Linear Algebra

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Outline

1. Introduction.
2. Matrix diagonalization.
3. Decoupling.

1. Introduction

Introduction

Suppose D is a diagonal matrix, say, $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Then

$$D^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$D^3 = DD^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}; \quad k \geq 1$$

Introduction

Computational advantages of diagonal matrices:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(1) D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D^T = D$$

$$(3) D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}, \quad d_i \neq 0$$

2. Matrix diagonalization

Diagonalization

- Several applications of linear algebra require the efficient computation of $\boldsymbol{A}^k$ for **large values of** $\boldsymbol{k}$.

- We can use our **eigenvalue/eigenvector** information to write a matrix A in the factored form $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$ where $\boldsymbol{D}$ stands for a **diagonal** matrix.

Now, if A can be written in the form $A = PDP^{-1}$, A^k is easy to compute.

sq. of diag. matrix

- $A^2 = \left(PDP^{-1}\right)\left(PDP^{-1}\right) = PD\left(\underline{P^{-1}P}\right)DP^{-1} = PDDP^{-1} = PD^2P^{-1}$

- $A^3 = \left(PDP^{-1}\right)A^2 = PDP^{-1}PD^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$

- In general,

    - $\boldsymbol{A^k = PD^kP^{-1}}$ $k \geq 1.$

$$\boldsymbol{A = P\,D\,P^{-1}}$$

How to get $\boldsymbol{P}$ and $\boldsymbol{D}$?

First → To get D

- Definition of similar matrices and related theorem.

Second → To get P

- Observe the relation $A = P\,D\,P^{-1}$ after getting D

$$A = P\,D\,P^{-1}$$

First → To get D

■Definition: Two square matrices A and B are said to be **similar** if

$$A = P\,B\,P^{-1} \text{ and } B = P\,A\,P^{-1}$$

■Thm: (Similar matrices have the same eigenvalues)

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf: A and B are similar $\Rightarrow P^{-1}AP = P^{-1}P\,B\,P^{-1}P$ or $\underbrace{B = P^{-1}\,A\,P}$

$|AB| = |A|\,|B|$

const. $\quad P^{-1}P$

$$\begin{aligned}|\lambda\mathrm{I} - B| &= |\lambda\mathrm{I} - P^{-1}AP| = |P^{-1}\lambda\mathrm{I}P - P^{-1}AP| = |P^{-1}(\lambda\mathrm{I} - A)P| \\ &= |P^{-1}||\lambda\mathrm{I} - A||P| = |P^{-1}||P||\lambda\mathrm{I} - A| = |\underbrace{P^{-1}P}_{|I|=1}||\lambda\mathrm{I} - A| \\ &= |\lambda\mathrm{I} - A|\end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

$$\boldsymbol{A = P\,D\,P^{-1}}$$

First → To get $\boldsymbol{D}$

Now,

→ A and D are similar matrices.

→ They have the same eigenvalues.

→ D is a diagonal matrix.

→ From last lecture, eigenvalues of diagonal or triangular matrices are the entries of their main diagonal.

If eigenvalues of $\boldsymbol{A}$ are $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$, $\boldsymbol{\lambda_3}$, …
then, D is calculated as

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$A = P\,D\,P^{-1}$$

Second → To get P

- Observe the relation $A = P\,D\,P^{-1}$ after getting D

Form for eigen values/vecs

$$A\,P = P\,D\,P^{-1}\,P$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$A\,P = P\,D$$

$$\begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix} \begin{array}{cc} P_1 & P_2 \\ \begin{bmatrix} \boxed{\begin{matrix} p_{11} \\ p_{12} \end{matrix}} & \boxed{\begin{matrix} p_{21} \\ p_{22} \end{matrix}} \end{bmatrix} \end{array} = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} P_{11}\lambda_1 & P_{21}\lambda_2 \\ P_{12}\lambda_1 & P_{22}\lambda_2 \end{bmatrix}$$

$AP_1 = \lambda P_1$

$$\begin{array}{cccc} \text{Col. 1} & \text{Col. 2} & & \\ [Ap_1 & Ap_2 & \ldots & Ap_n] \end{array} = \begin{array}{cccc} \text{Col. 1} & \text{Col. 2} & & \\ [\lambda_1 p_1 & \lambda_2 p_2 & \ldots & \lambda_n p_n] \end{array}$$

Conclusion:

Columns of P are the eigen vectors of A

Eigenvalues	Eigenvectors
$A \rightarrow \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$	$v_1, v_2, v_3, \ldots, v_n$

$$A = P\,D\,P^{-1}$$

$$P = [v_1 \quad v_2 \quad v_3 \quad \ldots \quad v_n]$$

Corresponding to λ_1

Corr. to λ_2

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Ex: Diagonalize the matrix $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix} = 0$

$\rightarrow \lambda s. = -1, -2$

@ $\lambda = -1$

$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix} \xrightarrow{\text{solve}} \bar{v}_1$

@ $\lambda = -2$

$\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix} \xrightarrow{\text{solve}} \bar{v}_2$

Sol: Recall from last lecture

$\lambda_1 = -1 \qquad \lambda_2 = -2$

$t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ t any value

$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\bar{v}_1 \Rightarrow \lambda_1 \quad \bar{v}_2 \Rightarrow \lambda_2 \quad \lambda_1$

$$A = P\,D\,P^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}^{-1}$$

λ_2

$\bar{v}_2 \quad \bar{v}_1 \quad \lambda_2 \quad \lambda_1$

$$\text{Or} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}$$

Can any matrix be diagonalized?

"No"

$$A_{n\times n} = P\,D\,P^{-1}$$

- A is a square matrix.
- P must be invertible.
  - $|P| \neq 0$
  - Row echelon of P has a pivot in each row/column.
  - A must have n independent eigenvectors.

Condition for diagonalization

An $n \times n$ matrix can be diagonalized if it has n independent eigenvectors

■Thm:

Eigenvectors of a matrix corresponding to different (unequal) eigenvalues are independent.

- If $A_{n \times n}$ has n different eigenvalues → Diagonalizable.
- If $A_{n \times n}$ has less than n different eigenvalues → Might be diagonalizable (Check the repeated eigenvalues)

eigen space of a single eigenvalue could include more than one vector

Ex: Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Triang.

eigenvalues = 1 "repeated"

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix} = (\lambda-1)^2 = 0$$

Eigenvalue : $\lambda_1 = 1$

Let $x_1 = t$, $x_2 = 0$ $\Rightarrow \bar{x} = \begin{bmatrix}t\\0\end{bmatrix} = t\begin{bmatrix}1\\0\end{bmatrix}$

Single Eigenvec

$P = [p_1 \; p_1]$

$|P| = 0$

$$\lambda I - A = I - A = \begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix} \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \end{array} \Rightarrow \text{Eigenvector}: \; p_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

For t=1

dim(eigenspace) corresp. to $\lambda = 1$ = 1

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

Ex: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Triang.

$\lambda s = 1, 0, -3$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable.

What is the process to find the factorization $A = PDP^{-1}$?

Step 1 Find the eigenvalues of $n \times n$ matrix A.

n diff λ's → diag.

Step 2 Find n linearly independent eigenvectors of A.

repetition check

Step 3 Construct the matrix P from the eigenvectors of A.

Step 4 Construct the diagonal matrix D from the corresponding eigenvalues of A.

Ex:

$$A=\begin{bmatrix}1 & -1 & -1\\ 1 & 3 & 1\\ -3 & 1 & -1\end{bmatrix}$$

$\overset{-1}{P}\,A\,P = \overset{-1}{P}\,P\,D\,P^{-1}\,P$

$\underbrace{\overset{-1}{P}\,A\,P}_{\text{Diag.}} = D$ diag.

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

steps skipped

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & 1 & 1\\ -1 & \lambda-3 & -1\\ 3 & -1 & \lambda+1\end{vmatrix}=(\lambda-2)(\lambda+2)(\lambda-3)=0$$

$$\text{Eigenvalues}:\ \lambda_1=2,\ \lambda_2=-2,\ \lambda_3=3$$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

dim(eig.space) = 1 = no. of free vars

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

dim = 1

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

dim = 1

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \ p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{array}{c} \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \\ \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \end{array}$$

$$\Rightarrow \quad D = \begin{array}{c} \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

Ex: (A diagonalizable matrix)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

repeated

$$\text{Eigenvalues}: \lambda_1 = 4, \lambda_2 = -2, \lambda_3 = -2$$

steps are skipped

$$(1)\lambda = 4 \Rightarrow \text{Eigenvector}: p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$s\begin{bmatrix}1\\-1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\end{bmatrix}$ independent

$$(2)\lambda=-2 \Rightarrow \text{ Eigenvector: } p_2=\begin{bmatrix}1\\-1\\0\end{bmatrix},\ p_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$P=[p_1 \quad p_2 \quad p_3]=\begin{array}{c}\begin{array}{ccc}v_1 & v_2 & v_3\end{array}\\ \begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array} \Rightarrow D=\begin{array}{c}\begin{array}{ccc}\lambda_1 & \lambda_2 & \lambda_3\end{array}\\ \begin{bmatrix}4 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix}\end{array}$$

- Notes:

$$(1)\ P=[p_2 \quad p_1 \quad p_3]=\begin{array}{c}\begin{array}{ccc}v_2 & v_1 & v_3\end{array}\\ \begin{bmatrix}1 & 1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array} \Rightarrow D=\begin{array}{c}\begin{array}{ccc}\lambda_2 & \lambda_1 & \lambda_3\end{array}\\ \begin{bmatrix}-2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -2\end{bmatrix}\end{array}$$

$$(2)\ P=[p_2 \quad p_3 \quad p_1]=\begin{array}{c}\begin{array}{ccc}v_2 & v_3 & v_1\end{array}\\ \begin{bmatrix}1 & 0 & 1\\ -1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\end{array} \Rightarrow D=\begin{array}{c}\begin{array}{ccc}\lambda_2 & \lambda_3 & \lambda_1\end{array}\\ \begin{bmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 4\end{bmatrix}\end{array}$$

Computing A^5

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^5 = P\, D^5\, P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (4)^5 & 0 & 0 \\ 0 & (-2)^5 & 0 \\ 0 & 0 & (-2)^5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$(A)^{-1} = (P\, D\, P^{-1})^{-1}$$

$$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = P\, D^{-1}\, P^{-1}$$

Decoupling

Another application for diagonalization
to solve a linear system of differential equations

$x^2 + 5x + 6 = 0$ Algebric equation Solution? $(x+2)(x+3)=0$

$x = -2, -3$

$|x-3| < 1$ Inequality Solution? $-1 < x-3 < 1$

$2 < x < 4$

$$\frac{dx}{dt} = t\frac{d^2x}{dt^2}$$ Differential equation

one indep. var.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$$ Differential equation

Ordinary ✓

Solution?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Differential equation Partial

More than one indep. var.

What is the solution of the differential equation?

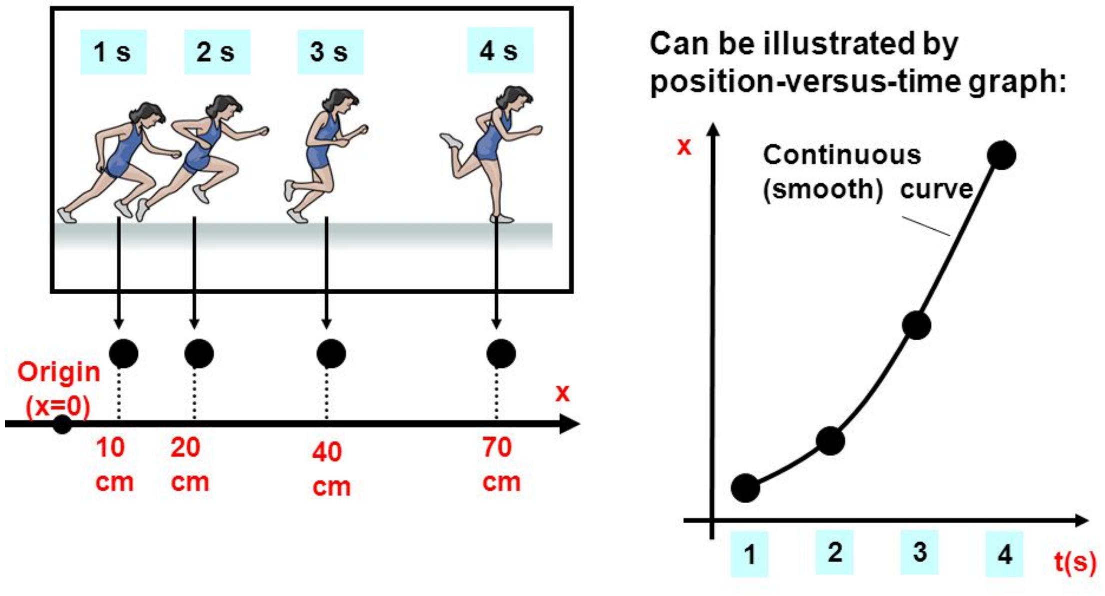
Ex: The kinematic equation $v = v_o + a\,t$

$$\frac{dx}{dt} = v_o + \frac{d^2x}{dt^2}\ t$$

Getting rid of the differentiation

$$x = f(t)$$

Motion along a straight line



Ordinary linear differential equation (OLDE) "with constant coefficients"

in one indep. var.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = f(x)$$

Constants

Examples:

$$y' = 1 \rightarrow \frac{dy}{dx} = 1$$

General sln.

$$\rightarrow y = x + c$$

$$\rightarrow 2 = 0 + c$$

$$\rightarrow c = 2$$

$$\rightarrow y = x + 2$$

Given initial conditions

$$\rightarrow y(0) = 2$$

@ x=0

y=2

There are different many techniques to solve differential equations

We will only study the simple technique of "separation of variables"

separation of variables → helps only solving DEs in the form

separated using multip./division

$$\frac{dy}{dx} = f(x)g(y)$$

$$\overbrace{\frac{1}{g(y)}dy}^{y\text{-only}} = \overbrace{f(x)dx}^{x\text{-only}}$$

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

Ex: $\boldsymbol{x' = a\,x} \rightarrow \boldsymbol{\frac{dx}{dt} = a\,x}$

$$\frac{1}{x}\,dx = a\,dt$$

$$\int \frac{1}{x}\,dx = \int a\,dt$$

could be $c, \frac{1}{2}, c^2, e^c, \ln c$

$$\ln(x) = a\,t + \overbrace{\ln(c)}$$

$$\ln(x) - \ln(c) = a\,t$$

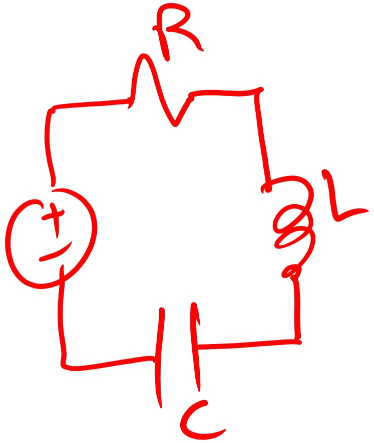
$$\ln\left(\frac{x}{c}\right) = a\,t$$

$$\frac{x}{c} = e^{a\,t}$$

$$x = c e^{a\,t}$$

$$\boxed{\begin{array}{l}\textit{Solution of } \boldsymbol{x' = a\,x} \\ \text{is } \boldsymbol{x(t) = c\,e^{a\,t}}\end{array}}$$

Assume two dependent variables $\boldsymbol{x(t)}$ and $\boldsymbol{y(t)}$



circuit of two loops

Coupled system

KVL loop1

$$\frac{dx}{dt} = 3x + y$$

KVL loop2

$$\frac{dy}{dt} = x - 4y$$

Decoupled system

$$\frac{dx}{dt} = -2\,x$$

$$\frac{dy}{dt} = y$$

$V_s = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$

$\frac{dv}{dt} = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i$

→ Get $i = f(t)$

"Transient response"

How to decouple?

Coupled system

$$\frac{dx_1}{dt} = a\,x_1 + b\,x_2$$

$$\frac{dx_2}{dt} = c\,x_1 + d\,x_2$$

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X' = A\ X$$

P^{-1}

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{Bmatrix} c_1x_1 + c_2x_2 \\ \end{Bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{Bmatrix} c_1x_1' + c_2x_2' \\ \end{Bmatrix}$$

Diagonalizing A:

$$A = P\,D\,P^{-1}$$

$$X' = PDP^{-1}X$$

$$P^{-1}X' = P^{-1}PDP^{-1}X$$

$$P^{-1}X' = DP^{-1}X$$

$$P^{-1}X' = DP^{-1}X$$

$$Y' = DY$$

$$\begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Decoupled system

$$\frac{dy_1}{dt} = \lambda_1\,y_1 \rightarrow y_1 = c_1 e^{\lambda_1 t}$$

$$\frac{dy_2}{dt} = \lambda_2\,y_2 \rightarrow y_2 = c_2 e^{\lambda_2 t}$$

To get X $\rightarrow P^{-1}X = Y$

$$\boxed{X = P\,Y}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= y_1\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} + y_2\begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

$$\boxed{X = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i}$$

\#

Ordinary differential equation (ODE)	Ordinary linear differential equation (OLDE)	Method of separation of variables to solve OLDE	Converting Coupled system of OLDEs to decoupled system of OLDEs (Decoupling)	Example

Example:

$$x_1' = x_1 - x_2 - x_3 \qquad x_1(0) = 0$$
$$x_2' = x_1 + 3x_2 + x_3 \qquad x_2(0) = -1$$
$$x_3' = -3x_1 + x_2 - x_3 \qquad x_3(0) = 10$$

Coeff. mat.

different, then diag

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = 0 \rightarrow \lambda = 2, -2, 3$$

	$\lambda I - A$		Eigenvector
$\rightarrow \lambda = 2$	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 3 & -1 & 3 & 0 \end{array}\right]$	➡	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
$\rightarrow \lambda = -2$	$\left[\begin{array}{ccc\|c} -3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 0 \\ 3 & -1 & -1 & 0 \end{array}\right]$	➡	$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$
$\rightarrow \lambda = 3$	$\left[\begin{array}{ccc\|c} 2 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right]$	➡	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$t[\;]$ select any t

t to simplify fractions

$$\boxed{X = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i}$$

$$t\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$t = 6$

Example: (Cont.)

$$
\begin{aligned}
x_1' &= x_1 - x_2 - x_3 & \qquad x_1(0) &= 0\\
x_2' &= x_1 + 3x_2 + x_3 & \qquad x_2(0) &= -1\\
x_3' &= -3x_1 + x_2 - x_3 & \qquad x_3(0) &= 10
\end{aligned}
$$

$$\boxed{\boldsymbol{X = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i}}$$

$$\lambda_1 = 2 \qquad \lambda_2 = -2 \qquad \lambda_3 = 3$$

$$p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{\overset{\lambda_1}{2t}} \overset{v_1}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} + c_2 e^{\overset{\lambda_2}{-2t}} \overset{v_2}{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}} + c_3 e^{\overset{\lambda_3}{3t}} \overset{v_3}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}} \;\Rightarrow\;
\begin{aligned}
x_1(t) &= -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t}\\
x_2(t) &= -c_2 e^{-2t} + c_3 e^{3t}\\
x_3(t) &= c_1 e^{2t} + 4c_2 e^{-2t} + c_3 e^{3t}
\end{aligned}$$

Example: (Cont.)

$$x_1' = x_1 - x_2 - x_3$$
$$x_2' = x_1 + 3x_2 + x_3$$
$$x_3' = -3x_1 + x_2 - x_3$$

$$x_1(0) = 0$$
$$x_2(0) = -1$$
$$x_3(0) = 10$$

t=0
x1=0
x2=-1
x3=10

General soln.

$$x_1(t) = -c_1e^{2t} + c_2e^{-2t} - c_3e^{3t}$$
$$x_2(t) = -c_2e^{-2t} + c_3e^{3t}$$
$$x_3(t) = c_1e^{2t} + 4c_2e^{-2t} + c_3e^{3t}$$

$$x_1(0) = 0 = -c_1 + c_2 - c_3$$
$$x_2(0) = -1 = -c_2 + c_3$$
$$x_3(0) = 10 = c_1 + 4c_2 + c_3$$

System of linear equations in c_1 , c_2 , and c_3

After solving using any method

$$c_1 = 1 , c_2 = 2 , c_3 = 1$$

Particular/specific soln.

$$\boxed{\begin{aligned} x_1(t) &= -e^{2t} + 2e^{-2t} - e^{3t} \\ x_2(t) &= -2e^{-2t} + e^{3t} \\ x_3(t) &= e^{2t} + 8e^{-2t} + e^{3t} \end{aligned}}$$

Final solution

Find the dimension & basis of the space spanned by the set of vectors

$$S = \{t^3 - 2t^2 + 3t \ , \ t^3 - 4t + 3 \ , \ 2t^2 + 5t - 1\}$$