Probability and Statistics

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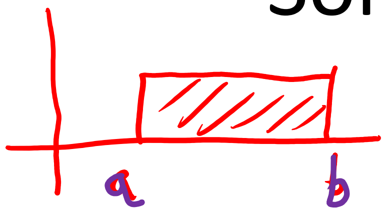
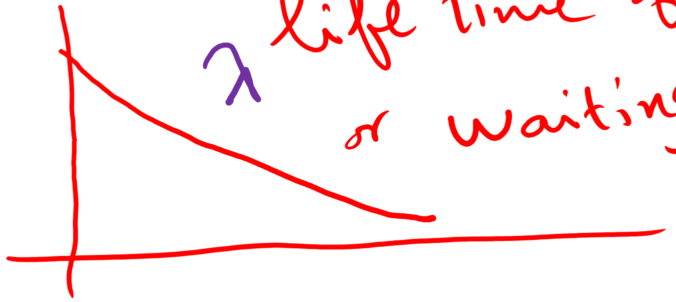
Outline

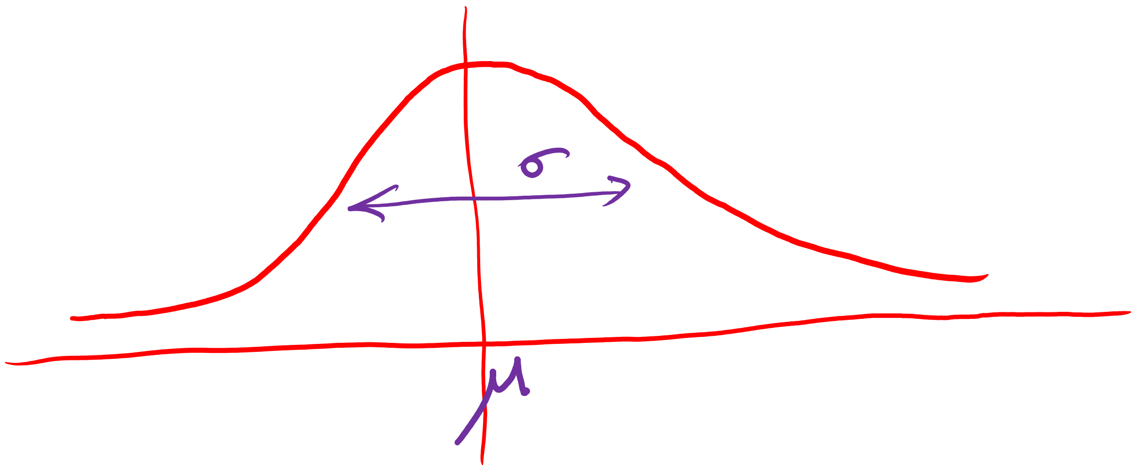
Some important continuous random variables



- Uniform PDF "completely random process"
- Exponential PDF
- Normal PDF



life time of devices or waiting time



Uniform random variable

"Completely random process"

$$E(X) = \int_a^b x \frac{1}{b-a}\,dx$$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$V(X) = E(X^2) - M_X^2$$

$$F_X(x) = \int_a^x \frac{1}{b-a}\,dx$$

PDF

$$f_X(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

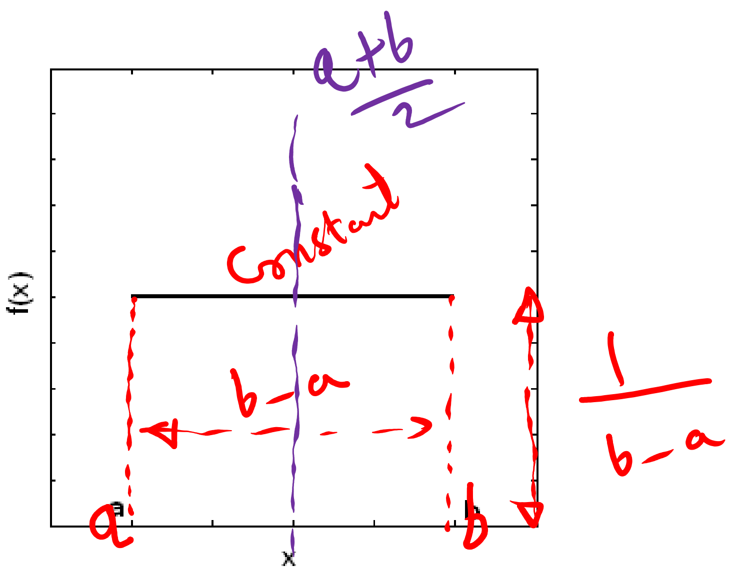
$$X \sim Uniform(a,b)$$

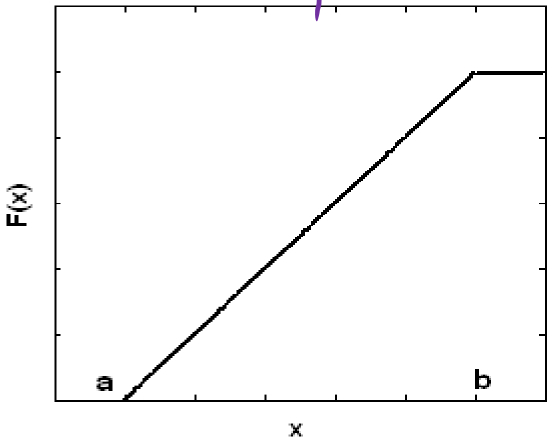
Mean & variance

$$E(X) = \frac{a+b}{2}, \qquad Var(X) = \frac{(b-a)^2}{12}$$

CDF

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$





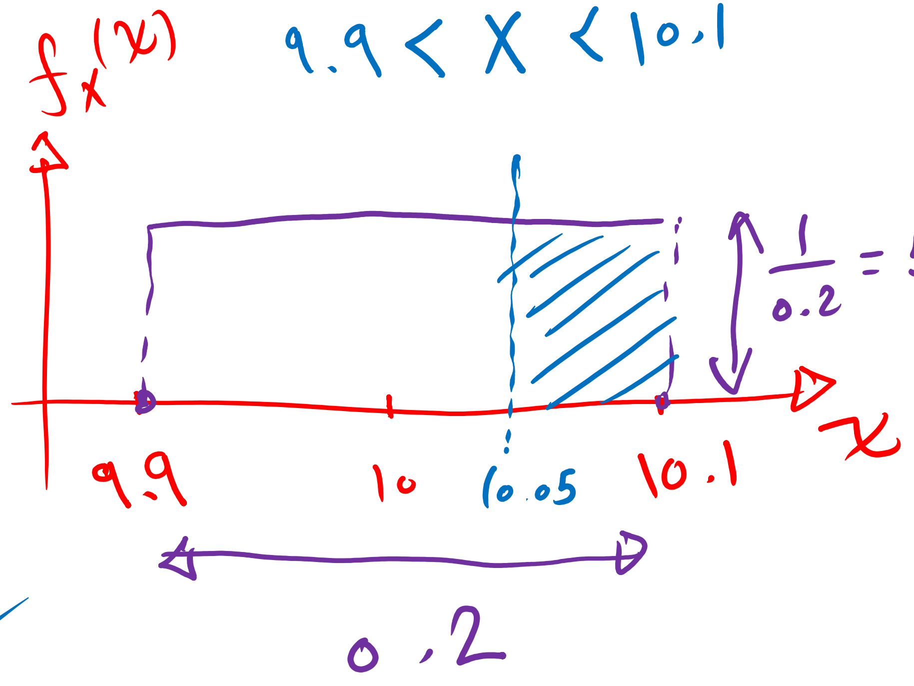
Example

The voltage across a resistor is *uniformally* distributed over the interval (10 ± 0.1).

$$X \sim \text{uniform}(\overset{a}{9.9}, \overset{b}{10.1})$$

What the probability that the voltage exceeds 10.05?

$$9.9 < X < 10.1$$



$$P(X > 10.05) = \int_{10.05}^{10.1} 5 \, dx$$

$$= (10.1 - 10.05) * 5 = \checkmark$$

Example

No info. @ the distⁿ. ≡ default uniform

Mr. Ali arrives at a bus station every day at 7:00 A.M. If a bus arrives at a random time between 7:00 A.M. and 7:30 A.M.

(a) Find the probability that he waits more than 10 minutes.

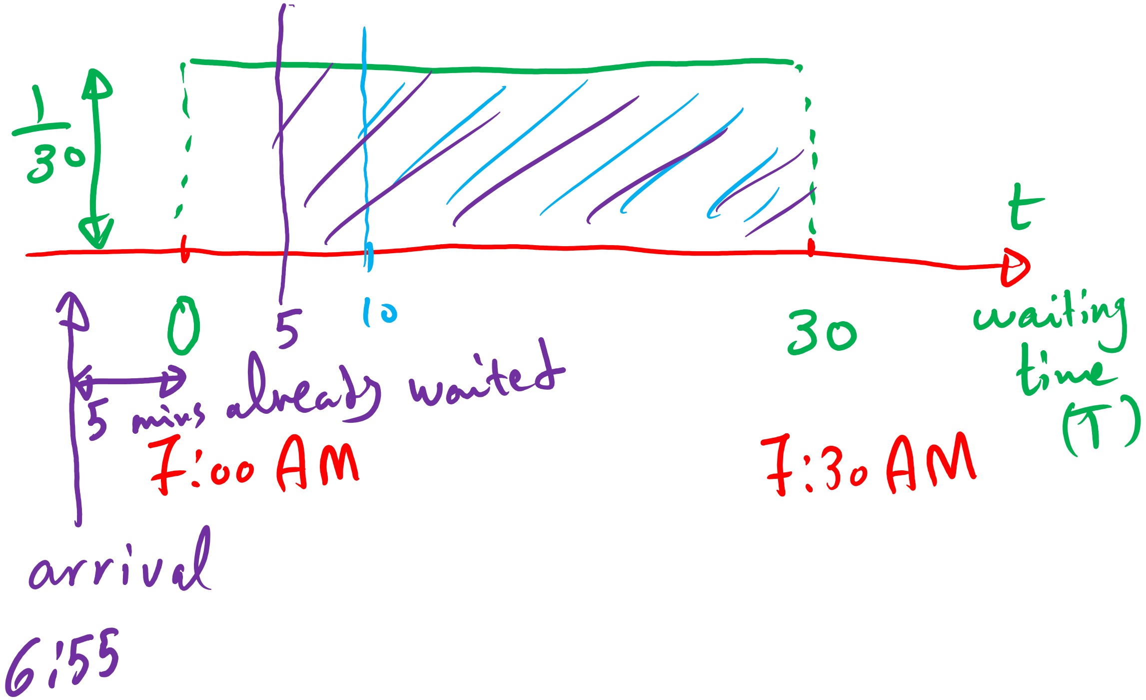
(b) Repeat (a) if Mr. Ali arrives at 6:55 A.M.

Condⁿ. question

(c) If it is known that the waiting time exceeds 10 minutes, what is the probability that it does not exceed 15 mins.

(a) $P(T>10) = (30-10)*\frac{1}{30} = \frac{2}{3}$

(b) $P(T>5) = (30-5)*\frac{1}{30} = \frac{25}{30} = \frac{5}{6}$



(c) $P(T<15 \mid T>10) = \frac{P(10<T<15)}{P(T>10)}$

$= \frac{(15-10)*\frac{1}{30}}{2/3} = ✓$

Example

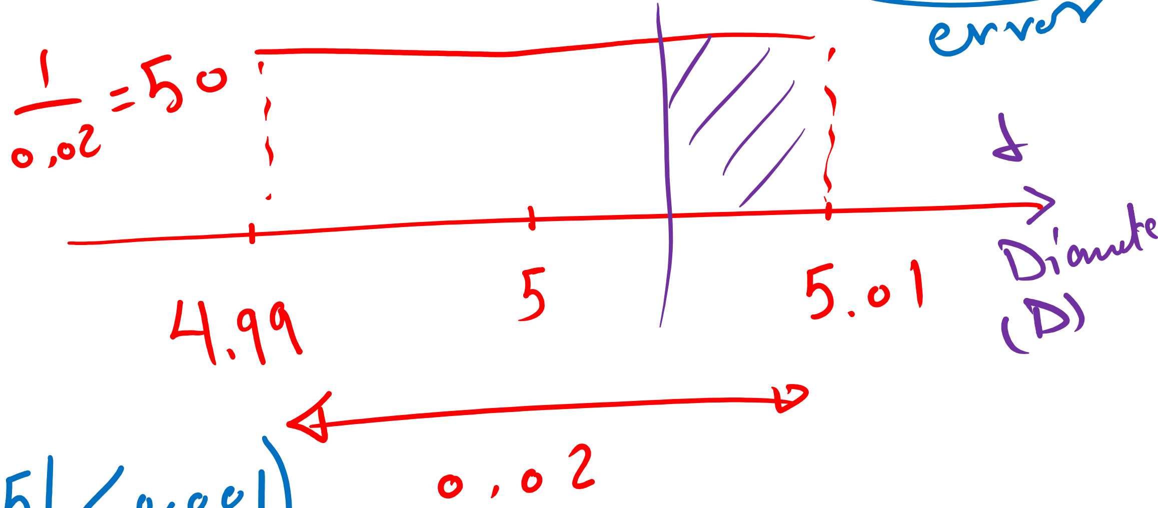
Target

A production line is expected to produce pipes of diameter 5 mm. Actual product is assumed to follow a uniform distribution over the interval [5 ± 0.01].

(a) Compute the probability that the diameter of a randomly chosen pipe is greater than 5.005.

(b) Compute the probability that the diameter of a randomly chosen pipe has a deviation greater than 0.001.

error



ⓐ $P(D > 5.005)$ by symmetry $= \frac{1}{4}$

$(5.01 - 5.005) * 50$

ⓑ $P(\underbrace{|X-5|}_{\text{deviation/error}} > 0.001) = 1 - P(|X-5| < 0.001)$

$= 1 - P(-0.001 < X-5 < 0.001) = 1 - P(4.999 < X < 5.001)$

$= 1 - (5.001 - 4.999) * 50 = \checkmark = 9r$

Example

A production line is expected to produce pipes of diameter 5 mm. Actual product is assumed to follow a uniform distribution over the interval [5 $\pm$ 0.01].

(a) Compute the probability that the diameter of a randomly chosen pipe is greater than 5.005.

(b) Compute the probability that the diameter of a randomly chosen pipe has a deviation greater than 0.001.

(b) A pipe is not accepted if its diameter has a deviation greater than 0.001. Prob. $= q$

In a lot of 100 items, what is the probability of getting at most 2 unaccepted items?

Y : no. of unaccepted items $\sim Bin(100, q) \Rightarrow P_Y(y) = {}^{100}C_y\, q^y (1-q)^{100-y}$

$y = 0, 1, 2, \dots, 100$

$$P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$$

sub.

Example

A production line is expected to produce pipes of diameter 5 mm. Actual product is assumed to follow a uniform distribution over the interval $[5 \pm 0.01]$.

(a) Compute the probability that the diameter of a randomly chosen pipe is greater than 5.005.

(b) Compute the probability that the diameter of a randomly chosen pipe has a deviation greater than 0.001.

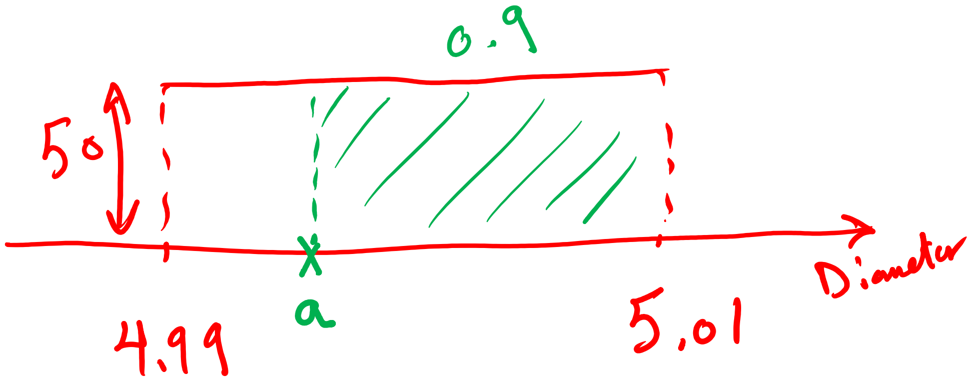
(b) A pipe is not accepted if its diameter has a deviation greater than 0.001. In a lot of 100 items, what is the probability of getting at most 2 unaccepted items?

d) Determine the pipe diameter that is exceeded in 90% of the production.

$$P(D > a) = 0.9$$

$$(5.01 - a) * 50 = 0.9$$

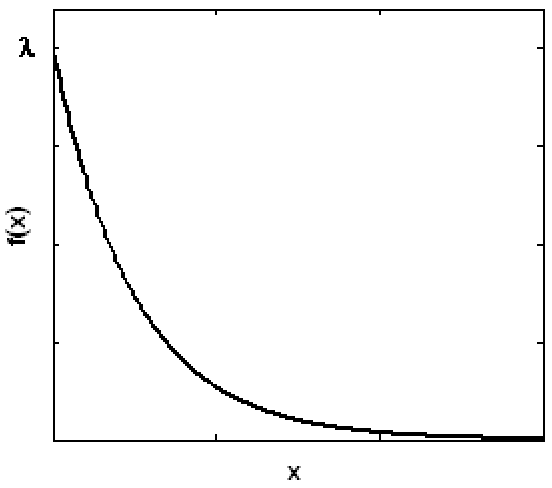
$$5.01 - a = \frac{9}{500} \Rightarrow a = 4.992 \text{ mm}$$



Exponential random variable

PDF

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \quad \lambda > 0$$



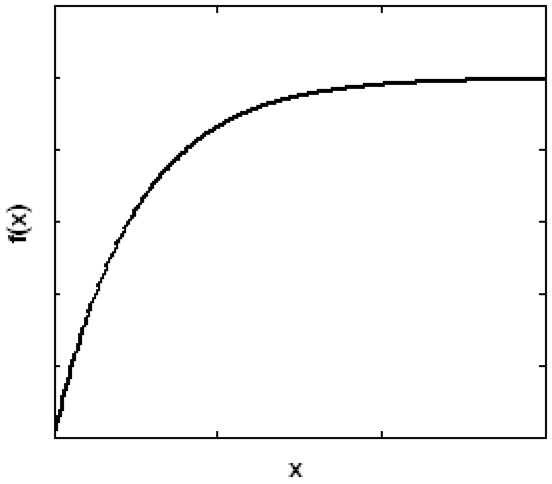
$$X \sim \text{Exponential}(\lambda)$$

Mean & variance

$$E(X) = 1/\lambda, \; Var(X) = 1/\lambda^2$$

CDF

$$\begin{aligned} F_X(x) &= 1 - e^{-\lambda x}, & x \geq 0, \\ &= 0, & x < 0 \end{aligned}$$



Example

The time to failure of a certain device is assumed to follow an exponential distribution of mean 3 years.

What is the probability that it fails during the first year of operation?

Number of events $\sim$ Poisson (λ)	$\longleftrightarrow$	*Time* between events $\sim$ Exponential (λ)

Example

The number of received telephone calls per minute is assumed to follow a Poisson distribution with mean 2 calls.

There is only one person to answer and each call is assumed to have a *fixed* duration of 0.5 minute.

If that person receives a call, what is the probability that the next call will be blocked?

Example

The number of received telephone calls per minute is assumed to follow a Poisson distribution with mean 2 calls.

There is only one person to answer and each call is assumed to have a *fixed* duration of 0.5 minute.

If that person receives a call, what is the probability that the next call will be blocked?

If No call arrived in the last 5 mins, what is the probability that the next call will arrive within the coming 3 mins?

Memoryless property of the exponential distribution

Assume that X follows an exponential distribution with rate λ.

$$\Pr(X \geq x + y | X \geq x) =$$

Normal (Gaussian) random variable

PDF

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

$$-\infty < \mu < \infty, \quad \sigma > 0$$

$$X \sim N(\mu, \sigma^2)$$

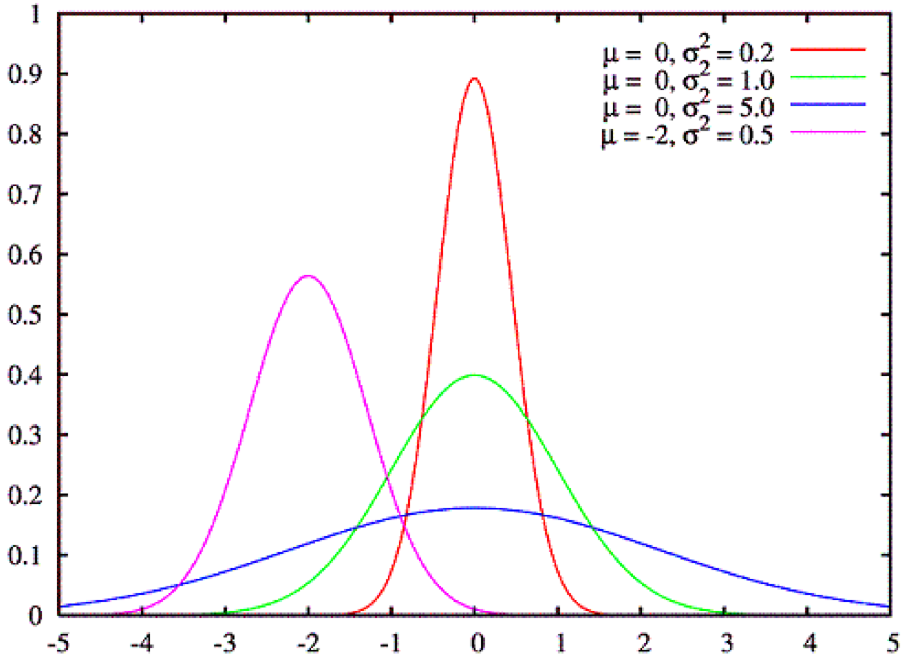
Mean & variance

$$E(X) = \mu,$$

$$Var(X) = \sigma^2$$

CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$



From http://en.wikipedia.org

Standard normal (Gaussian) RV

PDF $Z \sim N(0,1)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$$

CDF

$$\phi(z) = \int_{-\infty}^{z} f_Z(t)dt$$

z	$\phi(z)$
0.00	0.5000
0.01	0.5040
0.02	0.5080
...	...
2.99	0.9986

$\phi(0) =$

$\phi(-\infty) =$

$\phi(\infty) =$

$\phi(-2) =$

Example

Compute

(a) $P(Z < 1)$

(b) $P(Z > 2)$

(c) $P(1.5 < Z < 2.2)$

(d) $P(Z < -1)$

(e) $P(-2.2 < Z < -1.5)$

$$\boxed{X \sim N(\mu, \sigma^2)} \Rightarrow \boxed{\frac{X-\mu}{\sigma} \sim N(0,1)}$$

Example

Height of students in a certain college is assumed to follow a normal distribution with mean 170 cm and standard deviation 2 cm.

(a) What is the probability that the height of a student chosen at random will be greater than 175 cm?

(b) In a group of 50 students, what is the probability of getting at least 2 students with height greater than 175 cm? [$\phi(2.5) = 0.99$]

Example

Height of students in a certain college is assumed to follow a normal distribution with mean 170 cm and standard deviation 2 cm.

(a) What is the probability that the height of a student chosen at random will be greater than 175 cm?

(b) In a group of 50 students, what is the probability of getting at least 2 students with height greater than 175 cm?

(c) Determine the symmetric bounds around the mean that include 90% of all student lengths. $[\phi(1.645) = 0.95]$

Outline

Moment generating function

- Introduction
- Definition
- Proof
- Examples

Introduction

For a random variable X (discrete or continuous)

- The mean: $E(X)$
- The variance: $E(X^2) - E(X)^2$

- We need to calculate $E(X)$, $E(X^2)$, ….., $E(X^n)$, ….

1st moment 2nd moment nth moment

- Moment generating function (MGF): $M_X(t)$

Definition

Definition

$$M_X(t) = E(e^{tX})$$

Discrete **Continuous**

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$E(X^n) = \left[\frac{d^n}{dt^n} M_X(t)\right]_{t=0}$$

MGF for important RV's

$Binomial(n,p)$	$(1-p+pe^t)^n$
$Geometric(p)$	$\dfrac{pe^t}{1-(1-p)e^t}$
$Poisson(\lambda)$	$e^{\lambda(e^t-1)}$
$Uniform(a,b)$	$\dfrac{e^{bt}-e^{at}}{t(b-a)}$
$Exponential(\lambda)$	$\dfrac{\lambda}{\lambda-t}$
$N(\mu,\sigma^2)$	$e^{t\mu+\frac{1}{2}t^2\sigma^2}$

Proof

Example: The Binomial random variable

Binomial RV

X is the *number of successes* $\Rightarrow$ **X ~ Binomial (n, p)**

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, 2, \ldots, n$$

$$n = 1, 2, 3, \ldots, \quad 0 < p < 1$$

E(X) = n p,

Var(X) = n p (1 – p)

Example: The Geometric random variable

Geometric RV

X is the *number of trials until 1st success* $\Rightarrow$ **X ~ Geometric (p)**

$$P_X(x) = p\,(1-p)^{x-1}, \quad x = 1, 2, 3, \ldots$$

$$0 < p < 1$$

$$E(X) = \frac{1}{p}, \qquad \mathrm{Var}(X) = \frac{1-p}{p^2}$$

Example: The Exponential random variable

$$X \sim \text{Exponential}(\lambda)$$

PDF

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \quad \lambda > 0$$

$$E(X) = 1/\lambda, \; Var(X) = 1/\lambda^2$$

Example:

For a random variable with the following PMF

$$f_X(x) = \frac{1}{8}\binom{3}{x} \quad \text{for } x = 0\,,1\,,2\,,3$$

Find the MGF and use it to get $E(X)$ and $V(X)$

Exercise:

Consider the discrete random variable X with probability mass function

x	0	1	2	3
$P(X = x)$	$1/8$	$2/8$	$2/8$	$3/8$

1. Find the moment generating function of X
2. Using the obtained formula, find $E[X]$ and $V[X]$