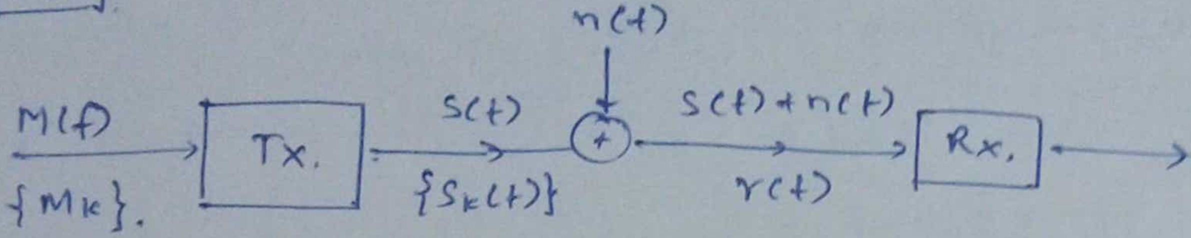
24/04/2023

M-ary



$k = 1, 2, \dots M$

$$S_k(t) = \sum_{j=1}^{N} S_{kj}\, \phi_j(t) \qquad \boxed{N \leq M}$$

$$r(t) = S_k(t) + n(t) \begin{cases} \to n'(t) \\ \to n''(t) \end{cases}$$

$$r(t) = \sum_{j=1}^{N} S_{kj} \cdot \phi_j(t) + \sum_{j=1}^{\infty} n_j\, \phi_j(t)$$

$$\boxed{r(t) = \underbrace{\sum_{j=1}^{N} (S_{kj} + n_j)\, \phi_j(t)}_{q(t) = S_k(t) + n'(t)} + \underbrace{\sum_{j=N+1}^{\infty} n_j\, \phi_j(t)}_{\text{irrelevant noise}}}$$

$$r(t) = S_k(t) + n'(t) + n''(t)$$

$\boxed{q = S_k + n'}$ where $S_k = [S_{k1}\ S_{k2} \dots S_{kN}]$

let $n = n'$ (Notation) $n' = [n_1'\ n_2' \dots n_N']$

$$S_{kj} = \int_0^{T_s} S_k(t) \cdot \phi_j(t)\, dt \quad \text{and} \quad n_j = \int_0^{T_s} n(t)\, \phi_j(t)\, dt$$

Now $\mathbb{E}[n_j] = 0$ (clearly).

$$\mathbb{E}[n_j n_k] = E\left[\int_0^{T_s} n(\tau)\, \phi_j(\tau)\, d\tau \int_0^{T_s} n(\beta)\, \phi_k(\beta)\, d\beta\right]$$

$$= E\left[\iint n(\tau)\, n(\beta)\, \phi_j(\tau)\, \phi_k(\beta)\, d\tau\, d\beta\right]$$

$$= \iint E[n(\tau)\, n(\beta)]\ \phi_j(\tau)\ \phi_k(\beta)\ d\tau\ d\beta$$

$$= \iint \underbrace{R_n(\tau-\beta)}_{\hookrightarrow \frac{N}{2}\delta(\tau-\beta)}\ \phi_j(\tau) \cdot \phi_k(\beta)\ d\tau\ d\beta$$

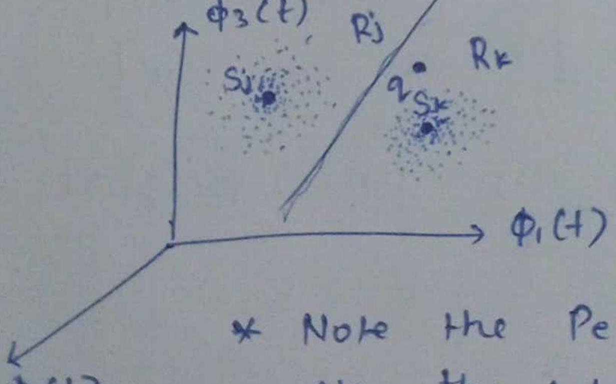
$$= \frac{N}{2}\int_0^{T_s} \phi_j(\tau)\ \phi_k(\tau)\ d\tau$$

$$E[n_j n_k] = \begin{cases} N/2 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases} \qquad C = \begin{bmatrix} N/2 & 0 & \cdots & 0 \\ 0 & N/2 & & \vdots \\ \vdots & & N/2 & \vdots \\ 0 & \cdots & & N/2 \end{bmatrix}$$

$$\underset{\substack{\uparrow \\ \text{joint dist}}}{P_n(n)} = \prod_{j=1}^{N} \frac{1}{\left(\sqrt{2\pi\frac{N}{2}}\right)^N} \exp\left(-\frac{\overset{\Sigma \hookrightarrow}{n_j^2}}{N}\right)$$

$$\boxed{P_n(n) = \frac{1}{(\pi N)^{N/2}} \exp\left(-\frac{\|n\|^2}{N}\right)}$$

Now suppose $N=3$ (3 basis functions)



$\boxed{q = s_k + n.}$

$$P_e(S_k) = \int_{R_j} P(q|s_k)\, dq$$

* Note the P_e directly increases with N

* Also, the total no. of Regions are M

if $q = s_k + n$

then $\mathbb{P}[c|q] = P[m_k|q]$

$$\mathbb{P}[c] = \int P(c|q) \cdot P(q)\,dq = \int P(m_k|q)\,P(q)\,dq$$

$\uparrow$ Probability of correct detection (maximize this)

since $P(q)$ is not in our hand, we max. $P(m_k|q)$

$$\max \mathbb{P}[c] \equiv \max P(m_k|q)$$

$\hookrightarrow$ Aposteriori Prob

$P(m_k) \Rightarrow$ Prior Prob.

Let $\hat{m}$ be estimated sym.

$\Rightarrow$ $\boxed{\hat{m} = m_k}$ if $P(m_k|q) \geq P(m_j/q)\ \forall\ j \in [1, M]$.

Thus, we would have to evaluate all $P(m_j|q)$

This decoder is called MAP decoder.

$$P(m_k|q) = \frac{P(q|m_k) \cdot \cancel{P(m_k)} \to \frac{1}{M}}{\cancel{P(q)} \to \text{const}}$$

$$\max_k P(m_k|q) \equiv \max_k P(q|m_k)$$

$\hookrightarrow$ M.L. decoder

* equivalent to MAP (under condition $P(m_k) = 1/M$)

Now we need to $\max\limits_k \mathbb{P}_q(q|m_k)$. we have

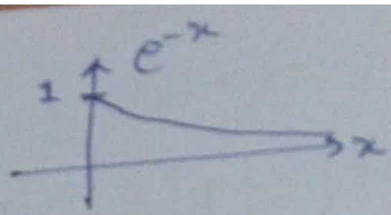
$$P_q(q|m_k) = P_n(\tilde{q} - s_k) \qquad q = s_k + n \quad \text{or} \quad n = q - s_k$$

$q = \tilde{q}$

$$\Rightarrow P_q(q|m_k) = \frac{1}{(\pi N)^{N/2}} \exp\left(-\frac{\|\tilde{q} - s_k\|^2}{N}\right)$$

$$P(m_k) \cdot P(q|m_k) = \frac{P(m_k)}{(\pi N)^{N/2}} \exp\left(-\frac{\|\tilde{q} - s_k\|^2}{N}\right) = v\ (\text{say})$$

e^{-x} is max. when x is close to 0



thus if $P(m_k) = 1/M$

we need to choose s_k closest to $\tilde{q}$

taking $\log(V)$

$$\log(V) = \log\left(\frac{P(m_k)}{(\pi N)^{N/2}}\right) - \frac{\|\tilde{q} - s_k\|^2}{N}$$

$$\frac{N}{2}\log(V) = \underbrace{\frac{N}{2}\log\left(\frac{P(m_k)}{(\pi N)^{N/2}}\right)}_{a_k \text{ (say)}} - \frac{\|\tilde{q} - s_k\|^2}{2}$$

$$\frac{N}{2}\log(V) = a_k - \frac{1}{2}\|\tilde{q} - s_k\|^2$$

$$= \underbrace{a_k - \frac{1}{2}\|\tilde{q}\|^2 - \frac{1}{2}\|s_k\|^2}_{b_k} + \langle \tilde{q}, s_k \rangle$$

$$= b_k + \langle \tilde{q}, s_k \rangle$$

b_k
- prob of m_k
- PSD of noise
- energy in s_k

$$\Rightarrow \boxed{\hat{m} = \arg\max_{m_k} \left(b_k + \langle \tilde{q}, s_k \rangle\right)}$$

optimal Reciever for AWGN.

25/04/2023.

we had got.

$$\hat{m} = \underset{m_k}{\operatorname{argmax}} \quad b_k + \langle q, s_k \rangle$$

Now we need to calculate $\langle q, s_k \rangle \ \forall \ k \in [1, M]$

$$= \int_{-\infty}^{\infty} q(\tau)\, h(T_m - \tau)\, d\tau$$

also we know that since vector spaces in $q_1(t)$ & n'' are ortho.
includes n'

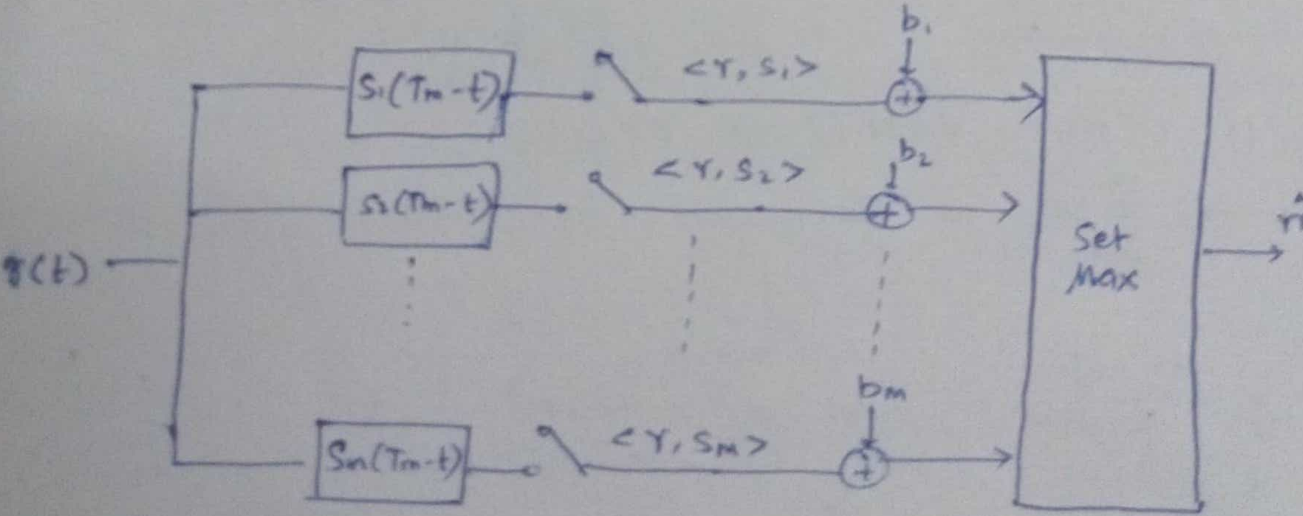
$$\Rightarrow \int n''(t) . s_k(t)\, dt = 0$$

we have $q(t) = r(t) + n''(t)$

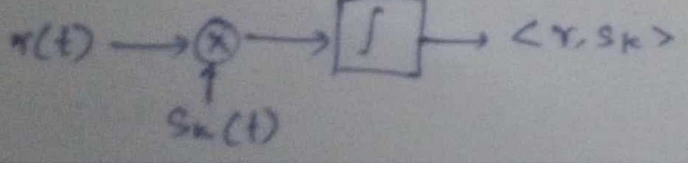
so we can say.

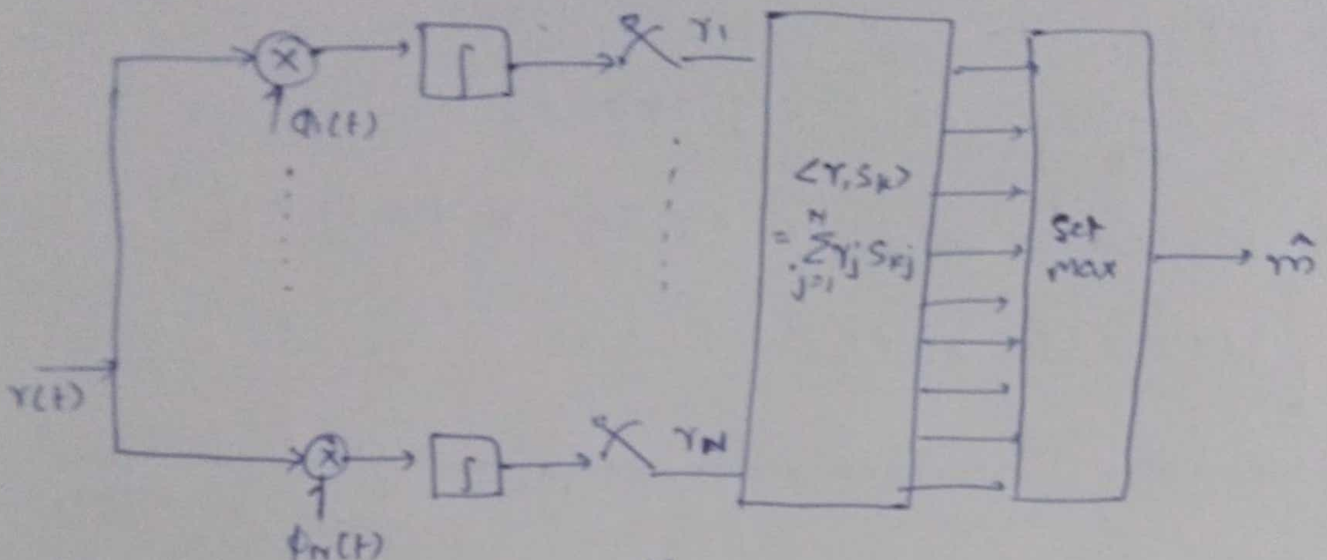
$$\langle q, s_k \rangle = \int_0^{T_m} q(\tau)\, s_k(\tau)\, d\tau = \int_0^{T_m} r(\tau) . s_k(\tau)\, d\tau = \langle r, s_k \rangle$$

we can implement this dot product using match filter.



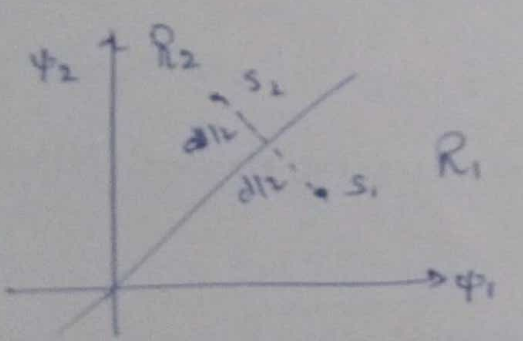
we can do dot product in an alternative way.





$\boxed{N = 2^m}$ $\quad r = \sum_{j=1}^{N} r_j \phi_j$

$$\langle r, s_k \rangle = \sum_{j=1}^{N} r_j s_{kj}$$



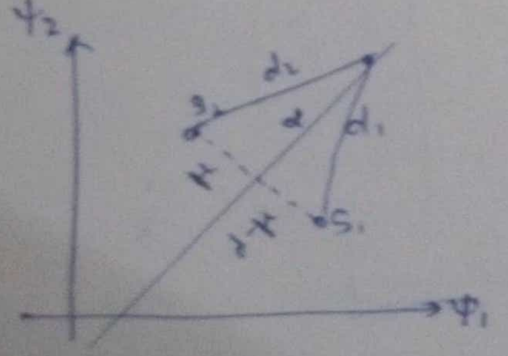
$$\frac{N_0}{2} \ln P(m_1) - \underbrace{\|q - s_1\|^2}_{d_1} \underset{m_2}{\overset{m_1}{\gtrless}} \frac{N}{2} \ln P(m_2) - \underbrace{\|q - s_2\|^2}_{d_2}$$

$$\Rightarrow -d_1^2 + d_2^2 \gtrless \underbrace{\frac{N}{2} \ln \left(\frac{P(m_2)}{P(m_1)} \right)}_{c}$$

$$\boxed{-d_1^2 + d_2^2 \gtrless c}$$

* Now if $P(m_1) = P(m_2) \Rightarrow c = 0$

then $d_1 \underset{m_2}{\overset{m_1}{\lessgtr}} d_2 \Rightarrow$ Perpendicular bisector



$$-d_1^2 + d_2^2 = c$$

and $d_2^2 = \alpha^2 + \mu^2$

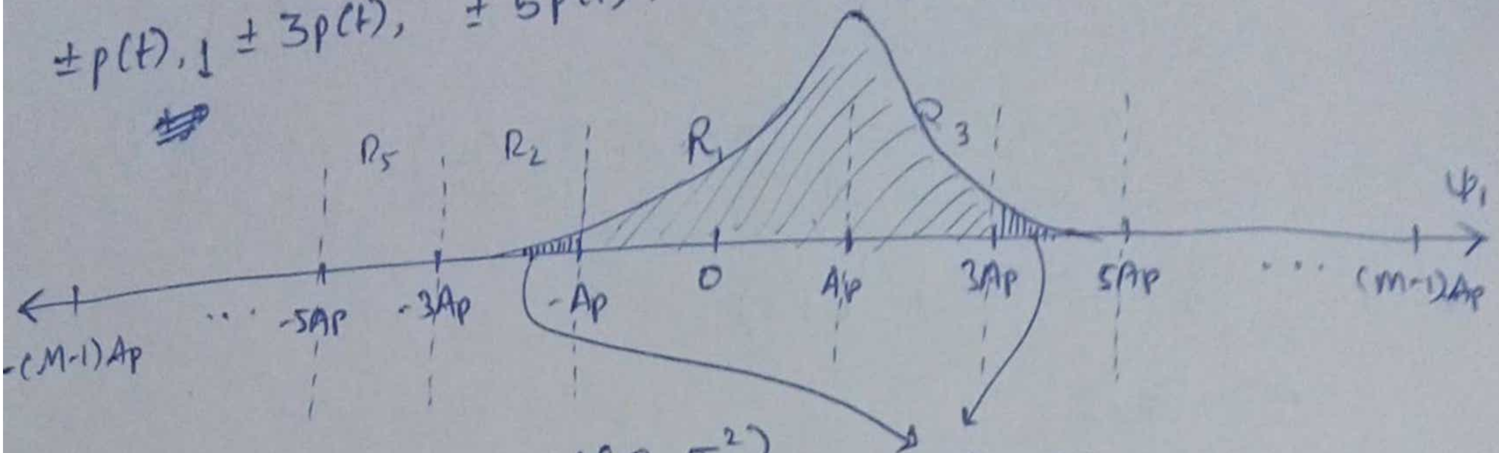
$\quad d_1^2 = \alpha^2 + (d - \mu)^2$

$\Rightarrow$ Solving, we get $\boxed{\mu = \frac{c + d^2}{2d}}$

where $c = \frac{N}{2} \ln \left(\frac{P(m_2)}{P(m_1)} \right)$

* M-ary P.A.M

$\pm p(t), \pm 3p(t), \pm 5p(t), \ldots \pm (M-1)p(t)$



$$q = \underbrace{\pm kA_p + n}_{\text{Normal dist}} \rightarrow N(0, \sigma^2)$$

$$P(c|m_1) = P[q \in [0, 2A_p]]$$

correct symbol recieved

$$= 1 - 2 \cdot P[q < 0]$$

$$= 1 - 2P[n \leq -A_p].$$

as ~~$(1 - Q(-x))$~~

$$= 1 - 2\left(1 - Q\left(\frac{-A_p - 0}{\sigma_n}\right)\right)$$

$$P(c|m_1) = 1 - 2 + 2Q\left(\frac{-A_p}{\sigma_n}\right) = 2Q\left(\frac{-A_p}{\sigma_n}\right) - 1$$

but as $1 - Q(-x) = Q(x)$

$$P(c|m_1) = 1 - 2Q\left(\frac{A_p}{\sigma_n}\right).$$

or in general $P(c|m_i) = 1 - 2Q\left(\frac{A_p}{\sigma_n}\right)$ for $i \in [1, M-2]$

but for $i = M-1$,

$$P(c|m_{M-1}) = 1 - Q\left(\frac{A_p}{\sigma_n}\right)$$

$$\Rightarrow P(e|m_i) = \begin{cases} 2Q\left(\frac{A_p}{\sigma_n}\right) & \text{for } i \in [1, \ldots M-2] \\ Q\left(\frac{A_p}{\sigma_n}\right) & \text{for } i = M-1, M \end{cases}$$

$$P_{em} = \sum_{j=1}^{M} P_e(m_j) \cdot P(m_j)$$

Now if uniform distribution, $p(m_i) = p(m_j)$

$$\Rightarrow P_{em} = \sum_{j=1}^{M} \frac{P_e(m_j)}{M} = \frac{1}{M}\left[(M-2)\, 2Q\left(\frac{A_p}{\sigma_n}\right) + 2Q\left(\frac{A_p}{\sigma_n}\right)\right]$$

$$= \frac{1}{M}\left[Q\left(\frac{A_p}{\sigma_n}\right)\right] \cdot (2M-4+2)$$

$$\cancel{= \frac{2}{M}(M-1}$$

$$P_{em} = \frac{2}{M}(M-1)\, Q\left(\frac{A_p}{\sigma_n}\right) \quad \text{—— (III)}$$

for $\{\pm A_p, \pm 3A_p, \ldots \pm (M-1)A_p\}$.

$$E_{avg} = A_p^2\left(1 + 9 + \ldots (M-1)^2\right)$$

$$E_{avg} = \frac{(M^2-1)}{3} E_p \qquad \text{where} \quad E_p = A_p^2$$

$$E_b = \frac{E_{avg}}{\log_2 M} = \frac{(M^2-1)\, E_p}{3 \log_2 M} = \frac{(M^2-1)}{3\log_2 M} \cdot A_p^2 \quad \text{—— (I)}$$

we know $\sigma_n = \sqrt{\frac{N}{2}}$ —— (II) $\Rightarrow$ using (I) & (II) in (III)

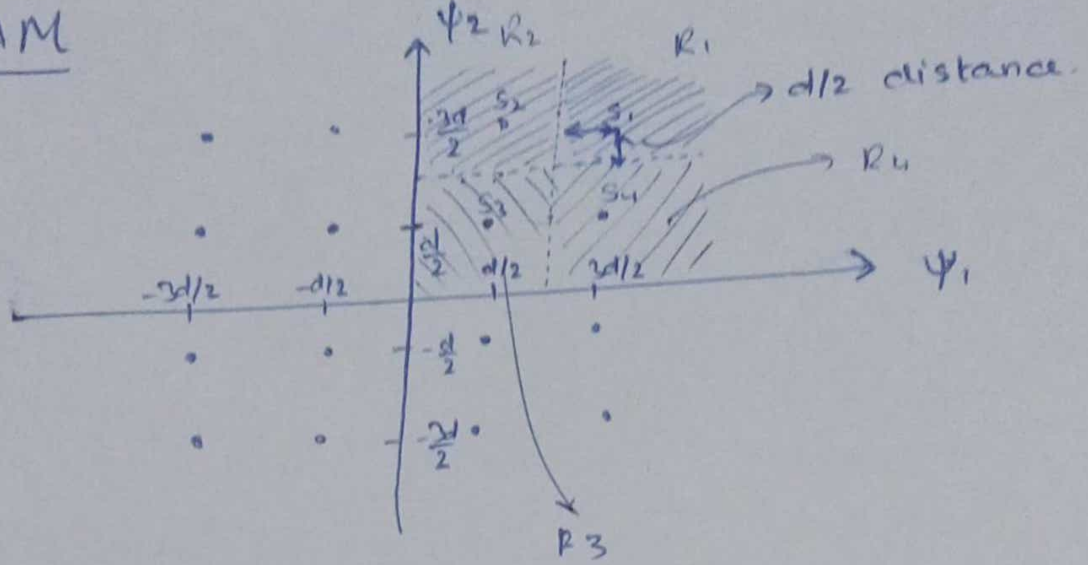
$\Rightarrow$ we can write:-

$$\frac{A_p}{\sigma} = \sqrt{\frac{2E_p}{N}}$$

$$P_{em} = 2 \cdot \frac{(M-1)}{M}\, Q\left(\sqrt{\frac{6 \log_2 M}{(M^2-1)} \cdot \frac{E_b}{N}}\right)$$

* 16-QAM is in syllabus

* 16 QAM



$$s_i(t) = a_i \psi_1(t) + b_i \psi_2(t).$$

$$q = s_i + n$$

$$q = \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$P(c|m_1) = P[q \in R_1] = P\left[n_1 > -\frac{d}{2},\ n_2 > -\frac{d}{2}\right].$$

average energy is calculated in form of d.

for Region 3

$$P(c|m_3) = P[q \in R_3] = P\left[|n_1| < \frac{d}{2},\ |n_2| < \frac{d}{2}\right]$$

for Region 2.

$$P(c|m_2) = P[q \in R_2] = P\left[|n_1| < \frac{d}{2},\ n_2 > -\frac{d}{2}\right]$$

and by symm.

$$P(c|m_3) = P(q \in R_4) = P\left[|n_2| < \frac{d}{2},\ n_1 > -\frac{d}{2}\right].$$