Probability and Statistics

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Outline

1.3 Probability computation

- Basics
- Counting techniques

1.4 Conditional probability

- Notion and definition
- Multiplication rule
- Law of total probability
- Bayes' theorem
- Independent events
- Some reliability problems

Two Lectures

Next

1.3 Probability computation

Basics - Simple problems

Equi-probable case

Non-equi-probable case

Advanced problems

Counting methods:

Multiplication/addition principals – Permutations – Combinations

Basics - Simple problems

Equi-probable case

Non-equi-probable case

Advanced problems

Counting methods:

Multiplication/addition principals – Permutations – Combinations

Basics

Discrete

→ Fair dice

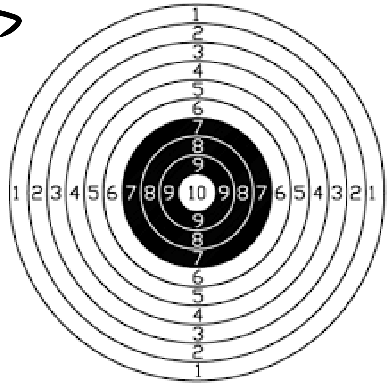
$S = \{1, 2, . . ., 6\}$

$P(1) = P(2) = . . . = 1/6$

$P(Even) = 3/6$

$P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$ ← N_{Even} (numerator), N_S (denominator)

Continuous



$S = \{(x, y); x^2 + y^2 \leq R\}$

Blind targeting

P(Hit 10) = Area10/Total area

If S is equi-probable, then

Discrete: **P(A) =** N_A/N_S

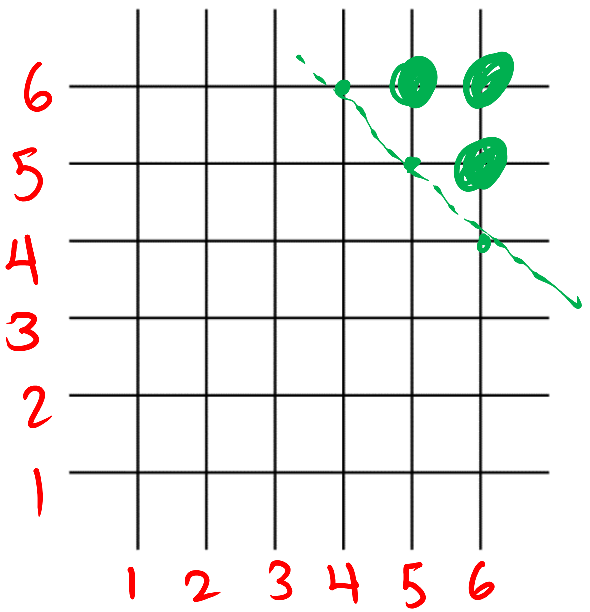
Continuous: **P(A) =** $Area_A/Area_S$ **or** $Length_A/Length_S$ or $\frac{\text{time}_A}{\text{times}}$

Example

Let a die be thrown twice. Find the probability of getting a sum greater than 10

> 10

Take care in discret
> 10
or
≥ 10

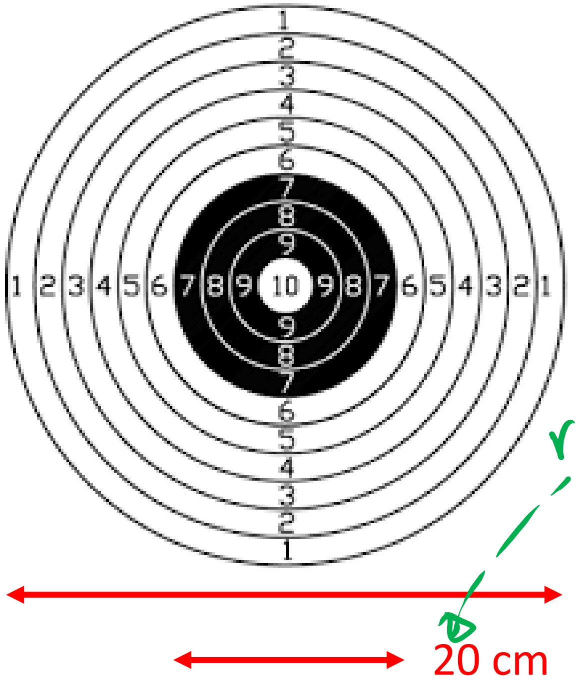


$$P(\text{Sum} > 10) = \frac{N_A}{N_s} = \frac{3}{36}$$

possible outcomes in one toss $\rightarrow a^{n} \leftarrow$ # of tosses

Example

Blind targeting a circle. Find the probability of hitting the 7,8,9,or 10 target circles



$$P\left(\begin{matrix}\text{Hitting}\\ 7,8,9,\text{or}\\ 10\end{matrix}\right) = \frac{\text{Area}_A}{\text{Area}_T} = \frac{\cancel{\pi}(10)^2}{\cancel{\pi}(50)^2} = \frac{100}{2500} = \frac{1}{25}$$

Basics - Simple problems → Equi-probable case

Basics - Simple problems → **Non-equi-probable case**

Advanced problems → Counting methods:

Multiplication/addition principals – Permutations – Combinations

Example (Non equi-probable sample space)

A die is weighted such that the probability of a number is proportional to its value. If the die is thrown once, what is the probability of getting

(a) an odd number?

(b) a number less than 3?

possible outcomes	1	2	3	4	5	6
probs	α	2α	3α	4α	5α	6α
	$1/21$	$2/21$	$3/21$	$4/21$	$5/21$	$6/21$

Rule: $\sum$ all probs. $= 1 \Rightarrow \alpha + 2\alpha + 3\alpha + 4\alpha + 5\alpha + 6\alpha = 1$

$21\alpha = 1 \Rightarrow \alpha = 1/21$

(a) $P(\text{odd number}) = P(1) + P(3) + P(5)$

$= 1/21 + 3/21 + 5/21 = 9/21 = 3/7$

(b) $P(\text{number} < 3) = P(1) + P(2) = 1/21 + 2/21 = 3/21 = 1/7$

Example (Non equi-probable sample space)

A die is weighted such that the probability of even twice the probability of odd. If the die is thrown once, what is the probability of getting

(a) an odd number?

(b) a number less than 3?

possible outcomes	1	2	3	4	5	6
probs	α	2α	α	2α	α	2α
	$1/9$	$2/9$	$1/9$	$2/9$	$1/9$	$2/9$

Rule: $\sum$ all probs. $= 1 \Rightarrow \alpha + 2\alpha + \alpha + 2\alpha + \alpha + 2\alpha = 1$

$9\alpha = 1 \Rightarrow \alpha = 1/9$

(a) $P(\text{odd number}) = P(1) + P(3) + P(5)$

$= 1/9 + 1/9 + 1/9 = 3/9 = 1/3$

(b) $P(\text{number} < 3) = P(1) + P(2) = 1/9 + 2/9 = 3/9 = 1/3$

Basics - Simple problems → Equi-probable case

Basics - Simple problems → Non-equi-probable case

Advanced problems → **Counting methods:**
Multiplication/addition principals – Permutations – Combinations

Counting methods

How many possible password?

00000
00001
00002
⋮
99998
99999

How many ?!

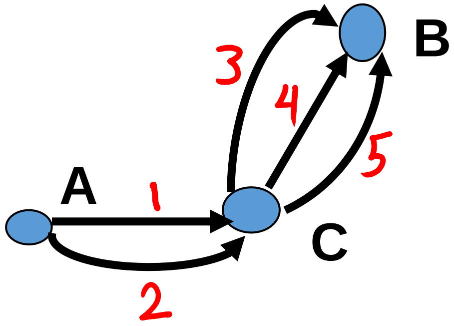
10	10	10	10	10

Stages

10^5

Techniques to determine the number of elements in a set without listing them

Multiplication Principle



In how many ways can one goes from A to B?

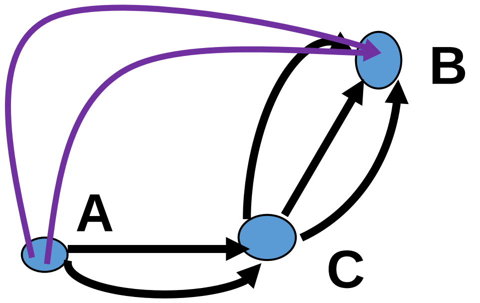
13
14
15
23
24
25

Stages

6 possibilities

$= \underbrace{2}_{\text{Possib. of stage 1}} \times \underbrace{3}_{\text{possib. of stage 2}}$

Addition Principle



In how many ways can one goes from A to B?

\# of possib.

6 + 2

Alternative

Example

not stated different "Default can repeat" — 5 possible numbers

Exercise different

A three-digit number is constructed at random from {2, 3, 5, 8, 9}. What is the probability that it is

(a) even?

(b) divisible by five?

Add 0

$\boxed{5}\boxed{5}\boxed{5}$ $\quad N_s = 5*5*5 = 5^3$

stages

(a) Even $\boxed{5}\boxed{5}\boxed{2}$ $\{2, 8\}$

$N_a = 5*5*2$

$P_a = \frac{N_a}{N_s} = \frac{5*5*2}{5*5*5} = 2/5$

(b) Divisible by 5 $\boxed{5}\boxed{5}\boxed{1}$ $\{5\}$

$P_b = \frac{5*5*1}{5*5*5} = 1/5$

$\{0, 2, 3, 5, 8, 9\} \Rightarrow$ 6 possibilities

$\boxed{5}\boxed{6}\boxed{6}$

0 is not allowed "3-digit no. cond."

(a) Even $\boxed{5}\boxed{6}\boxed{3}$ $\{0, 2, 8\}$

$P_a = \frac{5*6*3}{5*6*6} = 3/6 = 1/2$

(b) Divisible by 5

$\boxed{5}\boxed{6}\boxed{2}$ $\{0, 5\}$

$P_b = \frac{5*6*2}{5*6*6} = \frac{2}{6} = \frac{1}{3}$

Example

A password consists of 6 letters "Assume small letters".

5 or 6 letters

(a) How many passwords are there?

(b) What is the probability that a password does not contain *w* ?

Alternative

(a)

26	26	26	26	26	26

stages

$N_s = 26^6$

(b)

25	25	25	25	25

$N_b = 25^6$

$P_b = \frac{25^6}{26^6}$

(a) 5 letters or 6 letters

26				

26					

$N_s = 26^5 + 26^6$

(b) 5 letters or 6 letters

25				

25					

$N_b = 25^5 + 25^6$

$P_b = \frac{25^5 + 25^6}{26^5 + 26^6}$

Example

A password consists of ~~6 letters~~ "Assume small letters". ~~5 or 6 letters~~

(a) How many passwords are there?

(b) What is the probability that a password does not contain *w* ?

at least 3 letters

"Exercise"

→ Complement

Permutation and combination

$\boxed{A, B, C}$ → *In how many ways, can we select two* ***Different*** *items?*

Order of selection is important

Stages

$n=3$

$r=2$

$\boxed{3 \mid 2} \Longrightarrow 3 * 2 = 6$

{AB, BA, AC, CA, BC, CB}

{A,B,C} The remaining two possib

* Select 3 out of 5 $\boxed{5 \mid 4 \mid 3}$ (n, $n-1$, $n-r+1$)

* Select r out of n

$$\frac{n(n-1)\cdots(n-r+1)(n-r)\cdots *3*2*1}{(n-r)\cdots *3*2*1} = \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

Permutation

Order of selection is not important

{AB, AC, BC}

$\underbrace{AB, BA}_{\text{counted as one}}$, $\underbrace{AC, CA}_{\text{"}}$, $\underbrace{BC, CB}_{\text{"}}$

$N_s = \overset{\text{Total}}{6} * \frac{1}{2}$ ← repetitions $(r!)$

* select 3 out of 5

$\underbrace{ABC, ACB, BAC, BCA, CAB, CBA}$ 6 repetions

$r!$ ← $\boxed{3 \mid 2 \mid 1}$

${}^{5}P_{3} \frac{1}{6}$ ← $r!$

$${}^{n}C_{r} = \frac{1}{r!}\,{}^{n}P_{r}$$

Combination

Permutation and combination

A, B, C → *In how many ways, can we select two items?*

{AB, BA, AC, CA, BC, CB}

Permutation

- Permutation is the arrangement of items in which **order matters**
- Number of ways of **selection and arrangement of items** in which Order Matters

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

{AB, AC, BC}

Combination

- Combination is the selection of items in which **order does not matters** .
- Number of ways of **selection of items** in which Order does not Matters

$$^{n}C_{r} = \frac{1}{r!}\frac{n!}{(n-r)!} \quad {}^{n}P_{r}$$

Remove repetition

Example

A student is to answer 8 out of 10 questions in an exam.

(a) How many choices does he have?

(b) Repeat (a) if he must answer the first 2 questions?

(a) $\boxed{\text{10 qs.}} \xrightarrow{\text{select}} 8$

From the problem statement

→ select 8 out of 10 "different"

→ order does not matter

Combinations $^{n}C_{r}$

$^{10}C_{8}$

(b) 2 Qs. → 2 ; 8 Qs. → 6

Stages

$^{2}C_{2}$ * $^{8}C_{6}$

$\Rightarrow N_s = {}^{2}C_{2} * {}^{8}C_{6}$

Examples

Example

Three *different* components are to be put on a board having 10 slots.

How many choices does we have?

1 ... 10

Similar components?

→ select 3 positions out of 10

→ order matters

Permutations ^nP_r

$\# \text{Possibs.} = {}^{10}P_3$

→ order does not matter

combinations

$^{10}C_3$

Example

A binary code consists of 10 bits.

How many codes contain *exactly* TWO zeros?

1110111011

At least two zeros?

Think of it as follows

- Assume all 1's code
- select two random ones to convert to zero
- 0's are identical

Combinations

$$^{10}C_2 = \frac{10!}{2!(10-2)!}$$

Examples

0011111111

1001111111

1101011111

⋮

There could be a repetition

$$^{10}C_1 * {}^{9}C_1$$

$$\frac{10!}{1!\cancel{(10-1)!}} * \frac{\cancel{9!}}{1!(9-1)!}$$

Check

$$\boxed{2\,|\,|\,|\,|\,|\,|\,|\,|\,|\,1}$$

2 zeros or 3 zero or ------ 10 zeros

$$^{10}C_2 + {}^{10}C_3 + \cdots + {}^{10}C_{10}$$

Too many

Complement

No zero or one zero

$$N_{comp} = {}^{10}C_0 + {}^{10}C_1$$

$$N_s = \text{Total} - N_{comp}$$

$$= 2^{10} - [{}^{10}C_0 + {}^{10}C_1]$$

Example

A lot of 15 monitor contains 3 defective ones. Three monitors are chosen at random. What is the probability that

(a) none is defective?

(b) only one is defective?

$\boxed{15} \xrightarrow{3}$ $N_s = {}^{15}C_3$

(a) 3 D → 0, 12 G → 3; 3 stages

$$P_a = \frac{{}^{3}C_0 * {}^{12}C_3}{{}^{15}C_3}$$

(b) 3 D → 1, 12 G → 2

$$P_b = \frac{{}^{3}C_1 * {}^{12}C_2}{{}^{15}C_3}$$

At least one is defective?

1 D or 2 D or 3 D

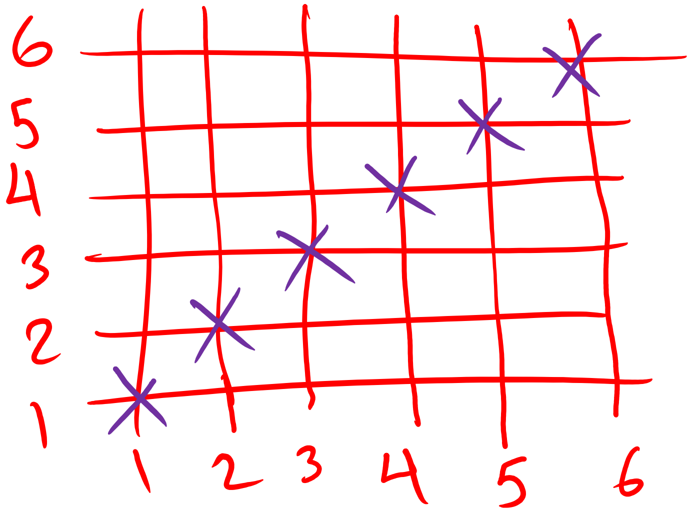
3 D → 1, 12 G → 2 | 3 D → 2, 12 G → 1 | 3 D → 3, 12 G → 0

$$\frac{{}^{3}C_1 * {}^{12}C_2}{{}^{15}C_3} + \frac{{}^{3}C_2 * {}^{12}C_1}{{}^{15}C_3} + \frac{{}^{3}C_3 * {}^{12}C_0}{{}^{15}C_3}$$

Exercise repeate assuming 15 monitors, 2 defective, select 3

Ex: A dice is tossed three times. Find the probability of obtaining different faces.

In case of tossing twice



$$\text{Prob.} = \frac{30}{36}$$

order of toss matters

select 2 out of 6 items

different

6	5

$\Rightarrow 6 * 5$ or $^6P_2$

For tossing 3 times

$\{1,2,3,4,5,6\}$

6	5	4

or

$^6P_3$

Ex: There are 7 novels, 4 dictionaries, and 3 catalogues to be arranged on a shelf. In how many ways can the 14 books be arranged on the shelf if

(a) They all are distinct?

(b) The novels are identical to each other, the dictionaries are identical to each other, and so are the catalogues?

(c) Repeat part (a) with the additional restriction that books belonging to the same category should follow each other?

(a) Select 14 out of 14 where order matters $\Rightarrow$ $^{14}P_{14} = \frac{14!}{\cancel{(14-14)!}} = 14!$

14	13	12	...	3	2	1

(b) Remove repetition $\Rightarrow$ $\underbrace{14!}_{\text{Total}} * \underbrace{\frac{1}{7!}}_{\text{novel}} * \underbrace{\frac{1}{4!}}_{\text{dict.}} * \underbrace{\frac{1}{3!}}_{\text{cata.}} = {}^{14}C_7 * {}^{7}C_4 * {}^{3}C_3$

14 positions, 14-7, 7-4

(c) distinct + books of same category follow each other

$\boxed{\text{novels } 7!}\ \boxed{\text{Dics. } 4!}\ \boxed{\text{cat. } 3!} = \overbrace{7! * 4! * 3!}^{\text{probability of any instance}} * 3!$

Stages

repeated 3! times

3! "Alternatives"

More problems

Example

An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly

a. All 5 problems?

b. At least 4 of the problems?

Ex: A conference on bioinformatics includes six talks for six different speakers (3 biologists and 3 engineers). However, the three biologists must speak first, in any order, followed by the three engineers, in any order.

How many different schedules are possible for the six talks in this conference?

How many schedules are possible if biologists can come first or second

Ex: Consider the experiment of flipping two coins then a fair dice is tossed. Determine the number of possible outcomes in this experiment.

Specific order of tossing

What if tossing is allowed in any order?

Ex: A driver from city A can directly reach city B using four different roads. Moreover, he can also reach city B by passing by city C first. In this latter case, he can drive along any of three roads connecting cities A and C, then choose any of the five roads from C to B.
How many different choices are available to the driver?

Ex: If each coded item in a catalogue begins with three distinct letters (a-z) followed by four distincit non-zero digits (1-9).
Find the probability of randomly selecting one of these codes with the first letter a vowel (a, e, I, o, u) or the last digit is even.

Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that

a. 3 red, 2 blue, and 2 green balls are withdrawn.
b. At least 2 red balls are withdrawn.
c. All withdrawn balls are the same color.
d. Either exactly 3 red balls or exactly 3 blue balls are withdrawn.