Unit -1

SAT

1) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$2p = \frac{2x}{a^2} \Rightarrow a^2 = \frac{x}{p}$

$2q = \frac{2y}{b^2} \Rightarrow b^2 = \frac{y}{q}$

$\therefore 2z = \frac{x^2}{x} p + \frac{y^2}{y} q$

$2z = px + qy$ //

2) $z = ax + by + a^2 + b^2$

$p = a$

$q = b$

$\therefore z = px + qy + p^2 + q^2$ //

3) $z = (x-a)^2 + (y-b)^2 + 1$

$p = 2(x-a) \Rightarrow x - a = \frac{p}{2}$

$q = 2(y-b) \Rightarrow y - b = \frac{q}{2}$

$\therefore z = \frac{p^2}{4} + \frac{q^2}{4} + 1$

$4z = p^2 + q^2 + 4$ //

4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$ — ① $\quad -\frac{x}{a^2} = \frac{zp}{c^2}$

$-\frac{c^2}{a^2} = \frac{zp}{x}$

$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$ — ②

$\frac{2}{a^2} + \frac{2p^2}{c^2} + \frac{2zr}{c^2} = 0$

$\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{zr}{c^2} = 0$

$-\frac{1}{a^2} = \frac{p^2 + zr}{c^2}$

$-\frac{c^2}{a^2} = p^2 + zr$

$\frac{zp}{x} = p^2 + zr$

$zp = p^2x + xzr$

5) $z = f(x^2 - y^2)$

$p = 2xf' \Rightarrow f' = \frac{p}{2x}$

$q = -2yf' \Rightarrow q = -2y\frac{p}{2x}$

$\Rightarrow qx = -py$

$\Rightarrow qx + py = 0$

6) $z = f(x+at) + g(x-at)$

$z_x = f' + g'$

$z_{xx} = f'' + g''$

$z_t = af' - ag' = a(f' - g')$

$z_{tt} = a(af'' + ag'')$

$= a^2(f'' + g'')$

$= a^2 z_{xx}$ //

7) $z = f(x) + e^y g(x)$

$p = f' + e^y g'$

$q = e^y g$

$r = f'' + e^y g''$

$t = e^y g$

$\therefore q = t$ //

8) $z = y^2 + 2f\left(\frac{1}{x} + \ln y\right)$

$p = \frac{-2}{x^2} f' \Rightarrow f' = -\frac{px^2}{2}$

$q = 2y + \frac{2}{y} f'$

$\Rightarrow q = 2y + \frac{2}{y}\left(-\frac{px^2}{2}\right)$

$\Rightarrow q = 2y - \frac{px^2}{y}$

$\Rightarrow qy = 2y^2 - px^2$

$\Rightarrow qy + px^2 = 2y^2$

9) $z = f(xy) + g(x+y)$

$p = yf' + g'$

$q = xf' + g'$

$r = y^2 f'' + g'' \Rightarrow r - y^2 f'' = g''$

$s = xyf'' + f' + g''$

$t = x^2 f'' + g''$

$p - q = yf' + \cancel{g'} - xf' - \cancel{g'}$

$p - q = f'(y - x)$

$f' = \frac{p-q}{y-x}$

$$r - t = (y^2 - x^2) f''$$

$$f'' = \frac{r-t}{y^2-x^2}$$

$$s = xy\frac{(r-t)}{y^2-x^2} + \frac{p-q}{y-x} + r - y^2\frac{(r-t)}{y^2-x^2}$$

$$\cancel{s = xy\frac{(r-t)}{y^2-x^2} + \frac{p-q}{y-x}} +$$

$$s(y^2-x^2) = xyr - xyt + (p-q)(y+x) + r(y^2-x^2)$$
$$- y^2 r + y^2 t$$

$$s(y^2-x^2) = xyr - xyt + py + px - qy - qx + ry^2 - rx^2$$
$$- y^2 r + y^2 t$$

$$s(y^2-x^2) = (p-q)(y+x) + rx(y-x) + yt(y-x)$$

LP

10) $y^2zp + x^2zq = xy^2$

Here, $P = y^2 z \quad Q = x^2 z \quad R = xy^2$

∴ The subsidiary eqn is :—

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Considering the eq^n $\frac{dx}{y^2z} = \frac{dy}{x^2z}$

$$x^2dx = y^2dy$$

$$x^3 - y^3 = c_1$$

$$\therefore\ u = x^3 - y^3$$

Considering the eq^n $\frac{dx}{y^2z} = \frac{dz}{xy^2}$

$$xdx = zdz$$

$$x^2 - z^2 = c_2$$

$$\therefore\ v = x^2 - z^2$$

$$\therefore f(x^3 - y^3,\ x^2 - z^2) = 0$$

11) $x^2p + y^2q = z^2$

Here, $P = x^2$ $\quad Q = y^2 \quad R = z^2$

$\therefore$ The subsidiary eq^n is —

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Considering the eq^n $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

$$\therefore\ u = \frac{1}{x} - \frac{1}{y}$$

Considering the eq^n $\frac{dy}{y^2} = \frac{dz}{z^2}$

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

$$c_2 = \frac{1}{y} - \frac{1}{z}$$

$$\therefore v = \frac{1}{y} - \frac{1}{z}$$

$$\therefore F\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$$

12) $z(p-q) = z^2 + (x+y)^2$

$zp - zq = z^2 + (x+y)^2$

Here, $P = z$ $\quad Q = -z$ $\quad R = z^2 + (x+y)^2$

$\therefore$ The Subsidiary eq^n is —

$$\frac{dx}{z} = -\frac{dy}{z} = \frac{dz}{z^2 + (x+y)^2}$$

Considering the eq^n $\frac{dx}{\cancel{z}} = -\frac{dy}{\cancel{z}}$

$$x + y = c_1$$

$$\therefore u = x + y$$

Considering the eq^n $-\frac{dy}{z} = \frac{dz}{z^2 + (x+y)^2}$

$$-dy = \frac{z\,dz}{z^2 + c_1^2}$$

$$-2y = \log(z^2 - c_1^2) + \log c_2$$

$$e^{-2y} = \log(z^2 - c_1^2)\, c_2$$

$$c_2 = e^{2y} \log(z^2 - (x+y)^2)$$

$$\therefore\ v = e^{2y} \log(z^2 - (x+y)^2)$$

13) $(y+z)p + (z+x)q = x+y$

Here, $P = y+z \quad Q = z+x \quad R = x+y$

$\therefore$ The subsidiary eqn is —

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Choosing $x, -y, -z$, as multiplier

14) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Here, $P = x^2(y-z)$ $Q = y^2(z-x)$ $R = z^2(x-y)$

∴ The subsidiary eqn is—

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Choosing $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multiplier we get

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\frac{-1}{x} - \frac{1}{y} - \frac{1}{z} = c_1$$

$$c_1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\therefore U = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier we get

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2$$

$$\therefore V = xyz$$

$$\therefore F\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$$

15) $\frac{p^2+qy}{2} = -(z+y^2)$

$p^2+qy = -2z-2y^2$

$p^2+qy+2z+2y^2 = 0$ — (1)

$f_x = 0$

$f_y = q+4y$

$f_p = 2p.$

$f_q = y$

$f_z = 2$

The subsidiary eqn is

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2+qy} = \frac{dp}{-0-2p} = \frac{dq}{-q-4y-2q}$$

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2+qy} = \frac{dp}{-2p} = \frac{dq}{-3q-4y}$$

Considering the eqn $\frac{dx}{2p} = \frac{dp}{-2p}$

$x = -p+a$

$p+x = a$

$p = a-x$

Now, putting the value of p in eq^n ①, we get

$$(a-x)^2 + qy + 2y^2 + 2z = 0$$

$$\Rightarrow qy = -2z - (a-x)^2 - 2y^2$$

$$\Rightarrow q = -\frac{2z}{y} - \frac{(a-x)^2}{y} - 2y$$

Now, the complete sol^n is —

$$dz = p\,dx + q\,dy$$

$$dz = (a-x)\,dx + \left[-\frac{2z}{y} - \frac{(a-x)^2}{y} - 2y\right]dy$$

$$dz + \frac{2z}{y}\,dy = (a-x)\,dx - \frac{(a-x)^2}{y}\,dy - 2y\,dy$$

Multiplying both sides with $2y^2$, we get

$$2y^2\,dz + 4yz\,dy = 2y^2(a-x)\,dx - (a-x)^2\,2y\,dy - 4y^3\,dy$$

$$d(2y^2 z) = -2y^2(x-a) - 2(x-a)^2 y\,dy - 4y^3\,dy$$

$$d(2y^2 z) = -d\left(y^2(x-a)^2\right) - 4y^3\,dy$$

$$2y^2 z = -y^2(x-a)^2 - y^4 + b^2$$

$$2y^2 z + y^2(x-a)^2 + y^4 = b^2$$

$$y^2\left(2z + (x-a)^2 + y^2\right) = b^2$$

6) $q - px - p^2 = 0$ —— (1)

$f_x = -p$

$f_y = 0$

$f_p = -x - 2p$

$f_q = 1$

$f_z = 0$

∴ The subsidiary eqⁿ is —

$$\frac{dx}{-x-2p} = \frac{dy}{1} = \frac{dz}{-px-2p^2+q} = \frac{dp}{p-0} = \frac{dq}{0+0}$$

$$\frac{dx}{-x-2p} = \frac{dy}{1} = \frac{dz}{-px-2p^2+q} = \frac{dp}{p} = \frac{dq}{0}$$

Considering the eqⁿ $\frac{dy}{1} = \frac{dp}{p}$

$$y = \log p + \log a$$

$$y = \log pa$$

$$e^y = pa$$

$$p = \frac{e^y}{a}$$

Now, Putting the value of p in eqn ①, we get

$$q - \frac{e^y x}{a} - \frac{e^{2y}}{a^2} = 0$$

$$q = \frac{e^y x}{a} + \frac{e^{2y}}{a^2}$$

Now, the complete soln is —

$$dz = p\,dx + q\,dy$$

$$dz = \frac{e^y}{a} dx + \frac{e^y x}{a} dy + \frac{e^{2y}}{a^2} dy$$

$$dz = d\left(\frac{xe^y}{a}\right) + \frac{e^{2y}}{a^2} dy$$

$$z = \frac{xe^y}{a} + \frac{1}{2}\frac{e^{2y}}{a^2} + b$$

17) $qz - p^2y - q^2y = 0$ — ①

$f_x = 0$
$f_y = -p^2 - q^2$

$f_p = -2py$

$f_q = z - 2qy$

$f_z = q$

The subsidiary eqⁿ is —

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{+2p^2y - zq + 2q^2y} = \frac{dp}{0-pq} = \frac{dq}{+p^2+q^2-q^2}$$

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{2p^2y+2q^2y-zq} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

Considering the eqⁿ $\frac{dp}{-pq} = \frac{dq}{p^2}$

$$p\,dp = -q\,dq$$

$$\frac{p^2}{2} + \frac{q^2}{2} = b$$

$$p^2+q^2 = 2b$$

$$p^2+q^2 = a^2$$

$$p = \sqrt{a^2-q^2}$$

Putting the value of p in eqⁿ ①, we get

$$qz - ya^2 = 0$$

$$qz = ya^2$$

$$q = \frac{ya^2}{z}$$

$$\therefore\ p^2 = a^2 - \frac{y^2a^4}{z^2} = \frac{a^2z^2 - y^2a^4}{z^2}$$

$$\Rightarrow p = \sqrt{\frac{a^2}{z^2}(z^2-y^2a^2)} = \frac{a}{z}\sqrt{z^2-y^2a^2}$$

Now the complete sol^n is —

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2-y^2a^2}\,dx + \frac{ya^2}{z}\,dy$$

$$z\,dz = a\sqrt{z^2-y^2a^2}\,dx + ya^2\,dy$$

$$z\,dz - ya^2\,dy = a\sqrt{z^2-y^2a^2}\,dx$$

$$\Rightarrow \frac{1}{a}\left(\frac{z\,dz - ya^2\,dy}{\sqrt{z^2-y^2a^2}}\right) = dx$$

$$\Rightarrow \frac{1}{2a}\left(\frac{2z\,dz - 2ya^2\,dy}{\sqrt{z^2-y^2a^2}}\right) = dx$$

$$\frac{1}{\cancel{2}a} \times \cancel{2}\sqrt{z^2-y^2a^2} = x+b$$

$$\sqrt{z^2-y^2a^2} = a(x+b)$$

On squaring both sides we get

$$z^2-y^2a^2 = a^2(x+b)^2$$

$$z = a^2\left(y^2+(x+b)^2\right)_{//}$$

18) $yz - p(xy+q) - qy = 0$

$f_x = -py$

$f_y = z - px - q$

$f_p = -xy - q$

$f_q = -p - y$

$f_z = y$

The subsidiary eq^n is —

$$\frac{dx}{-xy-q} = \frac{dy}{-p-y} = \frac{dz}{-pxy-pq-pq-yq} = \frac{dp}{\cancel{py} - py}$$

$$= \frac{dq}{-z+px+q-qy}$$

$$\frac{dx}{-xy-q} = \frac{dy}{-p-y} = \frac{dz}{-pxy-2pq-qy} = \frac{dp}{0} = \frac{dq}{-z+px+q-qy}$$

Considering the eq^n $\frac{dp}{0} = \frac{dx}{-xy-q}$

$$dp = 0$$

$$p = a$$

Now, Putting the value of p in eqn ①, we get

$$yz - axy - aq - qy = 0$$

$$\Rightarrow y(z-ax) - q(a+y) = 0$$

$$\Rightarrow q(a+y) = y(z-ax)$$

$$\Rightarrow q = \frac{y(z-ax)}{a+y}$$

Now the complete solⁿ is –

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + \frac{y(z-ax)}{a+y}\,dy$$

$$dz - a\,dx = \frac{y(z-ax)}{a+y}\,dy$$

$$\frac{dz - a\,dx}{z-ax} = \frac{y}{a+y}\,dy$$

$$\frac{d(z-ax)}{(z-ax)} = \left(1 - \frac{a}{a+y}\right)dy$$

$$\log(z-ax) = y - a\log(y+a) + \log b$$

$$\log(z-ax) + \log(y+a)^a - \log b = y$$

$$\log\frac{(z-ax)(y+a)^a}{b} = y$$

$$(z-ax)(y+a)^a = be^y \;//$$

9) $z - q^2y - p^2x = 0$

$f_x = -p^2$

$f_y = -q^2$

$f_z = 1$

$f_p = -2px$

$f_q = -2qy$

The subsidiary eq^n is

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q}$$

Choosing the $p^2, 2px, 0, q^2, 2qy$ as multiplier we get

$$\frac{p^2\,dx + 2px\,dp}{-\cancel{2p^3} + \cancel{2p^3x} - 2p^2x} = \frac{q^2\,dy + 2qy\,dy}{-\cancel{2q^3y} + \cancel{2q^3y} - 2q^2y}$$

$$\Rightarrow \frac{p^2\,dx + 2px\,dp}{+\cancel{2}p^2x} = \frac{q^2\,dy + 2qy\,dq}{+\cancel{2}q^2y}$$

$$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

$\Rightarrow \log p^2x = \log aq^2y$

$\Rightarrow p^2x = aq^2y$

Now, From eq^n (1)

$$z - q^2y - p^2x = 0$$

$$z - q^2y - aq^2y = 0$$

$$z - q^2y(1+a) = 0$$

$$q^2y(1+a)\ z$$

$$q^2 = \frac{z}{y(1+a)}$$

$$q = \left[\frac{z}{y(1+a)}\right]^{1/2}$$

$$\therefore\ p^2 = \frac{aq^2y}{x} = \frac{ay}{x}\ \frac{z}{y(1+a)}$$

$$p = \left[\frac{az}{x(1+a)}\right]^{1/2}$$

Now the complete sol^n is —

$$dz = p\,dx + q\,dy$$

$$dz = \left[\frac{az}{x(1+a)}\right]^{1/2} dx + \left[\frac{z}{y(1+a)}\right]^{1/2} dy$$

$$\sqrt{1+a}\ \frac{dz}{z^{1/2}} = \sqrt{a}\ \frac{1}{\sqrt{x}}\ dx + \frac{1}{\sqrt{y}}\ dy$$

$$2\sqrt{1+a}\ \sqrt{z} = 2\sqrt{a}\ \sqrt{x} + 2\sqrt{y} + 2b$$

$$\sqrt{1+a}\ \sqrt{z} = \sqrt{a}\ \sqrt{x} + \sqrt{y} + b$$