Date: / /

Subject: سری تمرین ۲

۱) نشان دهید [illegible]

$P_e = \sum_{i=1}^{M} \sum_{k=1}^{M} P(x \in R_i, \omega_k) P(\omega_k) = \sum_{i=1}^{M} \sum_{k=1}^{M} P(\omega_k) \int_{R_i} P(x|\omega_k)\, dx$

[illegible] $\to P_e = \sum_{i=1}^{M} \sum_{k=1}^{M} \int_{R_i} P(\omega_k|x) P(x)\, dx$

$R_i : P(\omega_i|x) = \max P(\omega_k|x) \quad ; \quad k = 1, \dots, M$

[illegible] P_e [illegible]

۲) [illegible] $N(0, \sigma^2)$, $N(1, \sigma^2)$

[illegible]

$\lambda_{11} = \lambda_{22} = 0 \quad , \quad x_0 = \frac{1}{2} - \sigma^2 \ln \frac{\lambda_{21} P(\omega_2)}{\lambda_{12} P(\omega_1)}$

$\begin{cases} l_1 = \lambda_{11} P(x|\omega_1) P(\omega_1) + \lambda_{21} P(x|\omega_2) P(\omega_2) \\ l_2 = \lambda_{12} P(x|\omega_1) P(\omega_1) + \lambda_{22} P(x|\omega_2) P(\omega_2) \end{cases}$

$l_1 \sim l_2 \quad \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{\sigma^2}}$

$\frac{P(x|\omega_1)}{P(x|\omega_2)} = \frac{\lambda_{21} P(\omega_2)}{\lambda_{12} P(\omega_1)}$

$\frac{1}{\sqrt{2\pi}\sigma} \frac{-(x-\mu)^2}{\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-1)^2}{\sigma^2}}$

$\frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{e^{-\frac{1}{2}\frac{(x-1)^2}{\sigma^2}}} = e^{-\frac{1}{2\sigma^2}(2x-1)}$

$\ln \frac{\lambda_{21} P(\omega_2)}{\lambda_{12} P(\omega_1)} = -\frac{1}{2\sigma^2}(2x-1)$

$\ln \lambda_{21} P(\omega_2) - \ln \lambda_{12} P(\omega_1) = -\frac{1}{2\sigma^2}(2x-1)$

$x = \frac{1}{2} - \sigma^2 \ln \left(\frac{\lambda_{21} P(\omega_2)}{\lambda_{12} P(\omega_1)} \right)$

$P(x \in R_2 | \omega_1) = \int_{R_2} P(x|\omega_1)\,dx$, $P(x \in R_1 | \omega_2) = \int_{R_1} P(x|\omega_2)\,dx$

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③ [illegible] (ε_1) [illegible]

[illegible] (ε_2) [illegible]

$r = r_1 P(\omega_1)\lambda_{11} + P(\omega_2)\lambda_{22} + P(\omega_1)(\lambda_{12} - \lambda_{11})\varepsilon_1 + P(\omega_2)(\lambda_{21} - \lambda_{22})\varepsilon_2$

$\varepsilon_1 = \int_{R_2} P(x|\omega_1)\,dx$

$r_k = \sum_{i=1}^{c} \lambda_{ki} \int_{R_i} P(x|\omega_k)\,dx$

$r_1 = \lambda_{11} \int_{R_1} P(x|\omega_1)\,dx + \lambda_{12} \int_{R_2} P(x|\omega_1)\,dx$

$r_2 = \lambda_{21} \int_{R_1} P(x|\omega_2)\,dx + \lambda_{22} \int_{R_2} P(x|\omega_2)\,dx$

$\int_{R_2} P(x|\omega_2)\,dx = 1 - \int_{R_1} P(x|\omega_2)\,dx = 1 - \varepsilon_2$

$r = \sum_{k=1}^{c} r_k P(\omega_k) \rightarrow r = r_1 P(\omega_1) + r_2 P(\omega_2)$

$r = \underbrace{\lambda_{11} P(\omega_1)}_{①} \underbrace{- \lambda_{11} P(\omega_1)\varepsilon_1 + \lambda_{12} P(\omega_1)\varepsilon_1 + \lambda_{21}\varepsilon_2 P(\omega_2)}_{③} + \underbrace{\lambda_{22} P(\omega_2) - \lambda_{22} P(\omega_2)\varepsilon_2}_{②}$

④ نشان دهید [illegible] احتمال خطای [illegible]

$P_e \leq \frac{m-1}{m}$

$\frac{1}{m} \leq (i = 1, \dots, m)\; P(\omega_i|x)$ [illegible]

$(\max \geq \frac{1}{m})$

$P(\omega_j|x) = \max P(\omega_i|x) \geq \frac{1}{m}$

$P_e = 1 - P = 1 - \max P(\omega_i|x) \geq 1 - \frac{1}{m}$

$\frac{1}{2} + \frac{1}{2}$

[illegible] max [illegible]

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۲) [illegible] Rayleigh [illegible]

$$p(x|\omega_1) = \begin{cases} \frac{x}{\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

[illegible]

$x \in \omega_1(\omega_2)$ if $\quad l_{12} = \frac{p(x|\omega_1)}{p(x|\omega_2)} >(<) \frac{p(\omega_2)(\lambda_{21}-\lambda_{22})}{p(\omega_1)(\lambda_{12}-\lambda_{11})}$

$l_{12}(\theta) =$ [illegible]

$$\frac{\frac{x}{\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}}}{\frac{x}{\sigma_2^2} e^{-\frac{x^2}{2\sigma_2^2}}} = \frac{\sigma_2^2}{\sigma_1^2} e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right)} = \frac{p(\omega_2)(\lambda_{21}-\lambda_{22})}{p(\omega_1)(\lambda_{12}-\lambda_{11})}$$

[illegible]

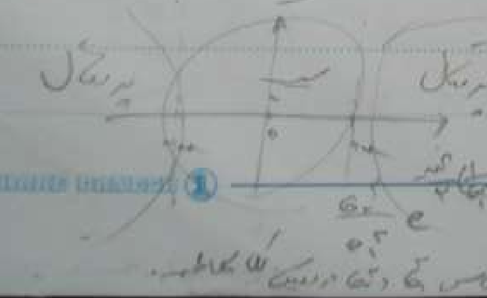
$$\ln \left| e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right)} = \frac{\sigma_1^2 p(\omega_2)(\lambda_{21}-\lambda_{22})}{\sigma_2^2 p(\omega_1)(\lambda_{12}-\lambda_{11})}\right.$$

$$\frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2\sigma_2^2}$$

$$-\frac{x^2}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right) = \ln\left(\frac{\sigma_1^2 p(\omega_2)(\lambda_{21}-\lambda_{22})}{\sigma_2^2 p(\omega_1)(\lambda_{12}-\lambda_{11})}\right)$$

$$x^2 = -\frac{2\sigma_1^2\sigma_2^2}{\sigma_2^2-\sigma_1^2} \ln\left(\frac{\sigma_1^2 \cancel{p(\omega_2)}(\lambda_{21}-\lambda_{22})}{\sigma_2^2 \cancel{p(\omega_1)}(\lambda_{12}-\lambda_{11})}\right)$$

[illegible] $= \pm \sqrt{x^2}$ [illegible]



$p(x|\omega_1)p(\omega_1) >(<) p(x|\omega_2)p(\omega_2)$

$\frac{p(x|\omega_1)}{p(x|\omega_2)} > 1 \sim \omega_1$

$\frac{\sigma_2^2}{\sigma_1^2} e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right)}$

$\frac{p(x|\omega_1)}{p(x|\omega_2)} < 1 \sim \omega_2$

[illegible]

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(۶) [illegible] $\varepsilon_2 = \varepsilon_0$

[illegible] اقل خطای [illegible]

تست نیمن - پیرسن decide x in ω_1 if $\dfrac{P(\omega_1|x)}{P(\omega_2|x)} > \theta$

[illegible]

[illegible] $q = \theta(\varepsilon_2 - \varepsilon_0) + \varepsilon_1$ [illegible]

$$\varepsilon_1 = \int_{R_2} P(x|\omega_1)P(\omega_1)\,dx$$

$$\varepsilon_2 = \int_{R_1} P(x|\omega_2)P(\omega_2)\,dx$$

[illegible] ε_2 [illegible]

$$q = \theta\left(P(\omega_2)\int_{x_0}^{\infty} P(x|\omega_2)\,dx - \varepsilon_0\right) + P(\omega_1)\int_{-\infty}^{x_0} P(x|\omega_1)\,dx$$

$$q_{x_0} = -\theta P(\omega_2)P(x_0|\omega_2) + P(\omega_1)P(x_0|\omega_1) = 0$$

$$\theta = \frac{P(\omega_1)P(x_0|\omega_1)}{P(\omega_2)P(x_0|\omega_2)} = \frac{P(\omega_1|x_0)}{P(\omega_2|x_0)}$$

[illegible] θ [illegible]

[illegible]

Subject : multi-dimensional

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(4) [illegible]

[illegible]

[illegible]

[illegible]

$$P_B = \int_{?}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-z^2/2)} dz$$

[illegible]

[illegible]

$$u = \ln P(x|\omega_1) - \ln P(x|\omega_2)$$

$N(\frac{1}{2}d_m^2, d_m^2)$ [illegible]

$N(-\frac{1}{2}d_m^2, d_m^2)$ if $x \in \omega_2$

[illegible]

$\frac{P(x|\omega_1)}{P(x|\omega_2)}$ [illegible]

[illegible] $P(\omega_1) = P(\omega_2) = \frac{1}{2}$ [illegible]

$$u = \ln(P(x|\omega_1)) - \ln(P(x|\omega_2)) = 0$$

$$u = \ln\left(\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{(-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1))}\right)$$

$$- \ln\left(\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{(-\frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2))}\right)$$

[illegible]

$\ln a e^b - \ln a e^c$

$= \ln \frac{a e^b}{a e^c} = \ln e^b - \ln e^c$

$= b - c$

$$= -\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1) + \frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)$$

$$= -\frac{1}{2}(\cancel{x^T \Sigma^{-1} x} - 2\mu_1^T \Sigma^{-1} x + \mu_1^T \Sigma^{-1} \mu_1 - \cancel{x^T \Sigma^{-1} x} + 2\mu_2^T \Sigma^{-1} x - \mu_2^T \Sigma^{-1} \mu_2)$$

$$= (\mu_1 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 \Leftarrow u$$

[illegible]

[illegible]

P_B [illegible]

[illegible]

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$\mu_1^T \Sigma^{-1} x - \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 - \mu_2^T\Sigma^{-1}x + \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2$

$u \sim N(\frac{1}{2} d_m^2, d_m^2) \quad \text{if} \quad x \in \omega_1$

$m_1 = (\mu_1 - \mu_2)^T \Sigma^{-1} \mu_1 - \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 + \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2$

$= \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 - \mu_2\Sigma^{-1}\mu_1 + \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2$

$= \frac{1}{2}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)$

$\sigma_1^2 = (\mu_1-\mu_2)^T \Sigma^{-1}\Sigma\Sigma^{-1}(\mu_1-\mu_2) = (\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)$

$u \sim N(-\frac{1}{2} d_m^2, d_m^2) \quad \text{if} \quad x \in \omega_2$

$m_2 = (\mu_1-\mu_2)^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 + \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2$

$= \mu_1^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_1\Sigma^{-1}\mu_1 - \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2$

$= -\frac{1}{2}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)$

$\boxed{x = \mu_2}$

$\sigma_2^2 = (\mu_1-\mu_2)^T\Sigma^{-1}\Sigma\Sigma^{-1}(\mu_1-\mu_2) = (\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)$

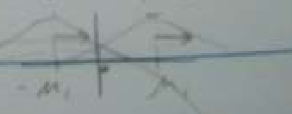
$\boxed{\sigma_1^2 = \sigma_2^2}$

$d_m^2 = (\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)$

$m_1 = \frac{1}{2} d_m^2 = -m_2$

$\sigma_1^2 = \sigma_2^2 = d_m^2 \rightarrow \sigma_1 = \sigma_2 = d_m$

$m_1 = 0$



$P_D = \int_{-\infty}^{0} P(u|\omega_1)P(\omega_1)\,du + \int_{0}^{\infty} P(u|\omega_2)P(\omega_2)\,du$

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(11) اگر $\Sigma_1 = \Sigma_2 = \Sigma$ نشان دهید معیار ساده‌تر این‌طور می‌شود

$$(\mu_1-\mu_2)^T \Sigma^{-1} x \ (<)\ c$$

$$c = \ln\theta + \frac{1}{2}\left(\|\mu_1\|_{\Sigma^{-1}} - \|\mu_2\|_{\Sigma^{-1}}\right)$$ که

چون براساس فرض

$$\ln\frac{|\Sigma_1|}{|\Sigma_2|} = 0$$

حل: $A = d_m(\mu_1, m|\Sigma_1)^2 - d_m(\mu_2, m|\Sigma_2)^2 =$

$$(m^T\Sigma^{-1}m - 2m^T\Sigma^{-1}\mu_1 + \mu_1^T\Sigma^{-1}\mu_1)$$

$$-(m^T\Sigma^{-1}m - 2m^T\Sigma^{-1}\mu_2 + \mu_2^T\Sigma^{-1}\mu_2)$$

$$= -2m^T\Sigma^{-1}(\mu_1-\mu_2) + \mu_1^T\Sigma^{-1}\mu_1 - \mu_2\Sigma^{-1}\mu_2$$

$$\Rightarrow (\mu_1-\mu_2)^T\Sigma^{-1}m > \ln\theta + \frac{1}{2}\|\mu_1\|^2_{\Sigma^{-1}} - \frac{1}{2}\|\mu_2\|^2_{\Sigma^{-1}}$$

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(11) [illegible]

$$p(x|\omega_1) = \frac{1}{(\sqrt{2\pi\sigma_1^2})^2} \exp\left(\frac{-1}{2\sigma_1^2}(x-\mu_1)^T(x-\mu_1)\right)$$

$$p(x|\omega_2) = \frac{1}{(\sqrt{2\pi\sigma_2^2})^2} \exp\left(\frac{-1}{2\sigma_2^2}(x-\mu_2)^T(x-\mu_2)\right)$$

$\mu_1 = [4, 1]^T \qquad \mu_2 = [1.5, 1.5]^T \qquad \sigma_1^2 = \sigma_2^2 = 0.2$

[illegible] $P(\omega_1) = P(\omega_2)$ [illegible]

a) [illegible]

b) [illegible] λ_{12} [illegible] λ_{21}

[illegible]

[illegible]

[illegible]

$$l_{12} = \frac{p(x|\omega_1)}{p(x|\omega_2)} \gtrless \frac{P(\omega_2)}{P(\omega_1)} \cdot \frac{(\lambda_{21} - \lambda_{22})}{(\lambda_{12} - \lambda_{11})}$$

$$\Rightarrow l_{12} = \frac{p(x|\omega_1)}{p(x|\omega_2)} \gtrless 1$$

$$l_{12} = \exp\left(-\frac{1}{2\sigma^2}\left((x-\mu_1)^T(x-\mu_1) - (x-\mu_2)^T(x-\mu_2)\right)\right)$$

اگر $A > 0$ → $B < 0$ → $l_{12} > 1$ → $x \in \omega_1$

اگر $A < 0$ → $B > 0$ → $l_{12} < 1$ → $x \in \omega_2$

ب : $x \in w_1 \Rightarrow (x-\mu_1)^T(x-\mu_1) - (x-\mu_2)^T(x-\mu_2) < 0$

$x^Tx - x^T\mu_1 - \mu_1^T x + \mu_1^T\mu_1 - (x^Tx - x^T\mu_2 - \mu_2^T x + \mu_2^T\mu_2)$

$x(\mu_2^T-\mu_1^T)^T + x^T(\mu_2-\mu_1) - \mu_2^T\mu_2 + \mu_1^T\mu_1 < 0$

$A^TB = AB^T$

$2x(\mu_2-\mu_1)^T < \mu_2^T\mu_2 - \mu_1^T\mu_1$

$x < \dfrac{\mu_2^T\mu_2 - \mu_1^T\mu_1}{2(\mu_2-\mu_1)^T}$ حل!

b) در این قسمت باید از تست ریسک مینیمم استفاده شود و حاصل میشود :

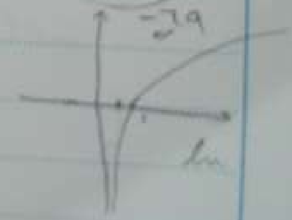
$\dfrac{P(x|w_1)}{P(x|w_2)} \gtrless \dfrac{P(w_2)}{P(w_1)} \dfrac{\lambda_{21}-\lambda_{22}}{\lambda_{12}-\lambda_{11}} = \dfrac{1}{2}$

$\exp\left(\dfrac{-1}{2\sigma^2}\left((x-\mu_1)^T(x-\mu_1) - (x-\mu_2)^T(x-\mu_2)\right)\right) \gtrless \frac{1}{2}$

از طرفین ln میگیریم

$\dfrac{-1}{2\sigma^2}\left[(x-\mu_1)^T(x-\mu_1) - (x-\mu_2)^T(x-\mu_2)\right] \gtrless \ln(\frac{1}{2})$

-0.69



$2x(\mu_1-\mu_2)^T + \mu_2^T\mu_2 - \mu_1^T\mu_1 \gtrless \ln(\frac{1}{2})\, 2\sigma^2$

$x \gtrless \dfrac{2\sigma^2\ln(1/2) - \mu_2^T\mu_2 - \mu_1^T\mu_1}{2(\mu_1-\mu_2)^T}$

حل

$$p(x|w_i) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1} (x-\mu_i)\right)$$

$$L_{12} = \frac{p(x|w_1)}{p(x|w_2)} \gtrless 1$$

$$L_{12} = \exp\left[-\frac{1}{2}\left((x-\mu_1)^T \Sigma^{-1} (x-\mu_1) - (x-\mu_2)^T \Sigma^{-1} (x-\mu_2)\right)\right] > (<) 1$$

$L_{12} > 1 \rightarrow \Theta < 0$ ... w_1

$L_{12} < 1 \rightarrow \Theta > 0$ ... w_2

$x \in w_1 \longrightarrow (x-\mu_1)^T \Sigma^{-1} (x-\mu_1) - (x-\mu_2)^T \Sigma^{-1} (x-\mu_2) < 0$

$$x < \frac{\mu_1^T\mu_1 - \mu_2^T\mu_2}{2(\mu_1-\mu_2)^T}$$

$$x \in w_1 (w_2) \quad L_{12} = \frac{p(x|w_1)}{p(x|w_2)} > (<) \frac{1}{2}$$

$$x > (<) \frac{2\ln(\frac{1}{2}) + \mu_1^T\mu_1 - \mu_2^T\mu_2}{2(\mu_1-\mu_2)^T}$$

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(۱۲) [illegible]

[illegible]

$$p(x|\omega_i) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1} (x-\mu_i)\right)$$

[illegible]

$\mu_1 = [1, 1]^T$ ، $\mu_2 = [1.5, 1.5]^T$ ، $\Sigma = \begin{bmatrix} 1.01 & 0.2 \\ 0.2 & 1.01 \end{bmatrix}$

$\begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$

[illegible]

Σ^{-1} ، $\begin{bmatrix} 1.01 & 0.2 \\ 0.2 & 1.01 \end{bmatrix}$ ، $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ ، $\begin{bmatrix} 1 & \\ 1 & \end{bmatrix}$

[illegible]

a) $l_{12} = \dfrac{p(x|\omega_1)}{p(x|\omega_2)} \gtrless 1$

$$l_{12} = \exp\left[-\frac{1}{2}\underbrace{\left(\overbrace{(x-\mu_1)^T \Sigma^{-1}(x-\mu_1) - (x-\mu_2)^T \Sigma^{-1}(x-\mu_2)}^{B}\right)}_{A}\right] \gtrless 1$$

اگر $A > 0 \rightarrow l_{12} > 1 \Rightarrow B < 0$ کلاس ω_1

اگر $A < 0 \rightarrow l_{12} < 1 \Rightarrow B > 0$ کلاس ω_2

[illegible] $(x-\mu_1)^T \Sigma^{-1} (x-\mu_1) - (x-\mu_2)^T \Sigma^{-1} (x-\mu_2) < 0 \rightarrow x \in \omega_1$

حل: $x < \dfrac{\mu_2^T \mu_2 - \mu_1^T \mu_1}{2(\mu_2 - \mu_1)^T}$

[illegible]

[illegible] $\frac{1}{2}$ [illegible] $\frac{1}{6}$

b) $x \in \omega_1 (\omega_2)$ $l_{12} = \dfrac{p(x|\omega_1)}{p(x|\omega_2)} \gtrless \dfrac{1}{2}$

$$x \gtrless \frac{2\ln(1/2) + \mu_1^T\mu_1 - \mu_2^T\mu_2}{2(\mu_1 - \mu_2)^T}$$

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[illegible]

$$\frac{P(\omega_i)}{P(\omega_j)}$$

[illegible]

$$\frac{1}{\sigma_{ii}(\mu_i,\mu_j)}\left((\Sigma^{-1}(\mu_i-\mu_j))^T(x-\mu_j)\right)$$

$$\left((\Sigma^{-1}(\mu_i-\mu_j))^T(x-x_0)\right)=0$$

[illegible]

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(10)

$p(x|\omega_1) \sim N(\mu, \sigma^2)$

$p(x|\omega_2) \sim$ $\int_a^b \frac{1}{b-a}$ $a<x<b$

$G\left(\frac{b-\mu}{\sigma}\right) - G\left(\frac{a-\mu}{\sigma}\right)$

$G(x) = P(y \leq x)$, $N(0,1)$

$x \in \omega_1$ if $\frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{P(\omega_2)}{P(\omega_1)}$

$a<x<b$ ، $= 0$

$p(x|\omega_1) > \frac{P(\omega_2)}{P(\omega_1)} p(x|\omega_2)$ $\to p(x|\omega_1) > 0$ true

$p(x|\omega_1) > \frac{P(\omega_2)}{P(\omega_1)} \frac{1}{b-a}$

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x_0-\mu)^2}{\sigma^2}} = \frac{P(\omega_2)}{P(\omega_1)} \frac{1}{b-a}$$

$$\ln\left(e^{-\frac{1}{2}\frac{(x_0-\mu)^2}{\sigma^2}}\right) = \ln\left(\frac{\sqrt{2\pi}\sigma\, P(\omega_2)}{P(\omega_1)(b-a)}\right)$$

$$\sqrt{\frac{(x_0-\mu)^2}{\sigma^2}} = \sqrt{-2\ln\frac{\sqrt{2\pi}\sigma\, P(\omega_2)}{P(\omega_1)(b-a)}}$$

$$\frac{x_0-\mu}{\sigma} = \pm\sqrt{-2\ln\frac{\sqrt{2\pi}\sigma}{b-a}\frac{P(\omega_2)}{P(\omega_1)}}$$

$$x_0 = \mu \pm \sigma\sqrt{-2\ln\left(\frac{\sqrt{2\pi}\sigma}{b-a}\frac{P(\omega_2)}{P(\omega_1)}\right)}$$

x_0 ، $(+,-)$

x_0 ، P_e

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۲- اگر فرض کنیم m ها معرفی شد ؛ نواحی تصمیم :

$$\begin{cases} R_1 = (-\infty, a) \cup (b, +\infty) \\ R_2 = (a, b) \end{cases}$$

$$P_e = P(\omega_1) \int_{R_2} p(x|\omega_1)\,dx + P(\omega_2) \int_{R_1} \overset{0}{p(x|\omega_2)}\,dx$$

$$= P(\omega_1) \int_a^b p(x|\omega_1)\,dx < \int_a^b p(x|\omega_1)\,dx \quad B$$

حالا x_0 ... $\rightarrow \ln\left(\frac{\sqrt{2\pi}\,\sigma}{b-a}\frac{P(\omega_2)}{P(\omega_1)}\right) > 0$

$$\rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} > \frac{P(\omega_2)}{P(\omega_1)} \frac{1}{b-a} \rightsquigarrow d = \mu \ldots$$

$x \in (m_1^-, m_1^+)$

$\omega_1 \leftarrow (m_1^-, m_1^+)$

$$\rightarrow \begin{cases} R_1 = (-\infty, a) \cup (b, +\infty) \cup (m_1^-, m_1^+) \\ R_2 = (a, b) \setminus (m_1^-, m_1^+) \end{cases}$$

$$P_e = P(\omega_1) \int_{R_2} p(x|\omega_1)\,dx + P(\omega_2) \int_{R_1} p(x|\omega_2)\,dx$$

$$= P(\omega_1) \int_{(a,b)\setminus(m_1^-, m_1^+)} p(x|\omega_1)\,dx + P(\omega_2) \int_{(a,b)\cap(m_1^-, m_1^+)} p(x|\omega_2)\,dx$$

$$P_e \le P(\omega_1) \int_{(a,b)\setminus(m_1^-, m_1^+)} p(x|\omega_1)\,dx + P(\omega_1) \int_{(a,b)\cap(m_1^-, m_1^+)} p(x|\omega_1)\,dx$$

$$= P(\omega_1) \int_a^b p(x|\omega_1)\,dx < \int_a^b p(x|\omega_1)\,dx \quad B$$

$$\int_a^b p(x|\omega_1)\,dx = \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz = \int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz - \int_{-\infty}^{\frac{a-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$x \sim N(\mu, \sigma^2)$

(۱۲) در آزمایش شیر یا خط برای یک سکه، احتمال رو آمدن شیر (۱) q است و خط (۰) $1-q$

x_i ، $i=1,\dots,N$ را در N آزمایش پی در پی نتیجه بگیرید $x_i \in \{0,1\}$.

نشان دهید که تخمین ML برای q به صورت زیر است:

$$q_{ML} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

راهنمایی: تابع درستنمایی : $P(X;q) = \prod_{i=1}^{N} q^{x_i}(1-q)^{(1-x_i)}$ → درستنمایی (احتمال) را ماکزیمم کنید

سپس نشان دهید ML از طریق مشتق گرفتن به دست می‌آید

$$q^{\sum_i x_i}(1-q)^{(N-\sum_i x_i)}\left(\frac{\sum_i x_i}{q} - \frac{N-\sum_i x_i}{1-q}\right) = 0$$

$$\text{log likelihood} \rightarrow L(q) = \underbrace{\ln\left(P(X;q)\right)}_{} = \sum_{i=1}^{N}\left(x_i \ln(q) + (1-x_i)\ln(1-q)\right)$$

$\max_q$ ← میتوان تخمین ML از q

$$\frac{dL}{dq} = \sum_{i=1}^{N}\left(\frac{x_i}{q} - \frac{1-x_i}{1-q}\right) = 0$$

$$\Rightarrow q = \frac{1}{N}\sum_{i=1}^{N} x_i$$

(۱۶) [illegible] N [illegible]

[illegible] Cramer Rao [illegible]

[illegible] $-E\left[\frac{\partial^2 L(m)}{\partial m^2}\right]$

[illegible]

حال:

[illegible]

[illegible] Log likelihood ($L(m)$) [illegible]

$$L(m) = -\frac{N}{2}\ln\left((2\pi)^P|\Sigma|\right) - \frac{1}{2}\sum_{k=1}^{N}(x_k - m)^T\Sigma^{-1}(x_k - m)$$

$$P=1, \quad \Sigma = \sigma^2$$

$$L(m) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\sum_{k=1}^{N}\frac{(x_k - m)^2}{\sigma^2}$$

[illegible]

① [illegible]

[illegible] $P(x;\theta)$ [illegible]

② [illegible] θ [illegible]

[illegible]

[illegible]

[illegible] θ [illegible]

$$L(\theta) = \ln\left(\prod_{k=1}^{N} P(x_k;\theta)\right)$$

[illegible]

[illegible]

$$I_{ij} = -E\left[\frac{\partial^2 L(\theta)}{\partial\theta_i\,\partial\theta_j}\right]$$

یعنی، واریانس $\hat{\theta}_i$ باید حداقل مقدار J^{-1}_{ii} باشد. consistent [illegible] زیاد است.

$$E[(\hat{\theta}_i-\theta_i)^2] \geq J^{-1}_{ii}$$

اگر تخمین $\hat{\theta}$ [illegible] efficient [illegible] از L نسبت به [illegible] مشتق L [illegible] داریم. [illegible]

$$\frac{\partial L}{\partial \mu}=\frac{1}{\sigma^2}\sum_{k=1}^{N}(x_k-\mu)$$

$$\frac{\partial^2 L}{\partial \mu^2}=\frac{1}{\sigma^2}\sum_{k=1}^{N}-1=\frac{-N}{\sigma^2}$$

پس:

$$-E\left[\frac{\partial^2 L}{\partial \mu^2}\right]=\frac{N}{\sigma^2}=\frac{1}{\frac{\sigma^2}{N}}=\frac{1}{var(\hat{\mu}_{ML})}$$

از آنجا که نشان دادیم $-E\left[\frac{\partial^2 L}{\partial \mu^2}\right]=\hat{\mu}^{-1}_{ML}$، تخمین μ_{ML} برای تخمین μ [illegible] است.
حالا حالتی که پارامتر مجهول σ^2 است [illegible]

$$\frac{\partial L}{\partial \sigma^2}=-\frac{N}{2}\frac{1}{\sigma^2}+\frac{1}{2}\sum_{k=1}^{N}\frac{(x_k-\mu)^2}{(\sigma^2)^2}$$

$$\frac{\partial^2 L}{\partial (\sigma^2)^2}=\frac{N}{2}\frac{1}{(\sigma^2)^2}-\sum_{k=1}^{N}\frac{(x_k-\mu)^2}{(\sigma^2)^3}$$

چون: $x_k \sim N(\mu,\sigma^2)$ ، $E[(x_k-\mu)^2]=\sigma^2$

حالا [illegible] σ^2 [illegible]

$$E\left[\frac{\partial^2 L}{\partial (\sigma^2)^2}\right]=\frac{N}{2}\frac{1}{(\sigma^2)^2}-\sum_{k=1}^{N}\frac{\sigma^2}{(\sigma^2)^3}=-\frac{N}{2}\frac{1}{(\sigma^2)^2}$$

بنابراین:

$$-E\left[\frac{\partial^2 L}{\partial (\sigma^2)^2}\right]=\frac{N}{2}\frac{1}{(\sigma^2)^2}=\frac{1}{\frac{(\sigma^2)^2}{N/2}}$$

(به طور مثال، ادامه‌ی مسأله با پاسخ دکتر راد)

سوال: در مورد تخمین ML در حالتی که μ را هم تخمین بزنیم، آیا تخمین بدون بایاس است؟
آیا عبارت کرامر-رائو مساوی $\frac{1}{var(\hat{\sigma}^2_{ML})}$ هست یا نه.

سپس با استفاده از مسأله ۱۹ $\hat{\sigma}^2_{ML}$ در حالت تخمین همزمان به صورت زیر داده می‌شود

$$\hat{\sigma}^2_{ML} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2$$

تخمین ML از میانگین $= \frac{1}{N}\sum_{k=1}^{N} x_k$

سپس می‌خواهد نشان دهد که $\hat{\sigma}^2_{ML}$ یک توزیع chi-squared با درجه آزادی $N-1$ دارد.

$$\hat{\sigma}^2_{ML} \sim \frac{\sigma^2}{N}\chi^2_{N-1}$$

می‌دانیم که امید ریاضی (میانگین) و واریانس یک توزیع chi-squared با $N-1$ درجه آزادی به ترتیب $N-1$ و $2(N-1)$ است

$$E[\hat{\sigma}^2_{ML}] = \frac{\sigma^2}{N}(N-1) = \frac{N-1}{N}\sigma^2$$

$$var(\hat{\sigma}^2_{ML}) = \frac{\sigma^4}{N^2}2(N-1) = \frac{2(N-1)}{N^2}\sigma^4$$

بنابراین $\Rightarrow$ بایاس دارد $\Rightarrow \frac{1}{var(\hat{\sigma}^2_{ML})} \neq -E\left[\frac{\partial^2 L}{\partial(\sigma^2)^2}\right]$

و بنابراین تخمین σ^2، کارا نیست.