COMMUNICATION THEORY (3-1-0)

Office hours: Thur., 10 to 11 Tutorial

Prerequisite: Signal and Systems, Probability and Random Process.

Reference Book:

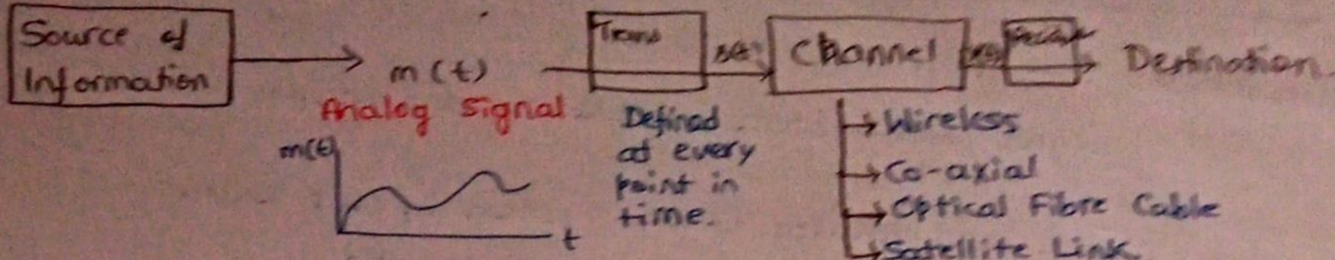
*① Modern Digital and Analog Comm. system by B. P. Lathi

② An intro to dig. & analog comm. by S. Hyakins.

③ Intro to comm. system by U Madow.

Evaluation	Grading: Absolute
End Sem – 40 %	$A \geq 70$
Mid Sem – 20 %	$A^{-1} \geq 63$
Quiz (2 of 3) – 20 %	$B \geq 57$
Surprise Test. – 10 %	$B^{-} \geq 50$
Assignment(6) – 10 %	$C \geq 45$
	$C^{-} \geq 40$
	$D \geq 35$

Basics



voice needs to be converted to electrical by transducer before transmission.

Each channel has different properties and prerequisites.

★ Signal might not have quality to transform in a specific channel. So, a transmitter is used. This makes signal suitable for transmission.

★ Now, we need to undo the effect of transmitter.

The device used is Receiver.

This will be done for a specific channel. We assume a very simple channel model for this and focus on Tx and Rx in this course.

Blocks in Transmitter (Tx)

→ Modulator – For both Digital and Analog Communication

It convert the BaseBand signal to passband signal.

Transform signal st. it becomes suitable for given type of channel.

To do this, the signal is superimposed on a carrier with high frequency.

base band → kHz, signal in original form

Base band Signal

→ A / D Converter

Encoder

→ Demodulator

02 Aug 2022

→ D/A Converter

→ Decoder.

Channel Impairments on signal

Due to some channel impairments, the signal sent by the transmitter and signal received by Receiver are different.

Some common Channel Impairments

① Attenuation:

Wireless P_S: Power of signal sent

Antenna 1

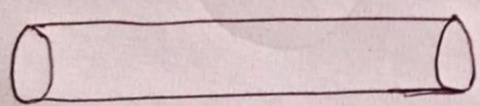
P_R: Power of signal received

Antenna 2

$$P_R \propto \frac{P_S}{d^{\alpha}}$$

'α' depends on environment

Wired



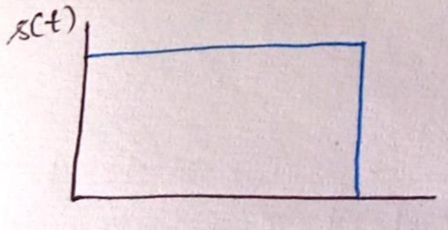
Co-axial Cable.

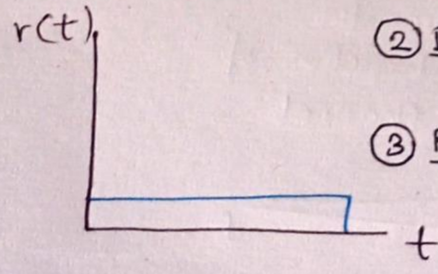
Common reasons for losses

① Diameter Length

② Dielectric of material

③ Radiation Losses



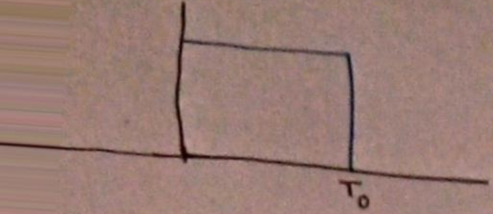


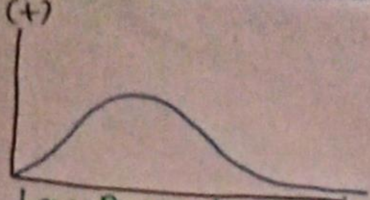
Distortion

Cut off freq. of LPF will be determined by co-axial cable.

width, T

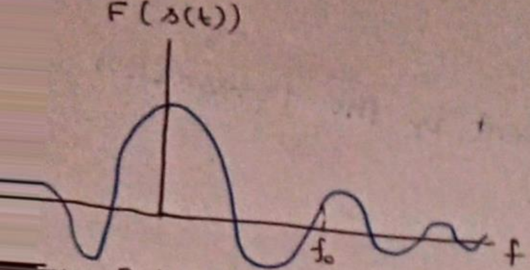
freq, f



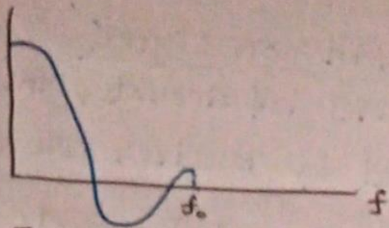


Square pulse tends to get elongated.

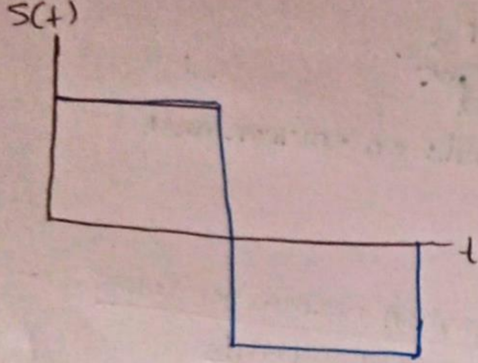
Low Pass representation of Pulse.

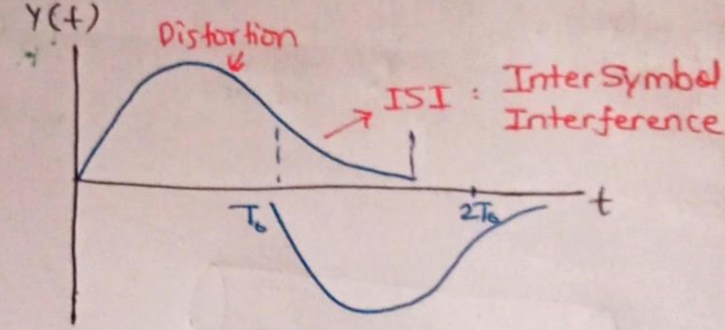


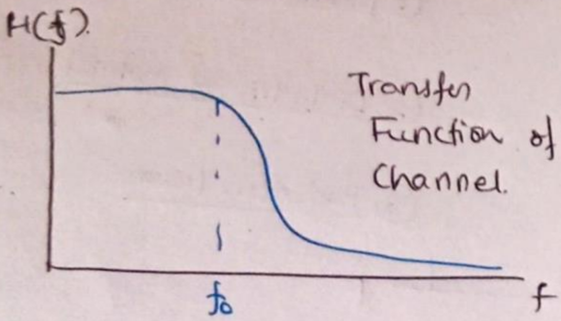
Freq. Reponse of $s(t)$



Freq response of $s(t)$ after Distortion

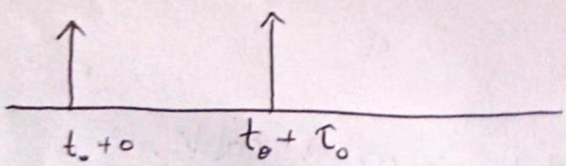


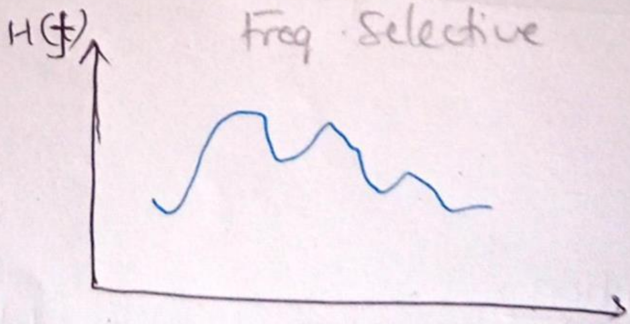




Transfer Function of Channel.

Wireless

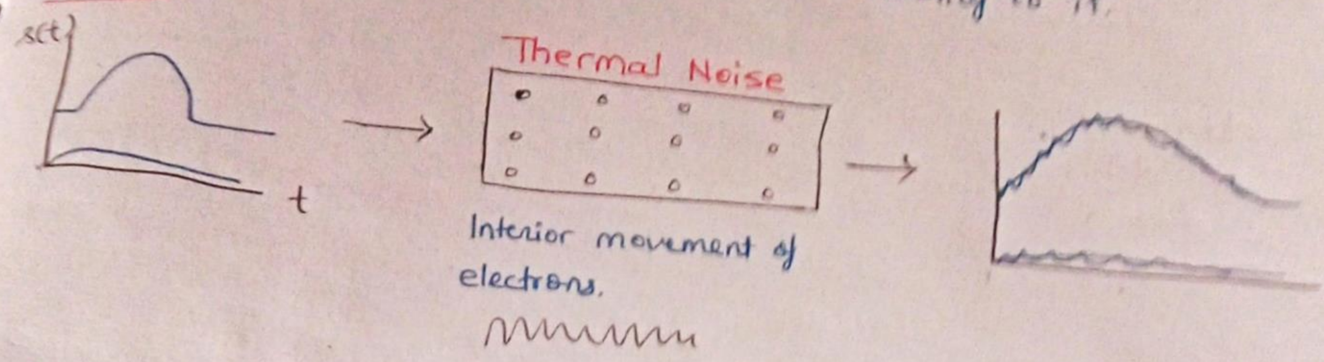




③ Noise

Not part of original signal but still adding to it.

Wired



If Amplitude of signal is large it will have large Signal to Noise Ratio

$SNR = \frac{P_S}{N_o}$: P_S : Power of signal

N_o : Power of Noise.

For low amplitude Signal, it will be just noise.

④ Delay

Modulator

Need

① For transforming signal to make it suitable for a given channel.

② Diameter of Receiver Antenna

RA design

$$\boxed{l_A \propto \lambda_{wave}}$$

l_A : Length / Dimension of Antenna

λ_{wave} : Wavelength of wave

3000 Hz $\quad \lambda = \frac{c}{f} = 10^5 = 100$ km ; size of antenna $= 0.1\lambda = 10$ km

30 MHz $\quad \lambda = \frac{c}{f} = 10$m ; size of antenna $= 0.1\lambda = 1$m

keep the size of antenna small, we need to increase the frequency of signal.

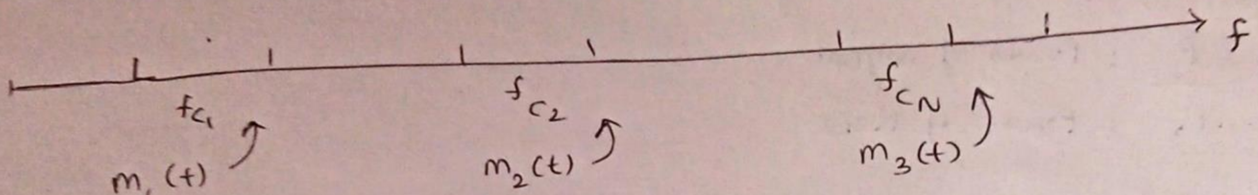
another advantage of modulation.

③ Multiplexing FDM Freq. Div. Multiplexing.

Transmitting multiple signal at diff freq. ...

Base Band → Pass Band.

put message into Diff. carrier freq ~~f~~



We are changing one feature of ~~...~~ carrier ~~freq~~ using $m(t)$ modulated signal

f_c decide location in spectrum

$m(t)$ — A or P.

$$c(t) = \underline{A} \cos(2\pi \underline{f_c} t + \underline{\theta})$$

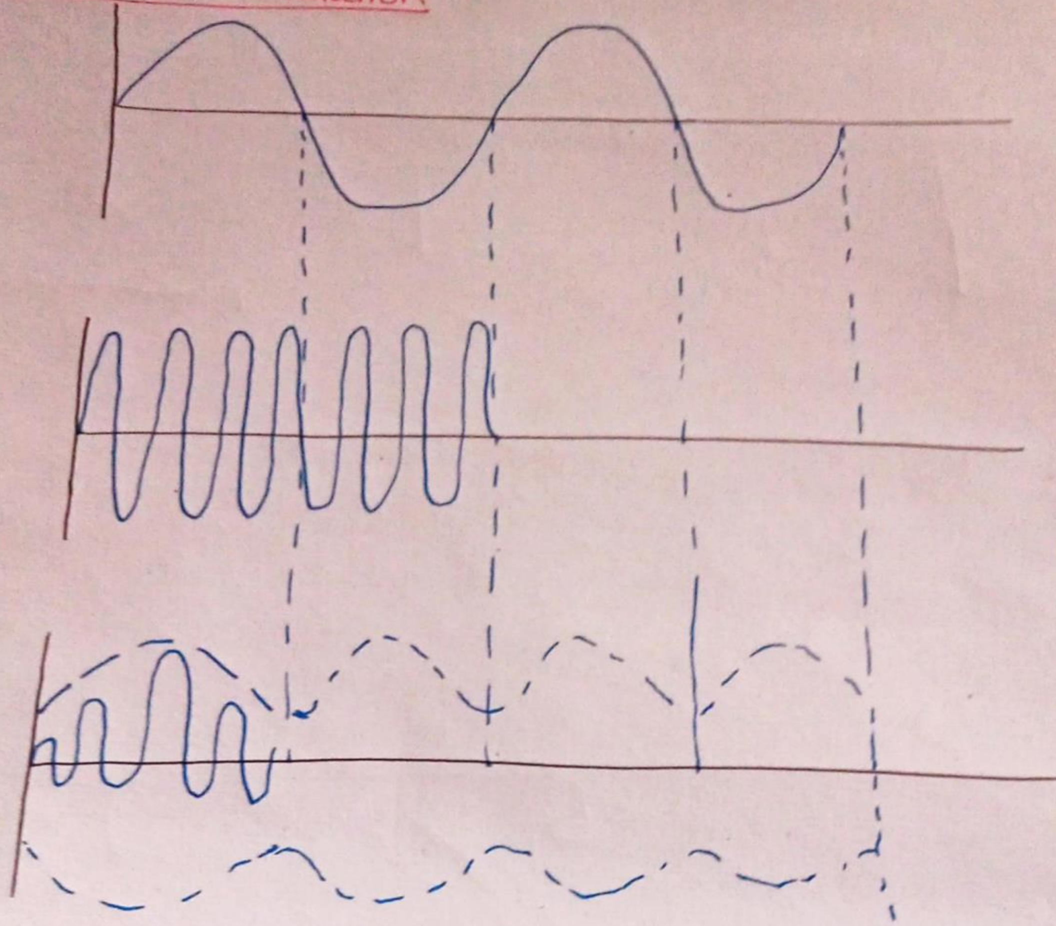
AM FM PM

ASK FSK PSK

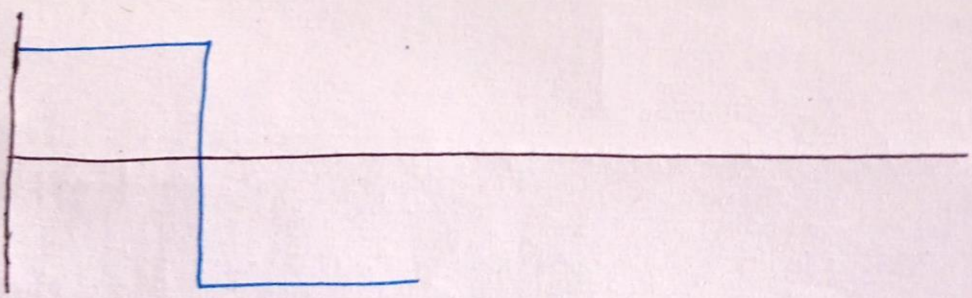
$\underbrace{\text{FM, FSK, PM, PSK}}_{\text{Angle Modulation}}$

$\underbrace{\text{AM, ASK, FM, FSK, PM, PSK}}_{\text{continuous wave Modulation.}}$

Amplitude Modulation



Amplitude Shift Keying : Digital Counterpart.



Jan 2022

SNRI   SNR$_o$

[S] — [Tx] — [Ch] — [Rx] — [D]

Tx:
- Modulator (mod.)
- A/D (D)
- Encoder (D)

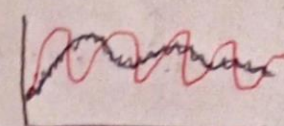
Ch:
- Attenuation
- Distortion
- Noise

Rx:
- Demodulator
- D/A Converter
- Demodulator.

Performance of Demodulator: BER/SER - For Digital

$SNR = \frac{P_s}{N_o}$ - For Analog. - How strong is un signal wrt. noise

We want high SNR output.



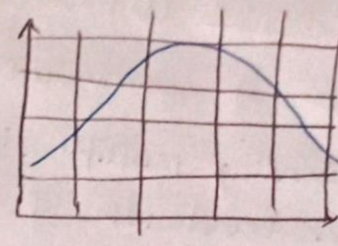
Power ↑ : Better representation of signal

$\frac{SNR_o}{SNR_i}$ = Noise figure of receiver.

Converting Analog to Digital Signal

1. sampling :

$T_s < \frac{1}{2B}$ Nyquist Theorem.



2. Quantisation: Approximating signal value to nearest level.

Quantisation Error: We show statistical description of error.

n level we can encode using $\log_2 n$ bits.

Symbol (Transmission) rate : $R_s = \frac{1}{T_s}$ Symbols sent over channel per unit time.

$$\boxed{R_b = \frac{\log_2 k}{T_s}}$$

Bits ~~per~~ of sample transmitted per unit time.

More R_b, better capacity of channel.

But R_b value is limited ∴ There is capacity of channels.

Encoders

Source Coding : Coding signal with lesser bits.

Error Correction Codes

$s_1 \to 0$, $\quad s_1 \to 0\ 0\ 0$ We introduce redundancy to decrease probababillity

$\hat{s}_1 \to 0$ $\quad \hat{s}_1 \to 1\ 0\ 0$ of Error.

S → Source Encoder → Channel Encoder → Modulator → Channel → Demodulator → Channel Decoder → Source Decoder → S

AWGN : Added to White Gaussian Noise

$Y(t) \longrightarrow \oplus \longrightarrow r(t) = y(t) + n(t)$

$N(t) \rightarrow$ Complex Gaussian $\underbrace{\text{White}}_{\text{Uncorrelated.}}$ Noise $\rightarrow CN(0, \sigma_n^2)$

Pr

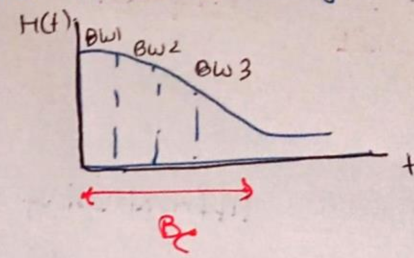
<u>Resources of Signal</u> ① Power, P_s ② Bandwidth, $B. < B_c$, channel bandwidth

$B_M > B$ B_M → Bandwidth of modulated signal

Transfer Function of communication channel. decides bandwidth of signal.

Multiplexing: Sending multiple signal over channel simultaneously

If we reduce bandwidth of modulated signal.



~~Construction~~

<u>Constraint</u> complexity of T_x / R_x.

<u>Representing a signal using vector</u>

Shannon Channel Chapparcity Theorem

$C = B_c \log_2 (1 + SNR)$ bit/sec

You can transfer C bit/sec, while keeping BER as small as possible

~~B is premodulated sig~~ → B_c is channel Bandwith.

Thus, we cannot have channel with capacity > C with small BER.

Digital Signal is sent using square pulses. Duration of these pulse and carrier freq. defines bandwidth of signal.

Energy

We deal with finite duration pulses / decreasing.

Energy dissipated by a 1 Ω resistor at time t is $x^2(t)$.

$x(t) \longrightarrow [t_1 \quad t_2]$

$$E_x = \int_{t_1}^{t_2} |x^2(t)| \, dt \quad < \infty$$

Power

If signal is increasing with time, we define power / infinite duration

So, when E is ∞, we define power.

$$P_S = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0^2/2}^{T_0^2/2} |x^2(t)| \, dt$$

If signal is complex, we take absolute value of signal.

Energy signal: If energy is finite. $P = 0$

Power Signal: If energy is infinite or zero, $P =$ finite.

All periodic signal are power signal

$x(t) \sim N(0, \sigma^2)$ Its power is variance.

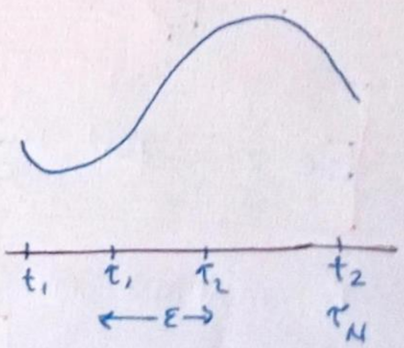
$$P_x = E\left[|x(t)|^2 \right] = \sigma^2$$

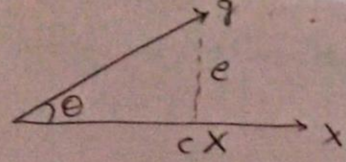
∴ Power can be characterised by variance and mean

Representing a signal using vector

$g(t) = [g(\tau_1) \ \ldots \ g(\tau_N)]$

$\lim_{N \to \infty} g \rightarrow g(t)$





1. Scalar Product $\langle g, g \rangle = \|g\|^2 = (\text{norm})^2$
2. Scalar g and x: $\langle g, x \rangle = \|x\| \, \|g\| \cos\theta$ Inner Product.
3. Component of g along x: $c\|x\| = \|g\| \cos\theta$ Projection of g along x.

$$g = cX + e$$

$$c = \frac{\|x\| \, \|g\| \cos\theta}{\|x\|^2} = \frac{\langle g, x \rangle}{\langle x, x \rangle}$$

4. If g and x are ortho $c = 0$ or $\theta = \frac{\pi}{2}$. $\langle g, x \rangle = 0$

$$\langle g, x \rangle = \sum g_i x_i$$

$$\langle g(t), x(t) \rangle = \sum_{i=0}^{\infty} g(\tau_i) \, x(\tau_i) = \int_{t_1}^{t_2} g(\tau) \, x(\tau) \, d\tau$$

$$\|x(t)\|^2 = \langle x(t), x(t) \rangle = \int_{t_1}^{t_2} |x^2(t)| \, dt = E_x$$ Energy is norm squared

If $g(t)$ and $x(t)$ are ortho $\boxed{\int_{t_1}^{t_2} g(t) \, x^*(t) \, dt = 0}$ No component of $g(t)$ falls on $x(t)$.

* $g(t) \simeq c\,x(t)$ to reduce to the energy of error, $g(t)$ is approximated by a factor of
$g(t) = c\underbrace{X(t) + e(t)}_{E_e}$ Approximating $g(t)$ with $x(t)$ linearly so that error is minimised.

Resolved by component orthogonal to x.

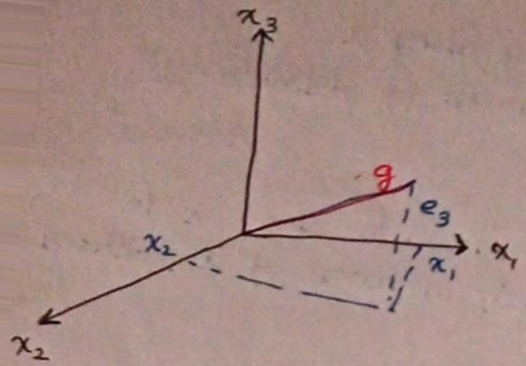
$$\boxed{E_e = \cancel{g(t)} \min_{c} \int_{t_1}^{t_2} [g(t) - c\,x(t)]^2 \, dt}$$ Error wrt. c.

To minimise error signal, optimum 'c' can be found by differentiating.

$$\frac{dE_e}{dc} = 0 \Rightarrow \boxed{c = \frac{\int_{t_1}^{t_2} g(t) \, x(t) \, dt}{\int_{t_1}^{t_2} x^2(t) \, dt} = \frac{\langle g(t), x(t) \rangle}{E_x}}$$

factor 2 ?

Auto-correlation Function



$$g = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$\langle x_1\ x_2 \rangle = 0 \quad \langle x_3, x_1 \rangle = 0 \quad \langle x_3, x_2 \rangle = 0$

Any 3D vector can be represented as linear combination of 3 orthogonal vectors.

$$c_i = \frac{\langle g, x_i \rangle}{\langle x_i, x_i \rangle}$$

$\{x_1, x_2, x_3\} \rightarrow$ Complete basis set.

Generalising to n dimensions

$$g(t) = \sum_{i=1}^{n} c_i\, x_i(t) + e_n$$

There would be some unresolved error for $x_{n+1}(t)$, $x_{n+2}(t)$ ..... components.

Basis set : $\{x_i(t)\}_{i=1}^{N}$

Generalising to infinite dimensions

$$g(t) = \sum_{i=-\infty}^{\infty} c_i\, x_i(t)$$

Decomposing signal to infinite sum of orthogonal basis functions.

Basis Set : $\{x_i(t)\}_{i=1}^{\infty}$

Generalised version of Fourier Series.

$\therefore \|e_n\|^2 \longrightarrow 0$ as $N \rightarrow \infty$

$$c_i = \frac{\langle g(t), x_i(t) \rangle}{\langle x_i(t), x_i(t) \rangle}$$

Fourier Series.

Complex Exponentials as Basis Function

$x_n(t) = \exp(j 2\pi n f_0 t) \qquad f_0 = \frac{1}{T_0}$

$n = -\infty$ to ∞.

$\rightarrow \int_0^{T_0} x_n(t)\, x_m^*(t)\, dt$.

To show the x_n form basis set, we need to show any pair of n, m, st. $n \neq m$ are orthogonal.

$$\rightarrow \int_0^{T_0} \exp(j 2\pi (n-m) f_0 t)\, dt = \begin{cases} 0, & m \neq n \\ T_0, & m = n \end{cases}$$

sum of sine and cos in exponential would be zero over a period.
And if $n = m$, $n - m = 0$, so $\int_0^{T_0} 1\, dt = T_0$.

$$\rightarrow g(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j 2\pi n f_0 t), \quad f_0 = \frac{1}{T_0}$$

IFT of $G(f)$

Expressing signal as linear combination of basis vector.

where, $c_n = \int_0^{T_0} g(t) \exp(-j 2\pi n f_0 t)\, dt = G(f)$ FT of $g(t)$

$$T_0 = \int_0^{T_0} \exp(j 2\pi n f_0 t) \exp(-j 2\pi n f_0 t)\, dt$$

$g(t) \xleftrightarrow{FT} g(f)$

★ $E_g = \sum_{n=-\infty}^{\infty} \|c_n\|^2$ Parseval's Theorem.

$$\|g(t)\|^2 = E_g = \sum_{n=-\infty}^{\infty} \|c_n \exp(j 2\pi n f_0 t)\|^2 = \sum_{n=-\infty}^{\infty} \|c_n\|^2 \cancel{\|\exp(j 2\pi n f_0 t)\|^2} \quad (\text{as } \|re^{i\theta}\| = r)$$

So, x_n are special basis function leading to Fourier Series

Addition of Orthogonal signals

$X = X_1 + X_2$

$\|X\|^2 = \|X_1\|^2 + \|X_2\|^2$

$$\boxed{\begin{aligned} X(t) &= X_1(t) + X_2(t) \\ E_X &= E_{X_1} + E_{X_2} \end{aligned}}$$

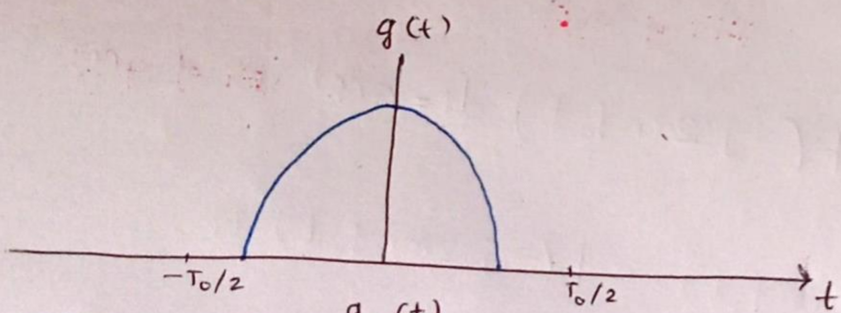
why?

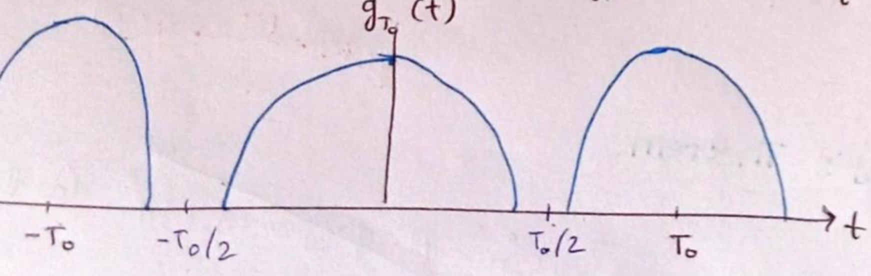
The Fourier Series is not applicable for aperiodic signals.

Fourier Series → Periodic → Finite Duration

Fourier Transform → Aperiodic → Infinite Duration.

How to extend it for longer duration or aperiodic signal?





$$\underbrace{\lim_{T_0 \to \infty} g_{T_0}(t)}_{\text{Periodic Signal.}} = g(t)$$

Fourier Transform / Fourier Spectrum

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j\,2\pi n f_0 t) \quad ; \quad f_0 = \frac{1}{T_0}$$

$$\text{where, } D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\, \exp(-j\,2\pi n f_0 t)\, dt$$

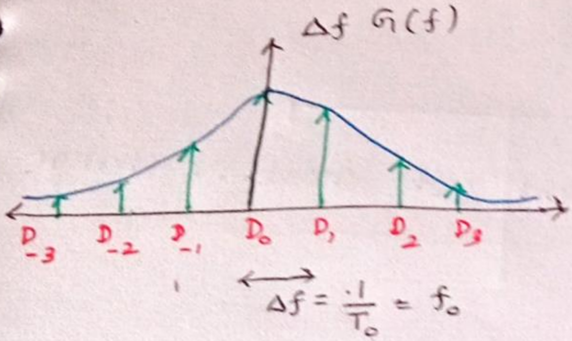
$$D_n = \frac{1}{T_0}\underbrace{\int_{-\infty}^{\infty} g(t)\,\exp(-j\,2\pi f t)\, dt \Big|_{f = nf_0}}_{G(f)}$$

As $g_{T_0}(t)$ and $g(t)$ are same in interval $\left(\frac{-T_0}{2}, \frac{T_0}{2}\right)$.

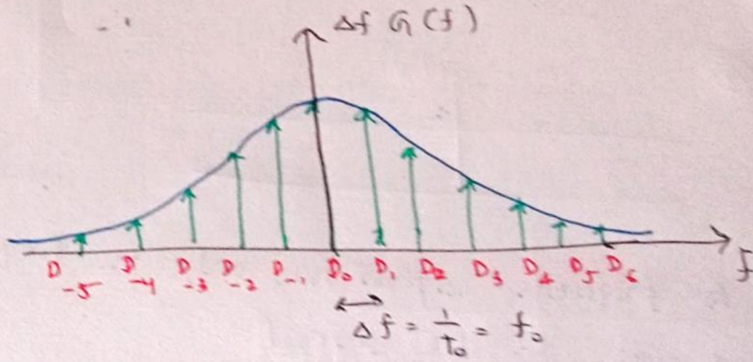
$$= \frac{1}{T_0}\, G(nf_0)$$

Let $\Delta f = f_0$ = spacing between frequency components

$$= \Delta f\ G(n\,\Delta f)$$



Smaller T_0



Larger T_0

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{T_0 \to \infty} \sum_{n=-\infty}^{\infty} \Delta f\ G(n\,\Delta f)\, e^{j 2\pi n \Delta f t}$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df \qquad G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j 2\pi f t}\, dt$$

$$|g(t)|\, e^{-j2\pi f t}$$

$$|G(f)|$$

…ies of Fourier Transform

1) $g(t)$ is real. $G(-f) = G^*(f)$

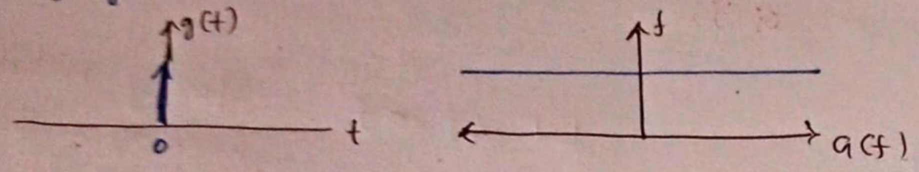
Magnitude $\rightarrow |G(-f)| = |G^*(f)| = |G(f)|$ Even Symmetric

Phase $\rightarrow \theta(G(-f)) = -\theta(g(f))$ Odd Symmetric

2) Linearity:

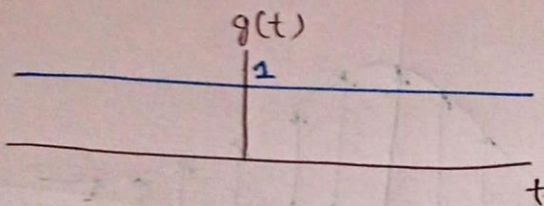
$\alpha_1 g_1(t) + \alpha_2 g_2(t) = \alpha_1 G_1(f) + \alpha_2 G_2(f)$

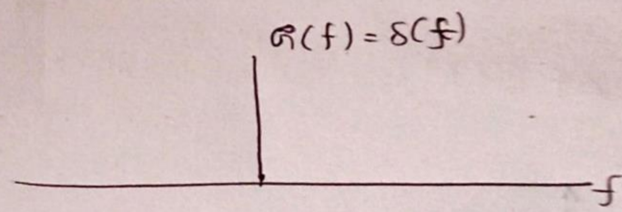
3) If $g(t) = \delta(t) \longleftrightarrow G(f) = 1 \quad \forall f$



All frequencies are present in equal magnitude.

4)

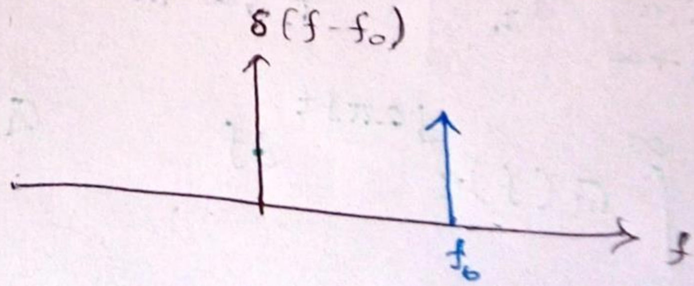




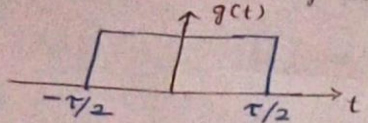
As frequency signal has only one freq. component, $g(t)$ shouldn't chang[e] at all. So, DC signal.

$g(t)\, e^{+j2\pi f_0 t} \xrightarrow{FT} \cancel{G(f)}\ G(f - f_0)$

$e^{j2\pi f_0 t} \xrightarrow{FT} \delta(f - f_0)$



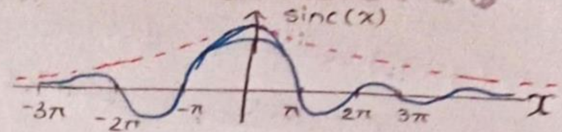
$g(t)$ is rectangular signal



$$G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t}\, dt = \tau \frac{\sin(\pi f \tau)}{\pi f \tau} = \tau \,\mathrm{sinc}(\pi f \tau)$$

$$= \left[\frac{e^{-j2\pi f t}}{j2\pi f} \right]_{-\tau/2}^{\tau/2}$$

Bandwidth
Range of spectrum where 90-95% of signal energy is concentrated



Sinc Function $\quad \mathrm{sinc}(x) = \dfrac{\sin x}{x}$

1). Even function : As it is product of two odd functions

2). Zero at $x = \pm K\pi$, $K > 0$

3). $\mathrm{sinc}(0) = 1 = \lim\limits_{x \to 0} \dfrac{\sin x}{x}$

4). Amplitude $\propto \dfrac{1}{x}$

So, we need to amplitude $\propto \left(\dfrac{1}{x}\right)^n$

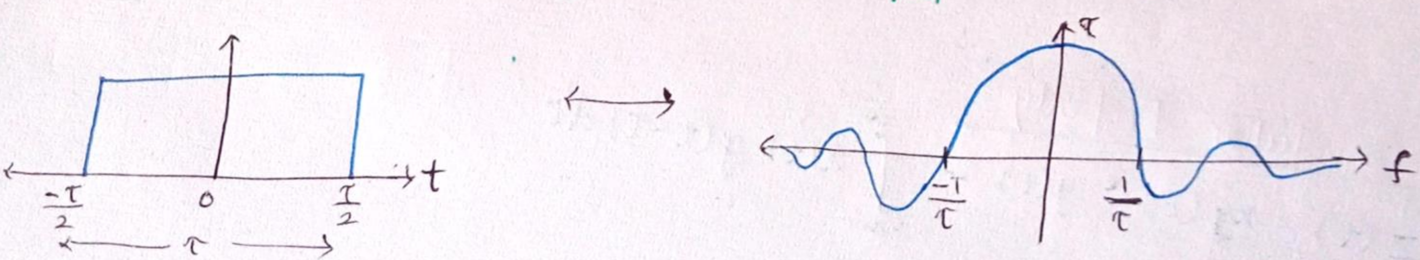
$n > 1$

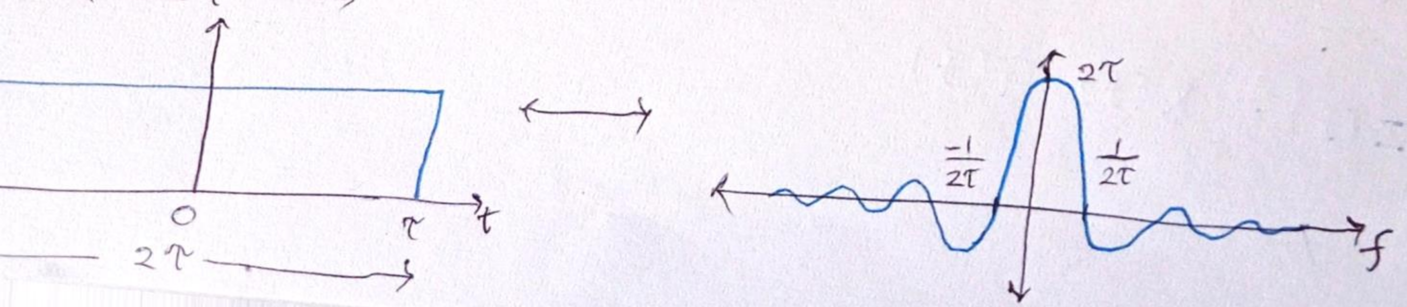
higher order.

Expansion in time
⇅
compression in freq domain
spectral Exp. comp.

All freq. component gets diff phase shift

5). $g(t) \xleftrightarrow{FT} G(f) \qquad g(at) \longleftrightarrow \dfrac{1}{|a|} G(f/a)$ Time Scaling





6) $g(t-t_0) \longleftrightarrow G(f)\,e^{-j2\pi f t_0}$

$g(t) \longleftrightarrow G(f)$ Time shifting

No change in amplitude of ~~phase~~ frequency spectrum

Amplitude spectrum remains unchanged.

Phase spectrum changes by $-2\pi f t_0$.

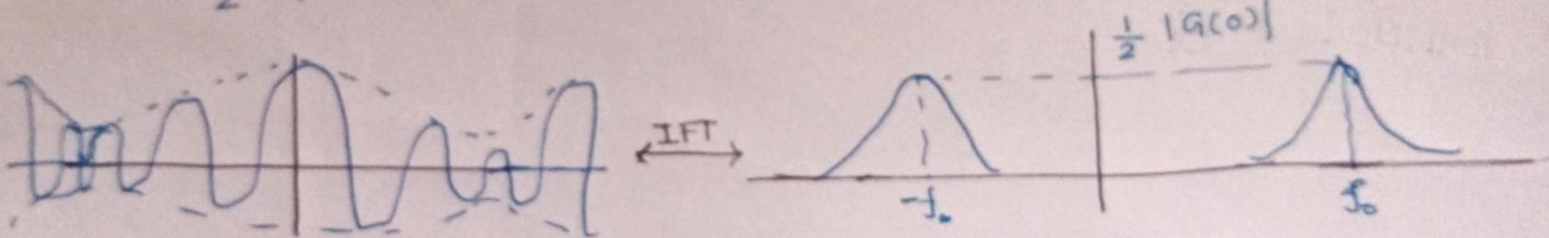
7) $G(f-f_0) \xleftrightarrow{IFT} g(t)\,e^{j2\pi f_0 t}$ Frequency shifting
Modulation Property

$\delta(f-f_0) \xleftrightarrow{IFT} e^{j2\pi f_0 t}$

Q. Find Fourier Transform of $g(t)\cos(2\pi f_0 t)$

Ans $\frac{g(t)}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) = \frac{1}{2}\left[g(t)\,e^{j2\pi f_0 t} + g(t)\,e^{-j2\pi f_0 t}\right]$

$= \frac{1}{2}\left[G(f-f_0) + G(f+f_0)\right]$



Modulation is changing frequency ~~&~~ using carrier.

8). Convolution Property

a). $z(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(t)\,g(t-\tau)\,d\tau$

$Z(f) = X(f) \cdot G(f)$

(b) $z(t) = x(t) * g(t)$

$Z(f) = X(f)\, G(f)$

Bandwidth B_z Bandwidth B_x Bandwidth B_g

$$\boxed{B_z = B_x + B_g}$$

Span of Convolution

$x(t) \longrightarrow$ [LTI $h(t)$] $\longrightarrow y(t)$

$y(t) = x(t) * h(t)$ → Impulse Response

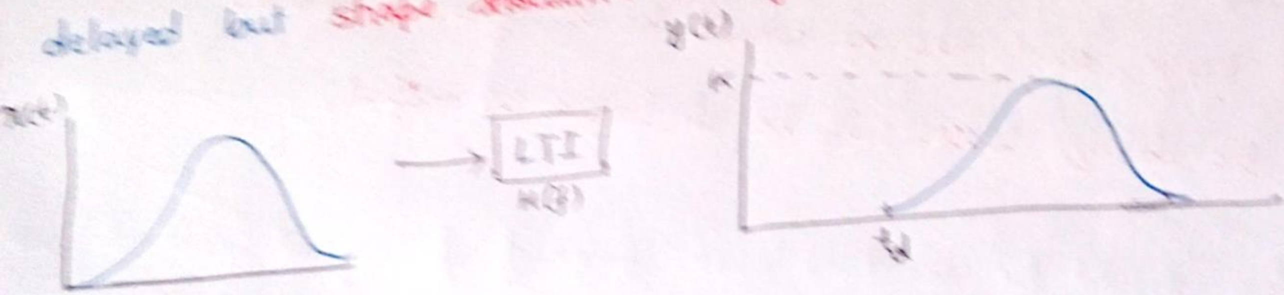
$Y(f) = X(f)\; H(f)$ → Transfer Function

FT pair are unique. So any LTI System can be described by $h(t)$ or $H(f)$ completely.

$H(f) = |H(f)|\, e^{-j\theta(f)}$

Distortionless System / Filter (DF)

Output doesn't distort the shape of input signal. It can get attenuated or delayed but shape shouldn't change.



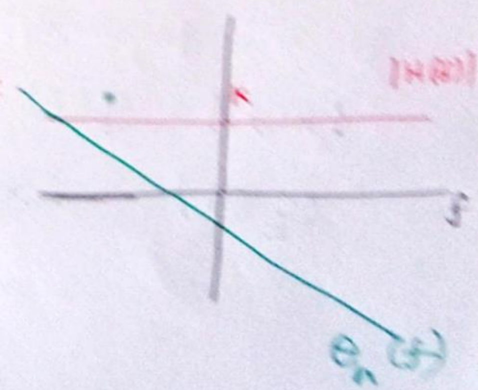
$y(t) = K\, x(t - t_d)$

$Y(f) = K\, X(f)\, e^{-j2\pi f t_d}$

For LTI System to be a DF

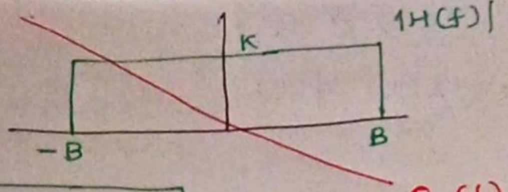
$$\boxed{H(f) = K e^{-j2\pi f t_d} \qquad h(t) = K\,\delta(t - t_d)}$$

Different frequency should have different phase shifts.



$x(t) \longrightarrow \boxed{LPF} \longrightarrow y(t)$

$B_x < B$



$\theta_h(f)$

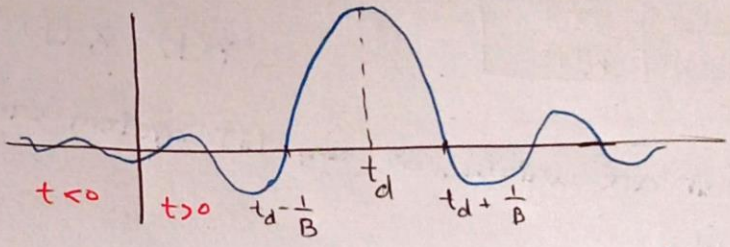
slope = $2\pi f\, t_d$

K: Factor of attenuation of input

$$\boxed{H(f) = \begin{cases} K e^{-j2\pi f t_d} & , \forall f \in [-B, B] \\ 0 & , \text{otherwise} \end{cases}}$$

We want LPF system to be distortionless. So, it should provide constant attenuation in Bandwidth of signal B_x. which is limited by B. So, we can draw spectrum and write $H(f)$ as above.

Corresponding $h(t)$ would be:



It depends on signal, $t<0$, which makes system non-causal.

So, we ignore part of signal for $t<0$.

$\hat{h}(t) = h(t)\, u(t)$

If t_d is large, most energy would be in positive part.

So, higher the delay, lesser will be distortion in output.

gnitude of

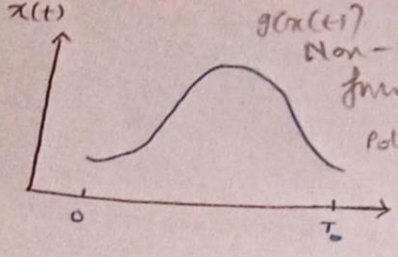
Transfer freq should be constant with freq. of i/p signal involved with phase $\propto$ frequency.

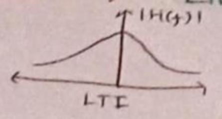
$|H(f)| = g(f)$

$\theta_h(f) = -2\pi f t_d$

If magnitude is not constant ? → Linear Distortion

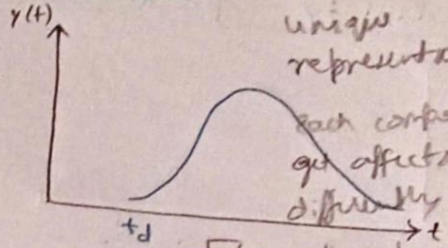
$g(x(t))$
Non-linear func





Polynomial fn of if

$\theta_h(f) = -2\pi f t_d$



unique representation

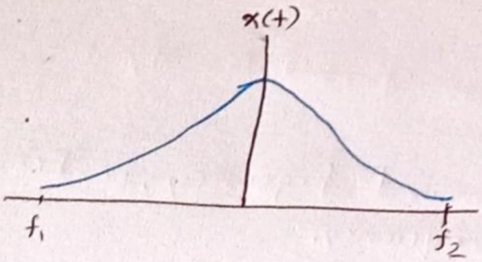
Each component get affects differently

Elongation of signal

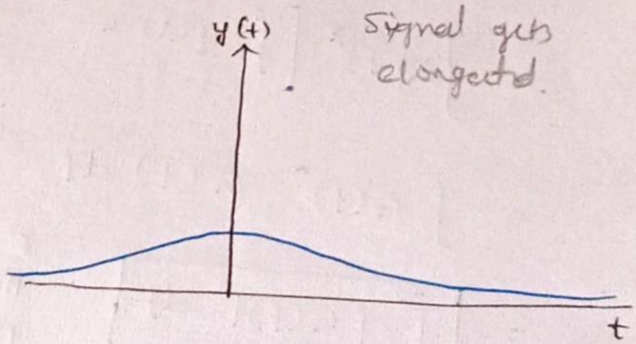
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_n t}$$

$$y(t) = \sum c_n' e^{j2\pi f_n t}$$

Non - Linear Distortion

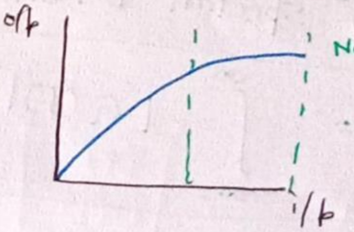


System
$y(t) = g(x(t))$
→



Signal gets elongated.

Eg Amplifier: Not linear above cut off voltage.



$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \dots$$

2B 3B B: Bandwidth.

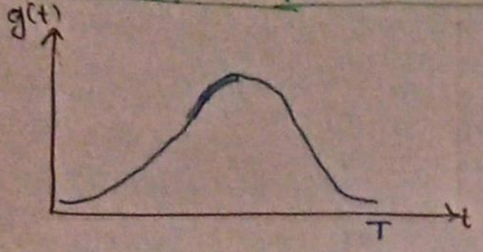
Energy/ power spectral density

For m(t) signal.

↳ Non- Linear Polynomial function of input.

Energy Spectral Density

Finite bounded.



$E_g =$

$$\boxed{E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt} = \int_{-\infty}^{\infty} g(t)\, g^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} G(f)\, e^{j 2\pi f t} \, df \right]^* dt$$

$$= \int_{-\infty}^{\infty} g(t) \left[\int G(f)^* \cdot e^{-2j\pi t f} \, df \right] dt$$

$$= \int G(f)^* \left[\int g(t)\, e^{-j 2\pi f t} \, dt \right] df \quad \text{(Replacing Integrals)}$$

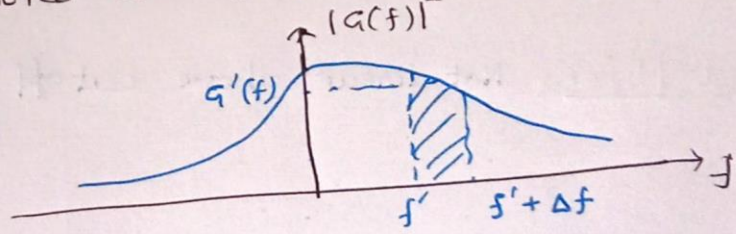
$$= \int G(f)^* \, G(f)\, df$$

$$= \boxed{\int_{-\infty}^{\infty} |G(f)|^2 \, df}$$

Energy Spectral Density

$$\boxed{E_g = \int_{-\infty}^{\infty} |G(f)|^2 \, df}$$

Area concentrated around frequency. i.e. Energy Distributed over frequency.



Energy of shaded frequency range: $|G'(f)|^2 \Delta f$

statistical matrix

Energy Spectrum Density : For Random signal ~~for random signal~~

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$ Auto - Correlation function.

$$E_g = R_g(0) = \int_{-\infty}^{\infty} |g(t)|^2\, dt$$

$$\Psi_g(f) = \int_{-\infty}^{\infty} R_g(\tau)\, e^{-2\pi j f \tau}\, d\tau$$

↳ Fourier Transfer of $R_g(\tau)$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt\right] e^{-2\pi j f \tau}\, d\tau = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt\right] e^{-j2\pi f \tau}$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g^*(t-\tau)\, e^{-j2\pi f \tau}\, d\tau\right] g(t)\, dt$$

$$= \int_{-\infty}^{\infty} G^*(t)\, e^{j2\pi f t}\, g(t)\, dt$$

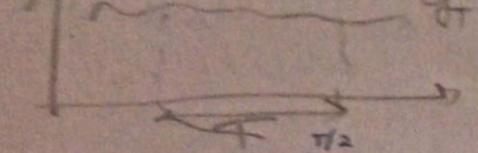
$$= G^*(f)\, G(-f) = |G(t)|^2$$

$$\Psi_g(f) \xleftrightarrow{FT} R_g(\tau)$$

$$R_g(\tau) = \int |G(f)|^2\, e^{j2\pi f \tau}\, df$$

$$|R_g(\tau)| \leq R_g(0) = \int |G(f)|^2\, df = E_g$$

Power Spectral Density



$\int_{-\infty}^{\infty} |G_T(f)|^2 \, df$ Infinite

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g_T(t)|^2 \, dt = \lim_{T\to\infty} \frac{E_{g_T}}{T}$$

Power of interval T

$T \to \infty \to$ Power constant or converging to constant

Both should converge at same rate

$\therefore$ E_g should incr with time

$\therefore$ spectral density ~~tcrd~~ should incr

Monotone convergence theorem $\to$ Exchange limit & $\int$.

Fraction of power conc around freq f.

$$P_g = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} \, df$$

$$\phi_g(t) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} \to \text{Power Spectral Density}$$

$$g(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H(f)\end{matrix}} \longrightarrow y(t)$$

Finite Duration pulse

$g(t) \downarrow$ ESD $= |G(f)|^2$

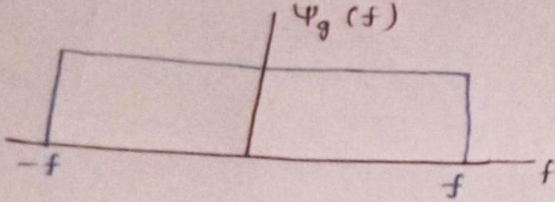
$y(t) \downarrow$ ESD $= |Y(f)|^2 = |G(f)H(f)|^2$

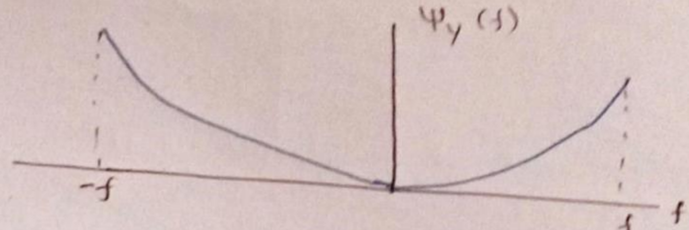
$= |G(f)|^2 \, |H(f)|^2$

$$\boxed{\Psi_y(f) = \Psi_g(f) \, |H(f)|^2}$$

LTI system Doesn't change spectral density of i/p either energy power

$g(t) \longrightarrow \underset{\boxed{\text{LTI}}}{\text{Differentiator}} \longrightarrow y(t)$





$$y(t) = \frac{d\,g(t)}{dt} = h(t) * g(t)$$

$$Y(f) = H(f) \quad G(f)$$

$\uparrow$

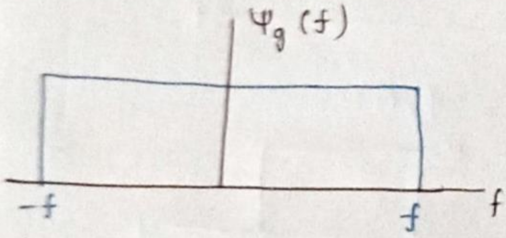
$j2\pi f \longrightarrow$ Fourier Transform of Differentiator.

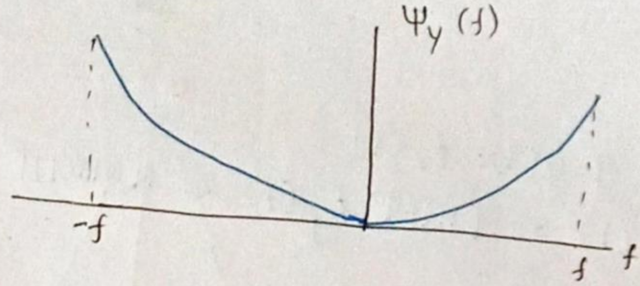
$$\boxed{\Psi_Y(f) = |j2\pi f|^2 \, \Psi_g(f)}$$

fferentiator system

$g(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$

Differentiator





$$y(t) = \frac{d\, g(t)}{dt} = h(t) * g(t)$$

$$Y(f) = \underset{\substack{\uparrow \\ j2\pi f}}{H(f)}\ G(f)$$

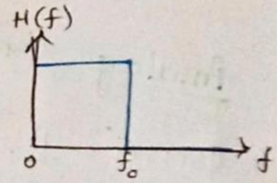
$j2\pi f \rightarrow$ Fourier Transform of Differentiator.

$$\boxed{\Psi_Y(f) = |j2\pi f|^2\, \Psi_g(f)}$$

0 mean noise

For ideal low pass Filter

We have an ideal low pass filter with transfer function:



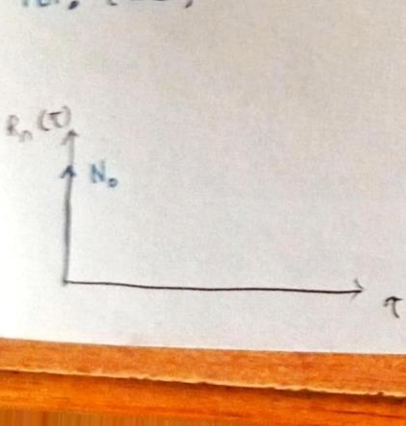
Let we give a random signal, i.e, 0 mean noise as input.

$$R_n(\tau) = E[g(t)\, g(t+\tau)] = \begin{cases} 0, & \tau \neq 0 \\ N_0, & \tau = 0 \end{cases}$$ Variance as mean is 0.

As $g(t)$ is uncorrellated, $E[g(t)\, g(t+\tau)] = \overset{0}{\cancel{E[g(t)]}}\, E[g(t+\tau)] = 0 \quad \forall\ \tau \neq 0$

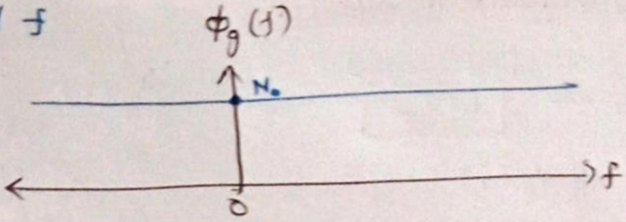
$\because$ 0 mean noise

For, $\tau = 0$,



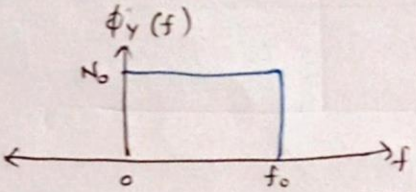
$$R_n(\tau) = E[g(\tau)\, g(t+\tau)] = N_0\, \delta(\tau)$$

ESD of $g(t)$: $\phi_g(f) = N_0 \ \forall f$



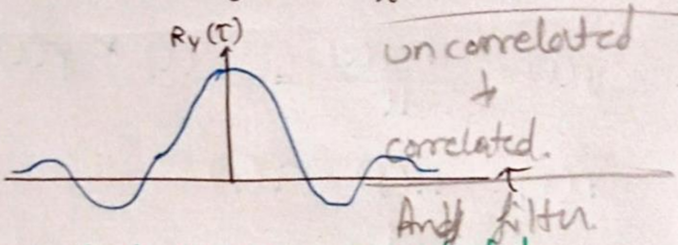
ESD of $y(t)$: $\phi_y(f)$

$$\phi_y(f) = |H(f)|^2 \phi_g(f) = N_0 |H(f)|^2$$



Auto-corr.

$R_y(\tau)$: Inverse FT of $\phi_y(f)$: sinc.



Conclusion:

Given uncorrelated function, the output becomes correlated regardless of type of filter used.

Analog Communication

Carrier: $\underset{\sim}{A} \cos(2\pi f_c t + \theta)$ Message: $m(t)$

$A(t) = g_1(m(t)) \rightarrow$ AM

$f(t) = g_2(m(t)) \rightarrow$ FM

$\theta(t) = g_3(m(t)) \rightarrow$ PM

FM and PM: Angle Modulation

AM, FM and PM: Carrier Modulation

Base band Communication

When message is transmitted as it is, i.e., without modulation

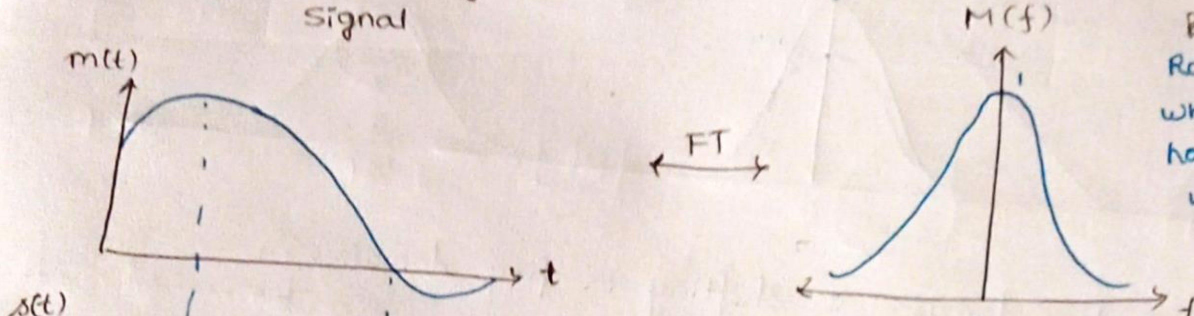
Amplitude Modulation: SC Double Side Band Modulation SC-DSB

Varying amplitude of carrier signal instantaneously w.r.t. message signal

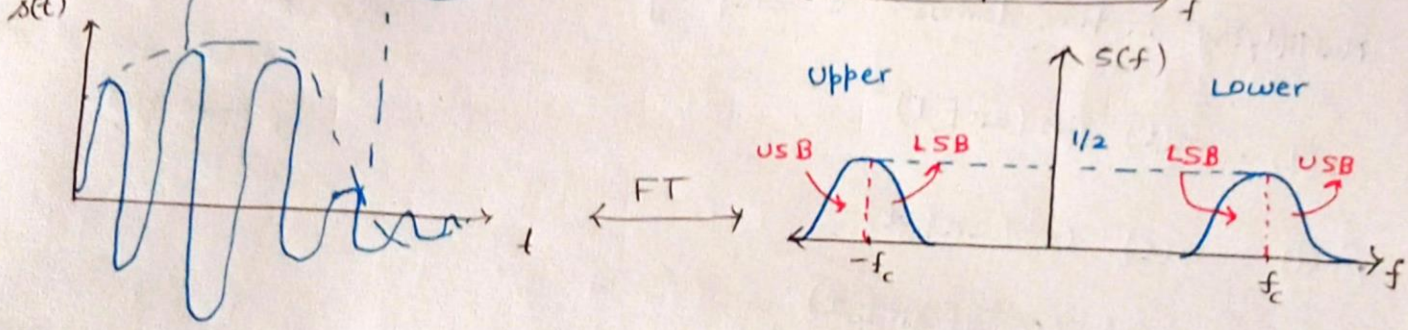
Suppressed carrier

$$s(t) = \underbrace{m(t)}_{\text{Modulating Signal}} \underbrace{\cos(2\pi f_c t)}_{\text{Carrier}}$$

shifts signal by $\pm f_c$ frequency



Bandwidth: Range of freq. where signal has significant value.



$$S(f) = \frac{1}{2}\left[M(f_c + f) + M(-f_c + f) \right]$$

$$E_m = \int_{-\infty}^{\infty} |M(f)|^2 \, df$$

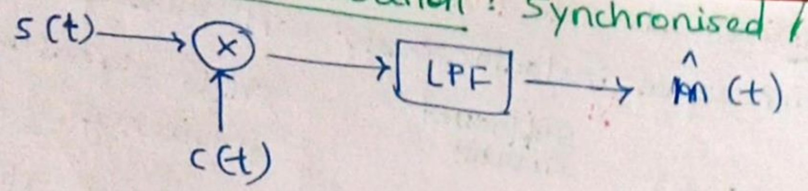
$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 \, df = \int_{-\infty}^{0} |S(f)|^2 \, df + \int_{0}^{\infty} |S(f)|^2 \, df$$

$$= \frac{1}{4} E_m + \frac{1}{4} E_m = \frac{1}{2} E_m \qquad \boxed{E_s = \frac{1}{2} E_m} \qquad \boxed{P_s = \frac{1}{2} P_m}$$

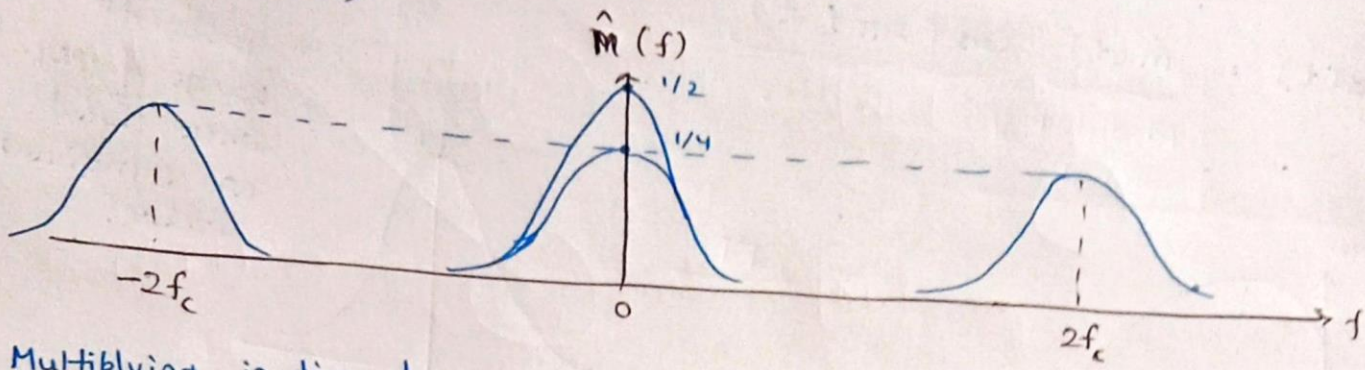
We are multiplying, but still energy of signal is decreasing as energy involves squaring for input which ~~twin~~ lies between (0,1) so squaring causes decrease in energy.

$$\boxed{B_s = 2 B_m}$$

Amplitude Demodulation : Synchronised / Coherent Demodulation



This again shifts signal by $\pm f_c$.



Multiplying in time domain is shifting in freq. domain.

$s(t) = m(t)\cos(2\pi f_c t)$

synchronised with carrier used at T_x to get same

$\hat{m}(t) = s(t)\cos(2\pi f_c t)$

$= m(t)\cos^2(2\pi f_c t)$

$m(t) = \frac{m(t)}{2} + \frac{m(t)}{2}\cos(4\pi f_c t)$

FT

$= \frac{m(t)}{2}$ (After LPF)

$\hat{M}(f) = \frac{1}{2}M(f) + \frac{1}{4}\left[M(f-2f_c) + M(f+2f_c)\right]$

Carrier way perfectly synchronised by frequency and phase

What if carrier at Receiver is not synchronised?

$s(t) \cos(2\pi(f_c + \Delta f)t + \theta)$ $\boxed{\Delta f << f_c}$

$$\hat{m}(t) = m(t) \cos(2\pi f_c t) \cos(2\pi(f_c + \Delta f)t + \theta)$$

$$\frac{m(t)}{2} \cos(2\pi \Delta f t + \theta) + \underbrace{\frac{m(t)}{2} \cos(2\pi(2f_c + \Delta f)t + \theta)}_{0,\ \text{when passed through LPF}}$$

CASE 1: $\Delta f = 0$ Constant phase shift..

constant phase shift $\downarrow$ attenuation

$$\hat{m}(t) = \frac{m(t)}{2} \cos(\theta)$$

So, attenuation in modulated signal is observed when there is a constant phase shift

$\uparrow \downarrow$ amplitude with time period $\frac{1}{\Delta f}$

CASE 2: $\theta = 0$

$$\hat{m}(t) = \frac{m(t)}{2} \cos(2\pi \Delta f t)$$

$\Delta f = 1\text{Hz}$

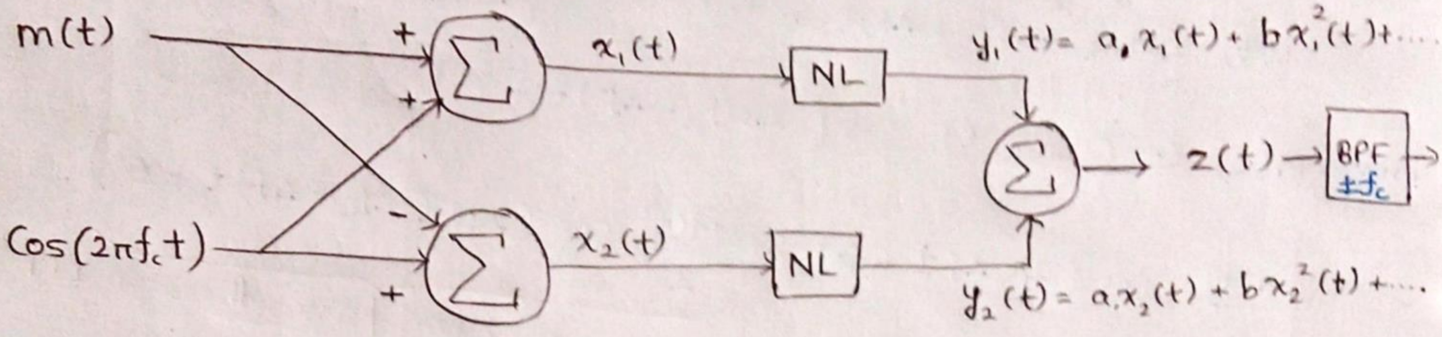
Modulator

(a) Non-Linear Modulator

06 Feb 2025

Types of modulator which gives product of two signals

Let say we have a Non-Linear System.



$$Z(t) = y_1(t) - y_2(t)$$

$$= a x_1(t) + b x_1^2(t) - a x_2(t) + b x_2(t)$$

$$= 2a\, m(t) + 4b\, m(t) \cos(2\pi f_c t)$$

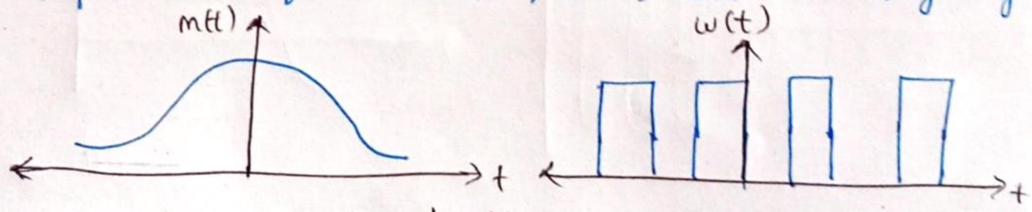
We need to pass it through a BPF, to get $4b\, m(t) \cos(2\pi f_c t)$

So, we can use a quadratic Non-Linear System as then, there would not be any higher terms.

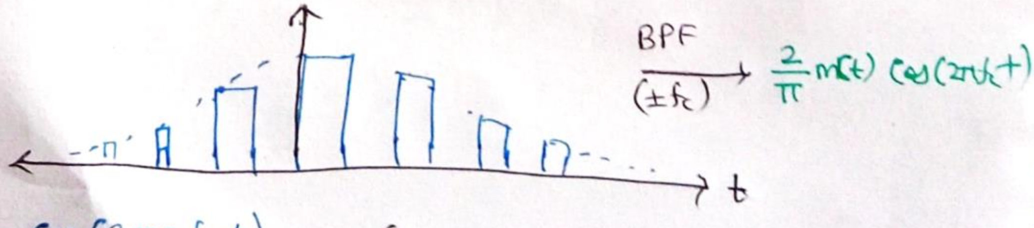
Switching

Let we multiply by a periodic square wave, $\omega(t)$, and the message signal.

$m(t) * \omega(t)$



switching on and off action



$$\omega(t) = \sum_{n=-\infty}^{\infty} C_n \cos(2\pi n f_c t)$$

(Fourier Transform)
(Sum of Cosines)

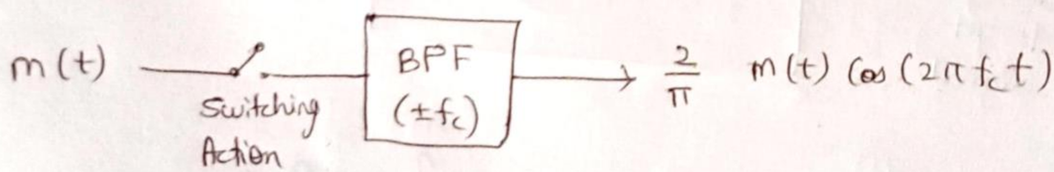
$$\rightarrow m(t)\,\omega(t) = \sum_{n=-\infty}^{\infty} m(t)\, C_n \cos(2\pi n f_c t)$$

Its Freq. spectrum will included scaled $M(f)$, $M(f \pm f_c)$, $M(f \pm 2f_c)$,
As we are interested in only $M(f \pm f_c)$ part of frequency component we can pass it through BPF.

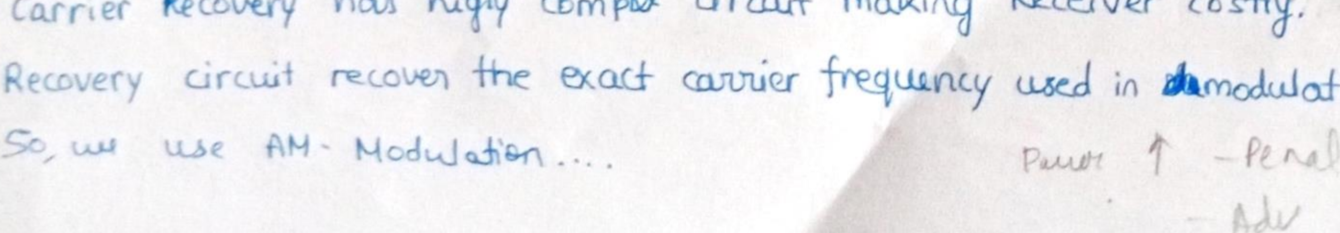
$$\rightarrow g(t) = m(t)\,\omega(t) = m(t)\left[\frac{1}{2} + \frac{2}{\pi}\left\{\cos(2\pi f_c t) - \frac{1}{3}\cos(3\pi f_c t) + \frac{1}{5}\cos(5\pi f_c\right.\right.$$

$$= \underbrace{\frac{1}{2}m(t)}_{\frac{1}{2}M(f)} + \underbrace{\frac{2}{\pi}m(t)\cos(2\pi f_c t)}_{\frac{1}{\pi}M(f\pm f_c)} - \underbrace{\frac{2m(t)}{3\pi}\cos(6\pi f_c t) + \ldots}_{\frac{1}{3\pi}M(f\pm 3f_c)\ \ldots}$$

→ We can pass this through BPF centered around $\pm f_c$ to get the modulated signal.

→ 

Demodulation in DSB-SC



Carrier Recovery has higly complex circuit making Receiver costly.
Recovery circuit recover the exact carrier frequency used in modulation.
So, we use AM - Modulation ....

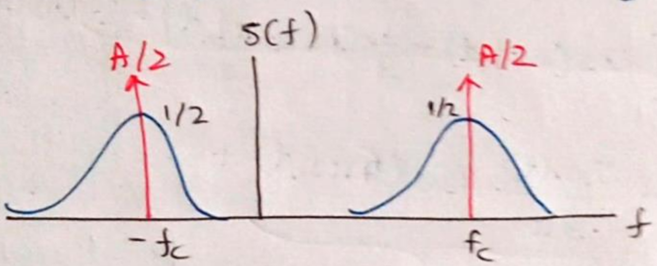
Power ↑ - Penalty
- Adv

Amplitude Modulation

As we need carrier signal for demodulation, we send carrier signal along with modulated signal also.

$$s(t) = \overbrace{A\cos(2\pi f_c t)}^{\text{carrier}} + \overbrace{m(t)\cos(2\pi f_c t)}^{\text{DSB}} = [A + m(t)]\cos(2\pi f_c t)$$

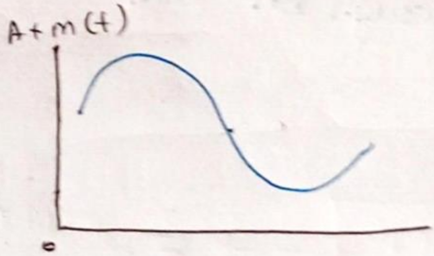
$$S(f) = \frac{A}{2}\left[\delta(f+f_c) + \delta(f-f_c)\right] + \frac{1}{2}\left[M(f+f_c) + M(f-f_c)\right]$$

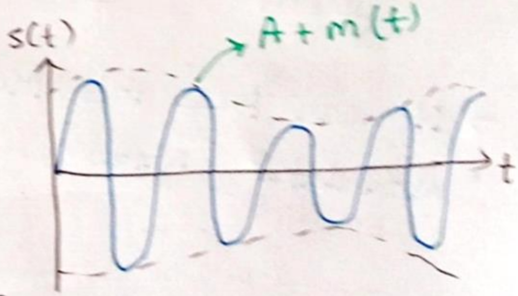


$$P_s = \frac{P_m}{2} + \underbrace{\frac{A^2}{2}}_{\text{Additional Power}}$$

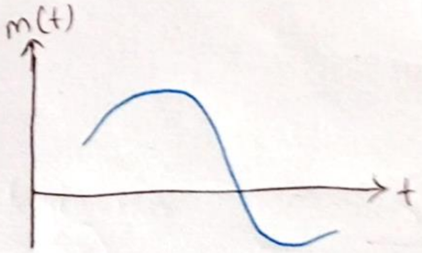
CASE 1: $A + m(t) \geq 0$

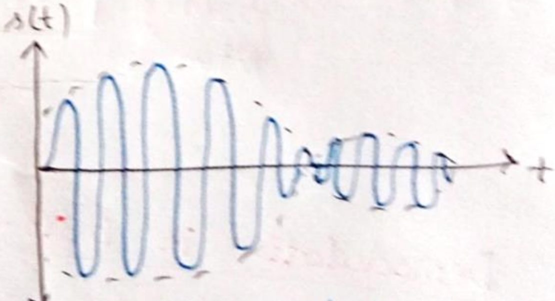
We can signal through Envelope Detector to get back the message signal.





CASE 2: $A + m(t) < 0$





Detecting / recovering the message using this signal would be difficulting. So, we can skip this case.

$A + m_p > 0$ m_p: Peak-to peak amplitude of modulated signal

$|m(t)| \leq m_p$

$\mu = \frac{m_p}{A}$ μ: Modulation Index : $0 < \mu < 1$

Tone Modulation

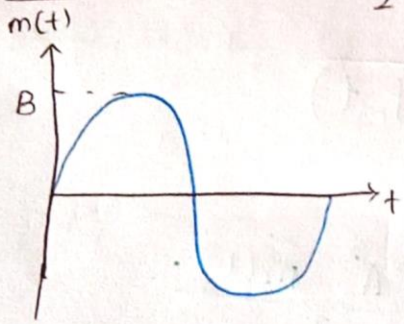
Tone Modulation.

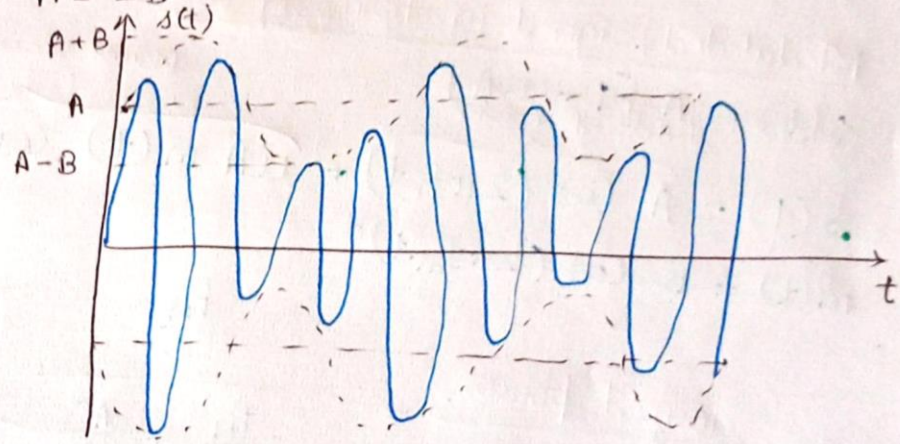
When the message signal consisting of only one frequency is used in modulation, it

$m(t) = B \cos(2\pi f_m t)$

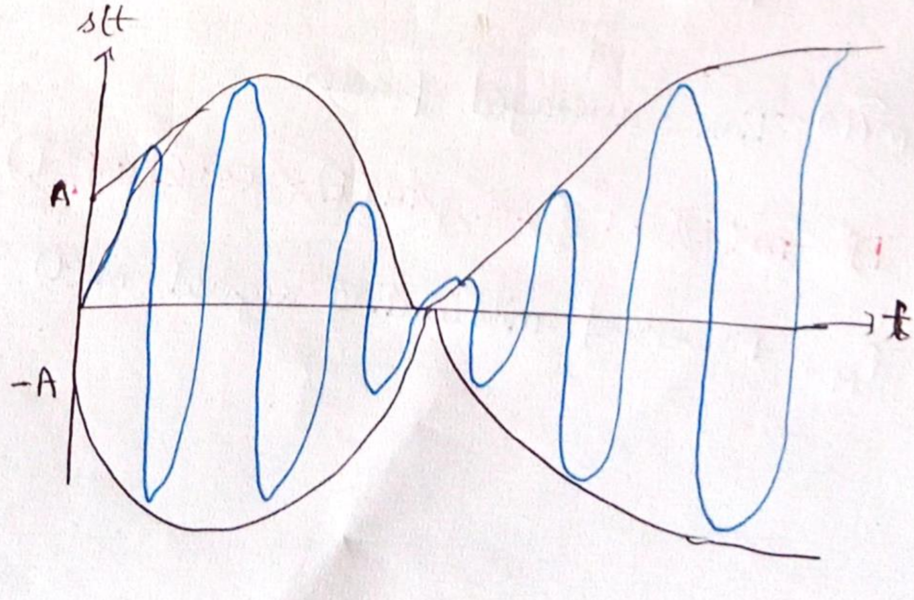
$\mu = \frac{B}{A}$

Example 1 $\mu = \frac{1}{2}$ i.e. $A = 2B$

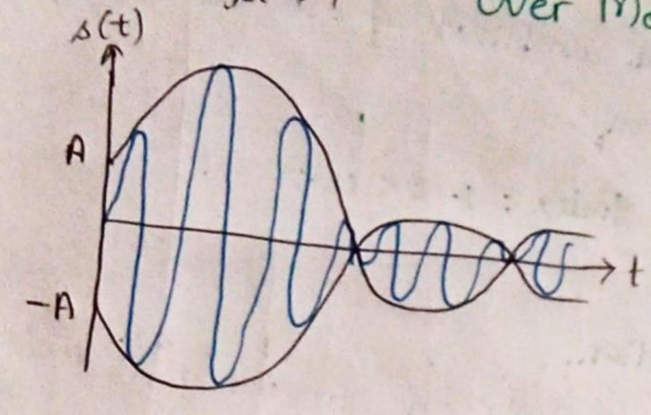




CASE2 $\mu = 1$ $A = B$



CASE 3 $\quad \mu > 1 \quad$ Over Modulation



If modulation index, $0 < \mu, 1$, then, we can easily recover the message signal using envelope detector.

What if $m(t) > 0$?

Then there is no need to add another DC component.

Modulated signal in terms of μ

$\cancel{s(t) = A\,[1 + \mu A]}$

$$s(t) = \overbrace{A\cos(2\pi f_c t)}^{\text{undesired Power}} + \overbrace{\mu A\, m(t)\cos(2\pi f_c t)}^{\text{Desired Power}}$$

$\cancel{m(t) = B\cos(2\pi f_m t)}$

$$\eta = \frac{\text{desired power}}{\text{total power}} = \frac{\frac{P_M}{2}}{\frac{P_M}{2} + \frac{A^2}{2}} = \frac{\frac{\mu^2 A^2\, \tilde{m}(t)}{2}}{\frac{\mu^2 A^2\, \tilde{m}(t)}{2} + \frac{A^2}{2}}$$

$\tilde{m}(t)$: Time - averaged power

~~If $\tilde{m}(t) = 1$, i.e., $m(t) = \cos(2\pi f_m t)$~~ $\quad \cancel{\eta = \frac{\mu^2}{\mu^2 + 1}}$

M : Power of modulating signal, $m(t)$.

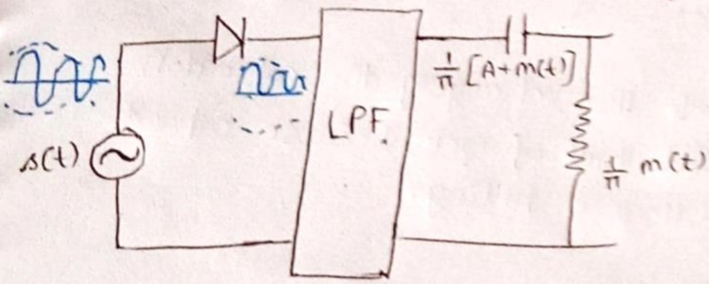
Eg. $m(t) = m_p \cos(2\pi f_m t) = \mu A \cos(2\pi f_m t) \rightarrow P_M = \frac{(\mu A)^2}{2}$

$$\boxed{\eta = \frac{\mu^2}{\mu^2+2}}$$

Max. value at $\mu = 1$: $\boxed{\eta = \frac{1}{3}}$ Max. efficiency for tone modulation

Rectifier Demodulator : Rectifier + LPF

AM helps to get demodulated signal easily at the cost of Power-efficiency.



Switching at regular intervals.

$s(t)\underbrace{w(t)}$ = AM × Switch

+ve cycle : 1 −ve cycle : 0

The output of low pass filter have a DC component 'A' and an attenuation of $\frac{1}{\pi}$. so, to block DC part a suitable capacitance can be used. so, output becomes $\frac{1}{\pi} m(t)$.

$s(t)\; w(t)$

$$= \underbrace{[A + m(t)] \cos(2\pi f_c t)}\left[\frac{1}{2} + \frac{2}{\pi}\left[\underbrace{\cos(2\pi f_c t)} - \frac{1}{3}\cos(6\pi f_c t) + \frac{1}{5}\cos(10\pi f_c t)\right]\right]$$

$$= \underbrace{\frac{1}{\pi}[A + m(t)]}_{\text{shift by } \pm f_c} + \cancel{\text{High frequency term}} \xrightarrow{\text{LPF}}$$

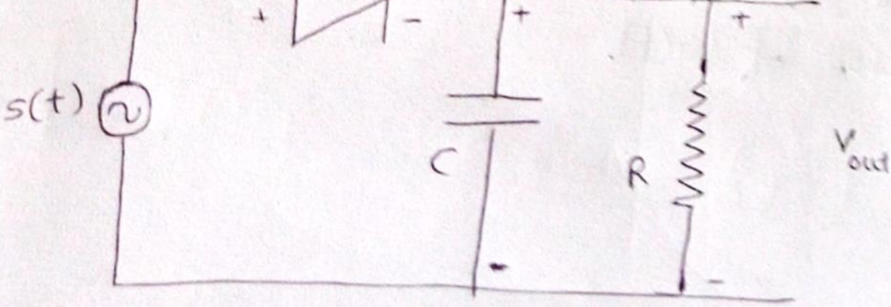
Cut off frequency of LPF should be freq of modulating signal, m(t).

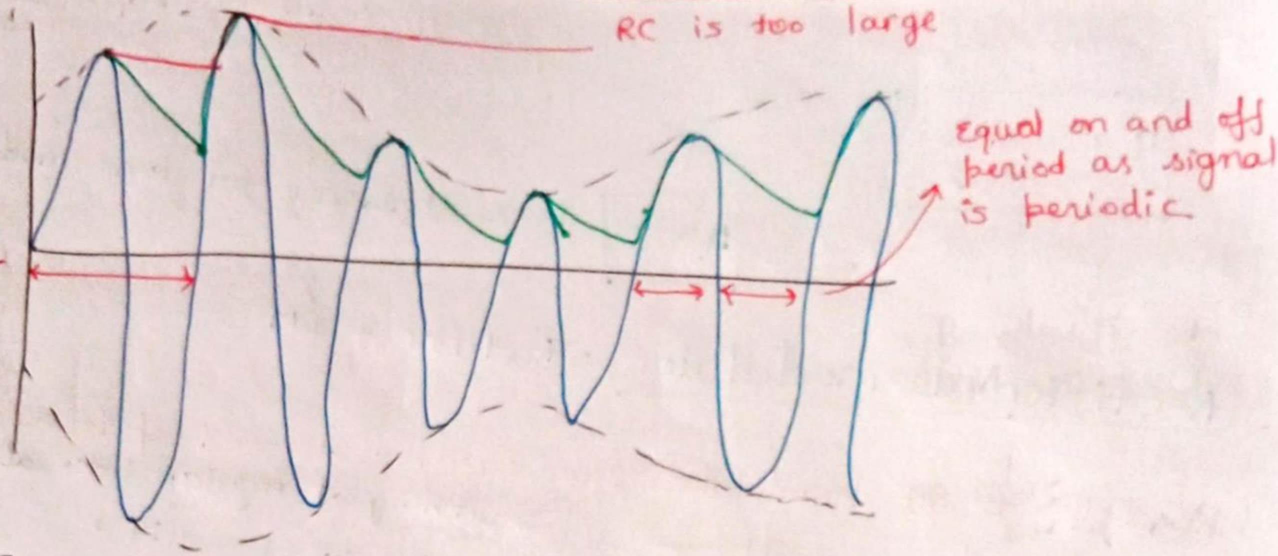
$$\frac{2}{\pi}[A + m(t)] \cos^2(2\pi f_c t) = \frac{2}{\pi}[A + m(t)]\left[\frac{1 + \cos(4\pi f_c t)}{2}\right]$$

$$= \frac{1}{\pi}[A + m(t)] + \text{Higher frequency terms}$$

Advantage: No synchronisation of carrier wave required

Envelope Demodulation





Working

Diode conducts in ~~+ve~~ incr half of cycle, thus, charging the capacitor.
Diode will be in reverse bias in ~~-ve~~ decr half of cycle, because capacitor is now charged and other node's voltage is falling.

Suitable values for R and C

→ We need to have large values for RC st. that there is a very large discharge time st. there is no drop in voltage across capacitor. This makes sure capacitor remains in reverse bias during decreasing half.

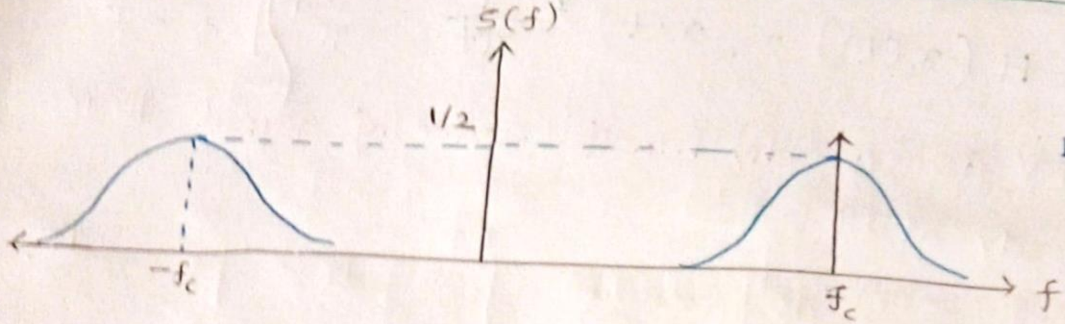
→ Also, we cannot make it too large as then it would be able to detect the forthcoming waveform.
So, its discharging time should be less than one period of modulating signal, $m(t)$.

→ Combining these two, we can write

$$\frac{1}{2\pi f_c} << RC << \frac{1}{2\pi f_m} \quad ??$$

Bandwidth of Amplitude Modulated signal $= 2B_m$
where, B_m is bandwidth of $m(t)$.

Single Side Band - Suppressed carrier (SSB - SC)

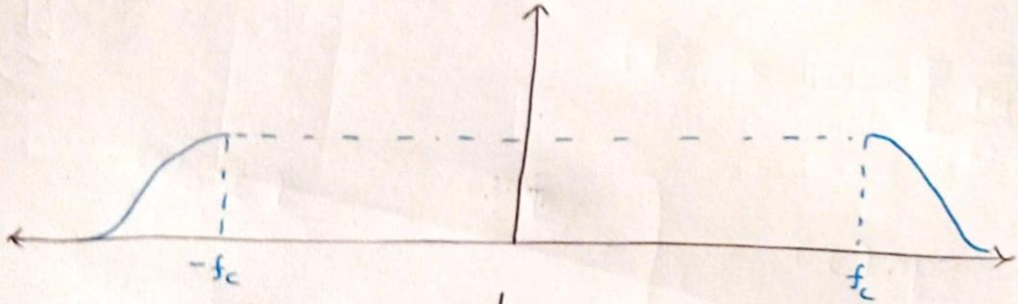


DSB-SC / AM

BW inefficient

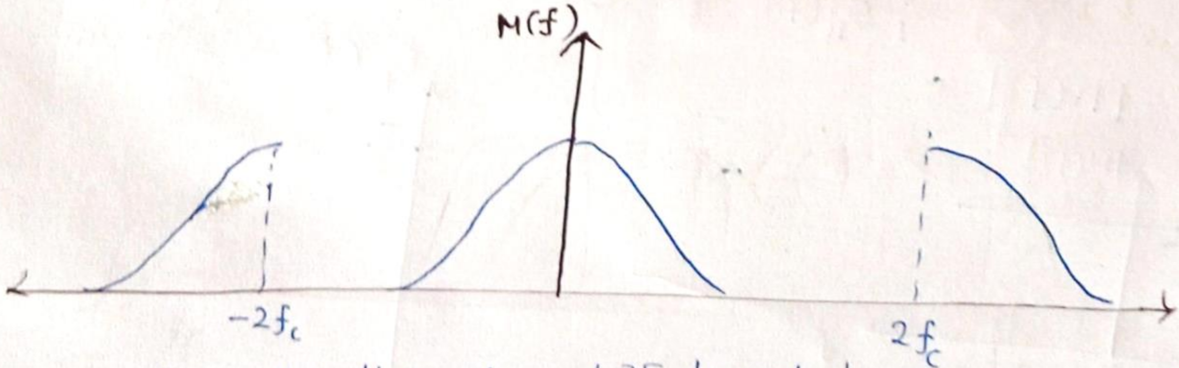
$B = 2B_M$

If somehow we can get the USB (Upper side Band), then, we can multiply with carrier wave to get back original signal.



USB

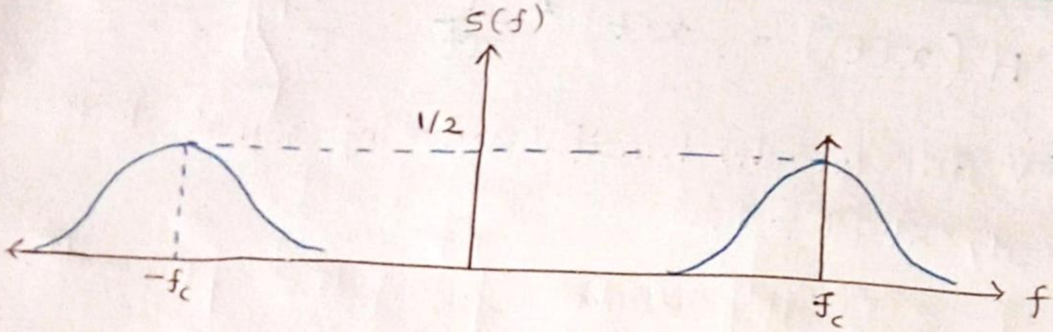
$\bigotimes \leftarrow \cos(2\pi f_c t)$ (Frequency shift of $\pm f_c$)



$\rightarrow$ LPF

We can pass this through a LPF to get back the signal.

Single Side Band- Suppressed carrier (SSB-SC)

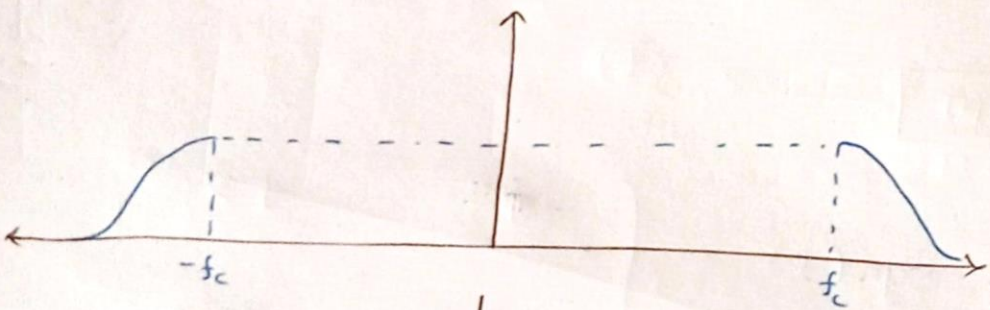


DSB-SC/AM

BW inefficient

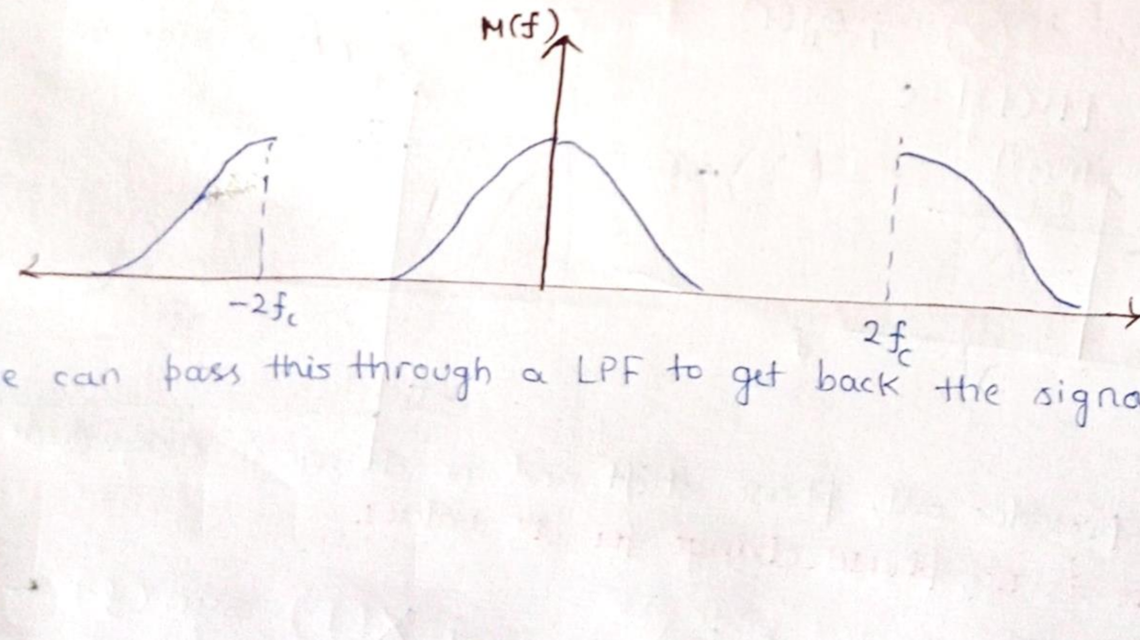
$B = 2B_M$

If somehow we can get the USB (Upper side Band), then, we can multiply with carrier wave to get back original signal.



USB

$\cos(2\pi f_c t)$ (Frequency shift of $\pm f_c$)



→ LPF

We can pass this through a LPF to get back the signal.

Hilbert Transform

$$x_h(t) = H(x(t)) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} PV\left[\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau\right]$$

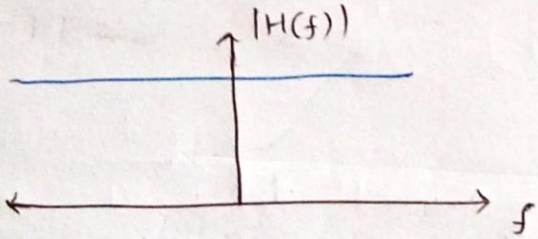
where PV is Cauchy Principal value of signal to remove singularity at $\tau = t$.

$$\frac{1}{\pi} PV\left[\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau\right] = \lim_{\epsilon \to 0} \int_{-\infty}^{t-\epsilon} \ldots d\tau + \int_{t+\epsilon}^{\infty} \ldots d\tau.$$

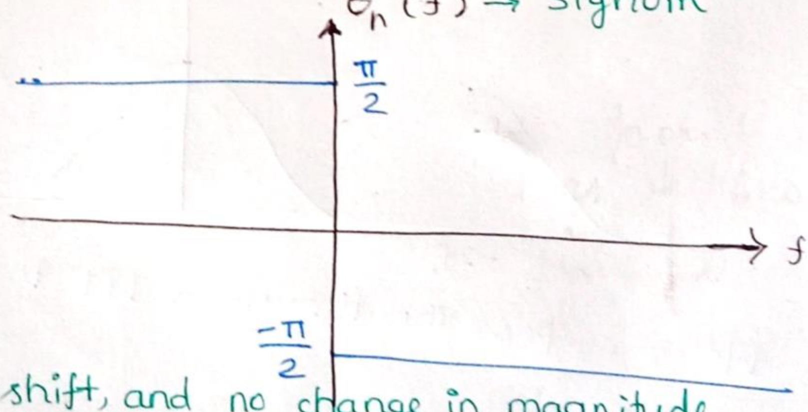
$$x(t) \longrightarrow \underset{H(f)}{\boxed{HT}} \longrightarrow x_h(t)$$

$$H(f) = -j\, \text{sgn}(f) = \begin{cases} e^{-j\frac{\pi}{2}}, & f > 0 \\ e^{j\frac{\pi}{2}}, & f < 0 \\ 0, & f = 0 \end{cases}$$

$$= |H(f)|\, e^{j\theta_h(f)}$$

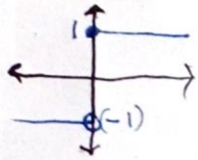


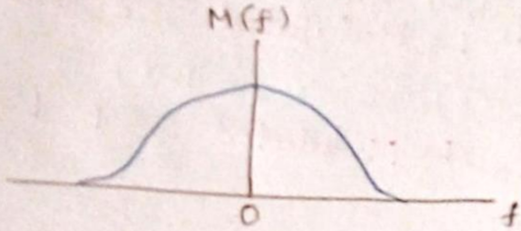
$\theta_h(f) \rightarrow$ Flipped signum

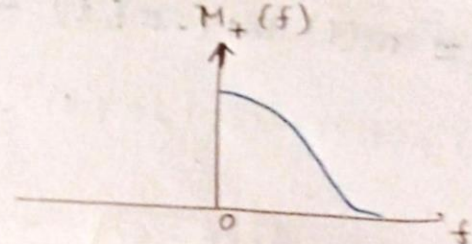


So, H(f) provides only phase shift, and no change in magnitude.
Also, there is no phase change for DC values.

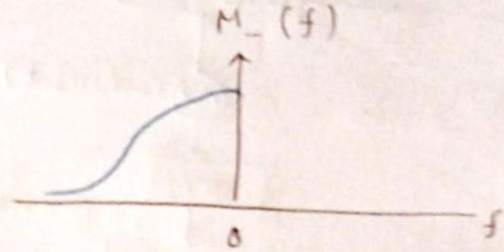
$$X_h(f) = X(f)\ FT\left(\frac{1}{\pi t}\right) = -j\, X(f)\, \underbrace{\text{sgn}(f)}$$







$$jX_h(f) = X(f)\,\mathrm{sgn}(f)$$



$$M_+(f) = M(f)\,u(f)$$

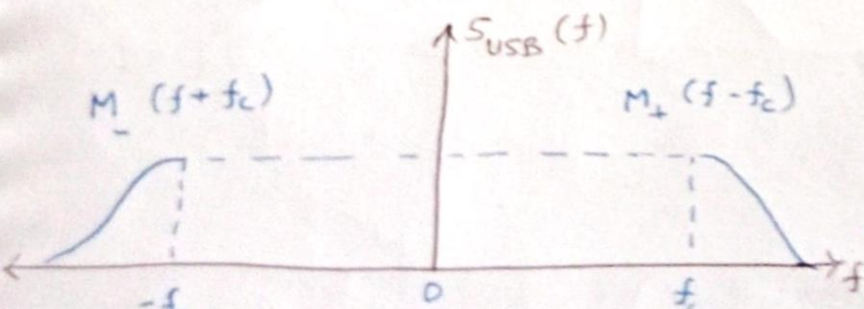
$$= M(f)\,\frac{1}{2}\left[1 + \mathrm{sgn}(f)\right]$$

$$= \frac{1}{2}\left[M(f) + j\,M_h(f)\right]$$

$$M_-(f) = M(f)\,u(-f)$$

$$= M(f)\,\frac{1}{2}\left[1 - \mathrm{sgn}(f)\right]$$

$$= \frac{1}{2}\left[M(f) - j\,M_h(f)\right]$$



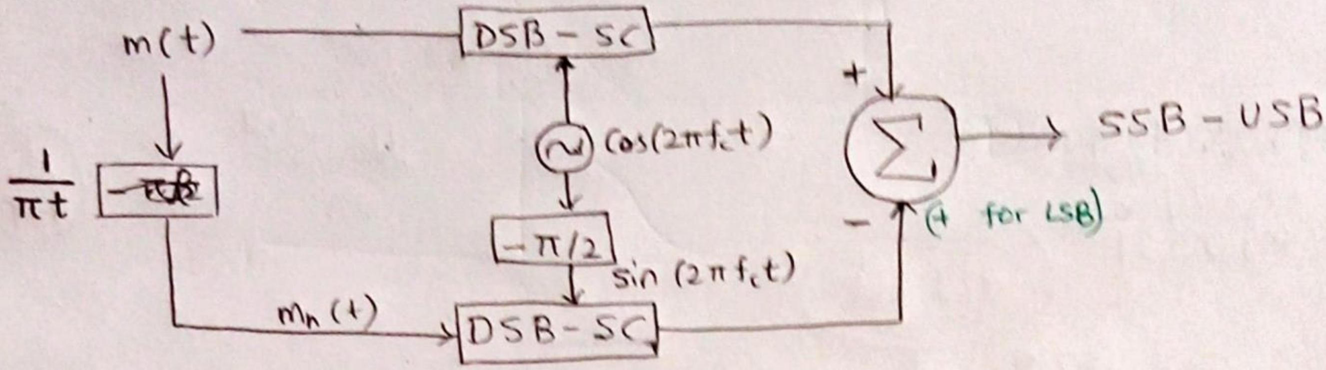
Repeat with LSB

$$S_{USB}(f) = M_+(f - f_c) + M_-(f + f_c)$$ SSB with USB

$$= \frac{1}{2}\left[M(f-f_c) + M(f+f_c)\right] - \frac{1}{2j}\left[M_h(f-f_c) - M_h(f+f_c)\right]$$

$$S_{USB}(t) = m(t)\cos(2\pi f_c t) - m_h(t)\sin(2\pi f_c t)$$
↳ Modulated SSB, USB

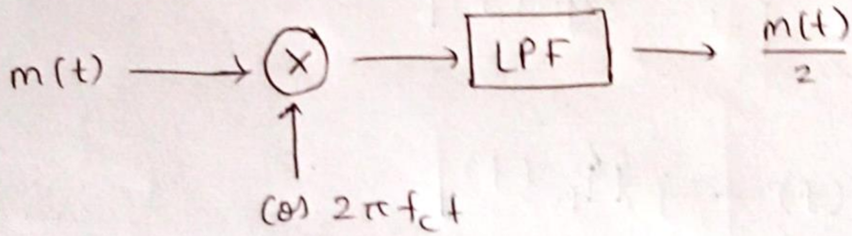
$$S_{LSB}(t) = m(t)\cos(2\pi f_c t) + m_h(t)\sin(2\pi f_c t)$$
↳ Modulated SSB, LSB



Demodulation

$s(t) * \cos(2\pi f_c t)$

$$= \frac{1}{2}m(t) + \cancel{\frac{1}{2}\cos(4\pi f_c t) - \frac{m_h(t)}{2}\sin(4\pi f_c t)} \rightarrow \text{LPF}$$



Advantage

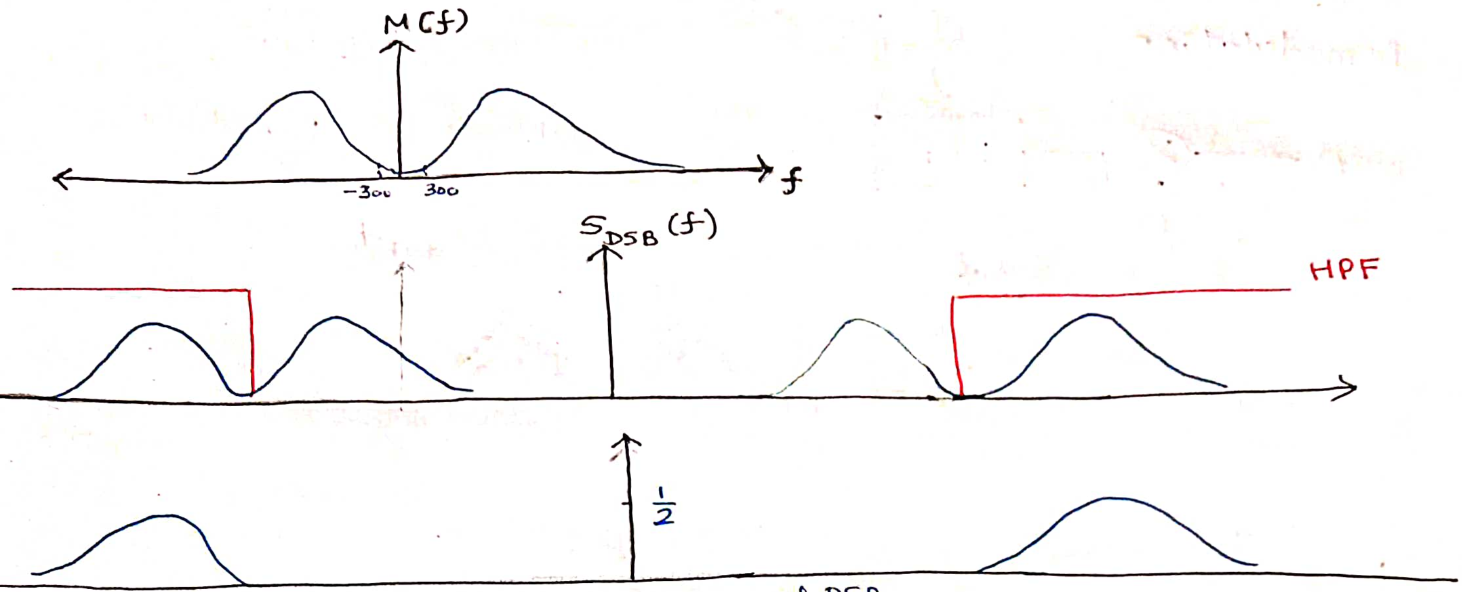
→ In SSB, Bandwidth is same

Disadvantage

→ Too much complexity at receiver
→ Hibert Transform is not realisable
→ Coherent Demodulation as we need to have same carrier frequency

Hilbert Transform is not causal ∵ $-\infty < t < \infty$.

Frequency Selective Modulator



→ We use an HPF to allow only USB of DSB to pass through. Even though the filter is real, it gives good results if lower frequency components aren't significant.

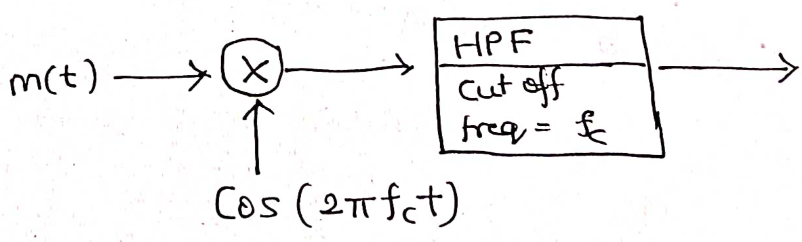
→ This method is particularly good for signals whose DSB is close to 0 at origin i.e. Frequency components nearby 0 has to have smaller components.

e.g. Audio Signal : Sound have freq. from 300 Hz – 3.4 kHz.

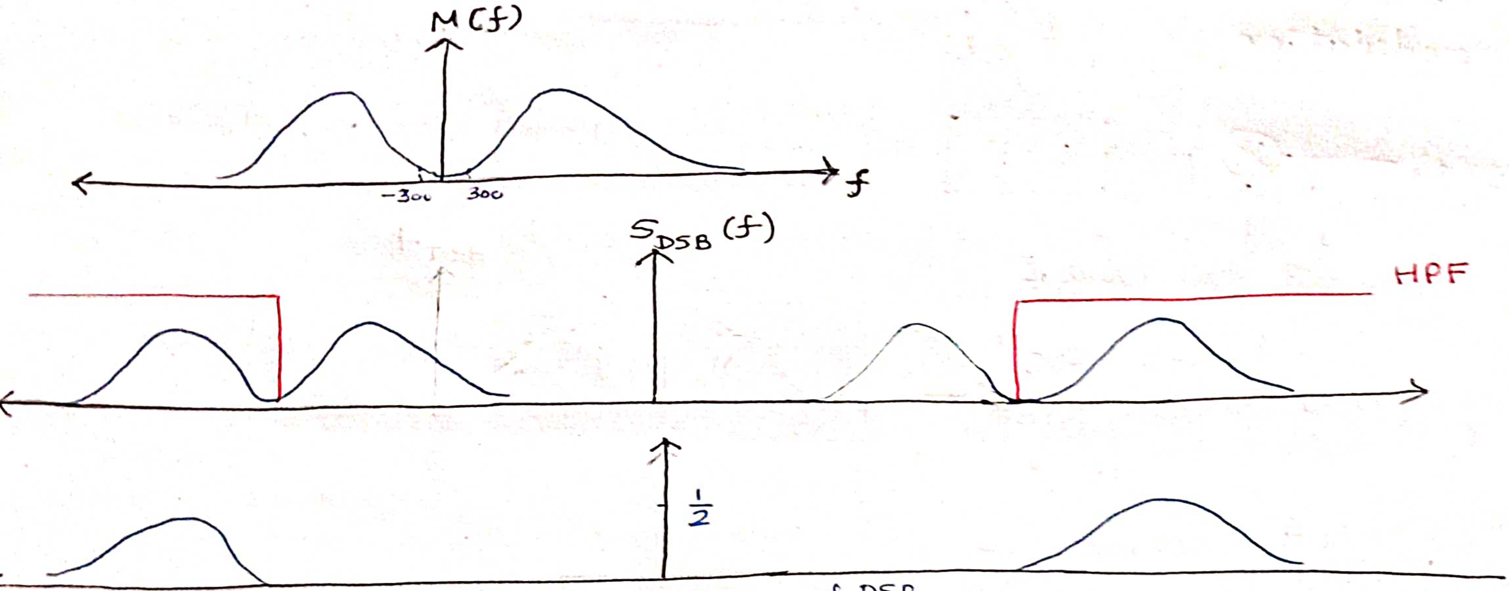
$B_S = B_M$ Simplicity ↑ Efficient ?

$P_S = \frac{P_M}{4}$

Block Diagram



Frequency Selective Modulator



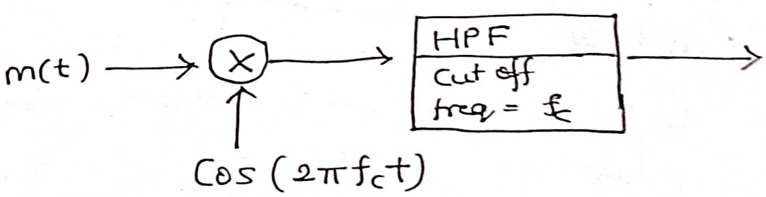
→ We use an HPF to allow only USB of DSB to pass through. Even though the filter is real, it gives good results if lower frequency components aren't significa

→ This method is particularly good for signals whose DSB is close to 0 at orig
i.e. Frequency components nearby 0 has to have smaller components
Eg. Audio Signal : Sound have freq. from 300 Hz – 3 4 kHz

$B_S = B_M$ Simplicity ↑ Efficient ?

$P_S = \frac{P_M}{4}$

Block Diagram



$$s(t) = m(t)\cos(2\pi f_c t) \mp m_h(t)\sin(2\pi f_c t)$$

(By LPF) ≈ 0

$$s(t)\cos(2\pi f_c t) = \frac{m(t)}{2} + \underbrace{\cancel{\frac{1}{2}m(t)\cos(4\pi f_c t) \mp \frac{m_h(t)}{2}\sin(4\pi f_c t)}}_{\text{SSB with Carrier as } 2f_c}$$

Demodulation

$$s(t) \longrightarrow \otimes \longrightarrow \boxed{\text{LPF}} \longrightarrow m(t)$$

$$\uparrow$$

$$2\cos 2\pi f_c t$$

Coherent DSB Demodulation

Disadvantage

→ Doppler's Effect Frequency shift in incoming signal

$$f_D = \frac{\lambda_c}{c}\cos\theta$$

$$f_R = f_c \pm f_D$$

λ_c: Carrier Wavelength

f_c: Carrier Frequency f_R: Received Frequency

f_D: Doppler's Frequency change

θ: Angle of Arrival

SSB with Carrier

$$s(t) = (A + m(t)) \cos 2\pi f_c t + m_h(t) \sin(2\pi f_c t)$$

Can we detect this using Envelope Detector ? Can we write this in terms of $m(t)$.
Yes, if we can write this

$$s(t) = \left[[A + m(t)]^2 + [m_h(t)]^2\right]^{1/2} \cos(2\pi f_c t + \theta)$$

$$a \cos x + b \sin x = R \cos(x - \alpha), \quad R = \sqrt{a^2 + b^2}$$

$$= A\left[1 + \underbrace{\frac{m^2(t)}{A^2}} + \frac{2m(t)}{A} + \underbrace{\frac{m_h^2(t)}{A^2}}_{\text{Square of}}\right]^{1/2} \cos(2\pi f_c t + \theta)$$

fractions are ignored assuming $\boxed{A >> m_p}$

$$\approx A\left[1 + \frac{2m(t)}{A}\right]^{1/2} \cos(2\pi f_c t + \theta)$$

Now we have Amplitude (Envelope) in term of m

$$\approx [A + m(t)] \cos(2\pi f_c t + \theta) \quad \text{(First Order Expansion)}$$

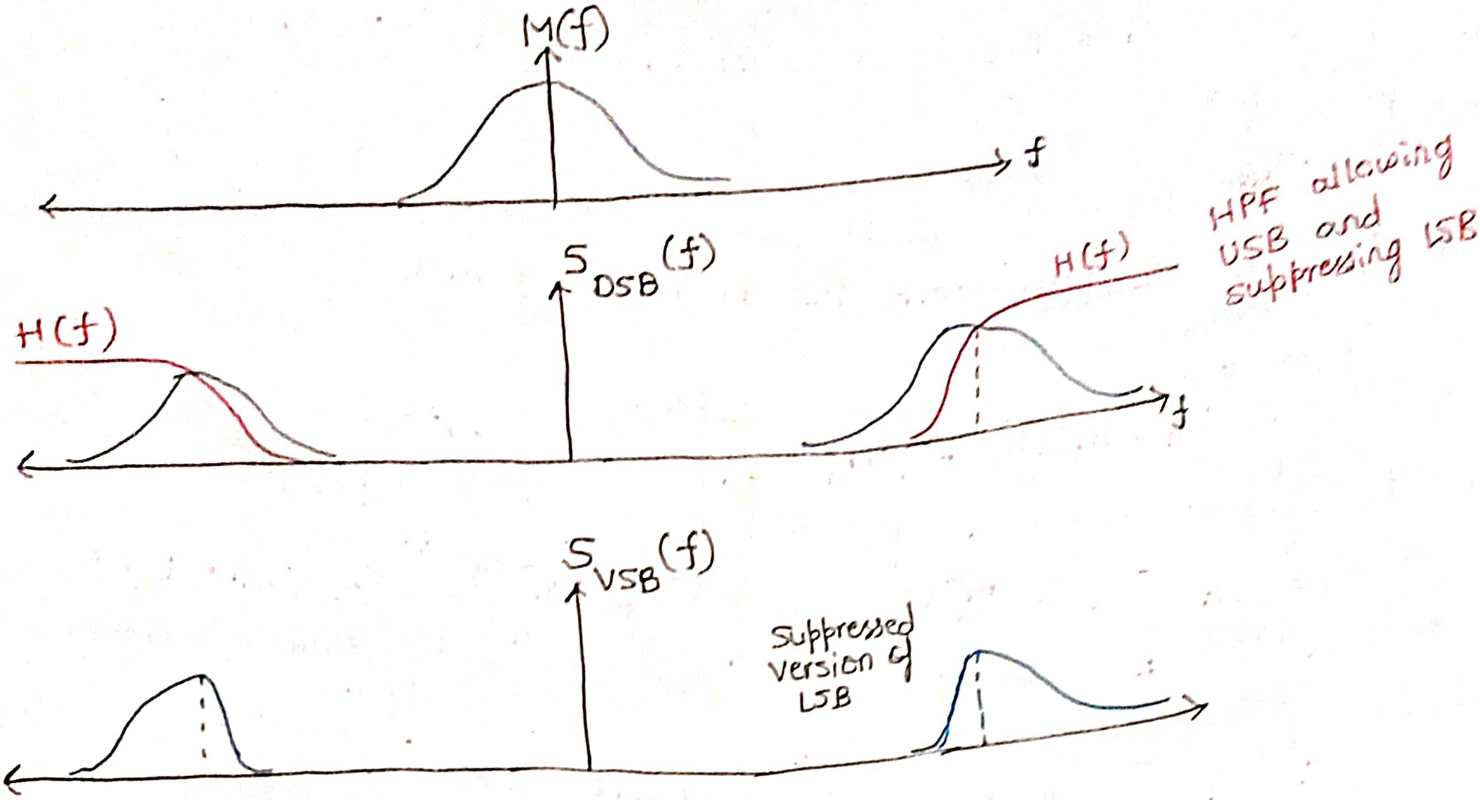
where, m_p : Peak-to-Peak amplitude of modulating signal.

As $A + m(t) > 0$ always as $A >> m_p$, we can use envelope detector for demodulation

Disadvantage Highly power inefficient as $A >> m_p$.

DSB $\rightarrow$ Simple Structure, High B.W. $= 2B_M$

SSB $\rightarrow$ High Complexity, Low B.W. $= B_M$



Advantage

→ Reduce Bandwidth: Depending on sharpness of HPF.

$B_S = 1.25 B_M$

Block Diagram

$m(t) \longrightarrow \boxed{DSB} \longrightarrow \boxed{HPF, H(f)} \longrightarrow S_{USB}(f)$

$$S_{VSB}(f) = \frac{1}{2}\left[M(f+f_c) + M(f-f_c) \right] H(f)$$

Coherent Demodulation Shift by $\pm f_c$.

$S_{VSB}(t) \longrightarrow \otimes \xrightarrow{X(t)} G(t) \longrightarrow m(t)$

$\uparrow$

$\cos(2\pi f_c t)$

$$X(f) = \frac{1}{2}\left[S_{VSB}(f+f_c) + S_{VSB}(f-f_c)\right]$$

(Shift of S_{VSB} by $\pm f_c$)

$$= \frac{1}{4}\left[\{M(f)+M(f-2f_c)\}H(f-f_c) + \{M(f)+M(f+2f_c)\}H(f+f_c)\right]$$

$$= \frac{M(f)}{2}\left[\frac{H(f+f_c)}{2} + \frac{H(f-f_c)}{2}\right] G(f)$$

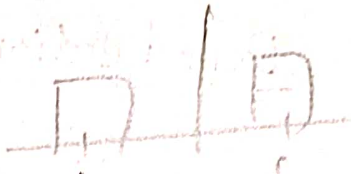
$G(f)$ → LPF

$$G(f) = \frac{1}{H(f+f_c)+H(f-f_c)} \quad \forall\ |f| < f_m$$

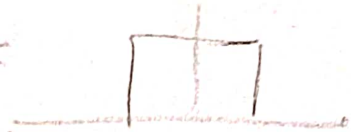
$H(f)$ – BPF

$H(f \pm f_c)$ – LPF

BPF



LPF



Quadrature Amplitude Modulation

incr BW

decr modulate demod structure

Transmitting 2 message signal over 2 BW.

∴ BW required for signal = B_M.

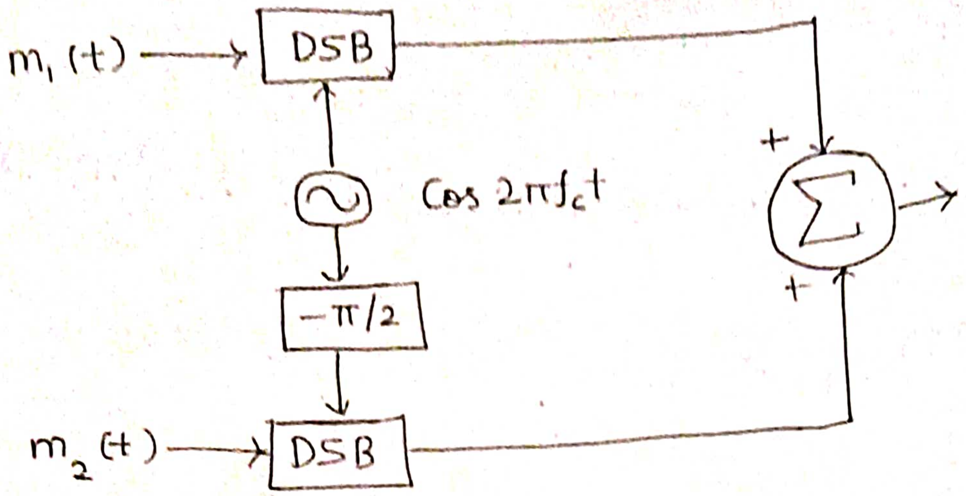
only if we can retrive signal

$$S_{SSB}(t) = m(t)\cos 2\pi f_c t + \hat{m}_h(t)\sin(2\pi f_c t)$$

$$S_{DSB}(t) = m(t)\cos(2\pi f_c t)$$

$$\underset{QAM}{S(f)} = \underbrace{m_1(t)\cos(2\pi f_c t)}_{\text{in phase component}} + \underbrace{m_2(t)\sin(2\pi f_c t)}_{\text{quadrature component}}$$

$2B_M$ $\quad$ $2B_M$ $\quad$ $2B_M$



Demodulation

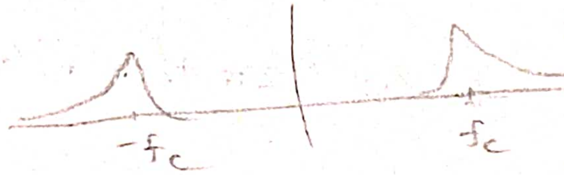
$$S_{QAM}(t)\ \sin(2\pi f_c t) = \frac{m_1(t)}{2}\ \sin(4\pi f_c t)$$

× cos

Sin

LPF

Demod

	P_S	B_S	Complexity.
DSB - SC	$\frac{P_M}{2}$	$2B_M$	Carrier Recovery required
AM / DSB - FC (Full Carrier)	$\frac{A^2}{2} + \frac{P_M}{2}$ $(A + m_p > 0)$	$2B_M$	Reduced complexity via use of envelope detector.
SSB	$\frac{P_M}{4}$	B_M	Modulation requires HT which increases complexity.
VSB (Vegistial Side Band)	$\frac{P_M}{2} > P_S > \frac{P_M}{4}$	$2B_M > B_S > B_M$	

PLL : Recovers carrier.

Generating FM signal

↳ VCO : voltage Controlled Oscillator

voltage ... change, const i/p voltage

Frequency Modulation

changing $f_i(t)$ linearly with modulating signal

$$f_i(t) = f_c + K_f\, m(t) \qquad -m_p < m(t) < m_p$$

$$= f_c \pm K_f\, m_p$$

$$A \cos(2\pi f_c' t + \theta_0)$$

$$A \cos(\theta(t))$$

$$f_i(t) = \frac{1}{2\pi} \frac{d\,\theta(t)}{dt} \qquad \theta(t) = 2\pi \int_{-\infty}^{t} f_i(t)\,dt$$

Int. Freq is a derivative of Int. angle

Phase Modulation

$$\theta(t) = 2\pi f_c t + \cancel{\theta_0}^{\,0} + \cancel{2\pi}\, K_p\, m(t)$$

$$S_{PM}(t) = A \cos(2\pi f_c t + \cancel{2\pi}\, K_p\, m(t)) \qquad \text{PM - SM}$$

$$f_i(t) = \frac{1}{2\pi} \frac{d\,\theta(t)}{dt} = f_c + \frac{K_p\, m(t)}{2\pi}$$

PM ⇒ int. freq $\propto$ m(t)

Frequency Modulation (FM)

$$f_i(t) = f_c + \frac{k_f}{2\pi} m(t)$$

$$\theta(t) = 2\pi \int_{-\infty}^{t} f_i(t)\,dt = 2\pi f_c t + k_f \int_{-\infty}^{t} m(\tau)\,d\tau$$

$$S_{FM}(t) = A \cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right)$$

$m'(t) = m(t)$

$m(t)$

FM : Int ~~Freq~~ Angle $\propto \int m(\tau)\,d\tau$

$$m(t) \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{m(t)} \boxed{FM} \longrightarrow S_{PM}(t)$$

$$m(t) \longrightarrow \boxed{\int (\,).dt} \xrightarrow{\int_{-\infty}^{t} m(\tau)d\tau} \boxed{PM} \longrightarrow S_{FM}(t)$$

FM: $\theta(t) \propto \int_{-\infty}^{t} m(\tau)\,d\tau = u(t) \circledast m(t)$

PM: $\theta(t) \propto m(t) = \delta(t) \circledast m(t)$

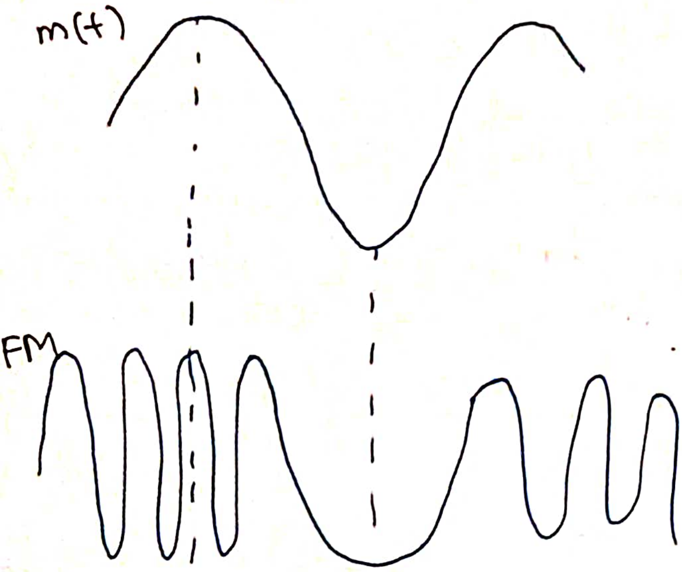
$$m(t) \longrightarrow \boxed{h(t)} \longrightarrow \theta(t) = m(t) \circledast h(t)$$

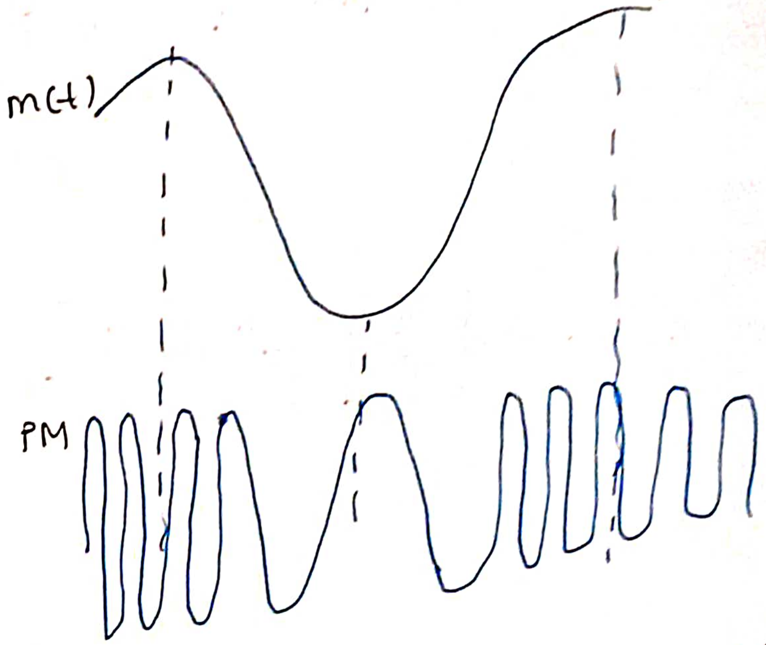
FM: $K_f \; u(t)$

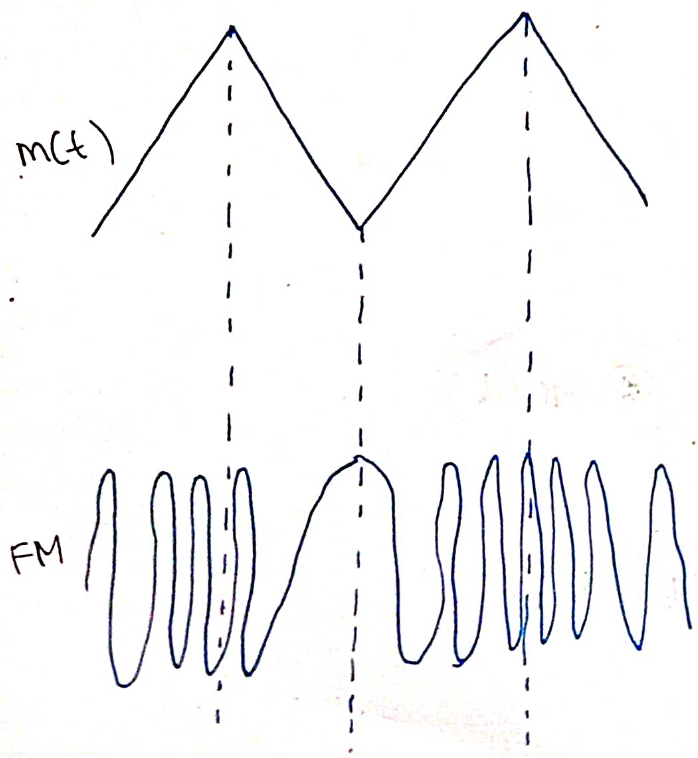
PM: $k_f \; \delta(t)$

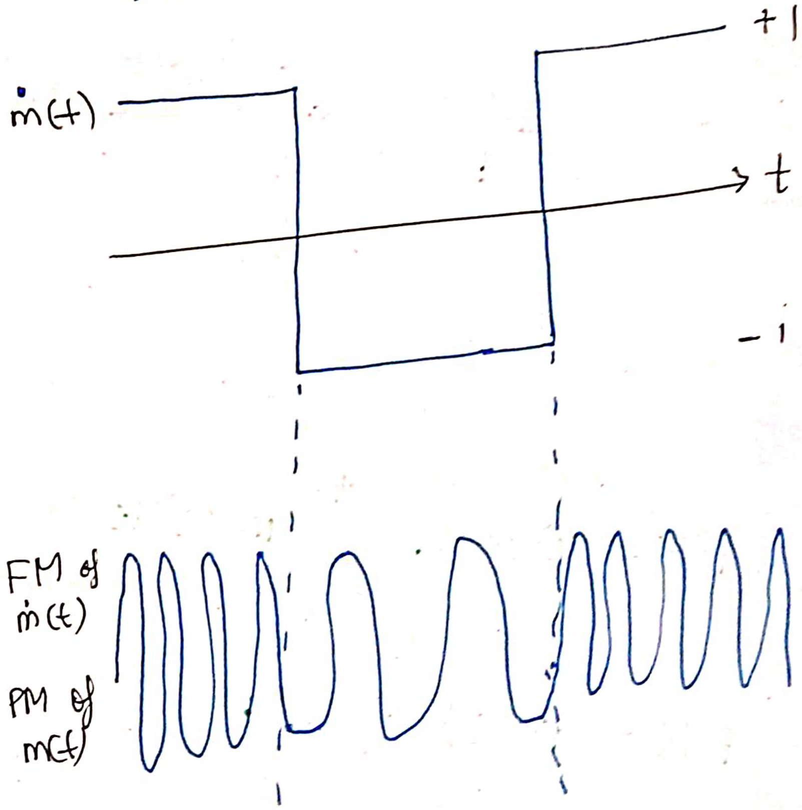
FM: $f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$

PM: $f_i(t) = f_c + K_p \, m(t)$









PM of $\dot{m}(t)$ At transition, 180° phase shift

is impulses

—

Non-linear mdt so, no FFT to determine freq components to find BW.

FM / PM : $\cancel{P_S = \frac{P_M}{2}}$?

$$\boxed{P_{FM} = P_{PM} = \frac{A^2}{2}}$$

$S_{FM}(t) =$

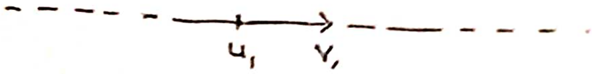
Tutorial 22 Feb 2023

Gram Schmidt's Orthogonalisation

$V = \{ v_1, v_2, \ldots\ldots, v_n \} \xrightarrow{GSO} \{ u_1, u_2, \ldots, u_n \}$

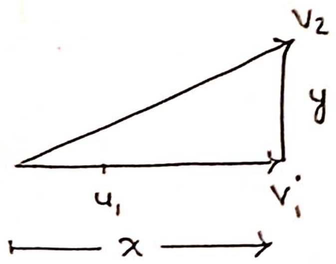
Linearly Independent Basis — Orthogonal Basis.

1-D



$$u_1 = \frac{\vec{v_1}}{\|v_1\|}$$

2-D



$v_2 = c u_1 + y$

$y = v_2 - x$

$= v_2 -$ proj. of v_2 on v_1

$= (v_2 \cdot v_1) \cdot v_1$

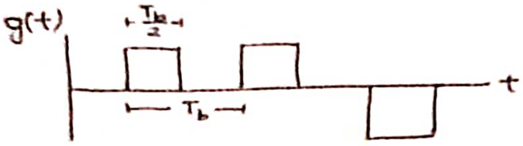
$$u_2 = \cancel{v_2}\, \frac{y}{\|y\|}$$

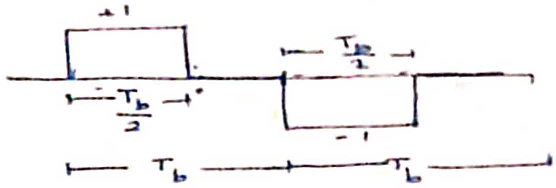
3-D

$x_3 - \cancel{x}$

Pulse Train, $g(t)$. Bit Time $\frac{T_b}{2}$, Bit changes every T_b sec.

$1 \rightarrow +1$
$0 \rightarrow -1$ } These two symbols are equally likely and occur randomly. Find PSD





Solⁿ:

$$R_g(\tau) \xleftrightarrow{FT} PSD$$

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int g(t)\, g(t-\tau)\, dt$$ Auto-correlation for random signal

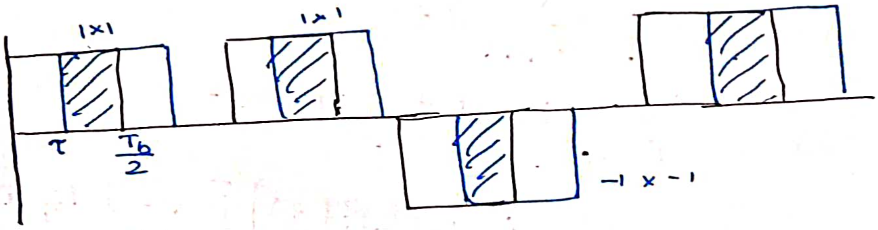
$T = N\, T_b$ where, N is number of pulses.

$T \rightarrow \infty$ as $N \rightarrow \infty$

$$R_g(\tau) = \lim_{N\to\infty} \frac{1}{N T_b} \int g(t)\, g(t-\tau)\, dt$$

CASE 1 $\tau < \frac{T_b}{2}$ CASE 2 $\tau > \frac{T_b}{2}$

CASE 1

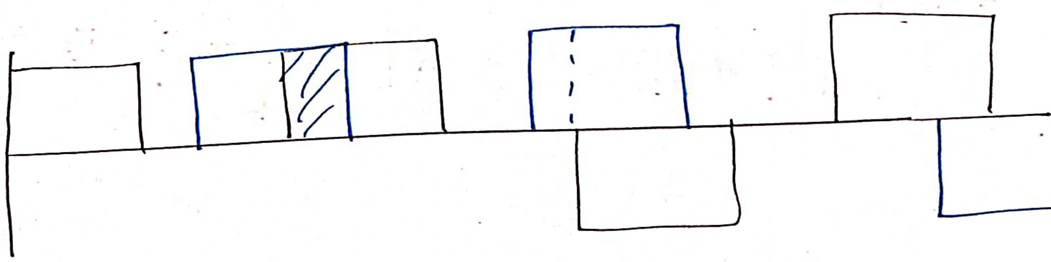


Same symbols are overlapping.

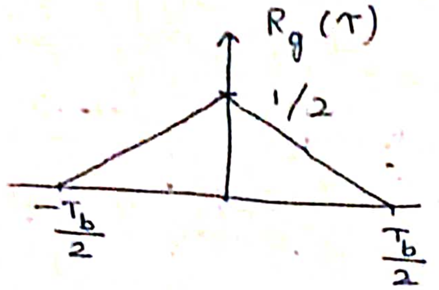
CASE 2

Next Symbol Overlap.

Hence, sum would average out to 0.



$$\frac{1}{T_b}\left(\frac{T_b}{2} - \tau\right) = \frac{1}{2} - \frac{\tau}{T_b} = \underbrace{\frac{1}{2}\left(1 - \frac{2\tau}{T_b}\right)}_{\text{Triangular pulse}}$$



can't distinguish b/w PM & FM

Discontinuous - changes derivative - makes PM, FM distinguishable.

$m(t) \to \boxed{\int} \to a(t)$

$A(f) = \frac{1}{j\pi f} M(f)$

Linear operator.

$B = B_M$

$$Q_{FM}(t) = A \cos(2\pi f_c t + K_f\, a(t)), \text{ where } a(t) = \int_{-\infty}^{t} m(\tau)\,d\tau$$

BM $\xleftrightarrow{\text{same}}$ BM

$$= \text{Re}\{\hat{\phi}_{FM}(t)\}.$$

(Representing in terms of complex exponentials)

here, $\hat{\phi}_{FM}(t) = A\, e^{j2\pi f_c t}\, e^{jK_f a(t)}$

$$= A\left[\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right]\left[1 + jK_f a(t) - \frac{(K_f a(t))^2}{2!} - j\frac{(K(f)a(t))^3}{3!} + \ldots\right]$$

$$_M(t) = \text{Re}[\phi_{FM}(t)] = A\Big[\underbrace{\cos(2\pi f_c t)}_{B} - \underbrace{K_f\, a(t)\sin(2\pi f_c t)}_{2B} \underbrace{- \frac{(K_f\, a(t))^2}{2!}\cos(2\pi f_c \ldots) + \ldots}_{4B}\Big]$$

DSB with a(t) as modulating signal centered around f_c

$K_f \cdot m_p$ (p-p amplitude) decides what frequency components would be present in FM signal.

$$\frac{[K_f\ a(f)]^n}{n!}$$

It has a bandwidth of $2nB$.

<u>CASE I:</u> $|K_f m_p| << 1$

$$\phi_{FM}(t) \simeq A \cos(2\pi f_c t) - A K_f\ a(t) \sin(2\pi f_c t)$$

Narrow Band FM Modulation (NBFM) Narrow Band

$$B_{FM} = 2 B_M$$

As the two terms in FM are out of phase, they cancel each other to give constant amplitude.

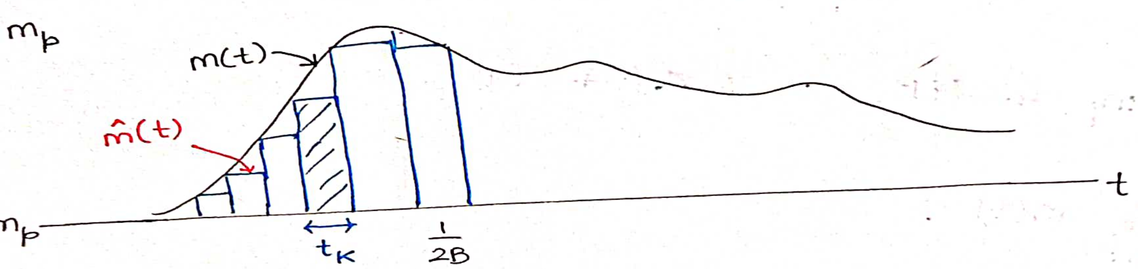
Wide Band Frequency Modulation (WBFM)

<u>CASE 2:</u> $|K_f m_p| >> 1$

Approx signal with known signal (rect

rect - sinc

Modulating carrier by approximated rect signal modulating signal



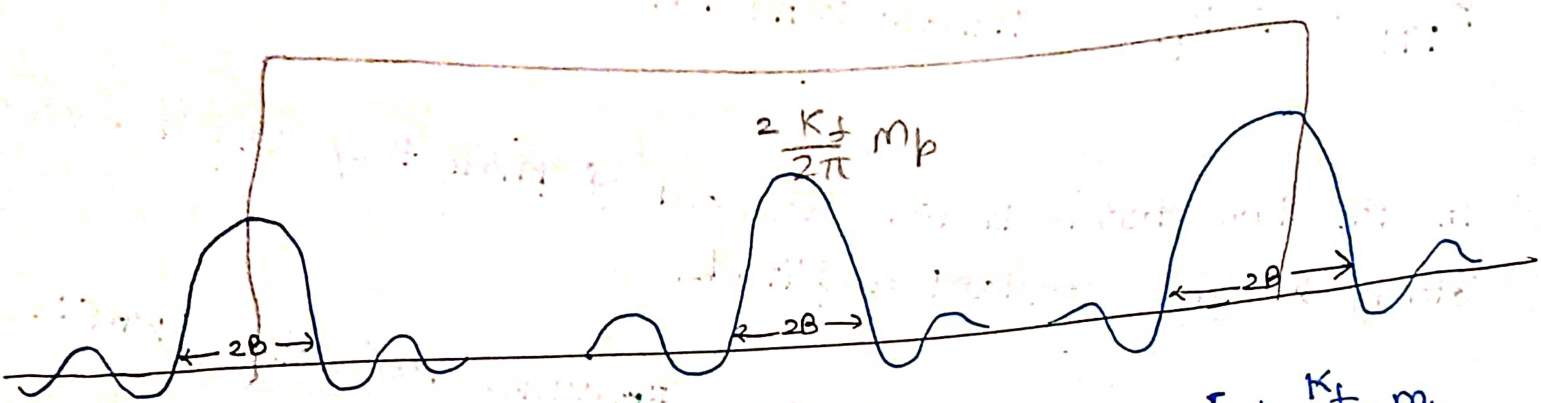
$$f_i(t_K) = f_c + \frac{K_f}{2\pi} \hat{m}(t_K)$$

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1, & \text{for } |t| < \tau \\ 0, & \text{otherwise} \end{cases} \longleftrightarrow \tau \,\text{sinc}\left(\frac{f\tau}{2}\right)$$

$$\Pi(t\,2B)\cos\left(2\pi f_c t + K_f \hat{m}(t)\right) \longleftrightarrow \frac{1}{2}\,\text{sinc}\left(2\pi f \pm 2\pi f_c + K_f \hat{m}(t_k)\right)$$

$$\boxed{\tau = \frac{1}{2B}}$$

$f_c - \frac{K_f}{2\pi} m_p$ $\qquad$ $f_c + \frac{K_f}{2\pi} m_p$



$f_c - \frac{K_f}{2\pi} m_p$ $\qquad$ $f_c + \frac{K_f}{2\pi}\hat{m}(t_k)$ $\qquad$ $f_c + \frac{K_f}{2\pi} m_p$

$$B_{\hat{\phi}} = \frac{2K_f}{2\pi} m_p + 4B = 2\Delta f + 4B$$

$$f_i(t) = f_c + \frac{K_f}{2\pi} m(t)$$

$$\Delta f = \frac{K_f}{2\pi}\frac{[m_p - (-m_p)]}{2} = \text{Frequency Deviation} = \frac{K_f m_p}{2\pi}$$

As we did stair case approximation, we overestimated bandwith.
$\because$ as rectangular pulse have more abrupt changes, it has more frequency components. Hence, overestimation.

Staircase Approx

$$B_{\phi} \in [\, 2\Delta f \text{ to } 2\Delta f + 4B \,]$$

CASE I : $\Delta f \approx 0 \Rightarrow B_{\hat{\phi}} = 4B$

$K_f m_p << 1$ NBFM $\phi_{NBFM} = 2B$

$$\hat{B}_{\phi} \approx 2\Delta f + 2B$$

$$= 2\beta B + 2B = 2B(\beta + 1)$$

Carson's Rule
Carson's Bandwidt

Modulation Index, $\boxed{\beta = \frac{\Delta f}{B}}$