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Q1.

for unbiased Dice

- probablity of each outcome = 1/6
- favourable outcome = 6
- no. of trials req. of 6 = x

we have to find.

$\Rightarrow H(x) = ?$

Now

$P(x=1)$ = prob. of getting 6 in 1st chance = 1/6

for

$x=2$, $P(x)$ = prob. of getting 6 in 2nd chance $= \frac{5}{6}\times\frac{1}{6}$

for $x=3$

$P(x)$ = ,, ,, ,, in 3rd $= \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

for $x=4$

$P(x)$ = ,, ,, ,, 4th $= \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

$P(x=n)$ = ,, ,, ,, n^{th} $= \left(\frac{5}{6}\right)^{n-1}\times\frac{1}{6}$

we know that

$$\because \quad H(x) = -\sum_{x} P(x)\log_2 P(x)$$

$$= -\left[\frac{1}{6}\log_2\frac{1}{6} + \frac{5}{6}\times\frac{1}{6}\log\frac{5}{6}\times\frac{1}{6} + \cdots + \left(\frac{5}{6}\right)^{n}\frac{1}{6}\log_2\left(\frac{5}{6}\right)^{n}\frac{1}{6}\right]$$

$$= -\frac{1}{6}\left[\frac{5}{6}\log\frac{5}{6} + 2\cdot\left(\frac{5}{6}\right)^2\log_2\frac{5}{6} + \cdots + n\left(\frac{5}{6}\right)^{n-1}\log_2\frac{5}{6}\right.$$

now we can say that

from here we have two geometric series

$H(n) = a + b$ where

$$a = -\frac{1}{6}\log_2\frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots + \left(\frac{5}{6}\right)^{n}\right].$$

$$b = -\frac{1}{6}\log_2\frac{5}{6}\left[\frac{5}{6} + 2\cdot\left(\frac{5}{6}\right)^2 + 3\cdot\left(\frac{5}{6}\right)^3 + \cdots + (n-1)\left(\frac{5}{6}\right)^{n-1}\right]$$

from here we can conclude that

a : Geometric series

b : arithmetic geometric series

Now for a,

first term = 1

and common ratio = 5/6

$a = -\frac{1}{6}\log\frac{1}{6}\times 6 = -\frac{1}{6}\log\left(\frac{1}{6}\right)\times 6$

$a = -\log_2\frac{1}{6} = \log_2 6 = \boxed{\log_2 6}$

Now for b

common ratio, $r = 5/6$

constant $k = -\frac{1}{6}\log\frac{5}{6}$

so $b = k[1r^1 + 2r^2 + 3r^3 + \dots + n-1\, r^{n-1}]$ ①

~~now~~

after dividing b by r

$\frac{b}{r} = k = [1 + 2r + 3r^2 + 4r^3 + \dots + n-1\, r^{n-2}]$ ②

⇒ the sum of RHS of eqn ② is derivative of geometric series summed from k = 1 to n-1.

and sum of geometric series $\frac{r^{n+1}-1}{r-1}$

$\Rightarrow \frac{b}{r} = k\frac{d}{dr}\left(\left[\frac{r^{(n-1)+1}-1}{r-1}\right] - 1\right)$

$= k\frac{d}{dr}\left[\frac{r^n - 1}{r-1}\right]$

$= k[(n-1)r^n - nr^{n-1} + 1] / (r-1)^2$

$b = k\left[(n-1)r^{n+1} - nr^{n} + r\right] / (r-1)^2$

$\because$ $r = \frac{5}{6} < 1$ so infinite sum will converge to $\frac{r}{(r-1)^2}$

$\therefore$ $b = -\frac{1}{6}\log\frac{5}{6} \times \frac{5/6}{1/36}$

$b = -\log 5/6 \times 5$

$b = 5\log_2 6/5$

$\therefore$ $H(x) = \log_2 6 + 5\log_2 6/5$

$= 6\log_2 6 - 5\log_2 5$

$= \log_2 \frac{6^6}{5^5} = \log_2\left(\frac{46656}{3125}\right)$

$= \log_2(14.89952)$

$= 3.91185$ bits

$\therefore$ $\boxed{H(x) = 3.91185 \text{ bits}}$

for Probablity $P(x) = \frac{1}{m}$ & x is no of trials for getting x

we can generalise $\because H(x) = m\log_2 m - (m-1)\log_2 m$

(b) ~~Now~~ Die is biased

given that

	1	2	3	4	5	6
Distribution	$\frac{1}{4}$	$1/4$	$1/8$	$1/8$	$1/8$	$1/8$

x_i : no. of trials needed to get ith value

we have to find $H(x_i)$

Now, as prob. distribution of (1,2) and (3,4,5,6) have same $1/4$, $1/8$ respectively

we can say that

$H(x_1) = H(x_2)$

$H(x_3) = H(x_4) = H(x_5) = H(x_6)$

now let's calculate for x_1

$p(x_1 = 1)$: prob. of getting 1 in 1st chance $= 1/4$

$p(x_1 = 2)$: " " 2nd $= \frac{1}{4} \times \frac{3}{4}$

$p(x_1 = n) =$ " " " nth $= \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$

$\therefore H(x_1) = -\sum_{x \in X_1} p(x) \log_2 p(x)$ where $x \in [1, \infty)$

$= -\sum_{n} \frac{1}{4}\left(\frac{3}{4}\right)^{n-1} \log\left[\frac{1}{4}\left(\frac{3}{4}\right)^{n-1}\right]$

Now, ~~for~~ from our generalised eq^n for prob

$p(x) = 1/m$ & if x is no. of trials required to get x, then

$$\boxed{H(x) = m \log_2 m - (m-1) \log_2 (m-1)} \quad eq \; (1)$$

(derived in last part)(a)

$\therefore\ H(X_1) = H(X_2) = 4\log_2 4 - 3\log_2 3$

$= 8 - 3(1.58)$

$= 3.2153$ bits

$\therefore\ \boxed{H(X_1) = H(X_2) = 3.2153 \text{ bits}}$

Now lets calculate entropy of X_3

$P(X_3 = 1)$ = prob. of getting 3 in 1st chance $= \frac{1}{8}$

For $x_3 = 2$. prob. of getting 3 in 2nd chance $= \frac{1}{8} \times \frac{7}{8}$

For $x_3 = n$ $P(x_3 = n) =$ " " " nth $= \frac{1}{8}\left(\frac{7}{8}\right)^n$

$\because H(X_3) = -\sum_{x \in X_1} P(x) \log_2 P(x)$; $x \in [1, \infty)$

$= -\sum_{x \in X_1} \frac{1}{8}\left(\frac{7}{8}\right)^{n-1} \log_2\left[\frac{1}{8}\left(\frac{7}{8}\right)^{n-1}\right]$

now putting $m = 8$ in eq^n 1 (Generalised for this eq.)

$= 8\log 8 - 7\log 7 = 24 - 7(2.8)$

$\boxed{\therefore\ H(X_4) = H(X_5) = H(X_6) = 4.4 \text{ bits}}$

Q.2 we know that

the axioms of uniqueness theorem are

cond. 1 : for a given n and prob. dist p. $(\sum_{i=1}^{n} P_i = 1)$. for the function $H(P_1, P_2 \dots P_n)$ which gives it highest value at $P_k = \frac{1}{n}$ $\forall$ $k \in \{1, 2, \dots n\}$

cond. 2 : let A & B be two r.v

then

$$H(A \cdot B) = H(A) + \sum_a P(A=a) \cdot H(B/A=a)$$

cond. 3 : $H(P_1, P_2, \dots P_n) = H(P_1, P_2, \dots P_n, 0)$.

now let $\overset{\text{cond 1}}{S_1}$ and $\overset{\text{cond 3}}{S_3}$ satisfy the condition for $H(x)$ but $\overset{\text{cond 2}}{S_2}$ does not. then function will have its maximum ~~distance~~ at uniform distance.

now

let $H(x) = F(x)$

we can find maxima of $f(x)$ by partially differentiating w.r.t $P_1, P_2, \dots P_n$ and will set derivative as 0

$$\Rightarrow \frac{\partial f}{\partial P_1} = \frac{\partial F}{\partial P_2} = \frac{\partial F}{\partial P_3} = \frac{\partial F}{\partial P_4} = \dots = \frac{\partial f}{\delta P_n} = 0$$

when $P_1 = P_2 = \dots P_n = \frac{1}{n}$ let $\frac{\partial f}{\partial P_i}$ be a linear eq^n

then

$$\frac{\partial f}{\partial P_i} = a \cdot P_i + b = 0 \Rightarrow P_i = -\frac{b}{a}$$

let $b = +1$, $a = -n$

$$\Rightarrow \frac{\partial f}{\partial P_i} = -nP_i + 1 = 0 \quad \text{(by substituting the value)}$$

using eq^n we can choose $a = -2$, $b = 2/n$ which satisfy $P_i = -b/a = 1/n$

$$\Rightarrow \boxed{\frac{\partial f}{\partial P_i} = \frac{2}{n} - 2P_i}$$

we have to note that this condition must hold for all i $(\forall i)$

and value of f can easily calculated ~~by~~ using

$$\boxed{f(n) = \sum_{k=1}^{n} \left(\frac{2P_k}{n} - P_k^2\right)} \text{ — (1)}$$

we can verify eq (1) for condition 1

since $\dfrac{\partial f}{\partial P_i} = \dfrac{\partial}{\partial P_i} \displaystyle\sum_{i=1}^{n} \left(\frac{2P_i}{n} - P_i^2\right)$

$$= \frac{2}{n} - 2P_i = 0 \Rightarrow \frac{2}{n} = 2P_i \text{ or } \boxed{P_i = \frac{1}{n}}$$

also here we can check condition

as $f(P_1, P_2, \ldots P_n) = \displaystyle\sum_{K=1}^{n} \left(\frac{2P_K}{n} - P_K^2\right)$

$$f(P_1, P_2, \ldots P_n, 0) = \sum_{K=1}^{n} \left(\frac{2P_k}{n} - P_k^2\right) + \left(\frac{2(0)}{n} - (0)^2\right) = f(P_1, P_2, \ldots P_n)$$

$\Rightarrow$ since $f(P_1, P_2, \ldots P_n) = \displaystyle\sum_{K=1}^{n} \left(\frac{2P_k}{n} - P_k^2\right)$ satisfy both the ~~conditions~~ ~~axioms~~ Condition1 and condition3 of uniqueness theorem but does not satisfy the axiom 2 (cond 2)

thus

$H(P_1, P_2, \ldots P_n) = \displaystyle\sum_{K=1}^{n} \left(\frac{2P_k}{n} - P_k^2\right)$ Can be one such measure of entropy.