14-03 Discrete RVs.

21-03 Special discrete distributions

28-03 Revision

30-03 Start of midterm

Probability and Statistics

DR. AHMED TAYEL

Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University

ahmed.tayel@alexu.edu.eg

Outline

Part 2 – Random variables

2.1 Introduction

2.2 Discrete random variables

- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation
- Variance and Standard deviation

2.3 Some important discrete PMF's

- Bernoulli PMF

2.1 Introduction

2.1 Introduction

What is a random variable (RV)

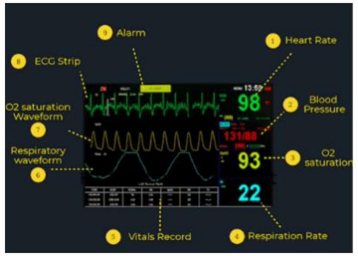
Random + Variable

- NOT based on a **physical law**
- Depends on the **outcome of a random experiment**

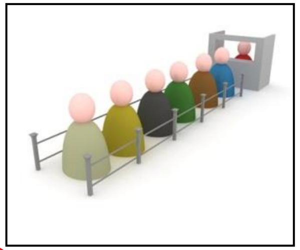
- $a \leq X \leq b$, **or** $X = x_1, x_2, \ldots$
- It takes different **numerical** values

Examples

Discrete



Heart rate



Queue length

Continuous

Call duration

Life time

From random experiment to RV’s

Example

A coin is tossed twice and the appearing sides are observed. **Let X be the number of appearing heads.**

Find the possible values of X and the probability of each value.

Observe appearing sides

Count the number of appearing heads

Capital "RV"

small "value taken by the RV"

$P(X = x)$

$\langle H \langle \begin{smallmatrix} H \\ T \end{smallmatrix} \quad T \langle \begin{smallmatrix} H \\ T \end{smallmatrix}$

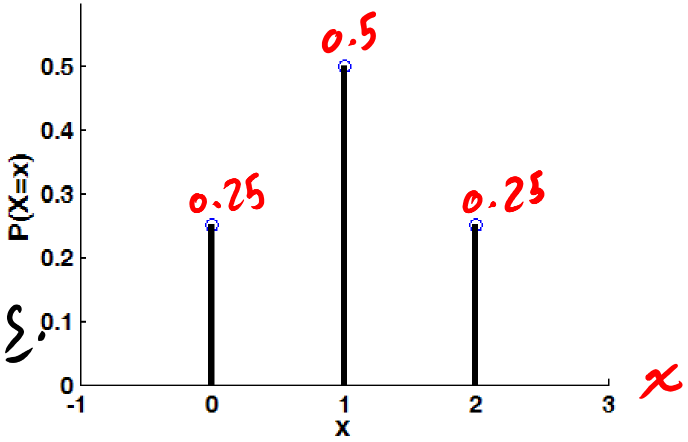
$S = \{ \overset{0}{TT}, \overset{1}{TH}, \overset{1}{HT}, \overset{2}{HH} \}$

x	0	1	2
P(X = x)	1/4	1/2	1/4

Σ Probs = 1

probability mass function "pmf"

→ Discrete RVs.



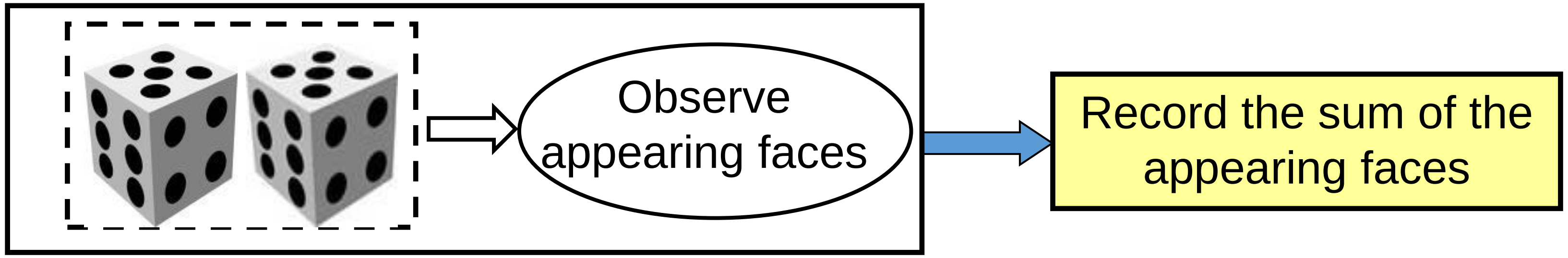
From random experiment to RV’s (Cont’d)

Example

A die is thrown twice.

Let X denotes the sum of the appearing numbers.

Find the possible values of X and the probability of each value.

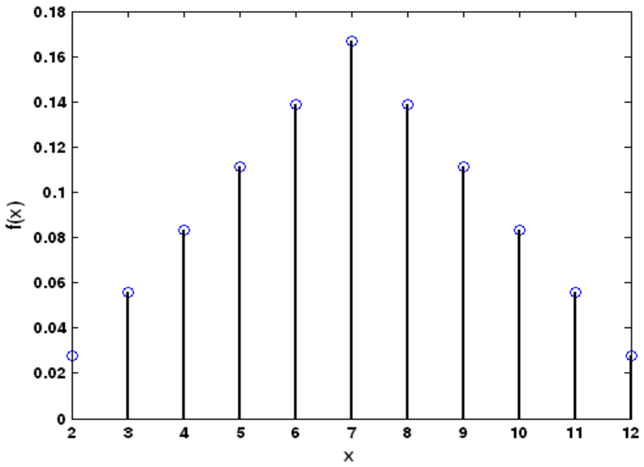


6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

each element is a subproblem

X	2	3	. . .	12
P(X = x)	1/36	2/36	. . .	1/36

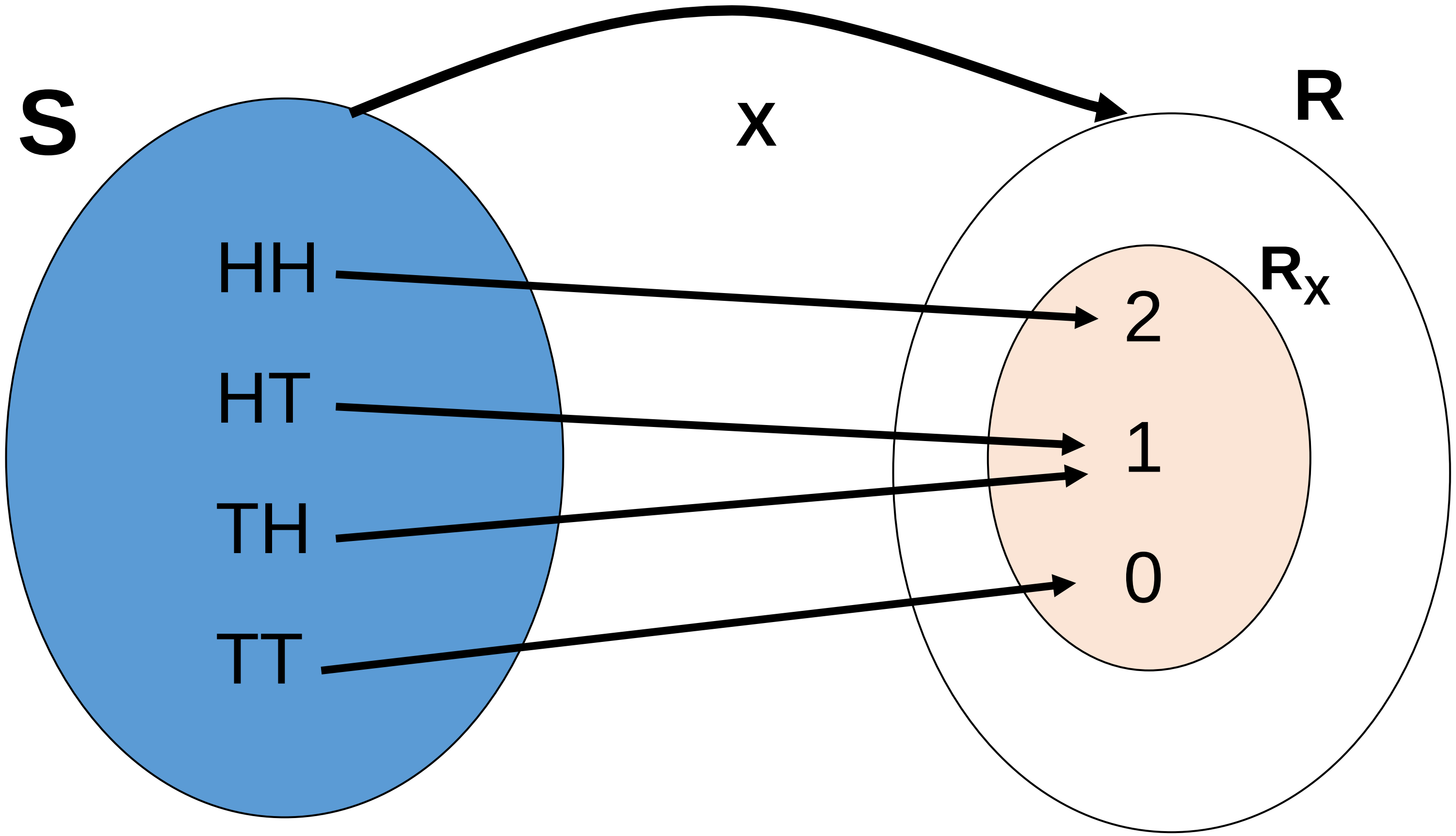
Σ probs. = 1



pmf of X

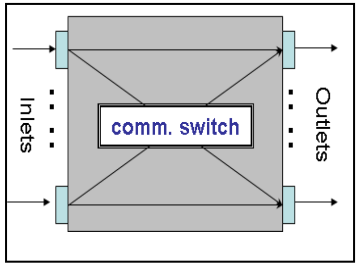
Mathematical definition of a RV

A *random variable* is a real function defined on the sample space.



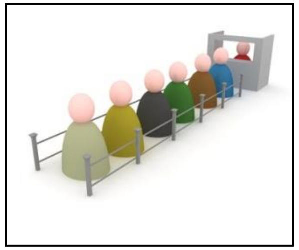
RV's classification

Discrete RV



X: Number of packets/min

$R_X = \{0, 1, 2, . . ., N\}$



X: Queue length

$R_X = \{0, 1, 2, . . .\}$

Before midterm

Continuous RV

X: Call duration

$R_X = [0, \infty]$

X: Life time

$R_X = [0, T]$

2.2 Discrete random variables

2.2.1 Probability Mass Function (PMF)

2.2 Discrete random variables

Probability mass function (PMF)

Example

50/50 V D

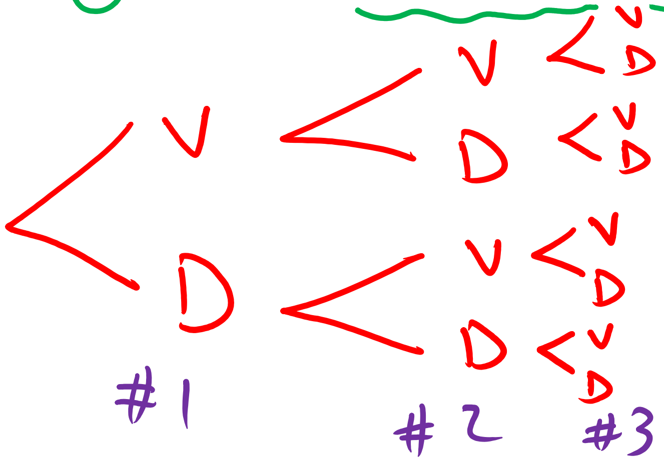
An arriving call is *equally likely* to be voice or data.

Observe three calls which are assumed to be independent.

Let X be the number of voice calls. Find the PMF of X.

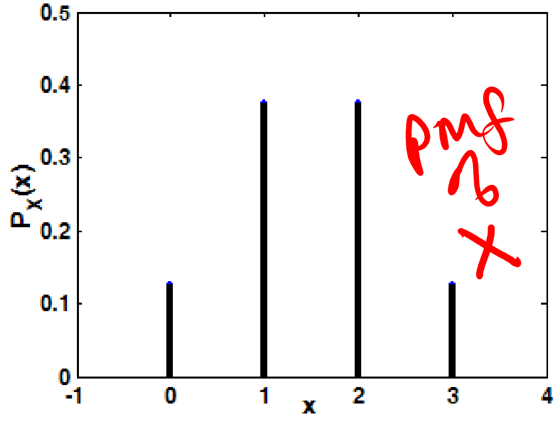
x	0	1	2	3
$P_X(x)$	1/8	3/8	3/8	1/8

DDD | VDD DVD DDV $^3C_1$ | DVV VDV VVD $^3C_2$ | VVV



$S = \{DDD, DDV, \ldots, VVV\}$

$N_s = 2^3 = 8$



Definition

capital

$P_X(x) = P(X = x)$

Capital small

General properties

(1) $P_X(x) \geq 0$, (2) $\sum_x P_X(x) = 1$

Example

A shipment of 8 microcomputers to retail outlet contains 3 that are defective. If a school makes a random purchase of two of these computers. **Find the probability distribution for the number of defectives.**

Total

D

8

5G
3D

select $\rightarrow$ 2 $\Rightarrow N_s = {}^8C_2$

X: no. of defectives

X	0	1	2
$P_X(x)$	${}^5C_2 / {}^8C_2$	$\dfrac{{}^5C_1 * {}^3C_1}{{}^8C_2}$	$\dfrac{{}^3C_2}{{}^8C_2}$

5G	$\rightarrow$ 2
3D	

5G	$\rightarrow$ 1
3D	$\rightarrow$ 1

stages

5G	
3D	$\rightarrow$ 2

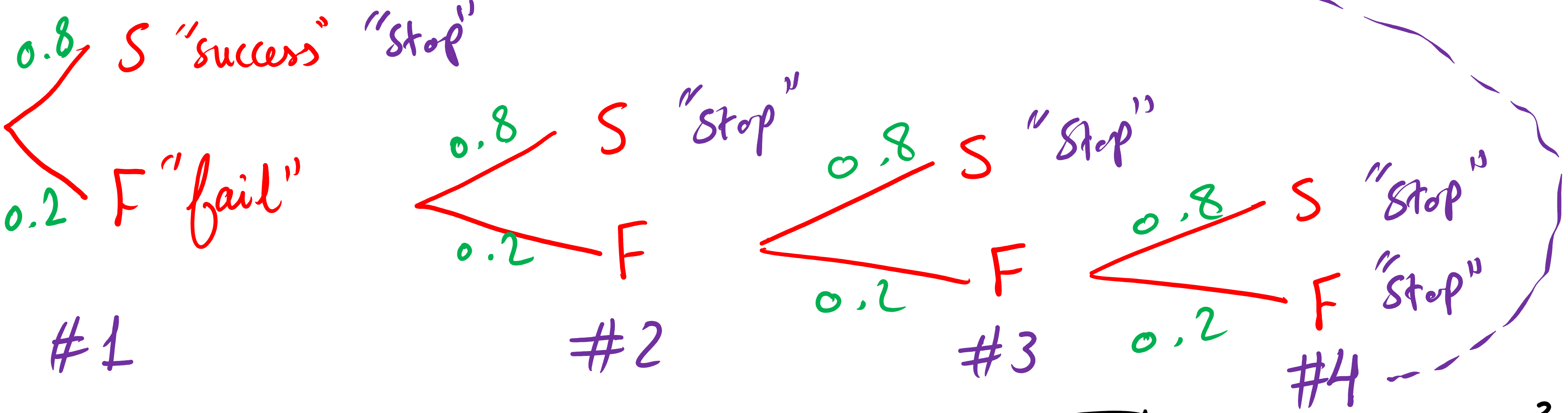
Example

The probability of a successful connection to a call center is 0.8.

You decided to try at most four times. Trials are independent.

Let X be the number of trials. Find $P_X(x)$.

Default: independent trials



X: number of trials

x	1	2	3	4
$P_X(x)$	0.8	$0.2*0.8$	$(0.2)^2*0.8$	
	S	FS	FFS	FFFS or FFFF

Stages

Alternatives: $(0.2)^3*0.8+(0.2)^4$

$\sum$ probs $=1$

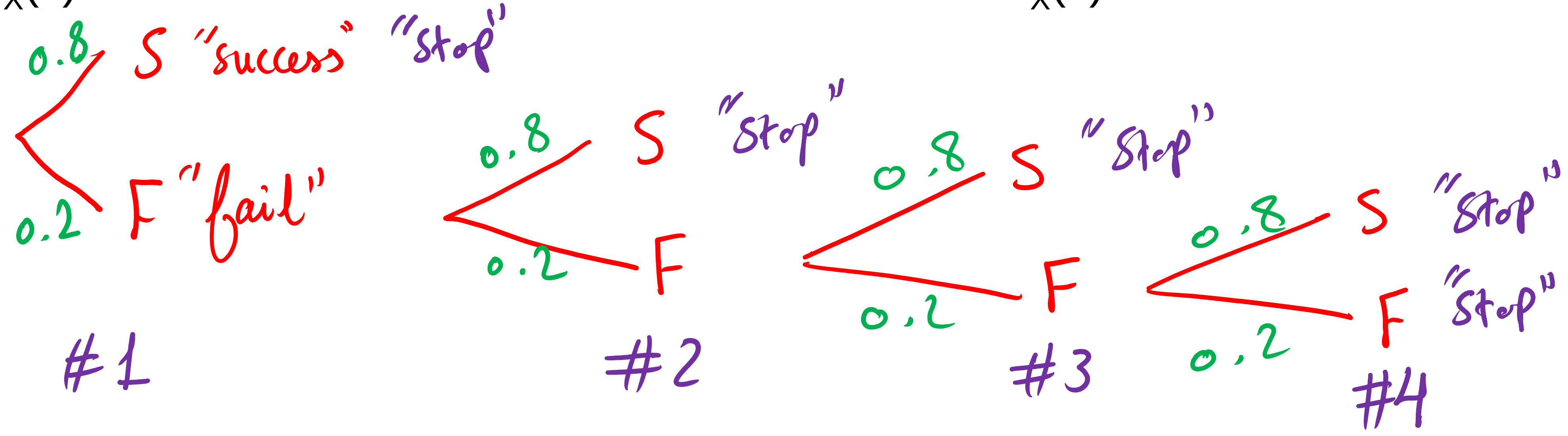
$1-[0.8+0.2*0.8+(0.2)^2*0.8]$

Example

The probability of a successful connection to a call center is 0.8.

You decided to try at most four times. Trials are independent.

	# success	# F
S	1	0
FS	1	1
FFS	1	2
FFFS	1	3
FFFF	0	4



Let X be the number of successes.

Find $P_X(x)$.

x	0	1
$P_X(x)$	$(0.2)^4$	$\rightarrow 1-(0.2)^4$

FFFF

Let X be the number of failures.

Find $P_X(x)$.

x	0	1	2	3	4
$P_X(x)$	0.8	$0.2 * 0.8$	$(0.2)^2 * 0.8$	$(0.2)^3 * 0.8$	$(0.2)^4$

Example

Let $P_X(x) = \cancel{k}\,4\,(0.2)^x$, $x = 1, 2, 3, \ldots$

(a) Find k.

Hint:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots$$

$$\sum_{\forall x} P_X(x) = 1$$

$$P(X=1) + P(X=2) + P(X=3) + \cdots = 1$$

$$\underbrace{K(0.2)^1}_{\text{common}} + K(0.2)^2 + K(0.2)^3 + \cdots = 1$$

$$K(0.2)\underbrace{\left[1 + (0.2) + (0.2)^2 + \cdots\right]}=1$$

$$K(0.2)\frac{1}{1-0.2} = 1$$

$$K\frac{0.2}{0.8} = 1$$

$$\boxed{K = 4}$$

Example

Let $P_X(x) = \cancel{k}^{\,4} (0.2)^x, \quad x = 1, 2, 3, \ldots$

(a) Find k.

(b) Compute

$P(X = 3)$,

$P(X \geq 2)$,

$P(X=3) = P_X(3) = 4\,(0.2)^3 = \checkmark$

$P(X \geq 2) = P(X=2) + P(X=3) + \cdots\cdots$

Or $= 1 - P(X<2) = 1 - P(X=1) = 1 - 4\,(0.2)^1$

$= 0.2$

Example

Let $P_X(x) = \overset{4}{\cancel{k}}(0.2)^x, \quad x = 1, 2, 3, \ldots$

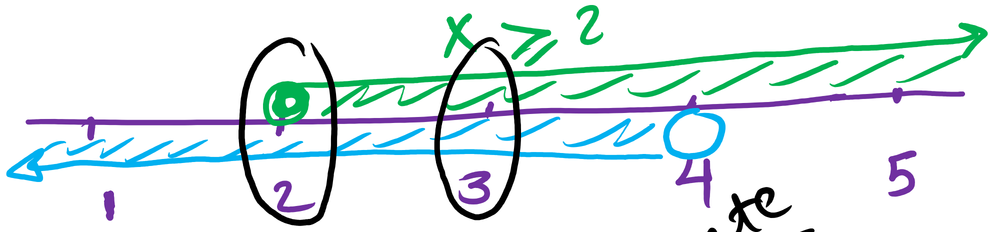
(a) Find k.

(b) Compute

$P(X < 4 \mid X \geq 2)$.

$$P(A|B) = \frac{\text{intersection}}{\text{condn.}} = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(X<4 \cap X \geq 2)}{P(X \geq 2)}$$



Discrete

$$= \frac{P(X=2) + P(X=3)}{1 - P(X=1)} = \frac{4(0.2)^2 + 4(0.2)^3}{1 - 4(0.2)^1}$$

2.2 Discrete random variables

2.2.2 Cumulative Distribution Function (CDF)

$P(X \leq 5) \Rightarrow F_X(5)$

$P(X \leq 100) \Rightarrow F_X(100)$

$P(X > 20) \Rightarrow 1 - P(X \leq 20) = 1 - F_X(20)$

$P(10 < X \leq 100) \ldots\ldots$

Easier calculations for the probabilities

$$F_X(x) = P(X \leq x)$$

Cumulative distribution function (CDF)

Definition $F_X(x) = P(X \le x)$

small

Capital

General properties of $F_X(x)$

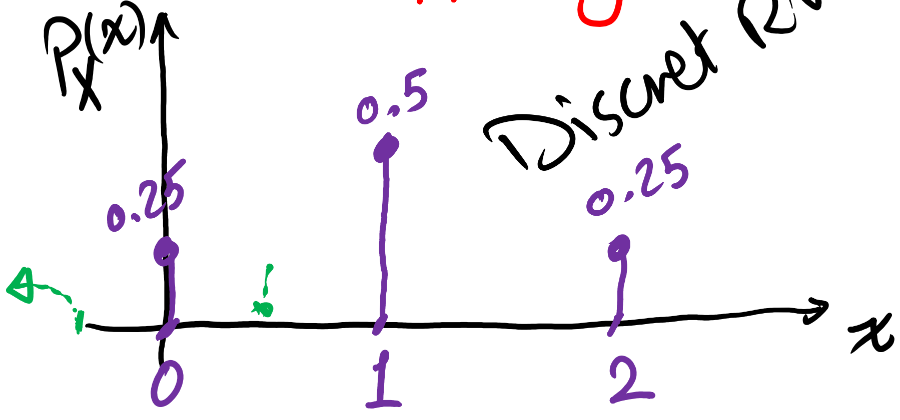
(1) Non-decreasing function

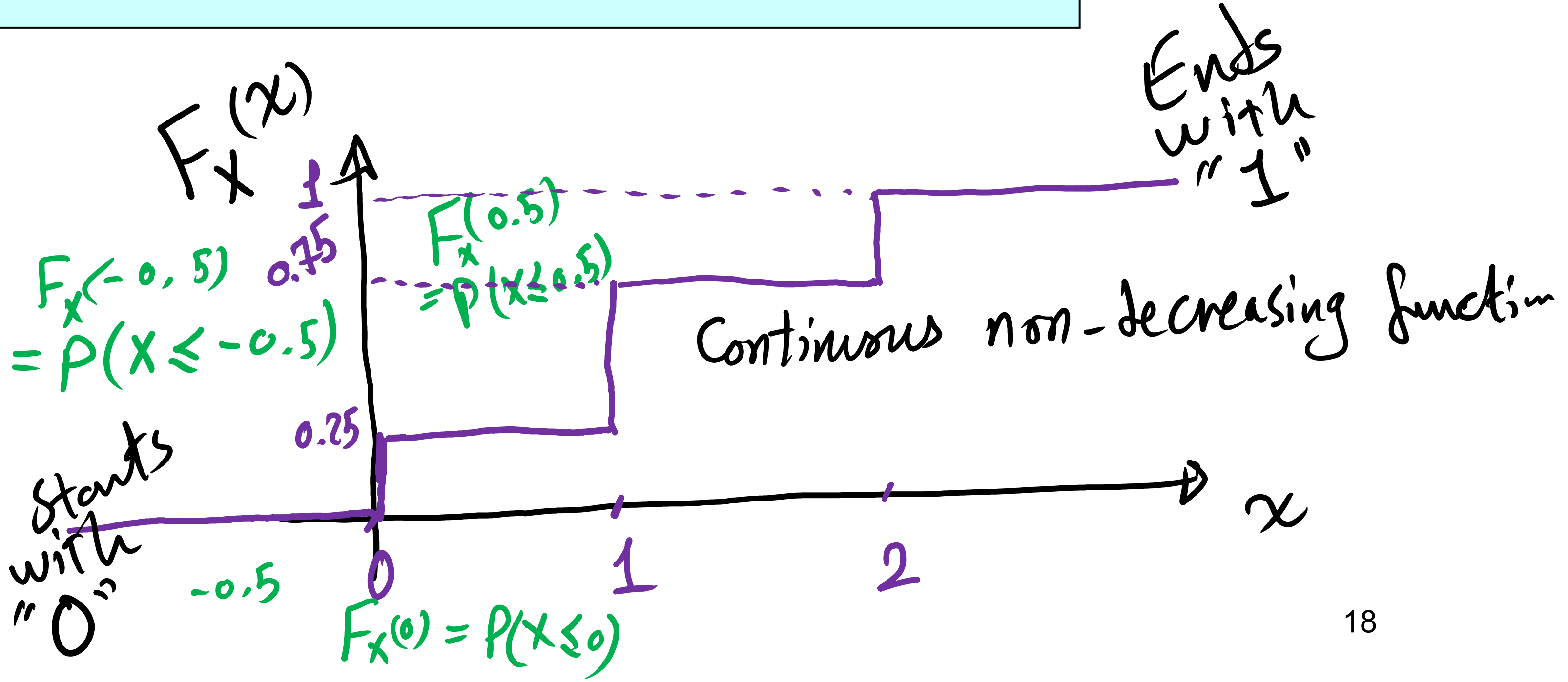
(2) $F_X(-\infty) = 0$

(3) $F_X(\infty) = 1$

Tossing a Coin twice

X: no. of appearing heads





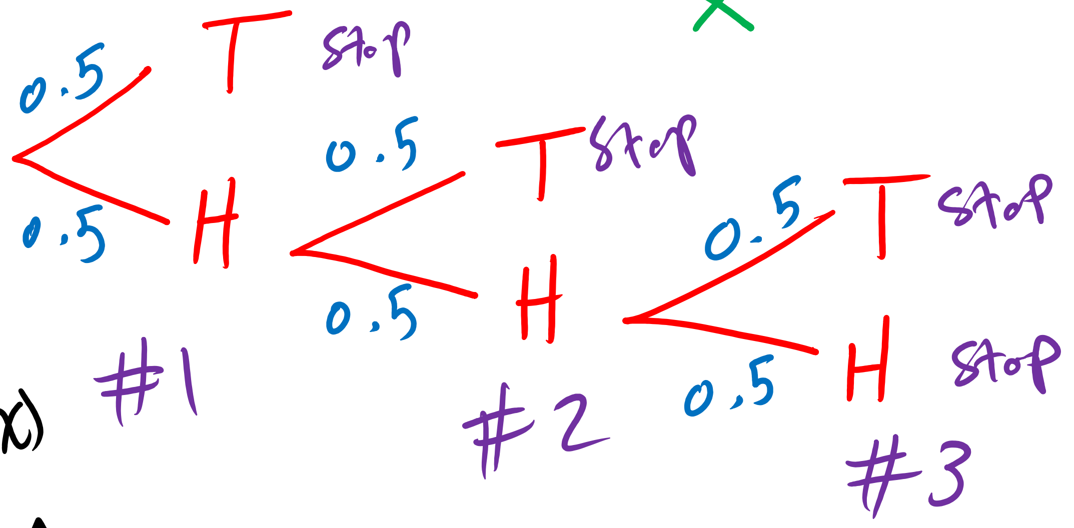
Example

Default: fair Coin

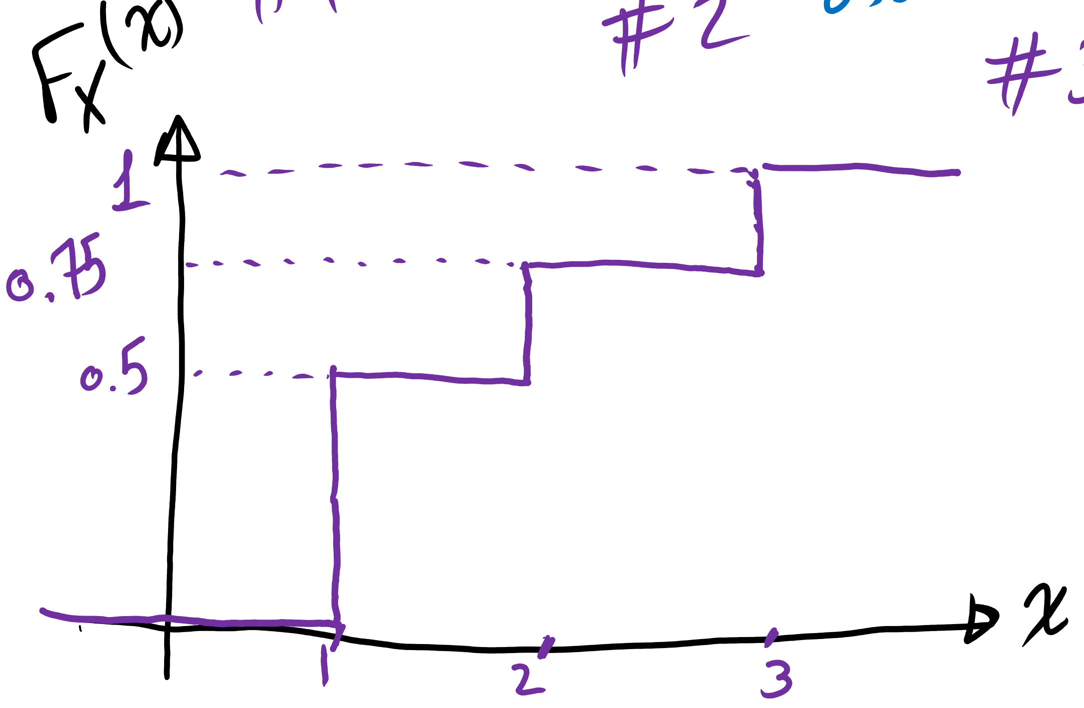
A coin is tossed until a tail appears or after three attempts have been made. Find the PMF and CDF of the number of trials made in this experiment and sketch their graphs

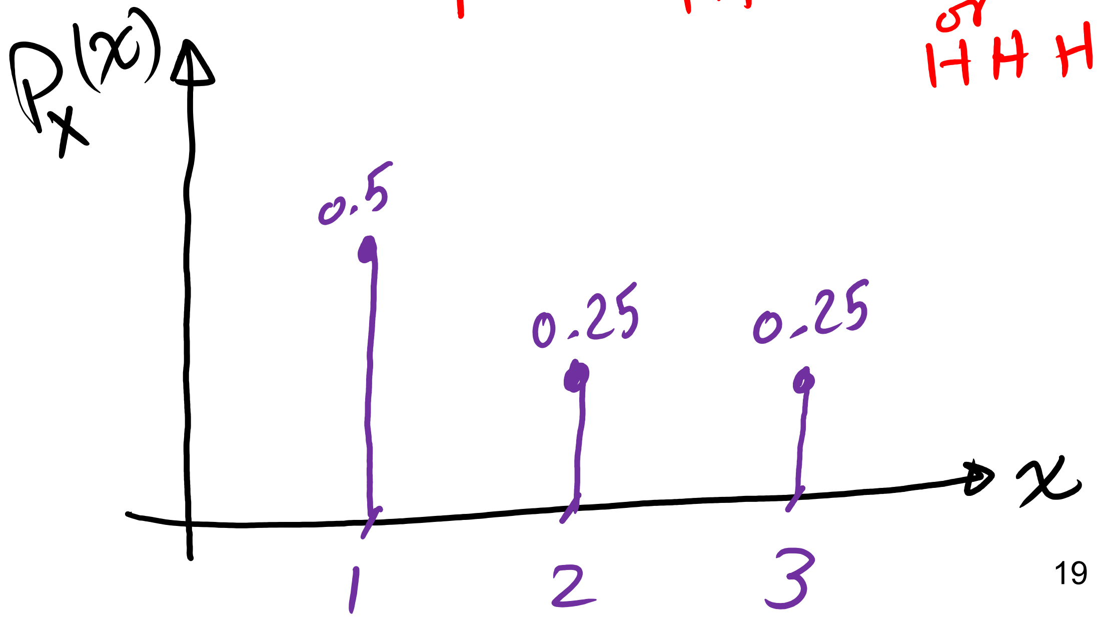
X

X: no. of trials



x	1	2	3
$P_X(x)$	0.5	0.25	$1-[0.5+0.25]$ $=0.25$
	T	HT	HHT or HHH





Example

Find the CDF of the RV X where

$P_X(x) = 4\,(0.2)^x, \qquad x = 1, 2, 3, \ldots$

Hint:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = 1 + a + a^2 + \cdots$$

$$P(X \le \cancel{10}\, x) = 1 - P(X > \cancel{10}\, x)$$

$$= 1 - \left[P(X = \cancel{11}\, x+1) + P(X = \cancel{12}\, x+2) + P(X = \cancel{13}\, x+3) + \cdots \right]$$

$$= 1 - \left[\underbrace{4\,(0.2)}_{\text{common}}{}^{\cancel{11}\, x+1} + 4(0.2)^{\cancel{12}\, x+2} + 4(0.2)^{\cancel{13}\, x+3} + \cdots \right]$$

$$= 1 - 4\,(0.2)^{\cancel{11}\, x+1} \left[\underbrace{1 + 0.2 + (0.2)^2 + \cdots}_{a = 0.2} \right]$$

$$= 1 - 4\,(0.2)^{\cancel{11}\, x+1} \frac{1}{1 - 0.2} = 1 - 5\,(0.2)^{\cancel{11}\, x+1} = F_X(x)$$

Example

Find the CDF of the RV X where

$P_X(x) = 4\,(0.2)^x, \qquad x = 1, 2, 3, \ldots$

then find: $P(X \le 5)$, $P(X \le 100)$, $P(X > 5)$, $P(X \ge 5)$, $P(4 \le X \le 20)$

$$F_X(x) = P(X \le x) = 1 - 5(0.2)^{x+1}$$

Note: equal is very important "especially in discrete RVs!"

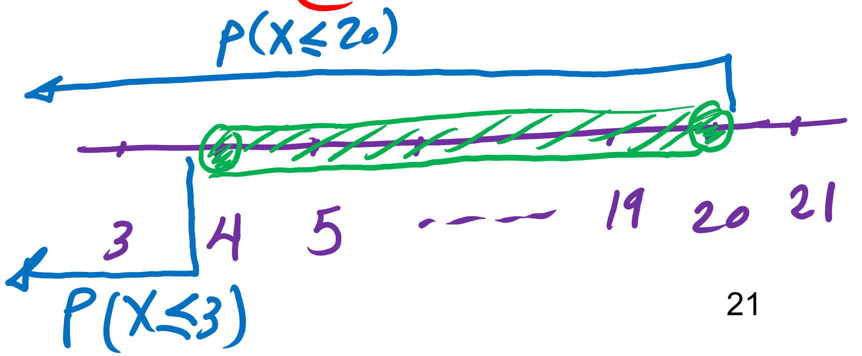
$$P(X \le 5) = F_X(5) = 1 - 5(0.2)^{5+1}$$

$$P(X \le 100) = F_X(100) = 1 - 5(0.2)^{100+1}$$

$$P(X > 5) = 1 - P(X \le 5) = 1 - F_X(5) = 1 - \left[1 - 5(0.2)^{5+1}\right]$$

$$P(X \ge 5) = 1 - P(X < 5) = 1 - P(X \le 4) = 1 - \left[1 - 5(0.2)^{4+1}\right]$$

$$P(4 \le X \le 20) = P(X \le 20) - P(X \le 3)$$
$$= F_X(20) - F_X(3)$$



Exercise in previous example

$P(4 < X \leq 20)$

2.2 Discrete random variables

2.2.3 Expectation

Introduction to expectation

Average age $= \frac{21+23+22}{3}$ مجموع الأعمار / عدد الطلبة

Example

A class of 100 students with ages ranging from 20 to 24 years old.

A record of the ages of students is introduced in the following table.

Age (x)	20	21	22	23	24
Number of students	5	20	50	15	10
$P_X(x)$	5/100	20/100	50/100	15/100	10/100

Q: How to average these numbers?

Motivation

Sum the ages of all students and divide by the number of students

$$\text{Average age} = \frac{20*5+21*20+22*50+23*15+24*10}{100}$$

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The average of X may be computed as follows:

$$\underbrace{20}_{x(1)} \times (5/100) + \underbrace{21}_{x(2)} \times (20/100) + \underbrace{22}_{x(3)} \times (50/100) + \underbrace{23}_{x(4)} \times (15/100) + \underbrace{24}_{x(5)} \times (10/100) = 22.05$$

$$\text{Average/mean value} = \text{Expectation of } X = E(X) = \sum_{\forall x} x\, P_X(x)$$

Expectation or mean ($E(X)$, μ_X)

رمز آخر

Definition $$E(X) = \sum_x x\,P_X(x)$$

What does E(X) represent?

- A *weighted sum* of the RV values.
- The *average value* of the RV over the long run.
- The *balancing point* of the PMF.

Example

Let X be the number of breakdowns/week of a certain machine. Find E(X)

x	0	1	2	3
$P_X(x)$	0.6	0.2	0.15	0.05

→ break downs/week

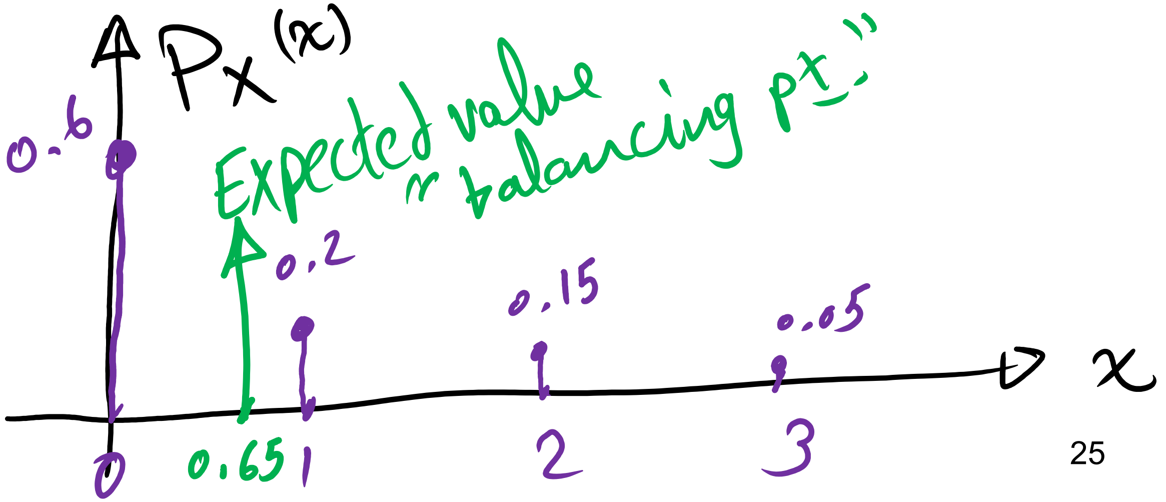
$$E(X) = \sum_{\forall x} x \, P_X(x)$$

$$E(X) = 0 * 0.6 + 1 * 0.2 + 2 * 0.15 + 3 * 0.05$$

$$= 0 + 0.2 + 0.3 + 0.15$$

$$= 0.65$$

breakdowns/week



E(X) properties

$E(\text{const.}) =$ the same const.

$$\boxed{E(aX + b) = aE(X) + b}$$

$$\boxed{E[g(X)] = \sum_x g(x) P_X(x)}$$

$$E(X) = \sum_{\forall x} x\, P_X(x)$$

$$E(aX+b) = \sum_{\forall x} (ax+b)\, P_X(x)$$

$$= \sum_{\forall x} a\, x\, P_X(x) + \sum_{\forall x} b\, P_X(x)$$

$$= a \underbrace{\sum_{\forall x} x\, P_X(x)}_{E(X)} + b \underbrace{\sum_{\forall x} P_X(x)}_{1} = aE(X) + b$$

E(X) properties

Example

For the following PMF, if Y = 100 X. Find E(Y).

If W = Y + 500. Find E(W).

	×100	×100	- - -	- - -
x	0	1	2	3
$P_X(x)$	0.6	0.2	0.15	0.05

$y = 100x$

1-to-1 fn.

$$E(aX + b) = aE(X) + b$$

$$E(X) = 0.65$$

$y = 100x$	0	100	200	300
$P_Y(y)$	0.6	0.2	0.15	0.05

$$E(Y) = \sum_{\forall y} y\, P_Y(y) = 0 * 0.6 + 100 * 0.2 + 200 * 0.15 + 300 * 0.05 = 65$$

$$E(Y) = E(100X) = 100\,E(X) = 100 * 0.65 = 65$$

$$E(W) = E(Y + 500) = E(100X + 500) = 100\,E(X) + 500 = 100 * 0.65 + 500 = 565$$

$$E(Y) + 500 = 65 + 500 = 565$$

Example

For the given PMF $Y = X^2$.

Find $\mu_{Y.}$

x	1	2	3	4
$P_X(x)$	0.8	0.16	0.032	0.008

$$E[\underset{\sim\sim}{g(X)}] = \sum_x \underset{\sim\sim}{g(x)} P_X(x)$$

✓ $E(X^2)$ is called the *second moment* of X.

$$E(Y) = E(X^2) = \sum_{\forall x} x^2 P_X(x)$$

$$= (1)^2 * 0.8 + (2)^2 * 0.16 + (3)^2 * 0.032 + (4)^2 * 0.008$$

Exercise

x	-2	-1	0	1	2
$P_X(x)$	0.05	0.2	0.5	0.2	0.05

$Y = X^2$ find pmf of Y & $E(Y)$

$E(X)$: 1^{st} moment

$E(x^2)$: 2^{nd} moment

⋮

$E(x^n)$: n^{th} moment

"Moment generating function"

Statistics

Variance

2^{nd} moment $-$ $(1^{st}$ moment$)^2$

2.2 Discrete random variables

2.2.4 Variance and Standard deviation

Introduction to variance (Var(X))

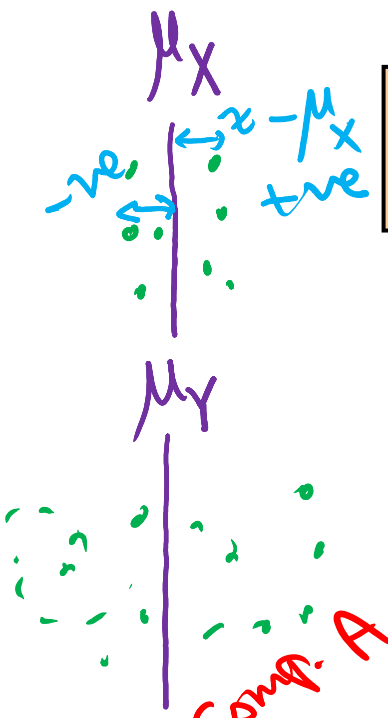
Example

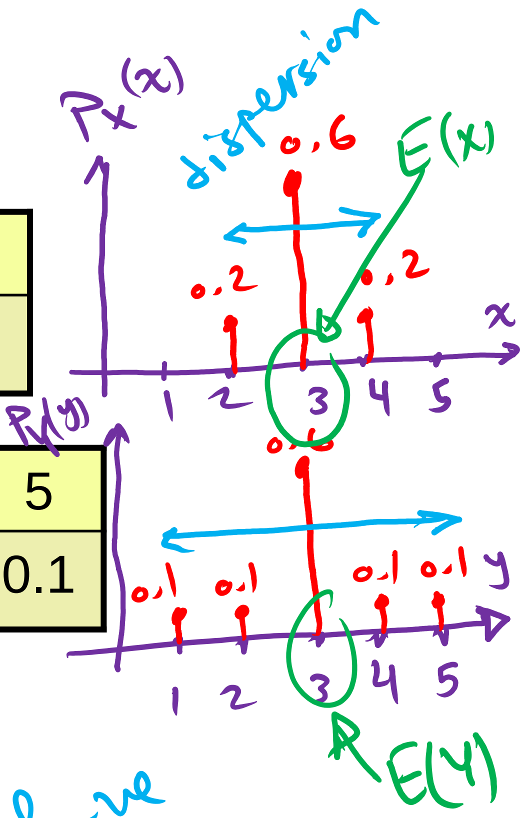
Repair time (in days) → Company A →

x	2	3	4
$P_X(x)$	0.2	0.6	0.2

Repair time (in days) → Company B →

y	1	2	3	4	5
$P_Y(y)$	0.1	0.1	0.6	0.1	0.1





Q: Which is better?

<u>Definition</u> **Var(X) = E(X – μ_X)^2**

to avoid adding +ve & -ve dispersion

$$= \sum_{\forall x} (x-\mu_x)^2 P_x(x)$$

Select Comp. A → less dispersion

E(X) = 3, E(Y) = 3

Var (X) = 0.4, Var (Y) = 1.0

Variance is a measure of dispersion

$$Var(X) = E(X-\mu_x)^2 = \sum_{\forall x} (x-\mu_x)^2 P_x(x) = (2-3)^2 * 0.2 + (3-3)^2 * 0.6 + (4-3)^2 * 0.2$$

$$= 0.4$$

More about variance

$E(\text{Const.}) =$ The same constant.

X : Random variable

μ_X : Constant

Var(X) = E(X – μ_X)^2

<u>Computational formula</u> **Var (X) = E(X^2) – (μ_X)^2** # $= 2^{nd}$ moment $- (1^{st}$ moment$)^2$

$$\mathrm{Var}(X) = E(X-\mu_X)^2 = \sum_{\forall x} \underbrace{(x-\mu_X)^2}\, P_X(x)$$

$$= \sum_{\forall x} \left(x^2 - 2\mu_X x + \mu_X^2\right) P_X(x)$$

$$= \underbrace{\sum_{\forall x} x^2 P_X(x)}_{E(X^2)} - 2\mu_X \underbrace{\sum_{\forall x} x\, P_X(x)}_{E(X) = \mu_X} + \mu_X^2 \underbrace{\sum_{\forall x} P_X(x)}_{1}$$

$$E(X^2) - 2\mu_X^2 + \mu_X^2$$

$$= E(X^2) - \mu_X^2$$

More about variance

$E(\text{const.}) = \text{The same const.}$

$Var(\text{Const}) = 0$

"No dispersion"

$$\boxed{Var(X) = E(X - \mu_X)^2}$$

Computational formula $\quad Var(X) = E(X^2) - (\mu_X)^2$

General property

$Var(a X + b) = a^2 Var(X)$ #

$$Var(aX+b) = E(aX+b)^2 - (E(aX+b))^2$$

$$= E[a^2X^2 + 2abX + b^2] - [a\mu_X + b]^2$$

$$= a^2E(X^2) + \cancel{2ab\mu_X} + \cancel{b^2} - a^2\mu_X^2 - \cancel{2ab\mu_X} - \cancel{b^2}$$

$$= a^2[E(X^2) - \mu_X^2] = a^2 Var(X)$$

More about variance

$$Var(X) = E(X - \mu_X)^2$$

Computational formula $\quad Var(X) = E(X^2) - (\mu_X)^2$

General property

$$Var(aX + b) = a^2 \, Var(X)$$

Standard deviation $\quad \sigma_X = \sqrt{Var(X)}$

water level in a tank
unit

$X : m$

$E(X) : m$

$Var(X) : m^2$

$\sigma_X = \sqrt{var(X)} : m$

Example

For the given PMF,

Find Var (X), Var (2 X)

x	0	1	2	3
$P_X(x)$	0.6	0.2	0.15	0.05

$E(X) = 0.65$

$$E(X^2) = (0)^2 * 0.6 + (1)^2 * 0.2 + (2)^2 * 0.15 + (3)^2 * 0.05$$

$$= \square$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \square - (0.65)^2 = \checkmark$$

Example

Two new product designs are to be compared on the basis of revenue potential on the long run. Marketing estimates that the revenue from design A can be predicted accurately to be 3 M \$. The revenue potential of design B is more difficult to asses. Marketing concludes that there is a probability of 0.3 that the revenue from design B will be 7 M \$. However, there is a probability of 0.7 that the revenue will be 2 M \$. Calculate the mean and standard deviation of the revenue from each design.

Exercise

In previous lectures $\longrightarrow$ General PMFs

x	0	1	2
$P_X(x)$	—	—	—

$P_X(x) = 4\,(0.2)^x \; ; \; x = 1, 2, \ldots$

2.3 Some important discrete PMFs

2.3.1 Bernoulli PMF

Bernoulli Trial

A random experiment with two possible outcomes that may be success or failure.

Bernoulli trial

p → Success $X=1$

$1-p$ → Fail $X=0$

x	0	1
$P_X(x)$	$1-P$	P

$$P_X(x) = \begin{cases} P, & X=1 \\ 1-P, & X=0 \end{cases}$$

$$E(X) = \sum_{\forall x} x \, P_X(x) = 0(1-P) + 1(P) = P$$

$$E(X^2) = (0)^2(1-P) + (1)^2 P = P$$

$$V(X) = E(X^2) - (E(X))^2 = P - P^2 = P(1-P)$$

E(X) = p,

Var(X) = p (1 – p)

Success: Event of interest

Random experiment could have more than 2 outcomes & still Bernoulli

Example

Rolling a die, where a six is "success" and everything else a "failure". Find the probability of success.

has

$X \sim \text{Bernoulli}(p_{\text{success}})$

6 appears

$1/6$