Linear Algebra

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Outline

1. Review.
2. Basis and dimension.

1. Review

In last lectures

- Vector space.
- Subspace of a vector space.
- Linear combination.
- Span of a vector set.
- Linear dependence/independence.

Vector Spaces

- Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) **u**+**v** is in V Closed

(2) **u**+**v**=**v**+**u**

(3) **u**+(**v**+**w**)=(**u**+**v**)+**w**

(4) V has a zero vector **0** such that for every **u** in V, **u**+**0**=**u**

(5) For every **u** in V, there is a vector in V denoted by –**u** such that **u**+(–**u**)=**0**

very important

Scalar multiplication:

(6) $c\mathbf{u}$ is in V. Closed

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Thm: (Test for a subspace)

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

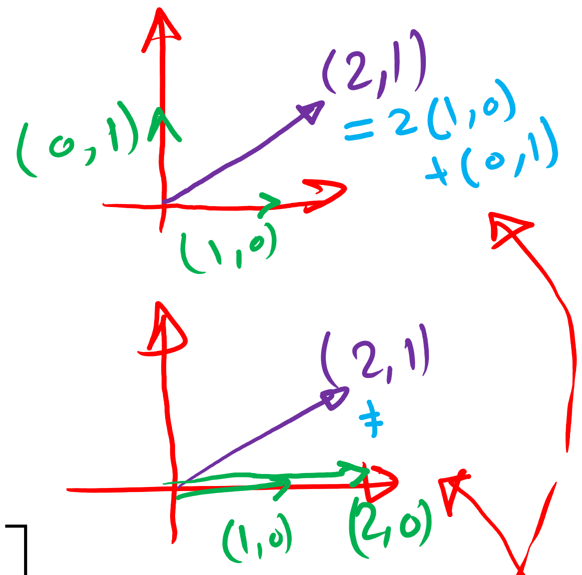
  (1) $\mathbf{0} \in W$.

  (2) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

  (3) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Writing a vector as a linear combination of other vectors

R^2 space



$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Test vec

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consistent

Another test vec.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Contrad.

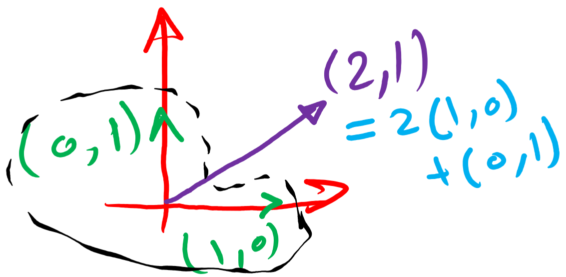
Inconsis.

Span of a set and spanning set of a vector space

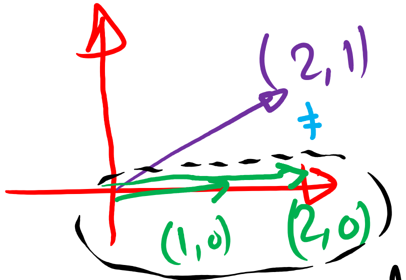
Spanning set for R^2 $\Leftarrow$ Span(S) : all vecs. in R^2

R^2 space

- the span of a set: span (S)



If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S



Span(S): all vecs on x-axis

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

Example: Determine whether the set S spans $\mathbb{R}^2$.

$$S = \{(-1,4),(4,-1),(1,1)\}$$

Test

$$4R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 4 & -1 & 1 & u_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 1 & u_1 \\ 0 & 15 & 5 & 4u_1 + u_2 \end{array}\right]$$

Consistent w/o Condn. on u_1, u_2

$\therefore$ S spans $\mathbb{R}^2$

Example: Determine whether the set S spans $\mathbb{R}^3$.

$$S = \{(1,-2,0),(0,0,1),(-1,2,0)\}$$

Test

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ -2 & 0 & 2 & u_2 \\ 0 & 1 & 0 & u_3 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 0 & 0 & 2u_1 + u_2 \\ 0 & 1 & 0 & u_3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 0 & u_3 \\ 0 & 0 & 0 & 2u_1 + u_2 \end{array}\right]$$

$$\text{span}(S) = \begin{bmatrix} u_1 \\ \cancel{u_2}\ -2u_1 \\ u_3 \end{bmatrix} = \underbrace{u_1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + u_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{plane in } \mathbb{R}^3}$$

Consistent under condn.

$2u_1 + u_2 = 0$

$\therefore$ S does not span $\mathbb{R}^3$

Example: Determine whether the set S spans R^3

$$S = \{(1,-2,0),(0,0,1)\}$$

S does not span R^3

$\#S = 2 < 3$

Linear dependence and Independence

L. Indep.

$S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

Test L.C.

homog. sys. of Linear eqs

unique zero soln $0=0$

Inf. soln including non-zero soln

$2\bar{v}_1 - \bar{v}_2 + \bar{v}_3 = 0$

L. Dep.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

Test

$[\bar{v}_2 \ \bar{v}_3 \mid \bar{v}_1]$

or $[\bar{v}_1 \ \bar{v}_3 \mid \bar{v}_2]$

or $[\bar{v}_1 \ \bar{v}_2 \mid \bar{v}_3]$

(1)" If the equation has only the ***trivial solution*** "($c_1 = c_2 = \cdots = c_k = 0$)" then "$S$" is called ***linearly independent***."

(2)" If the equation has a ***nontrivial solution*** "("i. e., not all zeros"), " then "S" is called ***linearly dependent***."

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Sol:

$$\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

Test

$$\Rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right]}_{\text{Square matrix}} \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

unique $\rightarrow$ L. Indep.

o.w. L. Dep.

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$\neq 0 \rightarrow$ invert. $\rightarrow$ uniq. $\rightarrow$ L. Indep

$= 0 \rightarrow$ inf $\rightarrow$ L. Dep.

- Ex: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$$

$\#S > n \Rightarrow$ L.Dep

Sol:

Test

max no. of pivots $= 3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{array}\right]$$

no. of vars $= 4$

always inf. no. of sols. $\rightarrow$ L.Dep.

Test → w as L.Comb. of v_1, v_2, v_3

Test → Test span

Test → Test indep.

$$\begin{array}{cccccc} v_1 & v_2 & v_3 & w & u & 0 \\ \end{array}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 1 & u_1 & 0 \\ 2 & 0 & 4 & 1 & u_2 & 0 \\ 3 & -1 & -5 & 1 & u_3 & 0 \end{array}\right]$$

Column w: Consist / Inconsi

Column u: consis without conds. or not

Column 0: uniq soln → Indep. ; Inf. soln → Dep.

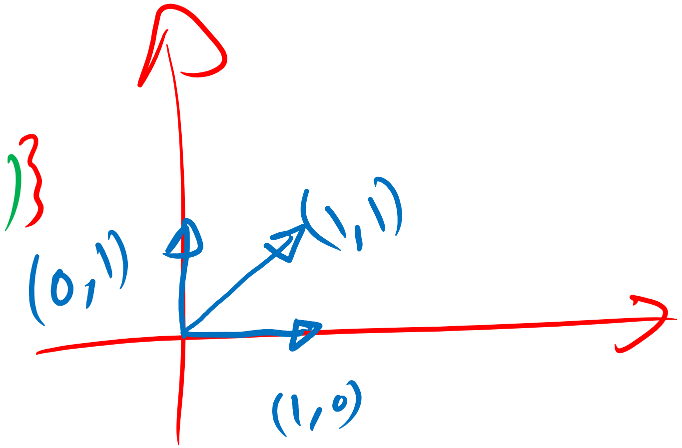
Shortcuts

R^n

- $\#S < n$ → Does not span
- $\#S > n$ → L.Dep.

R^2 space

$S: \{(1,0),(0,1),(1,1)\}$



Spanning set

→ more than enough

→ extra vecs.

Test independence

4. Basis and dimension

set of vecs.

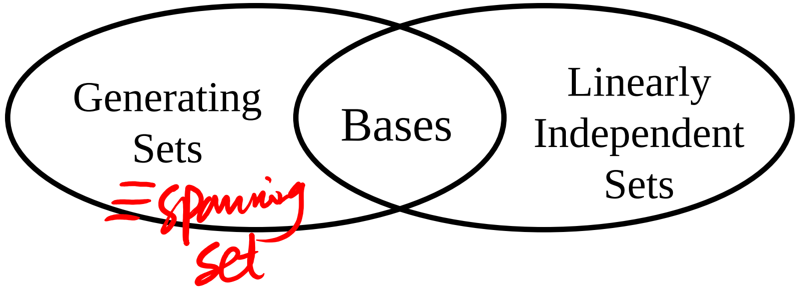
Spanning set + L. Indep.

Basis and Dimension

- Basis:

V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$



$$\begin{cases} (a) & S \text{ spans } V \text{ (i.e., } span(S) = V) \\ (b) & S \text{ is linearly independent} \end{cases}$$

$\Rightarrow$ S is called a **basis** for V

$W = (0.245, -0.0012, 0.45)$

$\rightarrow 0.245(1,0,0) - 0.0012(0,1,0) + 0.45(0,0,1)$

- Note:

The standard basis for R^3:

$\{i, j, k\}$ $\quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

3 vecs = no. of vecs in the basis

The standard basis for R^n:

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\quad \mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$, $\mathbf{e}_n=(0,0,\ldots,1)$

4 vecs. in basis

Ex: R^4 → $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

The standard basis for $m \times n$ matrix space:

$\{ E_{ij} \mid 1 \le i \le m\ ,\ 1 \le j \le n \}$

$M_{3\times4}$ → 12 vec. in basis

Ex: 2×2 matrix space:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The standard basis for $P_n(x)$:

→ n+1 vec- in basis

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\xrightarrow{3+1}$ $\{1, x, x^2, x^3\}$

$2+1$

$P_2 = \{1, x, x^2\}$

Ex: Show that $S = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ is a basis for R^2

$\#S = 2$ = need test

Sol:

However, if $|A| = 0$ $\xrightarrow{\text{conclude}}$ L.Dep. whatever spanning or not

S spans R^2 ✓

For any vector $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ it can be expanded as

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$$

→ $c_1 + c_2 = x_1$

→ $c_1 - c_2 = x_2$

$$\rightarrow \underbrace{\begin{array}{cc} v_1 & v_2 \\ \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \end{array}}_{A} \begin{bmatrix}c_1\\c_2\end{bmatrix} = \overset{\text{Test}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}}$$

→ $|A| \neq 0$

Consistent w/o condy.

S is linearly independent ✓

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{A}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

→ $|A| \neq 0$ → **Unique solution**

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$|A| \neq 0$ $\xrightarrow{\text{conclude}}$ L.Indep. & spanning set

Exercise Explain why S is not a basis for $\mathbb{R}^2$

$*2$

$S = \{(6, -5), (12, -10)\}$ ∴ L. Dep.

$-5*2=$

Solution

S is linearly dependent and does not span $\mathbb{R}^2$. For instance,$(1,1) \neq span(S)$.

Exercise Explain why S is not a basis for $\mathbb{R}^2$

$S = \{(-4, 5), (0, 0)\}$

Solution S is linearly dependent ($(0,0) \in S$) and does not span $\mathbb{R}^2$. For instance,$(1,1) \neq span(S)$.

A Nonstandard Basis for R^3

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans R^3 and is linearly independent. So, S is a basis for R^3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0 \qquad \Rightarrow span(S) = R^3$$

$$\Rightarrow S \text{ is linearly independent}$$

Spanning set + L. Indep.

To represent $w = (1, 1, 1)$ by this basis $\Rightarrow$ need to solve a sys. of linear eqs.

Exercise Explain why S is not a basis for $\mathbb{R}^3$

$$S = \{(2,1,-2),(-2,-1,2),(4,2,-4)\}$$

$$\because \ |A| = \begin{vmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{vmatrix} = 0 \quad \Rightarrow S \text{ is linearly dependent} \rightarrow$$ Not a basis

Ex: Show that the vector space P_3 has the following basis

$3+1=4$

need to test

$$S = \{1\ , x\ , x^2\ , x^3\}$$

$\#S=4$

<u>S spans P_3</u>

Any polynomial $p \in P_3$ can be written as

$$p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

→ Where $a_0, a_1, a_2, a_3 \in R$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|A| \neq 0$

<u>S is linearly independent</u>

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 = 0$$

→ By equating coefficients of both sides → $c_0 = c_1 = c_2 = c_3 = 0$

Ex: Explain why S is not a basis for P_2 $\leftarrow 3$ $\quad 2+1$

$$S = \{1 - x,\ 1 - x^2,\ 3x^2 - 2x - 1\}$$

<u>**Independence test**</u>

$$c_1(1 - x) + c_2(1 - x^2) + c_3(3x^2 - 2x - 1) = 0$$

$$[c_1 + c_2 - c_3] + [-c_1 - 2c_3]\,x + [-c_2 + 3c_3]\,x^2 = 0$$

→ By equating coefficients of both sides

→ $c_1 + c_2 - c_3 = 0\ ,\ -c_1 - 2c_3 = 0\ ,\ -c_2 + 3c_3 = 0$

$$\because |A| = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & -1 & 3 \end{vmatrix}\begin{matrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{matrix} = 0 \quad \Rightarrow S \text{ is linearly dependent} \rightarrow$$ Not a basis

$$|A| = (0 + 0 - 1) - (0 + 2 - 3) = -1 + 1 = 0$$

- Thm: (Uniqueness of basis representation)

To be a basis for ex for R^n → basis includes n vecs.

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every vector in *V* can be written in one and only one way as a linear combination of vectors in *S*.

Example

$|A| \neq 0$ ↓ invertible

Uniqueness of Basis Representation

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in R^3. Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$ → **Invertible** → **Unique solution for any $\bar{u}$**

- Dimension:

The **dimension** of a finite dimensional vector space V is defined to be the number of vectors in a basis for V.

V: a vector space $\qquad$ S: a basis for V

$\Rightarrow \dim(V) = \#(S)$ $\qquad$ (the number of vectors in S)

- Ex:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m,\ 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

$\dim(V) = n$

- Thm: (Bases and linear dependence)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

  [Bases are maximal linearly independent sets of V]

Thm: If <u>the homogeneous system</u> has <u>fewer equations than variables</u>, then it must have infinitely many solution. $\Rightarrow$ L. Dep.

$m > n \Rightarrow k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_m\mathbf{u}_m = \mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

Exercise Explain why S is not a basis for $\mathbb{R}^2$

$S = \{(-1, 2), (1, -2), (2, 4)\}$ $\quad \#S > \dim(\mathbb{R}^2)$

Solution

A basis for $\mathbb{R}^2$ can only have two vectors. Because S has three vectors, it is not a basis for $\mathbb{R}^2$.

Ex: R^3 has the basis $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ "Standard basis" $\dim(R^3) = 3$

$\#S = 4 > 3$

Then the elements in the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ -1 \end{bmatrix} \right\}$ must be linearly dependent.

Ex: P_3 has the basis $S = \{1, x, x^2, x^3\}$ "Standard basis" $\dim(P_3) = 4$

Then the elements in the set $\{1, \; 1 + x, \; 1 - x, \; 1 + x + x^2, \; 1 - x + x^2\}$ must be linearly dependent.

$\#S = 5 > 4$

- Thm: (Number of vectors in a basis)

  $\dim(V) = n$

  If a vector space V has one basis with n vectors, then every basis for V has n vectors. (All bases for a finite-dimensional vector space has the same number of vectors.)

Ex: True or False?

$$S = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} \right\} \text{ is not a basis for } R^3.$$

$\#S < \dim(R^3) \Rightarrow S$ does not span R^3

Exercise Explain why S is not a basis for $M_{2,2}$ → $\dim(M_{2\times 2}) = 4$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$\# S = 3 < 4$

Solution S does not span $M_{2,2}$, although it is linearly independent.

→ does not span

→ not a basis

Exercise Determine whether S is a basis for the indicated vector space.

$\frac{4}{3}$

$S = \{(3, -2), (4, 5)\}$ for R^2 $\therefore$ L. Indep.

$-2 * \frac{4}{3} \neq 5$

$$\begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} = 15 + 8 \neq 0$$

L. Indep. + span set

Solution

Because the vectors in S are not scalar multiples of one another, they are linearly independent. Because S consists of exactly two linearly independent vectors, it is a basis for R^2.

- Thm: (Basis tests in an n-dimensional space)

  Let V be a vector space of dimension n.

  (1) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

  (2) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ spans V, then S is a basis for V.

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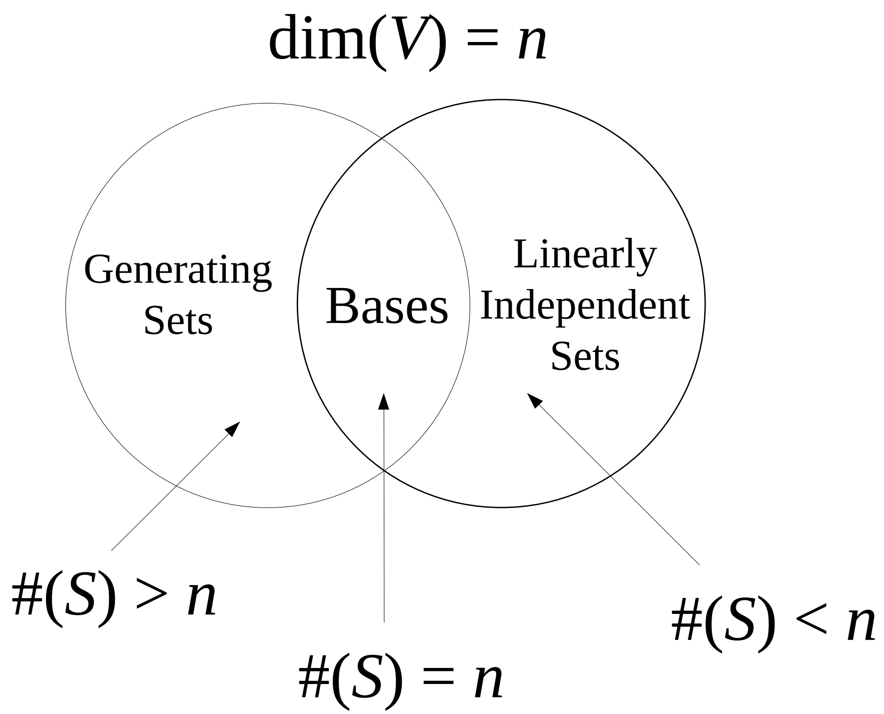

Dimension of the vector space → dim(V)

Number of vectors in S
#S < dim(v)

Does not span
→ Not a basis

Number of vectors in S
#S = dim(v)

Only one test is enough
→ Independence test is preferable

Number of vectors in S
#S > dim(v)

Dependent
→ Not a basis

Exercise Determine whether S is a basis for the indicated vector space.

$$S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\} \text{ for } R^4$$

1 2 3 4

Indep. test

Test

$\dim(R^4) = 4$

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 5 \end{vmatrix} \neq 0$$

→ S is linearly independent & contains 4 vectors

→ S is a basis for R^4

Exercise Determine whether S is a basis for the vector space $M_{2,2}$. $\rightarrow$ $\dim(M_{2\times 2}) = 4$

You could change the order

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

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$$S = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

1 2 3 4 # Test

Solution

. Form the equation

$$c_1 \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which yields the homogeneous system

$$\begin{aligned} 2c_1 + c_2 \qquad\qquad &= 0 \\ 4c_2 + c_3 + c_4 &= 0 \\ 3c_3 + 2c_4 &= 0 \\ 3c_1 + c_2 + 2c_3 \qquad &= 0. \end{aligned}$$

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 3 & 1 & 2 & 0 \end{vmatrix} \neq 0$$

This system has only the trivial solution. So, S consists of exactly four linearly independent vectors, and is therefore a basis for $M_{2,2}$.

Exercise Determine whether S is a basis for the vector space $M_{2,2}$. $\rightarrow$ dim = 4

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix} \right\}$$

1 2 3 4

$\Downarrow$ Test

Solution

Form the equation

$$c_1\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix} + c_2\begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix} + c_3\begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix} + c_4\begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which yields the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + 4c_3 + 12c_4 &= 0 \\ 2c_1 - 7c_2 - 9c_3 - 16c_4 &= 0 \\ -5c_1 + 6c_2 + 11c_3 + 17c_4 &= 0 \\ 4c_1 + 2c_2 + 12c_3 + 42c_4 &= 0. \end{aligned}$$

$| \quad | = 0$

Because this system has nontrivial solutions (for instance, $c_1 = 2$, $c_2 = -1$, $c_3 = 3$ and $c_4 = -1$), the set is linearly dependent, and is not a basis for $M_{2,2}$.

Exercise Determine whether S is a basis for the vector space P_3. $\rightarrow$ dim = 4

$$S = \{t^3 - 2t^2 + 1,\ t^2 - 4,\ t^3 + 2t,\ 5t\}$$

1 2 3 4 # Test

Solution

Form the equation

$$c_1\left(t^3 - 2t^2 + 1\right) + c_2\left(t^2 - 4\right) + c_3\left(t^3 + 2t\right) + c_4(5t) = 0 + 0t + 0t^2 + 0t^3$$

$$\begin{array}{l} \text{cons} \\ t \\ t^2 \\ t^3 \end{array} \begin{vmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{vmatrix} \neq 0$$

which yields the homogeneous system

$$\begin{aligned} c_1 \quad + c_3 \quad &= 0 \\ -2c_1 + c_2 \quad &= 0 \\ 2c_3 + 5c_4 &= 0 \\ c_1 - 4c_2 \quad &= 0. \end{aligned}$$

This system has only the trivial solution. So, S consists of exactly four linearly independent vectors, and is, therefore, a basis for P_3.

Exercise Determine whether S is a basis for the vector space P_3. $\rightarrow$ dim=4

$$S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$$

1 2 3 4

Solution

1. Form the equation

$$c_1(t^3 - 1) + c_2(2t^2) + c_3(t + 3) + c_4(5 + 2t + 2t^2 + t^3) = 0$$

which yields the homogeneous system

$$\begin{aligned} c_1 \qquad\qquad + \quad c_4 &= 0 \\ 2c_2 \qquad + 2c_4 &= 0. \\ c_3 + 2c_4 &= 0 \\ -c_1 \quad + 3c_3 + 5c_4 &= 0. \end{aligned}$$

$| \quad | = 0$

This system has nontrivial solutions (for instance, $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, and $c_4 = -1$). Therefore, S is not a basis for P_3 because the vectors are linearly dependent.

Exercise determine whether S is a basis for R^3 If it is, write

$u(8, 3, 8)$ as a linear combination of the vectors in S

dim = 3

$$S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$$

S = 3

Test

upper trian

$$\because |A| = \begin{vmatrix} 4 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 2 & 2 \end{vmatrix} \neq 0$$

$|A| = 4 * 3 * 2$ f.e

→ S **Linearly independent and contains 3 vectors**

→ S **is a basis**

special case
upper trian

In general
→ using Gauss
→ using inverse

$$\left[\begin{array}{ccc|c} \mathbf{4} & \mathbf{0} & \mathbf{0} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{8} \end{array}\right]$$

Forward sub.

→ **After solving the augmented matrix**

yields $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$. So,

$$\mathbf{u} = 2(4, 3, 2) - (0, 3, 2) + 3(0, 0, 2) = (8, 3, 8).$$

unique sol.

Exercise 52 determine whether S is a basis for R^3 If it is, write $u(8,3,8)$ as a linear combination of the vectors in S

$$S = \{(1, 0, 1), \cancel{(0, 0, 0)}, (0, 1, 0)\}$$

Solution

. The set S contains the zero vector, and is, therefore, linearly dependent.

$$0(1, 0, 1) + 1(0, 0, 0) + 0(0, 1, 0) = (0, 0, 0)$$

So, S is not a basis for R^3.

Example

Testing for a Basis in an n-Dimensional Space

Show that the set of vectors is a basis for $M_{5,1}$.

dim 5

upper trian

$$S = \left\{ \begin{array}{ccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \end{array} \right\}$$

$\#S = 5$

$\oplus$ Test

$|A| \neq 0$

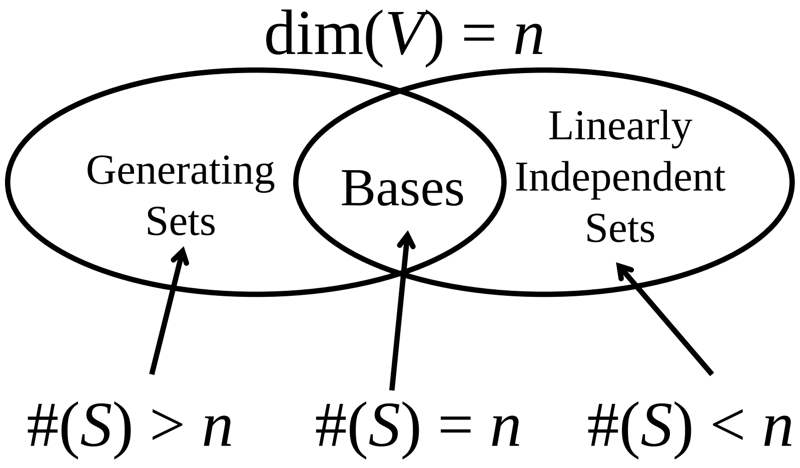
Because S has five vectors and the dimension of $M_{5,1}$ is five, you can apply Theorem to verify that S is a basis by showing either that S is linearly independent or that S spans $M_{5,1}$. To show the first of these, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0},$$

which yields the homogeneous system of linear equations shown below.

$$\begin{aligned} c_1 \qquad\qquad\qquad\qquad\qquad &= 0 \\ 2c_1 + c_2 \qquad\qquad\qquad\quad &= 0 \\ -c_1 + 3c_2 + 2c_3 \qquad\qquad &= 0 \\ 3c_1 - 2c_2 - c_3 + 2c_4 \qquad &= 0 \\ 4c_1 + 3c_2 + 5c_3 - 3c_4 - 2c_5 &= 0 \end{aligned}$$

Because this system has only the trivial solution, S must be linearly independent. So, by Theorem , S is a basis for $M_{5,1}$.



- **Notes:**

(1) $\dim(V) = n$, $S \subseteq V$

S : a generating set $\Rightarrow$ $\#(S) \geq n$

S : a L.I. set $\Rightarrow$ $\#(S) \leq n$

S : a basis $\Rightarrow$ $\#(S) = n$

(2) $\dim(V) = n$, W is a subspace of V $\Rightarrow$ $\dim(W) \leq n$

- Ex: (Finding the dimension of a subspace)

  (a) W={(d, c–d, c): c and d are real numbers}

  (b) W={(2b, b, 0): b is a real number}

Given W

ⓐ → p.t. W is a s.s. of R^3

ⓑ → Find the basis & dim. of W

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$ span def. → it remains to p.t. this set is L. indep.

$\times 0$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

$\times 0 \neq 1$

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2 \lneq \dim(R^3)$

(b) $\because (2b, b, 0) = b(2, 1, 0)$ → span def.

→ one vec. L. Indep.

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1 \lneq \dim(R^3)$

- Ex: (Finding the dimension of a subspace)

Represent a plane

$$W = \left\{ \begin{bmatrix} a-b+c \\ a-b-c \\ a-b \\ c \end{bmatrix}, a,b,c \in R \right\}$$

not scalar multiple

Sol:

$$\begin{bmatrix} a-b+c \\ a-b-c \\ a-b \\ c \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + c \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

by defn. spanning set

Test L. indep.

Test

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

or by inspection

$\bar{v}_1 = -\bar{v}_2$ $\therefore$ L. Dep.

$S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ not a basis

basis $\Rightarrow S = \{\bar{v}_1, \bar{v}_3\}$ Check only scalar multi.

$\dim(M_{2\times 2}) = 4$

- Ex: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

by defn.

spanning set for W

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

L. indep.

(positions of zeros)

$$\Rightarrow S \text{ is a basis for } W \quad \Rightarrow \dim(W) = \#(S) = 3 < 4$$

Exercise

3. Find a basis for $D_{3,3}$ (the vector space of all 3×3 diagonal matrices). What is the dimension of this vector space?

$\subset M_{3\times3}$

$\dim(M_{3\times3}) = 9$

Solution

. One basis for $D_{3,3}$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Because a basis for $D_{3,3}$ has 3 vectors,

no. of vectors in basis

$\dim(D_{3,3}) = 3.$

$$D_{3\times3} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

(a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace W of R^2.

69. $W = \{(2t, t): t \text{ is a real number}\}$

70. $W = \{(0, t): t \text{ is a real number}\}$

$t(2,1)$

$t(0,1)$ "y-axis"

69. (a) W is a line through the origin.

(b) A basis for W is $\{(2, 1)\}$.

(c) The dimension of W is 1.

70. (a) W is a line through the origin (the y-axis)

(b) A basis for W is $\{(0, 1)\}$.

(c) The dimension of W is 1.

Exercise 73,74

$$s\begin{bmatrix}2\\1\\0\\1\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

In Exercises 73–76, find (a) a basis for and (b) the dimension of the subspace W of R^4.

73. $W = \{(2s - t, s, t, s)$: s and t are real numbers$\}$

74. $W = \{(5t, -3t, t, t)$: t is a real number$\}$

$$t\begin{bmatrix}5\\-3\\1\\1\end{bmatrix}$$

73. (a) A basis for W is $\{(2, 1, 0, 1), (-1, 0, 1, 0)\}$.

(b) The dimension of W is 2.

74. (a) A basis for W $\{(5, -3, 1, 1)\}$.

(b) The dimension of W is 1.