7-3 Conditional prob. + independence

14-3
21-3 Discrete RVs.

Probability and Statistics

28-3 Revision

30-3 Midterm

DR. AHMED TAYEL

Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University

ahmed.tayel@alexu.edu.eg

1.4 Conditional probability

What is conditional probability

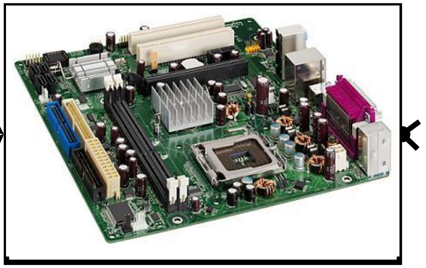
احتمال ب حدث المطلوب

condn.l info.

P(A given B) → P(A | B)

S1

S2



Good

Defective

	S_1	S_2	
G	99	96	195
D	1	4	5
	100	100	200

N_a

N_s

$N_{a\,new}$

$N_{s\,new}$

P(Defective) = ? $\frac{N_a}{N_s} = \frac{5}{200}$

P(Defective | S_1) = ? $\frac{N_{a\,new}}{N_{s\,new}} = \frac{1}{100}$

New info.

≡ Reduction to the sample space based on the new info.

Example

A die is thrown once. If the appearing number is even, what is the probability it is greater than 3?

info./condn.

$S = \{1, 2, 3, 4, 5, 6\}$

The new sample space

$P(\# > 3 \mid \# \text{ even}) = \frac{2}{3}$

Example

A die is thrown twice. If a six appears, what is the probability of getting a sum greater than 8?

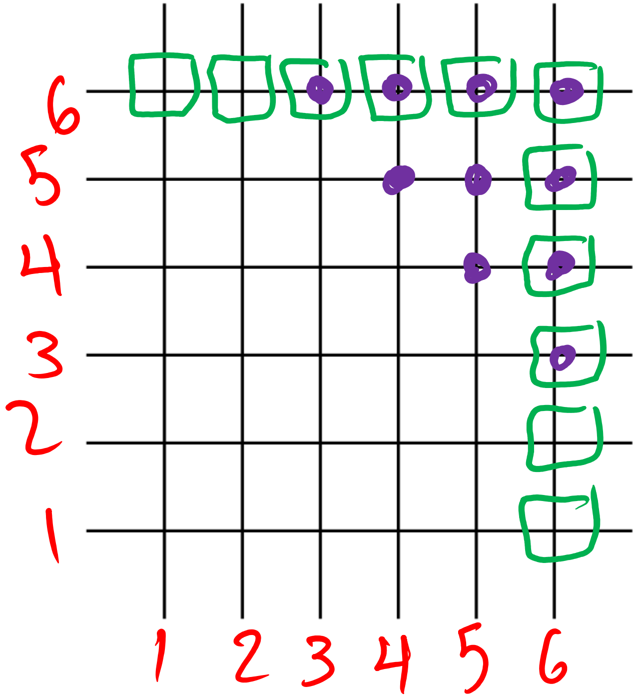
info. / cond^n

B

question

A

$\Box$: The new sample space



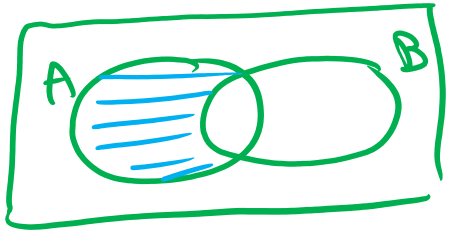
$$P(A|B) = \frac{7/36}{11/36}$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\text{intersection}}{\text{Condition}}$$

Conditional probability definition

$$\boxed{P(A\mid B)=\frac{P(A\cap B)}{P(B)}}\qquad \frac{\text{intersection}}{\text{condn.}}$$



Example

Given P(A) = 0.3, P(B) = 0.6, $P(A\cap B)$ = 0.1

Find $P(A\mid B)$, $P(B\mid A)$, $P(A\mid B^c)$

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{0.1}{0.6}=1/6$$

$$P(B\mid \underset{\text{condn.}}{A})=\frac{P(A\cap B)}{P(A)}=\frac{0.1}{0.3}=1/3$$

$$P(A\mid B^c)=\frac{P(A\cap B^c)}{P(B^c)}=\frac{P(A)-P(A\cap B)}{1-P(B)}=\frac{0.3-0.1}{1-0.6}=\frac{0.2}{0.4}=\frac{1}{2}$$

Multiplication rule

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

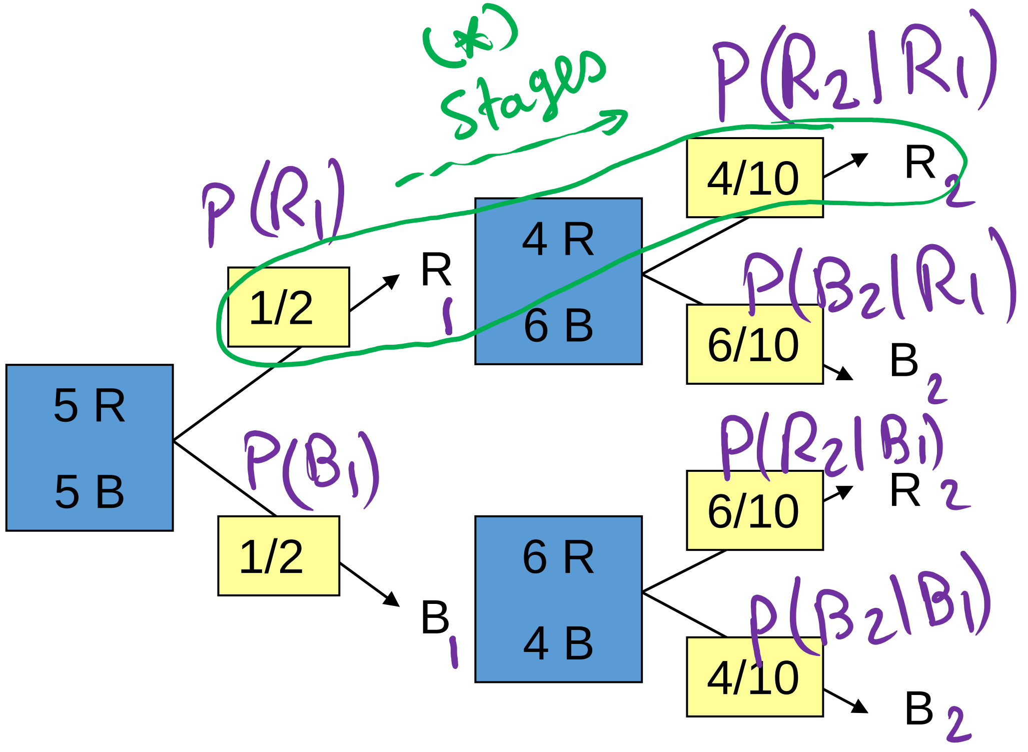
$$P(A\cap B) = P(B)\,P(A|B) = P(A)\,P(B|A)$$

if the balls is he without replacement
then the probability is 1/4
because there are equally likely

ple

A box contains 5 black balls and 5 red balls. A ball is selected at random and is replaced with a ball of the other color. Then a second ball is selected at random.

What is the probability that the two balls are red?



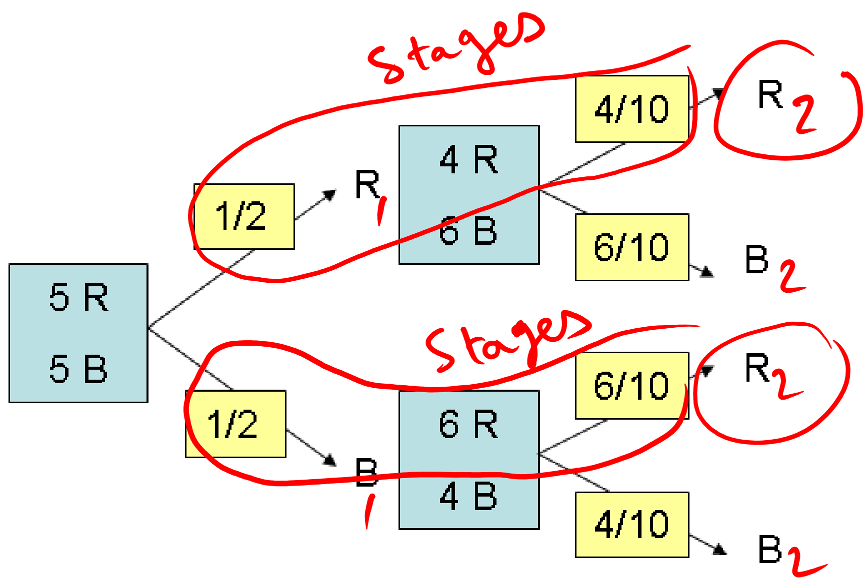
$$P(R_1\cap R_2) = P(R_1)\,P(R_2|R_1) = \frac{1}{2} * \frac{4}{10}$$

Example (cont'd)

(b) What is the probability that the second ball is red?

$$P(R_2) = P(R_1)P(R_2|R_1) + P(B_1)P(R_2|B_1)$$

$$= \frac{1}{2} * \frac{4}{10} + \frac{1}{2} * \frac{6}{10} = \checkmark$$



or Alternatively (+)

"Total probability law"

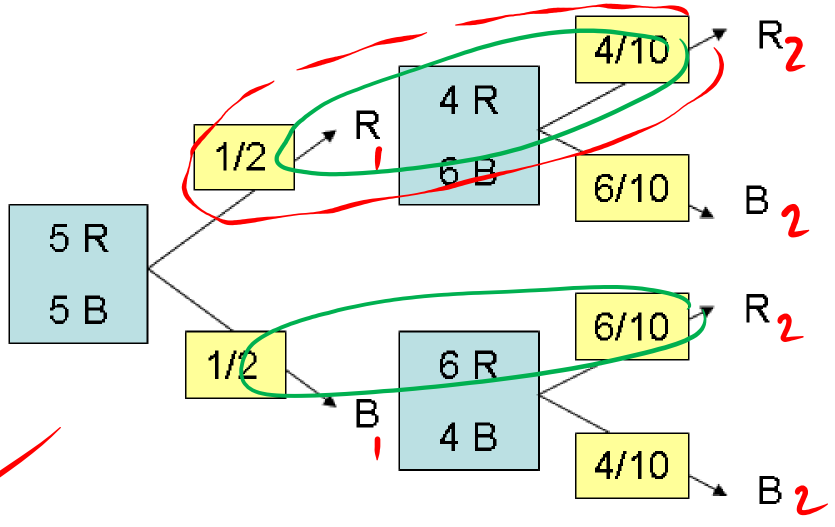
Example (cont'd)

Bayes' theorem

condn.l info.

(c) If the second ball is red, what is the probability that the first ball was also red?

$P(R_2|R_1)$



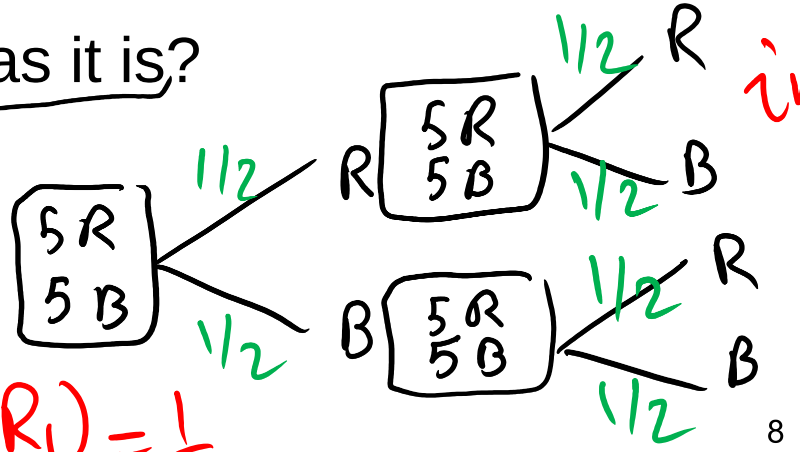
dep.

$$P(R_1 | R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)}$$

$$= \frac{\frac{1}{2} * \frac{4}{10}}{\frac{1}{2} * \frac{4}{10} + \frac{1}{2} * \frac{6}{10}} = \checkmark$$

(d) Repeat (c) if the first ball is replaced as it is?

indep.

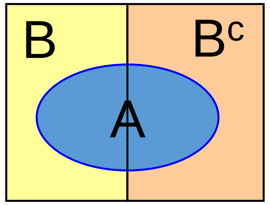


$$P(R_1|R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{1/2 * 1/2}{1/2 * 1/2 + 1/2 * 1/2} = \frac{1}{2}$$

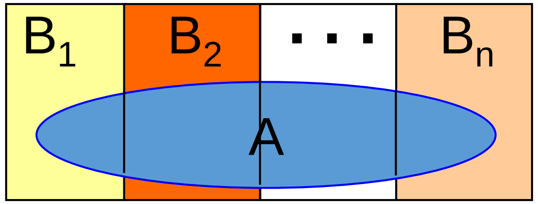
by logic → indep. trials "neglect the condn." $= P(R_1) = \frac{1}{2}$

Law of total probability

$P(R_2) = P(R_1)P(R_2|R_1) + P(B_1)P(R_2|B_1)$



$$P(A) = P(B)\,P(A\,|\,B) + P(B^c)\,P(A\,|\,B^c)$$



$$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A\,|\,B_i)$$

Example

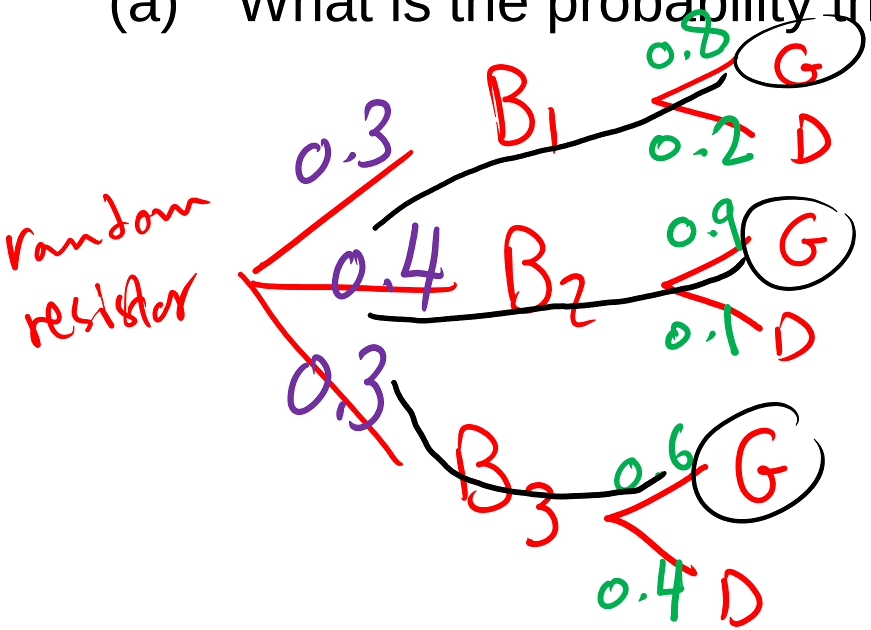
A company has three machines {B1, B2, B3} for making resistors.

80% of the resistors produced by B1 are acceptable. The percentages for B2, B3 are 90% and 60% respectively.

Each hour, B1 produces 3000 resistors, B2 produces 4000 resistors and B3 produces 3000 resistors.

$$P_{B_1} = \frac{3000}{3000+4000+3000} = 0.3$$

(a) What is the probability that a resistor selected at random is acceptable?



$$P(G) = 0.3 * 0.8 + 0.4 * 0.9 + 0.3 * 0.6$$

$= \checkmark$

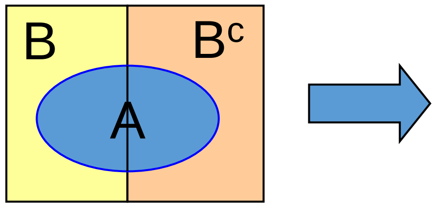
$$P(B_3 | G) = \frac{P(B_3 \cap G)}{P(G)}$$

$$= \frac{0.3 * 0.6}{}$$

Bayes' theorem

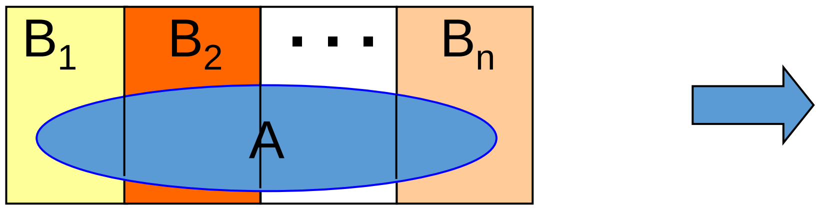
$$\frac{\text{intersection}}{\text{cond}^n.} = \frac{P(A \cap B)}{P(A)}$$

multip. Rule



$$P(B \mid A) = \frac{P(B)\,P(A \mid B)}{P(B)\,P(A \mid B) + P(B^c)\,P(A \mid B^c)}$$

Total prob. law



$$P(B_j \mid A) = \frac{P(B_j)\,P(A \mid B_j)}{\sum_{i=1}^{n} P(B_i)\,P(A \mid B_i)}$$

Example (cont'd)

(b) If a resistor selected at random is acceptable, what is the probability it is produced by B3?

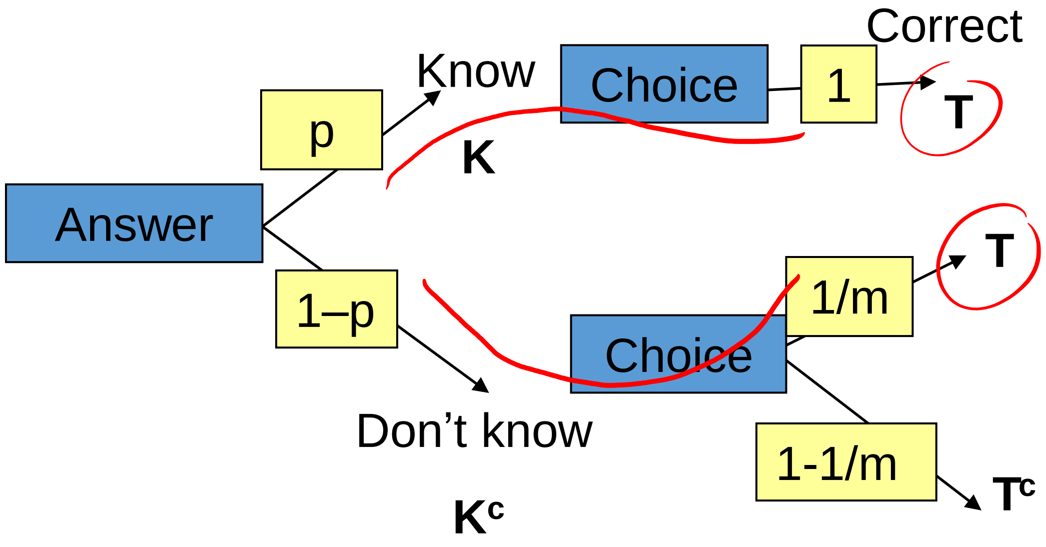
$$P(B_3 \mid G)$$

Example

In a multiple-choice test, the student either knows the answer (with probability p) or guesses (with probability 1 – p).
The number of choices for each question is m.

(a) What is the probability that a student answers a question correctly?

(b) If a student answered a question correctly, what is the probability he knew the answer?



ⓐ $P(T) = p * 1 + (1-p) * \frac{1}{m}$

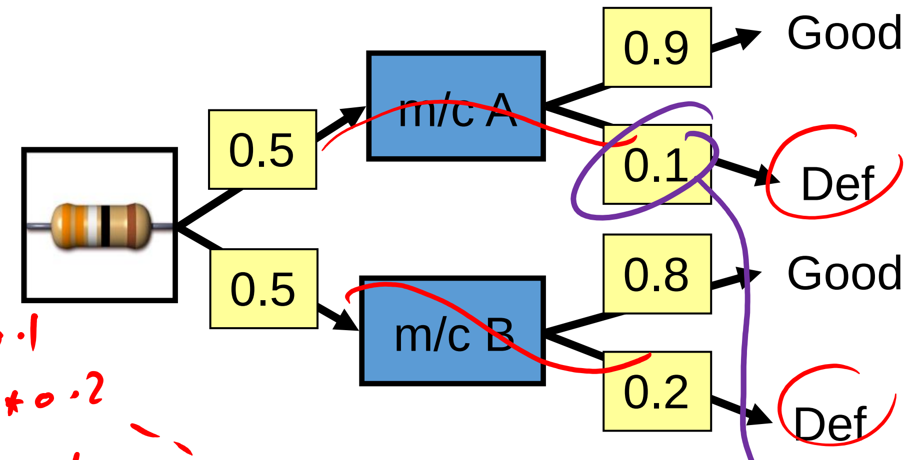
as $m \uparrow\uparrow$ $P(T) = p$

ⓑ $P(K|T)$ Bayes

$= \frac{P(K \cap T)}{P(T)} = \frac{p * 1}{p + \frac{1-p}{m}}$

as $m \uparrow\uparrow$ $P(K|T) = 1$

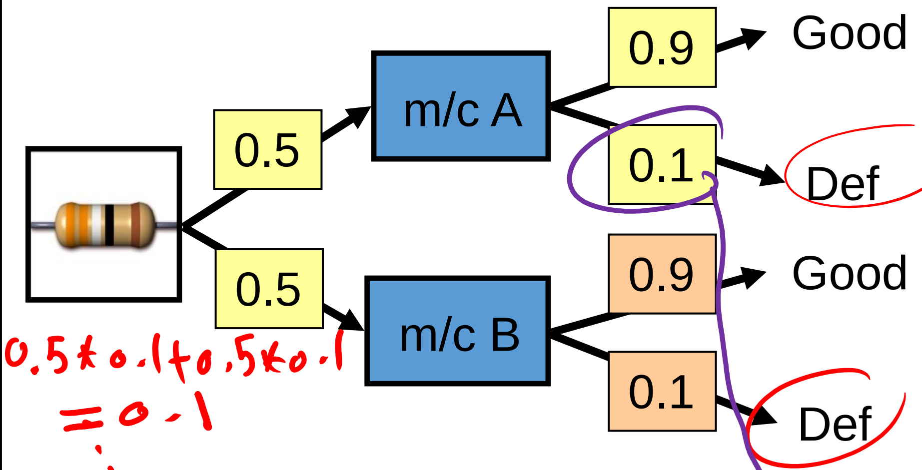
Introduction to **independence**



$P(D) = 0.5 * 0.1 + 0.5 * 0.2 = 0.05 + 0.1 = 0.15$

P(Def) = ? , P(Def | m/c A) = ?

0.1



$0.5 * 0.1 + 0.5 * 0.1 = 0.1$

indep.

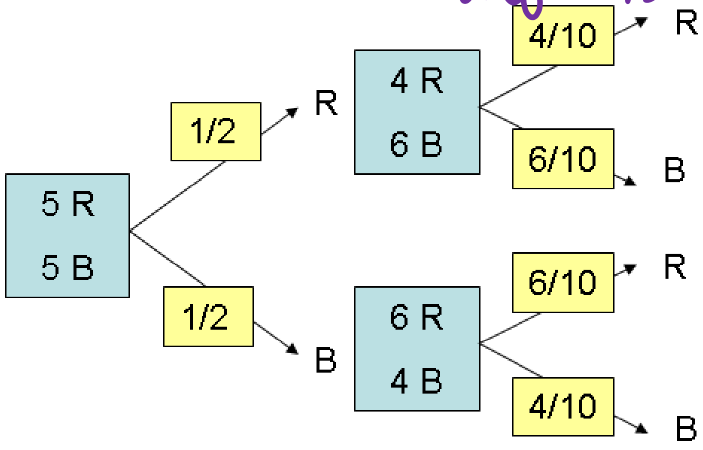
P(Def) = ? , P(Def | m/c A) = ? 0.1

info. is irrelevant

Are R_1 and R_2 independent? No

When are they independent?

Replace the ball as it is



Replacement with other color "Dependent"

Independent events

Also. $P(A|B^c) = P(A)$

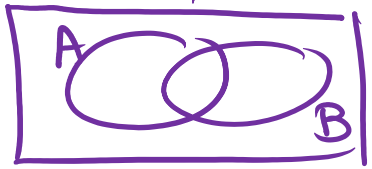
A are B independent if
$P(A \mid B) = P(A)$

⟺

A are B independent if
$P(A \cap B) = P(A)\,P(B)$

$P(A|\cancel{B}) = \frac{P(A \cap B)}{P(B)}$ indep

Note
intersection exist $\neq 0$



Example

Given that A and B are independent with

$P(A) = 1/2$, $P(A \cup B) = 2/3$.

A & b

Compute

(a) $P(A \mid B)$ $= \frac{P(A \cap B)}{P(B)} \xleftrightarrow{\text{indep.}} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$ direct

irrelevent

(b) $P(B)$

(c) $P(A \mid B^c)$

$$\begin{cases} P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}^{\,P(A)P(B)} \\ 2/3 = 1/2 + P_B - 1/2\, P_B \\ \frac{1}{2} P_B = \frac{1}{6} \rightarrow P_B = \frac{2}{6} = \frac{1}{3} \end{cases}$$

$$P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P_A - \cancel{P_{A\cap B}}^{\,P_A P_B}}{1 - P_B} = \frac{P_A(1-P_B)}{1-P_B} = P_A$$

Independent events vs Mutually exclusive events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Mutually exclusive events

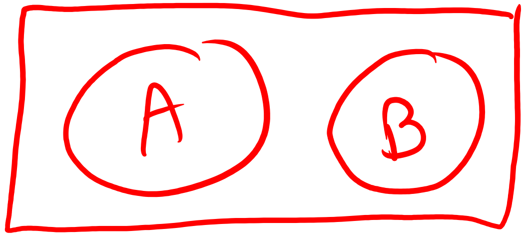
$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) - 0$

$\frac{\cancel{P(A \cap B)}}{P(B)} = P(A \mid B) = 0$

0

$P(B \mid A) = 0$



Independent events

$P(A \cap B) = P(A)\,P(B)$

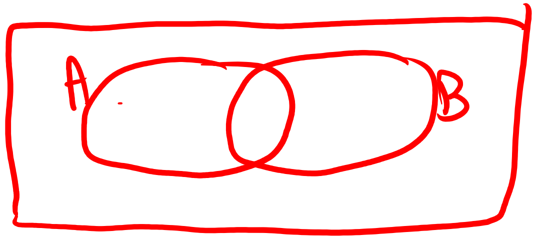
$P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$P(A \mid B) = P(A) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$

irrelevant

$P(B \mid A) = P(B)$

$P(A \mid B^c) = P(A)$



Some reliability problems

Example

What is the probability that a ***series*** operation succeeds?

[p : P(Work),
Elements behave independently]

operating

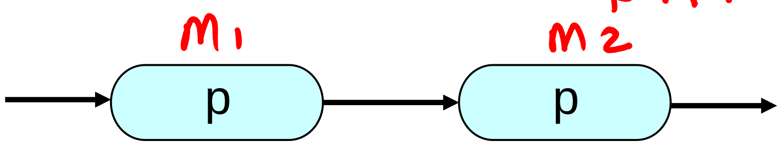
$m_{1_W} \cap m_{2_W}$

Failure

$m_{1_F} \cap m_{2_W}$

$m_{1_W} \cap m_{2_F}$

$m_{1_F} \cap m_{2_F}$



$P(m_{1_W} \cap m_{2_W}) = P(m_{1_W})\,P(m_{2_W})$

indep.

$= p * p = p^2$

Example

Parallel operation succeeds?

operating

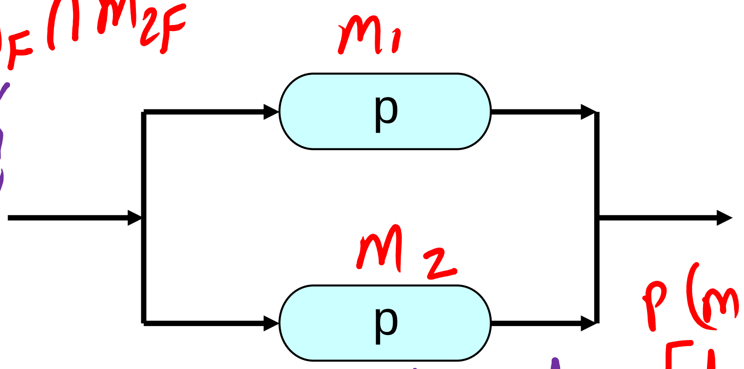
$m_{1_W} \cap m_{2_W}$

$m_{1_W} \cap m_{2_F}$

$m_{1_F} \cap m_{2_W}$

Failure

$m_{1_F} \cap m_{2_F}$



$P(m_{1_F} \cap m_{2_F})$

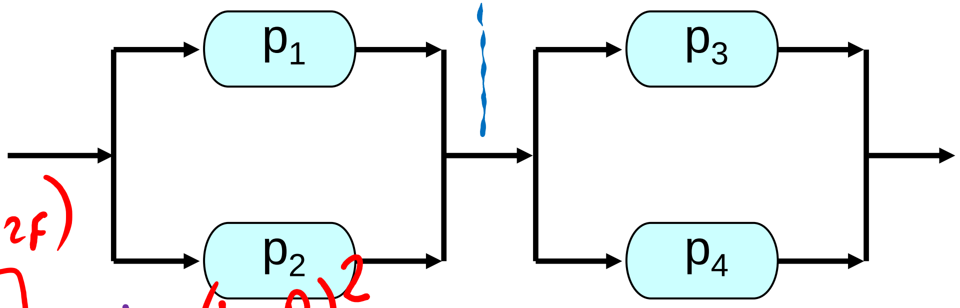
get Complement $= 1 - [\text{failure}] = 1 - (1-p)^2$

Example

Series-parallel operation succeeds?

series

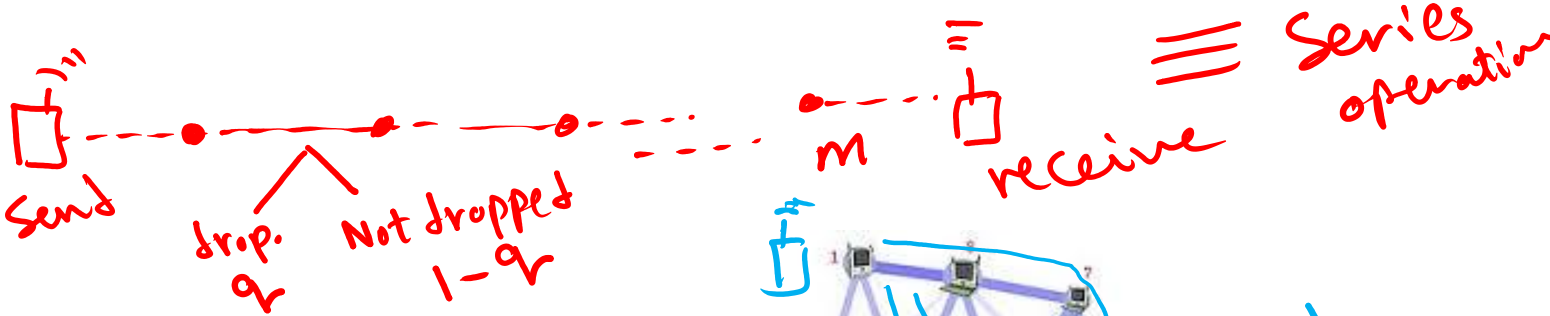
$[1-(1-p_1)(1-p_2)] \times [1-(1-p_3)(1-p_4)]$



More examples

dropped
many
altern.

Not dropped

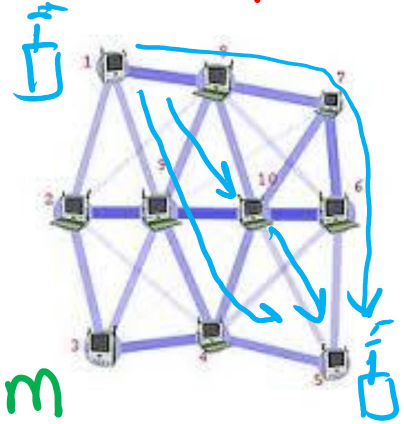


Example

In a path of **m** nodes, what is the probability that a packet is dropped?

one possibility

- Each node has a dropping probability **q**.
- Nodes behave independently.



select min. no. of towers

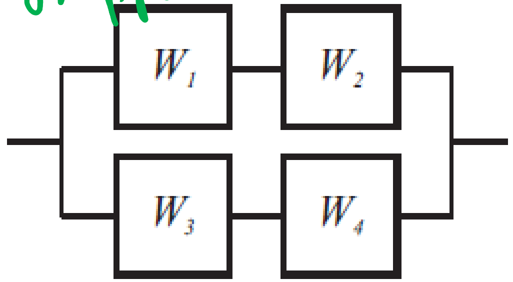
Not dropped at all nodes

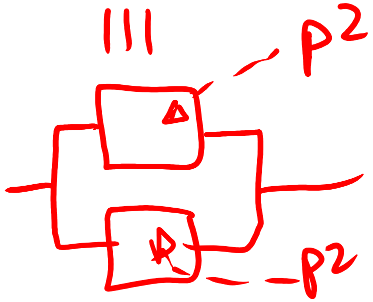
$$P_{drop} = 1-\underbrace{(1-q)^m}_{\text{not dropped at all nodes}}$$

Example

What is the probability that the shown operation succeeds?

- Each component succeeds with probability **p**.
- All components behave independently.





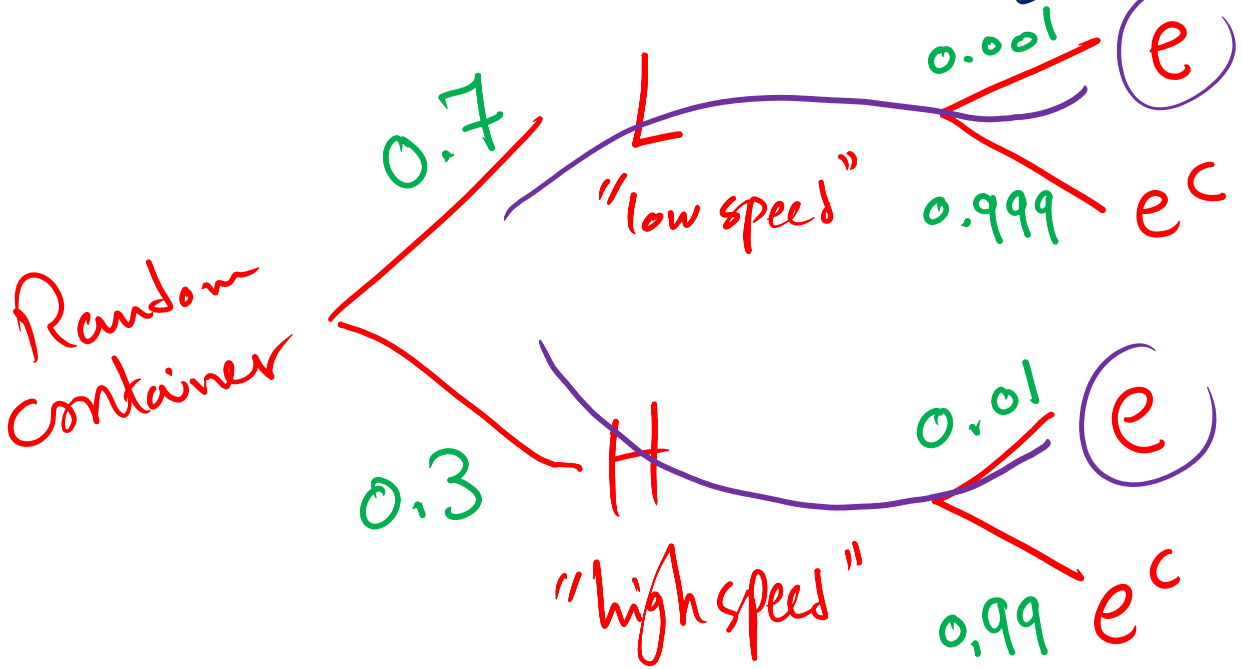
$$P_{success} = 1-(1-p^2)^2$$

More problems

Ex: In an automated filling operation, the probability of an incorrect fill when the process is operated at low speed is 0.001. When the process is operated at high speed, the probability of an incorrect fill is 0.01. Assume that 30% of the containers are filled when the process is operated at high speed and the remainder are filled when the process is operated at low speed.

(a) What is the probability of an incorrectly filled container?

(b) If an incorrectly filled container is found, what is the probability that it was filled during the high-speed operation?



(a) $P(e) = 0.7 * 0.001 + 0.3 * 0.01$

$= \checkmark$

(b) $P(H \mid e)$

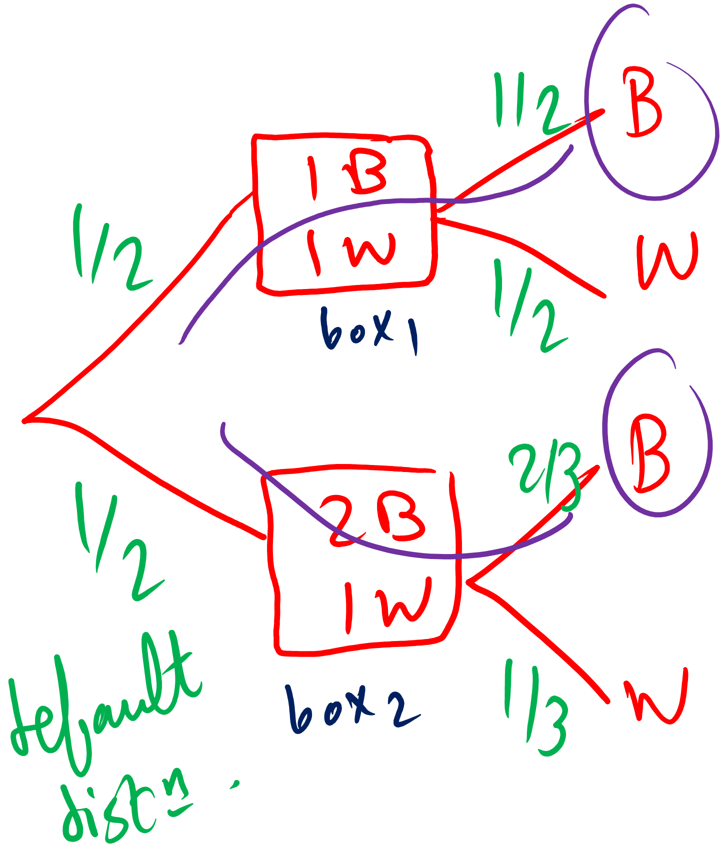
$= \frac{P(H \cap e)}{P(e)} = \frac{0.3 * 0.01}{\ }$

Ex: Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random.

a. What is the probability that the marble is black?

b. What is the probability that the first box was the one selected given that the marble is white?

Conds.



(a) $P(B) = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{2}{3}$

(b) $P(box_1 \mid W) = \dfrac{P(box_1 \cap W)}{P(W)}$

$$= \frac{\frac{1}{2} * \frac{1}{2}}{\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{3}} = \checkmark$$

Ex (a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$. Determine $P(A \cap B)$, $P(A^c \cap B)$.

(b) If $P(A|B) = 1$, does this imply that $A = B$? Draw a Venn diagram to explain your answer.

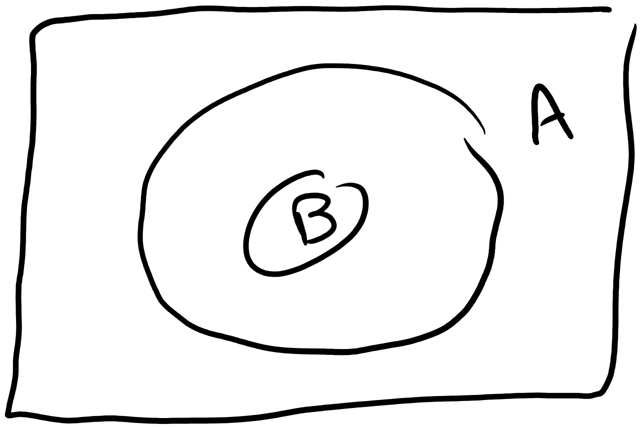
wrong

(a) $$P(A|B) = \frac{P(A \cap B)}{P(B)} \dashrightarrow 0.4 = \frac{P(A \cap B)}{0.5} \to P(A \cap B) = 0.2$$

$$P(A^c \cap B) = P(B) - P(A \cap B) = 0.5 - 0.2 = 0.3$$

(b) $$P(A|B) = \boxed{1 = \frac{P(A \cap B)}{P(B)}}$$

$$P(A \cap B) = P(B)$$



$$B \subseteq A$$

subset or equal to

Ex: Suppose *A* and *B* are mutually exclusive events.
Construct a Venn diagram that contains the three events *A*, *B*, and *C* such that P(A|C)=1 and P(B|C)=0 .

$$P(A|C)=1$$

$$\frac{P(A\cap C)}{P(C)}=1$$

$$P(A\cap C)=P(C)$$

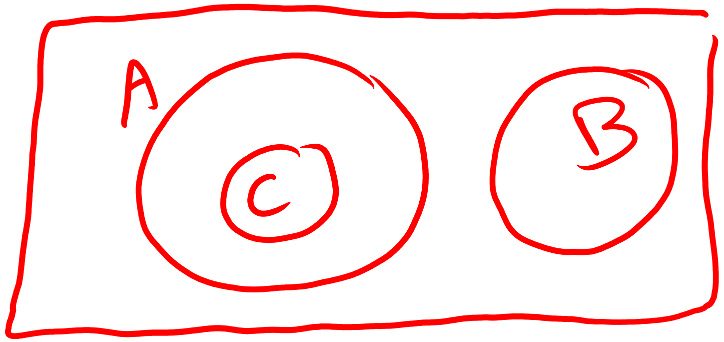
$$C\subseteq A$$

$$P(B|C)=0$$

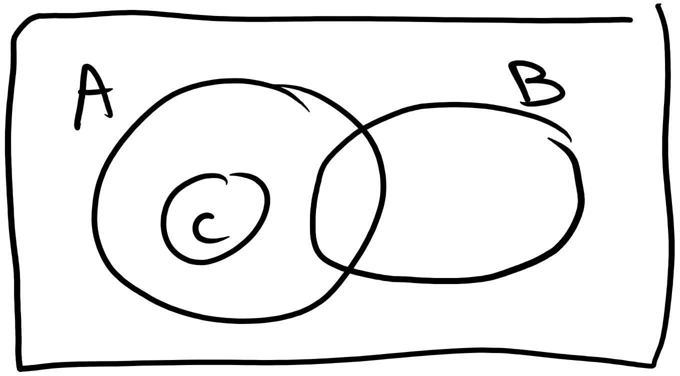
$$\frac{P(B\cap C)}{P(C)}=0$$

$$P(B\cap C)=0$$

B & C are mutually exclusive



if it is not stated that A & B are mutually exclusive



Ex: A Web ad can be designed from 4 different colors, 3 font types, 5 font sizes, 3 images, and 5 text phrases. A specific design in randomly generated by the Web server when you visit the site.

(a) Determine the probability that the ad color is red and the font size is not the smallest one.

(b) Let *A* denote the event that the design color is red and Let *B* denote the event that the font size is not the smallest one. Are *A* and *B* independent events? Explain why or why not.

4 Colors
3 F.type
5 F. Sizes
3 images
5 text

Stages

$$N_s = \underset{\text{Color}}{4} * 3 * \underset{\text{F.Size}}{5} * 3 * 5$$

(a) $$P_a = 1 * 3 * 4 * 3 * 5 = (A \cap B)$$

$$P(A \cap B) = \frac{1 * 3 * 4 * 3 * 5}{4 * 3 * 5 * 3 * 5} = \frac{1}{5} =$$

(b) By logic ⋯→ indep.

$$P_A P_B = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$$

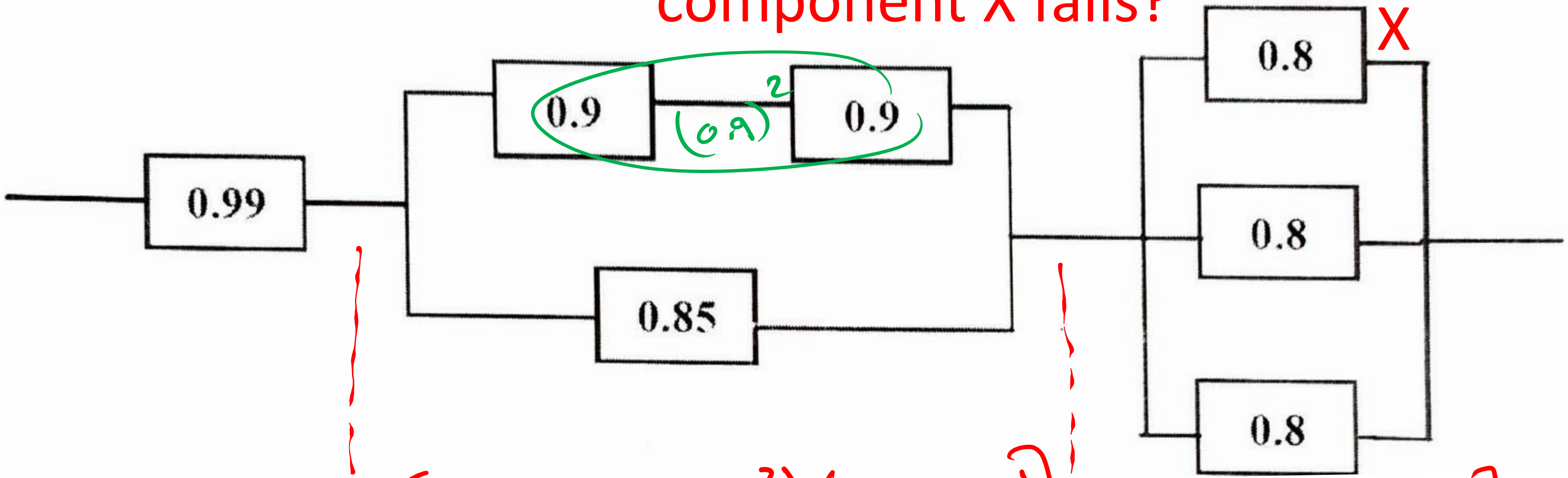
$$P(A) = \frac{1 * 3 * 5 * 3 * 5}{4 * 3 * 5 * 3 * 5} = \frac{1}{4}$$

$$P(B) = \frac{4 * 3 * 4 * 3 * 5}{4 * 3 * 5 * 3 * 5} = \frac{4}{5}$$

∴ indep.

The following circuit operates if and only if there is a path of functional devices from left to right. Assume that devices fail independently and the probability that each device operates is as shown. What is the probability that the circuit operates?

What if component X fails?



$$P_{success} = 0.99 * \left[1-(1-0.9^2)(1-0.85)\right] * \left[1-(1-0.8)^3\right]$$

$$= \checkmark * \checkmark * \left[1-(1-0.8)^2\right] = \checkmark <$$