Classical Mechanics Notes

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1 Kinematics

Define $\vec{v}$, the velocity of a particle to be the time derivative of its position vector $\vec{r}$. Define acceleration as the time derivative of $\vec{v}$. This is equivalent to $\ddot{\vec{r}}$. We can use these equations to determine $\vec{a}$, $\vec{r}$ or $\vec{r}$ given some conditions. The case where $\dot{a} = 0$ gives us useful equations. It is important to distinguish between $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$. The following example illustrates it nicely.

Example: A body moves with constant velocity $\vec{n}$. Find its velocity and speed when it reaches the point (x, y).

Since velocity $\vec{n}$ is constant, at (x, y) the velocity will remain $\vec{n}$. Speed is defined as the magnitude of the velocity vector. So in our case, speed is just n m/s. To find $\frac{\mathrm{d}r}{\mathrm{d}t}$, consider

$$r^2 = x^2 + y^2$$

If the particle will cross the point of maximum height, t decreases with a smaller height h. This is intuitive as it is expeditious for the particle to have a smaller velocity upwards to swiftly reach the maximum point before falling down. Otherwise if it has already passed or will not reach the point of maximum height, t decreases with a larger height h. This is also intuitive as it implies that the particle has a greater initial velocity and will reach the required point in a shorter time interval.

1.1 Equations of Motion Under Constant Acceleration

We begin by considering the definition of $\vec{a}$

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

Solving this differential equation (and letting the initial time be 0)

$$\int_0^t \vec{a}\mathrm{d}t = \int_{\vec{v}_0}^{\vec{v}_t} \mathrm{d}\vec{v}$$

$$\vec{a}t = \vec{v}_t - \vec{v}_0$$

$$\vec{v}(t) = \vec{v}_0 + at \tag{1}$$

If we use $\vec{v}(t) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$, and integrate,

$$\Delta x = \vec{v}_0 t + \frac{1}{2}at^2$$

Next we consider $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$ (by the chain rule)

$$a\mathrm{d}x = v\mathrm{d}v \iff a\int_{x_0}^{x} \mathrm{d}x = \int_{v_0}^{v} v\mathrm{d}v$$

$$a\Delta x = \frac{1}{2}(v^2 - v_0^2)$$

$$2a\Delta x = v^2 - v_0^2 \tag{2}$$

1.2 Kinematics in Polar Coordinates

1.2.1 Basis Vectors

Like $\hat{x}$ and $\hat{y}$ in the cartesian system, we have $\hat{r}$ and $\hat{\theta}$ as the orthogonal basis vectors in the polar system. It is important to note that $\hat{r}$ and $\hat{\theta}$ are both functions of time. $\hat{r}$ may also be dependent on θ or vice versa. Assume a point (r, θ) (or (x, y) in the cartesian system). Analyzing $\hat{r}$ and $\hat{\theta}$ results in the following equations

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{3}$$

$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \tag{4}$$

$\hat{r}$ points in the radial direction and $\hat{\theta}$ points in the direction of increasing θ (counter-clockwise taken to be positive). We can write everything in matrix form as

$$\begin{bmatrix}\hat{r} \\ \hat{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{x} \\ \hat{y}\end{bmatrix} \tag{5}$$

The matrix $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ is called the rotation matrix.

1.2.2 Velocity

The velocity $\dot{\vec{r}}$ is

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

To compute $\dot{\hat{r}}$, we use (3)

$$\dot{\hat{r}} = -\dot{\theta}\sin\theta\hat{x} + \dot{\theta}\cos\theta\hat{y} = \dot{\theta}(-\sin\theta\hat{x} + \cos\theta\hat{y}) \tag{6}$$

Similarly

$$\dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{7}$$

Noting that $\hat{\theta} = \dot{\theta}(-\sin\theta\hat{x} + \cos\theta\hat{y})$, we can write

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{8}$$

1.3 Acceleration

We simply compute the time derivative of (7)

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r(\ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\hat{\theta}}) = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r}$$

Combining the like terms gives us

$$\boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}} \tag{9}$$

2 Angular Kinematics

If, in a coordinate system of choice, a particle makes an angle θ w.r.t. an axis, then the **angular velocity** of the point about this axis is defined as

$$\vec{\omega} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\omega}$$

The direction of $\vec{\omega}$ is obtained as follows: curl your fingers (of your right hand) in the direction of rotation, then the direction of the thumb is the direction of $\hat{\omega}$. A particle having angular velocity $\vec{\omega}$ and position vector $\vec{r}$ w.r.t. a reference point moving with velocity $\vec{v}_{ref}$ has velocity

$$\vec{v} = \vec{v}_{\text{ref}} + \vec{\omega} \times \vec{r} \tag{10}$$

It is quite important to note that the operator that gives the rate of change of a vector is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}_{\text{rot. frame}} + \vec{\omega}\times \tag{11}$$

3 Statics

The laws of motion can be summarized as

1. A body moves with constant velocity unless acted upon by a force.
2. The rate of change of momentum of a body is equal to the force acting on it
3. For every force, there exists an equal and opposite force.

From this, we can define 0 force to be the force that results in no motion (of a body at rest). Consider a wedge with angle θ.

4 Momentum and Collisions

Recall newton's second law

$$\vec{F}^e = \dot{\vec{p}}$$

If there is no external force, then $\Delta\vec{p} = 0$. This tells us that in absence of external forces, the momentum of a system is conserved.

5 Energy

Say a body is under the influnce of a force $\vec{F}$ which causes it to be displaced by $\mathrm{d}\vec{r}$. Then define the work done in moving the body from $\vec{r}_0$ to $\vec{r}_f$ as

$$W \equiv \int_{\vec{r}_0}^{\vec{r}_f} \vec{F} \cdot \mathrm{d}\vec{r}$$

For a constant force $\vec{F}$, it boils down to

$$W = \vec{F} \cdot (\vec{r}_f - \vec{r}_i)$$

We can write this in a slightly different manner. Write the second law (in the x direction) and separate variables

$$F_x = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$F_x \mathrm{d}x = mv\mathrm{d}v$$

Notice that the LHS is inf. work, now integrate to get

$$W_x = \frac{1}{2}m(\Delta v_x)^2$$

We can define the RHS as the **kinetic energy**, the energy attributed to the motion of a body. Doing this for all directions and summing gives

$$\int_{r_0}^{r_f} \vec{F} \cdot \mathrm{d}\vec{r} = \frac{1}{2}m(v_f^2 - v_0^2)$$

Letting $E \equiv (1/2)mv_0^2$, $V(r) \equiv -\int_{r_0}^{r_f} \vec{F} \cdot \mathrm{d}\vec{r}$, $T(v) = (1/2)mv(t)^2$ gives

$$E = T + V = \text{const.}$$

This proves that **energy is conserved**. Forces for which we can write define a **potential** function are called **conservative**. The differential form is

$$\vec{F} = -\nabla V$$

The proof for this is that

$$\mathrm{d}V = F^i \mathrm{d}x_i$$

by definition. Using the chainrule,

$$\mathrm{d}V = \frac{\partial V}{\partial x_i}\mathrm{d}x_i$$

Thus,

$$F^i = \frac{\partial V}{\partial x_i}$$

Note that we have used the einstein summation. The interesting fact about conservative forces is that their potentials are strictly functions of position alone. Else, the force turns out to be non-conservative. Friction is an example, it depends on velocity. Let us consider the loop integral for a conservative force.

$$\oint \vec{F} \cdot \mathrm{d}\vec{r} = \int_A^B \vec{F} \cdot \mathrm{d}\vec{r} + \int_B^A \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

Then using stoke's theorem, we can also write

$$\nabla \times \vec{F} = 0$$

To summarize, the following are equivalent.

1. $\vec{F}$ is conservative
2. $\exists V(r)$ satisfying
$$\vec{F} = -\nabla V$$
3. $\nabla \times \vec{F} = 0$
4. $\oint \vec{F} \cdot \mathrm{d}\vec{r} = 0$
5. Cross derivatives of the components of $\vec{F}$ are equal. That is,
$$\frac{\partial F^i}{\partial x_j} = \frac{\partial F^j}{\partial x_i}$$
Where F_i is the ith component of the force vector. Note that this is just a restatement of the curl($\vec{F}$)=0 fact.

All this was pertaining only to the linear case. In general,

$$W = \int_{\vec{r}_{0CM}}^{\vec{r}_{fCM}} \vec{F} \cdot \mathrm{d}\vec{r}_{CM} + \int_{\theta_i}^{\theta_f} \vec{\tau} \mathrm{d}\theta$$

Where θ is measured about the CM.
Define **Power** as

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}$$

Let us write it in a more appealing way. Using $\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r}$

$$P = \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{F} \cdot \vec{v}$$

6 Angular Momentum

We define the angular momentum of a particle to be

$$\vec{L} = \vec{r} \times \vec{p} \tag{12}$$

Where $\vec{r}$ is the radius vector and $\vec{p}$ is the momentum vector respectively. We can also define the quantity torque, as

$$\sum \vec{\tau} \equiv \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

If we consider a differential mass element $\mathrm{d}m$ on a body (lying in the $x-y$ plane) rotating about an arbitrary axis parallel to the z axis with angular velocity ω, its $\vec{L}$ (about that axis) is

$$\vec{L} = \vec{r} \times \mathrm{d}m\vec{v} = r^2 \omega \mathrm{d}m\hat{z}$$

Now if we want to find $\vec{L}$ for the entire body, we just have to integrate over all the dm's

$$\vec{L} = \int \omega r^2 \mathrm{d}m$$

Since ω is usually constant, we can pull that out of the integral and define I to be

$$I \equiv \int r^2 \mathrm{d}m \tag{13}$$

Here, r is the perpendicular distance from the axis of rotation. If It is clear from the expression for I that it depends on the axis of rotation. For example, the moment of inertia of a sphere rotating about its diameter would not be the same as its moment of inertia about a point on its surface. If the moment of inertia takes the form $I = Mk^2$, then k is defined as the radius of gyration $k \equiv \sqrt{I/M}$. If the mass distribution is discrete, we can write (2) as

$$I = \sum_i r_i^2 m_i$$

With this definition, we can now write $\vec{L}$ as

$$\vec{L} = I\vec{\omega} \tag{14}$$

Since both I and ω vary with the choice of coordinates, so does $\vec{L}$. It is nice to note that this is the angular analogue for linear momentum $\vec{p} = m\vec{\dot{x}}$. The expression for kinetic energy for a rotating body is

$$\mathrm{d}K = \frac{1}{2} r^2 \omega^2 \mathrm{d}m$$

And so if we add the energies of all the differential masses, we get

$$K = \frac{1}{2} \int r^2 \omega^2 \mathrm{d}m$$

Noting that ω is usually constant, we get

$$K = \frac{1}{2} I \omega^2 \tag{15}$$

So far, we have only considered bodies executing only rotational motion. If we have a body that is both translating and rotating? We consider the motion as the sum of two distinct motions: the translation of the centre of mass and the rotation of the body about the centre of mass. We can do this thanks to **Chasle's Theorem**. Let $\vec{r}, \vec{r'}, \vec{R}$ represent the positions of a region in the body, the same region w.r.t the centre of mass and the centre of mass respectively. Clearly,

$$\vec{r} = \vec{r'} + \vec{R}$$

and so,

$$\vec{v} = \vec{v'} + \vec{V}$$

So $\vec{L}$ is

$$\begin{aligned}\vec{L} &= \int \mathrm{d}\vec{L} = \int \vec{r} \times \vec{p} \\ &= \int (\vec{r'} + \vec{R}) \times \mathrm{d}m(\vec{v'} + \vec{V}) \\ &= \int (\vec{r'} \times \vec{v'} + \vec{r'} \times \vec{V} + \vec{R} \times \vec{v'} + \vec{R} \times \vec{V}) \mathrm{d}m\end{aligned}$$

Using that fact that $\int \vec{r'} \mathrm{d}m = \int \vec{R} \times \vec{v'} = 0$ in the centre of mass frame, everything simplifies to

$$\vec{L} = \int (\vec{r'} \times \vec{v'}) \mathrm{d}m + \int (\vec{R} \times \vec{V}) \mathrm{d}m$$

$$\vec{L} = \vec{L}_{CM} + \vec{R} \times \vec{P}$$

Similarly, the total kinetic energy turns out to be

$$K = \frac{1}{2} M V^2 + \frac{1}{2} I_{CM} \omega^2$$

The Parallel Axis Theorem tells us

$$I_c = I_{CM} + MR^2 \tag{16}$$

For a planar body, The Perpendicular Axis Theorem tell us

$$I_z = I_x + I_y \tag{17}$$

Where x, y, z are mutually orthogonal and concurrent axes. Note that (14) is only valid for planar bodies. However, if we have a body that is rotating about only a single axis, then we can apply (14) to non planar objects as well.

Table 1: All objects have (uniform) total mass M

Moments of Inertia of Some Objects	
Ring (radius R,axis perp. to plane)	MR^2
Ring (radius R, axis through centre, parallel to plane)	$\frac{1}{2}MR^2$
Disk (radius R, axis through centre, perp. to plane)	$\frac{1}{2}MR^2$
Disk (radius R, axis through centre, parallel to plane)	$\frac{1}{4}MR^2$
Rod (length L, axis through centre, perp. to rod)	$\frac{1}{12}ML^2$
Rod (length L, axis through end, perp. to rod)	$\frac{1}{3}ML^2$
Spherical shell (radius R, axis through centre)	$\frac{2}{3}MR^2$
Solid sphere (radius R, axis through centre)	$\frac{2}{5}MR^2$
Isosceles Triangle (length L, vertex angle 2α, axis through vertex, perp. to plane)	$(1/2)ML^2[1-(2/3)\sin^2\alpha)]$
Regular N-gon (radius R, axis through centre, perp. to plane)	$(1/2)ML^2[1-(2/3)\sin^2(\pi/N)]$
Rectangle (dimensions a,b, axis through centre, perp. to plane)	$(1/12)M(a^2+b^2)$

6.1 Torque

Define torque as the time derivative of the angular momentum

$$\tau \equiv \dot{L} \tag{18}$$

We consider 3 cases.

6.1.1 Point mass, fixed origin

Since $\vec{L} = \vec{r} \times \vec{p}$, we have

$$\vec{\tau} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} \tag{19}$$

Thus, we can write $\vec{\tau} = \vec{r} \times \vec{F}$.

6.1.2 Extended mass, Fixed origin

Let $\vec{r}_i$ and $\vec{p}_i$ denote the position and momentum vectors respectively of the i-th particle. Then, we have

$$\vec{L} = \sum_i L_i = \sum_i \vec{r}_i \times \vec{p}_i \tag{20}$$

Thus, torque can be written as

$$\vec{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \vec{r}_i \times \vec{p}_i = \sum_i \left(\dot{\vec{r}}_i \times \vec{p}_i + \vec{r}_i \times \dot{\vec{p}}_i\right) = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} \tag{21}$$

6.1.3 Extended mass, Nonfixed origin

Let $\vec{r}_0$ be the position vector of a moving frame and $\vec{r}_i$ the position vectors of the i-th mass (in the fixed frame). Then we have

$$\vec{L} = \sum_i (\vec{r}_i - \vec{r}_0) \times m(\dot{\vec{r}}_i - \dot{\vec{r}}_0) \tag{22}$$

This gives us

$$\vec{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i (\vec{r}_i - \vec{r}_0) \times m(\dot{\vec{r}}_i - \dot{\vec{r}}_0) = \sum_i \left[(\dot{\vec{r}}_i - \dot{\vec{r}}_0) \times m(\dot{\vec{r}}_i - \dot{\vec{r}}_0) + (\vec{r}_i - \vec{r}_0) \times m(\ddot{\vec{r}}_i - \ddot{\vec{r}}_0)\right] \tag{23}$$

The first term is 0 because $\vec{a} \times \vec{a} = 0$.

$$\vec{\tau} = \sum_i (\vec{r}_i - \vec{r}_0) \times (\vec{F}_i^{\text{ext}} - m\ddot{\vec{r}}_0) = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{\text{ext}} - \sum_i (\vec{r}_i - \vec{r}_0) \times m\ddot{\vec{r}}_0 \tag{24}$$

The last term can be computed as follows

$$\sum_i \vec{r}_i \times m\ddot{\vec{r}}_0 - \sum_i \vec{r}_0 \times m\ddot{\vec{r}}_0 = M\vec{R}_{CM} \times \ddot{\vec{r}}_0 - \sum_i \vec{r}_0 \times m\ddot{\vec{r}}_0 = (M\vec{R}_{CM} - \vec{r}_0 \sum_i m) \times \ddot{\vec{r}}_0 = M(\vec{R}_{CM} - \vec{r}_0) \times \ddot{\vec{r}}_0 \tag{25}$$

This allows us to write

$$\boxed{\vec{\tau} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{\text{ext}} - M(\vec{R}_{CM} - \vec{r}_0) \times \ddot{\vec{r}}_0} \tag{26}$$

Note: if Angular impulse, $\mathcal{I}_\theta$ is defined as

$$\mathcal{I}_\theta \equiv \int_{t_1}^{t_2} \vec{\tau}(t)\mathrm{d}t \tag{27}$$

Then, using $\mathrm{d}\vec{L} = \vec{\tau}\mathrm{d}t$, we get

$$\int_{t_1}^{t_2} \vec{\tau}_{net}\mathrm{d}t = \vec{L}_2 - \vec{L}_1$$

If, for a constant $\vec{r}$, $\vec{\tau} = \vec{r} \times \vec{F}$

$$\int_{t_1}^{t_2} \vec{\tau}(t)\mathrm{d}t = \vec{r} \times \int_{t_1}^{t_2} \vec{F}(t)\mathrm{d}t = \vec{r} \times \mathcal{I}$$

Where $\mathcal{I} \equiv \int_{t_1}^{t_2} \vec{F}(t)\mathrm{d}t = \Delta\vec{p}$ is the regular impulse. We also get the result that

$$\mathcal{I}_\theta = \vec{r} \times \mathcal{I}$$

Note that this is only applicable when $F(t)$ is acting on a constant position $\vec{r}$, else we cannot pull it out of the integral like we did in this case. Combining all results, we get

$$\boxed{\Delta\vec{L} = \vec{r} \times \Delta\vec{p}} \tag{28}$$

7 Angular Momentum II

The velocity $\vec{v}$ of an object with angular velocity $\vec{\omega}$ with position vector $\vec{r}$ (relative to origin of rotation) is given by

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{29}$$

7.1 The Inertia Tensor

The expression for $\vec{L}$, as defined by (9) for a continuous mass distribution is

$$\vec{L} = \int \vec{r} \times (\vec{\omega} \times \vec{r})\mathrm{d}m \tag{30}$$

Computing this double cross product allows us to write

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{31}$$

Where $I_{ab} = -\int ab\mathrm{d}m$ and $I_{aa} = \int (b^2 + c^2)\mathrm{d}m$. The orthonormal basis vectors that make I diagonal are called the principal axes.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{32}$$

And each I_{ii} is called a principal moment.

7.2 Euler Equations

Consider a body revolving with angular velocity $\vec{\omega}$. If we consider the system in its principal axes, then we can write

$$\vec{\omega} = \langle \omega_1, \omega_2, \omega_3 \rangle$$

To compute $\vec{\tau} = \dot{\vec{L}}$, we simply compute the derivative of $\vec{L} = \langle I_1\omega_1, I_2\omega_2, I_3\omega_3 \rangle$. Doing so yields

$$\tau = \sum_{i=1}^{3} I_i \dot{\omega}_i \hat{x}_i + \sum_{i=1}^{3} I_i \omega_i \dot{\hat{x}}_i$$

Now we make use of the fact that (considering that a vector has fixed length) $\dot{A} = \vec{\omega} \times \vec{A}$ where $\vec{\omega}$ is its angular velocity. And thus, our torque equation is

$$\tau = \sum_{i=1}^{3} I_i \dot{\omega}_i \hat{x}_i + \vec{\omega} \times \vec{L}$$

Computing the cross product,

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{x_1} & \hat{x}_2 & \hat{x}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ L_1 & L_2 & L_3 \end{vmatrix} = (L_3\omega_2 - L_2\omega_3)\hat{x}_1 + (L_1\omega_3 - L_3\omega_1)\hat{x}_2 + (L_2\omega_1 - L_1\omega_2)\hat{x}_3$$

Equating the components of $\vec{\tau}$ with the components of the RHS,

$$\tau_i = I_i \dot{\omega}_i + (I_k - I_j)\omega_k \omega_j$$

These equations are called the Euler Equations.

8 Waves

8.1 Oscillation

8.1.1 Undamped

Applying newton's second law to a **harmonic oscillator** gives the following equation

$$m\ddot{x} = -kx$$

Which can be solved by defining $\omega \equiv \sqrt{k/m}$ as the **angular frequency** and writing

$$x = \mathrm{Re}[e^{i(\omega t + \varphi)}]$$

Where φ is a constant dependent on initial conditions. Analogously, for a **physical pendulum**, one obtains that

$$\omega = \sqrt{\frac{Mg\ell_{cm}}{I}}$$

8.1.2 Damped

Say now we also have a dissipative force proportional to $\dot{x}$.

$$f = -b\dot{x}$$

What is $x(t)$ now? Newton's second law reads

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

Where we have defined $\gamma \equiv b/m$. We guess that $x(t) = \mathrm{Re}(e^{i\alpha t})$ and thus,

$$-\alpha^2 + i\alpha\gamma + \omega^2 = 0$$

Solving for α,

$$\alpha = \frac{i\gamma}{2} \pm \sqrt{\omega^2 - \frac{\gamma^2}{4}}$$

The nature of the determinant plays an important role.

Underdamped $\omega^2 > \gamma^2/4$

This means that the drag force is small. For brevity, define $\omega' \equiv \sqrt{\omega^2 - \frac{\gamma^2}{4}}$

$$x_1(t) = \mathrm{Re}[e^{-\gamma t/2} e^{i\omega' t}] = e^{-\gamma t/2} \cos \omega' t$$

$$x_2(t) = \mathrm{Re}[e^{-\gamma t/2} e^{-i\omega' t}] = e^{-\gamma t/2} \sin \omega' t$$

Critically Damped $\omega^2 = \gamma^2/4$

This means that $\omega' = 0$ and thus

$$x(t) = Ae^{-(\gamma/2)t}$$

The motion is not periodic!

Overdamped $\omega^2 < \gamma^2/4$

$$\alpha = i\left(\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2}\right)$$

Then,

$$x_1(t) = \mathrm{Re}\left[e^{-(\gamma/2)t} \exp\left\{\left(\frac{\gamma^2}{4} - \omega^2\right)\right\}\right]$$

8.1.3 Driven Oscillations

Say we have a **driving force**

$$f_d = f_0 \cos \omega_d t$$

Our $f = ma$ equation is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos \omega_d t$$

To solve this, we use complex numbers and guess

$$x = \mathrm{Re}(z) = \mathrm{Re}(Ae^{i\omega_d t})$$

$$Ae^{i\omega_d t}(-\omega_d^2 + i\gamma\omega_d + \omega_0^2) = f_0 e^{i\omega_d t}$$

Which gives

$$A = \frac{f_0}{-\omega_d^2 + i\gamma\omega_d + \omega_0^2}$$

So our solution is

$$x = \mathrm{Re}\left[\frac{f_0}{-\omega_d^2 + i\gamma\omega_d + \omega_0^2} e^{i\omega_d t}\right]$$

We still need to condense the expression into something nicer. Let us define the denominator as

$$I \equiv \omega_0^2 - \omega_d^2 + i\gamma\omega_d$$

Then notice that we can write

$$I = \sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2} e^{i\varphi}$$

Where

$$\varphi = \arctan\left(\frac{\gamma\omega_d}{\omega_0^2 - \omega_d^2}\right)$$

Combining all this,

$$z(t) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}} e^{i(\omega_d t - \varphi)}$$

Therefore

$$\boxed{x(t) = \frac{f_0 \cos(\omega_d t - \varphi)}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}}}$$

8.2 Normal Modes

Assume two masses m_1 , m_2 between two walls, each connected to the nearest wall by a spring of spring constant k and connected to each other by another spring of spring constant κ. Let the displacement from equilibrium of m_i be $x_i(t)$. Then

$$m_1\ddot{x}_1 = -kx_1 - \kappa(x_1 - x_2)$$
$$m_2\ddot{x}_2 = -kx_2 - \kappa(x_2 - x_1)$$

How do we solve this? Define

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

So then we can write

$$-m_1A_1\omega^2 + kA_1 + \kappa(A_1 - A_2 = 0$$
$$-m_2A_2\omega^2 + kA_2 + \kappa(A_2 - A_1) = 0$$

Conveniently, in matrix form,

$$\begin{bmatrix} -m_1\omega^2 + k + \kappa & \kappa \\ \kappa & -m_2\omega^2 + k + \kappa \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We assume that the determinant is 0.

$$\begin{vmatrix} -m_1\omega^2 + k + \kappa & -\kappa \\ -\kappa & -m_2\omega^2 + k + \kappa \end{vmatrix} = 0$$

Let us assume $m_1 = m_2 = m$ for an enlightening result

$$-m\omega^2 + k + \kappa = \pm\kappa$$

This gives us

$$\omega^2 = \frac{k}{m}$$
$$\omega^2 = \frac{2\kappa + k}{m}$$

Plugging the first value to our original matrix gives

$$\kappa \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

Which implies $(A_1, A_2) = (\tilde{\zeta}, \tilde{\zeta})$. Where $\tilde{\zeta} \equiv \zeta e^{i\varphi}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} e^{i\omega t + \varphi}$$

Taking the real part gives

$$x_1(t) = \zeta\cos(\omega_a t + \varphi_a)$$
$$x_2(t) = \zeta\cos(\omega_a t + \varphi_a)$$

8.3 Mechanical Waves

Waves can be thought of as an array of infinite coupled oscillator. We denote the wavefunction $\Psi(x, t)$ to denote the displacement from equilibrium at position x at time t. There are two types of waves:

- Longitudinal: Displacement is along direction of propagation
- Transverse: displacement is perpendicular to direction of propagation

8.4 The One-Dimensional Wave Equation

We simply apply newton's second law to a taut string. Let the tension in the string be T and its linear mass density μ. Assume we displace a section of length $\mathrm{d}x$ at x by $\Psi(x,t)$. The forces that act on this section are the restoring tensions. Decompose the forces, and let $\theta_2 = \theta_1 + \mathrm{d}\theta$. Thus, in the y and x directions respectively, we have

$$T\sin(\theta_1 + \mathrm{d}\theta) - T\sin\theta_1 = m\ddot{y} \tag{33}$$

$$T\cos\theta_2 - T\cos\theta_1 = m\ddot{x} = 0 \tag{34}$$

Now we use the small angle approximation $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. We can thus, safely abandon the second equation and modify the first one and write

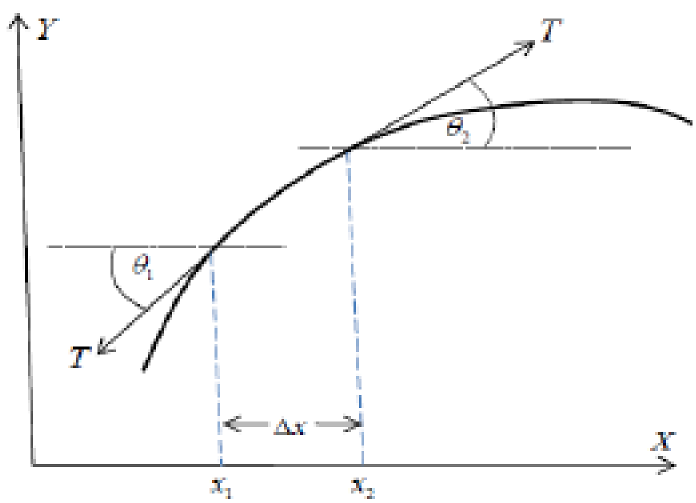
$$T\mathrm{d}\theta = \mu \mathrm{d}x\ddot{\Psi} \tag{35}$$

where we used the fact that y represents Ψ. Rearranging,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mu}{T}\frac{\partial^2\Psi}{\partial t^2} \tag{36}$$

Recognizing that $\mathrm{d}\theta = \mathrm{d}\Psi/\mathrm{d}x$, we finally get

$$\boxed{\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}} \tag{37}$$



8.4.1 Solution of the Wave Equation

Define $\xi = x - vt$ and $\eta = x + vt$. Thus,

$$\Psi(x,t) = \Psi\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2v}\right)$$

Using the multivariate chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = -v\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}$$

The following can be verified

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial^2}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial \eta^2} + 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \xi^2}\right)$$

Thus, plugging into the wave equation,

$$\left(\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\eta\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)\Psi=\frac{v^2}{v^2}\left(\frac{\partial^2}{\partial\eta^2}+2\frac{\partial^2}{\partial\eta\partial\xi}+\frac{\partial^2}{\partial\xi^2}\right)\Psi$$

This gives us

$$\frac{\partial^2\Psi}{\partial\eta\partial\xi}=0$$

It is thus clear that

$$\frac{\partial\Psi}{\partial\eta}=h(\eta)$$

where h is an arbitrary function dependent only on η. We can thus integrate and get

$$\Psi=\int h(\eta)\mathrm{d}\eta+g(\xi)$$

$$\Psi=f(x+vt)+g(x-vt)$$

Notice that this tells us that v must be the "velocity" of the wave. As t increases, the argument of f increases, and that of g decreases. It is well known that decreasing the argument of a single variate function is equivalent to shifting the graph by that amount in the positive direction. This allows us to write the general solution for a wave moving in the +x direction as

$$\Psi=A\cos(k(x-vt)+\varphi)$$

We have included the constants A,φ,k to maintain generality. In reality, they are determined by initial conditions. We could, alternatively write

$$\Psi=A\cos(kx-\omega t+\varphi)$$

Yet again, we can also write

$$\Psi=\mathrm{Re}(Ae^{i(kx-\omega t+\varphi)})$$

Where $\omega=kv$ is the angular frequency of oscillations. We also note that

$$k=\frac{2\pi}{\lambda}$$

8.4.2 Energetics of String Waves

8.4.3 Kinetic Energy

It is more suitable to consider the linear kinetic energy density ε instead.

$$\mathrm{d}K=\frac{1}{2}\mu\mathrm{d}x\left(\frac{\partial\Psi}{\partial t}\right)^2$$

$$\varepsilon_K=\frac{1}{2}\mu\left(\frac{\partial\Psi}{\partial t}\right)^2$$

8.4.4 Potential Energy

Again, we are more interested in the Potential energy density. The amount by which a string stretches is

$$\sqrt{1+\left(\frac{\partial\Psi}{\partial x}\right)^2}\mathrm{d}x-\mathrm{d}x$$

Using $(1+x)^n\approx 1+nx$, and noting that the work done is the amount by which the string stretches times the tension,

$$\mathrm{d}U=\frac{1}{2}T\left(\frac{\partial\Psi}{\partial x}\right)^2\mathrm{d}x$$

Thus the potential energy density is

$$\varepsilon_P=\frac{1}{2}T\left(\frac{\partial\Psi}{\partial x}\right)^2$$

Thus the total energy density is

$$\boxed{\varepsilon(x,t) = \frac{\mu}{2}\left(\left(\frac{\partial \Psi}{\partial t}\right)^2 + v^2\left(\frac{\partial \Psi}{\partial x}\right)^2\right)}$$

8.4.5 Power

The force that a section experiences is

$$-T\frac{\partial \Psi}{\partial x}$$

Taking the time derivative gives us

$$P = -T\frac{\partial \Psi}{\partial t}\frac{\partial \Psi}{\partial x}$$

8.4.6 Sound Waves

Consider a tube of gas with uniform (inital) density ρ and cross sectional area A. Define $\Psi(x,t)$ as the displacement of a molecule at equilibrium position x. Let the equilibrium pressure be p_0. Also call the excess pressure at x,t as $\Psi_p(x,t)$. Consider the section from x to $x+\mathrm{d}x$. After a small time t, it will be between $x+\Psi$ and $x+\mathrm{d}x+\Psi(x+\mathrm{d}x)$. The volume of this section is

$$A[\Psi(x+\mathrm{d}x) - \Psi + \mathrm{d}x]$$

The inital volume is $A\mathrm{d}x$, thus the change in volume is given by

$$\mathrm{d}V = A[\Psi(x+\mathrm{d}x) - \Psi(x)]$$

Now, this change in volume is due to excess pressure, so it is safe to write

$$\mathrm{d}V = -\kappa V \Psi_p$$

Here κ is the compressibility of the medium. We have used the negative symbol to account for the fact that increase in pressure causes a decrease in volume. Plugging in the equations gives us

$$\frac{\Psi(x+\mathrm{d}x) - \Psi(x)}{\mathrm{d}x} = -\kappa\Psi_p \iff \frac{\partial \Psi}{\partial x} = -\kappa \Psi_p$$

Again, we consider the section between $x+\Psi$ and $x+\mathrm{d}x+\Psi(x+\mathrm{d}x)$. The $F = ma$ equations reads

$$-A[\Psi_p(x+\mathrm{d}x) - \Psi_p(x)] = \rho A\mathrm{d}x\frac{\partial^2 \Psi}{\partial t^2}$$

$$-\frac{\partial \Psi_p}{\partial x} = \rho\frac{\partial^2 \Psi}{\partial t^2}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \rho\kappa\frac{\partial^2 \Psi}{\partial t^2}$$

Comparing this to the original wave equation, we see that the velocity of the wave is

$$v = \sqrt{\frac{1}{\rho\kappa}} = \sqrt{\frac{B}{\rho}}$$

To arrive at an expression for κ, we take the total derivative of the adiabatic equation

$$\mathrm{d}pV^\gamma + p\gamma V^{\gamma-1}\mathrm{d}V = 0$$

$$\mathrm{d}V = -\frac{V}{\gamma p}\mathrm{d}p$$

Thus, $\kappa = \frac{1}{\gamma p_0}$.

8.4.7 Volume Energy Density

$$\varepsilon = \frac{1}{2}\rho\left(\frac{\partial\Psi}{\partial t}\right)^2$$

$$\varepsilon = \frac{1}{2}\kappa\Psi_p = \frac{1}{2\kappa}\left(\frac{\partial\Psi}{\partial x}\right)^2$$

8.5 Three Dimensional Waves

8.5.1 Plane Waves

In the spirit of our earlier discussions, we say that the three dimensional wave equation is

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

As is the procedure in solving Poisson's equation, we assume $\Psi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$ and write

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{v^2}XYZT''$$

Divide through by $XYZT$ to get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{v^2}\frac{T''}{T}$$

Noting that each term should remain constant, we can write $X = A_x e^{ik_x x}$ and $T = A_t e^{-i\omega t}$ and define the wave vector $\vec{k}$ as

$$\vec{k} = \langle k_x, k_y, k_z\rangle$$

and also note that $|\vec{k}| = \frac{\omega^2}{v^2}$. Using our solutions, we can write Ψ as

$$\Psi = Ae^{i(k_x x + k_y y + k_z z - \omega t) + \phi}$$

Taking the real part,

$$\Psi = A\cos\Big(\vec{k}\cdot\vec{r} - \omega t + \phi\Big)$$

Where we note that $A = A_x A_y A_z A_t$.

8.5.2 Spherical Waves

The wave equation is

$$\nabla^2 = \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

We introduce $\Psi' = r\Psi$ to get rid of the first degree term. $\partial_r\Psi' = \Psi + r\partial_r\Psi$, $\partial_{rr}\Psi' = 2\Psi + r\partial_{rr}\Psi \implies \partial_{rr}\Psi = (1/r)\partial_{rr}\Psi' - (2/r)\partial_r\Psi$. We get the equation

$$\frac{\partial^2\Psi'}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2\Psi'}{\partial t^2}$$

From which we conclude that

$$\Psi = \frac{A}{r}\cos(kr - \omega t + \phi)$$

So a spherical wave's amplitude decays linearly radially.

9 Special Relativity

9.1 Relativistic Kinematics

Galilean Transforms

Let the position vector of a frame S' w.r.t a frame S be $\vec{r}$. Assume an event took place and that the position vector (in the S' frame) is denoted by $\vec{R}_{S'}$. What is the position vector in the S frame? If $\dot{\vec{r}} = 0$, then simple vector algebra yields

$$R_S = \vec{r} + \dot{\vec{r}}t + \vec{R}_{S'} = \vec{r} + \vec{R}_{S'}$$

But if $\dot{\vec{r}} \neq 0$, then we will also have to consider the motion of S'. Then we have

$$R_S = \vec{r}(0) + \dot{\vec{r}}t + \vec{R}_{S'}$$

Most of the time $\vec{r}(0) = 0$, so we can write

$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x - \dot{x}t' \\ y - \dot{y}t' \\ z - \dot{z}t' \\ t \end{pmatrix}}$$

This is also known as the galilean transform. As a result, we can say that if a frame moves with a constant velocity (w.r.t an inertial frame), then it, too, is inertial. This can be proven by simply taking two time derivatives of both matrices.

10 Central Forces

A force that depends only on the distance between the bodies. We will start with discussing a single body that is in a potential $V(r)$ that is due to the centre of the coordinate system. Our main goal is to solve for potentials. And it suffices to work with potentials because the force is derivable

$$\vec{F} = -\nabla V$$

Consider a body of mass m and velocity $\vec{v}$ in this field. We know that $\vec{\tau} = 0 \implies \dot{\vec{L}} = 0$.
Theorem: A body affected only by a central force field has its angular momentum conserved.
Proof 1: By definition,

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{r} \times \vec{F}$$

Now, $\vec{F} \parallel \vec{r}$ because $\vec{F} = -(\partial V/\partial r)\hat{r}$. So $\dot{\vec{L}} = 0$, so $\vec{L}$ is conserved over time.
Proof 2: By Definition,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{v} + \vec{r} \times \dot{\vec{p}} = 0$$

As the cross product of parallel vectors is 0.

10.1 Effective Potential

Consider the lagrangian in polar coordinates.

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{38}$$

The Lagrange equations for $\vec{r}$ and θ are,

$$m\ddot{r} = m\dot{\theta}^2 r - V'(r) \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}mr\dot{\theta}^2 = 0 \tag{40}$$

Notice that the first equation is Newton's second law in the radial direction and the second one is the conservation of L. To proceed, we eliminate θ by writing $\dot{\theta} = L/mr^2$.

$$m\ddot{r} = \frac{L^2}{mr^3} - V'(r)$$

Multiply by $\dot{r}$,

$$m\dot{r}\ddot{r} = \dot{r}\frac{L^2}{mr^3} - V'(r)\dot{r}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{r}^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{L^2}{2mr^2} - V(r)\right)$$

Let the constant of integration be C and write

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = C$$

It turns out that C is actually the total energy. And since $E = T + V$, the term after the kinetic energy term in the previous equation is like the effective potential.

$$V_{\text{eff}} \equiv \frac{L^2}{2mr^2} + V(r)$$

11 Fluid Mechanics

We define the pressure to be the *normal* force per unit area on a body

$$P = \frac{\mathrm{d}F}{\mathrm{d}A} \tag{41}$$

In case the fluid is in equilibrium, then the net force at all locations should equal 0. To that end, consider an imaginary box with base area A, thickness $\mathrm{d}z$ at a distance z from $z = 0$ (the base). Then, writing $F = ma$ yields

$$A\mathrm{d}p = -A\rho g\mathrm{d}z$$

Where we have defined ρ as the density of the fluid. Thus,

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g \tag{42}$$

So the pressure at any height z from the base is (after solving (41))

$$p(z) = p_0 - \rho g\Delta z \tag{43}$$

Where p_0 is the pressure at the reference point. (42) holds for any shape of container. There is a negative potential energy between two adjacent molecules in a fluid. This energy is porportional to the area that the liquid is exposed to.

11.1 Surface Energy

Thus, we define the surface energy γ as

$$\gamma = \frac{\mathrm{d}U}{\mathrm{d}A} \tag{44}$$

We can determining the restoring fore in the following manner. Consider an arbitrary fluid-atmosphere interface and draw a line L that divides it into 2 parts. Define $\hat{n}$ as the vector normal to the surface at every point on L. Then, by definition,

$$\vec{F} = \int_L \gamma\mathrm{d}\vec{\ell} \times \hat{n} \tag{45}$$

Where $\mathrm{d}\vec{\ell}$ is an infinitesimal segment along L. The direction of $\hat{n}$ is chosen so as to appropriately satisfy the signs. The proof is as follows. Apply an external force on the left hand portion so as to balance the restoring surface tension. Now to use virtual work, assume that we displace the left hand portion by $\delta\vec{x}$ to the left due to an external force $\vec{F}^{(e)}$. Then,

$\vec{F}^{(e)} \cdot \delta\vec{x} = -\vec{F} \cdot \delta\vec{x}$. The LHS is just the virtual work due to external forces. The RHS is also equal to the change in surface energy, which is just the change in surface area multiplied by the surface energy. The area carved out by the displacement of $\mathrm{d}\vec{\ell} \times \hat{n}$ by a distance $\delta\vec{x}$ is $\delta\vec{x} \cdot (\mathrm{d}\vec{\ell} \times \hat{n})$. So the work done by the right hand region is equal to the negative work done by the external forces, which lets us to write

$$\vec{F}^{(e)} \cdot \delta\vec{x} = -\int_L \gamma\delta\vec{x} \cdot (\mathrm{d}\vec{\ell} \times \hat{n})$$

We can safely drop $\delta\vec{x}$, and make use of the fact that $\vec{F}^{(e)} = -\vec{F}$ and thus write

$$\vec{F} = \int_L \gamma \mathrm{d}\vec{\ell} \times \hat{n}$$

Example: Find the pressure difference across the boundary of a spherical bubble of radius r and surface tension γ.

We use the principle of virtual work. Let the inner pressure be p_1 and the outer pressure be p_0. We want $\Delta p = p_1 - p_0$. Assuming a radial expansion of δx, the virtual work is

$$\delta W = \Delta p 4\pi r^2 \delta r$$

This is the work due the pressure gradient. The change in surface energy should equal to exactly this (by conservation of energy). What is the change in surface energy?

$$\delta U = \gamma\delta A = \gamma 8\pi r^2 \delta r$$

Thus,

$$\Delta p 4\pi r^2 \delta r = \gamma 8\pi r^2 \delta r$$

Which gives us

$$\Delta p = \frac{2\gamma}{r}$$

We can write this result more generally.

Young-Laplace Equation

$$\Delta p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{46}$$

11.2 Angle of Contact

12 Lagrangian Mechanics