Linear Algebra

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Outline

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2. Motivational example.
3. Possible solutions to a system of linear equations.
4. Row Echelon form.
5. Matrix and vector forms.
6. Solution of a system of linear equations using Gauss elimination.
7. Gauss-Jordan elimination.

1. Introduction

What is linear algebra?

Branch of mathematics concerning **linear equations**.

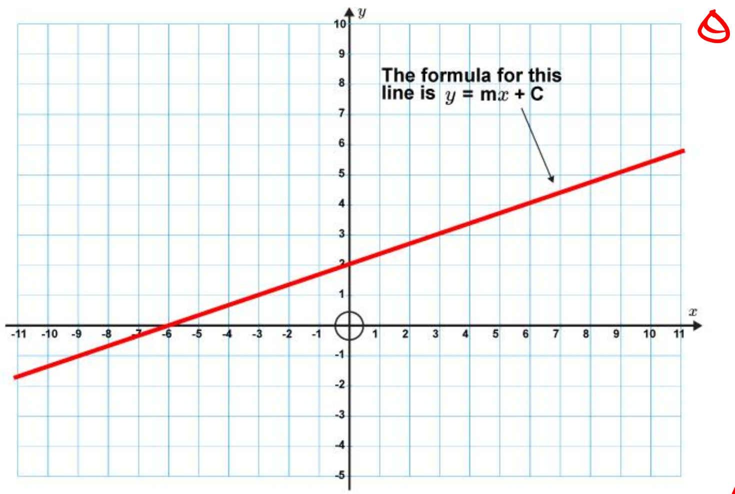
- Solution techniques/algorithms to solve system of linear equations.
- Important applications.

Complicated system $\xrightarrow{\textbf{Converted to}}$ Simpler linear system

- Could be of **hundreds** or **thousands** of linear equations
- Need **efficient algorithms** to solve them

What is linear equation?

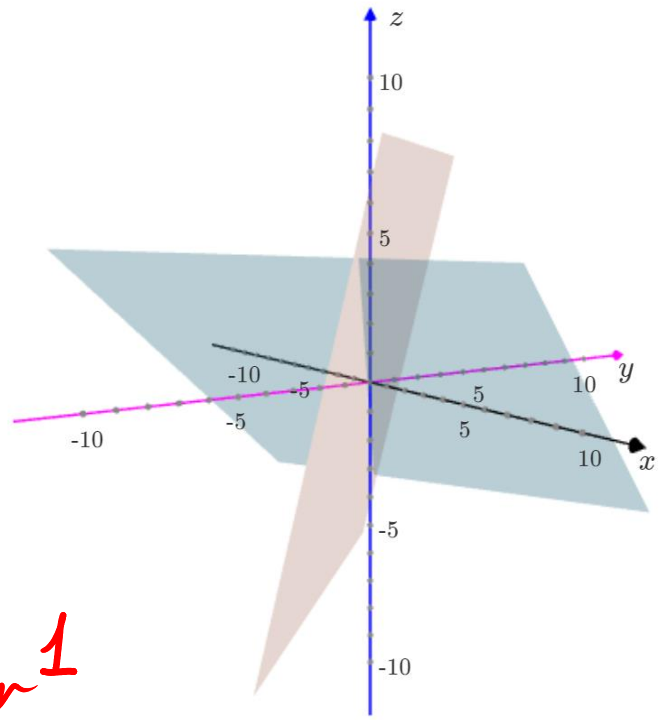
- A linear equation is an equation in which the **highest power** of the variable is **always 1**.
- It is also known as a **one-degree equation**.
- The graph of a linear equation always forms **a straight line, a plane**, etc. (the name **linear**)



$$y = mx + b$$

$$ax + by + cz = d$$

- a, b, c, and m are constants.
- x, y, and z are variables/unknowns.



Any term
+
−
=

Const.

Const. * Var^1

e^x, Sinx, xy, x^2 لا يوجد

Linear or not?

Any term $\begin{cases} \text{Const} \\ \text{Const} * \text{Var}^1 \end{cases}$

$xy + 5yz = 13$ Non-linear

$x_1 - 2\,x_2 + 10\,x_3 + x_4 = 0$ $\boxed{\text{Linear}}$

$x_1 + x_2^3/x_4 - x_3 x_4 x_5^2 = 0$ Non-linear

$\underbrace{\sin\left(\frac{\pi}{2}\right)}_{\text{Const.}} x_1 - 4\,x_2 = e^2$ $\boxed{\text{Linear}}$, if $\sin(x_1)$ → Non-linear

$\frac{1}{x} + \frac{1}{y} = 4$ Non-linear

$2\,e^x + x = 5$ Non-linear

Example: The non-linear equation is

ⓐ $x - 2y - z = 3$

ⓑ $x + y = 0$

ⓒ $2\sqrt{x} - y - z = 1$

ⓓ $x + z = 5$

what is a soln. of any eqn.

for ex. $x + y = 1$

$(1, 0)$
$(0, 1)$
$(0.5, 0.5)$
$\vdots$

All pts. (x, y) satisfying the eqn. (Contd.)

عدد لا نهائي من الحلول

What is a system of linear equations?

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Find (x, y, z) satisfying

Cond1 & $x + y + z = 6$

Cond2 & $2x - y - z = -3$

Cond3 $3y - 2z = 0$

at the same time

Our target is to find the values of the variables that satisfy all the equations at the same time

Motivational example: **Example: 4.16.** *Balance the following chemical equation:*

$$\mathbf{Cu + HNO_3 \longrightarrow Cu(NO_3)_2 + H_2O + NO}$$

Solution.

$$x_1\mathbf{Cu} + x_2\mathbf{HNO_3} \longrightarrow x_3\mathbf{Cu(NO_3)_2} + x_4\mathbf{H_2O} + x_5\mathbf{NO}$$

Cu: $x_1 = x_3$

H: $x_2 = 2x_4$

N: $x_2 = 2x_3 + x_5$

O: $3x_2 = 6x_3 + x_4 + x_5$

This leads to the following homogeneous system

$$x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 = 0$$

$$0x_1 + x_2 + 0x_3 - 2x_4 + 0x_5 = 0$$

$$0x_1 + x_2 - 2x_3 + 0x_4 + x_5 = 0$$

$$0x_1 + 3x_2 - 6x_3 - x_4 - x_5 = 0$$

Find $(x_1, x_2, x_3, x_4, x_5)$ satisfying all 4 eqs. at the same time

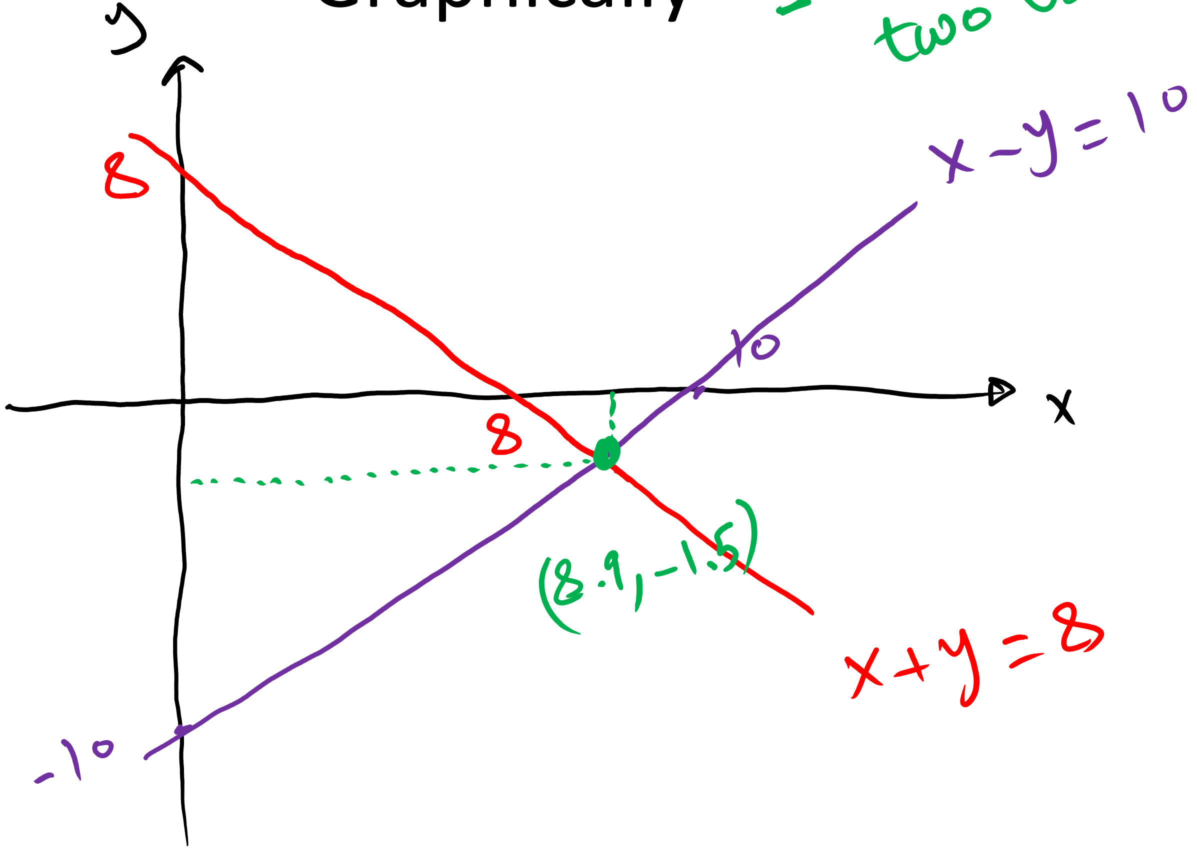
Motivational example: Solve the system of linear equations

$x + y = 8$

$x - y = 10$

Problems in Graphical soln
- Accuracy issue
- Limited to two vars.

Graphically



Analytically ✓

$$
\begin{aligned}
x + \cancel{y} &= 8 \\
x - \cancel{y} &= 10 \\
\hline
2x &= 18
\end{aligned}
$$

الحل

$\Rightarrow \boxed{x = 9}$

Sub. in any eqn.

$9 + y = 8 \Rightarrow \boxed{y = -1}$

very accurate

Real life problems

- Contains **hundreds/thousands** of simultaneous **linear equations**.
- The **need to study linear equations** (linear algebra).
- Study **efficient techniques/algorithms** to solve **large systems of linear equations**.
- The ability for these **algorithms to be programmed**.

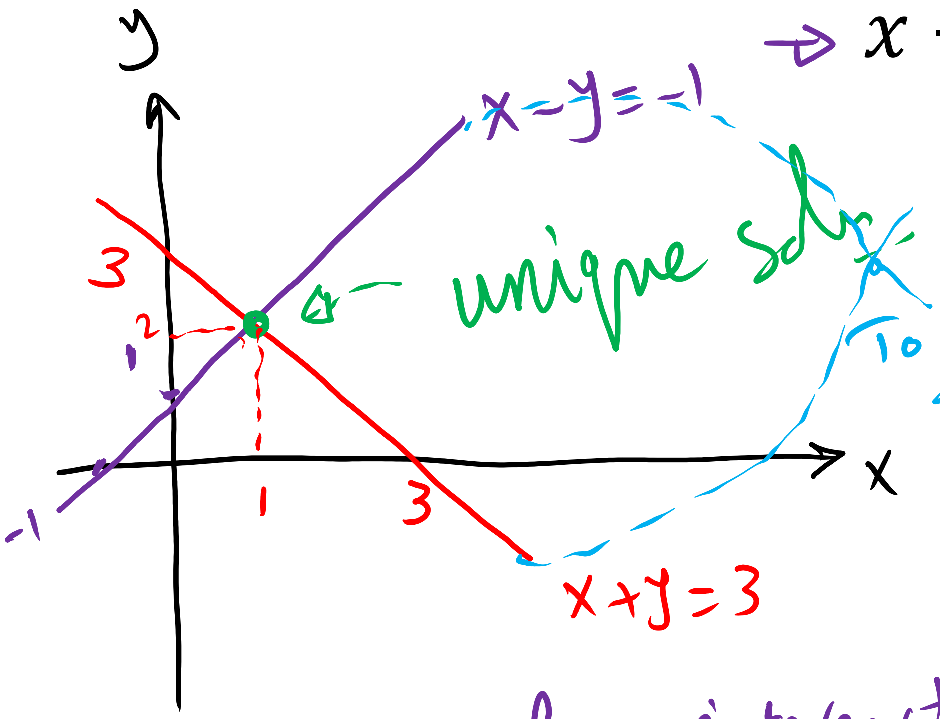
3. Possible solutions to a system of linear equations

System of two equations in two unknowns

Case 1: Solve

$$x + y = 3$$

$$x - y = -1$$



To meet again the eqs. would be non-linear

$$\begin{aligned} x + \cancel{y} &= 3 \\ x - \cancel{y} &= -1 \end{aligned}$$
بالجمع

$$2x = 2$$

$$\boxed{x = 1}$$

Sub. in any eqn.

$$1 + y = 3 \Rightarrow \boxed{y = 2}$$

Can two st. lines intersect in more than one pt.?

unique — inf. → Nothing in betn.

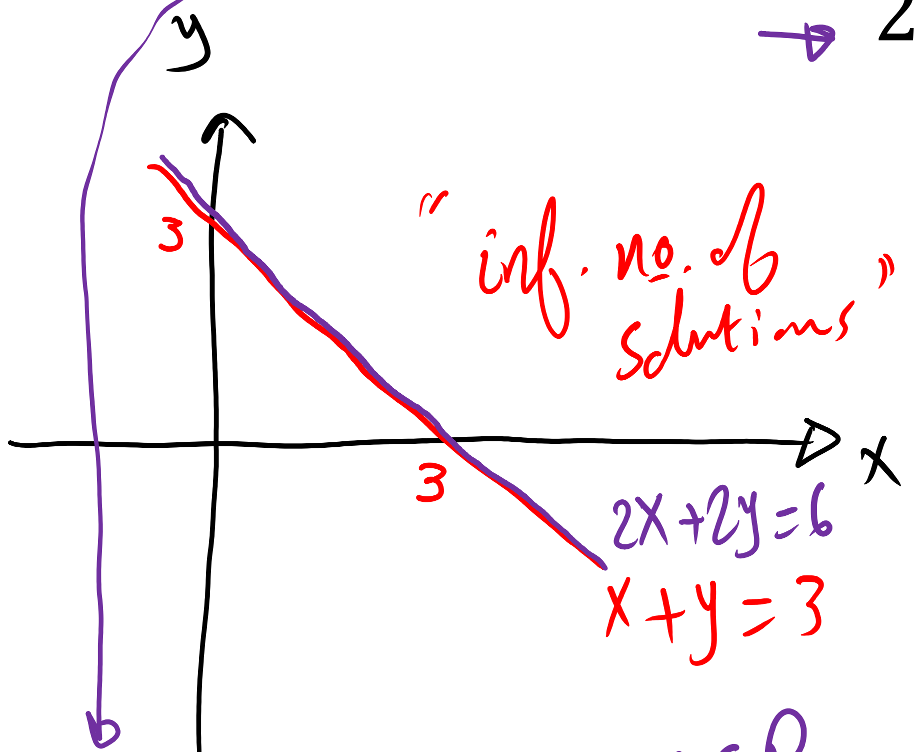
System of two equations in two unknowns

الأصل معادلة واحدة

Case 2: Solve

$x + y = 3$

$2x + 2y = 6$ “Redundant eqn.!” "معادلة مكررة"



$x + y = 3 \to$ let $y = t$, $x = 3 - t$, $t \in \mathbb{R}$

$(x,y) = (3-t, t)$

$$x + y = 3$$
$$2x + 2y = 6$$

$-2E_1 + E_2$

$$-2x - 2y = -6$$
$$2x + 2y = 6$$
$$0 = 0$$

الجمع

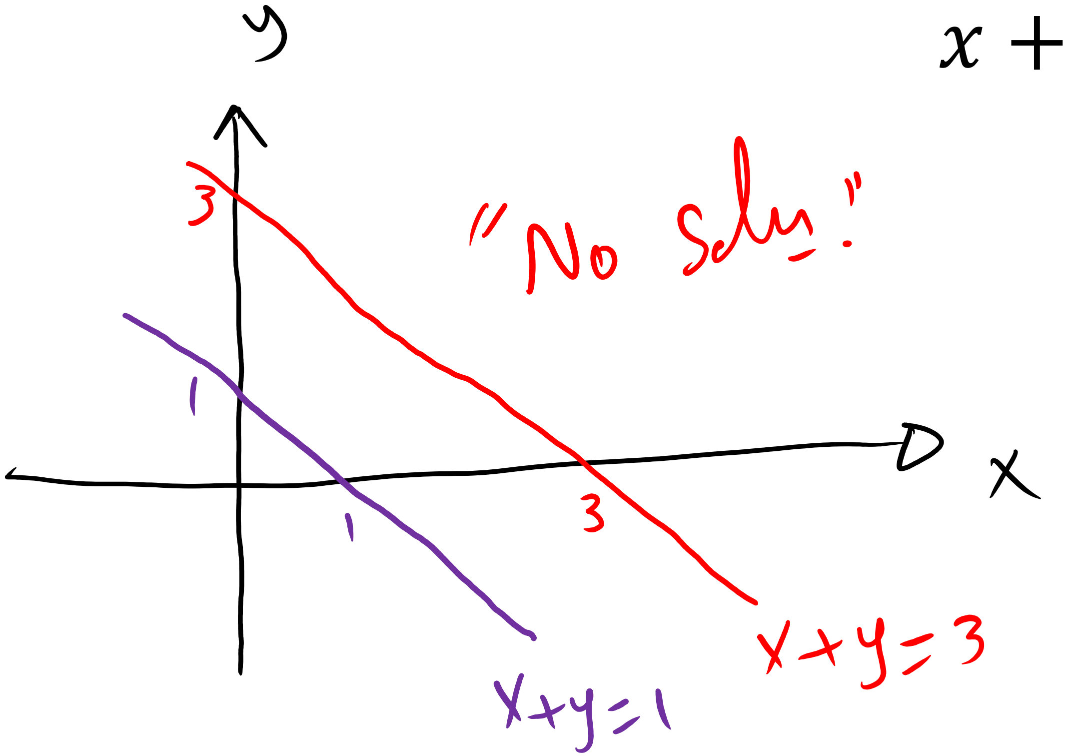
redundant eqn.

System of two equations in two unknowns

Case 3: Solve

$$x + y = 3$$

$$x + y = 1$$



$$\begin{array}{r} x + y = 3 \\ x + y = 1 \\ \hline 0 = 2 \end{array}$$

بالطرح

Contradiction

System of two equations in two unknowns

Solution cases

Unique solution ← Nothing in between → Infinite number of solutions

$\underbrace{\text{Unique solution} \qquad \text{Infinite number of solutions}}$

متوافق Consistent

No solution

↓

غير متوافقين Inconsistent

4. Row Echelon form

Idea of row Echelon form

Consider the system of equations

Is it easy to solve it? Why?

Row Echelon form (RE)

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Back substitution

E_3: $\boxed{z = 2}$

E_2: $y + 3(2) = 5 \rightarrow \boxed{y = -1}$

E_1: $x - 2(-1) + 3(2) = 9$

$x + 2 + 6 = 9 \rightarrow \boxed{x = 1}$

Any system of linear equation $\xrightarrow{\text{Convert to}}$ Echelon form

Two systems of linear equations are called equivalent if they have precisely the same solution set.

Mathematical operations that results in equivalent systems of equations

- Interchange two equations.
- Multiply any equation by non-zero constant.
- Add a multiple of an equation to another equation.

$$x + y = 3$$
$$x - y = -1$$
$$\downarrow$$
$$x - y = -1$$
$$x + y = 3$$

$$x + y = 3$$
$$\Big\downarrow \times 2$$
$$2x + 2y = 6$$

$$x + y = 4 \xrightarrow{\times 2} 2x + 2y = 8$$
$$x - 2y = -2 \longrightarrow x - 2y = -2$$

$$3x \qquad = 6$$

Replace any of the original equations

Replace eqn. 2 by the new eqn.

$$x + y = 4$$
$$3x \qquad = 6$$

Replace eq1 by the new eqn

$$3x \qquad = 6$$
$$x - 2y = -2$$

Example: Rewrite the following system in RE form

$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17$$

$E_1 + E_2 \rightarrow E_2$, $-2E_1 + E_3 \rightarrow E_3$

$$x - 2y + 3z = 9$$
$$0 + y + 3z = 5$$
$$0 - y + z = -1$$

unique Sol.

$E_2 + E_3 \rightarrow E_3$

$$x - 2y + 3z = 9$$
$$y + 3z = 5$$
$$+ 2z = 4$$

E_3: $z = 2$

E_2: $y + 3(2) = 5 \rightarrow y = -1$

E_1: $x - 2(-1) + 3(2) = 9$

$\rightarrow x = 1$

Example: Rewrite the following system in RE form

$$x - 3y + z = 1$$
$$2x - y - 2z = 2$$
$$x + 2y - 3z = -1$$

$-2E_1 + E_2 \to E_2$, $-E_1 + E_3 \to E_3$

$$x - 3y + z = 1$$
$$5y - 4z = 0$$
$$5y - 4z = -2$$

$-E_2 + E_3 \to E_3$

$$x - 3y + z = 1$$
$$5y - 4z = 0$$
$$\boxed{0 = -2}$$

Contradiction $\to$ No Soln.

Example: Rewrite the following system in RE form

$$\begin{aligned} y - z &= 0 \\ x \quad - 3z &= -1 \\ -x + 3y \quad &= 1 \end{aligned}$$

$R_1 \leftrightarrow R_2$

$$\begin{aligned} x \quad\quad\; - 3z &= -1 \\ y - z &= 0 \\ -x + 3y \quad &= 1 \end{aligned}$$

Notice the empty space

$E_1 + E_3 \to E_3$

$$\begin{aligned} x \quad - 3z &= -1 \\ y - z &= 0 \\ 3y - 3z &= 0 \end{aligned}$$

$-3E_2 + E_3$

$$\left\{\begin{aligned} x \quad - 3z &= -1 \\ y - z &= 0 \end{aligned}\right.$$

2 eqs. in 3 unkn

$\boxed{0 = 0}$ Redundant eq.

$E_2: y - z = 0$

Let $z = t \Rightarrow y = t$

$E_1: x - 3t = -1 \Rightarrow x = 3t - 1$ inf. no. of solns.

$(x, y, z) = (3t-1, t, t);\ t \in \mathbb{R}$

Example: determine whether each statement is true or false

1. A system of one linear equation in two variables is always consistent. True, Any single eqn. has inf. no. of solns.

2. A system of two linear equations in three variables is always consistent. False, two plane could be parallel

3. If a linear system is consistent, then it has an infinite number of solutions False, it could have a unique soln.

4. A system of linear equations can have exactly two solutions. False, either unique or inf.

5. Two systems of linear equations are equivalent if they have the same solution set. True

6. A system of three linear equations in two variables is always inconsistent False, 3 st. lines could be consis. or inconsis.

Example: The equation $3x + y = 2$is

ⓐ Consistent with one solution

ⓑ Inconsistent

ⓒ Consistent with infinite solution

Example: The system

$$3x + 2y = 1$$
$$6x + 4y = 1$$

is

$-2E_1 + E_2 \to E_2$

$3x + 2y = 1$

$0 = -1$

contrad.

ⓐ Consistent with one solution

ⓑ Inconsistent

ⓒ Consistent with infinite solution

Example: The system

OR $-\frac{5}{3}E_1 + E_2 \to E_2$ [fractions]

$-5E_1 + 3E_2 \to E_2$

$$3x + 2y = 1$$
$$5x + 3y = 0$$

$-15x - 10y = -5$

$15x + 9y = 0$

$E_1 + E_2$ $\quad -15x - 10y = -5$

$y = -5$

is

(a) Consistent with one solution

(b) Inconsistent

(c) Consistent with infinite solution

Requires one multip. + one addn. instead of two multip. + one addn.

Example: The system

$-2E_1 + E_2 \to E_2$

$\boxed{x + 2y = 4}$

$0 = 0$

$$x + 2y = 4$$
$$2x + 4y = 8$$

is

ⓐ Only one solution

ⓑ no solutions

ⓒ Infinite solutions ✓

ⓓ two solutions

5. Matrix and vector forms

Consider the system of equations

$$
\begin{aligned}
x - 4y + 3z &= 5 \\
-x + 3y - z &= -3 \\
2x \qquad - 4z &= 6
\end{aligned}
$$

Matrix form

$$
\begin{array}{ccc} x & y & z \end{array} \qquad \text{Free term}
$$

$$
\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}
$$

Coefficient matrix
A

Unknown
X

Constant
B

$$A X = B$$

Consider the system of equations

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad\;\; - 4z &= 6 \end{aligned}$$

Vector form

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} x + \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} y + \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} z = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$$

6. Solution of a system of linear equations using Gauss elimination

Consider the system of equations

$$
\begin{aligned}
x - 4\,y + 3\,z &= 5 \\
-\,x + 3\,y - z &= -3 \\
2\,x \qquad - 4\,z &= 6
\end{aligned}
$$

Matrix form

$$
\begin{array}{c}
\quad x \qquad y \qquad z \\
\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}
\end{array}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}
$$

Coefficient matrix A | Unknown X | Constant B

$$A_{m \times n} X = B$$

m = # Rows = # of equations

n = # Columns = # of variables

$B \neq 0$ — **Non- Homogeneous**

$B = 0$ — **Homogeneous**

Definition "Augmented Matrix"

$$\begin{aligned} x - 4\,y + 3\,z &= 5 \\ -\,x + 3\,y \;-\; z &= -3 \\ 2\,x \qquad\quad - 4\,z &= 6 \end{aligned}$$

Matrix form

$$\boxed{\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}}$$

$$A\,X = B$$

$$\begin{array}{cccc} x & y & z & c \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$$

Augmented Matrix

المصفوفة الموسعة

Augmented Matrix ≡ Aug A = [A|B]

$C_{3\times5}$

	no. of eqs.	no. of vars.
C coeft.	3	5
C aug.	3	4

Revisit "Row Echelon form (RE)"

First non-zero entry in each row

اول عنصر

The row Echelon form

- Leading entries (pivots) move to the right.
- Elements below leading elements = 0.
- Leading entries = 1.
- Zero rows at the bottom.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Not 1

Should be at the bottom

Should be zero

Pivot moved left

Elementary row operations

Any operation affects the whole row

- Interchange two rows.
- Multiply a row by a non-zero constant.
- Add multiple of a row to another (and replace any of them).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Revisit "Elementary row operations"

$$\begin{bmatrix}1&2&3\\0&1&-1\\0&0&2\\0&0&0\end{bmatrix} \xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix}1&2&3\\0&1&-1\\0&0&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&3&0&9\\0&0&0&0\\0&1&1&4\\0&0&0&1\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1&3&0&9\\0&1&1&4\\0&0&0&0\\0&0&0&1\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1&3&0&9\\0&1&1&4\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&3&0&9\\0&\boxed{1}&2&0\\0&2&1&4\\0&0&0&1\end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix}1&3&0&9\\0&1&2&0\\0&0&-3&4\\0&0&0&1\end{bmatrix} \xrightarrow{\frac{1}{-3}R_3 \to R_3} \begin{bmatrix}1&3&0&9\\0&1&2&0\\0&0&1&-4/3\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&3&0&9\\0&1&3&0\\0&0&0&1\\0&0&3&0\end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1&3&0&9\\0&1&3&0\\0&0&3&0\\0&0&0&1\end{bmatrix} \xrightarrow{\frac{1}{3}R_3 \to R_3} \begin{bmatrix}1&3&0&9\\0&1&3&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Gauss elimination method

- Form Augmented matrix.
- Augmented matrix → RE matrix (using elementary row operations) .
- Solve the system (using back substitution).

$$\begin{aligned} x - 4\,y + 3\,z &= 5 \\ -\,x + 3\,y - \,z &= -3 \\ 2\,x \qquad - 4\,z &= 6 \end{aligned}$$

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-4} & \mathbf{3} & \mathbf{5} \\ \mathbf{-1} & \mathbf{3} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{0} & \mathbf{-4} & \mathbf{6} \end{array}\right]$$

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix} \xrightarrow[-2\,R_1 + R_3 \;\to\; R_3]{R_1 + R_2 \;\to\; R_2} \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 8 & -10 & -4 \end{bmatrix} \xrightarrow{-R_2 \;\to\; R_2} \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 8 & -10 & -4 \end{bmatrix}$$

$$\xrightarrow{-8\,R_2 + R_3 \;\to\; R_3} \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & 12 \end{bmatrix} \xrightarrow{\frac{1}{6} R_3 \;\to\; R_3} \begin{array}{c} \begin{array}{cccc} x & y & z & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

$E_3: z = 2$

$E_2: y - 2(2) = -2 \to y = 2$

$E_1: x - 8 + 6 = 5 \to x = 7$

Gauss elimination method (case of non-homogeneous system)

Example:

$$\begin{aligned} x_2 + x_3 - 2\,x_4 &= -3 \\ x_1 + 2\,x_2 - x_3 \qquad &= 2 \\ 2\,x_1 + 4\,x_2 + x_3 - 3\,x_4 &= -2 \\ x_1 - 4\,x_2 - 7\,x_3 - x_4 &= -19 \end{aligned}$$

" unique sol. "

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \end{array} \underset{\text{equiv.}}{\overset{R_1 \leftrightarrow R_2}{\sim}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \overset{\substack{-2R_1 + R_3 \to R_3 \\ -R_1 + R_4 \to R_4}}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right] \overset{6\,R_2 + R_4 \to R_4}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

$$\overset{\frac{1}{3}R_3 \leftrightarrow R_3}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right] \overset{\frac{1}{-13}R_4 \leftrightarrow R_4}{\sim} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

$E_4:\ x_4 = 3$

$E_3:\ x_3 - 3 = -2 \to x_3 = 1$

$E_2:\ x_2 + 1 - 6 = -3 \to x_2 = 2$

$E_1:\ x_1 + 4 - 1 = 2 \to x_1 = -1$

Gauss elimination method (case of non-homogeneous system)

Example:

$$\begin{aligned} x_1 \quad -2x_2 \quad +3x_3 \quad +2x_4 \quad +x_5 &= 10 \\ 2x_1 \quad -4x_2 \quad +8x_3 \quad +3x_4 \quad +10x_5 &= 7 \\ 3x_1 \quad -6x_2 \quad +10x_3 \quad +6x_4 \quad +5x_5 &= 27 \end{aligned}$$

Let $x_2 = s$, $x_5 = t$; $s \,\&\, t \in \mathbb{R}$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$ Aug.

; row operations

$$\sim \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & c \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$ Echelon form

E_3: $x_4 - t = 7$

$\Rightarrow x_4 = 7 + t$

E_2: $x_3 + 2t = -3$

$\Rightarrow x_3 = -3 - 2t$

E_1: $x_1 - 2s + 3(-3-2t) + 2(7+t) + t = 10$

$\Rightarrow x_1 = 5 + 2s + 3t$

3 eqs. , 5 vars.

$(x_1, x_2, x_3, x_4, x_5) = (5+2s+3t, s, -3-2t, 7+t, t)$

inf. no. of solns.

$$\begin{aligned} 3x_1 \quad +x_2 \quad -3x_3 &= -4 \\ x_1 \quad +x_2 \quad +x_3 &= 1 \\ 5x_1 \quad +6x_2 \quad +8x_3 &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 8 & 8 \end{array}\right]$$ Aug.

; row operations

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$ Echelon

$0 = -1$

contrad.

"No soln."

7. Gauss-Jordan elimination

The row Echelon form (RE)

- Leading entries (pivots) move to the right.
- Elements below leading elements = 0.
- Leading entries = 1.
- Zero rows at the bottom.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The reduced row Echelon form (RRE)

- Leading entries (pivots) move to the right.
- Elements above and below leading elements = 0.
- Leading entries = 1.
- Zero rows at the bottom.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \to x = -4 \\ \to y = 3 \\ \to z = -2 \end{array}$$

Example Solve the system of linear equations

$$\begin{aligned} x \quad - 4y \quad + 3z &= 5 \\ -x \quad + 3y \quad - z &= -3 \\ 2x \quad\quad\quad - 4z &= 6 \end{aligned}$$

we can apply both Forward & backward elimination at the same time

Forward elimination

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix} \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \sim \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 8 & -10 & -4 \end{bmatrix} \xrightarrow{-R_2 \to R_2} \sim \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 8 & -10 & -4 \end{bmatrix}$$

$4R_2 + R_1 \to R_1$ & $-8R_2 + R_3 \to R_3$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & 12 \end{bmatrix} \xrightarrow{\frac{1}{6}R_3 \to R_3} \sim \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[-3R_3 + R_1 \to R_1]{2R_3 + R_2 \to R_2} \sim \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{4R_2 + R_1 \to R_1} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Backward elimination

$(x, y, z) = (7, 2, 2)$

Example Solve the system of linear equations

3 eqs., 5 unknowns

No soln. inf.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + x_4 &= 0\\
4x_1 + 5x_2 + 6x_3 + x_5 &= 0\\
7x_1 + 8x_2 + 9x_3 &= 1
\end{aligned}
$$

3 eqs. in 3 unknowns

unique inf. no soln.

$$
\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0\\ 4 & 5 & 6 & 0 & 1 & 0\\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right]
\underset{\substack{-4R_1+R_2 \to R_2\\ -7R_1+R_3 \to R_3}}{\sim}
\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0\\ 0 & -3 & -6 & -4 & 1 & 0\\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]
\underset{(-1/3)R_2 \to R_2}{\sim}
\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & 2 & 4/3 & -1/3 & 0\\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]
\underset{6R_2+R_3 \to R_3}{\sim}
$$

$$
\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & 2 & 4/3 & -1/3 & 0\\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]
\underset{\substack{-R_3+R_1 \to R_1\\ -(4/3)R_3+R_2 \to R_2}}{\sim}
\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 2 & -1\\ 0 & 1 & 2 & 0 & 7/3 & -4/3\\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]
\underset{-2R_2+R_1 \to R_1}{\sim}
$$

Pivots: 1st non-zero entry in each row

$$
\begin{array}{cccccc}
x_1 & x_2 & \boxed{x_3} & x_4 & \boxed{x_5} & c\\
\end{array}
$$
$$
\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -8/3 & 5/3\\ 0 & 1 & 2 & 0 & 7/3 & -4/3\\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]
$$

Let $x_3 = s$, $x_5 = t$

E_3: ✓

E_2: ✓

E_1: ✓

$x + 2y + 3z = 0, \quad x + y + z = 0, \quad x + y + 2z = 0, \quad x + 3y + 3z = 0$

homog. sys.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 3 & 0 \end{bmatrix} \underset{\substack{-R_1+R_2 \\ -R_1+R_3 \\ -R_1+R_4}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} -R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underset{\substack{R_2+R_3 \\ -R_2+R_4 \\ -2R_2+R_1}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \underset{\substack{R_3+R_1 \\ -2R_3+R_2 \\ 2R_3+R_4}}{\sim} \begin{array}{c} \begin{array}{cccc} x & y & z & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$E_3: z=0, \quad E_2: y=0, \quad E_1: x=0$

$$x - 2y + 3z = 4, \quad 2x - y - 3z = 5, \quad 3x + z = 2, \quad 3x - 3y = 7$$

h.w. No soln.