Probability and Statistics

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Part 1: Elements of probability theory

1.1 Basic definitions

1.2 Axioms of probability theory

This lecture

1.3 Probability computations

1.4 Conditional probability

Next lecture

Reading

Counting techniques

- Combinations

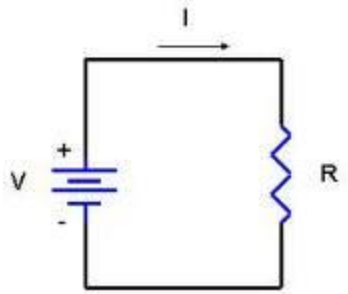
- Permutations

1.1 Basic definitions

1. Random experiment

An experiment

Procedure

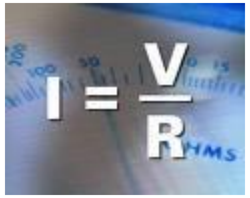


Connect a circuit

Observation

Measure current

Deterministic experiment



Current?

Random experiment

Which side?

Waiting time?

2. Sample space

Tossing a coin once

Which side?

Head or tail; {H, T}

Tossing a die once

Which face?

{1, 2, . . ., 6}

Waiting time?

[0, T] some max. waiting time

The sample space **(S or Ω)** of a random experiment is the **set of all possible outcomes**

Classification

Tossing a coin twice

Tossing a Coin until a Head appears

Sample space

Discrete

•S = {HH, HT, TH, TT}

•S = {H, TH, TTH, . . . }

Finite

Infinite

Continuous

•S = [0, T]

•S = [0, ∞[

More about sample space

$\#S = a^n$; a: no. of possible outcomes in one toss
n: no. of tosses

Tossing a coin twice

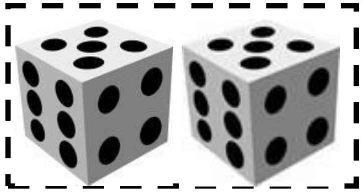
Example 1.1

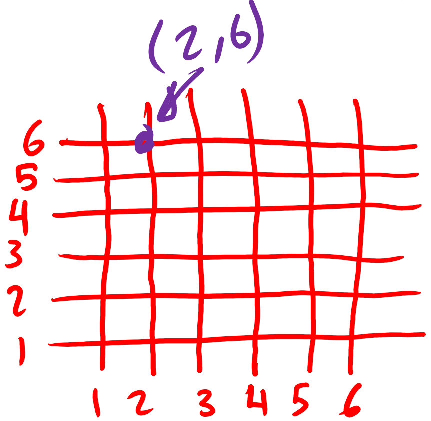
$S = \{HH, HT, TH, TT\}$

no. of tosses	1	2	3	--	n
#S	2	4	8	--	2^n

Tossing a die twice

Example 1.2





$S = \{(1,1), (1,2), \ldots, (1,6)$
$\vdots$
$(6,1), (6,2), \ldots, (6,6)\}$

no. of tosses	1	2	----	n
#s	6	36		6^n

Tossing a coin until a Head appears

Example 1.3

. . . <until H appears>

$S = \{H, TH, TTH, \ldots\}$

$\#S = \infty$

3. Events

- {Served in less 10 min} $\rightarrow$ $[0, 10[\subset S$
- {Wait for more than 30 min} $\rightarrow$ $]30, T] \subset S$

- {Even number appears} $\rightarrow$ $\{2,4,6\} \subset S = \{1,2,3,4,5,6\}$

An event is a subset of the sample space ($A \subseteq S$)

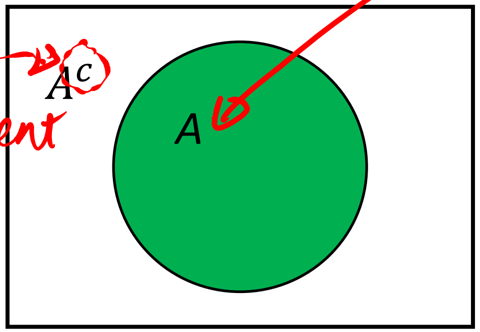
Operations on events

Sample space

u or *s*

event

Complement



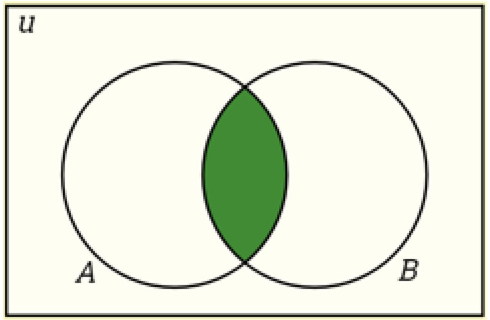
$$(A^c)^c = A$$

and / both

$$A^c \cap A = \phi$$

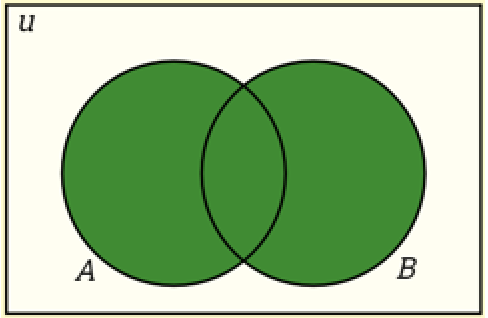
$$A^c \cup A = S$$

or / either



Both A and B $A \cap B$

A intersect *B*

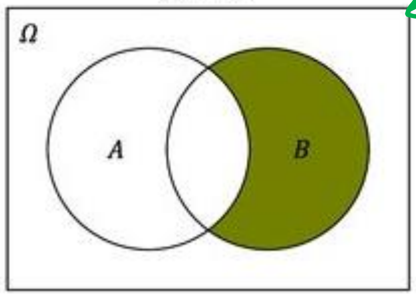


Either A or B $A \cup B$

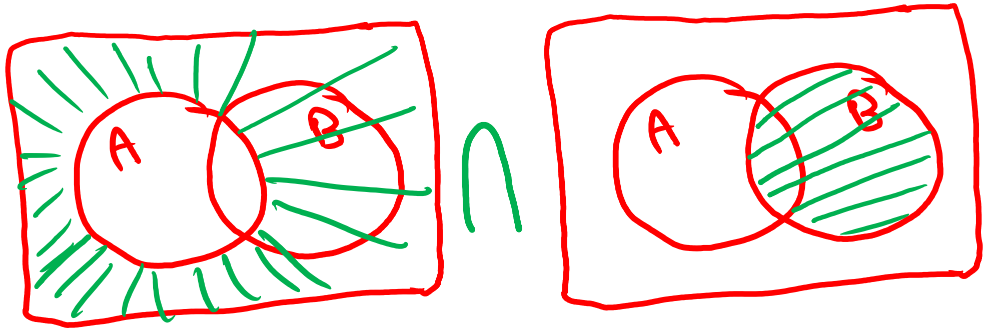
A union *B*

Operations on events

$A' \cap B$



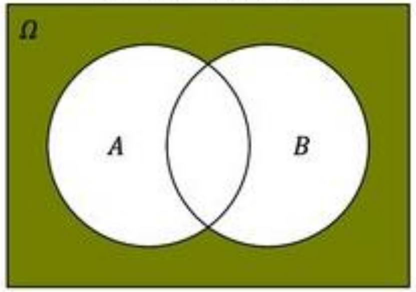
$A^c \cap B =$



$B - (A \cap B)$

or $(A \cup B) - A$

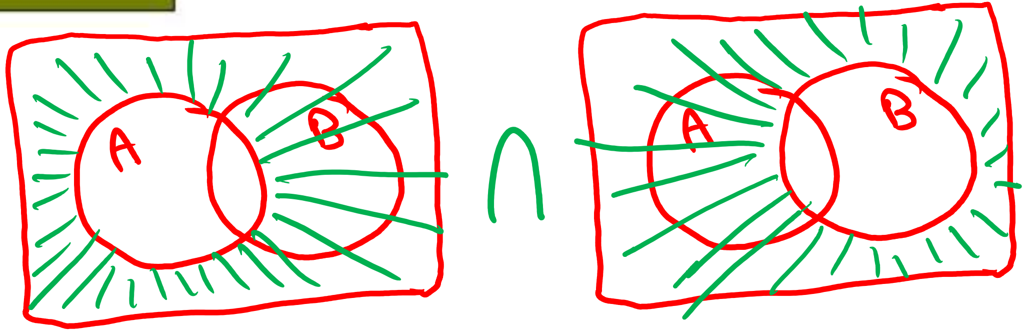
$A' \cap B' = (A \cup B)'$



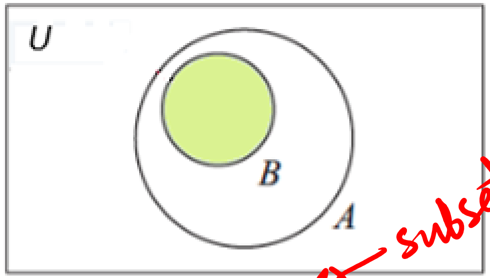
DE Morgan's law

$(A \cup B)^c = A^c \cap B^c$

also $(A \cap B)^c = A^c \cup B^c$



Operations on events (special cases)



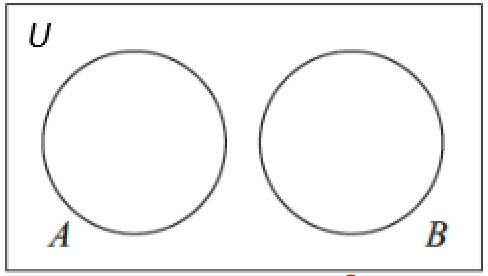
B is subset of *A*

$B \subset A$ ← subset

$\not\subset$ ← Not subset

$A \cap B = B$

$A \cup B = A$



A and *B* are disjoint sets

Mutually exclusive events

$A \cap B = \emptyset$

The same as even # $\cap$ odd # $= \emptyset$

Example 1.4

Throw a die once and observe the output face. $S=\{1,2,3,4,5,6\}$

Let A = {Even number appears}, $=\{2,4,6\}$

B = {Odd number appears}, $=\{1,3,5\}$

Not equal

D = {A number greater than 4 appears}. $=\{5,6\}$

Find and interpret $A \cup D$, $B \cap D$, D^c, $A \cap D^c$, $A \cap B$.

$A \cup D = \{2,4,5,6\}$

Event of an even # appears or a # > 4

$B \cap D = \{5\}$

Event of an odd # appears and a # > 4

$D^c = \{1,2,3,4\}$

Event of a # less than or equal to 4

$A \cap D^c = \{2,4\}$

Event of an even # appears & a # ≤ 4

$A \cap B = \phi$ "Mutually exclusive / disjoint events"

1.2 Axioms of probability theory

Probability interpretation

I. P(A) → chance of appearance (degree of belief)

Fair coin

P(H) = 0.5,

P(T) = 0.5

Equal chance

P(Served in less 10 min) = 0.9

Highly expected

pretty sure

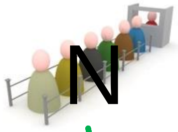
H, T, T, T, H, H - - - - -

II. P(A) → Long-run relative frequency

. . .

Heads appear approximately for 50% of the trials.

. . .



Served in less than 10 min in approximately 90% of the visits.

pretty sure

complete randomness

P(A) 0 0.5 1

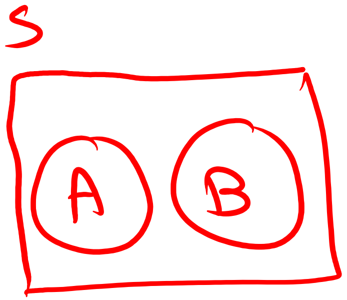
more random

Axiomatic definition of probability

For any event A, P(A) satisfies the following axioms:

1. **$P(A) \geq 0$**
2. **$P(S) = 1$**
3. **If A, B are mutually exclusive (*disjoint*), then**

$$\mathbf{P(A \cup B) = P(A) + P(B)}$$



$A \subseteq S$; $P(S)=1$

- $0 \leq P(A) \leq 1$
- If A, B and C are *mutually exclusive* (*disjoint*) then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Deductions from the axioms

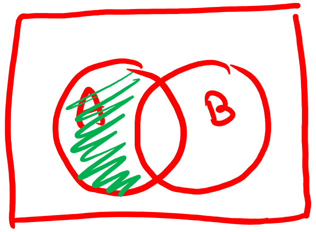
If P(A) satisfies the axioms, then:

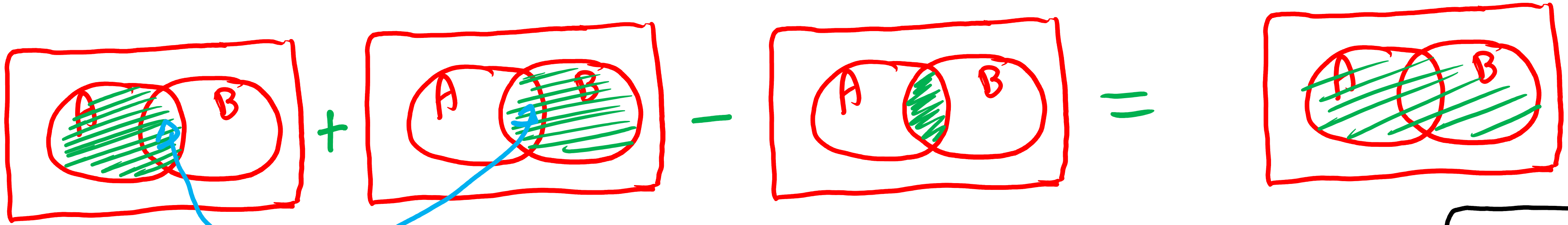
1. $P(\phi) = 0$
2. $P(A^c) = 1 - P(A)$ $\quad P(A^c) = \overset{1}{\cancel{P(S)}} - P(A)$
3. $P(A \cap B^c) = P(A) - P(A \cap B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For disjoint events

$A \cap B = \phi$

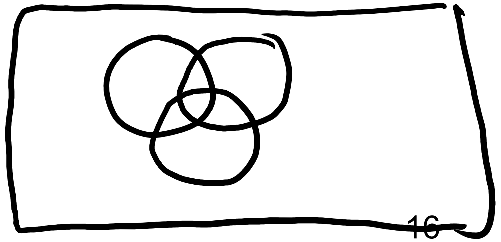
$P(A \cap B) = 0$





intersection area is counted twice

Exercise: Deduce $P(A \cup B \cup C)$



Example 1.5

From data record of years

The probability that a scheduled flight departs on time is 0.95, arrives on time is 0.85, and departs and arrives on time is 0.80.

A

B

A ∩ B

$P(A) = 0.95$

$P(B) = 0.85$

$P(A \cap B) = 0.8$

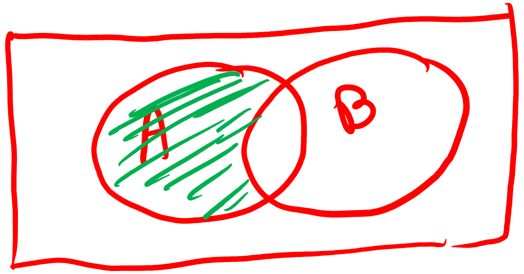
What is the probability that it departs on time but not arrives on time?

A

B^c

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$= 0.95 - 0.8 = 0.15$$



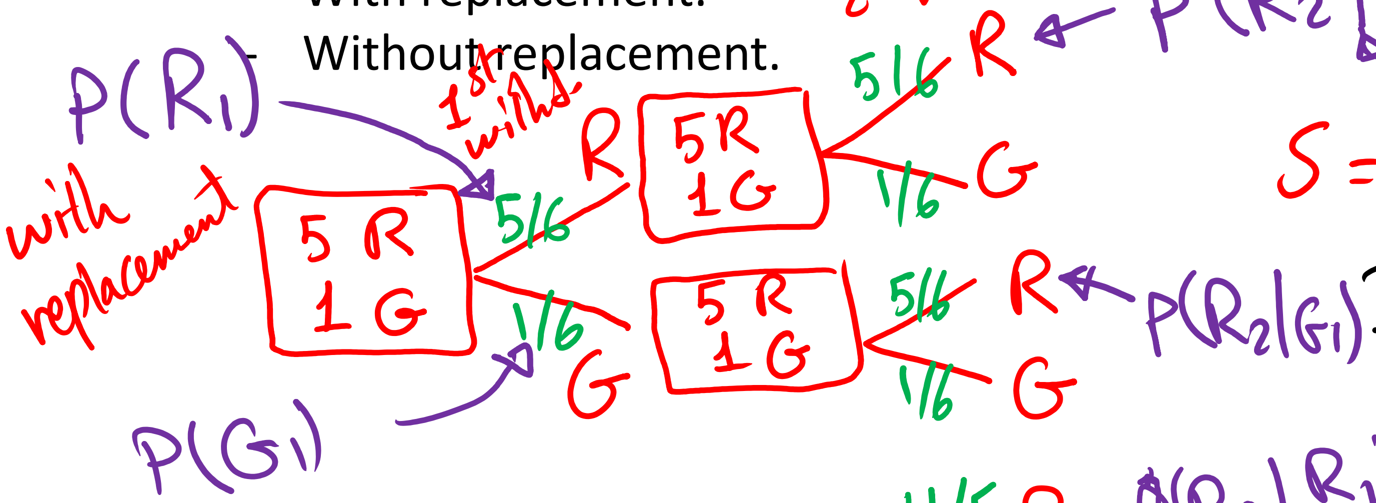
More examples

irrelevant in case of indep. exp.

Describe the sample space of:

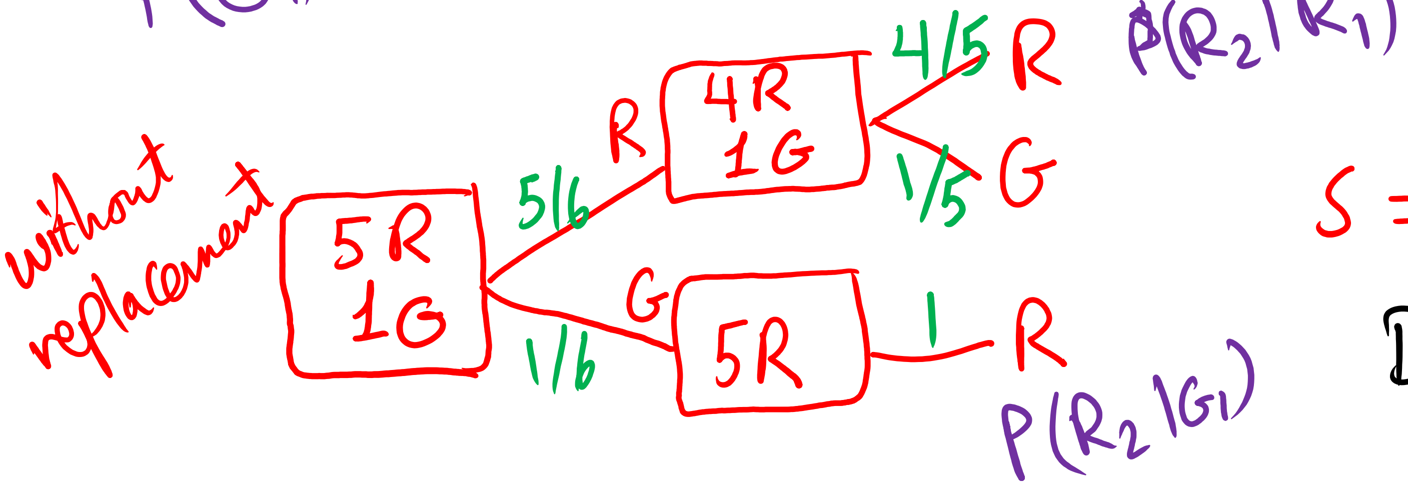
Choosing two balls from a box containing 5 red balls and 1 green ball in the two cases:

- With replacement.
- Without replacement.



$S = \{RR, RG, GR, GG\}$

Independent experiment



$S = \{RR, RG, GR\}$

Dependent experiment

Selecting a positive real number at random which is less than 10.

$S =]0, 10[$ open less than not equal

Selecting a positive real number at random ~~between~~ 3 and 7.

$S =]3, 7[$

Selecting a positive integer number at random which is less than 10.

$S = \{1,2,3,4,5,6,7,8,9\}$

Selecting a positive integer number at random between 3 and 7.

$S = \{4,5,6\}$

Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		Shock resistance	
		High B^c	Low B
Scratch resistance	High A	70 $A \cap B^c$	9 $A \cap B$
	Low A^c	16 $A^c \cap B^c$	5 $A^c \cap B$

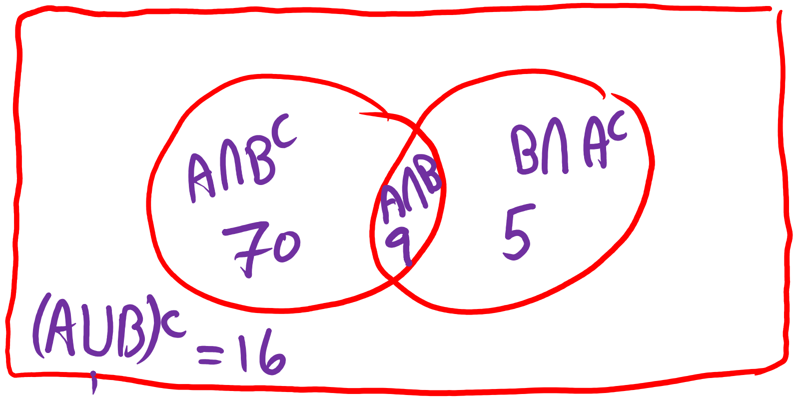
Let A be the event that the sample has high scratch resistance and B be the event that the sample has low shock resistance, find the number of samples of:

A → $70+9 = 79$

B → $9+5 = 14$

$A \cap B^c$ → 70

$A \cup B$ → $70+9+5$ or $100-16=84$



De Morgan

$A^c \cap B^c$

Reading

Counting techniques

- Combinations
- Permutations