Linear Algebra

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Outline

1. Matrices.
2. Matrix operations.

1. Matrices

What is a matrix?

A **rectangular** array of elements, **arranged in rows and columns**.

$a_{1\,2}$

row

col.

$$A_{m\times n} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

no. of rows

no. of cols.

Types of matrices

Row vector

$$\begin{bmatrix} 1 & -3 & 17 \end{bmatrix}$$

Column vector

$$\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

Square matrix

$$\begin{bmatrix} 2 & 7 & 9 \\ 1 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$$

3x3

Main diagonal

Order = 3

Types of matrices (special types of square matrices)

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Triangular matrix

Upper Triangular matrix U

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$1*1*9$ $\qquad$ 0

Echelon form

Lower Triangular matrix L

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 9 \end{bmatrix}$$

-5 $\qquad$ 0

$|U|, |L|, |D| =$ prod. of main diagonal elements

Diagonal matrix D

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

45 $\qquad$ 0 $\qquad$ 0

Types of matrices

Zero matrix

$$O_{3\times 3} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$O_{2\times 3} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Additive identity

Identity matrix (**square diagonal** matrix)

$$I_3 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplicative identity

2. Matrix operations

Matrix operations

- Equality
- Addition/Subtraction
- Scalar Multiplication
- Matrix Multiplication
- Matrix Power
- Matrix Transpose
- Matrix Determinant
- Matrix inverse

Matrix operations (Equality)

Two matrices are **equal** iff:

- Have the **same dimension** or order.
- The **corresponding elements** are **identical**.

$$A_{m\times n} = B_{p\times q}$$

$$m = p \quad \text{and} \quad n = q$$

$$a_{ij} = b_{ij}$$

$$i = 1,2,\dots,m$$

$$j = 1,2,\dots,n$$

Example: if

$$\underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}}_{2\times 2} = \underbrace{\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \mathbf{4} & \mathbf{6} \end{bmatrix}}_{2\times 2}$$

Find $a,\ b,\ c,$ and d

$a=1$

$b=2$

$c=4$

$d=6$

Matrix operations (Addition/Subtraction)

Addition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$[a_{ij}]_{m \times n}$ $\qquad$ $[b_{ij}]_{m \times n}$ $\qquad$ $[a_{ij} + b_{ij}]_{m \times n}$

Subtraction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{bmatrix}$$

$[a_{ij}]_{m \times n}$ $\qquad$ $[b_{ij}]_{m \times n}$ $\qquad$ $[a_{ij} - b_{ij}]_{m \times n}$

$$C_{m\times n} = A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})_{m\times n}$$

Note: Matrices of different size cannot be added or subtracted.

Matrix operations (Addition/Subtraction)

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} \text{ order } 3 \quad \text{and} \quad B = \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix} \text{ order } 3$$

$$A + B = \begin{bmatrix} 2+4 & -1+7 & 3+(-8) \\ 0+9 & 4+3 & 6+5 \\ -6+1 & 10+(-1) & -5+2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{bmatrix} \text{ order } 3$$

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \end{bmatrix}$$

Can not be added

Matrix operations (Scalar Multiplication)

Scalar c

$$cA = c \cdot \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

$$cA = \begin{bmatrix} c(a_1) & c(a_2) & c(a_3) \\ c(a_4) & c(a_5) & c(a_6) \\ c(a_7) & c(a_8) & c(a_9) \end{bmatrix}$$

Scalar c is being multiplied to each entry or element of Matrix A

Examples:

$$2\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{4} & \mathbf{8} \\ \mathbf{-4} & \mathbf{-12} \end{bmatrix} = 4\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

Matrix operations (Matrix Multiplication)

$$A_{m\times n}\ B_{n\times p} = \underline{\underline{C_{m\times p}}}$$

$$(a_{ik})\ (b_{kj})$$

$$A\,B = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

C_{23}

A B C

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{bmatrix}$$

$m \times n \quad n \times p$

Inner dimension

Outer dimension

$m \times p$

Result

Matrix operations (Matrix Multiplication)

Example:

$A = \begin{bmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -4 & -3 \\ 2 & 0 \end{bmatrix}$

3x2 2x2

$AB = \begin{bmatrix} \checkmark & \checkmark \\ \checkmark & \checkmark \\ \checkmark & \square \end{bmatrix}$ 3x2

row 3 col. 2

$2*(-3) + 7*0 = -6$

$BA =$ Can not multiply

2x2 3x2

Example:

$A = \begin{bmatrix} -2 & 4 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ 2x2

2x2 2x2

row 1 col 2

$-2*6 + 4 \times 4 = -12 + 16 = 4$

$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 2 \end{bmatrix}$

$= BA = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

2x2 2x2

$3*4 + 6*2 = 24$

Matrix operations (Matrix Multiplication)

Properties:

$A\,B \neq B\,A$ Ex: $A_{3\times 2}\;B_{2\times 2}$

$(A\,B)C = A\,(B\,C)$ مع مراعاة الترتيب

$A\,(B + C) = A\,B + A\,C$

$A\,C + A\,B = A\,(C + B)$ $A\,C + B\,A \neq A(C + B)$

$A\,B + C\,B = (A + C)\,B$

$A\,I = I\,A = A$

$(c\,A)B = c\,(A\,B)$

Matrix operations (Matrix Multiplication)

Common misunderstanding in matrices:

$$A\,B = O \quad \textbf{does not imply} \quad A = O \ \text{ or } \ B = O$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A\,C = B\,C \quad \textbf{does not imply} \quad A = B$$

A C ✓ D ✓

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

B C ✓ D ✓

$$\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

Matrix operations (Matrix Power)

$$A^i A^k = A^{i+k}$$

$$\left(A^i\right)^k = A^{i\,k}$$

$$A^2 = A\,A$$

$$A^3 = A\,A\,A$$

$$\vdots$$

Square

Matrix multiplication

$$A^2 = \underset{m \times n}{A}\ \underset{m \times n}{A}$$

$m = n$

Examples:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$A^0 = I$$

$$I^2 = I \;\rightarrow\; I^n = I$$

Matrix operations (Matrix Transpose)

$$[A^t]_{ij} = [A]_{ji}$$

If A is $m \times n$

then A^t is $n \times m$

Examples:

row 1 , Col. 2

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{3} & \mathbf{7} \end{bmatrix}^t =$$

2x3

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 7 \end{bmatrix}$$

row2, Col1

3x2

Matrix operations (Matrix Transpose)

Properties:

$$(A^t)^t = A$$

$$(A \pm B)^t = A^t \pm B^t$$

$$(A\,B)^t \neq A^t\,B^t \qquad\qquad (A\,B)^t = B^t\,A^t$$

$$(c\,A)^t = c\,A^t$$

Matrix operations (Matrix Transpose)

Definitions:

Symmetric matrix (square)	Skew Symmetric matrix (square)
$A^t = A$	$A^t = -A$
$a_{ij} = a_{ji}$	$a_{ij} = -a_{ji}$

$$P^t = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 5 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

$=$

$$P = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 5 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

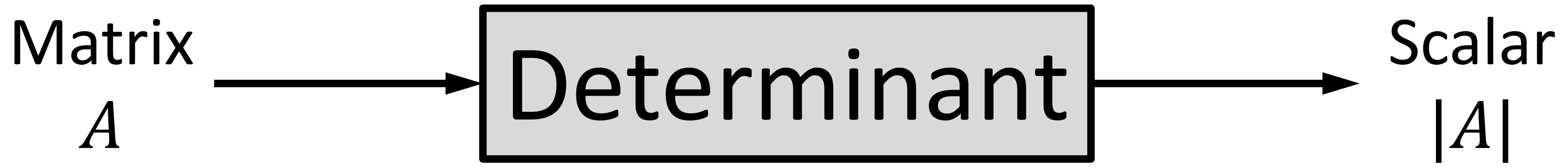
$$Q = \begin{bmatrix} \boxed{0} & 1 & -3 \\ -1 & \boxed{0} & 4 \\ 3 & -4 & \boxed{0} \end{bmatrix}$$

Main diagonal elements are zeros

Diagonal matrices are symmetric

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a matrix



Determinant of a 2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Examples:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -5 \end{bmatrix} \rightarrow det(A) = |A| = \begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix} = (2)(-5) - (-1)(3) = -10 + 3 = -7.$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \rightarrow det(A) = |A| = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = (2)(5) - (1)(3) = 10 - 3 = 7$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \rightarrow det(A) = |A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = (2)(2) - (1)(4) = 4 - 4 = 0.$$

determinant not absolute

$$A = [-5] \rightarrow det(A) = |A| = -5$$

Note:

1. The determinant of a matrix can be positive, zero, or negative.
2. The determinant of a matrix of order 1 is defined simply as the entry of the matrix.

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Top left element

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 *matrix*

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 *matrix*

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ *matrix*

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 2 $= \begin{vmatrix} 4 & 0 \\ 5 & 2 \end{vmatrix} = 8 - 0 = 8$

Cofactor of 2 $= +8$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 1 $= \begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = 4 - 0 = 4$

Cofactor of 1 $= -4$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of $-1 = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 10 - (-4) = 14$ Cofactor of $-1 = 14$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & \cancel{4} & \cancel{0} \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 2 $= \begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = 2 - (-5) = 7$

Cofactor of 2 $= -7$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 4 $= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$

Cofactor of 4 $= +3$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of $0 = \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} = 10 - (-1) = 11$

Cofactor of $0 = -11$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of $-1 = \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} = 0 - (-4) = 4$

Cofactor of $-1 = +4$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 5 $= \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 0 - (-2) = 2$

Cofactor of 5 $= -2$

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Minors and Cofactors of a Matrix

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor of 2 $= \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 8 - 2 = 6$

Cofactor of 2 $= +6$

Example

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, find the cofactors of the elements a_{23} and a_{31}

Solution

$$\text{Cof. of } a_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\left[a_{11}a_{32} - a_{31}a_{12}\right]$$

[illegible]

Cof. of a_{31} - - - -

Determinant of a $n \times n$ matrix where $n > 2$

Definition: The Determinant of a Matrix

If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors. That is,

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}.$$

Note: Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding by any row or column.

Determinant of a $n \times n$ matrix where $n > 2$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

قاعدة الإشارات

Example:

$$\begin{array}{c} + \\ - \\ + \end{array} \begin{vmatrix} 1 & 3 & -2 \\ 0 & 5 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 5 & 1 \\ -1 & 2 \end{vmatrix} - 0 + 4 \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= (10+1) - 0 + 4(3+10) = 11 + 72$$

$$= 83$$

Example: Find the determinant of

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & \end{array} \\ \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \end{array}$$

Solution

$$|A| = +3 \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = 3\left\{ 0 + 2 \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right\}$$

$$= 3\left[2(2-6) - 3(-4-3) \right]$$

$$=$$

Determinant of a 3×3 matrix – Special for 3×3 matrices

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Another method using row operations to find det.

$$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right.$$

Example:

$$(-6)+(0)+(-1) = -7$$

$$A = \left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ -2 & 4 & 1 & -2 & 4 \end{array}\right.$$

$$2+(0)+(-12) = -10$$

$$|A| = -10 - (-7) = -3$$

Inverse of a matrix

Definition: The inverse of an $n \times n$ matrix **A** is an $n \times n$ matrix **B** having the property that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$$

$\boldsymbol{B}$ is called the *inverse* of $\boldsymbol{A}$ and is usually denoted by $\boldsymbol{A}^{-1}$.

If a square matrix has an inverse, it is said to be *invertible* or *nonsingular* ($|\boldsymbol{A}| \neq \boldsymbol{0}$).

If it doesn't possess an inverse, it is said to be *singular*. [$|A| = 0$]

Example: The inverse of $\overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$ is $\overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$ because

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Calculating the inverse of a matrix

Inverse of a 2×2 matrix

لا تنسى

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} = \frac{1}{1*4-2*3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Calculating the inverse of a matrix

Lec 3

Inverse of a $n \times n$ matrix

Another method to find A^{-1}

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$$

$$\mathrm{adj}(A) = \mathrm{cof}(A)^t$$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$

To find the determinant, choose any row or column and multiply each element by its cofactor. The determinant is the sum of these.

Step 1: Find the determinant

Expanding by the second row:

element * Cofactor

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Determinant = $-2\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 0$

$$= -2(1\times 2-(-1)\times 5)+4(2\times 2-(-1)\times(-1))-0$$

$$= -14+12$$

$$= -2 \neq 0$$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

$\boxed{\text{Not}}$ multip.

Cofactor of 2 $= + \begin{vmatrix} 4 & 0 \\ 5 & 2 \end{vmatrix} = 8 - 0 = 8$

Matrix of cofactors = $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 1 $= -\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = -(4-0) = -4$

Matrix of cofactors = $\begin{pmatrix} 8 & & \\ & & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Step 2: Replace each element by its cofactor.

Cofactor of -1 $= +\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 10 - (-4) = 14$

Matrix of cofactors $= \begin{pmatrix} 8 & -4 & \\ & & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 2 $= -\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = -(2-(-5)) = -7$

Matrix of cofactors $= \begin{pmatrix} 8 & -4 & 14 \\ & & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 4 $= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$

Matrix of cofactors = $\begin{pmatrix} 8 & -4 & 14 \\ -7 & & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 0 $= -\begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} = -(10-(-1)) = -11$

Matrix of cofactors $= \begin{pmatrix} 8 & -4 & 14 \\ -7 & 3 & \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of -1 $= \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} = 0 - (-4) = 4$

Matrix of cofactors = $\begin{pmatrix} 8 & -4 & 14 \\ -7 & 3 & -11 \\ & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 5 $= -\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = -(0-(-2)) = -2$

Matrix of cofactors = $\begin{pmatrix} 8 & -4 & 14 \\ -7 & 3 & -11 \\ 4 & & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$ $\qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Step 2: Replace each element by its cofactor.

Cofactor of 2 $= \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 8 - 2 = 6$

Matrix of cofactors $= \begin{pmatrix} 8 & -4 & 14 \\ -7 & 3 & -11 \\ 4 & -2 & \end{pmatrix}$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

Find the inverse of $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 5 & 2 \end{pmatrix}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Matrix of cofactors $= \begin{pmatrix} 8 & -4 & 14 \\ -7 & 3 & -11 \\ 4 & -2 & 6 \end{pmatrix}$
$\mathrm{cof}(A)$

Determinant $= -2$
$\det(A)$

transp.

Step 3:

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{-2}\begin{bmatrix} 8 & -7 & 4 \\ -4 & 3 & -2 \\ 14 & -11 & 6 \end{bmatrix}$$

$$\mathrm{adj}(A) = \mathrm{cof}(A)^t$$

Calculating the inverse of a matrix

Inverse of a $n \times n$ matrix

$$A = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} \begin{matrix} -1 & 2 \\ 2 & 1 \\ 4 & -2 \end{matrix}$$

$$|A| = (-5+0+12) - (-12+0+20)$$
$$= 7 - 8 = -1 \neq 0$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} & +\begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^{t} = -1 \begin{bmatrix} 5 & -10 & -8 \\ & & \\ & & \end{bmatrix}^{t}$$

Calculating the inverse of a matrix

<u>Inverse of a $n \times n$ matrix</u>

$$A = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-7} & \mathbf{3} \end{bmatrix} \rightarrow |A| = 6 - 7 = -1$$

det.

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} +|3| & -|-7| \\ -|-1| & +|2| \end{bmatrix}^t = -\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}^t = \begin{bmatrix} -3 & -1 \\ -7 & -2 \end{bmatrix}$$

$$\equiv \frac{1}{-1} \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$$

Matrix operations (Inverse)

Properties:

- A is a square matrix
- $(A^{-1})^{-1} = A$
- $(A\,B)^{-1} = B^{-1}\,A^{-1}$
- $(A^t)^{-1} = (A^{-1})^t$
- $\left(A^k\right)^{-1} = (A^{-1})^k$

- **If $A^{-1} = A^t$ then A is called "*orthogonal matrix*".**