Baby Steps in Geometric Measure Theory

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Abstract

These are notes for a reading course in Geometric Measure Theory I did under Dr. Natasa Sesum at Rutgers University - New Brunswick.

Contents

Introduction

The notes that follow are initially compiled from Stein & Shakarchi's Measure Theory text to build the foundation of Hausdorff measure/dimension. Then, we switch mainly to Frank Morgan's book on Geometric Measure Theory for the crux of the material.
It goes without saying that as these are notes for a reading course, they are prone to error. So take what is written with half a grain of the finest Himalayan salt. If you are so inclined, if you find any errors, you can contact me at yas33@scarletmail.rutgers.edu and I will fix them, mention you, and re-upload a corrected version.

Chapter 1

Measures

We begin this reading course with a discussion on measures. In particular, we discuss Hausdorff measure, integral-geometric measure, and densities.

1.1 Hausdorff measure & Hausdorff dimension

This draws heavily on Stein & Shakarchi's Real Analysis book.

1.1.1 Hausdorff measure

Hausdorff measure can be seen as a new way to measure volume/mass (as we are introducing a measure of course). Hausdorff measure is very much also related with the idea of dimension. For some appropriate set E and for a fixed $\alpha > 0$, $m_\alpha(E)$ can be thought of as the α-dimensional mass of E among sets of "dimension" α. The quotations around "dimension" will become clear as we get deeper in the material. If α is bigger than the dimension of E, $m_\alpha(E) = 0$. If α is smaller than the dimension of E, $m_\alpha(E) = \infty$. When α hits the dimension of E itself, then $m_\alpha(E)$ describes the actual α-dimensional size of the set.

It is also helpful to view Hausdorff measure as providing us more detail than the Lebesgue measure does. For example, consider the rectifiable curve Γ in $\mathbb{R}^2$. The Lebesgue measure is 0, so clearly information is lost about the curve using Lebesgue measure. On the other hand, the Hausdorff measure $m_1(\Gamma)$ actually gives us the length of Γ. Note that in $\mathbb{R}^2$, the usual way to measure length required differentiability everywhere, however here we only require rectifiability, which implies bounded variation, therefore differentiability *almost everywhere*.

As was the case with Lebesgue measure, we first define an outer measure. Let $E \subseteq \mathbb{R}^d$. Then, let the **exterior α-dimensional Hausdorff measure** of E be

$$m_\alpha^*(E) = \lim_{\delta \to 0} \inf \left\{ \sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k,\ \operatorname{diam} F_k \leq \delta\, (\forall k) \right\}.$$

For the sake of simplicity, we will denote

$$\mathcal{H}_\alpha^\delta(E) = \inf\left\{\sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k,\ \operatorname{diam} F_k \le \delta\ (\forall k)\right\}.$$

We now make note of three of things:

(1) As δ increases, $\mathcal{H}_\alpha^\delta(E)$ increases as well, and so $\lim_{\delta\to 0}\mathcal{H}_\alpha^\delta(E)$ exists but could be infinite.

(2) $\mathcal{H}_\alpha^\delta(E) \le m_\alpha^*(E)$ for all $\delta > 0$.

(3) Scaling is consistent here. If F is scaled by r, then $(\operatorname{diam} F)^\alpha$ scales by r^α.

Hausdorff measure enjoys the same properties of Lebesgue measure: monotonicity, subadditivity, and additivity of sets E_1, E_2 if $d(E_1, E_2) > 0$. Therefore m_α^* satisfies the same properties as a metric Carathéodory exterior measure, and so when we restrict ourselves to Borel sets, we obtain a countably additive measure m_α, the **α-dimensional Hausdorff measure**.
We now turn our attention to some interesting properties of the Hausdorff measure which are different from those exhibited by the Lebesgue measure:

(1) $m_0(E)$ counts the number of points in E, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$. In general, d-dimensional Hausdorff measure in $\mathbb{R}^d$ is equal to Lebesgue measure (up to a constant multiple).

(2) First, note that the iso-diametric inequality states the following: If $E \subset \mathbb{R}^d$ bounded and $\operatorname{diam} E = \sup\{|x-y| : x, y \in E\}$, then

$$m(E) \le v_d\left(\frac{\operatorname{diam} E}{2}\right)^d.$$

Then, we can deduce the following: If E is a Borel subset of $\mathbb{R}^d$, then $c_d m_d(E) = m(E)$ for $c_d = m(B)/(\operatorname{diam} B)^d = v_d/2^d$, where B is the unit ball, and v_d is its volume.

(3) In the same vein as (2): If E is a Borel subset of $\mathbb{R}^d$ and $m(E)$ is its Lebesgue measure, then $m_d(E) \approx m(E)$ in the sense that

$$c_d m_d(E) \le m(E) \le 2^d c_d m_d(E).$$

(4) If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. Also, if $m_\alpha^*(E) > 0$ and $\beta > \alpha$, then $m_\beta^*(E) = \infty$.

Proof. If $\operatorname{diam} F \le \delta$ and $\beta > \alpha$, then

$$(\operatorname{diam} F)^\beta = (\operatorname{diam} F)^{\beta-\alpha}(\operatorname{diam} F)^\alpha \le \delta^{\beta-\alpha}(\operatorname{diam} F)^\alpha.$$

Therefore, we conclude

$$\mathcal{H}^{\delta}_{\beta}(E) \leq \delta^{\beta-\alpha}\mathcal{H}^{\delta}_{\alpha}(E) \leq \delta^{\beta-\alpha} m^*_{\alpha}(E).$$

Since $m^*_{\alpha}(E) < \infty$ and $\beta - \alpha > 0$, as $\delta \to 0$, $m^*_{\beta}(E) = 0$.

The rest follows by the contrapositive. □

We can already feel the power of the Hausdorff measure in comparison with the Lebesgue measure. There are definitely rich geometric properties associated with this, and we will see in the following section the Hausdorff measure captures more information about sets than the Lebesgue measure does. Exciting!

1.1.2 Hausdorff dimension

Consider $E \subset \mathbb{R}^d$ which is open in the topology induced by the standard metric. From Property (4) above, we have that there exists a unique α such that

$$m_{\beta}(E) = \begin{cases} \infty & \beta < \alpha \\ 0 & \beta > \alpha. \end{cases}$$

At α, we say that E has **Hausdorff dimension** α, i.e. $\dim E = \alpha$.
What happens at α? Well, in general, $0 \leq m_{\alpha}(E) \leq \infty$. If E is bounded, the inequalities would be strict, and so E is said to have **strict Hausdorff dimension** α.

The Cantor Set

Let $\mathcal{C}$ be the Cantor set. Since $3\mathcal{C}$ exists in $[0, 3]$ and this interval contains 2 copies of $\mathcal{C}$, $2 = 3^d$, where d is the dimension of $\mathcal{C}$. Therefore $d = \log(2)/\log(3)$. We will show this is actually the strict Hausdorff dimension of $\mathcal{C}$. But first we need some more machinery.

Lemma 1.1.1. *Suppose a function f defined on a compact set E satisfies the γ-Hölder condition. Then,*

(i) $m_{\beta}(f(E)) \leq M^{\beta} m_{\alpha}(E)$ *if* $\beta = \alpha/\gamma$.

(ii) $\dim f(E) \leq \frac{1}{\gamma} \dim E$.

Proof. Suppose $\{F_k\}$ is a countable family of sets covering E. Then $\{f(E \cap F_k)\}$ covers $f(E)$, and $\operatorname{diam} f(E \cap F_k) \leq M(\operatorname{diam} F_k)^{\gamma}$ via the Hölder condition. Therefore,

$$\sum_k (\operatorname{diam} f(E \cap F_k))^{\alpha/\gamma} \leq M^{\alpha/\gamma} \sum_k (\operatorname{diam} F_k)^{\alpha}.$$

This establishes part (i). Part (ii) follows from part (i). □

We take the following lemma without proof.

Lemma 1.1.2. *The Cantor-Lebesgue function F on $\mathcal{C}$ satisfies the γ-Hölder condition with $\gamma = \log(2)/\log(3)$.*

Now we turn our attention to $\mathcal{C}$. We want to prove that $0 < m_\alpha(\mathcal{C}) < \infty$ for some α. If we do this, then we are done because $\mathcal{C}$ would have strict Hausdorff dimension α.
$m_\alpha(\mathcal{C}) \leq 1$ since $\mathcal{C} \subset [0,1]$ and by monotonicity.
In order to prove $0 < m_\alpha(\mathcal{C})$, we use the two lemmas above. Let $\alpha = \log(2)/\log(3)$. Now, observe that in the first lemma, if we let $E = \mathcal{C}$, f the Cantor-Lebesgue function, and $\gamma = \alpha$, we have $m_1([0,1]) \leq M^\beta m_\alpha(\mathcal{C})$. Clearly, this implies $1 \leq M m_\alpha(\mathcal{C})$, and so $m_\alpha(\mathcal{C}) > 0$. Therefore, $\dim \mathcal{C} = \log(2)/\log(3)$. In fact, the $\log(2)/\log(3)$-dimensional Hausdorff measure of $\mathcal{C}$ is 1.
The following draws heavily from Frank Morgan's GMT book.

1.2 Integral-Geometric Measure

Like the m-dimensional Hausdorff measure, which we will from now on denote $\mathscr{H}^m$ (either because I like that notation more or because I switched reference books and changing the 39 instances of m_α in the preceding notes was too daunting...one of the two), there is another m-dimensional measure on $\mathbb{R}^n$ ($m = 0, 1, ..., n$), namely the integral-geometric measure. We denote it by $\mathscr{I}^m$.

Chapter 2

Lipschitz Functions & Rectifiable Sets

Chapter 3

Chapter 4