Linear Algebra

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Outline

1. Matrices.
    1. Basic operations and properties.
2. System of linear equations.
    1. Echelon form/Gauss/Gauss-Jordan.
    2. Rank and Solution nature.
    3. LU decomposition.
3. Vector spaces.
    1. Linear combination/Span/independence.
    2. Basis and dimension.

Review on selected Matrix operations

Types of matrices

Row vector $\begin{bmatrix} 1 & -3 & 17 \end{bmatrix}$

1×3

$1 \times n$

Column vector $\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$

3×1

$n \times 1$

Square matrix $\begin{bmatrix} 2 & 7 & 9 \\ 1 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$

3×3

$n \times n$

Order = 3 or n

Zero matrix

$$O_{3\times 3} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Types of matrices (special types of square matrices)

الاسم مرتبط بمكان وجود الأرقام

RE form

Triangular matrix

Upper Triangular matrix U

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$1*1*9$ $\qquad$ $1*0*0$

Lower Triangular matrix L

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 9 \end{bmatrix}$$

$1*5*-1$

$|U|, |L|, |D| =$ prod. of main diagonal elements

Diagonal matrix D

diag(1 , 5 , 9) $\qquad$ **diag(1 , 5 , 0)** $\qquad$ **diag(1 , 0 , 0)**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RRE form

Matrix Multiplication

Matrix Transpose

Matrix Determinant

Matrix Inverse

$$A = \begin{bmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -4 & -3 \\ 2 & 0 \end{bmatrix}$$

3x2 = 2x2

o/p

row1 Col.2

$$A\,B = \begin{bmatrix} \square & \square \\ \square & \square \\ \square & \square \end{bmatrix}$$

3x2

row1, Col.2

5x(-3) + 8(0) = -15

row3, Col.1

2*(-4) + 7*2 =

Properties:

$$A\,B \neq B\,A$$

2x2 ≠ 3x2

Ex: $A_{3\times2}\ B_{2\times2}$

$$(A\,B)C = A\,(B\,C)$$

مع مراعاة الترتيب

التوزيع

$$A\,(B + C) = A\,B + A\,C$$

المشترك - العامل

$$A\,C + A\,B = A\,(C + B)$$

$$A\,C + B\,A \neq A(C + B)$$

$$A\,B + C\,B = (A + C)\,B$$

$$A\,I = I\,A = A$$

$$(c\,A)B = c\,(A\,B)$$

Matrix Multiplication

Matrix Transpose

Matrix Determinant

Matrix Inverse

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 7 \end{bmatrix}_{2\times3}^{t} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 7 \end{bmatrix}_{3\times2}$$

Properties:

$$(A^t)^t = A$$

$$(A \pm B)^t = A^t \pm B^t$$

$$(A\,B)^t = B^t\,A^t$$

$$(c\,A)^t = c\,A^t$$

Symmetric matrix (square)

$$A^t = A \ \#$$

$$\begin{bmatrix} 2 & 3 & 7 \\ 3 & 5 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

Skew Symmetric matrix (square)

$$A^t = -A \ \#$$

$$\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$

Main diagonal elements are zeros

To prove A symm/skew/otherwise

Get A^t
- $= A$ → Symmetric
- $= -A$ → Skew-symmetric
- $= otherwise$ → Neither

Matrix Multiplication

Matrix Transpose

Matrix Determinant

Matrix Inverse

Special for 2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Special for 3×3 matrix

gec+hfa+idb

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

aei+bfg+cdh

Now, subtract the sums: (aei+bfg+cdh) - (gec+hfa+idb)

General for $n \times n$ matrix

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d & e & f \\ g & h & i \end{vmatrix} = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example:

$$\begin{vmatrix} +1 & 3 & -2 \\ -0 & 5 & 1 \\ +4 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 5 & 1 \\ -1 & 2 \end{vmatrix} - 0 \begin{vmatrix} & \\ & \end{vmatrix} + 4 \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= 10 - (-1) + 0 + 4(3 + 10) = ✓$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 0 & 5 & 1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 1 & 3 \\ 0 & 5 \\ 4 & -1 \end{matrix}$$

$$(10 + 12 + 0) - (-40 - 1 + 0)$$

$$= ✓$$

Special for 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \frac{1}{1 \times 4 - 2 * 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \checkmark$$

Inverse of a $n \times n$ matrix

$$A^{-1} = \frac{1}{|A|}\,\mathbf{adj}(A) \quad \text{, where } \mathbf{adj}(A) = \mathbf{cof}(A)^t$$

$$|A| = \begin{matrix} -1 & 2 & -3 & -1 & 2 \\ 2 & 1 & 0 & 2 & 1 \\ 4 & -2 & 5 & 4 & -2 \end{matrix}$$

$$(-5+0+12) - (-12+0+20) = 7 - 8 = -1 \neq 0$$

Example:

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} & +\begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^{t}$$

لا تنسى

Properties:

- A is a square matrix
- $(A^{-1})^{-1} = A$
- $(A\,B)^{-1} = B^{-1}\,A^{-1}$ (Same as transpose)
- $(A^t)^{-1} = (A^{-1})^t$
- $(A^k)^{-1} = (A^{-1})^k$
- **If $A^{-1} = A^t$ then A is called "*orthogonal matrix*".**

Examples on matrices

Ex: Use matrix inversion to solve the system of linear equations:

$$2x - y + 3z = 9 \ , \ x + y + z = 6 \ , \ x - y + z = 2$$

Matrix form

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$A\,X = B$$

$$A^{-1}A\,X = A^{-1}B$$

$$X = A^{-1}B$$

inverse

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = A^{-1}B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0.5 & -0.5 \\ 1 & -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1\,(9) + 1(6) + 2(2) \\ 0\,(9) + 0.5\,(6) - 0.5(2) \\ 1\,(9) - 0.5(6) - 1.5(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Only used when the system is non-homogeneous and in case of having unique solution.

Ex: Find the values of x and y such that the matrix A is orthogonal:

$A^{-1} = A^t$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}} \overset{A^t}{\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$x + 4 + 2y = 0$ ← sub.

$2x + 2 - 2y = 0$

(+) ______________

$3x + 6 = 0 \rightarrow 3x = -6$

$\boxed{x = -2}$

$-2 + 4 + 2y = 0$

$2y = -2$

$\boxed{y = -1}$

Review on solving a system of linear equations

Consider the system of equations

$$
\begin{aligned}
x - 4y + 3z &= 5 \\
-x + 3y - z &= -3 \\
2x \qquad - 4z &= 6
\end{aligned}
$$

Matrix form

$$
\begin{array}{c|c}
 & \begin{array}{ccc} x & y & z \end{array} \\
\begin{array}{c} \text{eq}_1 \to \\ \text{eq}_2 \to \\ \text{eq}_3 \to \end{array} & \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}
\end{array}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}
$$

Coefficient matrix   Unknown   Constant

m = # Rows = # of equations

n = # Columns = # of variables

$$A_{m \times n} X = B$$

$B \neq 0$ — **Non- Homogeneous** "general case"

$B = 0$ — **Homogeneous** special case of non-homog.

Definition “Augmented Matrix”

$$\begin{aligned} x - 4\,y + 3\,z &= 5 \\ -\,x + 3\,y\; -\; z &= -3 \\ 2\,x \qquad\quad - 4\,z &= 6 \end{aligned}$$

Matrix form

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$$

$$A\,X = B$$

$$\begin{array}{c} \begin{array}{cccc} x & \quad y & \quad z & \quad c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right] \end{array}$$

Augmented Matrix

Augmented Matrix ≡ Aug A = [A|B]

Gauss/Gauss-Jordan

Rank

Solution nature

Inverse using G-J

LU decomp.

The row Echelon form (RE)

1^{st} non-zero elements in each row

- Leading entries (pivots) move to the right.
- Elements below leading elements = 0.
- Leading entries = 1. ← Std.
- Zero rows at the bottom.

$$\begin{array}{cccc} x & y & z & c \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$x + 2(3) - (-2) = 4$

$x = \checkmark$

$y = 3$

$z = -2$

Gauss elimination method

The reduced row Echelon form (RRE)

- Leading entries (pivots) move to the right.
- Elements above and below leading elements = 0.
- Leading entries = 1.
- Zero rows at the bottom.

$$\begin{array}{cccc} x & y & z & c \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

→ $x = -4$

→ $y = 3$

→ $z = -2$

Gauss-Jordan method

Elementary row operations

- Interchange two rows.
- Multiply a row by a non-zero constant.
- Add multiple of a row to another (and replace any of them).

Gauss/Gauss-Jordan

Rank

Solution nature

Inverse using G-J

LU decomp.

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & -1\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix} \xrightarrow{\frac{1}{2} R_3 \to R_3} \begin{bmatrix}1 & 2 & 3\\0 & 1 & -1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 0 & 0 & 0\\0 & 1 & 1 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 1 & 4\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 1 & 4\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 2 & 1 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-2 R_2 + R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 0 & -3 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{\frac{1}{-3} R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 0 & 1 & -4/3\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 3 & 0\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{\frac{1}{3} R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Gauss/Gauss-Jordan

Rank

Solution nature

Inverse using G-J

LU decomp.

Definition: The rank of a matrix $A \rightarrow \rho(A)$

The number of pivots (non-zero rows) in the **row echelon form of matrix** A

could be non-std. form
pivots could be ≠1

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank = 4

not in Echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -3 & 4 \end{bmatrix} \xrightarrow{-2R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -3 & 4 \end{bmatrix} \xrightarrow{-R_3 + R_4 \rightarrow R_4} \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank ≠ 4

Rank = 3

Gauss/Gauss-Jordan
Rank
Solution nature
Inverse using G-J
LU decomp.

Solution cases (Non-homogeneous)

rank ≡ no. of actual eqs.

Unique solution

$\rho(A) = \rho(aug\ A) = n$ or no. of vars # cols. of A

$$\underbrace{\underbrace{\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]}_{A}}_{aug\ A}$$

4 4 4

Infinite number of solutions

$\rho(A) = \rho(aug\ A) < n$

3 3 < 5

$$\underbrace{\underbrace{\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]}_{A}}_{aug\ A}$$

Consistent

No solution

$\rho(A) \neq \rho(aug\ A)$

$2 \neq 3$

$$\underbrace{\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]}_{A}}_{aug(A)}$$

0 = −1 Contrad.

Inconsistent

Solution cases (Homogeneous)

$\rho(A) = 2$
$\rho(aug A) = 2$
$n = 3$

$$\underbrace{\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{A}}_{aug\ A}$$

Unique solution
(The zero solution)
$\rho(A) = \rho(aug\ A) = n$

Infinite number of solutions ∞
$\rho(A) = \rho(aug\ A) < n$

Always consistent

- Gauss/Gauss-Jordan
- Rank
- Solution nature
- **Inverse using G-J**
- LU decomp.

Inverse of a 3×3 matrix (method 2) using Gauss-Jordan

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$-R_1 + R_2 \rightarrow R_2$
$6\,R_1 + R_3 \rightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$R_2 + R_1 \rightarrow R_1$
$4\,R_2 + R_3 \rightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$-R_3 \rightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$R_3 + R_2 \rightarrow R_2$
$R_3 + R_1 \rightarrow R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$\overbrace{}^{I}$ (identity block on the left)

$$A^{-1} = \boxed{\begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}}$$

if $=0$

If the coefficient matrix A is **square**

$|A|$

- $\neq 0$ → A **Invertible** → Unique solution

  $X = A^{-1}B$

- $= 0$ (Row/col is multiple or linear comb. of others)
  - **Non-homogeneous** → No solution
  - **Non-homogeneous** → Infinite number of solutions
  - **homogeneous** → Infinite number of solutions

If $|A| = 0$
→ $|RE(A)| = 0$

If $|A| \neq 0$
→ $|RE(A)| \neq 0$

Examples on Solving a system of linear equations

Example Solve the system of linear equations

$$
\begin{aligned}
x \quad - 4y \quad + 3z &= 5 \\
-x \quad + 3y \quad - z &= -3 \\
2x \quad\quad\quad - 4z &= 6
\end{aligned}
$$

Using Gauss-Jordan

$$
\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]
\underset{\substack{R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3}}{\sim}
\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 8 & -10 & -4 \end{array}\right]
\underset{-R_2 \to R_2}{\sim}
\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 8 & -10 & -4 \end{array}\right]
\underset{\substack{4R_2 + R_1 \to R_1 \\ -8R_2 + R_3 \to R_3}}{\sim}
\left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & 12 \end{array}\right]
$$

$$
\underset{\frac{1}{6}R_3 \to R_3}{\sim}
\left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]
\underset{\substack{2R_3 + R_2 \to R_2 \\ 5R_3 + R_1 \to R_1}}{\sim}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]
$$

$x = 7$

$y = 2$

$z = 2$

Example Solve the system of linear equations

$-\frac{3}{2}R_1+R_2$

$-\frac{5}{2}R_1+R_3$

or

No fractions

$$\begin{aligned} 2x \quad\quad\quad & -1z &= 0 \\ 3x + 3y & - z &= -3 \\ 5x + y & - 4z &= 6 \end{aligned}$$

$\begin{cases} -3R_1+2R_2 \rightarrow R_2 \\ -5R_1+2R_3 \rightarrow R_3 \end{cases}$ $\quad\quad -R_2+3R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 3 & 3 & -1 & -3 \\ 5 & 1 & -4 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 6 & 1 & -6 \\ 0 & 2 & -3 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 6 & 1 & -6 \\ 0 & 0 & -10 & 42 \end{array}\right]$$

Eq3: $-10z=42 \rightarrow z=-4.2$

Eq2: $6y-4.2=-6 \rightarrow y=$ ✓

Eq1: $2x+0y-(4.2)=0 \rightarrow x=2.1$

Example Solve the system of linear equations

$$\begin{array}{rrrrrl} x_1 & +2x_2 & +3x_3 & +x_4 & & =0 \\ 4x_1 & +5x_2 & +6x_3 & & +x_5 & =0 \\ 7x_1 & +8x_2 & +9x_3 & & & =1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \overset{\substack{-4R_1 + R_2 \to R_2 \\ -7R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \overset{(-1/3)R_2 \to R_2}{\sim} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$$

$6R_2 + R_3 \to R_3$

$$\sim \begin{array}{c} \begin{array}{cccccc} \# & \# & & \# & & \\ x_1 & x_2 & \boxed{x_3} & x_4 & \boxed{x_5} & c \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] \end{array}$$

Let $x_3 = s$, $x_5 = t$

Eq3: $x_4 - 2t = 1 \to x_4 = 1 + 2t$

Eq2: $3x_2 + \cancel{2}6s + \frac{4}{\cancel{3}}(1+2t) - \frac{1}{\cancel{3}}t = 0$

$3x_2 + 6s + 4 + 8t - t = 0 \to 3x_2 = -6s - 7t - 4$

Eq1: $x_1 + 2\left(-2s - \frac{7}{3}t - \frac{4}{3}\right) + 3s + 1(1+2t) = 0$

$\to x_1 =$

Example Solve the system of linear equations

$$3x_1 + x_2 - 3x_3 = -4$$
$$x_1 + x_2 + x_3 = 1$$
$$5x_1 + 6x_2 + 8x_3 = 8$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 8 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & -4 \\ 5 & 6 & 8 & 8 \end{array}\right] \xrightarrow[-5R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & -7 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & -2 & -6 & -7 \end{array}\right]$$

$$\xrightarrow{2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & \boxed{0} & \boxed{-1} \end{array}\right]$$

$\rho(A) = 2$

$\rho(\text{aug } A) = 3$

$0 = -1$ Contrad.

Sys. is inconsistent

Example Find all values of α for which the following system of equations has:
(a) no solution, (b) a unique solution, (c) infinitely many solutions

$$
\begin{aligned}
x &\quad +y &\quad -z &\quad = 2 \\
x &\quad +2y &\quad +z &\quad = 3 \\
x &\quad +y &\quad +(b-5)z &\quad = a
\end{aligned}
$$

ممكن السؤال يكون
صح فيه a,b
في آخر صف

$n=3$

$$
\left[\begin{array}{ccc|c}
1 & 1 & -1 & 2 \\
1 & 2 & 1 & 3 \\
1 & 1 & b-5 & a
\end{array}\right]
\begin{array}{c}
-R_1+R_2 \to R_2 \\
-R_1+R_3 \to R_3 \\
\sim
\end{array}
\left[\begin{array}{ccc|c}
1 & 1 & -1 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & b-4 & a-2
\end{array}\right]
$$

0 or ?

Contrad.
$0=1$

no sol.
$\rho(A) \neq \rho(\text{aug } A)$
should be 2, 3
$b-4=0$ & $a-2 \neq 0$
$b=4$ & $\underbrace{a \neq 2}_{a = R - \{2\}}$

unique sol.
$\rho(A) = \rho(\text{aug } A) = \cancel{n}\ 3$
should be 3, 3
$b-4 \neq 0 \rightarrow b \neq 4$
$b = R - \{4\}$

inf. no. of sols.
$\rho(A) = \rho(\text{aug } A) < \cancel{n}\ 3$
should be 2, 2
$b-4=0$ & $a-2=0$
$b=4$ & $a=2$

Find the rank of the matrix A in the following situations:

homog.

(1) A is a 7×6 matrix such that $\underset{7\times 6}{A}\,\underset{6\times 1}{X} = 0$ has only the trivial solution.

unique soln.

$\rho(A) = \rho(\text{aug } A) = \cancel{n}\ 6$

$\rho(A) = 6$

(2) A is a 5×7 matrix and the number of free variables in the general solution of $AX = 0$ is four.

المتغيرات التي يتم فرضها

t, s, r

المتغيرات التي لا تحتوي pivot

$$A \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

free var.

Given 4

~~# Free vars~~. + no. of pivots = n ~~no. of cols.~~ 7 $\Rightarrow$ no. of pivots $= 7 - 4 = 3 = \rho(A)$

Find the value of k such that the following system has a non-trivial solution $\equiv$ inf. no. of sln

$$\begin{aligned} x \quad +2y \quad -3z &= 0 \\ 2x \quad +y \quad +z &= 0 \\ x \quad -y \quad +kz &= 0 \end{aligned}$$

homog.

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & 1 \\ 1 & -1 & k \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{matrix} \rightarrow (k+2+6)-(-3-1+4k)=0 \rightarrow -3k+12=0$$

$3k=12$

$\boxed{k=4}$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & k & 0 \end{bmatrix} \underset{-R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 7 & 0 \\ 0 & -3 & 3+k & 0 \end{bmatrix} \overset{-R_2+R_3}{\sim} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & k-4 & 0 \end{bmatrix}$$

لا يجوز القسمة على

inf. no. of sln

$\overset{2}{\rho(A)} = \overset{2}{\rho(\text{aug } A)} < \cancel{n}\ 3$

$k-4=0$

$\boxed{k=4}$

special for homog. & square coeff. mat.

$$|A| \begin{cases} =0 \rightarrow \text{inf. no. of slns} \\ \neq 0 \rightarrow \text{unique zero sln} \end{cases}$$

Ex: Solve the system of linear equations by first evaluating the inverse using Gauss-Jordan elimination

$$\begin{aligned} x_1 - x_2 \qquad\qquad &= 1 \\ x_1 \qquad\quad - x_3 &= 2 \\ -6\,x_1 + 2\,x_2 + 3\,x_3 &= -1 \end{aligned}$$

$$A\,X = B$$

$$A^{-1}A\,X = A^{-1}B$$

$$X = A^{-1}B$$

$$\left[\begin{array}{ccc|ccc} \overbrace{1 & -1 & 0}^{A} & \overbrace{1 & 0 & 0}^{I} \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-R_1 + R_2 \to R_2 \\ 6R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \underset{\substack{R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\underset{-R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \underset{\substack{R_3 + R_2 \to R_2 \\ R_3 + R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|ccc} \overbrace{1 & 0 & 0}^{I} & \overbrace{-2 & -3 & -1}^{A^{-1}} \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = \underbrace{A^{-1}}_{A^{-1}}B$$

$$= \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -8 \\ -9 \end{bmatrix}$$

Gauss/Gauss-Jordan
Rank
Solution nature
Inverse using G-J
LU decomp.

Ex: Find the LU factorization of $A = L\,U$

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ \blacksquare & \blacksquare & 1 \end{bmatrix}$$

$0\,R_1 + R_2 \rightarrow R_2$

$-2\,R_1 + R_3 \rightarrow R_3$

$4\,R_2 + R_3 \rightarrow R_3$

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}}\ U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \blacksquare & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

Remember

No row switching or **scaling**

only Add multiple of a row to another

Solving system of linear equation using LU decomposition

Matrix form

$$\begin{aligned} 3x - 7y - 2z &= 7 \\ -3x + 5y + z &= -5 \\ 6x - 4y \phantom{{}+z} &= 2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix}}_{B}$$

$$\boldsymbol{A\,X = B}$$

$$\boldsymbol{L(U\,X) = B} \qquad \text{Let } \boldsymbol{U\,X = Z}$$

$$\boldsymbol{L\,Z = B}$$ → Get "$\boldsymbol{Z}$" **by Forward substitution**

$$\boldsymbol{U\,X = Z}$$ → Get "$\boldsymbol{X}$" **by Backward substitution**

Example Use the LU decomposition to solve

$$
\begin{aligned}
3\,x - 7\,y - 2\,z &= 7 \\
-3\,x + 5\,y + \quad z &= -5 \\
6\,x - 4\,y \qquad &= 2
\end{aligned}
$$

1st Factorize the coefficient matrix into $\boldsymbol{L}$ and $\boldsymbol{U}$

$$
A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}
\underset{\substack{\boldsymbol{R_1 + R_2 \to R_2} \\ \boldsymbol{-2\,R_1 + R_3 \to R_3}}}{\sim}
\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix}
\underset{\boldsymbol{5\,R_2 + R_3 \to R_3}}{\sim}
\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & \blacksquare & 1 \end{bmatrix}
\qquad
L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}
$$

Example (Cont.)

Use the LU decomposition to solve

$AX=B$
$L(UX)=B$
$LZ=B$ &
$UX=Z$

$$\begin{aligned} 3x-7y-2z&=7\\ -3x+5y+z&=-5\\ 6x-4y&=2 \end{aligned}$$

$$L=\begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}$$

$$U=\begin{bmatrix}3&-7&-2\\0&-2&-1\\0&0&-1\end{bmatrix}$$

2[nd] Solve the system $LZ=B$ using **forward sub.**

$$\begin{array}{ccc|c} z_1 & z_2 & z_3 & \\ 1 & 0 & 0 & 7\\ -1 & 1 & 0 & -5\\ 2 & -5 & 1 & 2 \end{array}$$

$Eq_1: z_1 = 7$

$Eq_2: -(7)+z_2=-5 \rightarrow z_2 = 2$

$Eq_1: 2(7)-5(2)+z_3=2 \rightarrow z_3 = -2$

3[rd] Solve the system $UX=Z$ using **backward sub.**

$$\begin{array}{ccc|c} x & y & z & \\ 3 & -7 & -2 & 7\\ 0 & -2 & -1 & 2\\ 0 & 0 & -1 & -2 \end{array}$$

$E_3: -z=-2 \rightarrow \boxed{z=2}$

$E_2: -2(y)-2=2 \rightarrow -2y=4 \rightarrow \boxed{y=-2}$

$E_1: 3x-7(-2)-2(2)=7 \rightarrow 3x+14-4=7$

$3x=-3 \rightarrow \boxed{x=-1}$

Determinant using LU decomposition

- For two square matrices A and B → $|A\,B| = |A|\,|B|$
- For diagonal matrices D , upper triangular matrices U , and lower triangular matrices L

  → $|D|$, $|U|$, $|L|$ = product of main diagonal elements

- Given $A = LU$ → $|A| = |L||U|$, where both L *and* U *are triangular matrices*

In last Example

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$|A| = \overset{\text{always } 1}{|L|}\,|U| = 1 * 3*(-2)*(-1)$$

$$= 6$$

Review on Vector spaces

Vector space/ subspace

Linear comb./span /indep.

Basis and dimension

Test for a Vector Space

c: is any real number
+ve/-ve
Rational/decimal/integer

Start with these properties

Addition:

(1) **u**+**v** is in V

(2) **u**+**v**=**v**+**u**

(3) **u**+(**v**+**w**)=(**u**+**v**)+**w**

(4) V has a zero vector **0** such that for every **u** in V, **u**+**0**=**u**

(5) For every **u** in V, there is a vector in V denoted by –**u** such that **u**+(–**u**)=**0**

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

Test for a subspace

(1) $\mathbf{0} \in W$.

(2) If **u** and **v** are in W, then **u**+**v** is in W.

(3) If **u** is in W and c is any scalar, then c**u** is in W.

Vector Spaces

Ex: Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: : $(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

+ the remaining 8 conds.

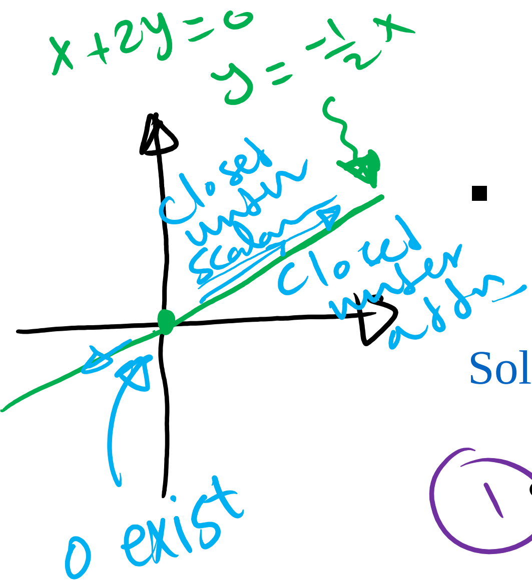
$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

Ex: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underbrace{(\tfrac{1}{2})}_{\text{scalar}}\underbrace{(1)}_{\text{integer}} = \underbrace{\tfrac{1}{2}}_{\text{noninteger}} \notin V$$ (it is not closed under scalar multiplication)



النقاط الموجودة على الخط المستقيم

- Ex: $W = \{(x, y), x + 2\,y = 0\}$ Is W s.s. of $\mathcal{R}^2$?

subspace

Let $y=t$, $x=-2y=-2t$

Sol:

1. $\bar{0} \in W$

-2* نفس الرقم

أي رقم

$$W = \{\boxed{(-2\,t, t)}, t \in \mathcal{R}\}$$

2. Let $u = \{-2\,u_1, u_1\}$ and $v = \{-2\,v_1, v_1\}$

$$u + v = \{\underbrace{-2u_1-2v_1}_{-2(u_1+v_1)}, \underbrace{u_1+v_1}_{\text{أي رقم}}\} \in W$$

3. Let $u = \{-2\,u_1, u_1\}$

$$c\,u = \{\underbrace{c(-2u_1)}_{-2(cu_1)}, \underbrace{cu_1}_{\text{أي رقم}}\} \in W$$

$\therefore$ W is a s.s. of R^2

Vector space/ subspace
Linear comb./span /indep.
Basis and dimension

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

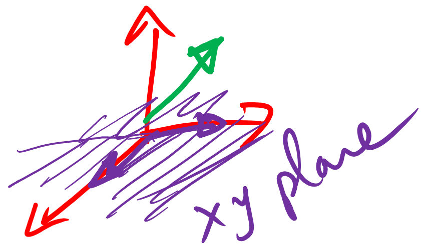
$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$

Writing a vector as a linear combination

$$\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \text{Test} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

Consistent or not



Test Span

Test

$$\begin{bmatrix} 1 & 0 & -1 & u_1 \\ 2 & 1 & 0 & u_2 \\ 3 & 2 & 1 & u_3 \end{bmatrix}$$

Consistent without condition → spanning set

or with condition

Not a spanning set

Test independence

$$\begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \text{Test} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

- Trivial zero solution
→ L. Independent
- Infinite number of solutions
→ L. Dependent

For P_n → $1+x^2+x^3$

$M_{m\times n}$ → $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

⇒ R^n

$$\begin{array}{c} c \\ x \\ x^2 \\ x^3 \end{array}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

Writing a vector as a linear combination of other vectors

$\mathbf{v}_1 = (1,2,3)$ $\mathbf{v}_2 = (0,1,2)$ $\mathbf{v}_3 = (-1,0,1)$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ✓

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ✗

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Test

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consist or Not ✓

new Test

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Contrad.

Consist. or Not ✓

Span of a set and spanning set of a vector space

- the span of a set: span (S)

  If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S

- a spanning set of a vector space:

  If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Example: Determine whether the set S spans $\mathbb{R}^2$.

$$S = \{(-1,4),(4,-1),(1,1)\}$$

Test

$$4R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 4 & -1 & 1 & u_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 0 & 15 & 5 & 4u_1 + u_2 \end{array}\right]$$

Consistent without conds. on (u_1, u_2)

Example: Determine whether the set S spans $\mathbb{R}^3$.

S is not a spanning set for R^3

$$S = \{(1,-2,0),(0,0,1),(-1,2,0)\}$$

$2R_1 + R_2 \rightarrow R_2$ $\qquad$ $R_2 \leftrightarrow R_3$

Test

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ -2 & 0 & 2 & u_2 \\ 0 & 1 & 0 & u_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 0 & 0 & 2u_1 + u_2 \\ 0 & 1 & 0 & u_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 0 & u_3 \\ 0 & 0 & 0 & 2u_1 + u_2 \end{array}\right]$$

Consistent with Condn.

$$\boxed{2u_1 + u_2 = 0}$$

$$u_2 = -2u_1$$

$$\text{span}(S) = \begin{bmatrix} u_1 \\ -2u_1 \\ u_3 \end{bmatrix} = \underbrace{u_1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + u_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{plane in } R^3}$$

Linear dependence and Independence

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

Homog. sys.

(1)" If the equation has only the ***trivial solution*** "($c\_1 = c\_2 = \cdots = c\_k = 0$)" then "$S$" is called ***linearly independent***."

(2)" If the equation has a ***nontrivial solution*** "("i. e., not all zeros"), " then "S" is called ***linearly dependent***."

if A "coeff. mat." is square

unique
L.I.
$|A| \neq 0$

inf.
L.D.
$|A| = 0$

اختصار

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1, 2, 3)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

Test

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Square matrix

L. Indep.

unique ✓

inf. no. of sols.

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{matrix} = (1+0-8)-(-6)$$

$$= -7+6 = -1 \neq 0$$

If no. of vecs. in S > dim(V) $\Rightarrow$ S is L.D.

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$$

no. of vars = 4 → #5

free var.

Sol:

3 rows

max. no. of Pivots = 3

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{0} & \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-6} & \mathbf{-3} & \mathbf{0} \end{array}\right]$$

Already in Echelon form

by default has inf no. of sln

inf no. of sln

→ Linearly dep.

Vector space/ subspace
Linear comb./span /indep.
Basis and dimension

Generating Sets | Bases | Linearly Independent Sets

$\Rightarrow \dim(R^n) = n$

$\Rightarrow \dim(M_{m\times n})=mn$

$\Rightarrow \dim(P_n(x)) = n+1$

R^3

only 2 vecs.

Not a spanning set

R^2

Dimension of the vector space → dim(V)

Number of vectors in S
#S < dim(v)

Does not span
→ Not a basis

if span & indep.

Number of vectors in S
#S = dim(v)

if indep. & span

Only one test is enough
→ Independence test is preferable "using det."

$|A| \begin{cases} \neq 0 & L.I. \to \text{basis} \\ = 0 & L.D. \to \text{Not a basis} \end{cases}$

Number of vectors in S
#S > dim(v)

L.Dependent
→ Not a basis

Exercise Explain why S is not a basis for $\mathbb{R}^2$

$S = \{(-1,2),(1,-2),(2,4)\}$

Solution

$\#S > \dim(V) \rightarrow$ L.D. $\rightarrow$ Not a basis

A basis for $\mathbb{R}^2$ can only have two vectors. Because S has three vectors, it is not a basis for $\mathbb{R}^2$.

Exercise Explain why S is not a basis for $M_{2,2}$

$\dim(M_{2\times 2}) = 4$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$\underset{3}{\#S} < 4$

Solution S does not span $M_{2,2}$, although it is linearly independent.

Not a basis $\rightarrow$ S is not a spaning set

Ex: Determine whether $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ -1 \end{bmatrix} \right\}$ is a basis for R^3 or not and why?

$\#S = 4 < \dim(\mathbb{R}^3) \Rightarrow$ L.D.

$\Rightarrow$ Not a basis

Ex: Determine whether $S = \{1, 1 + x, 1 - x, 1 + x + x^2, 1 - x + x^2\}$ is a basis for P_2 or not and why?

$\dim(P_2) = 2 + 1 = 3$

$\#S = 5 > \dim P_2$

$\Rightarrow$ L.D.

$\Rightarrow$ Not a basis

Exercise Determine whether S is a basis for the indicated vector space.

dim = 4 = #S

$$S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\} \text{ for } R^4$$

1 2 3 4

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 5 \end{vmatrix} \neq 0$$

→ S is linearly independent & contains 4 vectors

invertib.

→ S is a basis for R^4.

unique sol.

↳ L. Indep. → basis

Exercise determine whether S is a basis for R^3 If it is, write $u(8,3,8)$ as a linear combination of the vectors in S

$\to$ dim $=3$

$$S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$$

$\# S = 3$

low triang.

$$\because |A| = \begin{vmatrix} 4 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 2 & 2 \end{vmatrix} \neq 0$$

$24 \neq 0$

→ **S Linearly independent and contains 3 vectors**

→ **S is a basis**

$$\begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \text{Test} \\ \end{array}$$

$$\left[\begin{array}{ccc|c} \mathbf{4} & \mathbf{0} & \mathbf{0} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{8} \end{array}\right]$$

invert.

Forward sub.

→ **After solving the augmented matrix**

yields $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$. So,

$$\mathbf{u} = 2(4, 3, 2) - (0, 3, 2) + 3(0, 0, 2) = (8, 3, 8).$$

$\dim(V) = n$, W is a subspace of $V \Rightarrow \dim(W) \le n$

- Ex: Given $W=\{(d, c-d, c)$: c and d are real numbers$\}$

(a) Prove that W is a subspace of R^3

(b) Find the dimension of W

Sol: (a) at $c=0$, $d=0 \rightarrow (0,0,0) \in W$ ①

Let (d_1, c_1-d_1, c_1), $(d_2, c_2-d_2, c_2) \in W$

$\bar{v}_1 + \bar{v}_2 = (d_1+d_2,\ c_1-d_1+c_2-d_2,\ c_1+c_2)$ # ②

check 3rd − 1st ?! $(c_1+c_2)-(d_1+d_2)$ ✓

Let (d_1, c_1-d_1, c_1)

$k(d_1, c_1-d_1, c_1)$

(kd_1, kc_1-kd_1, kc_1) # ③

(b) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$ by def.

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and ~~S spans~~ spanning set W)

one is not a scalar multip. of the other

check indep.

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ uniq. → L.I. ; inf. → L.D.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2 \le 3$

- Ex: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$. ← dim = 4

  What is the dimension of W?

Sol:

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

by defn spanning set

Test indep.

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ spans W and S is L.I.

Test

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow S$ is a basis for W $\quad\Rightarrow \dim(W) = \#(S) = 3 \leq 4$

L.I. unique

L.D. inf