Linear Algebra

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Outline

1. Review
2. Linear combination.
3. Spanning set of a vector space.
4. Linear dependence and independence.

1. Review

In last lecture

- Vector space.
- Subspace of a vector space.
- Linear combination “Relation between vector spaces and solution of systems of linear equations”.

Vector Spaces

- ## Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

Vector Spaces

Ex: Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: : $(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

Ex: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{(\tfrac{1}{2})}\underset{\text{integer}}{(1)} = \underset{\text{noninteger}}{\tfrac{1}{2}} \notin V$$ (it is not closed under scalar multiplication)

Subspaces of Vector Spaces

- Subspace:

$(V,+,\bullet)$: a vector space

$$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Thm: (Test for a subspace)

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

  (1) $\mathbf{0} \in W$.

  (2) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

  (3) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- Ex: $W = \{(x, y), x + 2y = 0\}$ Is W s.s. of $\mathcal{R}^2$?

$x = -2y$

Let $y = t$

Sol:

$x = -2t$

① $\bar{0} \in W$

-2^* The same value

any value

$$W = \{(-2t, t), t \in \mathcal{R}\}$$

② Let $u = \{-2u_1, u_1\}$ and $v = \{-2v_1, v_1\}$

$$u + v = \{-2u_1 - 2v_1, u_1 + v_1\} \in W$$

$-2(u_1 + v_1)$ any value

-2^* The same val

$\therefore W$ is a s.s. of R^2

③ Let $u = \{-2u_1, u_1\}$

$$cu = \{c(-2u_1), cu_1\} \in W$$

$-2(cu_1)$ any val

-2^* The same val

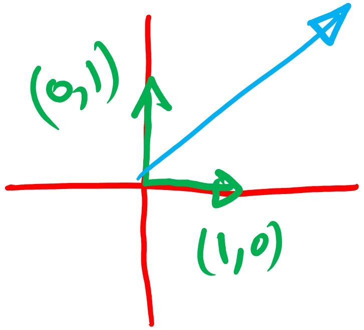
2. Linear combination

Relation between vector spaces and solution of a system of linear equations

R^2 space

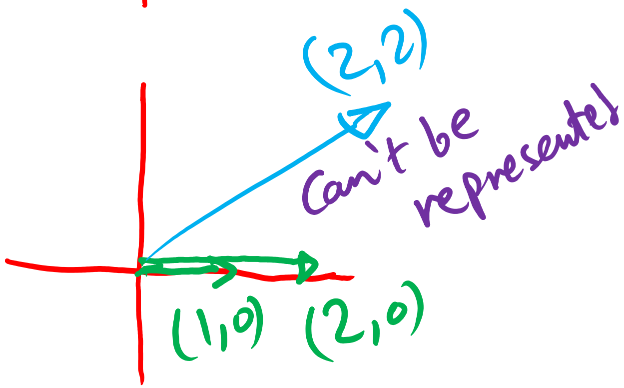
- Linear combination:

$$\underset{}{(2,2)} = \overset{c_1}{2}(1,0) + \overset{c_2}{2}(0,1)$$



A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$



▪ Ex: (Finding a linear combination)

$\mathbf{v}_1 = (1,2,3)$ $\mathbf{v}_2 = (0,1,2)$ $\mathbf{v}_3 = (-1,0,1)$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Test vecs. (b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$$

$$(1,1,1) = (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$$

$$\Rightarrow \left.\begin{aligned} c_1 \qquad - c_3 &= 1 \\ 2c_1 + c_2 \qquad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}\right\} \text{system of linear eqs.}$$

Consistent

w can be written as a L.C. of $v_{1,2,3}$

Inconsistent

w can not be w

$$
\begin{array}{cccc}
& & A & \\
\bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{w}\text{: test vector}
\end{array}
$$

$$\Rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right]}_{\text{aug } A} \xrightarrow{\text{Guass−Jordan Elimination}} \begin{array}{cccc} c_1 & c_2 & c_3 & c \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$|RE(A)| = 0 \Rightarrow |A| = 0$

Consistent with inf. no. of solutions

$$\Rightarrow c_1 = 1 + t,\ c_2 = -1 - 2t,\ c_3 = t$$

Let $c_3 = t$

E_2: $c_2 + 2(t) = -1$

$\longrightarrow c_2 = -1 - 2t$

E_1: $c_1 - (t) = 1$

$\longrightarrow c_1 = 1 + t$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

When A is a square matrix

& $|A|$
- $\neq 0$ Consistent with unique soln.
- $= 0$
  - Non-homog.
    - Inf. solns
    - No. solns
  - homog. — Inf. solns. $\longrightarrow$ Consis.

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \begin{array}{c} \begin{array}{cccc} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{w}_{\text{new}} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \end{array} \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Contrad.

$\therefore$ Inconsis.

$\Rightarrow$ this system has no solution $(\because 0 \neq 7)$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

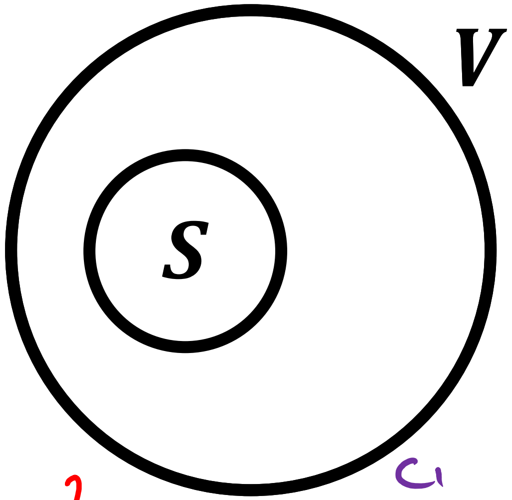
3. Spanning set of a vector space

- the span of a set: span (S) General defns.

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

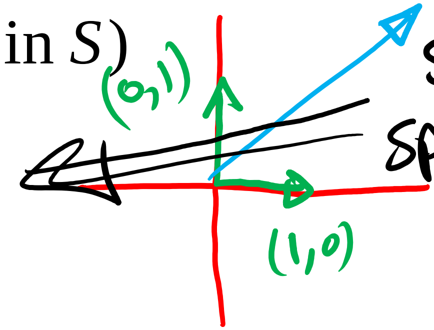
$$span(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)



R^2 space

$(2,2) = \overset{c_1}{2}(1,0) + \overset{c_2}{2}(0,1)$

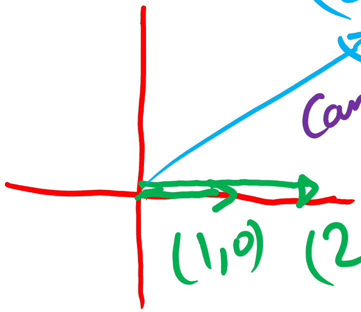


$S=\{(1,0),(0,1)\}$

Span(S): all vectors in R^2

- a spanning set of a vector space: Spanning set of R^2

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.



can't be represented

$S=\{(1,0),(2,0)\}$

Span(S) all vecs. on x-axis

$$
\begin{array}{cccc|c|c}
v_1 & v_2 & v_3 & & \text{Test vector } w & \text{Test vect.} \\
1 & 2 & 0 & \vdots & 1 & u_1 \\
0 & 1 & 1 & \vdots & 1 & u_2 \\
1 & 0 & 1 & \vdots & 1 & u_3
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & u_1 \\
0 & 1 & 0 & fn(u_1,2,3) \\
0 & 0 & 1 & fn(u_1,u_2,u_3)
\end{array}\right]
\qquad
\left[\begin{array}{ccc|c}
1 & 0 & 0 & u_1 \\
0 & 1 & 0 & fn(u_1,2,3) \\
0 & 0 & 0 & fn(u_1,2,3)
\end{array}\right]
$$

$fn(u_1,2,3) = 0$

In case of testing Linear Comb.

- Consistent — W can be written as a L.C.
- InConsistent — W can't

General vec. — Testing span — Spanning set — span subspace

- consistent without conds.
- Consis. under conds.

- Ex: The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 ⟵ **Standard spanning set of R^3**

For every vector u in R^3, it can be written as

$$u = c_1\,(1,0,0) + c_2\,(0,1,0) + c_3\,(0,0,1)$$

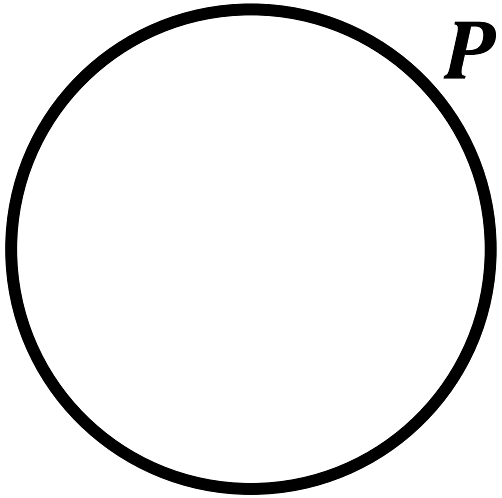
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \end{array}\right]$$

Ex: Let $u = (0.5, 0.75, 1)$

Then $u = 0.5\,(1,0,0) + 0.75\,(0,1,0) + 1\,(0,0,1)$ without solving aug. mat.

$\equiv (1,0,0)$ $\quad \equiv (0,1,0)$ $\quad \equiv (0,0,1)$

- Ex: The set $S = \{1 \;,\; x \;,\; x^2\}$ spans the set of all polynomials of degree at most 2 "P^2"



$\boldsymbol{P^2}$

$$P(x) = a_0\,1 + a_1\,x + a_2\,x^2$$

Where a_0 , a_1 , and a_2 are real vales set of vecs.

ex: $S = \{(1,0),(0,1)\}$

vector space

ex: $\mathbb{R}^2$

includes all points (x,y)

$x,y \in \mathbb{R}$

$\bar{i} = (1,0,0)$

$\bar{j} = (0,1,0)$

$\bar{k} = (0,0,1)$

- Notes:

$$\text{span}(S) = V$$

span(S): all possible vecs. that can be written as a L.C. of vecs. in S

Given $\Rightarrow$ S spans (generates) V

V is spanned (generated) by S

S is a spanning set of V

$\bar{v} = c_1(1,0) + c_2(0,1)$

$c_1 \& c_2 \in \mathbb{R}$

- Notes:

(1) $S \subseteq span(S)$

(2) $S_1, S_2 \subseteq V$

$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$

all 3D space

$\bar{V} = \{\bar{i}, \bar{j}, \bar{k}, (1,1,0), \ldots \ldots\}$

S_2

$\{\bar{i}, \bar{j}, \bar{k}\}$

span all 3D space including xy plane

S_1

$\{\bar{i}, \bar{j}\}$

span xy plane

at $c_3 \bar{k}$

$=0$

& it can take non-zero values

S_2 includes only 3 vecs

span(S_2) include all $\bar{v}$

- Ex: (A spanning set for R^3)

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ sapns R^3

Test vec.

Sol:

We must determine whether an arbitrary vector $\mathbf{u}=(u_1,u_2,u_3)$ General vec

in R^3 can be as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u}\in R^3 \Rightarrow \mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$$

Test in span

$$\Rightarrow \begin{aligned} c_1 \qquad\quad -2c_3 &= u_1 \\ 2c_1+\ c_2 \qquad\quad &= u_2 \\ 3c_1+\ 2c_2+\ c_3 &= u_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & u_1 \\ 2 & 1 & 0 & u_2 \\ 3 & 2 & 1 & u_3 \end{array}\right]$$

method 1

use Gauss

The problem thus reduces to determining whether this system is consistent for all values of u_1, u_2, and u_3.

Coeff mat: "A" square

method 2

$|A| <$ $=0$ Consistent under Condn.

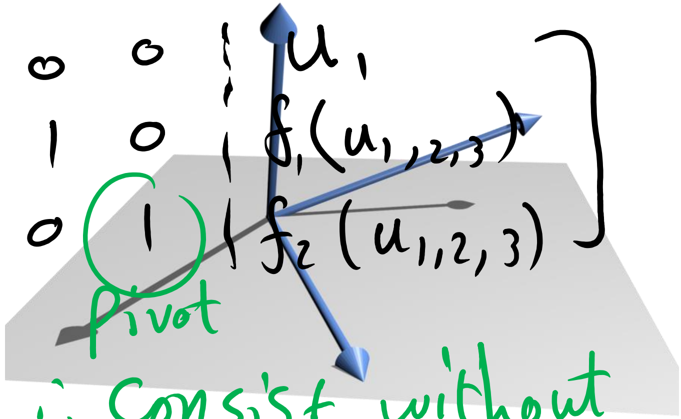
$\neq 0$ Consist without Condn.

ex:

$S=\{(1,0),(0,1),(1,1)\}$

$span(S) = R^2$

$$\because |A| = \begin{vmatrix}1 & 0 & -2\\ 2 & 1 & 0\\ 3 & 2 & 1\end{vmatrix} \neq 0 \quad \xrightarrow[\text{G.J.}]{\text{using}} \quad \left[\begin{array}{ccc|c}1 & 0 & 0 & u_1\\ 0 & 1 & 0 & f_1(u_{1,2,3})\\ 0 & 0 & 1 & f_2(u_{1,2,3})\end{array}\right]$$



Pivot

$\therefore$ consist. without conds. on $u_{1,2,3}$

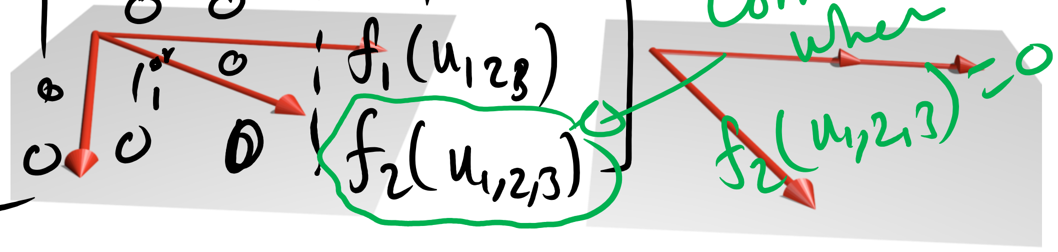
$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution for every u.

$\Rightarrow span(S) = R^3$

- Note

The set $S = \{(1,2,3), (0,1,2), (-1,0,1)\}$ does not sapn R^3

$$\because |A| = \begin{vmatrix}1 & 0 & -1\\ 2 & 1 & 0\\ 3 & 2 & 1\end{vmatrix} = 0 \quad \longrightarrow \quad \left[\begin{array}{ccc|c}1 & 0 & 0 & u_1\\ 0 & 1 & 0 & f_1(u_{1,2,3})\\ 0 & 0 & 0 & f_2(u_{1,2,3})\end{array}\right]$$



consis. when $f_2(u_{1,2,3}) = 0$

$$\mathbf{Col\,1 = 2\,Col\,2 - Col\,3}$$

Example: Determine whether the set S spans $\mathbb{R}^2$. $S = \{(5,0),(5,4)\}$

$0 * 1 \neq 4$

No vec. is scalar multiple of the other

$\therefore$ the 2 vecs. are in diff direcs. in R^2

$\therefore$ span R^2

Solution

Let $\mathbf{u} = (u_1, u_2)$ be any vector in R^2. Solving the equation

$$c_1(5,0) + c_2(5,-4) = (u_1, u_2)$$

for c_1 and c_2 yields the system

$$\begin{aligned} 5c_1 + 5c_2 &= u_1 \\ -4c_2 &= u_2. \end{aligned}$$

This system has a unique solution because the determinant of the coefficient matrix is nonzero. So, S spans R^2.

$$\left[\begin{array}{cc|c} 5 & 5 & u_1 \\ 0 & 4 & u_2 \end{array}\right]$$

already in Echelon form

$\Rightarrow$ Consistent w/o condn

or

$A = \begin{bmatrix} 5 & 5 \\ 0 & 4 \end{bmatrix}$ sq. mat

$|A| = 20 \neq 0$

consis. w/o condn.

Example: Determine whether the set S spans $\mathbb{R}^2$.

If # of vecs in S < $\vec{R}^n$

∴ S does not span R^n

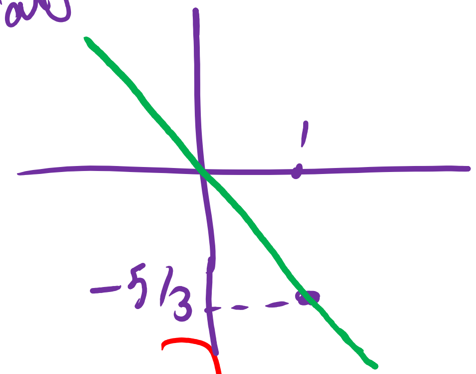
$$S = \{(-3, 5)\}$$

∴ S does not span R^2

Solution

S does not span R^2 because only vectors of the form $t(-3, 5)$, are in span(S). For example, $(0, 1)$ is not in span(S). S spans a line in R^2.

vecs. on straight line



if # of vecs. ≥ $\vec{R}^n$ need test

$$5R_1+3R_2 \to R_2$$

$$\left[\begin{array}{c|c} -3 & u_1 \\ 5 & u_2 \end{array}\right] \sim \left[\begin{array}{c|c} -3 & u_1 \\ 0 & 5u_1+3u_2 \end{array}\right]$$

Consist. under cond^{n}. "$\underbrace{5u_1+3u_2=0}$"

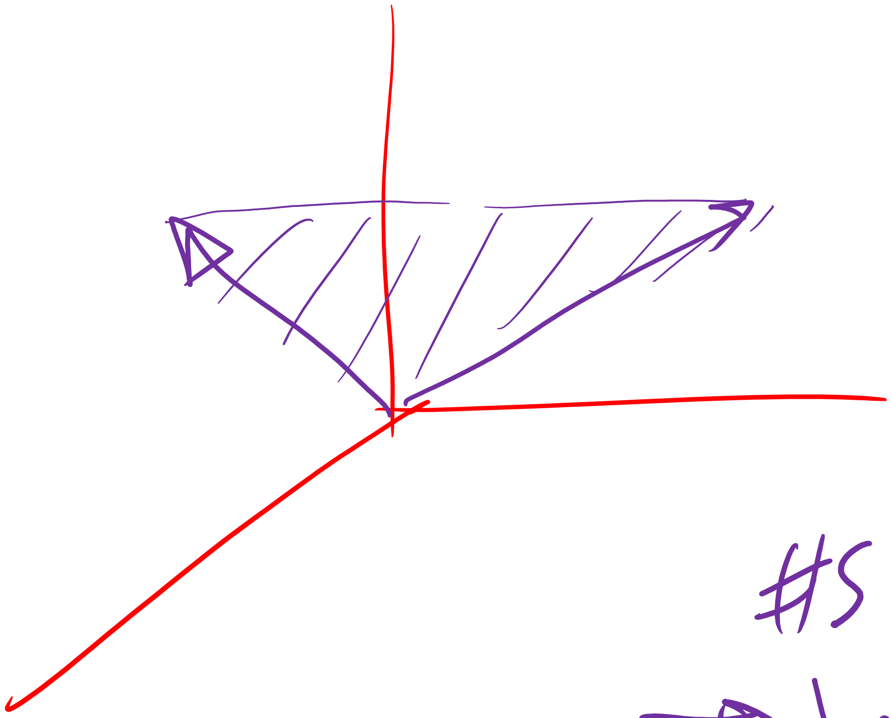
$$u_2 = \frac{-5}{3}u_1$$

$$\text{Span}(S) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ \frac{-5}{3}u_1 \end{bmatrix} = u_1\begin{bmatrix} 1 \\ -5/3 \end{bmatrix}$$

Example: Determine whether the set S spans $\mathbb{R}^3$.

$$S = \{(-2,5,0),(4,6,3)\}$$ 2 vecs $<$

Solution This set does not span $\mathbb{R}^3$. S spans a plane in $\mathbb{R}^3$



$\mathbb{R}^n$

# of vectors in S

$\#S < n$ $\Rightarrow$ doesn't span $\mathbb{R}^n$

$\#S \geq n$ may be req. test

Example:

Determine whether the set S spans $\mathbb{R}^2$.

$*-2$

$$S = \{(-1, 2), (2, -4)\}$$

$2*(-2)=-4$ ok

$\therefore$ 2 vecs. are multiple of each other

$\Rightarrow$ in the same line

$\Rightarrow$ Represent only vecs. on the same line

Solution

S does not span R^2 because only vectors of the form $t(1, -2)$, are in span(S). For example, $(0, 1)$ is not in span(S). S spans a line in R^2.

Example: Determine whether the set S spans $\mathbb{R}^2$.

$$S = \{(-1,4),(4,-1),(1,1)\}$$

$$4R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 4 & -1 & 1 & u_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 0 & 15 & 5 & 4u_1 + u_2 \end{array}\right]$$

Consistent w/o condn.

$\therefore$ S spans $\mathbb{R}^2$

Example: Determine whether the set S spans $\mathbb{R}^3$.

$$S = \{(1,-2,0),(0,0,1),(-1,2,0)\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 0 & 0 & u_3 \\ 0 & 0 & 0 & 2u_1+u_2 \end{array}\right]$$

2 Conds

$u_3 = 0$ &

$2u_1 + u_2 = 0$

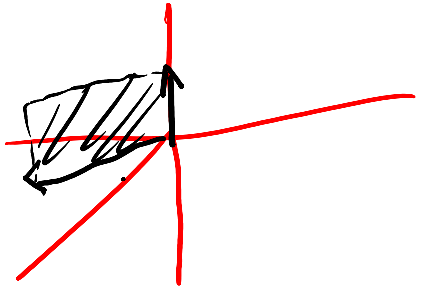
$$\begin{bmatrix} u_1 \\ -2u_1 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & u_1 \\ -2 & 0 & 2 & u_2 \\ 0 & 1 & 0 & u_3 \end{bmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & -1 & u_1 \\ 0 & 0 & 0 & 2u_1 + u_2 \\ 0 & 1 & 0 & u_3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -1 & u_1 \\ 0 & 1 & 0 & u_3 \\ 0 & 0 & 0 & 2u_1 + u_2 \end{bmatrix}$$

Consis. under cond $\leftarrow 2u_1 + u_2 = 0$

$$\text{Span}(S) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_1 \\ u_3 \end{bmatrix} = \underbrace{u_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Plane}}$$

$u_2 = -2u_1$



S does not span $\mathbb{R}^3$

4. Linear dependence and independence

Let $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

- Linear Independent (L.I.) and Linear Dependent (L.D.):

unique zero soln

Inf. soln including non-zero solns

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ Homog. sys. of linear eqs. "Always Consistent"

$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = 0$

$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = 0$

(1)" If the equation has only the trivial solution "($c\_1 = c\_2 = \cdots = c\_k = 0$)" then "$S$" is called linearly independent."

$c_1 = c_2 = c_3 = 0$

$0 = 0$

$\Rightarrow$ Linearly indep.

let $c_1 = 1, c_2 = -1, c_3 = 1$

(2)" If the equation has a nontrivial solution "("i.e., not all zeros")," then "S" is called linearly dependent."

$\bar{v}_1 - \bar{v}_2 + \bar{v}_3 = 0$

$\Rightarrow \bar{v}_2 = \bar{v}_1 + \bar{v}_3$

Another soln. if A is a sq. mat.

$\Rightarrow$ Test $\rightarrow$ Homog. sys.

$|A| <$ $= 0$ Inf. $\rightarrow$ L.D.

$\neq 0$ unique zero soln $\rightarrow$ L.I.

$\rightarrow$ vecs اي يمكن كتابة [illegible] L.C. [illegible]

$\therefore$ L.Dep.

w as l. Comb. of v_1, v_2, v_3 — Test span — Test indep.

$$
\begin{array}{cccccc}
 & & & \text{Test } w & \text{Test} & \text{Test} \\
v_1 & v_2 & v_3 & & & \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
1 & 1 & 2 & 1 & u_1 & 0 \\
2 & 0 & 4 & 1 & u_2 & 0 \\
3 & -1 & -5 & 1 & u_3 & 0
\end{array}\right]
$$

Consist / Inconsi — consis without conds. or not

0 → uniq soln → Indep.

→ Inf. soln → Dep.

- Thm: (A property of linearly dependent sets)

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\boldsymbol{v}_j$ in S can be written as a linear combination of the other vectors in S.

$S = \{(1,0), (0,1), (2,0)\}$

$(2,0) = 2(1,0)$

- Corollary to the Theorem:

  Two vectors **u** and **v** in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

for any R^n ← any n

Exercise determine whether the set S is linearly independent or linearly dependent.

$*6$

$$\boxed{S = \{(1, -4, 1), (6, 3, 2)\}}$$

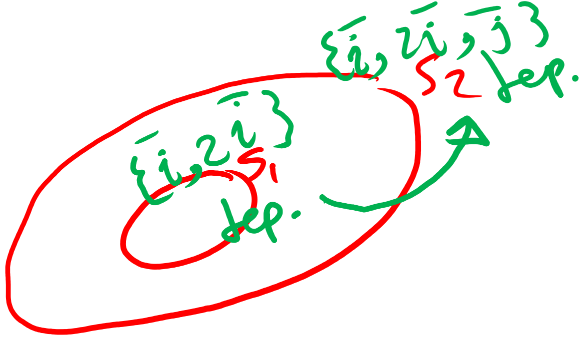
$-4 * 6 \neq 3$

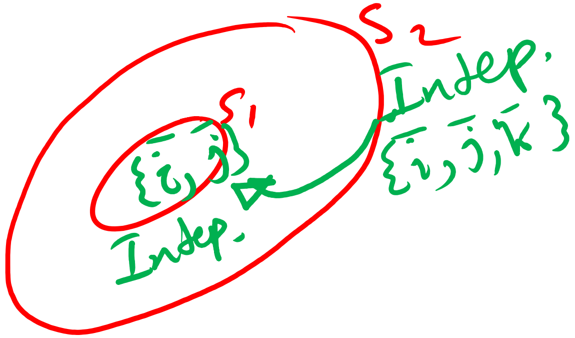
not scalar multiple → In diff. direcs.

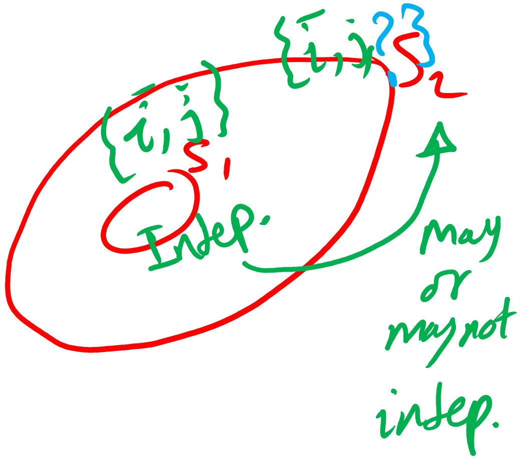
Solution Because $(1, -4, 1)$ is not a scalar multiple of $(6, 3, 2)$, the set S is linearly independent.

→ Indep.

- Notes:







If $S_1 \subseteq S_2$

$$S_1 \text{ is linearly dependent} \Rightarrow S_2 \text{ is linearly dependent}$$

$$S_2 \text{ is linearly independent} \Rightarrow S_1 \text{ is linearly independent}$$

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

$$\mathbf{v}_1 \qquad \mathbf{v}_2 \qquad \mathbf{v}_3$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

Test

sq. mat "A"

$|A| \neq 0$

unique zero sol

L.I.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

uniq sol

$\rightarrow$ L.I.

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$\equiv(1,1,-2) \qquad \equiv(2,5,-1) \qquad \equiv(0,1,1)$$

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2},\ \underset{\mathbf{v}_2}{2+5x-x^2},\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0}$$

i.e. $c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0+0x+0x^2$

$$(c_1+2c_2) + (c_1+5c_2+c_3)\,x + (-2c_1-c_2+c_3)\,x^2 = 0+0x+0x^2$$

Test

$$\Rightarrow \begin{aligned} c_1+2c_2 \quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Inf. sols.

$\therefore$ L.D.

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

  Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S=\left\{\begin{bmatrix}2&1\\0&1\end{bmatrix},\begin{bmatrix}3&0\\2&1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix}\right\}$$

$$\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3$$

$\mathbf{v}_1 = (2,1,0,1)$, $\mathbf{v}_2 = (3,0,2,1)$, $\mathbf{v}_3 = (1,0,2,0)$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

$$\left[\begin{array}{ccc|c}2&3&1&0\\1&0&0&0\\0&2&2&0\\1&1&0&0\end{array}\right]$$

$$c_1\begin{bmatrix}2&1\\0&1\end{bmatrix}+c_2\begin{bmatrix}3&0\\2&1\end{bmatrix}+c_3\begin{bmatrix}1&0\\2&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+c_2 \qquad &= 0 \end{aligned}$$

aug A

A

$\rho(A) = \rho(\text{aug}A) = 3$

no. of var $= 3$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

unique soln.

$\rightarrow$ L.I.

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

Example: Determine whether the set of vectors in $M_{4,1}$ is linearly independent or linearly dependent.

$$S = \left\{ \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \end{array} \right\}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} c_1 + c_2 \qquad\qquad\qquad &= 0 \\ c_2 + 3c_3 + c_4 &= 0 \\ -c_1 \qquad + c_3 - c_4 &= 0 \\ 2c_2 - 2c_3 + 2c_4 &= 0 \end{aligned}$$

Test

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

unique

L.I.

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$$

4 vecs. in R^3

Sol:

Test

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{array}\right]$$

max. no. of pivots $= 3$

R^n

no. of vars $= 4$ = no. of vecs. in S

by default no. of pivots < no. of vars $\Longrightarrow$ Inf. sols. $\Rightarrow$ L.D.

For $\mathbb{R}^n$ very important

Short Cut
2 vecs. in $\mathbb{R}^n$
test indep. by scalar multip

Testing span

# of vecs. in $S < n$

$\Rightarrow$ does not span

Testing Indep.

# of vecs. in $S > n$

$\Rightarrow$ L. Dep.

otherwise Test

Example:

For which values of t is each set linearly independent?

(a) $S = \{(t, 1, 1), (1, t, 1), (1, 1, t)\}$

(b) $S = \{(t, 1, 1), (1, 0, 1), (1, 1, 3t)\}$

$$\left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 1 & t & 1 & 0 \\ 1 & 1 & t & 0 \end{array}\right]$$

The same steps.

sq. mat.

$$\left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3t & 0 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|cc} t & 1 & 1 & t & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 3t & 1 & 1 \end{array}\right] \rightarrow |A| = (0+1+1) - (0+t+3t)$$

$$= 2-4t \neq 0 \text{ to be L.I.}$$

$$2-4t \neq 0 \rightarrow 4t \neq 2 \rightarrow \boxed{t \neq 2}$$

In $\mathbb{R}^2$

$S = \{(1,0), (0,1), (1,1)\}$

→ span $\mathbb{R}^2$

→ more than enough → L.D.

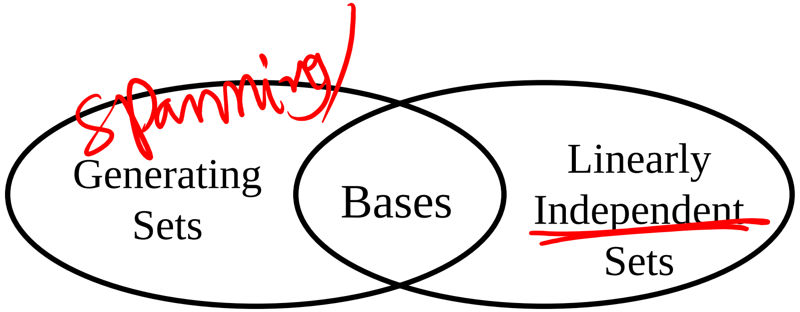
Intro. to next lecture

Basis and Dimension

- Basis:

  V : a vector space

  $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$



$$\begin{cases} (a) & S \text{ spans } V \ (\text{i.e., } span(S) = V) \\ (b) & S \text{ is linearly independent} \end{cases}$$

$\Rightarrow S$ is called a **basis** for V

- Notes:

  (1) Ø is a basis for $\{\mathbf{0}\}$

  (2) the standard basis for R^3:

  $\{i, j, k\} \quad i = (1, 0, 0), \ j = (0, 1, 0), \ k = (0, 0, 1)$

Exercise

Find the values of $\boldsymbol{k}$ that makes the following set linearly independent

$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ -1 & k \end{bmatrix} \right\}$$