Linear Algebra

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Outline

1. Definition.
2. Matrix transformation.
3. Matrix transformation and bases.
4. Kernal of the transformation (Null space).

1. Definition

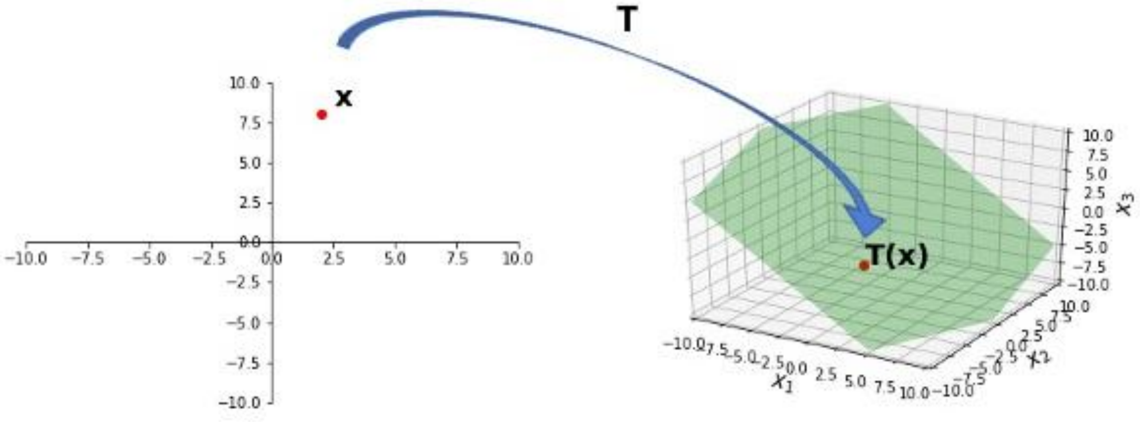
Introduction to Linear Transformations

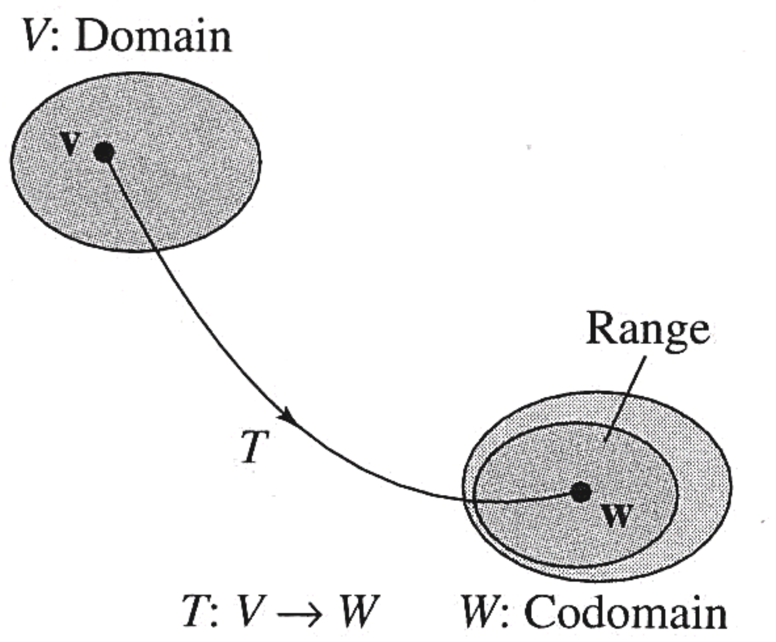
■ Function ***T*** that maps a vector space ***V*** into a vector space ***W***:

$$\boldsymbol{T}:\boldsymbol{V}\xrightarrow{\ \boldsymbol{mapping}\ }\boldsymbol{W},\qquad \boldsymbol{V},\boldsymbol{W}:\textbf{vector space}$$

V: the domain of T

W: the codomain of T





Definitions

- Image of $\mathbf{v}$ under T:

  If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

  $$T(\mathbf{v}) = \mathbf{w}$$

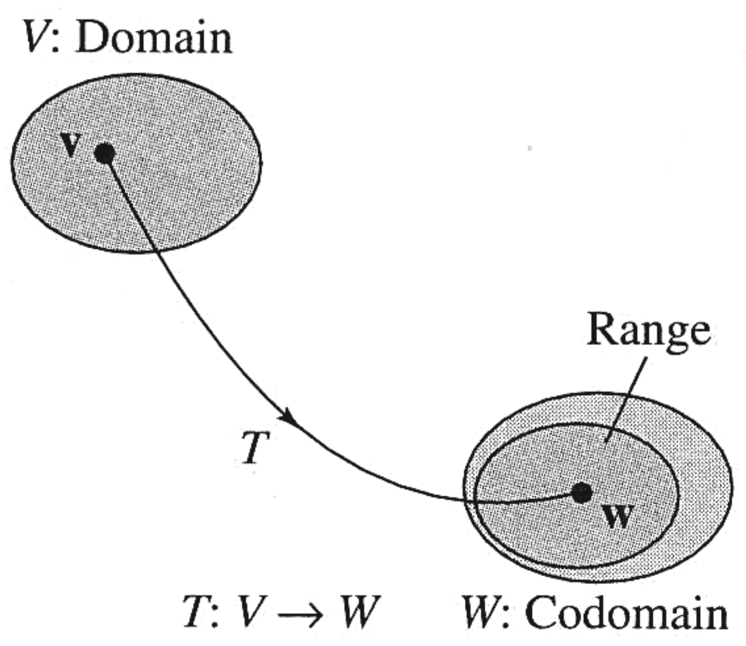
  Then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T.

- the range of T:

  The set of all images of vectors in V.

- the preimage of $\mathbf{w}$:

  The set of all $\mathbf{v}$ in V such that $T(\mathbf{v})=\mathbf{w}$.



■Ex: (A function from R^2 into R^2)

Domain Codomain

$$T: R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

fn. of transformation

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

x comp. v_1 y comp. v_2

Sol:

$$(a)\ \mathbf{v} = (\overset{v_1}{-1}, \overset{v_2}{2})$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

Sys. of linear eqs.

$$\begin{array}{ccc} v_1 & v_2 & c \end{array} \qquad \begin{bmatrix} 1 & -1 & \vdots & -1 \\ 1 & 2 & \vdots & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \vdots & -1 \\ 0 & 3 & \vdots & 12 \end{bmatrix} \quad -R_1 + R_2 \rightarrow R_2$$

E_2: $3v_2 = 12$ $\boxed{v_2 = 4}$

E_1: $v_1 - (4) = -1$ $\boxed{v_1 = 3}$

$$\Rightarrow v_1 = 3,\ v_2 = 4$$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

■Linear Transformation (L.T.):

V, W : vector space

$T: V \to W$: V to W linear transformation

$$(1)\ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

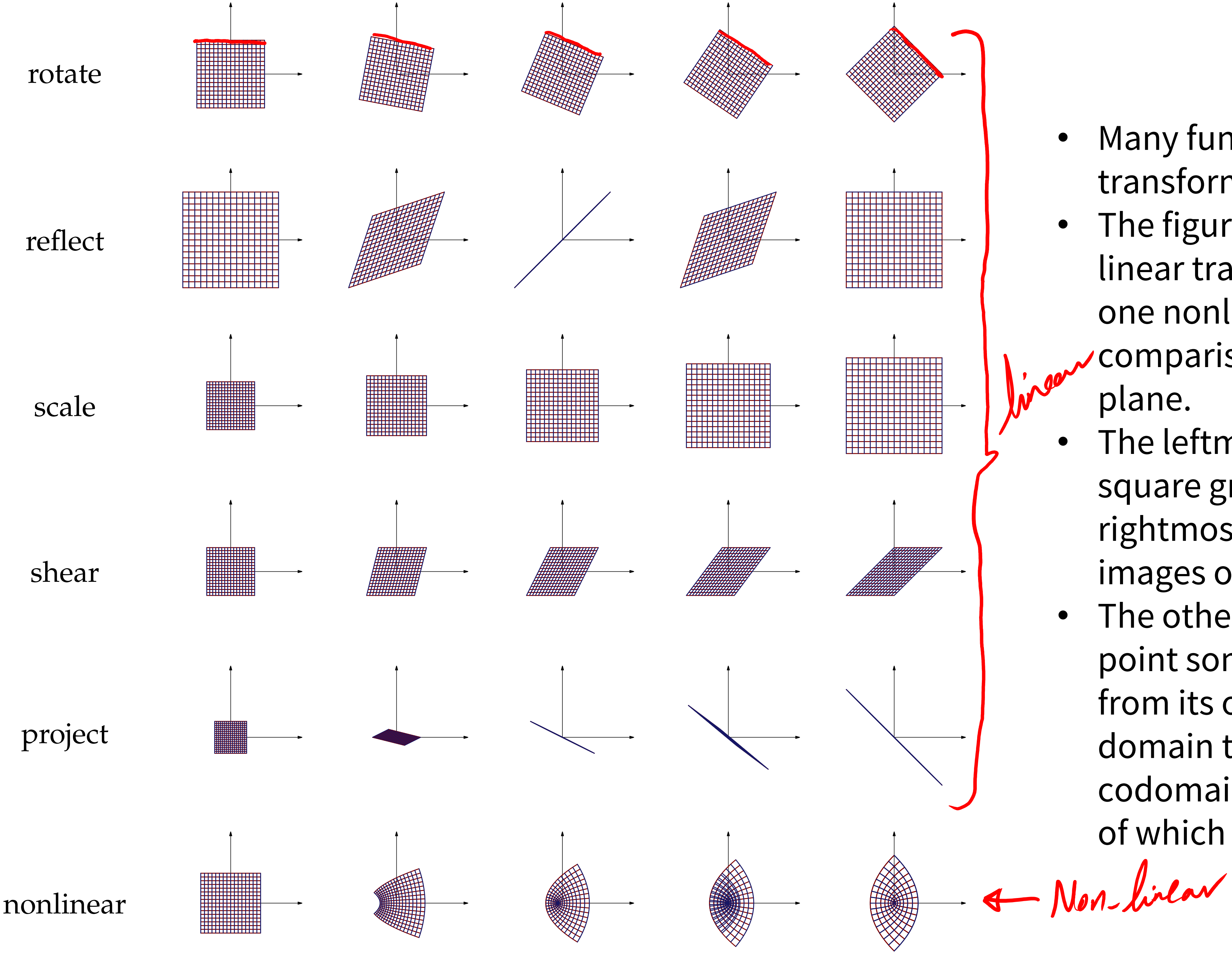
Or in **one condition**

$$T(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) = \alpha\,T(\boldsymbol{u}) + \beta\,T(\boldsymbol{v})$$

- Note:

A linear transformation is said to be operation preserving.

$$T(\mathbf{u}\underset{\text{Addition in }V}{\uparrow{+}}\mathbf{v}) = T(\mathbf{u})\underset{\text{Addition in }W}{\uparrow{+}}T(\mathbf{v}) \qquad T(\underset{\text{Scalar multiplication in }V}{c\mathbf{u}}) = \underset{\text{Scalar multiplication in }W}{cT}(\mathbf{u})$$



- Many fundamental geometric transformations are linear.
- The figure below illustrates several linear transformations (as well as one nonlinear one, for comparison) from the plane to the plane.
- The leftmost column shows a square grid of points, and the rightmost column shows the images of those points.
- The other columns show each point somewhere along the path from its original location in the domain to its final location in the codomain, to help you get a sense of which points go where.

Verifying a linear transformation

$\alpha\bar{u} + \beta\bar{v}$

$\alpha(u_1, u_2) + \beta(v_1, v_2)$

$(\alpha u_1, \alpha u_2)$
$+ (\beta v_1, \beta v_2)$

$(\alpha u_1 + \beta v_1,\ \alpha u_2 + \beta v_2)$

■Ex: (Verifying a linear transformation T from R^2 into R^2)

$$T(\overset{x}{v_1}, \overset{y}{v_2}) = (\overset{x-y}{v_1 - v_2}, \overset{x+2y}{v_1 + 2v_2})$$ fn. of transf.

Pf:

$\mathbf{u} = (\overset{x}{u_1}, \overset{y}{u_2})$, $\mathbf{v} = (v_1, v_2)$: vector in R^2, α, β: any real number

We need to prove that $T(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = \alpha\,T(\mathbf{u}) + \beta\,T(\mathbf{v})$

$$LHS = T(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = T(\overbrace{\alpha\,u_1 + \beta\,v_1}^{x}, \overbrace{\alpha\,u_2 + \beta\,v_2}^{y})$$

$$= (\overset{x}{(\alpha\,u_1 + \beta\,v_1)} \overset{-}{-} \overset{y}{(\alpha\,u_2 + \beta\,v_2)}, \overset{x}{(\alpha\,u_1 + \beta\,v_1)} \overset{+2}{+ 2}\overset{y}{(\alpha\,u_2 + \beta\,v_2)})$$

check

$$= (\alpha\,u_1 + \beta\,v_1 - \alpha\,u_2 - \beta\,v_2, \alpha\,u_1 + \beta\,v_1 + 2\,\alpha\,u_2 + 2\,\beta\,v_2)$$

$$RHS = \alpha\,T(\mathbf{u}) + \beta\,T(\mathbf{v}) = \alpha\,(u_1 - u_2, u_1 + 2u_2) + \beta(v_1 - v_2, v_1 + 2v_2)$$

$$= (\alpha\,u_1 - \alpha\,u_2 + \beta\,v_1 - \beta\,v_2, \alpha\,u_1 + 2\,\alpha\,u_2 + \beta\,v_1 + 2\,\beta\,v_2) = LHS$$ ✓

Therefore, T is a linear transformation.

Verifying a linear transformation

■Ex: Let $L: R^3 \to R^3$, $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{y}, \boldsymbol{x}-\boldsymbol{z})$ fn. of transf.

Is L a linear transformation ?

Pf:

Let $\boldsymbol{u}, \boldsymbol{v} \in R^3$, $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$

α, β: any real number

We need to prove that $L(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = \alpha\,L(\mathbf{u}) + \beta\,L(\mathbf{v})$

$$LHS = L(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = L(\overbrace{\alpha\,u_1 + \beta\,v_1}^{x}, \overbrace{\alpha\,u_2 + \beta\,v_2}^{y}, \overbrace{\alpha\,u_3 + \beta\,v_3}^{z})$$

check

$$= (\overset{x+y}{\alpha\,u_1 + \beta\,v_1 + \alpha\,u_2 + \beta\,v_2}, \overset{y}{\alpha\,u_2 + \beta\,v_2}, \overset{x-z}{\alpha\,u_1 + \beta\,v_1 - \alpha\,u_3 - \beta\,v_3})$$

$$RHS = \alpha\,L(\mathbf{u}) + \beta\,L(\mathbf{v}) = \alpha\,(u_1 + u_2, u_2, u_1 - u_3) + \beta(v_1 + v_2, v_2, v_1 - v_3)$$

$$= (\alpha\,u_1 + \alpha\,u_2 + \beta\,v_1 + \beta\,v_2, \alpha\,u_2 + \beta\,v_2, \alpha\,u_1 - \alpha\,u_3 + \beta\,v_1 - \beta\,v_3) = LHS\ \checkmark$$

Therefore, T is a linear transformation.

Verifying a linear transformation

■Ex: Let $L: \overset{\text{Domain}}{R^2} \rightarrow \overset{\text{Co-domain}}{R^3}$, $L(x, y) = (x+1, y, x+y)$

Is L a linear transformation ?

Pf:

Let $u, v \in R^2$, $u = (u_1, u_2)$, $v = (v_1, v_2)$

α, β: any real number

We need to prove that $L(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = \alpha\,L(\mathbf{u}) + \beta\,L(\mathbf{v})$

$$LHS = L(\alpha\,\mathbf{u} + \beta\,\mathbf{v}) = L(\overbrace{\alpha\,u_1 + \beta\,v_1}^{x}, \overbrace{\alpha\,u_2 + \beta\,v_2}^{y})$$

check

$$= (\overbrace{\alpha\,u_1 + \beta\,v_1 + \boxed{1}}^{x+1}, \overbrace{\alpha\,u_2 + \beta\,v_2}^{y}, \overbrace{\alpha\,u_1 + \beta\,v_1 + \alpha\,u_2 + \beta\,v_2}^{x+y})$$

$$RHS = \alpha\,L(\mathbf{u}) + \beta\,L(\mathbf{v}) = \alpha\,(u_1 + 1, u_2, u_1 + u_2) + \beta(v_1 + 1, v_2, v_1 + v_2)$$

$$= (\alpha\,u_1 + \boxed{\alpha} + \beta\,v_1 + \boxed{\beta}, \alpha\,u_2 + \beta\,v_2, \alpha\,u_1 + \alpha\,u_2 + \beta\,v_1 + \beta\,v_2) \neq \boldsymbol{LHS}$$

Therefore, T is **not** a linear transformation.

2. Matrix transformation

Matrix transformation

$$\boxed{T(x) = A\,x}$$

Every **matrix transformation** $\boldsymbol{T}: \boldsymbol{x} \to \boldsymbol{Ax}$ is a **linear** transformation.

Domain Co-domain

$\boldsymbol{A}_{\boldsymbol{m}\times\boldsymbol{n}}\,\boldsymbol{x}_{\boldsymbol{n}\times\mathbf{1}} = \boldsymbol{b}_{\boldsymbol{m}\times\mathbf{1}}$ transforms $\boldsymbol{x}$ (in $\boldsymbol{R}^{\boldsymbol{n}}$) into $\boldsymbol{b}$ (in $\boldsymbol{R}^{\boldsymbol{m}}$)

Proof that Ax is a linear transformation:

Let $\boldsymbol{T}(\boldsymbol{x}_1) = \boldsymbol{A}\boldsymbol{x}_1$ and $\boldsymbol{T}(\boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_2$

We need to prove that

$$T(\alpha\, x_1 + \beta\, x_2) = \alpha\, T(x_1) + \beta\, T(x_2)$$

$\alpha\,, \beta$: any real number

$$LHS = T(\alpha\, x_1 + \beta\, x_2) = A(\alpha\, x_1 + \beta\, x_2)$$

$$= \alpha\, A\, x_1 + \beta\, A\, x_2$$

$$= \alpha\, T(x_1) + \beta\, T(x_2) = RHS$$

- **Ex: (A linear transformation defined by a matrix)**

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

3x2

Find $T(\mathbf{v})$,where $\mathbf{v} = (2, -1)$

Sol: $\mathbf{v} = (2, -1)$

R^2 vector col. vec. R^3 vector

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore T(2, -1) = (6, 3, 0)$

Ex: Let $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$ and let

2×5

$T: R^5 \to R^2$ be a linear transformation such that $T(X) = A\,X$

Find (a) $T(1\,,0\,,-1\,,3\,,0)$ (b) $T^{-1}(-1\,,8)$

(a) $T(1\,,0\,,-1\,,3\,,0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+0-1+9+0 \\ 0+0-2-3+0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$

(b) Let $(u_1, u_2, u_3, u_4, u_5)$ be the pre-image of $(-1\,,8)$ under $T: T(X) = A\,X$

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

System of linear equations

The augmented matrix

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 & -1 \\ 0 & 0 & 2 & -1 & 0 & 8 \end{bmatrix}$$

Ex:(Continued) Let $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$ and let

$T: R^5 \to R^2$ be a linear transformation such that $T(X) = A\,X$

Find (a) $T(1\,,0\,,-1\,,3\,,0)$ (b) $T^{-1}(-1\,,8)$

The augmented matrix

$$\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & c \end{array}$$
$$\left[\begin{array}{cccccc} -1 & 2 & 1 & 3 & 4 & -1 \\ 0 & 0 & 2 & -1 & 0 & 8 \end{array}\right]$$

Already in Echelon form

Let $u_2 = t_1$, $u_4 = t_2$, $u_5 = t_3$

$E_2: \; 2u_3 - t_2 = 8 \Rightarrow u_3 = 4 + \frac{1}{2}t_2$

$E_1: \; -u_1 + 2t_1 + 4 + \frac{1}{2}t_2 + 3t_2 + 4t_3 = -1$

$u_1 = 2t_1 + \frac{7}{2}t_2 + 4t_3 + 5$

$$T^{-1}(-1,8) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 2t_1 + \frac{7}{2}t_2 + 4t_3 + 5 \\ t_1 \\ 4 + \frac{1}{2}t_2 \\ t_2 \\ t_3 \end{pmatrix}$$

inf. solutions

$\rightarrow$ no. of rows or max no. of pivots $= 2 <$ no. of vars.

$\rightarrow T: R^5$ "high dim." $\rightarrow R^2$ "low dim."

3. Matrix transformation and bases

- **Ex: (Linear transformations and bases)**

Tranf. given as
→ fn. of transf.
→ Matrix of transf.
→ Basis of domain

Let $T: \underbrace{R^3}_{\text{Domain}} \rightarrow R^3$ be a linear transformation such that

$$\left\{\begin{array}{l} T(1,0,0) = (2,-1,4) \\ T(0,1,0) = (1,5,-2) \\ T(0,0,1) = (0,3,1) \end{array}\right.$$

Find $T(2, 3, -2)$.

Transformation of the standard basis

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 5 & 3 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \text{Test} \end{array}$$

std. basis

Sol:

Linear combination in terms of the basis

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

Sub.

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.})$$

$$= 2(\underset{x}{2}, \underset{y}{-1}, \underset{z}{4}) + 3(\underset{x}{1}, \underset{y}{5}, \underset{z}{-2}) - 2(\underset{x}{0}, \underset{y}{3}, \underset{z}{1})$$

$$= (\underset{x}{7}, \underset{y}{7}, \underset{z}{0})$$

Matrix transformation

To represent a linear transformation $\boldsymbol{T}$ by a matrix $\boldsymbol{A}$

1. Apply the **transformation** to each vector in the **standard bases** of the domain.
2. Place the resulting vectors in the **columns of $\boldsymbol{A}$**

$$A = [T(e_1)\,, T(e_2)\,, T(e_3)\,, \ldots, T(e_n)]$$

$$\begin{bmatrix}1\\0\\0\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\0\\0\\\vdots\\1\end{bmatrix}$$

Ex: Find the matrix $\boldsymbol{A}$ of the following transformation such that $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{A}\,\boldsymbol{x}$

$$\boldsymbol{T}\left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{x}+\boldsymbol{y} \end{bmatrix}$$

fn. of transf.

Domain R^2

Transformation is from $\boldsymbol{R}^2$ to $\boldsymbol{R}^3$ → Let $\boldsymbol{A} = [\boldsymbol{T}(\boldsymbol{e}_1), \boldsymbol{T}(\boldsymbol{e}_2)]$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

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$$\therefore\ \boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

mat. of transf.

• **Ex: (Linear transformations and bases)**

Standard Basis

Let $T: R^3 \rightarrow R^2$ be a linear transformation such that

$$T(1,0,0) = (2,-4)$$

$$T(0,1,0) = (3,-5)$$

$$T(0,0,1) = (2,3)$$

$$A = \begin{bmatrix} 2 & 3 & 2 \\ -4 & -5 & 3 \end{bmatrix}$$

Find (a) $T(1,-2,3)$ (b) $T(a,b,c)$ fn. of trans.

Sol: (a)

$$\begin{bmatrix} 2 & 3 & 2 \\ -4 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

$$\therefore T(1,-2,3) = (2,15)$$

(b)

$$\begin{bmatrix} 2 & 3 & 2 \\ -4 & -5 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b+2c \\ -4a-5b+3c \end{bmatrix}$$

$$\therefore T(a,b,c)$$

$$= (2a+3b+2c, -4a-5b+3c)$$

Ex: Let $L: R^3 \to R^2$ be a linear transformation for which

$L(1,\overset{\bar{v}_1}{1},0) = (2,1)$, $L(1,\overset{\bar{v}_2}{0},1) = (1,-1)$, $L(0,\overset{\bar{v}_3}{1},1) = (0,0)$ <u>Non</u> standard Basis

Find $L(\overset{\text{Test}}{\overline{2,-4,1}})$

First, write $(2,-4,1)$ as a linear combination of $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$

$\rightarrow (2,-4,1) = \alpha\,(1,1,0) + \beta\,(1,0,1) + \gamma\,(0,1,1)$

$$-R_1 + R_2 \to R_2 \qquad\qquad R_2 + R_3 \to R_3$$

$$\begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \text{Test} \end{array} \qquad\qquad\qquad\qquad\qquad \begin{array}{cccc} \alpha & \beta & \gamma & c \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$

From **row 3** $\rightarrow 2\gamma = -5 \rightarrow \gamma = -2.5$

From **row 2** $\rightarrow -\beta + (-2.5) = -6 \rightarrow \beta = 6 - 2.5 \rightarrow \beta = 3.5$

From **row 1** $\rightarrow \alpha + (3.5) + (0) = 2 \rightarrow \alpha = 2 - 3.5 \rightarrow \alpha = -1.5$

<u>Ex:(Continued)</u> Let $\boldsymbol{L}: \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^2$ be a linear transformation for which

$$\boldsymbol{L}(\mathbf{1}, \mathbf{1}, \mathbf{0}) = (\mathbf{2}, \mathbf{1}) \quad , \quad \boldsymbol{L}(\mathbf{1}, \mathbf{0}, \mathbf{1}) = (\mathbf{1}, -\mathbf{1}) \quad , \quad \boldsymbol{L}(\mathbf{0}, \mathbf{1}, \mathbf{1}) = (\mathbf{0}, \mathbf{0})$$

Find $\quad \boldsymbol{L}(\mathbf{2}, -\mathbf{4}, \mathbf{1})$

First, write $(\mathbf{2}, -\mathbf{4}, \mathbf{1})$ as a linear combination of $(\mathbf{1}, \mathbf{1}, \mathbf{0})$, $(\mathbf{1}, \mathbf{0}, \mathbf{1})$, and $(\mathbf{0}, \mathbf{1}, \mathbf{1})$

$\rightarrow (\mathbf{2}, -\mathbf{4}, \mathbf{1}) = -\mathbf{1.5}\,(\mathbf{1}, \mathbf{1}, \mathbf{0}) + \mathbf{3.5}\,(\mathbf{1}, \mathbf{0}, \mathbf{1}) - \mathbf{2.5}\,(\mathbf{0}, \mathbf{1}, \mathbf{1})$

Sub.

$\rightarrow \boldsymbol{L}(\mathbf{2}, -\mathbf{4}, \mathbf{1}) = -\mathbf{1.5}\,\boldsymbol{L}(\mathbf{1}, \mathbf{1}, \mathbf{0}) + \mathbf{3.5}\,\boldsymbol{L}(\mathbf{1}, \mathbf{0}, \mathbf{1}) - \mathbf{2.5}\,\boldsymbol{L}(\mathbf{0}, \mathbf{1}, \mathbf{1})$

$\rightarrow \boldsymbol{L}(\mathbf{2}, -\mathbf{4}, \mathbf{1}) = -\mathbf{1.5}\,(\mathbf{2}, \mathbf{1}) + \mathbf{3.5}\,(\mathbf{1}, -\mathbf{1}) - \mathbf{2.5}\,(\mathbf{0}, \mathbf{0})$

$\rightarrow \boldsymbol{L}(\mathbf{2}, -\mathbf{4}, \mathbf{1}) = (\mathbf{0.5}, -\mathbf{5})$

Ex: Let $L: P_2 \to P_2$ be a linear transformation for which

$$L(x+1) = x \quad , \quad L(x-1) = 1 \quad , \quad L(x^2) = 0$$

Non standard Basis

Find $L(-x^2 + 3x + 2)$

First, write $(-x^2 + 3x + 2)$ as a linear combination of $(x+1)$, $(x-1)$, and (x^2)

Test $(2,3,-1)$ $(1,1,0)$ $(-1,1,0)$ $(0,0,1)$

$\rightarrow (-x^2 + 3x + 2) = \alpha\,(x+1) + \beta\,(x-1) + \gamma\,(x^2)$

$\rightarrow (-x^2 + 3x + 2) = [\alpha - \beta] + [\alpha + \beta]\,x + [\gamma]\,x^2$

$\rightarrow \alpha - \beta = 2$

$\rightarrow \alpha + \beta = 3$

$\rightarrow$ Solve

$2\alpha = 5 \rightarrow \alpha = 2.5$

$\beta = 3 - 2.5 = 0.5$

$\rightarrow \gamma = -1$

OR

Test

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Ex:(Continued) Let $L: P_2 \to P_2$ be a linear transformation for which

$$L(x+1) = x \quad , \quad L(x-1) = 1 \quad , \quad L(x^2) = 0$$

Find $\quad L(-x^2 + 3x + 2)$

First, write $(-x^2 + 3x + 2)$ as a linear combination of $(x+1)$, $(x-1)$, and (x^2)

→ $(-x^2 + 3x + 2) = 2.5\,(x+1) + 0.5\,(x-1) - 1\,(x^2)$

Sub,

→ $L(-x^2 + 3x + 2) = 2.5\; L(x+1) + 0.5\; L(x-1) - 1\; L(x^2)$

→ $L(-x^2 + 3x + 2) = 2.5\;(x) + 0.5\;(1) - 1\;(0)$

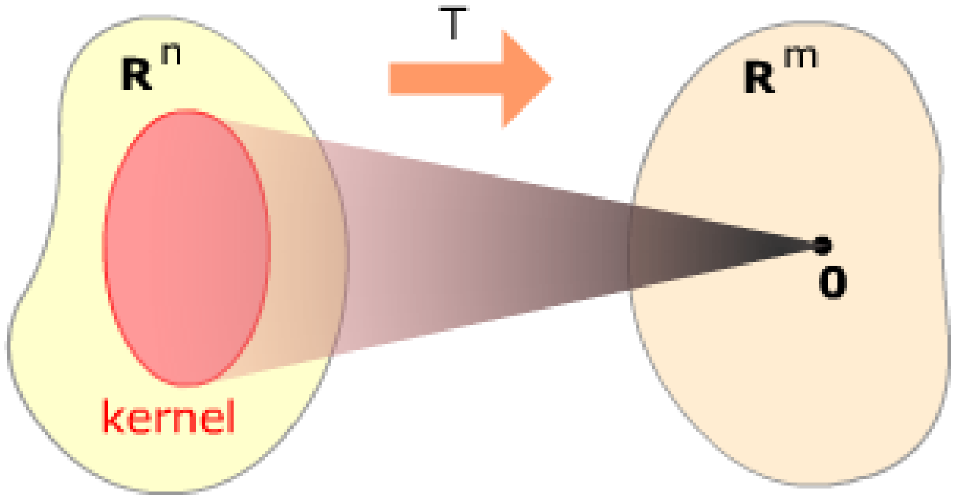
→ $L(-x^2 + 3x + 2) = 2.5\,x + 0.5$

4. Kernal of the transformation (Null space)

Kernal of the transformation

It the set of all vectors that map into the zero vector upon transformation.

$$\boxed{\ker(T) = \{x; T(x) = 0\}}$$



Given $A_{m\times n}$ the matrix of transformation T

$\ker(T)$ is the solution set of the homogeneous system $\underset{m\times n}{A}\ \underset{n\times 1}{x} = 0$ → $x \in R^n$

→ Also called "**Nullspace**" of A

Ex: Find the kernal of the transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + z \end{bmatrix}$$

fn. of transf.

$\ker(T)$ is the set of all vectors x where $T(x) = 0$

$\rightarrow \begin{bmatrix} x - y \\ x + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow x - y = 0$

$\rightarrow x + z = 0$

$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$A\ X = 0$

aug(A)

The matrix of transformation

$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \begin{array}{c} \begin{array}{cccc} \quad & y & z & \quad \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$

Let $z = t$

E2: $y + t = 0 \rightarrow y = -t$

E1: $x - (-t) = 0 \rightarrow x = -t$

one free var.

$\ker(T) = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$; $t \in R$

$\mathrm{Dim}(NS(A)) = 1$

$=$ no. of free vars.

Null space : $NS(A)$

- Thm : (Solutions of a homogeneous system)

  If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the nullspace of A.

- Notes: The nullspace of A is also called the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

- Nullity:

  The dimension of the nullspace of A is called the nullity of A.

$$\text{nullity}(A) = \dim(NS(A))$$

- Ex: (Finding the solution space of a homogeneous system)

Find the nullspace of the matrix A.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}$$

Rank = 2 → Nullity = 4 − 2 = 2

no. of cols

3x4

∈ R^4

Sol: The nullspace of A is the solution space of $A\mathbf{x}=\mathbf{0}$.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}\begin{matrix}0\\0\\0\end{matrix}\xrightarrow{G.J.E}\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix}0\\0\\0\end{matrix}$$

Let $x_2=s,\ x_4=t$

$E_2:\ x_3+t=0 \rightarrow x_3=-t$

$E_1:\ x_1+2s+3t=0$

$x_1=-2s-3t$

$$\Rightarrow x_1=-2s-3t,\ x_2=s,\ x_3=-t,\ x_4=t$$

$$\Rightarrow \mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2s-3t\\ s\\ -t\\ t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}=s\mathbf{v}_1+t\mathbf{v}_2$$

$$\Rightarrow NS(A)=\{s\mathbf{v}_1+t\mathbf{v}_2 \mid s,t\in R\}$$

Basis to $NS(A)=\{(-2,1,0,0),(-3,0,-1,1)\}$; Nullity(A) = 2

- Thm : (Dimension of the solution space)

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is

$$n = \text{rank}(A) + \text{nullity}(A)$$

- Notes:

(1) rank(A): The number of leading variables in the solution of $A\mathbf{x}=\mathbf{0}$.

(The number of nonzero rows in the row-echelon form of A)

(2) nullity (A): The number of free variables in the solution of $A\mathbf{x} = \mathbf{0}$.

- Ex: (Rank and nullity of a matrix)

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$$

4×5

$T: A x$

$R^5 \to R^4$

(a) Find the rank and nullity of A.

(b) Find nullspace of A.

rank = 3
Nullity = 5 - 3 = 2

Sol: Let B be the reduced row-echelon form of A.

$$A=\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{G.J.} B=\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) rank(A) = 3 (the number of nonzero rows in B)

$$\text{nuillity}(A) = n - \text{rank}(A) = 5 - 3 = 2$$

$$B = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$$

(b) Find nullspace of A.

Let $x_3 = s$, $x_5 = t$

E_3: $x_4 - t = 0 \rightarrow x_4 = t$

E_2: $x_2 + 3s - 4t = 0$

$\rightarrow x_2 = -3s + 4t$

E_1: $x_1 - 2s + t = 0 \rightarrow x_1 = 2s - t$

$$\mathrm{Ker}(T) = NS(A) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - t \\ -3s + 4t \\ s \\ t \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

All vecs. on this plane transforms into zero vec. in cs-domain