Linear Algebra

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Outline

1. Review.
2. Range of transformation (Column space).
3. Basis to a column space.
4. Special types of linear transformation.
   1. Onto transformation.
   2. 1-to-1 transformation.
   3. Isomorphic transformation (Onto+1-to-1).

1. Review

Linear Transformation (L.T.): Condition $T(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) = \alpha\, T(\boldsymbol{u}) + \beta\, T(\boldsymbol{v})$

L.T. is given as

Transformation is from $\boldsymbol{R^2}$ to $\boldsymbol{R^3}$

Transformation function	Transformation matrix	Transformation of standard basis vectors	Transformation of non-standard basis vectors

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2} \quad X_{2\times 1} = b_{3\times 1}$$

$$\boxed{T(x) = A\,x}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A = [\,T(e_1)\;, T(e_2)\,]$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \; ?!$$

$\ker(T)$ is the set of all vectors x where $T(x) = 0$ or $A\,x = 0$. Also known as $NS(A)$

Coefficient matrix

$$\begin{aligned} x \quad\quad &= 0 \\ y &= 0 \\ x + y &= 0 \end{aligned}$$

$\text{nullity}(A) = \dim(NS(A))$ = no. of free vars.

$n = \text{rank}(A) + \text{nullity}(A)$

n → no. of cols of A

Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$C_2 = -1$

$C_1 - 1 = 1 \rightarrow C_1 = 2$

Given   Given

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - 1T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2. Range of transformation (Column space)

Introduction

Consider the system of equations

$$x - 4y + 3z = b_1$$
$$-x + 3y - z = b_2$$
$$2x - 4z = b_3$$

Matrix form

Domain Co-domain

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Vector form

$$x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

→ **For the system to be consistent $\bar{b}$ should be a linear combination of the columns of A.**

→ $\bar{b} \in span(columns\ of\ A)$.

→ $span(columns\ of\ A) \equiv Column\ space\ of\ A\ "CS(A)"$.

→ $CS(A) \equiv Range\ of\ transformation$ **represented by** A.

Range of transformation (Column space)

Null space $NS(A)$

$A\,x = 0$

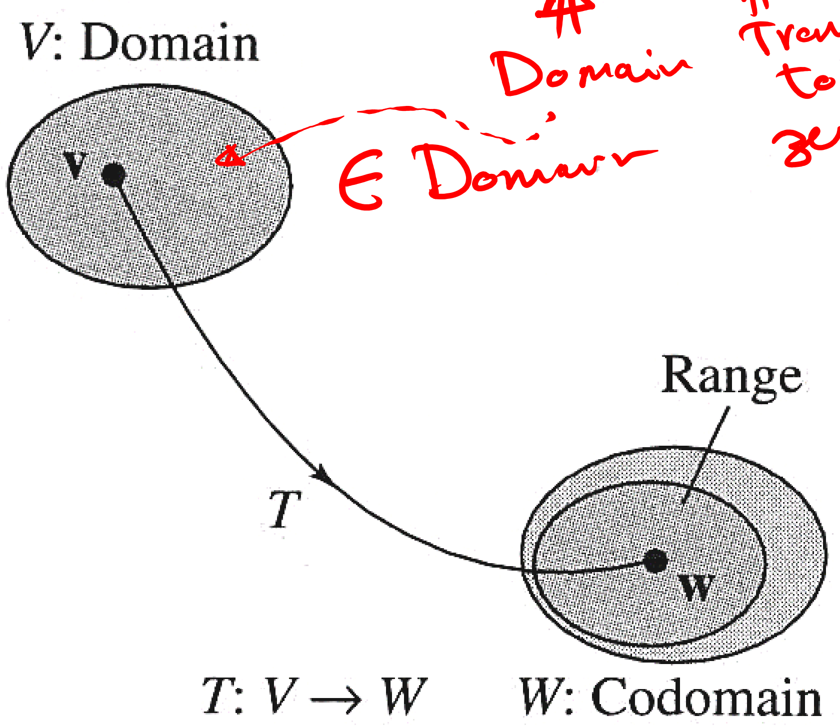
Domain

Transform to zero

$$T : V \xrightarrow{\ mapping\ } W, \qquad V, W : \text{vector space}$$

- The range of T:

the set of all vectors in the co-domain that can be reached under the transformation T



∈ Domain

$$Range(T) = \{v\,; T(x) = v\}$$

For $T(x) = A\,x$, $Range(T) = CS(A)$

- **So, vectors in the range should be a linear combination of the columns of A**
- **OR: vectors "b" in the range should make the system Ax=b consistent (has a solution)**

$CS(A) \equiv Range(T)$

∈ Co-domain

All vecs. in Co-domain that can be reached by T

Range of transformation (Column space)

0 ∈ closed(+) closed(·)

The range of transformation is a subspace of the co-domain.

Proof:

- Zero vector is in the range since $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{0}$ is always consistent.
- Assuming two vectors in the range of the transformation where

$$\boldsymbol{A}\,\boldsymbol{x_1} = \boldsymbol{b_1} \quad \text{and} \quad \boldsymbol{A}\,\boldsymbol{x_2} = \boldsymbol{b_2}$$

Consistent

- $b_1 + b_2 \in Range(T)$ since $A\,(x_1 + x_2) = b_1 + b_2$ Consistent
- $cb_1 \in Range(T)$ since $A\,(c\,x_1) = c\,Ax_1 = cb_1$ Consistent

Range of transformation (Column space)

$T: R^3 \to R^2$

large space   smaller space

Ex 01: Find the range of the transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+z \end{bmatrix}$

Get A → Get $CS(A) \equiv Span(columns\ of\ A)$

$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+z \end{bmatrix} \rightarrow \begin{matrix} x-y=0 \\ x+z=0 \end{matrix} \rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

span test → general vec

Consist. w/o cond.

" w cond.

$\rightarrow Range(T) = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$*-1$ $*1$

$-R_1 + R_2$

$\begin{bmatrix} 1 & -1 & 0 & b_1 \\ 1 & 0 & 1 & b_2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 0 & b_1 \\ 0 & \boxed{1} & 1 & b_2 \end{bmatrix}$

Consistent for any vector $\bar{b}$ w/o cond.

$Range(T) = all\ of\ R^2$

= all of the Co-domain

Redundant vector → Dependent set

This is not a basis

→ How to get a basis to the $CS(A)$?

3. Basis to a column space

Not Column spac

- Thm : (Row-equivalent matrices have the same row space)

span{rows of A}

If an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B,

then the row space of A is equal to the row space of B.

row Echelon

$$\boldsymbol{A} \xrightarrow[\text{Linear combination of the rows}]{\text{Row operations}} \boldsymbol{r}(\boldsymbol{A}) \quad \Longrightarrow \quad \text{Any row in } \boldsymbol{r}(\boldsymbol{A}) \in \boldsymbol{RS}(\boldsymbol{A})$$

$$\boldsymbol{r}(\boldsymbol{A}) \xrightarrow[\text{Linear combination}]{\text{Row operations}} \boldsymbol{A} \quad \Longrightarrow \quad \text{Any row in } \boldsymbol{A} \in \boldsymbol{RS}(\boldsymbol{r}(\boldsymbol{A}))$$

$$\boxed{\boldsymbol{RS}(\boldsymbol{A}) \equiv \boldsymbol{RS}(\boldsymbol{r}(\boldsymbol{A}))}$$

- Note:

Elementary row operations can change the column space.

- Thm : (Basis for the row space of a matrix)

  If a matrix A is row equivalent to a matrix B in row-echelon form, then the **nonzero row vectors of *B*** form a basis for the row space of A.

$$A = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} \boxed{\blacksquare \quad \blacksquare \quad \blacksquare} \\ \boxed{0 \quad \blacksquare \quad \blacksquare} \\ 0 \quad 0 \quad 0 \end{bmatrix}$$

Non-zero rows (pivot rows)
Has distinct zeros and
independent
→ Basis to $RS(A)$

▪ Ex Find a basis of row space of $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Sol:

$-2a_1 + a_2 = a_3$ $\qquad\qquad$ $-2b_1 + 1b_2 = b_3$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \end{array} \xrightarrow{\text{G.E.}} B = \begin{array}{cc} \begin{array}{c} \begin{bmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{array} \end{array} & \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \\ \end{array} \end{array}$$

Basis to $CS(A)$

a basis for $RS(A)$ = {the nonzero row vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

• Notes:

(1) $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2$

$\Rightarrow \mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$

(2) $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is L.I.

$\Rightarrow \underbrace{\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}}_{\text{Basis to } CS(A)}$ is L.I.

Conclusion

Elementary row operations

- **preserve the row space not the column space.**
- **Preserve linear relations in columns.**
- **Preserve linear dependence/independence in columns.**
- **Rank=dim(RS(A))=dim(CS(A))**

Nullity = n − Rank

To get a basis for row/column spaces of matrix A

For RS(A) or CS(A)

$A \longrightarrow Echelon(A) \longrightarrow$ Identify the pivots

→ Basis to RS(A) $\equiv$ $Pivot\ rows$

→ Basis to CS(A) → Identify the pivot columns → Select the corresponding columns from A

* Basis from the same colum
* Echelon of (A) req. for NS(A)

Method 2 for CS(A)

$A \longrightarrow A^t \longrightarrow Echelon(A^t) \longrightarrow$ Select the non-zero rows

To get a basis for a subspace

- → First, put the vectors in the **rows** of a matrix.
- → Continue similar to row space

OR

- → First, put the vectors in the **columns** of a matrix.
- → Continue similar to column space

Ex 02: Find a basis for the range and kernal of T and state their dimensions.

$$\boldsymbol{T(x) = A\,x}$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{5} & \mathbf{-5} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{5} \\ \mathbf{2} & \mathbf{-4} & \mathbf{4} & \mathbf{-4} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \xrightarrow{-2\,R_1 + R_3} \sim \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 0 & 2 & -6 & 6 \end{bmatrix} \xrightarrow{-2\,R_2 + R_3} \sim \begin{bmatrix} \boxed{1} & -3 & 5 & -5 \\ 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & \boxed{-4} \end{bmatrix}$$

Basis to $CS(A)$

Pivot columns

$$Range(T) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ -4 \end{bmatrix} \right\} \qquad \boxed{dim = 3}$$

Ex 02: Find a basis for the range and kernal of T and state their dimensions.

Cont

$$T(x) = A\,x$$

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$$

To get the kernal of T
Solve the homogeneous system

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 0 & 2 & -6 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \begin{array}{cccc|c|c} x_1 & x_2 & x_3 & x_4 & & \\ \boxed{1} & -3 & 5 & -5 & 0 & Eq1 \\ 0 & \boxed{1} & -3 & 5 & 0 & Eq2 \\ 0 & 0 & 0 & \boxed{-4} & 0 & Eq3 \end{array}$$

Pivot columns

Let $x_3 = t$

From $Eq3$: $-4x_4 = 0 \rightarrow x_4 = 0$

From $Eq2$: $x_2 - 3t = 0 \rightarrow x_2 = 3t$

From $Eq1$: $x_1 - 3(3t) + 5t = 0 \rightarrow x_1 = 4t$

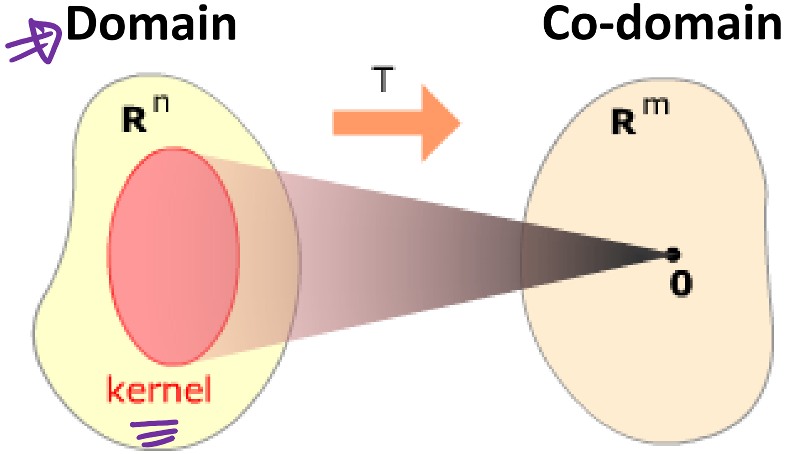
$$Kernal(T) = \begin{bmatrix} 4\,t \\ 3\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{dim = 1}$

= no. of free vars.

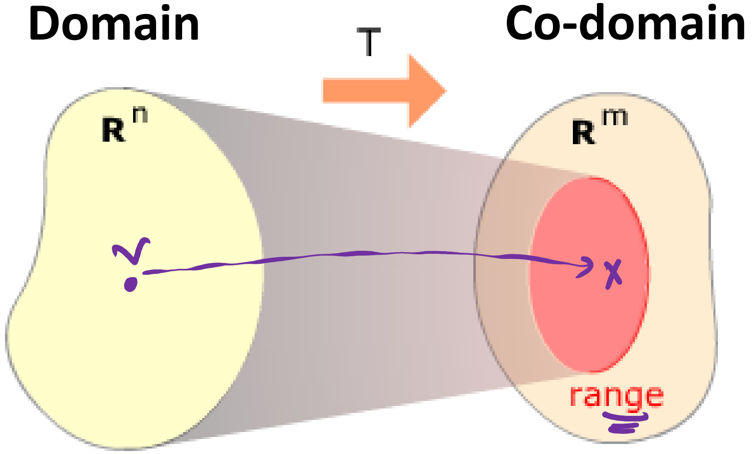
For $T(x) = A\,x$

$\text{Kernal}(T) \equiv NS(A)$



- **Solution** of $A\,x = 0$
- **dim = nullity of** A
- **Subspace** of **domain**
- Vectors x in the kernal satisfy $A\,x = 0$

$\text{Range}(T) \equiv CS(A)$

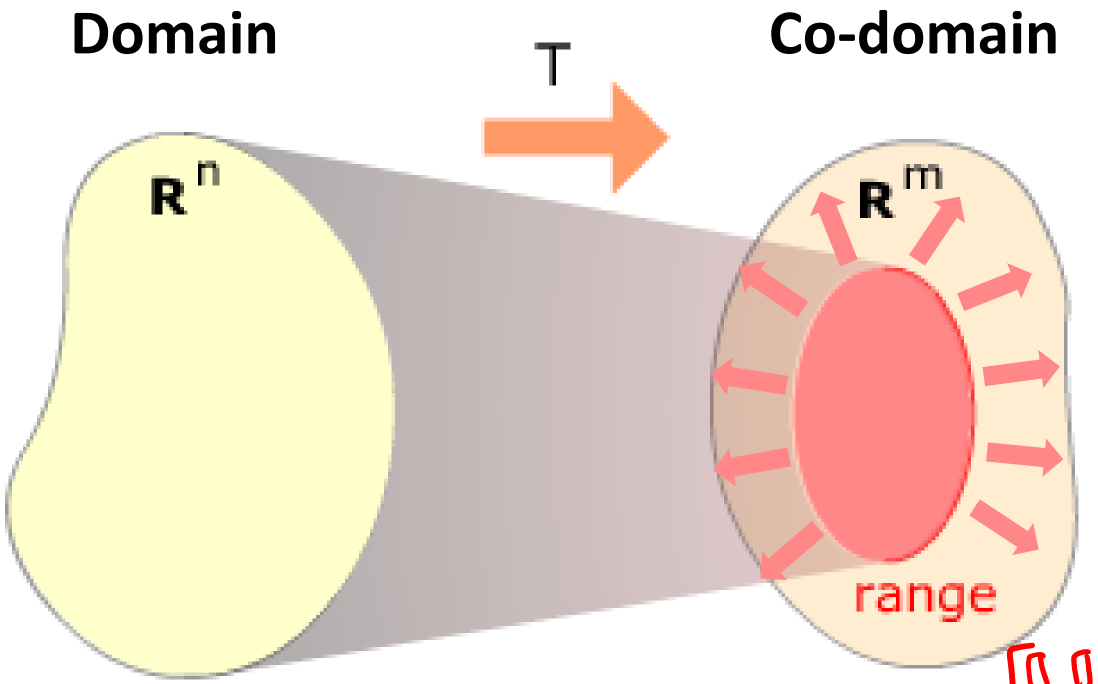


- **Span**$\{$**Columns of** $A\}$
- **dim = rank**(A) $= \dim(CS(A)) = \dim(RS(A))$
- **Subspace** of **co-domain**
- Vectors x in the range satisfy $A\,v = x$

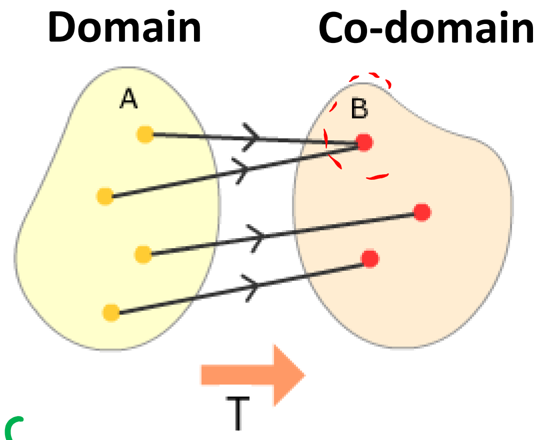
\# of cols of A $\rightarrow$ ***n* = rank(*A*) + nullity(*A*)**

4. Special types of linear transformation

Onto transformation



- Range ≡ Co-domain
- Every vector in the co-domain has a preimage.
- $Ax = b$ is always consistent. last row of A has a pivot
- A has a pivot in each row (in echelon form).
- For $A_{m\times n}$, it must have $m \le n$



Onto

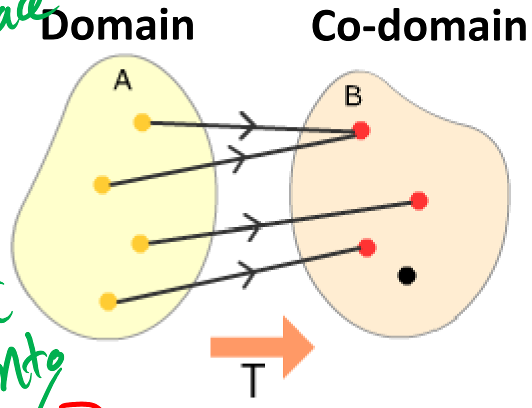
Not 1-to-1

By logic

$T: \mathbb{R}^2 \to \mathbb{R}^3$

small space

larger space



Not 1-to-1

Not onto

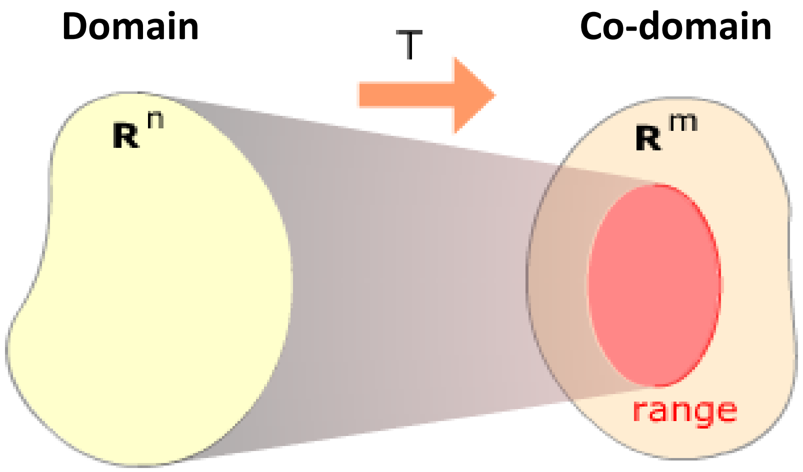
Not onto

3x2

has chance

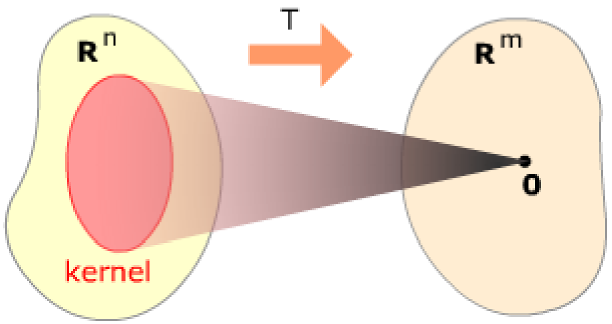
2x3

1-to-1 transformation



- Every vector in the range has **only one preimage**.
- $A\,x = b$ has a unique solution.
- A has no free variables $\equiv$ A has a pivot in each **column** (in echelon form).
- For $A_{m \times n}$, it must have $m \geq n$.
- $\mathrm{Kern}(T)$ has only the zero vector.

Nullity = 0



has a chance

3x2

Not 1-to-1

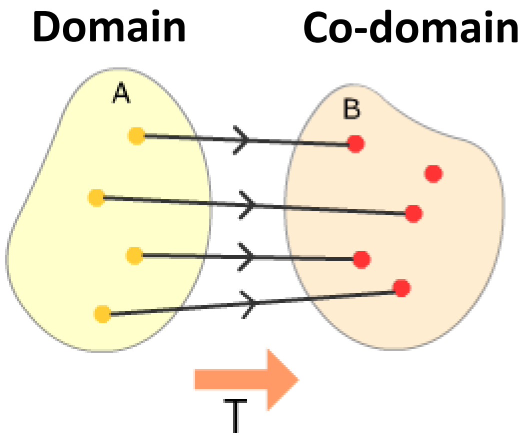
2x3

max Pivots = 2

$R^2 \to R^3$

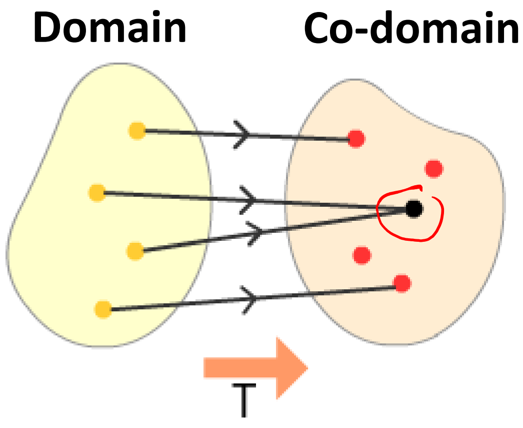
small space

larger space



1-to-1

Not onto



Not 1-to-1

Not onto

Ex 03: For each of the following transformation, determine whether T is onto and/or 1-to-1?

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y+z \end{bmatrix}$$

Already in echelon form

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \end{bmatrix}$$

- Pivot in each row.
- Always consistent.
- Range(T) is the whole R^2

Onto

- Has a free variable.
- Infinite number of solutions.

Not 1-to-1

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

$R^3 \rightarrow R^4$

$A_{4\times 3}$

Already in echelon form

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Not onto

1-to-1

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

$R^2 \rightarrow R^2$

$A_{2\times 2}$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & \boxed{2} \end{bmatrix}$$

$$\left.\begin{array}{l} \textbf{Onto} \\ \textbf{1-to-1} \end{array}\right\} \textbf{Isomorphic}$$

<u>Ex 04:</u> If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

$$v = [6 \quad 16 \quad 2 \quad 27]^T$$

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix} \underset{\sim}{\text{Gauss}} \begin{bmatrix} \boxed{1} & 3 & 2 & -1 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

- Pivot in each row → **Onto.**
- Pivot in each column → **1-to-1.**

Isomorphic

Ex 04: If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T,
(Cont.) state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

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Basis to $CS(A)$

$$Range(T) = Span\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ 1 \\ 7 \end{bmatrix}\right\}$$

$$dim = rank = 4$$

Ex 04: If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T,
(Cont.) state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

$$v = [6 \quad 16 \quad 2 \quad 27]^T$$

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix} \underset{\sim}{\text{Gauss}} \left[\begin{array}{cccc|c} \boxed{1} & 3 & 2 & -1 & 0 \\ 0 & \boxed{1} & 3 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Homog. + unique sln.

Has only the zero solution

$$Ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$dim = 0$$

<u>Ex 04:</u> If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T,
<u>(Cont.)</u> state whether T is onto and/or 1 to 1

Domain Codomain

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix} \quad R^4 \to R^4 \quad 4\times 4$$

$\subseteq$ Codomain R^4 $\qquad$ $\subseteq$ Domain R^4 $\qquad$ dim $= 0$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

Yes $\qquad$ No

transform $\quad v = [6 \quad 16 \quad 2 \quad 27]^T$

$Ax = v$?!

$$A = \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 6 \\ 0 & 2 & 6 & 8 & 16 \\ -1 & 1 & 2 & 1 & 2 \\ 3 & 9 & 5 & 7 & 27 \end{array}\right] \text{ Gauss } \sim \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & \blacksquare \\ 0 & 1 & 3 & 4 & \blacksquare \\ 0 & 0 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & 1 & \blacksquare \end{array}\right] \begin{array}{c} Eq1 \\ Eq2 \\ Eq3 \\ Eq4 \end{array}$$

After solving the system using back substitution

The pre-image of v is $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Ex 05: If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & -1 & 1 \\ 2 & 4 & 6 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & -1 & 1 \\ 2 & 4 & 6 & -2 \end{bmatrix} \underset{\sim}{\text{Gauss}} \begin{array}{cccc|c|c} x_1 & x_2 & x_3 & x_4 & & \\ \boxed{1} & 2 & 3 & -1 & 0 & Eq1 \\ 0 & \boxed{-3} & -4 & 2 & 0 & Eq2 \\ 0 & 0 & 0 & 0 & 0 & Eq3 \end{array}$$

Basis to $CS(A)$

- Not all rows have pivots
  → **Not onto.**
- Has free variables
  → **Not 1-to-1.**

$$Range(T) = Span\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right\}$$

$$dim = rank = 2$$

Let $x_3 = s$ and $x_4 = t$

From $Eq2$: $-3x_2 - 4s + 2t = 0 \rightarrow x_2 = \frac{-4}{3}s + \frac{2}{3}t$

From $Eq1$: $x_1 + 2\left(\frac{-4}{3}s + \frac{2}{3}t\right) + 3s - t = 0 \rightarrow x_1 = \frac{-1}{3}s - \frac{1}{3}t$

$$Kernal(T) = \begin{bmatrix} \frac{-1}{3}s - \frac{1}{3}t \\ \frac{-4}{3}s + \frac{2}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{dim = 2}$$

Ex 05: (Cont.) If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & -1 & 1 \\ 2 & 4 & 6 & -2 \end{bmatrix}$$

3x4

$T: \underset{\text{Domain}}{R^4} \to \underset{\text{Codomain}}{R^3}$

$$Range(T) = Span\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right\}$$

$$Kernal(T) = s\begin{bmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

then determine whether the following vectors belongs to the range and/or kernel. If it belongs to the range find its pre image:

$v_2 = [6 \quad 16 \quad 2]^T$

3 components → $\notin NS(A)$, Could $\in CS(A)$

Consist. or not

$$\begin{bmatrix} 1 & 2 & 6 \\ 1 & -1 & 16 \\ 2 & 4 & 2 \end{bmatrix}$$

Check linear combination

$v_1 = [6 \quad 16 \quad 2 \quad 27]^T$

4 components → Could $\in NS(A)$, $\notin CS(A)$

Consist. or not

$$\begin{bmatrix} -1/3 & -1/3 & 6 \\ -4/3 & 2/3 & 16 \\ 1 & 0 & 2 \\ 0 & 1 & 27 \end{bmatrix}$$

Check linear combination