{m}(t) \\ V(t) &= \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)} \\ \Theta(t) &= \arctan\left(-\frac{\hat{m}(t)}{m(t)}\right) \end{aligned}$$

c) If conventional AM is employed, then

$$\begin{aligned} u(t) &= A(1 + m(t))\cos(2\pi f_0 t + \phi) \\ &= A(1 + m(t))\cos(\phi)\cos(2\pi f_0 t) - A(1 + m(t))\sin(\phi)\sin(2\pi f_0 t) \end{aligned}$$

Hence,

$$\begin{aligned} x_c(t) &= A(1 + m(t))\cos(\phi) \\ x_s(t) &= A(1 + m(t))\sin(\phi) \\ V(t) &= \sqrt{A^2(1 + m(t))^2(\cos^2(\phi) + \sin^2(\phi))} = A|(1 + m(t))| \\ \Theta(t) &= \arctan\left(\frac{A(1 + m(t))\cos(\phi)}{A(1 + m(t))\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi \end{aligned}$$

d) A PM modulated signal has the form

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + k_p m(t)) \\ &= \mathrm{Re}\left[Ae^{j2\pi f_c t}e^{jk_p m(t)}\right] \end{aligned}$$

From the latter expression we identify the lowpass equivalent signal as

$$u_l(t) = Ae^{jk_p m(t)} = x_c(t) + jx_s(t)$$

Thus,

$$\begin{aligned} x_c(t) &= A\cos(k_p m(t)) \\ x_s(t) &= A\sin(k_p m(t)) \\ V(t) &= \sqrt{A^2(\cos^2(k_p m(t)) + \sin^2(k_p m(t)))} = A \\ \Theta(t) &= \arctan\left(\frac{A\cos(k_p m(t))}{A\sin(k_p m(t))}\right) = k_p m(t) \end{aligned}$$

e) To get the expressions for an FM signal we replace $k_p m(t)$ by $2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$ in the previous relations. Hence,

$$\begin{aligned} x_c(t) &= A\cos(2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ x_s(t) &= A\sin(2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ V(t) &= A \\ \Theta(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau \end{aligned}$$

Problem 3.23

1) If SSB is employed, the transmitted signal is

$$u(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Provided that the spectrum of $m(t)$ does not contain any impulses at the origin $P_M = P_{\hat{M}} = \frac{1}{2}$ and

$$P_{SSB} = \frac{A^2 P_M}{2} + \frac{A^2 P_{\hat{M}}}{2} = A^2 P_M = 400\frac{1}{2} = 200$$

The bandwidth of the modulated signal $u(t)$ is the same with that of the message signal. Hence,

$$W_{SSB} = 10000 \text{ Hz}$$

2) In the case of DSB-SC modulation $u(t) = Am(t)\cos(2\pi f_0 t)$. The power content of the modulated signal is

$$P_{DSB} = \frac{A^2 P_M}{2} = 200\frac{1}{2} = 100$$

and the bandwidth $W_{DSB} = 2W = 20000$ Hz.

3) If conventional AM is employed with modulation index $\alpha = 0.6$, the transmitted signal is

$$u(t) = A[1 + \alpha m(t)]\cos(2\pi f_0 t)$$

The power content is

$$P_{AM} = \frac{A^2}{2} + \frac{A^2\alpha^2 P_M}{2} = 200 + 200 \cdot 0.6^2 \cdot 0.5 = 236$$

The bandwidth of the signal is $W_{AM} = 2W = 20000$ Hz.

4) If the modulation is FM with $k_f = 50000$, then

$$P_{FM} = \frac{A^2}{2} = 200$$

and the effective bandwidth is approximated by Carson's rule as

$$B_c = 2(\beta + 1)W = 2\left(\frac{50000}{W} + 1\right)W = 120000 \text{ Hz}$$

Problem 3.24

1) Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_{-\infty}^{t} \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is

$$\Delta f_{max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule, the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6 + 1)200 = 2800 \text{ Hz}$$

Problem 3.25

1) The maximum phase deviation of the PM signal is

$$\Delta\phi_{max} = k_p \max[|m(t)|] = k_p$$

The phase of the FM modulated signal is

$$\begin{aligned} \phi(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = 2\pi k_f \int_{0}^{t} m(\tau)d\tau \\ &= \begin{cases} 2\pi k_f \int_0^t \tau d\tau = \pi k_f t^2 & 0 \leq t < 1 \\ \pi k_f + 2\pi k_f \int_1^t d\tau = \pi k_f + 2\pi k_f(t-1) & 1 \leq t < 2 \\ \pi k_f + 2\pi k_f - 2\pi k_f \int_2^t d\tau = 3\pi k_f - 2\pi k_f(t-2) & 2 \leq t < 3 \\ \pi k_f & 3 \leq t \end{cases} \end{aligned}$$

The maximum value of $\phi(t)$ is achieved for $t = 2$ and is equal to $3\pi k_f$. Thus, the desired relation between k_p and k_f is

$$k_p = 3\pi k_f$$

2) The instantaneous frequency for the PM modulated signal is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}k_p\frac{d}{dt}m(t)$$

For the m(t) given in Fig. P-3.25, the maximum value of $\frac{d}{dt}m(t)$ is achieved for t in [0, 1] and it is equal to one. Hence,

$$\max(f_i(t)) = f_c + \frac{1}{2\pi}$$

For the FM signal $f_i(t) = f_c + k_f m(t)$. Thus, the maximum instantaneous frequency is

$$\max(f_i(t)) = f_c + k_f = f_c + 1$$

Problem 3.26

1) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P = \frac{A_c^2}{2} \Longrightarrow P = \frac{100^2}{2} = 5000$$

The same result is obtained if we use the expansion

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

along with the identity

$$J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1$$

2) The maximum phase deviation is

$$\Delta\phi_{max} = \max|4\sin(2000\pi t)| = 4$$

3) The instantaneous frequency is

$$\begin{aligned} f_i &= f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \\ &= f_c + \frac{4}{2\pi}\cos(2000\pi t)2000\pi = f_c + 4000\cos(2000\pi t) \end{aligned}$$

Hence, the maximum frequency deviation is

$$\Delta f_{max} = \max|f_i - f_c| = 4000$$

4) The angle modulated signal can be interpreted both as a PM and an FM signal. It is a PM signal with phase deviation constant $k_p = 4$ and message signal $m(t) = \sin(2000\pi t)$ and it is an FM signal with frequency deviation constant $k_f = 4000$ and message signal $m(t) = \cos(2000\pi t)$.

Problem 3.27

The modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

The power in the frequency component $f = f_c + kf_m$ is $P_k = \frac{A_c^2}{2}J_n^2(\beta)$. Hence, the power in the carrier is $P_{carrier} = \frac{A_c^2}{2}J_0^2(\beta)$ and in order to be zero the modulation index β should be one of the roots of $J_0(x)$. The smallest root of $J_0(x)$ is found from tables to be equal 2.404. Thus,

$$\beta_{min} = 2.404$$

Problem 3.28

1) If the output of the narrowband FM modulator is,

$$u(t) = A\cos(2\pi f_0 t + \phi(t))$$

then the output of the upper frequency multiplier ($\times n_1$) is

$$u_1(t) = A\cos(2\pi n_1 f_0 t + n_1\phi(t))$$

After mixing with the output of the second frequency multiplier $u_2(t) = A\cos(2\pi n_2 f_0 t)$ we obtain the signal

$$\begin{aligned} y(t) &= A^2\cos(2\pi n_1 f_0 t + n_1\phi(t))\cos(2\pi n_2 f_0 t) \\ &= \frac{A^2}{2}\left(\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t)) + \cos(2\pi(n_1-n_2)f_0 + n_1\phi(t))\right) \end{aligned}$$

The bandwidth of the signal is $W = 15$ KHz, so the maximum frequency deviation is $\Delta f = \beta_f W = 0.1 \times 15 = 1.5$ KHz. In order to achieve a frequency deviation of $f = 75$ KHz at the output of the wideband modulator, the frequency multiplier n_1 should be equal to

$$n_1 = \frac{f}{\Delta f} = \frac{75}{1.5} = 50$$

Using an up-converter the frequency modulated signal is given by

$$y(t) = \frac{A^2}{2}\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t))$$

Since the carrier frequency $f_c = (n_1+n_2)f_0$ is 104 MHz, n_2 should be such that

$$(n_1+n_2)100 = 104 \times 10^3 \Longrightarrow n_1 + n_2 = 1040 \text{ or } n_2 = 990$$

2) The maximum allowable drift (d_f) of the 100 kHz oscillator should be such that

$$(n_1+n_2)d_f = 2 \Longrightarrow d_f = \frac{2}{1040} = .0019 \text{ Hz}$$

Problem 3.29

The modulated PM signal is given by

$$\begin{aligned} u(t) &= A_c\cos(2\pi f_c t + k_p m(t)) = A_c \mathrm{Re}\left[e^{j2\pi f_c t}e^{jk_p m(t)}\right] \\ &= A_c \mathrm{Re}\left[e^{j2\pi f_c t}e^{jm(t)}\right] \end{aligned}$$

The signal $e^{jm(t)}$ is periodic with period $T_m = \frac{1}{f_m}$ and Fourier series expansion

$$
\begin{aligned}
c_n &= \frac{1}{T_m}\int_0^{T_m} e^{jm(t)} e^{-j2\pi n f_m t} dt \\
&= \frac{1}{T_m}\int_0^{\frac{T_m}{2}} e^{j} e^{-j2\pi n f_m t} dt + \frac{1}{T_m}\int_{\frac{T_m}{2}}^{T_m} e^{-j} e^{-j2\pi n f_m t} dt \\
&= -\frac{e^{j}}{T_m j 2\pi n f_m} e^{-j2\pi n f_m t}\Big|_0^{\frac{T_m}{2}} - \frac{e^{-j}}{T_m j 2\pi n f_m} e^{-j2\pi n f_m t}\Big|_{\frac{T_m}{2}}^{T_m} \\
&= \frac{(-1)^n - 1}{2\pi n} j(e^{j} - e^{-j}) = \begin{cases} 0 & n = 2l \\ \frac{2}{\pi(2l+1)}\sin(1) & n = 2l+1 \end{cases}
\end{aligned}
$$

Hence,

$$
e^{jm(t)} = \sum_{l=-\infty}^{\infty} \frac{2}{\pi(2l+1)} \sin(1) e^{j2\pi l f_m t}
$$

and

$$
\begin{aligned}
u(t) &= A_c \mathrm{Re}\left[e^{j2\pi f_c t} e^{jm(t)}\right] = A_c \mathrm{Re}\left[e^{j2\pi f_c t} \sum_{l=-\infty}^{\infty} \frac{2}{\pi(2l+1)} \sin(1) e^{j2\pi l f_m t}\right] \\
&= A_c \sum_{l=-\infty}^{\infty} \left|\frac{2\sin(1)}{\pi(2l+1)}\right| \cos(2\pi(f_c + l f_m)t + \phi_l)
\end{aligned}
$$

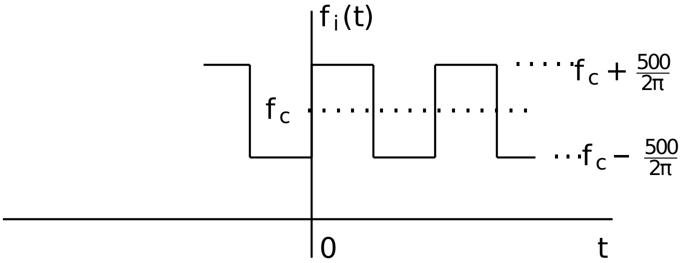
where $\phi_l = 0$ for $l \geq 0$ and $\phi_l = \pi$ for negative values of l.

Problem 3.30

1) The instantaneous frequency is given by

$$
f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi} 100 m(t)
$$

A plot of $f_i(t)$ is given in the next figure



2) The peak frequency deviation is given by

$$
\Delta f_{max} = k_f \max[|m(t)|] = \frac{100}{2\pi} 5 = \frac{250}{\pi}
$$

Problem 3.31

1) The modulation index is

$$
\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{max}}{f_m} = \frac{20 \times 10^3}{10^4} = 2
$$

The modulated signal u(t) has the form

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t + \phi_n)$$
$$= \sum_{n=-\infty}^{\infty} 100 J_n(2)\cos(2\pi(10^8 + n10^4)t + \phi_n)$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$P_{f_c+kf_m} = \frac{100^2 J_k^2(2)}{2}$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	$10^8 + 10^4$	57.67	1663.1
2	.3528	$10^8 + 2\times 10^4$	35.28	622.46
3	.1289	$10^8 + 3\times 10^4$	12.89	83.13
4	.0340	$10^8 + 4\times 10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2\times 10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2\times 10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2\times 10^4$, $10^8 - 2\times 10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$B_c = 2(\beta + 1)f_m = 2(2 + 1)10^4 = 6\times 10^4 \text{ Hz}$$

Problem 3.32

1)

$$\beta_p = k_p \max[|m(t)|] = 1.5\times 2 = 3$$
$$\beta_f = \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000\times 2}{1000} = 6$$

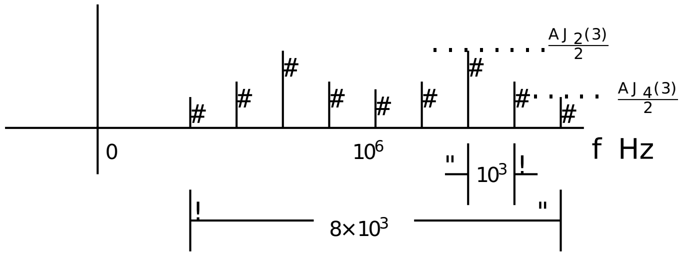
2) Using Carson's rule we obtain

$$B_{PM} = 2(\beta_p + 1)f_m = 8\times 1000 = 8000$$
$$B_{FM} = 2(\beta_f + 1)f_m = 14\times 1000 = 14000$$

3) The PM modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} AJ_n(\beta_p)\cos(2\pi(10^6 + n10^3)t)$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 4\times 10^3, 10^6 + 4\times 10^3]$. Note that $J_0(3) = -.2601$, $J_1(3) = 0.3391$, $J_2(3) = 0.4861$, $J_3(3) = 0.3091$ and $J_4(3) = 0.1320$.

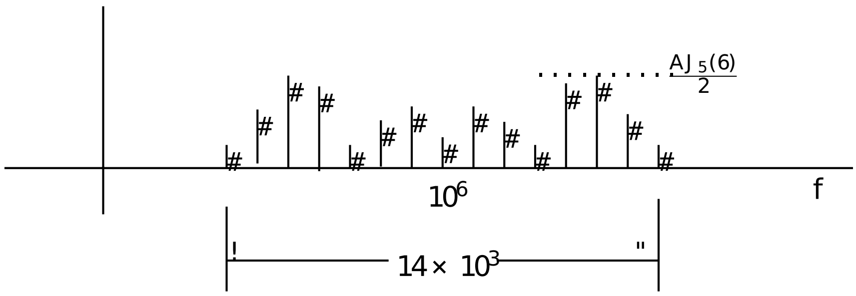


In the case of the FM modulated signal

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + \beta_f \sin(2000\pi t)) \\ &= \sum_{n=-\infty}^{\infty} AJ_n(6)\cos(2\pi(10^6 + n10^3)t + \phi_n) \end{aligned}$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 7\times 10^3, 10^6 - 7\times 10^3]$. The values of $J_n(6)$ for $n = 0, \ldots, 7$ are given in the following table.

n	0	1	2	3	4	5	6	7
$J_n(6)$	.1506	-.2767	-.2429	.1148	.3578	.3621	.2458	.1296



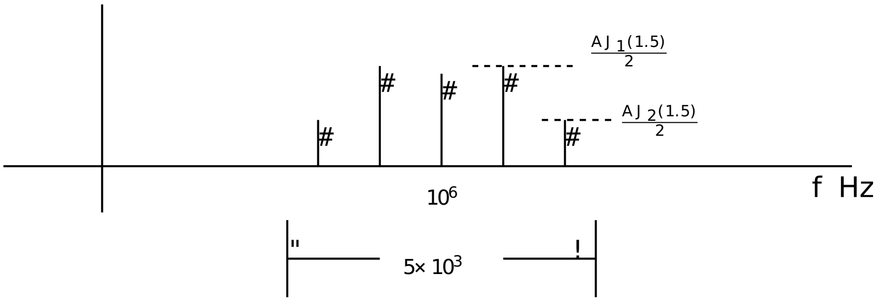
4) If the amplitude of $m(t)$ is decreased by a factor of two, then $m(t) = \cos(2\pi 10^3 t)$ and

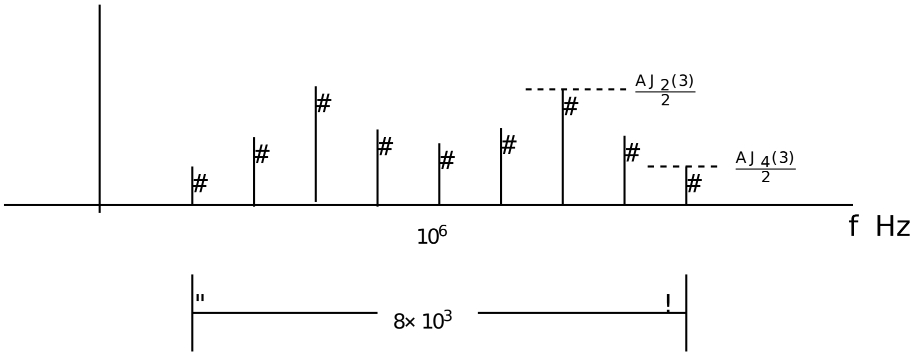
$$\begin{aligned} \beta_p &= k_p \max[|m(t)|] = 1.5 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000}{1000} = 3 \end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned} B_{PM} &= 2(\beta_p + 1)f_m = 5\times 1000 = 5000 \\ B_{FM} &= 2(\beta_f + 1)f_m = 8\times 1000 = 8000 \end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that $J_0(1.5) = .5118$, $J_1(1.5) = .5579$ and $J_2(1.5) = .2321$.





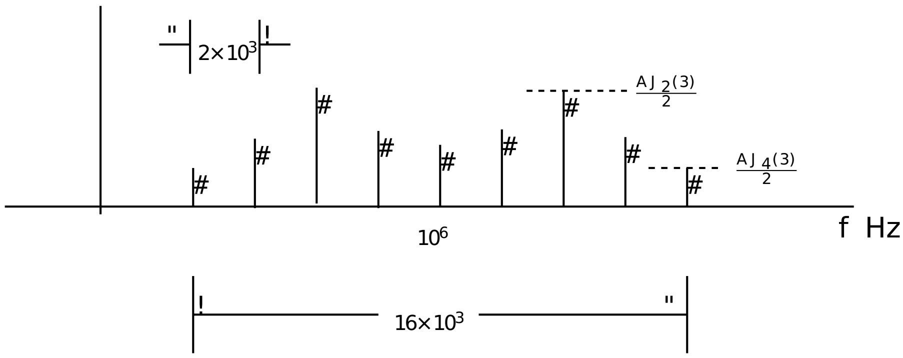
5) If the frequency of m(t) is increased by a factor of two, then $m(t) = 2\cos(2\pi 2\times 10^3 t)$ and

$$\beta_p = k_p \max[|m(t)|] = 1.5\times 2 = 3$$
$$\beta_f = \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000\times 2}{2000} = 3$$

The bandwidth is determined using Carson's rule as

$$B_{PM} = 2(\beta_p + 1)f_m = 8\times 2000 = 16000$$
$$B_{FM} = 2(\beta_f + 1)f_m = 8\times 2000 = 16000$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that doubling the frequency has no effect on the number of harmonics in the bandwidth of the PM signal, whereas it decreases the number of harmonics in the bandwidth of the FM signal from 14 to 8.



Problem 3.33

1) The PM modulated signal is

$$
\begin{aligned}
u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000 t)) \\
&= \sum_{n=-\infty}^{\infty} 100 J_n(\frac{\pi}{2})\cos(2\pi(10^8 + n10^3)t)
\end{aligned}
$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

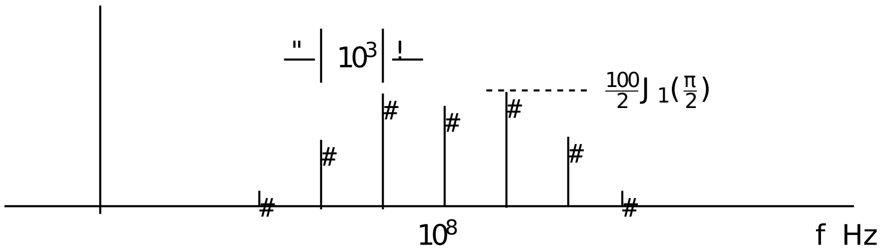
The total power of the modulated signal is $P_{tot} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$
\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2(\frac{\pi}{2}) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2(\frac{\pi}{2}) \geq 0.99
$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$
B_{\mathrm{eff}} = 4 \times 10^3 = 4000
$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.



2) Using Carson's rule, the approximate bandwidth of the PM signal is

$$
B_{PM} = 2(\beta_p + 1)f_m = 2(\frac{\pi}{2} + 1)1000 = 5141.6
$$

As it is observed, Carson's rule overestimates the effective bandwidth allowing in this way some margin for the missing harmonics.

Problem 3.34

1) Assuming that u(t) is an FM signal it can be written as

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} \alpha \cos(2\pi f_m \tau) d\tau) \\ &= 100\cos(2\pi f_c t + \frac{k_f \alpha}{f_m} \sin(2\pi f_m t)) \end{aligned}$$

Thus, the modulation index is $\beta_f = \frac{k_f \alpha}{f_m} = 4$ and the bandwidth of the transmitted signal

$$B_{FM} = 2(\beta_f + 1) f_m = 10 \text{ KHz}$$

2) If we double the frequency, then

$$u(t) = 100\cos(2\pi f_c t + 4\sin(2\pi 2 f_m t))$$

Using the same argument as before we find that $\beta_f = 4$ and

$$B_{FM} = 2(\beta_f + 1) 2 f_m = 20 \text{ KHz}$$

3) If the signal u(t) is PM modulated, then

$$\beta_p = \Delta\phi_{max} = \max[4\sin(2\pi f_m t)] = 4$$

The bandwidth of the modulated signal is

$$B_{PM} = 2(\beta_p + 1) f_m = 10 \text{ KHz}$$

4) If f_m is doubled, then $\beta_p = \Delta\phi_{max}$ remains unchanged whereas

$$B_{PM} = 2(\beta_p + 1) 2 f_m = 20 \text{ KHz}$$

Problem 3.35

1) If the signal $m(t) = m_1(t) + m_2(t)$ DSB modulates the carrier $A_c \cos(2\pi f_c t)$ the result is the signal

$$\begin{aligned} u(t) &= A_c m(t) \cos(2\pi f_c t) \\ &= A_c (m_1(t) + m_2(t)) \cos(2\pi f_c t) \\ &= A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \cos(2\pi f_c t) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where $u_1(t)$ and $u_2(t)$ are the DSB modulated signals corresponding to the message signals $m_1(t)$ and $m_2(t)$. Hence, AM modulation satisfies the superposition principle.

2) If m(t) frequency modulates a carrier $A_c \cos(2\pi f_c t)$ the result is

$$\begin{aligned} u(t) &= A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} (m_1(\tau) + m_2(\tau)) d\tau) \\ &\neq A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_1(\tau) d\tau) \\ &\quad + A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_2(\tau) d\tau) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where the inequality follows from the nonlinearity of the cosine function. Hence, angle modulation is not a linear modulation method.

Problem 3.36

The transfer function of the FM discriminator is

$$H(s) = \frac{R}{R + Ls + \frac{1}{Cs}} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Thus,

$$|H(f)|^2 = \frac{4\pi^2 \left(\frac{R}{L}\right)^2 f^2}{\left(\frac{1}{LC} - 4\pi^2 f^2\right)^2 + 4\pi^2 (\frac{R}{L})^2 f^2}$$

As it is observed $|H(f)|^2 \leq 1$ with equality if

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Since this filter is to be used as a slope detector, we require that the frequency content of the signal, which is $[80 - 6, 80 + 6]$ MHz, to fall inside the region over which $|H(f)|$ is almost linear. Such a region can be considered the interval $[f_{10}, f_{90}]$, where f_{10} is the frequency such that $|H(f_{10})| = 10\% \max[|H(f)|]$ and f_{90} is the frequency such that $|H(f_{10})| = 90\% \max[|H(f)|]$.

With $\max[|H(f)| = 1$, $f_{10} = 74 \times 10^6$ and $f_{90} = 86 \times 10^6$, we obtain the system of equations

$$\begin{aligned}
4\pi^2 f_{10}^2 + \frac{50 \times 10^3}{L} 2\pi f_{10} [1 - 0.1^2]^{\frac{1}{2}} - \frac{1}{LC} &= 0 \\
4\pi^2 f_{90}^2 + \frac{50 \times 10^3}{L} 2\pi f_{90} [1 - 0.9^2]^{\frac{1}{2}} - \frac{1}{LC} &= 0
\end{aligned}$$

Solving this system, we obtain

$$L = 14.98 \text{ mH} \quad C = 0.018013 \text{ pF}$$

Problem 3.37

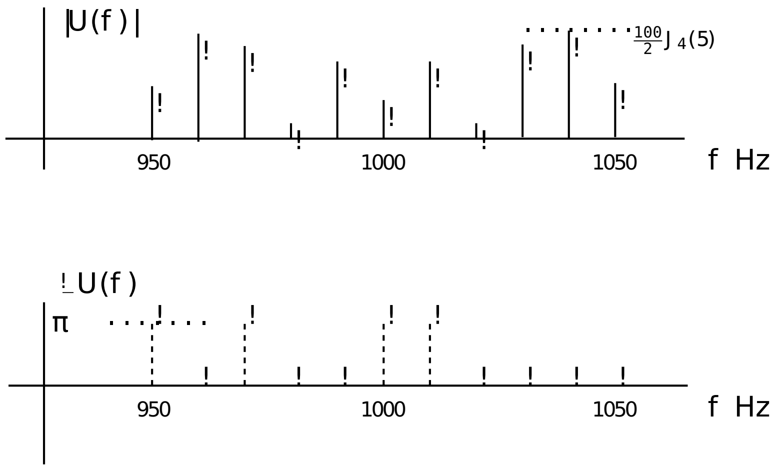
The case of $\phi(t) = \beta\cos(2\pi f_m t)$ has been treated in the text (see Section 3.3.2). the modulated signal is

$$\begin{aligned}
u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi(f_c + nf_m)) \\
&= \sum_{n=-\infty}^{\infty} 100 J_n(5) \cos(2\pi(10^3 + n10))
\end{aligned}$$

The following table shows the values of $J_n(5)$ for $n = 0, \ldots, 5$.

n	0	1	2	3	4	5
$J_n(5)$	-.178	-.328	.047	.365	.391	.261

In the next figure we plot the magnitude and the phase spectrum for frequencies in the range [950, 1050] Hz. Note that $J_{-n}(\beta) = J_n(\beta)$ if n is even and $J_{-n}(\beta) = -J_n(\beta)$ if n is odd.



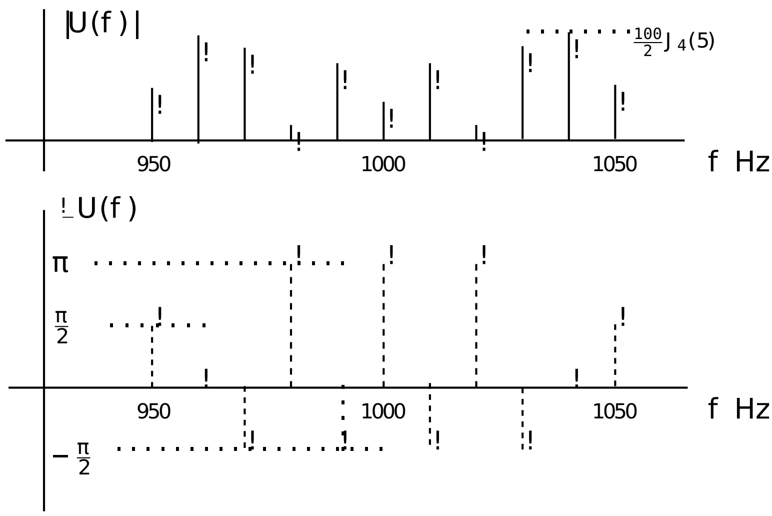
The Fourier Series expansion of $e^{j\beta\sin(2\pi f_m t)}$ is

$$\begin{aligned}
c_n &= f_m \int_{\frac{1}{4f_m}}^{\frac{5}{4f_m}} e^{j\beta\sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{j\beta\cos u - jnu} e^{j\frac{n\pi}{2}} du \\
&= e^{j\frac{n\pi}{2}} J_n(\beta)
\end{aligned}$$

Hence,

$$\begin{aligned}
u(t) &= A_c \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_c t} e^{j2\pi n f_m t}\right] \\
&= A_c \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} e^{j2\pi(f_c + n f_m)t + \frac{n\pi}{2}}\right]
\end{aligned}$$

The magnitude and the phase spectra of $u(t)$ for $\beta = 5$ and frequencies in the interval [950, 1000] Hz are shown in the next figure. Note that the phase spectrum has been plotted modulo 2π in the interval $(-\pi, \pi]$.



Problem 3.38

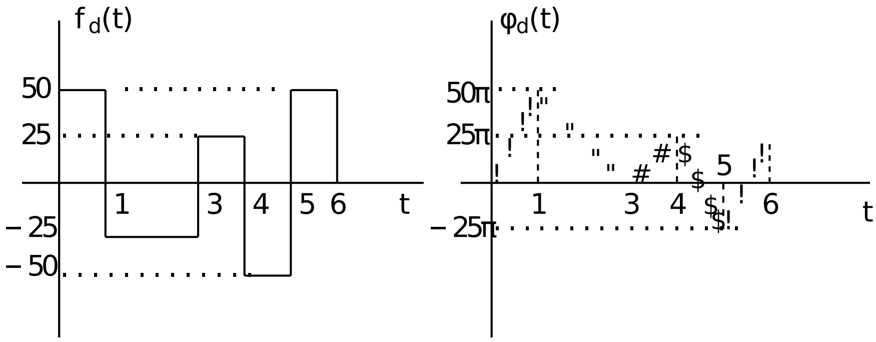
The frequency deviation is given by

$$f_d(t) = f_i(t) - f_c = k_f m(t)$$

whereas the phase deviation is obtained from

$$\phi_d(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$$

In the next figure we plot the frequency and the phase deviation when m(t) is as in Fig. P-3.38 with $k_f = 25$.



Problem 3.39
Using Carson's rule we obtain

$$B_c = 2(\beta+1)W = 2(\frac{k_f \max[|m(t)|]}{W}+1)W = \begin{cases} 20020 & k_f = 10 \\ 20200 & k_f = 100 \\ 22000 & k_f = 1000 \end{cases}$$

Problem 3.40
The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{10 \times 10}{8} = 12.5$$

The output of the FM modulator can be written as

$$\begin{aligned} u(t) &= 10\cos(2\pi 2000t + 2\pi k_f \int_{-\infty}^{t} 10\cos(2\pi 8\tau)d\tau) \\ &= \sum_{n=-\infty}^{\infty} 10 J_n(12.5)\cos(2\pi(2000+n8)t + \phi_n) \end{aligned}$$

At the output of the BPF only the signal components with frequencies in the interval $[2000-32, 2000+32]$ will be present. These components are the terms of u(t) for which $n = -4, \ldots, 4$. The power of the output signal is then

$$\frac{10^2}{2}J_0^2(12.5) + 2\sum_{n=1}^{4}\frac{10^2}{2}J_n^2(12.5) = 50 \times 0.2630 = 13.15$$

Since the total transmitted power is $P_{tot} = \frac{10^2}{2} = 50$, the power at the output of the bandpass filter is only 26.30% of the transmitted power.

Problem 3.41
1) The instantaneous frequency is

$$f_i(t) = f_c + k_f m_1(t)$$

The maximum of $f_i(t)$ is

$$\max[f_i(t)] = \max[f_c + k_f m_1(t)] = 10^6 + 5 \times 10^5 = 1.5 \text{ MHz}$$

2) The phase of the PM modulated signal is $\phi(t) = k_p m_1(t)$ and the instantaneous frequency

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{k_p}{2\pi}\frac{d}{dt}m_1(t)$$

The maximum of $f_i(t)$ is achieved for t in $[0,1]$ where $\frac{d}{dt}m_1(t) = 1$. Hence, $\max[f_i(t)] = 10^6 + \frac{3}{2\pi}$.

3) The maximum value of $m_2(t) = \text{sinc}(2\times 10^4 t)$ is 1 and it is achieved for $t = 0$. Hence,

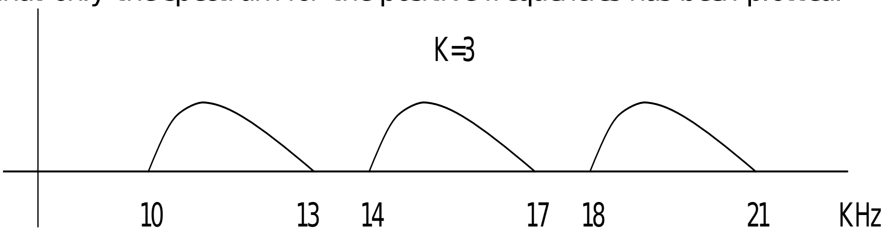
$$\max[f_i(t)] = \max[f_c + k_f m_2(t)] = 10^6 + 10^3 = 1.001 \text{ MHz}$$

Since, $\mathcal{F}[\text{sinc}(2\times 10^4 t)] = \frac{1}{2\times 10^4}\Pi(\frac{f}{2\times 10^4})$ the bandwidth of the message signal is $W = 10^4$. Thus, using Carson's rule, we obtain

$$B = 2(\frac{k_f \max[|m(t)|]}{W} + 1)W = 22 \text{ KHz}$$

Problem 3.42

1) The next figure illustrates the spectrum of the SSB signal assuming that USSB is employed and K=3. Note, that only the spectrum for the positive frequencies has been plotted.



2) With $LK = 60$ the possible values of the pair (L, K) (or (K, L)) are $\{(1,60),(2,30),(3,20),(4,15),(6,10)\}$. As it is seen the minimum value of $L + K$ is achieved for $L = 6$, $K = 10$ (or $L = 10$, $K = 6$).

3) Assuming that $L = 6$ and $K = 10$ we need 16 carriers with frequencies

$$\begin{array}{ll} f_{k_1} = 10 \text{ KHz} & f_{k_2} = 14 \text{ KHz} \\ f_{k_3} = 18 \text{ KHz} & f_{k_4} = 22 \text{ KHz} \\ f_{k_5} = 26 \text{ KHz} & f_{k_6} = 30 \text{ KHz} \\ f_{k_7} = 34 \text{ KHz} & f_{k_8} = 38 \text{ KHz} \\ f_{k_9} = 42 \text{ KHz} & f_{k_{10}} = 46 \text{ KHz} \end{array}$$

and

$$\begin{array}{ll} f_{l_1} = 290 \text{ KHz} & f_{l_2} = 330 \text{ KHz} \\ f_{l_3} = 370 \text{ KHz} & f_{l_4} = 410 \text{ KHz} \\ f_{l_5} = 450 \text{ KHz} & f_{l_6} = 490 \text{ KHz} \end{array}$$

Problem 3.43

Since $88 \text{ MHz} < f_c < 108 \text{ MHz}$ and

$$|f_c - f_c''| = 2f_{IF} \quad \text{if } f_{IF} < f_{LO}$$

we conclude that in order for the image frequency f_c'' to fall outside the interval [88, 108] MHZ, the minimum frequency f_{IF} is such that

$$2f_{IF} = 108 - 88 \Longrightarrow f_{IF} = 10 \text{ MHz}$$

If $f_{IF} = 10$ MHz, then the range of f_{LO} is $[88 + 10, 108 + 10] = [98, 118]$ MHz.
