Linear Algebra

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Outline

1. Introduction.
2. Vector space.
3. Subspace of a vector space.

1. Introduction

Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $\quad (x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space

= set of all real number

$n = 2$ R^2 = 2-space

= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ R^4 = 4-space

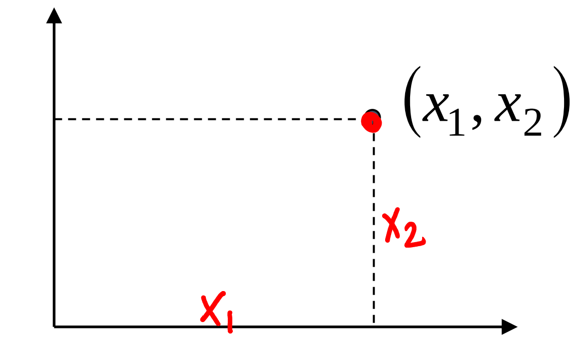
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

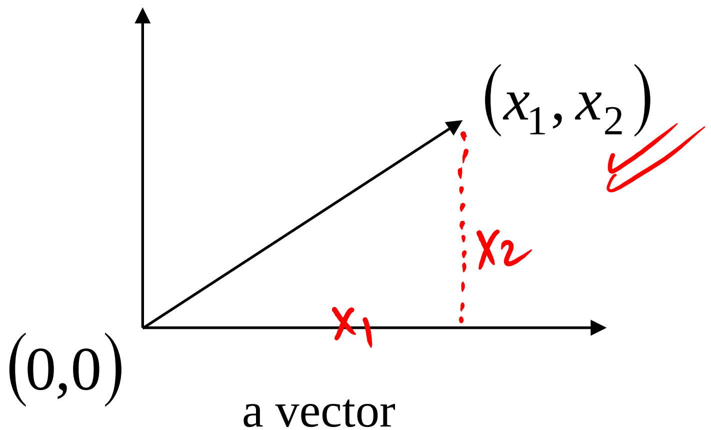
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

<u>Vector notation:</u> $\vec{v}$, $\bar{v}$, $\boldsymbol{v}$

$(1,2),\ (3,-1) \in R^2$
space

$$\mathbf{u} = (u_1, u_2, \cdots, u_n),\ \ \mathbf{v} = (v_1, v_2, \cdots, v_n) \qquad \text{(two vectors in } R^n\text{)}$$

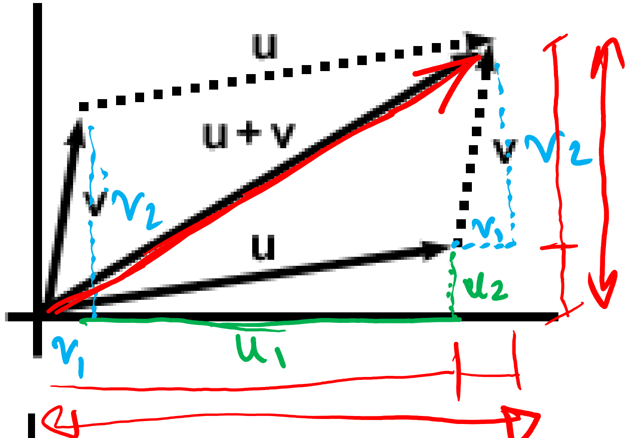
- Equal:

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1,\ u_2 = v_2, \cdots, u_n = v_n$$

- Vector addition (the sum of **u** and **v**):

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

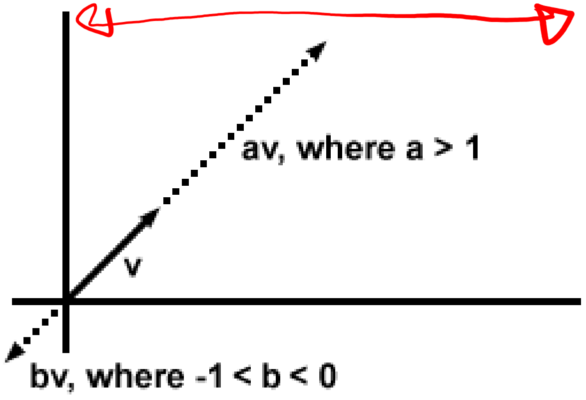
Graphically



- Scalar multiplication (the scalar multiple of **u** by *c*):

$$c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$$

Graphically



$\bar{u} - \bar{v} = \bar{u} + (-\bar{v})$

- Notes:

The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

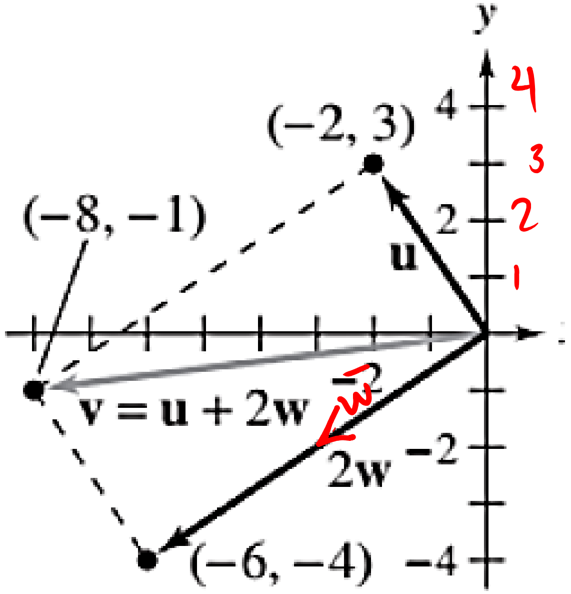
$\bar{v}$ $\bar{u}$ $\bar{v}$ $-\bar{v}$ $\bar{u}+(-\bar{v})$

Ex: Given $\vec{u} = (-2, 3)$ and $\vec{w} = (-3, -2)$ $\in R^2$; Find $\vec{v}$

$$\mathbf{v} = \mathbf{u} + 2\mathbf{w}$$

Solution

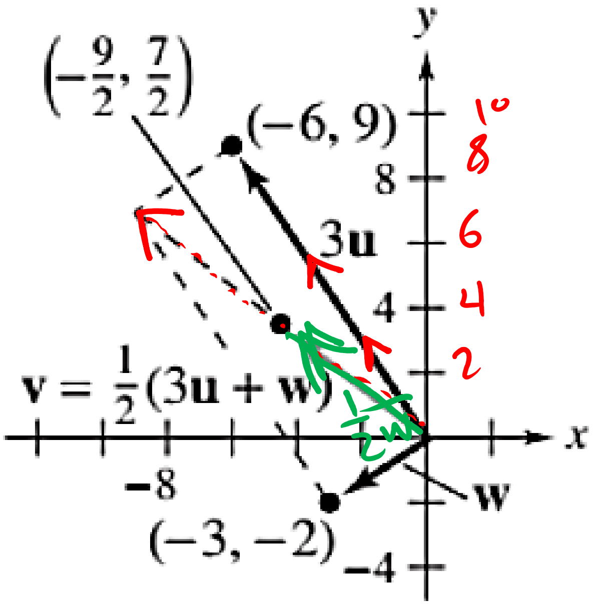
$$= (-2, 3) + 2(-3, -2)$$
$$= (-2, 3) + (-6, -4)$$
$$= (-2 - 6, 3 - 4)$$
$$= (-8, -1)$$



$$\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$$

Solution

$$= \frac{1}{2}(3(-2, 3) + (-3, -2))$$
$$= \frac{1}{2}((-6, 9) + (-3, -2))$$
$$= \frac{1}{2}(-9, 7) = \left(-\frac{9}{2}, \frac{7}{2}\right)$$



Vector Operations in R^3

Provided that $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in R^3, find each vector.

(a) $\mathbf{u} + \mathbf{v}$ (b) $2\mathbf{u}$ (c) $\mathbf{v} - 2\mathbf{u}$

Solution

(a) $\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$

(b) $2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$ Scalar multiple of $\bar{u}$

(c) Using the result of part (b), you have

$$\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3).$$

Ex: Given $\vec{u} = (2,-1,5,0)$, $\vec{v} = (4,3,1,-1)$ and $\vec{w} = (-6,2,0,3)$; Solve for $\vec{x}$ $\in R^4$

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$ (b) $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

Explicit - صريح

Implicit - ضمني

Solution

(a) Using the properties listed in Theorem , you have

$$
\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
&= (18, -11, 9, -8).
\end{aligned}
$$

(b) Begin by solving for $\mathbf{x}$ as follows.

$$
\begin{aligned}
3(\mathbf{x} + \mathbf{w}) &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} + 3\mathbf{w} &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} - \mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
\mathbf{x} &= \tfrac{1}{2}(2\mathbf{u} - \mathbf{v} - 3\mathbf{w})
\end{aligned}
$$

Using the result of part (a) produces

$$
\begin{aligned}
\mathbf{x} &= \tfrac{1}{2}(18, -11, 9, -8) \\
&= \left(9, -\tfrac{11}{2}, \tfrac{9}{2}, -4\right).
\end{aligned}
$$

Which of the vectors below are scalar multiples of $\mathbf{z} = (3, 2, -5)$?

$*-2$ check

$*-2$ check

$*-2$

(a) $\mathbf{u} = (-6, -4, 10)$ ✓ 2 times the original vector in opposit direc.

$*\frac{2}{3}$ check

(b) $\mathbf{v} = \left(2, \frac{4}{3}, -\frac{10}{3}\right)$ ✓ 2/3 the original vect. in the same direc.

$*2$

(c) $\mathbf{w} = (6, 4, 10)$ X

should be -10

Solution

(a) Because $(-6, -4, 10) = -2(3, 2, -5)$, $\mathbf{u}$ is a scalar multiple of $\mathbf{z}$.

(b) Because $\left(2, \frac{4}{3}, -\frac{10}{3}\right) = \frac{2}{3}(3, 2, -5)$, $\mathbf{v}$ is a scalar multiple of $\mathbf{z}$.

(c) Because $(6, 4, 10) \neq c(3, 2, -5)$ for any c, $\mathbf{w}$ is *not* a scalar multiple of $\mathbf{z}$.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

المحايد الجمعي

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

المعكوس الجمعي

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

$$\bar{v} + (-\bar{v}) = \bar{0}$$

Properties of additive identity and additive inverse

1. The additive identity is unique.

    1. If $\vec{u} + \vec{v} = \vec{v}$ then $\vec{u} = \vec{0}$

2. The additive inverse of $\vec{v}$ is unique.

    1. If $\vec{u} + \vec{v} = \vec{0}$ then $\vec{u} = -\vec{v}$

scalar

3. $0\,\vec{v} = \vec{0}$

scalar

4. $c\,\vec{0} = \vec{0}$

scalar vector zero vec.

5. If $c\,\vec{v} = \vec{0}$ then $c = 0$ or $\vec{v} = \vec{0}$

6. $-(-\vec{v}) = \vec{v}$

Matrix Matrix Zero mat.

However, $AB = 0$ does not imply that $A = 0$ or $B = 0$

Properties of vector addition and scalar multiplication

Addn

1. $\vec{u} + \vec{v}$ "Closed under addition".
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ "Commutative".
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ "Associative".
4. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ "Addition identity".
5. $\vec{u} + (-\vec{u}) = 0$ "Addition inverse".

scalar multip-

6. $c\,\vec{u}$ "Closed under scalar multiplication".
7. $c\,(\vec{u} + \vec{v}) = c\,\vec{u} + c\,\vec{v}$
8. $(c + d)\,\vec{v} = c\,\vec{v} + d\,\vec{v}$
9. $c\,(d\,\vec{v}) = (c\,d)\,\vec{v}$

"Distributive" (7–9)

General rule
any vec. space must pass through the origin

scalar

10. $1\,\vec{v} = \vec{v}$ "Multiplicative identity".

Operation is closed in some space ⋯→ Result of the operation still in the same space

$\underset{\in \mathbb{R}^2}{(1,2)} + \underset{\in \mathbb{R}^2}{(2,-1)} = \underset{\in \mathbb{R}^2}{(3,1)}$

$2\,\underset{\in \mathbb{R}^2}{(1,2)} = \underset{\in \mathbb{R}^2}{(2,4)}$

2. Vector space

Vector Spaces

- Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

تجمع لأي
set of objects satisfying the 10 conds.

$c \in \mathbb{R}$ any real value integer or fraction +ve or -ve

Most important conds. "Start with them"

Addition:

(1) **u**+**v** is in V Closed

(2) **u**+**v**=**v**+**u**

(3) **u**+(**v**+**w**)=(**u**+**v**)+**w**

(4) V has a zero vector **0** such that for every **u** in V, **u**+**0**=**u**

(5) For every **u** in V, there is a vector in V denoted by –**u** such that **u**+(–**u**)=**0**

Scalar multiplication:

(6) $c\mathbf{u}$ is in V. Closed

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V : nonempty set

c : scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}$: vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}$: scalar multiplication

$(V, \ +, \ \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$: zero vector space "Most trivial vector space"

- Examples of vector spaces:

(1) n-tuple space: R^n

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$ vector addition Closed

$$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n)$$ scalar multiplication Closed

Also, satisfy the remain 10 conds

(2) Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: : $(m = n = 2)$ "set of all matrices of size 2x2"

$$\left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -0.1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \cdots \right\}$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition Closed

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication Closed

Also, satisfies the rem 10 conds

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

P_3 : "set of all polynomials of degree 3 or less

ex: $x+x^3$, $2+x$, 2, $0+0x+0x^2+0x^3$

x^3+4

$\equiv$

$(0, 0, 0, 0)$

$x^0 \quad x^1 \quad x^2 \quad x^3$

Summary of Important Vector Spaces

R = set of all real numbers
R^2 = set of all ordered pairs
R^3 = set of all ordered triples
R^n = set of all n-tuples
$C(-\infty, \infty)$ = set of all continuous functions defined on the real number line
$C[a, b]$ = set of all continuous functions defined on a closed interval $[a, b]$
P = set of all polynomials
P_n = set of all polynomials of degree $\leq n$
$M_{m,n}$ = set of all $m \times n$ matrices
$M_{n,n}$ = set of all $n \times n$ square matrices

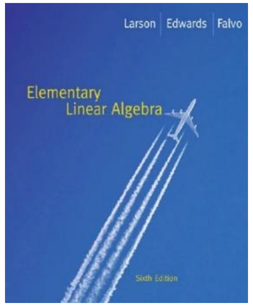
All satisfy the 10 conds.

For further reading about:

$C(-\infty, \infty)$ and $C[a, b]$ → go to the reference

Elementary Linear Algebra, by Larson et. al.

[Files section of the team in references – example 5 – Pages 193-194]



- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex: The set of all integer is not a vector space.

$S = \{1, 2, 3, 4, \cdots\}$

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\substack{\uparrow \\ \text{scalar}}}{(\tfrac{1}{2})}\underset{\substack{\uparrow \\ \text{integer}}}{(1)} = \underset{\substack{\uparrow \\ \text{noninteger}}}{\tfrac{1}{2}} \notin V$$

(it is not closed under scalar multiplication)

Not integer

- Ex: The set of all second-degree polynomials is not a vector space.

only "Not less"

Note

$P_2 \ni n \leq 2$

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

$S = \left\{ \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \dots \right\}$

- Ex:

$$S = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix}, a, b, c \in \mathcal{R} \right\}$$

Verify S is not a vector space.

What about

$$S = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, a, b, c \in \mathcal{R} \right\}$$

Sol:

شرط واحد يكفي

1. $\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S$

2. $\begin{bmatrix} a_1 & b_1 \\ c_1 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 2 \end{bmatrix} \notin S$

not 1

3. $2 \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} 2\,a & 2\,b \\ 2\,c & 2 \end{bmatrix} \notin S$

$\bar{0} \in S$

$\begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & 0 \end{bmatrix} \in S$

$k \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & 0 \end{bmatrix} \in S$

+ يجب التحقق بالباقي ال 10 شروط

- Ex:

$S = \{(x, y), x \geq 0, y \in \mathcal{R}\}$ Verify S is not a vector space.

Solution

This set is *not* a vector space. The set is not closed under scalar multiplication. For example,
$(-1)(3, 2) = (-3, -2)$ is not in the set.

any Scalar

$x \geq 0$

any number

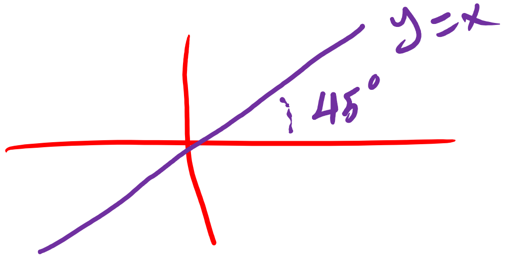
- Ex:

$S = \{(x, y), x \geq 0, y \geq 0\}$ Verify S is not a vector space.

- Ex:

$$S = \{(x, x), x \in \mathcal{R}\}$$

Verify S is a vector space.



(1) **u**+**v** is in *V*

(2) **u**+**v**=**v**+**u**

(3) **u**+(**v**+**w**)=(**u**+**v**)+**w**

(4) *V* has a zero vector **0** such that for every **u** in *V*, **u**+**0**=**u**

(5) For every **u** in *V*, there is a vector in *V* denoted by –**u** such that **u**+(–**u**)=**0**

(6) $c\mathbf{u}$ is in *V*.

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(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

① $\bar{0} \in S$ where $x=0$

let $\bar{v}_1 = (x_1, x_1) \in S$

$\bar{v}_2 = (x_2, x_2) \in S$

② $\bar{v}_1 + \bar{v}_2 = (x_1+x_2, x_1+x_2) \in S$

نفسه

closed

③ $c\bar{v}_1 = (cx_1, cx_1) \in S$

نفسه

close

+ باقي الـ 10 conds

- Ex:

$$S = \left\{ \left(x, \tfrac{1}{2}x \right), x \in \mathcal{R} \right\}$$ Verify S is a vector space.

زوجي

زوجي $\frac{1}{2}$

(1) **u**+**v** is in *V*

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(10) $1(\mathbf{u}) = \mathbf{u}$

10 conds الباقي +

① $\bar{0} \in S$ where $x=0$

Let $\bar{v}_1 = (x_1, \frac{1}{2}x_1)$

$\bar{v}_2 = (x_2, \frac{1}{2}x_2)$

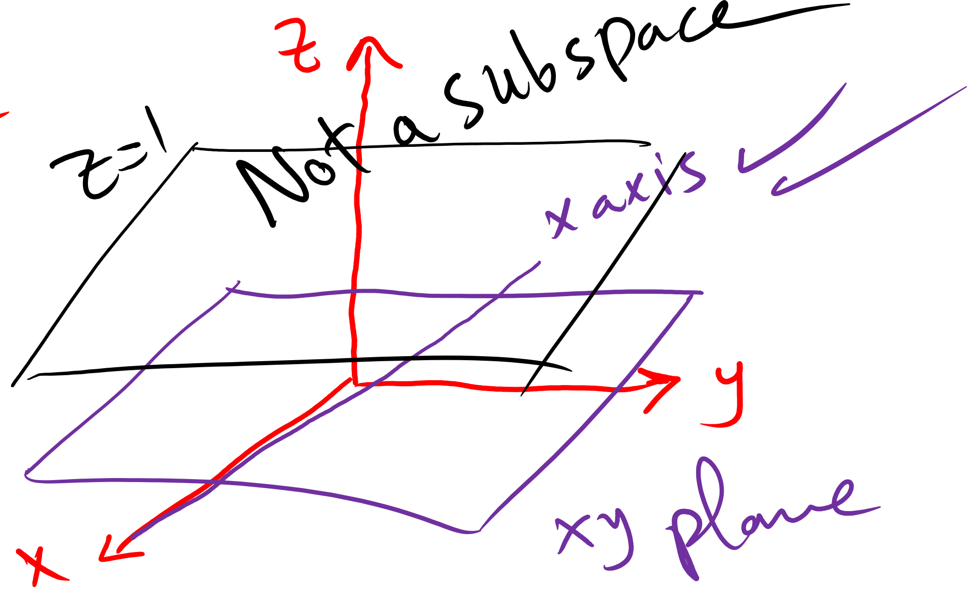
② $\bar{v}_1 + \bar{v}_2 = (x_1 + x_2, \underline{\frac{1}{2}x_1 + \frac{1}{2}x_2})$

زوجي $\rightarrow \frac{1}{2}(x_1 + x_2) \in S$ Closed

③ $c\bar{v}_1 = (cx_1, \underline{c\frac{1}{2}x_1}) \in S$ closed

نصفه $\rightarrow \frac{1}{2}(cx_1)$

R3 space
vector space



3. Subspace of a vector space

جزء من الكل

بشرط الجزئي حد ذاته vec. space
يحقق ال ١٠ شروط

بما ان subspace جزء من vec. space
بيحقق ال ١٠ شروط

بيحقق جزء من ال ١٠
وباقي فقط التأكد من 3 شروط خاصة

closed +, closed scalar mul, $\bar{0}$ ident

Subspaces of Vector Spaces

verify

S

vector space → 10 conds

subspace of a vec. space → only 3 main conds

- Subspace:

$(V,+,\bullet)$: a vector space

$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Other "non-trivial subspaces" are called Proper subspace.

- A subspace is also a vector space.
  - It satisfies all 10 conditions of a vector space.

- Thm: (Test for a subspace)

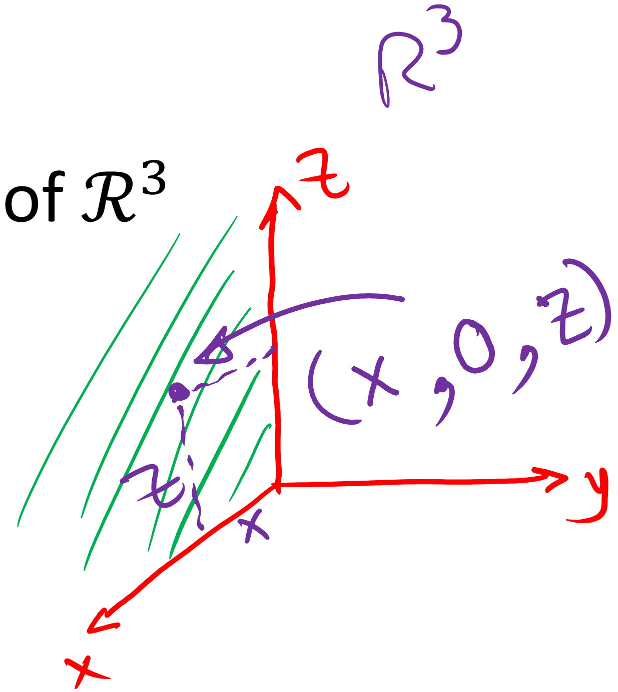
  If W is a <u>nonempty subset</u> of a vector space V, then W is a subspace of V if and only if the following conditions hold.

  (1) $\mathbf{0} \in W$.

  (2) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

  (3) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- Ex: Show that the xz plane is a subspace (s.s.) of $\mathcal{R}^3$



Sol: $W = \{(x, 0, z), x, z \in \mathcal{R}\}$

1. • $\bar{0} \in W$

2. • Let $u = \{u_1, 0, u_3\}$ and $v = \{v_1, 0, v_3\}$

   $u + v = \{u_1 + v_1, 0 + 0, u_3 + v_3\} \in W$ closed

3. • Let $u = \{u_1, 0, u_3\}$

   $c\,u = \{cu_1, c0, cu_3\} \in W$ closed

$\therefore$ W is a s.s. to R^3

- Ex: $W = \{(x_1, x_2, 1), x_1, x_2 \in \mathcal{R}\}$ Is W s.s. of $\mathcal{R}^3$?

Sol:

- $\bar{0} \notin W$

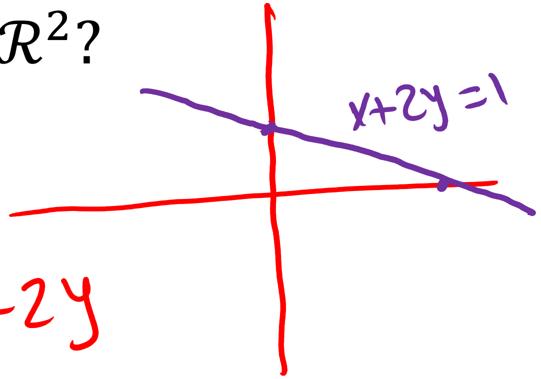
$0 + 2(0) \neq 1$

- Ex: $W = \{(x, y), x + 2\,y = 1\}$ Is W s.s. of $\mathcal{R}^2$?

$x=0 \rightarrow y=\frac{1}{2}$

The set of points on the line $x + 2y = 1$

$y=0 \rightarrow x=1$



Sol:

$x+2y=1 \rightarrow x = 1-2y$

- $\bar{0} \notin W$

$(1-2y, y)\,;\, y \in R$

- Ex: $W = \{(x, y), x + 2y = 0\}$ Is W s.s. of $\mathcal{R}^2$?

$x = -2y$

Sol:

Let $y = t$

- $\bar{0} \in W$

$-2 *$ عوض

[illegible]

$$W = \{(-2t, t), t \in \mathcal{R}\}$$

- Let $u = \{-2u_1, u_1\}$ and $v = \{-2v_1, v_1\}$

$u + v = \{-2u_1 - 2v_1, u_1 + v_1\} \in W$

$-2(u_1 + v_1)$

- Let $u = \{-2u_1, u_1\}$

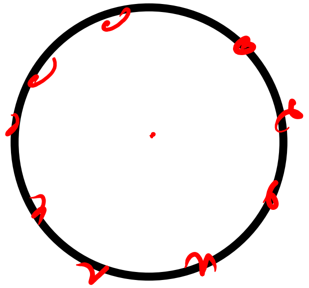
$cu = \{c(-2u_1), cu_1\} \in W$

$-2(cu_1)$

$\therefore W$ is a s.s. of R^2

- Ex: $W = \{(x, y), \underbrace{x^2 + y^2 = 1}\}$ Is W s.s. of $\mathcal{R}^2$?

All points $\boxed{\text{on}}$ the circle of radius 1 & center $(0,0)$



Sol:

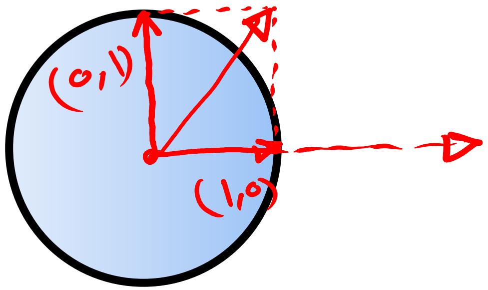
- $\bar{0} \notin W$

- Ex: $W = \{(x, y), x^2 + y^2 \leq 1\}$ Is W s.s. of $\mathcal{R}^2$?

on & inside

Sol:

- $\bar{0} \in W$



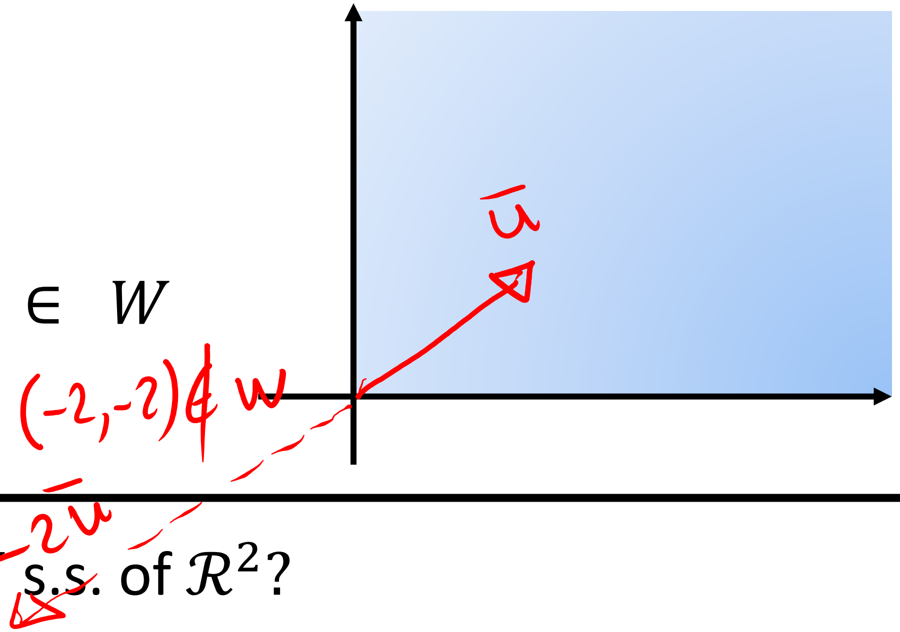
- Let $u = \{0, 1\}$ and $v = \{1,0\}$

  $u + v =$ $\{1,1\} \notin W$

- Ex: $W = \{(x, y), x \geq 0, y \geq 0\}$ Is W s.s. of $\mathcal{R}^2$?

Sol:

- $\bar{0} \in W$
- For $u, v \in W \rightarrow u + v \in W$
- Let $u = (1,1) \rightarrow -2u = (-2,-2) \notin W$



- Ex: $W = \{(x, y), x\,y \geq 0\}$ Is W s.s. of $\mathcal{R}^2$?

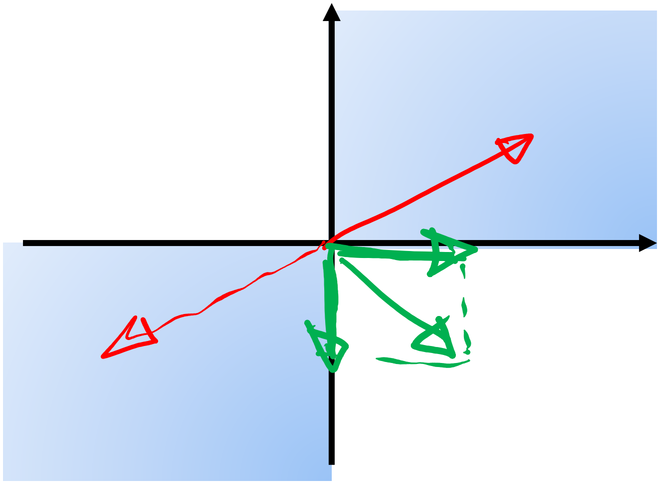
$+\ +$

$-\ -$

Sol:

- $\bar{0} \in W$
- Let $u = (1,0)$ and $v = (0,-1)$

$u + v = (1,-1) \notin W$



- Ex: $W = \{(x, x\,z, z); x, z \in \mathcal{R}\}$ Is W s.s. of $\mathcal{R}^3$?

Sol:

- $\bar{0} \in W$

- Let $u = \{u_1, u_1 u_3, u_3\}$ and $v = \{v_1, v_1 v_3, v_3\}$

  $u + v = \{u_1+v_1, u_1u_3+v_1v_3 \neq, u_3+v_3\} \notin W$

  Check

  product

  $(u_1+v_1)(u_3+v_3) = u_1u_3 + v_1v_3 + u_3v_1 + u_1v_3$

- Let $u = \{u_1, u_1 u_3, u_3\}$

  $c\,u = \{cu_1, cu_1u_3 \neq, cu_3\} \notin W$

  Check

  $c^2u_1u_3$

- Ex: $W = \{(x, y, z, 0); x, y, z \in \mathcal{R}\}$ Is W s.s. of $\mathcal{R}^4$?

Sol:

- $\bar{0} \in W$

- Let $u = \{u_1, u_2, u_3, 0\}$ and $v = \{v_1, v_2, v_3, 0\}$

  $u + v = \{u_1+v_1, u_2+v_2, u_3+v_3, 0\} \in W$

- Let $u = \{u_1, u_2, u_3, 0\}$

  $c\,u = \{cu_1, cu_2, cu_3, 0\} \in W$

$\therefore$ W is a s.s. of R^4

Determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations.

$$W = \{(a, b, a + 2b) : a \text{ and } b \text{ are real numbers}\}$$

الأول الثاني

Solution

1 $\bar{0} \in W$

Note that $W \subset R^3$ and W is nonempty. If $(a_1, b_1, a_1 + 2b_1)$ and $(a_2, b_2, a_2 + 2b_2)$ are vectors in W, then their sum

الأول الثاني الأول + 2 (الثاني)

2 $$(a_1, b_1, a_1 + 2b_1) + (a_2, b_2, a_2 + 2b_2) = (a_1 + a_2, b_1 + b_2, (a_1 + a_2) + 2(b_1 + b_2))$$

is also in W. Furthermore, for any real number c and $(a, b, a + 2b)$ in W,

الأول الثاني الأول + 2 الثاني

3 $$c(a, b, a + 2b) = (ca, cb, ca + 2cb)$$

Determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations.

$$W = \{(s, s-t, t) : s \text{ and } t \text{ are real numbers}\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s-t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solution

1 $\bar{0} \in W$ @ $s=t=0$

W is a subspace of R^3. Note first that $W \subset R^3$ and W is nonempty

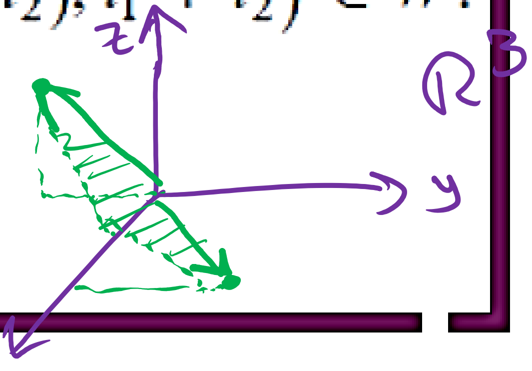
If $(s_1, s_1 - t_1, t_1)$ and $(s_2, s_2 - t_2, t_2)$ are in W, then their sum is also in W.

2 $(s_1, s_1 - t_1, t_1) + (s_2, s_2 - t_2, t_2) = (s_1 + s_2, (s_1 + s_2) - (t_1 + t_2), t_1 + t_2) \in W.$

Furthermore, if c is any real number,

3 $c(s_1, s_1 - t_1, t_1) = (cs_1, cs_1 - ct_1, ct_1) \in W.$

Represent the plane including the 2 vecs.



Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$ with the standard operations of matrix addition and scalar multiplication

$M_{m\times n}$ set of all $m\times n$ matrices

$M_{n\times n}$ set of all square matrices

Matrix space $\Rightarrow M_{2\times 2}$ set of all sq. 2×2 matrices

$$W = \left\{ \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \right\} \Rightarrow W = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} ; a, b, \& c \in \mathbb{R} \right\}$$

(1) $\bar{0} \in \bar{W}$ where $a=b=c=0$

Let $\bar{u}_1 = \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix} \in W$, $\bar{u}_2 = \begin{bmatrix} a_2 & c_2 \\ c_2 & b_2 \end{bmatrix} \in W$

(2) $\bar{u}_1 + \bar{u}_2 = \begin{bmatrix} a_1+a_2 & c_1+c_2 \\ c_1+c_2 & b_1+b_2 \end{bmatrix} \in W$ Closed

(3) $k\bar{u}_1 = \begin{bmatrix} ka_1 & kc_1 \\ kc_1 & kb_1 \end{bmatrix} \in W$ Closed

$\therefore W$ is a s.s. of $M_{2\times 2}$

Symm mat. subspace

not sym.

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$ with the standard operations of matrix addition and scalar multiplication

Condn. for symm $A^t = A$

Condn. for sk. sm $A^t = -A$

Solution

Zero mat. is a symm mat.

Let $W = \{A \in M_2(\mathbb{R}) | A = A^t\} \subset M_2(\mathbb{R})$. Then

1 $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$, for $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}^T$. Let $A, B \in M_2(\mathbb{R})$. Then

Req. proof

Given

$\boxed{A = A^T}, \boxed{B = B^T}$, and 2 $(A+B)^T = A^T + B^T = A + B$. Thus (Given; $\therefore$ symm)

$A + B \in W$. If $\alpha \in \mathbb{R}$, then 3 $(\alpha A)^T = \alpha A^T = \alpha A$. Thus $\alpha A \in W$ (Given; $\therefore$ sym)

A+B symm

αA symm

. Therefore, $W = \{A \in M_2(\mathbb{R}) | A = A^t\} \subset M_2(\mathbb{R})$ is a subspace of $V = M_2(\mathbb{R})$.

Calculat A^t

A — symm

$-A$ — sk.

o.w.

has no inverse

det. = 0

The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$

singular

non-singular

Let W be the set of singular matrices of order 2. Show that is not a subspace of $M_{n,n}(M_n)$ with the standard operations.

Solution

you can show that a subset W is not a subspace by showing that W is empty, W is not closed under addition, or W is not closed under scalar multiplication. For this particular set, W is nonempty and closed under scalar multiplication, but it is not closed under addition. To see this, let A and B be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \in W$$

$|A| = 0$ $\qquad$ $|A| = 0$

Then A and B are both singular (noninvertible), but their sum

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad |A+B| = 1 \neq 0$$

is nonsingular (invertible). So W is not closed under addition, and by Theorem 4.5 you can conclude that it is not a subspace of $M_{2,2}$.

Verify that W is a subspace of V. In each case, assume that V has the standard operations

W is the set of all 3×2 matrices of the form $\begin{bmatrix} a & b \\ a+b & 0 \\ 0 & c \end{bmatrix}$ where

$V = M_{3\times2}$

1 ŌEW

Solution

Because W is nonempty and $W \subset M_{3,2}$, you need only check that W is closed under addition and scalar multiplication. Given

$$\begin{bmatrix} a_1 & b_1 \\ a_1 + b_1 & 0 \\ 0 & c_1 \end{bmatrix} \in W \quad \text{and} \quad \begin{bmatrix} a_2 & b_2 \\ a_2 + b_2 & 0 \\ 0 & c_2 \end{bmatrix} \in W$$

it follows that

2

$$\begin{bmatrix} a_1 & b_1 \\ a_1 + b_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ a_2 + b_2 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ (a_1 + a_2) + (b_1 + b_2) & 0 \\ 0 & c_1 + c_2 \end{bmatrix} \in W.$$

Furthermore, for any real number d,

3

$$d\begin{bmatrix} a & b \\ a + b & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} da & db \\ da + db & 0 \\ 0 & dc \end{bmatrix} \in W.$$

$AX = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$2\times3 \quad 3\times1 \quad 2\times1$

$X_{3\times1} \in R^3$

Let A be a fixed 2×3 matrix. Prove that the set

$$W = \left\{ \mathbf{x} \in R^3 : A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is not a subspace of R^3

$W = \{(x,y) ; x^2+y^2=1\}$

Solution

- $\bar{0} \notin W$

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\square * 0 + \square * 0 + \square * 0 = 0 \neq 1$

$X A = A X$

$\square \times 2 \quad 2\times 2 \quad 2\times 2 \quad 2\times\square$

Let A be a fixed 2×2 matrix. Prove that the set

$W = \{X : XA = AX\}$ Condn

is a subspace of $M_{2,2}$.

Solution ① $\bar{0} \in W$

Req proof:

$(X+Y)A = A(X+Y)$

$(cX)A = A(cX)$

The set W is a nonempty subset of $M_{2,2}$. (For instance, $A \in W$)

To show closure, let $X, Y \in W \Rightarrow \boxed{AX = XA}$ and Given

Given
$\boxed{AY = YA}$. ① Then, $(X + Y)A = XA + YA = AX + AY$ from right common to left

$= A(X + Y) \Rightarrow X + Y \in W$. Similarly, if c is a scalar, then

② $(cX)A = c(XA) = c(AX) = A(cX) \Rightarrow cX \in W$

Given

- Ex: Verify whether the following subset of R^3 is a subspace

V = The points on the plane $x+y-z=0$

vector representation of the same plane

Sol:

→ Homogeneous system of 1 equation in 3 unknowns

→ Let $y=s$ and $z=t$ → $x=t-s$

$$V=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}t-s\\s\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\1\end{bmatrix}$$

1 $\begin{bmatrix}0\\0\\0\end{bmatrix}\in V$ when $t=0$ and $s=0$

2 Let $u_1=\begin{bmatrix}t_1-s_1\\s_1\\t_1\end{bmatrix}$ and $u_2=\begin{bmatrix}t_2-s_2\\s_2\\t_2\end{bmatrix}$

→ $u_1+u_2=\begin{bmatrix}(t_1+t_2)-(s_1+s_2)\\s_1+s_2\\t_1+t_2\end{bmatrix}\in V$

3 Let $u_1=\begin{bmatrix}t_1-s_1\\s_1\\t_1\end{bmatrix}$

→ $c\,u_1=\begin{bmatrix}ct_1-cs_1\\cs_1\\ct_1\end{bmatrix}\in V$

∴ V is a subspace of R^3

- **Ex:** Prove that the solution set of any homogeneous system $\mathbf{A}\,\mathbf{x} = \mathbf{0}$ for $\mathbf{A}_{m\times n}$ is a subspace of $\mathbf{R}^n$

Sol:

Homog. → Always consis. < unique zero soln. / inf. solns. including zero soln

$2x + y = 0$
$-2x - 3y = 0$

$$\mathbf{A}_{m\times n}\,\mathbf{x}_{n\times 1} = \mathbf{0}_{m\times 1}$$

$$\mathbf{x} \in \mathbf{R}^n$$

① $\mathrm{x} = O_{n\times 1} \in R^n$ is a solution to the system $A\,\mathrm{x} = 0$

② Let x_1 and x_2 be solutions to the system $A\,\mathrm{x} = 0$

→ $\mathrm{A}\,\mathrm{x}_1 = 0$ and $\mathrm{A}\,\mathrm{x}_2 = 0$

→ **Check** $\mathrm{x}_1 + \mathrm{x}_2$ → $A\,(\mathrm{x}_1 + \mathrm{x}_2) = A\,\mathrm{x}_1 + A\,\mathrm{x}_2 = 0 + 0 = 0$

③ **Check** $c\,x_1$ → $A\,(c\,\mathrm{x}_1) = c\,A\,\mathrm{x}_1 = c\,0 = 0$

∴ x , the solution space of the system $\mathbf{A}_{m\times n}\,\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbf{R}^n$

The solution set of any homogeneous system of linear equations $AX = 0$ forms a subspace.

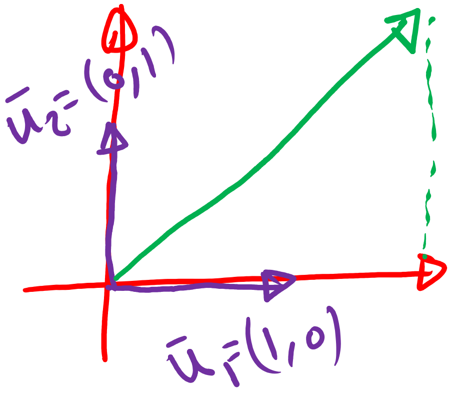
Intro. to next lecture

Spanning Sets and Linear Independence

R^2 space

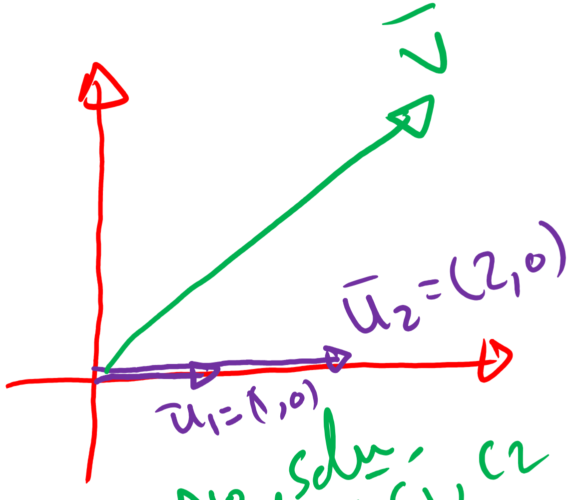
$\bar{v}=(2,2) = 2\bar{u}_1 + 2\bar{u}_2$

- Linear combination:



A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$



No soln. for c_1, c_2

- Ex: (Finding a linear combination)

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$$

$$(1,1,1) = (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$$

$$\Rightarrow \left.\begin{aligned} c_1 \qquad - c_3 &= 1 \\ 2c_1 + c_2 \qquad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}\right\}$$

sys. of linear eqs.

Consistent

There exist values for c_1, c_2, c_3

unique inf.

Inconsistent

No soln

Shortcut only for $\mathbb{R}^n$

Aug. mat.

$$\begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{w} \end{array}$$

test vector

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Eliminatio n}} \begin{array}{cccc} c_1 & c_2 & c_3 & c \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consist. with inf.

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

Let $c_3 = t$

E_2: $c_2 + 2t = -1 \rightarrow c_2 = -1 - 2t$

E_1: $c_1 - t = 1 \rightarrow c_1 = 1 + t$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \begin{array}{cccc} v_1 & v_2 & v_3 & w \\ \end{array} \text{ Test}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Contrad.

No Soln.

$$\Rightarrow \text{ this system has no solution } (\because 0 \neq 7)$$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Exercise:

[Q1] Verify whether the following subset of R^3 is a subspace $\boldsymbol{V} = \{< \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{0} >, \boldsymbol{x} > \boldsymbol{y}\}$

[Q2] Which of these are subspaces?

$$\boldsymbol{W_1} = \left\{ \begin{bmatrix} \boldsymbol{a} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{b} \end{bmatrix} ; \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R} \right\}$$

$\boldsymbol{W_2} =$ **The set of polynomials of the form** $\boldsymbol{a} + \boldsymbol{b}\, \boldsymbol{x}^2$ **where** $\boldsymbol{a}$ **and** $\boldsymbol{b} \in \boldsymbol{R}$