Vieta's Relations

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What is a Polynomial?

Take a couple variables. Do algebraic operations on them. As long as you don't put any variable in the denominator, your resulting expression is a polynomial.

Definition

A polynomial in the variable x is an expression of the form

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

Where $n \geq 0$ is called the degree of the polynomial

For example, the polynomial $P(x) = 2x + 3$ can be obtained by setting $n = 1, a_1 = 2, a_0 = 3$. But $2/x + 3$ is not a polynomial because $n = -1$, which disagrees with the $n \geq 0$ condition.

Roots of a Polynomial

Definition

A root/zero of a function $f(x)$ is a number α that satisfies $f(\alpha) = 0$

Examples:

- The root of $f(x) = 2x + 3$ is $x = -3/2$.

Finding Roots of Polynomials

We basically set $x = \alpha$ and solve the equation $f(\alpha) = 0$. For example, say we want to find the roots of $f(x) = x^2 - 5$. We simply let it equal zero which gives

$$\alpha^2 = 5 \implies \alpha = \pm\sqrt{5}$$

If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the n roots of a polynomial g, we can write

$$g(x) = k\prod_{i=1}^{n}(x - \alpha_i)$$

Which is basically fancy notation for

$$g(x) = k(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$$

Every field of math has a "fundamental theorem" (Here's a nice list.) Algebra is no exception.

Fundamental Theorem of Algebra

A polynomial of the nth degree has n (not necessarily distinct) roots

Proof. Beyond the scope of this class. See this if you're interested though.

Example:

- The polynomial $x^2 - 2x + 1 = 0$ is a 2nd degree polynomial that can be factored as $(x-1)^2$. Setting it to 0 gives $x = 1$. Which may seem like a dilemma because there's only one root. We reconcile by saying that the second root is also $x = 1$.

Appetizer

Consider the polynomial

$$h = 5x^2 + 5x - 30$$

To find its roots we factor it

$$h = 5(x^2 + x - 6) = 5(x^2 + 3x - 2x - 6) = 5(x - 2)(x + 3)$$

Thus, the roots are $2, -3$. The sum of roots is $-3 + 2 = -1$. But if you take the coefficient of x and divide by the leading coefficient, you get the same answer. Product of roots $= -6$.

Generalized Result for Quadratics

Theorem

In a quadratic polynomial $g(x) = kx^2 + \ell x + \gamma$

$$\ell = -k(\alpha_1 + \alpha_2)$$

$$\gamma = k\alpha_1\alpha_2$$

Where α_1, α_2 are the roots of g.

Any quadratic polynomial $ax^2 + bx + c$ can be written as

$$h = a(x - \alpha_1)(x - \alpha_2)$$

Expanding using FOIL, we get

$$h = a(x^2 - x\alpha_2 - x\alpha_1 + \alpha_1\alpha_2) = ax^2 - a(\alpha_1 + \alpha_2)x + a\alpha_1\alpha_2$$

Proof Continued

Comparing coefficients, we see

$$b = -a(\alpha_1 + \alpha_2)$$

$$c = a\alpha_1\alpha_2$$

□

This is written more commonly as

$$\boxed{\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}}$$

Example

(Canada 1988)

For some integer a, the equations $1988x^2 + ax + 8891 = 0$ and $8891x^2 + ax + 1988 = 0$ share a common root. Find a.

Solution. Let α be the common root. We'll first find it

$$(1988 - 8891)\alpha^2 + (8891 - 1988) = 0$$

You'll get

$$\alpha^2 - 1 = 0 \implies \alpha = \pm 1$$

That's nice! Now we plug $x = \pm 1$ into one of the above equations

$$1998(1) \pm a + 8891 = 0 \iff \boxed{a = \pm 10879}$$

Generalizing for nth degree polynomials

Vieta's Relations

For the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Define S_k as the sum of all the roots taken k at a time. Then,

$$S_k = (-1)^k \frac{a_{n-k}}{a_n}$$

Proof. Expand $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$. □

S_k

When we say "sum of roots take k roots at a time", we mean "take any k out of n roots you have, multiply them together, then add that product with every other possible product of k roots."

Example: If you have roots $\alpha_1, \alpha_2, \alpha_3$ of $ax^3 + bx^2 + cx + d$

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 = \frac{-b}{a}$$

$$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = \frac{c}{a}$$

$$S_3 = \alpha_1\alpha_2\alpha_3 = \frac{-d}{a}$$

Exercise: For a biquadratic polynomial, with roots a, b, c, d, find S_3.

Example

Example 1

For the polynomial

$$3x^3 - 2x + 1$$

Compute the sum and product of roots.

Answer. Sum is 0, Product is $1/3$.

Example

Example 2

For the polynomial

$$2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7$$

Find the sum of roots taken 4 at a time.

Answer. 3

Example

(HMMT 1998)

Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $2, -3$ and 5. Find the value of $a + b + c$.

You could either use vieta's relations and solve for a, b, c individually and then add them, or you could be clever. The problem is: we only know three roots. But we notice that the sum of roots is 0. That means the 4th root is $0 - (2 + 5 - 3) = -4$. This means that the polynomial is basically

$$x^4 + ax^2 + bx + c = (x - 2)(x + 3)(x - 5)(x + 4)$$

We can easily find $a + b + c$ by plugging $x = 1$

$$1 + a + b + c = (1 - 2)(1 + 3)(1 - 5)(1 + 4) = 80$$

Thus $a + b + c = \boxed{79}$.