Baby Steps in Geometric Measure Theory

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Abstract

These are notes for a reading course in Geometric Measure Theory I did under Dr. Natasa Sesum at Rutgers University - New Brunswick.

Contents

Introduction

The notes that follow are initially compiled from Stein & Shakarchi's Measure Theory text to build the foundation of Hausdorff measure/dimension. Then, we switch mainly to Frank Morgan's book on Geometric Measure Theory for the crux of the material.
It goes without saying that as these are notes for a reading course, they are prone to error. So take what is written with half a grain of the finest Himalayan salt.

Chapter 1

Hausdorff Measure, Hausdorff Dimension, & Rectifiability

We begin this reading course with two preliminaries: Hausdorff measure and rectifiability. These are foundational to studying GMT.

1.1 Hausdorff measure & Hausdorff dimension

1.1.1 Hausdorff measure

Hausdorff measure can be seen as a new way to measure volume/mass (as we are introducing a measure of course). Hausdorff measure is very much also related with the idea of dimension. For some appropriate set E and for a fixed $\alpha > 0$, $m_\alpha(E)$ can be thought of as the α-dimensional mass of E among sets of "dimension" α. The quotations around "dimension" will become clear as we get deeper in the material. If α is bigger than the dimension of E, $m_\alpha(E) = 0$. If α is smaller than the dimension of E, $m_\alpha(E) = \infty$. When α hits the dimension of E itself, then $m_\alpha(E)$ describes the actual α-dimensional size of the set.

It is also helpful to view Hausdorff measure as providing us more detail than the Lebesgue measure does. For example, consider the rectifiable curve Γ in $\mathbb{R}^2$. The Lebesgue measure is 0, so clearly information is lost about the curve using Lebesgue measure. On the other hand, the Hausdorff measure $m_1(\Gamma)$ actually gives us the length of Γ. Note that in $\mathbb{R}^2$, the usual way to measure length required differentiability everywhere, however here we only require rectifiability, which implies bounded variation, therefore differentiability *almost everywhere.*

As was the case with Lebesgue measure, we first define an outer measure. Let $E \subseteq \mathbb{R}^d$. Then, let the **exterior α-dimensional Hausdorff measure** of E be

$$m_\alpha^*(E) = \lim_{\delta \to 0} \inf \left\{ \sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k,\, \operatorname{diam} F_k \leq \delta \, (\forall k) \right\}.$$

For the sake of simplicity, we will denote

$$\mathcal{H}_\alpha^\delta(E) = \inf \left\{ \sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k,\, \operatorname{diam} F_k \leq \delta \, (\forall k) \right\}.$$

We now make note of three of things:

(1) As δ increases, $\mathcal{H}_\alpha^\delta(E)$ increases as well, and so $\lim_{\delta\to 0}\mathcal{H}_\alpha^\delta(E)$ exists but could be infinite.

(2) $\mathcal{H}_\alpha^\delta(E) \le m_\alpha^*(E)$ for all $\delta > 0$.

(3) Scaling is consistent here. If F is scaled by r, then $(\operatorname{diam} F)^\alpha$ scales by r^α.

Hausdorff measure enjoys the same properties of Lebesgue measure: monotonicity, subadditivity, and additivity of sets E_1, E_2 if $d(E_1, E_2) > 0$. Therefore m_α^* satisfies the same properties as a metric Carathéodory exterior measure, and so when we restrict ourselves to Borel sets, we obtain a countably additive measure m_α, the **α-dimensional Hausdorff measure**.

We now turn our attention to some interesting properties of the Hausdorff measure which are different from those exhibited by the Lebesgue measure:

(1) $m_0(E)$ counts the number of points in E, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$. In general, d-dimensional Hausdorff measure in $\mathbb{R}^d$ is equal to Lebesgue measure (up to a constant multiple).

(2) First, note that the iso-diametric inequality states the following: If $E \subset \mathbb{R}^d$ bounded and $\operatorname{diam} E = \sup\{|x-y| : x, y \in E\}$, then

$$m(E) \le v_d \left(\frac{\operatorname{diam} E}{2}\right)^d.$$

Then, we can deduce the following: If E is a Borel subset of $\mathbb{R}^d$, then $c_d m_d(E) = m(E)$ for $c_d = m(B)/(\operatorname{diam} B)^d = v_d/2^d$, where B is the unit ball, and v_d is its volume.

(3) In the same vein as (2): If E is a Borel subset of $\mathbb{R}^d$ and $m(E)$ is its Lebesgue measure, then $m_d(E) \approx m(E)$ in the sense that

$$c_d m_d(E) \le m(E) \le 2^d c_d m_d(E).$$

(4) If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. Also, if $m_\alpha^*(E) > 0$ and $\beta > \alpha$, then $m_\beta^*(E) = \infty$.

Proof. If $\operatorname{diam} F \le \delta$ and $\beta > \alpha$, then

$$(\operatorname{diam} F)^\beta = (\operatorname{diam} F)^{\beta-\alpha}(\operatorname{diam} F)^\alpha \le \delta^{\beta-\alpha}(\operatorname{diam} F)^\alpha.$$

Therefore, we conclude

$$\mathcal{H}_\beta^\delta(E) \le \delta^{\beta-\alpha}\mathcal{H}_\alpha^\delta(E) \le \delta^{\beta-\alpha} m_\alpha^*(E).$$

Since $m_\alpha^*(E) < \infty$ and $\beta - \alpha > 0$, as $\delta \to 0$, $m_\beta^*(E) = 0$.

The rest follows by the contrapositive. □

1.1.2 Hausdorff dimension

1.2 Rectifiability

Chapter 2

Chapter 3

Chapter 4