# Linear Algebra

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# Outline

1. Rank.
2. Homogeneous linear system.
3. Inverse using Gauss-Jordan elimination.
4. LU decomposition.

# 0. Review

Definition “Augmented Matrix”

$$
\begin{aligned}
x - 4\,y + 3\,z &= 5 \\
-\,x + 3\,y \;-\; z &= -3 \\
2\,x \qquad\quad - 4\,z &= 6
\end{aligned}
$$

**Matrix form**

$$
\boxed{\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}}
$$

$$
A\,X = B
$$

$$
\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]
$$

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**Augmented Matrix**

Augmented Matrix ≡ Aug $A$ = [A|B]

$1^{st}$ non-zero entry in each row

## The row Echelon form (RE)

- Leading entries (pivots) move to the right.
- Elements below leading elements $= 0$.
- Leading entries $= 1$.
- Zero rows at the bottom.

$$\begin{array}{cccc} x & y & z & c \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$E_3$: $z=-2$

$E_2$: $y=3$

$E_1 = x+6+2=4$

$\Rightarrow x=-4$

## The reduced row Echelon form (RRE)

- Leading entries (pivots) move to the right.
- Elements above and below leading elements $= 0$.
- Leading entries $= 1$.
- Zero rows at the bottom.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \end{array} \begin{array}{l} x \\ y \\ z \end{array}$$

Revisit "Elementary row operations"

## Elementary row operations

- Interchange two rows.
- Multiply a row by a non-zero constant.
- Add multiple of a row to another (and replace any of them).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

# Revisit "Elementary row operations"

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & -1\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix} \xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix}1 & 2 & 3\\0 & 1 & -1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

no pivot

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 0 & 0 & 0\\0 & 1 & 1 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 1 & 4\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 1 & 4\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 2 & 1 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-2\,R_2 + R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 0 & -3 & 4\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{\frac{1}{-3}R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 2 & 0\\0 & 0 & 1 & -4/3\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 3 & 0\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{\frac{1}{3}R_3 \to R_3} \begin{bmatrix}1 & 3 & 0 & 9\\0 & 1 & 3 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

# Gauss elimination method (case of non-homogeneous system)

**Example:**

$$
\begin{aligned}
0x_1 + 1\,x_2 + 1\,x_3 - 2\,x_4 &= -3 \\
x_1 + 2\,x_2 - x_3 &= 2 \\
2\,x_1 + 4\,x_2 + x_3 - 3\,x_4 &= -2 \\
x_1 - 4\,x_2 - 7\,x_3 - x_4 &= -19
\end{aligned}
$$

(Unique soln.)

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array} \\
\left[\begin{array}{cccc|c}
0 & 1 & 1 & -2 & -3 \\
1 & 2 & -1 & 0 & 2 \\
2 & 4 & 1 & -3 & -2 \\
1 & -4 & -7 & -1 & -19
\end{array}\right]
\end{array}
\xrightarrow{R_1 \leftrightarrow R_2}
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
2 & 4 & 1 & -3 & -2 \\
1 & -4 & -7 & -1 & -19
\end{array}\right]
\xrightarrow[-R_1 + R_4 \,\to\, R_4]{-2R_1 + R_3 \,\to\, R_3}
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 3 & -3 & -6 \\
0 & -6 & -6 & -1 & -21
\end{array}\right]
$$

$$
\xrightarrow{6\,R_2 + R_4 \,\to\, R_4}
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 3 & -3 & -6 \\
0 & 0 & 0 & -13 & -39
\end{array}\right]
\xrightarrow{\frac{1}{3}R_3 \,\leftrightarrow\, R_3}
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & -13 & -39
\end{array}\right]
\xrightarrow{\frac{1}{-13}R_4 \,\leftrightarrow\, R_4}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array} \\
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & 1 & 3
\end{array}\right]
\end{array}
$$

$E_4:\ \boxed{x_4 = 3}$

$E_3:\ x_3 - 3 = -2 \rightarrow \boxed{x_3 = 1}$

$E_2:\ x_2 + 1 - 6 = -3 \rightarrow \boxed{x_2 = 2}$

$E_1:\ x_1 + 4 - 1 = 2 \rightarrow \boxed{x_1 = -1}$

# Gauss elimination method (case of non-homogeneous system)

**Example:**

$$
\begin{aligned}
x_1 \quad -2\,x_2 \quad +3\,x_3 \quad +2\,x_4 \quad +x_5 \quad &= 10\\
2\,x_1 \quad -4\,x_2 \quad +8\,x_3 \quad +3\,x_4 \quad +10\,x_5 \quad &= 7\\
3\,x_1 \quad -6\,x_2 \quad +10\,x_3 \quad +6\,x_4 \quad +5\,x_5 \quad &= 27
\end{aligned}
$$

Let $\boxed{x_2 = s}$, $\boxed{x_5 = t}$

$E_3$: $x_4 - 4t = 7 \rightarrow \boxed{x_4 = 7+4t}$

$E_2$: $x_3 + 2t = -3 \rightarrow \boxed{x_3 = -3-2t}$

$E_1$: $x_1 - 2s - 9 - 6t + 14 + 8t + t = 10$

$\rightarrow \boxed{x_1 = 5 - 3t + 2s}$

$$
\left[\begin{array}{ccccc|c}
1 & -2 & 3 & 2 & 1 & 10\\
2 & -4 & 8 & 3 & 10 & 7\\
3 & -6 & 10 & 6 & 5 & 27
\end{array}\right]
$$

augA

$$
\sim
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & C\\
\left[\begin{array}{ccccc|c}
1 & -2 & 3 & 2 & 1 & 10\\
0 & 0 & 1 & 0 & 2 & -3\\
0 & 0 & 0 & 1 & -4 & 7
\end{array}\right]
\end{array}
$$

Echelon

Inf. no. of solutions

$$
\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}
=
\begin{bmatrix} 5-3t+2s\\ s\\ -3-2t\\ 7+4t\\ t \end{bmatrix}
=
\begin{bmatrix} 5\\ 0\\ -3\\ 7\\ 0 \end{bmatrix}
+ t \begin{bmatrix} -3\\ 0\\ -2\\ 4\\ 1 \end{bmatrix}
+ s \begin{bmatrix} 2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix}
$$

vector form

---

$$
\begin{aligned}
3\,x_1 \quad +x_2 \quad -3\,x_3 &= -4\\
x_1 \quad +x_2 \quad +x_3 &= 1\\
5\,x_1 \quad +6\,x_2 + 8\,x_3 &= 8
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
3 & 1 & -3 & -4\\
1 & 1 & 1 & 1\\
5 & 6 & 8 & 8
\end{array}\right]
$$

augA

$$
\sim
\begin{array}{cccc}
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 1\\
0 & 1 & 3 & 3\\
0 & 0 & 0 & -1
\end{array}\right]\\
x_1 \quad x_2 \quad x_3 \quad c
\end{array}
$$

Echelon

$E_3$: $0 = -1$

Contradiction

"No Soln."

1. Rank
2. Homogeneous system of linear equations

# Gauss elimination method (case of non-homogeneous system)

| Unique solution | Infinite number of solutions | No solution |
|---|---|---|
| $\left[\begin{array}{cccc\|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$ | $\left[\begin{array}{ccccc\|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$ | $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$ |

A · aug A

أي رقم
صفر [illegible]

| | Unique solution | Infinite number of solutions | No solution |
|---|---|---|---|
| no. of pivots in A | $\rho_A \Rightarrow 4$ = | Rank of matrix $\Rightarrow 3$ = | $\rho_A$ 2 ≠ |
| no. of pivots in aug A | $\rho_{augA} \Rightarrow 4$ = | $\Rightarrow 3$ < | $\rho_{augA}$ 3 |
| no. of vars | $n$ 4 | $< n$ 5 | Don't Care |

Consistent

Inconsistent

**Definition: The rank of a matrix $A \rightarrow \rho(A)$**

- The size of the largest non-zero sub-determinant.
- The number of pivots (non-zero rows) in the row echelon form of matrix $A$

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# Gauss elimination method (case of non-homogeneous system)

Solution cases
(Non-homogeneous)

Unique solution
$\boldsymbol{\rho(A) = \rho(aug\ A) = n}$

no. of vars.

Infinite number of solutions
$\boldsymbol{\rho(A) = \rho(aug\ A) < n}$

No solution
$\boldsymbol{\rho(A) \neq \rho(aug\ A)}$

Consistent

Inconsistent

# Gauss elimination method (case of homogeneous system)

$C_1 X + C_2 y = 0$

Unique solution

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

0 → Don't care zero

Unique solution

$\rho(A) = \rho(aug\ A) = n$ ✓

$E_4: X_4 = 0$

$E_3: X_3 = 0$

$X_2 = 0$

$x_1 = 0$ What is the solution?

Trivial Zero sol.

Infinite number of solutions

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right]$$

Infinite number of solutions

$\rho(A) = \rho(aug\ A) < n$ ✓

$X_2 = s,\ X_5 = t$

$A X = 0$

$E_3: X_4 - 4t = 0 \rightarrow X_4 = 4t$

$E_2: X_3 + 2t = 0 \rightarrow X_3 = -2t$

$E_1: X_1 - 2s - 6t + 8t + t = 0 \rightarrow X_1 = -t$

No solution

$x = y = 0$ Trivial sol.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0 = \cancel{3}$ / zero

No solution

$\rho(A) \neq \rho(aug\ A)$

**Impossible to have no solution**

[Q] In homog. Sys.

→ Find $\underline{k}$ to have non-zero sol.

∞ Inf. sol.

# Gauss elimination method (case of homogeneous system)

Solution cases
(Homogeneous)

Unique solution
(The zero solution)
$\boldsymbol{\rho(A) = \rho(aug\ A) = n}$

Infinite number of solutions
$\boldsymbol{\rho(A) = \rho(aug\ A) < n}$

including the trivial zero soln.

$$\underbrace{\hspace{10em}}_{\text{Always consistent}}$$

# Gauss elimination method (case of homogeneous system)

**Example:**

$$\begin{aligned} x_1 \quad - x_2 \quad +3\,x_3 &= 0 \\ 2\,x_1 \quad + x_2 \quad +3\,x_3 &= 0 \end{aligned}$$

plane 1

plane 2

intersect in a St. line

Homog. → Always Consist. → 2 eqs. 3 vars. → Inf. sols.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \underset{-2R_1+R_2 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \underset{\frac{1}{3}R_2 \leftrightarrow R_2}{\sim} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array}$$

2 actual eqs.

Let $\boxed{x_3 = t}$

$E_2: x_2 - t = 0 \rightarrow \boxed{x_2 = t}$, $E_1: x_1 - t + 3t = 0 \rightarrow \boxed{x_1 = -2t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

[Q] 3 homog. eqs. in 3 vars

→ sols. nature $\begin{cases} \text{unique if actual eqs. are 3} \\ \text{inf. if redundant eqs. exist} \end{cases}$

# More examples

**Example** Find all values of $k$ for which the following system of equations has:
(a) no solution, (b) a unique solution, (c) infinitely many solutions

$$\begin{aligned} x \quad +2y \quad +z \quad &= 3 \\ 2x \quad -y \quad -3z \quad &= 5 \\ 4x \quad +3y \quad -z \quad &= k \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & -3 & 5 \\ 4 & 3 & -1 & k \end{array}\right] \xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -1 \\ 0 & -5 & -5 & k-12 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2 \to R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1/5 \\ 0 & -5 & -5 & k-12 \end{array}\right] \xrightarrow{5R_2+R_3 \to R_3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & 0 & k-11 \end{array}\right]$$

$\boxed{\text{Do not}}$ eliminat the unknown consts, in last row

$\boxed{\text{Do}}$ not divide by $k-11$

No Soln

$\rho(A) \neq \rho(\text{aug } A)$

$\rho(A) = 2$, $\rho(\text{aug } A)$ needs to be 3

$k-11 \neq 0$

$k \in \mathbb{R} - \{11\}$

$\boxed{k \neq 11}$

Unique Soln.

$\rho(A) = \rho(\text{aug } A) = n$

$2 \neq 3$

Impossible case

No value for $k$

Inf. no. of Solns

$\rho(A) = \rho(\text{aug } A) < n$

$2$, needs to be 2, $3$

$k-11=0$

$\boxed{k=11}$

**Example** Find all values of $\alpha$ for which the following system of equations has:

(a) no solution, (b) a unique solution, (c) infinitely many solutions

$$
\begin{aligned}
x &+ y &- z &= 2\\
x &+ 2y &+ z &= 3\\
x &+ y &+ (\alpha^2-5)\,z &= \alpha
\end{aligned}
$$

Do not divide by any unknown

last row $\rightarrow [\,0\ \ 0\ \ \square \mid \triangle\,]$ — Don't Care

Pivot

$-R_1 + R_2 \rightarrow R_2$

$-R_1 + R_3 \rightarrow R_3$

$$
\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & \alpha^2-5 & \alpha \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \alpha^2-4 & \alpha-2 \end{array}\right]
$$

| | $\alpha^2-4$ | $\alpha-2$ |
|---|---|---|
| unique | | |
| No soln. | Zero | |
| unique | | Zero |
| Inf. | Zero | Zero |

Don't Care

**No soln**

$\rho(A) \neq \rho(\text{aug } A)$

$\alpha^2-4=0$ & $\alpha-2\neq 0$

$\alpha=\pm 2$ & $\alpha\neq 2$

$\boxed{\alpha=-2}$

**Unique soln**

$\rho(A) = \rho(\text{aug } A) = n$

$n = 3$

$\alpha^2-4\neq 0$

$\boxed{\alpha\neq\pm 2}$

$\alpha \in \mathbb{R} - \{\pm 2\}$

**Inf. no. of solns**

$\rho(A) = \rho(\text{aug } A) < n$

$n = 3$

$\alpha^2-4=0$ & $\alpha-2=0$

$\alpha=\pm 2$ & $\alpha=2$

$\boxed{\alpha=2}$

# 3. Inverse using Gauss-Jordan elimination

Definition: The inverse of an $n \times n$ matrix **A** is an $n \times n$ matrix **B** having the property that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$$

$\boldsymbol{B}$ is called the *inverse* of $\boldsymbol{A}$ and is usually denoted by $\boldsymbol{A}^{-1}$. Not Power

If a square matrix has an inverse, it is said to be *invertible* or *nonsingular* $(|\boldsymbol{A}| \neq \boldsymbol{0})$.

If it doesn't possess an inverse, it is said to be *singular*. $(|A| = 0)$

Example: The inverse of $\overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$ is $\overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$ because

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}\overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}\overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

# Calculating the inverse of a matrix

**<u>Inverse of a $2 \times 2$ matrix</u>**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

---

$$\underset{A}{\text{Matrix}} \longrightarrow \boxed{\text{Determinant}} \longrightarrow \underset{|A|}{\text{Scalar}}$$

**<u>Determinant of a $2 \times 2$ matrix</u>**

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

# Calculating the inverse of a matrix

**Inverse of a $2 \times 2$ matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Example:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \frac{1}{1*4-2*3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Calculating the inverse of a matrix

**Inverse of a $n \times n$ matrix**

**Using Gauss-Jordan method**

**This lecture**

**Using Adjoint matrix**

**In lecture 01**

# Calculating the inverse of a matrix

## Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan

- To find the inverse to an nxn matrix A:
  - Adjoin the identity matrix I to the right side of A, thereby producing a matrix of the form

$$\left[ A \mid I \right]$$

  - Apply row operations to this matrix until the left side is reduced to I. If successful, these operations will convert the right side to $A^{-1}$ ,so that the final matrix will have the form

G.J.

$$\left[ I \mid A^{-1} \right]$$

A is a square matrix + $|A| \neq 0$

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

$$\begin{bmatrix} 1 & \square & \square \\ 0 & \square & \square \\ 0 & \square & \square \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \square \\ 0 & 1 & \square \\ 0 & 0 & \square \end{bmatrix}$$

$$I \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Calculating the inverse of a matrix

## Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan

- **Example** :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

# Calculating the inverse of a matrix

## <u>Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan</u>

- **Example**: Do row operations to get upper triangular form: (Like Gaussian Elimination)

Forward

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

# Calculating the inverse of a matrix

**<u>Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan</u>**

- **Example**: Continue doing row operations to get 0's in columns **above** the pivots:

Backward

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\underbrace{\quad}_{I} \qquad \underbrace{\quad}_{A^{-1}}$$

# Calculating the inverse of a matrix

**Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan**

- **Example**: At this point the last matrix on the left is the Identity. Thus, the right matrix must be the inverse to A:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \left| \underbrace{\begin{matrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{matrix}}_{A^{-1}} \right]$$

# Calculating the inverse of a matrix

## Inverse of a $3 \times 3$ matrix (method 2) using Gauss-Jordan

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-R_1 + R_2 \to R_2 \\ 6R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \underset{\substack{R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$-R_3 \to R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \underset{\substack{R_3 + R_2 \to R_2 \\ R_3 + R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$I$

$$A^{-1} = \boxed{\begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}}$$

$A^{-1}$ or not

row 1 & 2 multiple $\Rightarrow |A| = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \underset{-2R_1 + R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Not $I$

# Application of inverse

**Solving $n$ equations in $n$ unknowns**

$$2x + 3y + z = -1$$
$$3x + 3y + z = 1$$
$$2x + 4y + z = -2$$

only used when

→ A square

→ $|A| \neq 0$

**Matrix form**

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Coefficient matrix

Unknown

Constant

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

# Application of inverse

**Solving $n$ equations in $n$ unknowns**

$$2x + 3y + z = -1$$
$$3x + 3y + z = 1$$
$$2x + 4y + z = -2$$

**Matrix form**

using adj.
~ G.J.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

inverse

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = A^{-1}B = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} -1\times-1+1\times1+0\times-2 \\ 1+0-2 \\ -6-2+6 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}}_{3\times 1}$$

$$\begin{bmatrix} \text{Same } A^{-1} \end{bmatrix} \begin{bmatrix} \text{Diff const} \end{bmatrix} = \text{new sols.}$$

**Only used when the system is non-homogeneous and in case of having unique solution.**

square

no. of eqs = no. of vars

Ex: Solve the system of linear equations

$$
\begin{aligned}
x_1 - x_2 \qquad\qquad &= 1 \\
x_1 \qquad\qquad - x_3 &= 2 \\
-6\,x_1 + 2\,x_2 + 3\,x_3 &= -1
\end{aligned}
$$

$$A\,X = B$$

$$A^{-1}A\,X = A^{-1}B$$

$$X = A^{-1}B$$

$$
\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}
\overset{\substack{-R_1 + R_2 \to R_2 \\ 6R_1 + R_3 \to R_3}}{\sim}
\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{bmatrix}
\overset{\substack{R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3}}{\sim}
\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{bmatrix}
$$

$$
\overset{-R_3 \to R_3}{\sim}
\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{bmatrix}
\overset{\substack{R_3 + R_2 \to R_2 \\ R_3 + R_1 \to R_1}}{\sim}
\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\;\;\underbrace{\begin{matrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{matrix}}_{A^{-1}}\right]
$$

Ex (cont.): Solve the system of linear equations

$$
\begin{aligned}
x_1 - x_2 \qquad\qquad &= 1 \\
x_1 \qquad\qquad - x_3 &= 2 \\
-6\,x_1 + 2\,x_2 + 3\,x_3 &= -1
\end{aligned}
$$

$$A\,X = B$$

$$A^{-1}A\,X = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = A^{-1}B = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x = ✓ \\ y = ✓ \\ z = ✓ \end{bmatrix}$$

# Matrix operations (Inverse)

**Properties:**

- $A^{-1}$ is unique. → For a matrix $A_{n\times n}$ either a single inverse exist or no inverse.
- $(A^{-1})^{-1} = A$
- $(A\,B)^{-1} = B^{-1}\,A^{-1}$ Similar to $(AB)^t = B^t A^t$
- $(A^t)^{-1} = (A^{-1})^t$
- $\left(A^k\right)^{-1} = (A^{-1})^k$
- (scalar or const) $(k\,A)^{-1} = \frac{1}{k}\,A^{-1}$ where $k \neq 0$ $\quad (\cancel{k}A)(\frac{1}{\cancel{k}}A^{-1}) = I$
  - **Example:** $(2\,A)^{-1} = \frac{1}{2}\,A^{-1}$
- $|A^{-1}| = \frac{1}{|A|}$ $\Rightarrow |AA^{-1}| = |I| \Rightarrow |A|\,|A^{-1}| = \cancel{|I|}\ 1$

If the **coefficient matrix** $\boldsymbol{A}$ is **square**

- $|A|$
  - $\neq 0$ → $A$ **Invertible** → Unique solution

    non-homog
    or
    homog. zero soln.
  - $= 0$

    row or col. is multiple or linear comb. of others
    - **Non-homogeneous**
      - → No solution
      - → Infinite number of solutions
    - **homogeneous** → Infinite number of solutions

Homog. + $A$ square
- $|A| \neq 0 \rightarrow$ unique zero soln.
- $|A| = 0 \rightarrow$ Inf. solns.

## 3 basic row operations

## In determinants

→ interchange 2 rows - - - - → $*(-1)$

→ multiply a row by non-zero constant - - → $\frac{1}{\text{non-zero con}}$

→ Add a multiple of a row to another → No effect

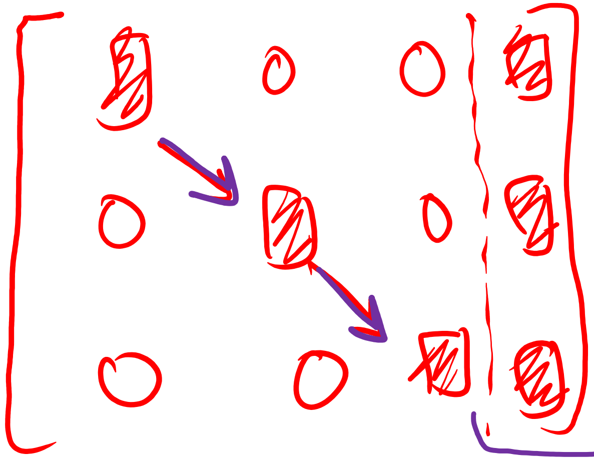
If $|A| = 0 \rightarrow |RE(A)| = 0$

$|A| \neq 0 \rightarrow \underbrace{|RE(A)|}_{\text{Triangular matrix}} \neq 0$

if $|A| = 0 \rightarrow$ last row(s) in Echelon $= 0$

$\Rightarrow |\text{Triang mat}| =$ prod. of main diag. elements

RE (aug A)



Full rank

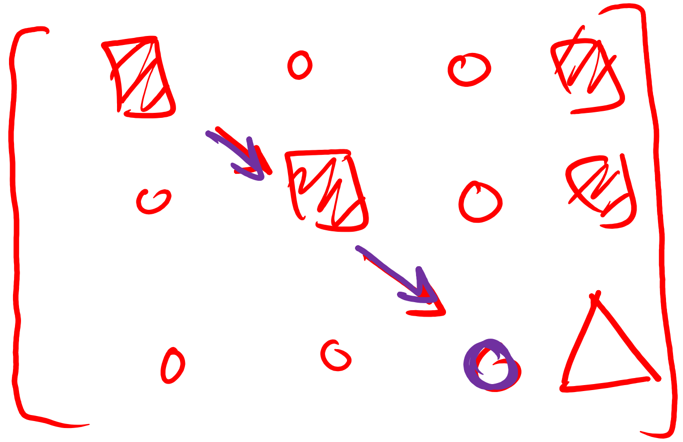
$\rightarrow$ Det $\neq 0$

unique sol.

$\rho(A) = 3 = \rho(\text{aug } A) = 3$ no. of vars.

$|A| \neq 0$

$|Ech(A)| \neq 0$



Det. $A = 0$

or Det. $(REA) = 0$

$\Delta$ ب

$= 0$ Inf.

$\neq 0$ no. sol.

# 5. LU decomposition

## Easy to solve system

Some linear system that can be easily solved

Diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$a_{11}x_1 = b_1 \rightarrow x_1 = \frac{b_1}{a_{11}}$

$x_2 = \frac{b_2}{a_{22}}$

The solution: $\begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$

# Easy to solve system (Cont.)

## Lower triangular matrix:

Forward Sub.

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$E_1: a_{11} x_1 = b_1$

$\rightarrow x_1 = \frac{b_1}{a_{11}}$

$E_2: a_{21}\left(\frac{b_1}{a_{11}}\right) + a_{22} x_2 = b_2$

$x_2 = \checkmark$

## Solution:

## This system is solved using forward substitution

$$x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij} . x_j\right) / a_{ii}$$

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# Easy to solve system (Cont.)

## Upper Triangular Matrix:

Echelon form

Backward sub.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$E_{n-1}$ $X_{n-1}$

$E_n : a_{nn} X_n = b_n$

**Solution:**

**This system is solved using Backward substitution**

$$x_i = (b_i - \sum_{j=i+1}^{n} a_{ij}.x_j)/a_{ii}$$

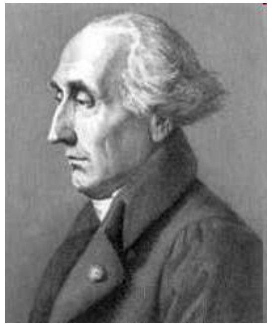
# LU Decomposition

Lagrange, in the very first paper in his collected works( 1759) derives the algorithm we call Gaussian elimination. Later Turing introduced the *LU* decomposition of a matrix in 1948 that is used to solve the system of linear equation.

Let $A$ be a $m \times m$ with nonsingular square matrix. Then there exists two matrices $L$ and $U$ such that, where $L$ is a lower triangular matrix and $U$ is an upper triangular matrix.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{mm} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix}$$

Where, $A = LU$



J-L *Lagrange*
(1736 –1813)

A. M. Turing
(1912-1954)

# LU Decomposition by Gaussian elimination

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

look like Echelon form

[*U*] is the same as the coefficient matrix at the end of the forward elimination step **[No row switching or scaling]**. only add multiple of a row to another

[*L*] is obtained using the ***multipliers*** that were used in the forward elimination process [$\mathbf{-1 \times multiplier}$]

# Ex: Find the LU factorization of $A$

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ \blacksquare & \blacksquare & 1 \end{bmatrix}$$

لو كان بدي 2 برضو 1 يعني لازم هو لا يمكن الضرب في ثابت

$$0\,R_1 + R_2 \rightarrow R_2$$

$$-2\,R_1 + R_3 \rightarrow R_3$$

$$4\,R_2 + R_3 \rightarrow R_3$$

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}} \; U$$

غير مسموح بالضرب في ثابت

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \blacksquare & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

**Remember**

**No row switching** or **scaling**

# Solving system of linear equation using LU decomposition

$$
\begin{aligned}
3x - 7y - 2z &= 7 \\
-3x + 5y + z &= -5 \\
6x - 4y \quad &= 2
\end{aligned}
$$

**Matrix form**

$$
\underbrace{\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X}
=
\underbrace{\begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix}}_{B}
$$

$$\boldsymbol{A\,X = B}$$

$$\boldsymbol{L\,U\,X = B}$$ Let $\boldsymbol{U\,X = Z}$

$\boldsymbol{L\,Z = B}$ → Get "$\boldsymbol{Z}$" **by Forward substitution**

$\boldsymbol{U\,X = Z}$ → Get "$\boldsymbol{X}$" **by Backward substitution**

**Example** Use the LU decomposition to solve

$$\begin{aligned} 3\,x - 7\,y - 2\,z &= 7 \\ -3\,x + 5\,y + \phantom{2}\,z &= -5 \\ 6\,x - 4\,y \phantom{{}-2\,z} &= 2 \end{aligned}$$

## 1st Factorize the coefficient matrix into $\boldsymbol{L}$ and $\boldsymbol{U}$

$1\,\boldsymbol{R_1} + \boldsymbol{R_2} \rightarrow \boldsymbol{R_2}$

$-2\,\boldsymbol{R_1} + \boldsymbol{R_3} \rightarrow \boldsymbol{R_3}$

$5\,\boldsymbol{R_2} + \boldsymbol{R_3} \rightarrow \boldsymbol{R_3}$

يبقى كما هو

لا داعي للعمود الـ 3rd

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & \blacksquare & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

**Example (Cont.)** Use the LU decomposition to solve

$\underbrace{A}X=B$

$\underbrace{L\underbrace{UX}_{Z}}=B$ $UX=Z$

$$\begin{aligned} 3x-7y-2z&=7\\ -3x+5y+z&=-5\\ 6x-4y&=2 \end{aligned}$$

$$L=\begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}$$

$$U=\begin{bmatrix}3&-7&-2\\0&-2&-1\\0&0&-1\end{bmatrix}$$

2nd Solve the system $\boldsymbol{L}\,\boldsymbol{Z}=\boldsymbol{B}$ using **forward sub.**

$$\begin{array}{ccc|c} z_1 & z_2 & z_3 & B \\ \hline 1 & 0 & 0 & 7 \\ -1 & 1 & 0 & -5 \\ 2 & -5 & 1 & 2 \end{array}$$

$L$

Forward sub.

$E_1:$ $\boxed{z_1=7}$

$E_2: -7+z_2=-5 \rightarrow \boxed{z_2=2}$

$E_3: 2(7)-5(2)+z_3=2 \rightarrow \boxed{z_3=-2}$

3rd Solve the system $\boldsymbol{U}\,\boldsymbol{X}=\boldsymbol{Z}$ using **backward sub.**

$$\begin{array}{ccc|c} x & y & z & \\ \hline 3 & -7 & -2 & 7 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & -2 \end{array}$$

$U$ $Z$

Backward sub.

$E_3: -z=-2 \rightarrow \boxed{z=2}$

$E_2: -2y-2=2 \rightarrow \boxed{y=-2}$

$E_1: 3x+14-4=7 \rightarrow \boxed{x=-1}$

# Determinant using LU decomposition

- For two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ → $|\boldsymbol{A}\,\boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}|$
- For diagonal matrices $\boldsymbol{D}$ , upper triangular matrices $\boldsymbol{U}$ , and lower triangular matrices $\boldsymbol{L}$

  → $|\boldsymbol{D}|$ , $|\boldsymbol{U}|$ , $|\boldsymbol{L}|$ = product of main diagonal elements

- Given $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ → $|\boldsymbol{A}| = |\boldsymbol{L}||\boldsymbol{U}|$ , where both $\boldsymbol{L}$ *and* $\boldsymbol{U}$ *are triangular matrices*

## In last Example

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$3 * -2 * -1$

$$|A| = \overset{1}{\cancel{|L|}}\,|U| = |U|$$

$$= 6$$