Linear Algebra

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Outline

1. Linear transformation.
2. Eigenvalues and eigenvectors.
3. Diagonalization.
4. Decoupling.

1. Linear Transformation

Linear Transformation (L.T.): Condition $T(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) = \alpha\, T(\boldsymbol{u}) + \beta\, T(\boldsymbol{v})$

L.T. is given as

Transformation is from R^2 to R^3

Transformation function	Transformation matrix	Transformation of standard basis vectors	Transformation of non-standard basis vectors
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$	$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ 3x2	$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
	$\boxed{T(x) = A\, x} = b$ (3x1; 3x2, 2x1)	$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
	Coefficient matrix	$A = [\boxed{T(e_1)}, \boxed{T(e_2)}]$	$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$? !

$\ker(T)$ is the set of all vectors x where $T(x) = 0$ or $A\,x = 0$
Also known as $NS(A)$

$$\begin{aligned} x \quad\quad &= 0 \\ y &= 0 \\ x + y &= 0 \end{aligned}$$

nullity$(A) = \dim(NS(A))$

$n = \text{rank}(A) + \text{nullity}(A)$

Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \overset{-R_1 + R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$C_2 = -1$

$C_1 - 1 = 1 \rightarrow C_1 = 2$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2\, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - 1 T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Similarly, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Ex 02: Find a basis for the range and kernal of T and state their dimensions.

$$T(x) = A\,x$$

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \sim \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 0 & 2 & -6 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \sim \begin{bmatrix} \boxed{1} & -3 & 5 & -5 \\ 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & \boxed{-4} \end{bmatrix}$$

Basis to $CS(A)$

Pivot columns

$$Range(T) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ -4 \end{bmatrix} \right\}$$

$\boxed{dim = 3}$ $= \rho(A)$

$$\begin{array}{c} t^2 \\ t \\ t^0 \end{array} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 2 & 0 & 0 \\ 0 & -1 & 2 & -4 \end{bmatrix}$$

Find a basis for the span of S where $S = \{ t^2+2t, 2t-1, 2, 3t^2-4 \}$

To get a basis for a subspace

→ First, put the vectors in the columns of a matrix.

→ Continue similar to column space

Ex 02: Find a basis for the range and kernal of T and state their dimensions.

Cont

$$T(x) = A\,x$$

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$$

To get the kernal of T
Solve the homogeneous system

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \sim \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 0 & 2 & -6 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \sim \begin{array}{cccc|cc} x_1 & x_2 & x_3 & x_4 & & \\ \boxed{1} & -3 & 5 & -5 & 0 & Eq1 \\ 0 & \boxed{1} & -3 & 5 & 0 & Eq2 \\ 0 & 0 & 0 & \boxed{-4} & 0 & Eq3 \end{array}$$

Pivot columns

Let $x_3 = t$

From $Eq3$: $-4x_4 = 0 \rightarrow x_4 = 0$

From $Eq2$: $x_2 - 3t = 0 \rightarrow x_2 = 3t$

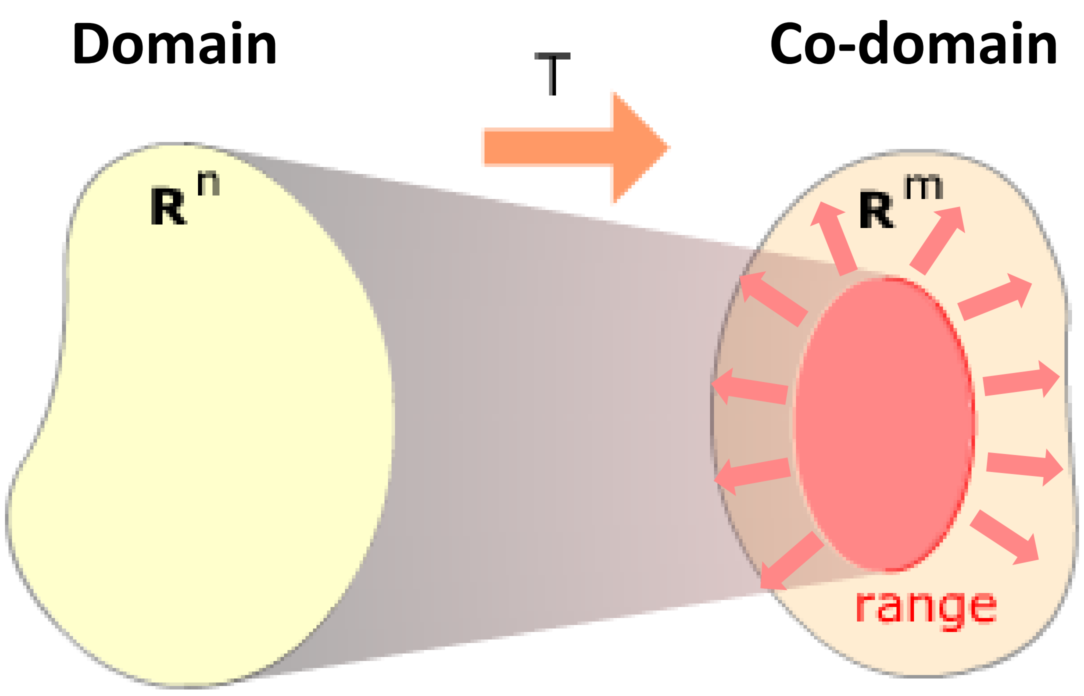
From $Eq1$: $x_1 - 3(3t) + 5t = 0 \rightarrow x_1 = 4t$

$$\Rightarrow \quad Kernal(T) = \begin{bmatrix} 4t \\ 3t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{dim = 1}$

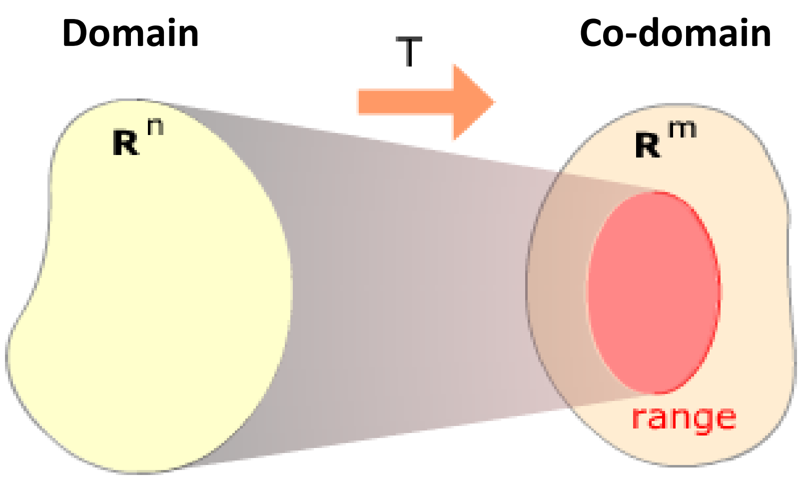
= no. of free vars.

Onto transformation

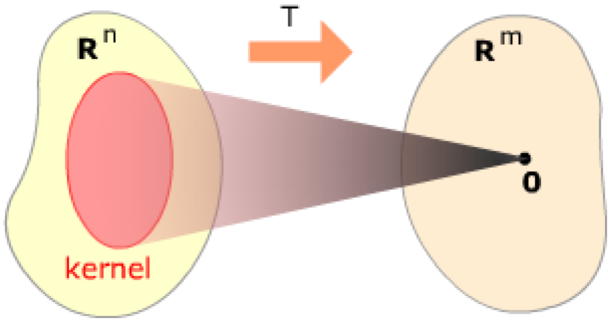


- Range $\equiv$ Co-domain
- Every vector in the co-domain has a preimage.
- $A\,x = b$ is always consistent.
- A has a pivot in each row (in echelon form).
- For $A_{m \times n}$, it must have $m \leq n$

1-to-1 transformation



- Every vector in the range has only one preimage.
- $A\,x = b$ has a unique solution.
- A has no free variables $\equiv$ A has a pivot in each column (in echelon form).
- For $A_{m \times n}$, it must have $m \geq n$.
- Kern(T) has only the zero vector.



Ex 03: For each of the following transformation, determine whether T is onto and/or 1-to-1?

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y + z \end{bmatrix}$$

Already in echelon form

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \end{bmatrix}$$

- Pivot in each row.
- Always consistent.
- Range(T) is the whole R^2

Onto

- Has a free variable.
- Infinite number of solutions.

Not 1-to-1

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

$R^3 \to R^4$

$A_{4\times 3}$

Already in echelon form

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Not onto

1-to-1

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$$

$R^2 \to R^2$

$A_{2\times 2}$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2} \sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & \boxed{2} \end{bmatrix}$$

Onto

1-to-1

$\Big\}$ **Isomorphic**

Ex 04: If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

$$v = [6 \quad 16 \quad 2 \quad 27]^T$$

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix} \underset{\sim}{\text{Gauss}} \begin{bmatrix} \boxed{1} & 3 & 2 & -1 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

- Pivot in each row → **Onto.**
- Pivot in each column → **1-to-1.**

Isomorphic

<u>Ex 04:</u> If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T,
<u>(Cont.)</u> state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

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Basis to $CS(A)$

$$Range(T) = Span\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ 1 \\ 7 \end{bmatrix}\right\}$$

$$dim = rank = 4$$

Ex 04: (Cont.) If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

$$v = [6 \quad 16 \quad 2 \quad 27]^T$$

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix} \underset{\sim}{\text{Gauss}} \left[\begin{array}{cccc|c} \boxed{1} & 3 & 2 & -1 & 0 \\ 0 & \boxed{1} & 3 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Has only the zero solution

$$Ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$dim = 0$$

Ex 04: (Cont.) If **T: x → Ax** find the dimension and a suitable basis for the range and kernel of T, state whether T is onto and/or 1 to 1

Given $A_{3\times 4}$

$R^4 \rightarrow R^3$

kernel Range

&

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 2 & 6 & 8 \\ -1 & 1 & 2 & 1 \\ 3 & 9 & 5 & 7 \end{bmatrix}_{4\times 4}$$

then determine whether the following vector belongs to the range and/or kernel. If it belongs to the range find its pre image:

Nullity = 0

$A\boxed{X} = v$

?!

$$v = [6 \quad 16 \quad 2 \quad 27]^T$$

T is onto then v must have a preimage

$$A = \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 6 \\ 0 & 2 & 6 & 8 & 16 \\ -1 & 1 & 2 & 1 & 2 \\ 3 & 9 & 5 & 7 & 27 \end{array}\right] \overset{\text{Gauss}}{\sim} \begin{bmatrix} \boxed{1} & 3 & 2 & -1 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \begin{array}{l} \blacksquare \; Eq1 \\ \blacksquare \; Eq2 \\ \blacksquare \; Eq3 \\ \blacksquare \; Eq4 \end{array}$$

After solving the system using back substitution

The pre-image of v is $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

unique soln.

as T is 1-to-1

2. Eigenvalues and Eigenvectors

EXAMPLE Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

- Reverse the sign of all elements
- Add λ to the main diagonal elements

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \right| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

SOLUTION The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) - (-12) \\ &= \lambda^2 + 3\lambda - 10 + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) = 0 \end{aligned}$$

Eigen value

$$A\,x = \lambda\,x$$

Square matrix

Eigen vector

Non-zero vector

$\lambda x - Ax = 0$

$(\lambda I - A) x = 0$

$(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$

EXAMPLE Finding Eigenvalues and Eigenvectors

(Continued)

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

Let $\boxed{x_2 = t}$

Eq1 : $x_1 - 4t = 0$

$\boxed{x_1 = 4t}$

SOLUTION

For $\lambda_1 = -1$,

$$(-1)I - A = \begin{bmatrix} -1-2 & 12 \\ -1 & -1+5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{array}{cc|c} x_1 & x_2 & \\ \hline 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

dim = 1 = no. of free vars of $\lambda I - A$,

eigenspac of $\lambda = -1$

for $t = 1$

All scalar multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are eigenvectors of $\lambda = -1$

$\equiv$ eigenspace of $\lambda = -1$

EXAMPLE **Finding Eigenvalues and Eigenvectors**

(Continued)

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

SOLUTION

For $\lambda_2 = -2$,

$$(-2)I - A = \begin{bmatrix} -2-2 & 12 \\ -1 & -2+5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow[\textbf{Elimination}]{\textbf{Gauss}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

dim=1

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

THEOREM 7.3 Eigenvalues of Triangular Matrices

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

or diagonal

EXAMPLE Finding Eigenvalues of Diagonal and Triangular Matrices

Find the eigenvalues of each matrix.

$$\text{(a)}\quad A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\lambda s. = 2, 1, -3$

$\lambda s. = -1, 2, 0, -4, 3$

SOLUTION (a) Without using Theorem 7.3, you can find that

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

Properties of eigenvalues and eigenvectors:

If $\boldsymbol{A}$ is **singular** i.e. has no inverse,
$\boldsymbol{A^{-1}}$ does not exist $\leftrightarrow$ $\boldsymbol{\lambda = 0}$

Kernal
$A x = 0 = \cancel{\lambda}^{0} x$
بشرو
$x \neq 0 \dashrightarrow$ Not 1-to-1

If $\boldsymbol{A}$ is **invertible** i.e. has inverse,

	Eigenvalues	Eigenvectors
$\boldsymbol{A} \rightarrow$	$\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \ldots}$	$\boldsymbol{v_1, v_2, v_3, \ldots}$
$\boldsymbol{A^{-1}} \rightarrow$	$\boldsymbol{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots}$	$\boldsymbol{v_1, v_2, v_3, \ldots}$

	Eigenvalues	Eigenvectors
$\boldsymbol{A} \rightarrow$	$\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \ldots}$	$\boldsymbol{v_1, v_2, v_3, \ldots}$
$\boldsymbol{A^n} \rightarrow$	$\boldsymbol{(\lambda_1)^n, (\lambda_2)^n, (\lambda_3)^n, \ldots}$	$\boldsymbol{v_1, v_2, v_3, \ldots}$

$\boldsymbol{A}$ and $\boldsymbol{A^T}$ have the **same eigenvalues**

Example: If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ has the eigen values $\lambda = 7, -4$. Find the eigen values of A^{-1}, A^T and A^2.

eigenvalues

A $\quad 7, -4$

A^{-1} $\quad \frac{1}{7}, \frac{1}{-4}$ + same eigvecs.

A^T $\quad 7, -4$

A^2 $\quad (7)^2, (-4)^2$ + same eigvecs.

CAYLEY-HAMILTON THEOREM:

- **Statement:** Every square matrix satisfies its own characteristic equation

1. Verify that A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ **satisfies its own characteristic equation and hence find** A^4

$|\lambda I - A| = 0$

Solution:Given A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \; \#$$

To find A^4: From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

3. Find A^{-1} if A = $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem

$A^3 = 2A^2 + 5A - 6I$ "iterative"

Solution: The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ---------- (1)

To find A^{-1}: Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I) \text{ ---------- (2)}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

3. Diagonalization

Eigenvalues	Eigenvectors
$A \rightarrow \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$	$v_1, v_2, v_3, \dots, v_n$

$$A = P\, D\, P^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$P = [v_1 \;\; v_2 \;\; v_3 \;\; \dots \;\; v_n]$$

Corresp. to λ_1 $\quad \lambda_2$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Condition for diagonalization

An $n \times n$ matrix can be diagonalized if it has n independent eigenvectors

■Thm:

Eigenvectors of a matrix corresponding to different (unequal) eigenvalues are independent.

- If $A_{n\times n}$ has n different eigenvalues → Diagonalizable.
- If $A_{n\times n}$ has less than n different eigenvalues → Might be diagonalizable (Check the repeated eigenvalues)

Ex: Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Triang.
$\lambda s = 1, 1$ ~ "repeated"

$A_{2\times 2}$ ↓ reqs. 2 indep. eigvectors

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix}\lambda - 1 & -2\\ 0 & \lambda - 1\end{vmatrix} = (\lambda - 1)^2 = 0$$

Eigenvalue : $\lambda_1 = 1$

let $x_1 = t$
Eq1: $x_2 = 0$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}t\\ 0\end{bmatrix} = t\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\lambda I - A = I - A = \begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix} \sim \begin{array}{cc} x_1 & x_2 \\ \left[\begin{matrix}0 & \boxed{1}\\ 0 & 0\end{matrix}\right] & \end{array} \begin{matrix}0\\ 0\end{matrix} \Rightarrow \text{Eigenvector}: \; p_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$$

@ t=1 selected to avoid fractions

dim = 1

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

$P^{-1}AP = P^{-1}PDP^{-1}P$
diag.

$\underbrace{P^{-1}AP}_{\text{diag.}} = D$
diag.

Ex:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+2)(\lambda-3) = 0$$

$$\text{Eigenvalues}: \lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{t=1} \Rightarrow \text{Eigenvector}: \ p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \frac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \xrightarrow{t=4} \Rightarrow \text{Eigenvector}: \ p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{t = 1} \Rightarrow \text{Eigenvector}: \ p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

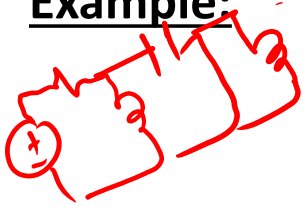
$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} \boxed{\begin{matrix} -1 \\ 0 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \\ 4 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 1 \\ 1 \end{matrix}} \end{bmatrix} \\ \begin{matrix} \text{corresp. to } \lambda_1 & \lambda_2 & \lambda_3 \end{matrix} \end{array}$$

$$\Rightarrow \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

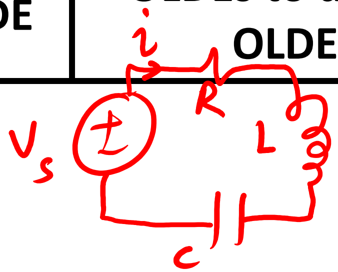
$\lambda_1 \quad \lambda_2 \quad \lambda_3$

4. Decoupling

Example:



$$x_1' = x_1 - x_2 - x_3 \qquad x_1(0) = 0$$
$$x_2' = x_1 + 3x_2 + x_3 \qquad x_2(0) = -1$$
$$x_3' = -3x_1 + x_2 - x_3 \qquad x_3(0) = 10$$



$V_s = V_R + V_L + V_C$

$V_s = iR + L\frac{di}{dt} + \frac{1}{C}q$

Single loop $\frac{dV_s}{dt} = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}i$

$i(t)$ transient resp.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = 0 \rightarrow \lambda = 2, -2, 3$$

$\lambda I - A$ — Eigenvector

$$\rightarrow \lambda = 2 \qquad \left[\begin{matrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{matrix}\right] \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \lambda = -2 \qquad \left[\begin{matrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{matrix}\right] \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\rightarrow \lambda = 3 \qquad \left[\begin{matrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{matrix}\right] \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{X = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i}$$

Example: (Cont.)

$$x_1' = x_1 - x_2 - x_3 \qquad x_1(0) = 0$$
$$x_2' = x_1 + 3x_2 + x_3 \qquad x_2(0) = -1$$
$$x_3' = -3x_1 + x_2 - x_3 \qquad x_3(0) = 10$$

$$\boxed{\boldsymbol{X = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i}}$$

$$\lambda_1 = 2 \qquad \lambda_2 = -2 \qquad \lambda_3 = 3$$

$$p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{\overset{\lambda_1}{2t}} \overset{p_1}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} + c_2 e^{\overset{\lambda_2}{-2t}} \overset{p_2}{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}} + c_3 e^{\overset{\lambda_3}{3t}} \overset{p_3}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

General soln.

$$x_1(t) = -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t}$$
$$x_2(t) = -c_2 e^{-2t} + c_3 e^{3t}$$
$$x_3(t) = c_1 e^{2t} + 4c_2 e^{-2t} + c_3 e^{3t}$$

Ordinary differential equation (ODE)	Ordinary linear differential equation (OLDE)	Method of separation of variables to solve OLDE	Converting Coupled system of OLDEs to decoupled system of OLDEs (Decoupling)	Example

Example: (Cont.)

$$x_1' = x_1 - x_2 - x_3$$
$$x_2' = x_1 + 3x_2 + x_3$$
$$x_3' = -3x_1 + x_2 - x_3$$

$\frac{dx_1}{dt}$

$$x_1(0) = 0$$
$$x_2(0) = -1$$
$$x_3(0) = 10$$

$t=0$ $x_1=0$

General soln.

$$x_1(t) = -c_1e^{2t} + c_2e^{-2t} - c_3e^{3t}$$
$$x_2(t) = -c_2e^{-2t} + c_3e^{3t}$$
$$x_3(t) = c_1e^{2t} + 4c_2e^{-2t} + c_3e^{3t}$$

$$x_1(0) = 0 = -c_1 + c_2 - c_3$$
$$x_2(0) = -1 = -c_2 + c_3$$
$$x_3(0) = 10 = c_1 + 4c_2 + c_3$$

linear system of eqs.

System of linear equations in c_1 , c_2 , and c_3

After solving

$$c_1 = 1 , c_2 = 2 , c_3 = 1$$

Final solution

Particular soln.

$$\boxed{\begin{aligned} x_1(t) &= -e^{2t} + 2e^{-2t} - e^{3t} \\ x_2(t) &= -2e^{-2t} + e^{3t} \\ x_3(t) &= e^{2t} + 8e^{-2t} + e^{3t} \end{aligned}}$$

Next week

Office hours: Tuesday 9:00 AM to 11:00 AM
مبنى اعدادي – الدور الأخير – الجهة البحرية