Linear Algebra

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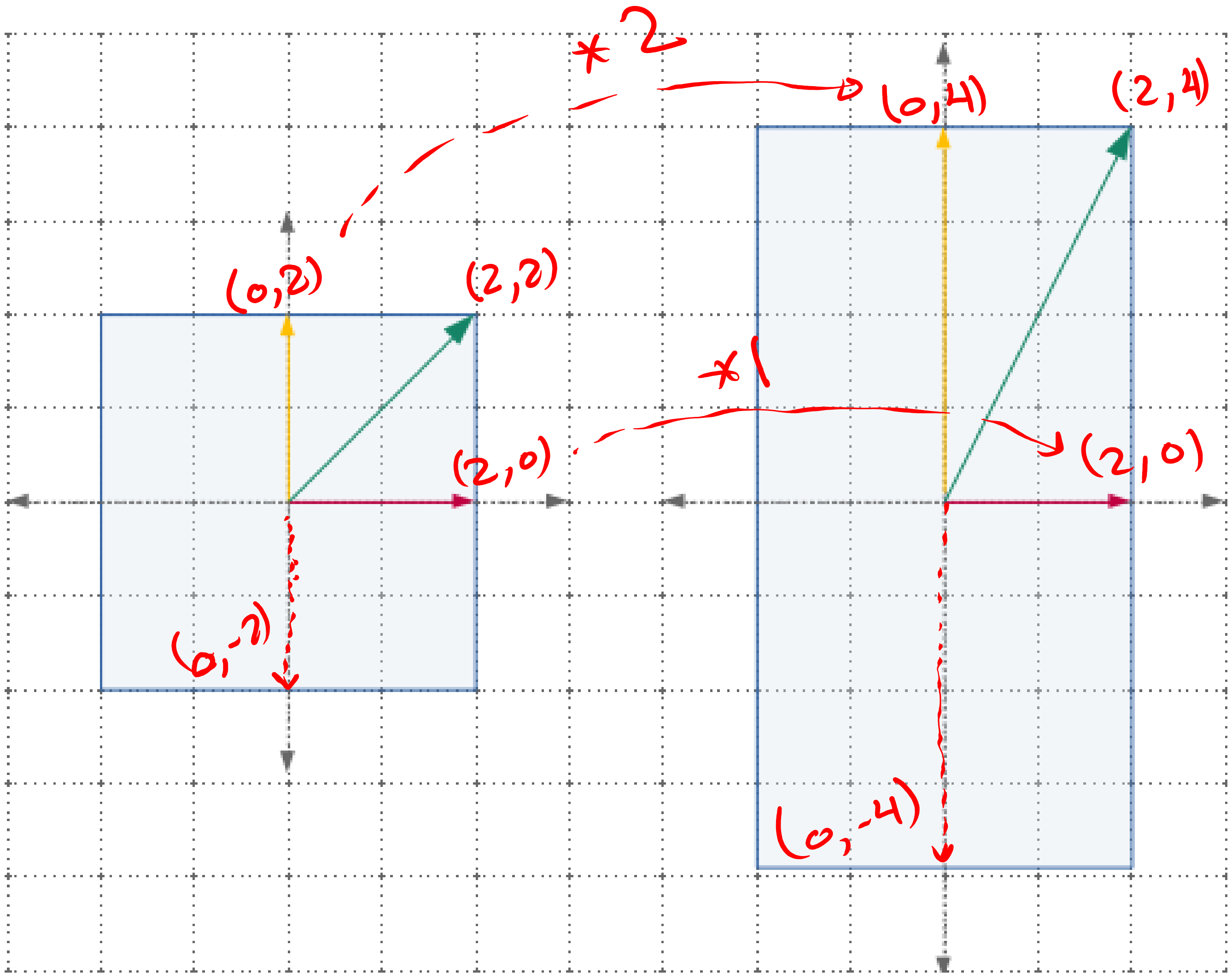
Outline

1. Eigenvalues and eigenvectors.
2. Properties of eigenvalues and eigenvectors.
3. Cayley-Hamilton theorem.

Eigen Values and Eigen Vectors

Introduction

Assume the linear transformation ($R^2 \rightarrow \mathrm{R}^2$) of
→ vertical scaling of +2 to every vector of a square.
→ It will transform the square into a rectangle.



Note that during these transformations, some of the vectors (red and yellow) **remain on the same line (span)** as they were earlier. **[Eigenvectors]**

• The horizontal vector remains **unchanged** (same direction, same length). **[Eigenvalue= 1]**
• The vertical vector has same direction, but doubled in length. **[Eigenvalue= 2]**
• The diagonal vector has changed its angle (direction) as well as length.

For visual interactive version of the transformation, check the following link"
https://www.geogebra.org/m/mdvN0HTt

Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is called an **eigenvalue** of A if there is a *nonzero* vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Eigen value

$$A\,x = \lambda\,x$$

Square matrix

Eigen vector

Non-zero vector

EXAMPLE Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$,

$$Ax_1 = \lambda x_1$$

SOLUTION

$$A\mathbf{x}_1 = \overset{A}{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}} \overset{x_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underset{A}{\overset{\lambda}{2}} \overset{x_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

Eigenvalue Eigenvector

EXAMPLE Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

verify that

$$\mathbf{x}_1 = (-3, -1, 1) \quad \text{and} \quad \mathbf{x}_2 = (1, 0, 0)$$

are eigenvectors of A and find their corresponding eigenvalues.

Except the zero vector

SOLUTION

$$\underset{A}{} \quad \underset{x_1}{} \quad \lambda_1 \; x_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{0}_{\text{eigenvalue}}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

Any vector $\in$ $\boldsymbol{Kern}(\boldsymbol{T})$ represented by A is and **eigenvector** with **eigenvalue of zero**.

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{1}_{\text{eig. val.}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Exercise

determine whether **x** is an eigenvector of A.

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$$

(a) $\mathbf{x} = (2, -4, 6)$

(b) $\mathbf{x} = (2, 0, 6)$

(a) Because

$*4$

$$A\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 24 \end{bmatrix} = 4\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$*4$ $*4$ check

x *is* an eigenvector of A (with a corresponding eigenvalue 4).

(b) Because

$*2$

$$A\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ \overset{\neq}{-16} \\ 12 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

$*2$

x is *not* an eigenvector of A.

EXAMPLE Verifying Eigenvalues and Eigenvectors

eigenvalue

Is $\begin{bmatrix}1\\1\\1\end{bmatrix}$ an eigenvector of $A = \begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 3 & 2\end{bmatrix}$? Yes $\begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\6\\6\end{bmatrix} = 6\begin{bmatrix}1\\1\\1\end{bmatrix}$

eigenvec

When the sum of each row of the matrix is equal to some constant value k, then we know that one of its eigenvectors is an all-ones vector and its corresponding eigenvalue is k

$$\begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 3 & 2\end{bmatrix}\begin{matrix}\to sum = 6\\ \to sum = 6\\ \to sum = 6\end{matrix} \quad \to \quad \text{Eigenvector} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \text{ eigenvalue} = 6$$

$$\begin{bmatrix}1 & 8\\5 & 4\end{bmatrix}\begin{matrix}\to sum = 9\\ \to sum = 9\end{matrix} \quad \to \quad \text{Eigenvector} = \begin{bmatrix}1\\1\end{bmatrix}, \text{ eigenvalue} = 9$$

Finding the eigenvalues and eigenvectors:

$$A\,x = \lambda\,x$$ X non-zero vec

?! Important

$$\lambda I x - A\,x = 0$$ common

?!

$$\underbrace{(\lambda\,I - A)}_{A^*}\,x = 0$$

Trivial zero soln. X

Non-zero soln

Homogeneous system of linear equations

→ To get **non-zero** value for $\boldsymbol{x}$

→ We need to have **infinite**

number of solutions

only unknown

$$\boxed{|\lambda\,I - A| = 0}$$

Characteristic equation

If $\boldsymbol{A}$ is $\boldsymbol{n} \times \boldsymbol{n}$ matrix,

Then the ccs eqn. is

***n* degree polynomial**

→ n **eigen value**

Substitute in $\overbrace{(\boldsymbol{\lambda\,I - A})}^{\text{Now is known}}\ \boldsymbol{x} = \boldsymbol{0}$

with the obtained λs and

solve the **homogeneous**

system of equations.

EXAMPLE Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

- Reverse the sign of all elements
- Add λ to the main diagonal elements

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

SOLUTION The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) - (-12) \\ &= \lambda^2 + 3\lambda - 10 + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2). \end{aligned}$$

$(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$

EXAMPLE **(Continued)**

Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

$$A\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

$$A\begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$$

Coeff. mat. for the homog. sys.

Let $\boxed{x_2 = t}$

$x_1 - 4t = 0$

$\boxed{x_1 = 4t}$

SOLUTION

For $\lambda_1 = -1$,

$$(-1)I - A = \begin{bmatrix} \overset{\lambda}{-1} - 2 & 12 \\ -1 & \overset{\lambda}{-1} + 5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow[\textbf{Elimination}]{\textbf{Gauss}} \begin{array}{cc} x_1 & x_2 \\ \end{array} \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

All scalar multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are eigenvectors of $\lambda = -1$

$\equiv$ eigenspace of $\lambda = -1$

one free var

dim = 1

EXAMPLE **Finding Eigenvalues and Eigenvectors**

(Continued)

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

SOLUTION

For $\lambda_2 = -2$,

$$(-2)I - A = \begin{bmatrix} \overset{\lambda}{-2} - 2 & 12 \\ -1 & \overset{\lambda}{-2} + 5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow[\textbf{Elimination}]{\textbf{Gauss}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

Example

(a) Find the eigen values and its corresponding eigen vectors of the matrix.

(b) What is the dimension of the eigen space corresponding to each eigen value.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix} = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = (\lambda + 3)(\lambda - 3)^2 = 0.$$

$\lambda = -3, 3, 3$

repeated

For $\lambda_1 = -3$, $\begin{bmatrix} \lambda_1 - 1 & -2 & 2 \\ 2 & \lambda_1 - 5 & 2 \\ 6 & -6 & \lambda_1 + 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

after solving the homog. sys.

The solution is $\{(t, t, 3t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -3$ is $(1, 1, 3)$.

dim = 1

span of

Example(continued)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

(a) Find the eigen values and its corresponding eigen vectors of the matrix.

(b) What is the dimension of the eigen space corresponding to each eigen value.

For $\lambda_2 = 3$, $\begin{bmatrix} \lambda_2 - 1 & -2 & 2 \\ 2 & \lambda_2 - 5 & 2 \\ 6 & -6 & \lambda_2 + 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

after solving

The solution is $\{(s - t, s, t) : s, t \in R\}$. So, two eigenvector corresponding to $\lambda_2 = 3$ are $(1, 1, 0)$ and $(1, 0, -1)$.

dim = 2

Example

Is $\mathbf{3}$ an eigenvalue of $A = \begin{bmatrix} 1 & -1 \\ -6 & 0 \end{bmatrix}$?

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 \\ 6 & \lambda \end{vmatrix}$$

$$|3I - A| = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = (2)(3) - (1)(6) = 0$$

$\therefore$ $\mathbf{3}$ an eigenvalue of $A = \begin{bmatrix} 1 & -1 \\ -6 & 0 \end{bmatrix}$

Since it satisfies its characteristic equation.

To calculate the corresp. eigenvec.

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 6 & 3 & 0 \end{array}\right]$$

$\downarrow$

Solve

EXAMPLE Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

Triangular mat.

$$A = \begin{bmatrix} \cancel{2}\,1 & 1 & 0 \\ 0 & \cancel{2}\,2 & 0 \\ 0 & 0 & \cancel{2}\,3 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

SOLUTION

$$\underbrace{|\lambda I - A|}_{\text{Triang.}} = \begin{vmatrix} \lambda - \cancel{2}\,1 & -1 & 0 \\ 0 & \lambda - \cancel{2}\,2 & 0 \\ 0 & 0 & \lambda - \cancel{2}\,3 \end{vmatrix} = (\lambda - 2)^3. \longrightarrow 2I - A = \begin{array}{c} \begin{matrix} s & & t \\ x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\lambda = 1, 2, 3$

$-x_2 = 0$

$\rightarrow \boxed{x_2 = 0}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$ s and t not both zero.

dim = 2

THEOREM 7.3
Eigenvalues of Triangular Matrices

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

or diagonal

EXAMPLE **Finding Eigenvalues of Diagonal and Triangular Matrices**

Find the eigenvalues of each matrix.

$$\text{(a)}\ A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad \text{(b)}\ A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 2, 1, -3$

$\lambda = -1, 2, 0, -4, 3$

SOLUTION (a) Without using Theorem 7.3, you can find that

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

Example Triang.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$ $\lambda = 2, 3, 1$

(a) Find the eigen values and its corresponding eigen vectors of the matrix.

(b) What is the dimension of the eigen space corresponding to each eigen value.

Triangular matrix → $\lambda = 2\,, 3\,, 1$

For $\lambda_1 = 2$, $\begin{bmatrix} \lambda_1 - 2 & 0 & -1 \\ 0 & \lambda_1 - 3 & -4 \\ 0 & 0 & \lambda_1 - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xRightarrow[\text{ops.}]{\text{row}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \\ t & & \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(t, 0, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is $(1, 0, 0)$. span

For $\lambda_2 = 3$, $\begin{bmatrix} \lambda_2 - 2 & 0 & -1 \\ 0 & \lambda_2 - 3 & -4 \\ 0 & 0 & \lambda_2 - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} \begin{array}{ccc} & t & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

The solution is $\{(0, t, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 3$ is $(0, 1, 0)$.

Example(continued)

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigen values and its corresponding eigen vectors of the matrix.

(b) What is the dimension of the eigen space corresponding to each eigen value.

Triangular matrix → $\lambda = 2\,,3\,,1$

t

$$\text{For } \lambda_3 = 1, \begin{bmatrix} \lambda_3 - 2 & 0 & -1 \\ 0 & \lambda_3 - 3 & -4 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is $\{(-t, -2t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is $(-1, -2, 1)$.

Example Diag.

Find the dimension of the eigenspace corresponding to the eigenvalue $\boxed{\lambda = 3}$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Ax = \underbrace{(3I)}_{A}x = 3(Ix) = 3x$$

$$\underbrace{\begin{bmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{bmatrix}}_{\lambda I - A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{ccc} t_1 & t_2 & t_3 \\ \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$\rightarrow$ All vecs. in $\mathbb{R}^3$ are eigen vecs.

Properties of Eigen Values and Eigen Vectors

Properties of eigenvalues and eigenvectors:

$$\boxed{\begin{array}{c}\text{If } \boldsymbol{A} \text{ is } \textbf{singular} \text{ i.e. has no inverse,} \\ \boldsymbol{A^{-1}} \text{ does not exist } \leftrightarrow \boldsymbol{\lambda = 0}\end{array}}$$

However, $\bar{x} \neq 0$

Proof:

Suppose A is square matrix and has an eigenvalue of 0 .

→ $Ax = \lambda x$ with $\lambda = 0$

The system $Ax = 0$ has a non-trivial solution when

- The system has infinite number of solutions.
- i.e. no pivot in last row(s) of A
- i.e. $|A| = 0$.
- i.e. A is singular.

Example of kernal

$|A| = 0$

0 eigenvalue

Non-zero eigvec

$$Ax_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

Properties of eigenvalues and eigenvectors:

If A is **invertible** i.e. has inverse,

	Eigenvalues	Eigenvectors
$A \rightarrow$	$\lambda_1, \lambda_2, \lambda_3, \ldots$	$v_1, v_2, v_3, \ldots$
$A^{-1} \rightarrow$	$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots$	$v_1, v_2, v_3, \ldots$

Proof:

$$Ax = \lambda x$$

$\rightarrow A^{-1}Ax = \lambda A^{-1}x$

$\rightarrow Ix = \lambda A^{-1}x$

$\rightarrow A^{-1}x = \frac{1}{\lambda}x$ The same eigen vector x with eigenvalue $\frac{1}{\lambda}$

Properties of eigenvalues and eigenvectors:

	Eigenvalues	Eigenvectors
$A \rightarrow$	$\lambda_1, \lambda_2, \lambda_3, \ldots$	$v_1, v_2, v_3, \ldots$
$A^n \rightarrow$	$(\lambda_1)^n, (\lambda_2)^n, (\lambda_3)^n, \ldots$	$v_1, v_2, v_3, \ldots$

Proof:

$$Ax = \lambda x$$

$$\rightarrow AAx = \lambda \, \cancel{Ax} \; \lambda x$$

$$\rightarrow A^2x = \lambda \, \lambda \, x$$

$$\rightarrow A^2x = \lambda^2 \, x$$

Repeating $\rightarrow A^n x = \lambda^n x$ The same eigen vector x with eigenvalue λ^n

Properties of eigenvalues and eigenvectors:

A and A^T have the same eigenvalues

Proof:

Starting from the characteristic equation

$$|\lambda\, I - A| = 0$$

Since $|B| = |B^T|$

$$|\lambda\, I - A| = |(\lambda\, I - A)^T| = 0$$

$$= |\lambda\, I^T - A^T|$$

$$= |\lambda\, I - A^T|$$

Since A and A^T have the same characteristic equation

→ then they have the same eigenvalues

Theorem

- If λ is an eigen value for A

Then

- λ is an eigen value for A^t
- $\frac{1}{\lambda}$ is an eigen value for A^{-1}
- λ^m is an eigen value for A^m

with the same eigenvecs. of A

Example: If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ has the eigen values $\lambda = 7, -4$. Find the eigen values of A^{-1}, A^T and A^2.

steps skipped

Eigenvalues

$A \longrightarrow 7, -4$

$A^{-1} \longrightarrow \frac{1}{7}, \frac{1}{-4}$

$A^T \longrightarrow 7, -4$

$A^2 \longrightarrow (7)^2, (-4)^2$

Cayley-Hamilton theorem

CAYLEY-HAMILTON THEOREM:

- **Statement:** Every square matrix satisfies its own characteristic equation

  The characteristic equation: $\boldsymbol{f(\lambda) = |\lambda\, I - A| = 0}$

  Replace $\boldsymbol{\lambda}$ with the matrix $\boldsymbol{A}$ → $\boldsymbol{f(A) = 0}$

- **Uses of Cayley-Hamilton theorem:**
- (1) To calculate the positive integral powers of A.
- (2) To calculate the inverse of a square matrix A.

1. Verify that A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4

$\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix}$

Solution:Given A =$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$--------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find A^4:

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

2. find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ steps are skipped

Solution: The characteristic equation of A is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To be calculated

To find A^4:

$$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I \text{ ------------ } (2)$$

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

$$\text{Hence, } A^4 = 28\underbrace{\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}}_{A^2} - 45\underbrace{\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}}_{A} + 18\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}: $A^3 - 6A^2 + 8A - 3I = 0$ ------------ (1)

Multiplying (1) by A^{-1}, $A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

3. Find A^{-1} if A = $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem

Solution: The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ---------- (1)

To find A^{-1}: Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I) \text{ ---------- } (2)$$

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$