Logarithms

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§1 Introduction

Just as subtraction is the inverse of addition, division is the inverse of multiplication, logarithm is the inverse of exponentiation.

Definition 1.1.

$$x^{\log_x a} = a$$

To be more specific, the logarithm of a number k to a base b is just the exponent you put onto b to make the result equal k. Thus,

$$b^x = k \iff x = \log_b k$$

Here, b is known as the **base** and x, the **exponent**. If the base is 10, we call it the common logarithm. We usually omit the base when we write common logarithms. For example, $\log_{10} 23 = \log 23$ is the common logarithm of 23. If the base is Euler's number $e = 2.71828$, we call it the natural logarithm (also written as ln, that's a lowercase L and N not uppercase I and lowercase N) In this handout we'll discuss a few laws of logarithms. All of them can be derived from the laws of exponents, which you should be thorough with before proceeding.

§2 Laws of logarithms

Theorem 2.1 (Identity)

$$\log_a a = 1$$

Proof. Let $\alpha = \log_a a$. Consider a^α. By definition we have

$$a^{\log_a a} = a$$

Comparing exponents trivially results in our desired expression □

Theorem 2.2 (Sum Rule)

$$\log_a x + \log_a y = \log_a(xy)$$

Proof. Let $\alpha = \log_a x$, $\beta = \log_a y$. This gives us $x = a^\alpha$, $y = a^\beta$ Since we have an xy term, we might as well multiply them both. That results in

$$xy = a^{\alpha+\beta}$$

Taking the logarithm with base a on both sides gives

$$\log_a(xy) = \alpha + \beta$$

Recalling the definition of α, β, we get

$$\log_a x + \log_a y = \log_a(xy)$$

□

Example:
1)$\log_2 5 + \log_2 4 = \log_2(5 \times 4) = \log_2 20$
2)$\log_{10} 6 + \log_{10} 3 = \log_{10}(6 \times 3) = \log_{10} 18$

Theorem 2.3 (Difference Rule)

$$\log_a x - \log_a y = \log_a(x/y)$$

Proof. Left as an exercise to reader.

Example 2.4
Prove that $\log_n 1 = 0\ \forall n \in \mathbb{R}$.

Solution. There are many ways to do this, the simplest being the realization that $n^0 = 1$ holds $\forall n \in \mathbb{R}$ (for all real numbers n). Since we're discussing the difference of logs, lets do it that way.
Consider, for some $m > 0$, $\log_n m$.

$$\log_n m = \log_n(m/1) = \log_n m - \log_n 1$$

Cancelling $\log_n m$ on both sides gives us $\log_n 1 = 0$.

Theorem 2.5 (Power Rule)

$$\log_a x^n = n \log_a x$$

Proof. By now, you should be able to guess that the first step is to write $\alpha = \log_a x^n$. We then consider a^α

$$a^\alpha = a^{\log_a x^n} = x^n = (a^{\log_a x})^n$$

Comparing the coefficients,

$$\alpha = \log_a x^n = n \log_a x$$

□

Corollary 2.6

$$\log_a a^m = m \log_a a = m$$

Example 2.7 1. $\log_{10} 5^3 = 3 \log_{10} 5$

2. $\log x^2 = 2 \log x$

Theorem 2.8

$$\log_{a^n} b = \frac{1}{n} \log_a b$$

Proof. Let $\xi = \log_{a^n} b$. Then, by definition

$$(a^n)^\xi = b$$

But $b = a^{\log_a b}$.

$$a^{n\xi} = a^{\log_a b}$$

Comparing exponents,

$$n\xi = \log_a b \implies \log_{a^n} b = \frac{1}{n} \log_a b$$

□

Corollary 2.9

$$\log_{a^n} b^n = \log_a b$$

Example 2.10
Compute

$$\log_{\sqrt{2}} 4$$

Solution. Since $\sqrt{2} = 2^{1/2}$,

$$\log_{\sqrt{2}} 4 = \frac{1}{2} \log_2 4 = \boxed{1}$$

Theorem 2.11 (Base Change or Chain Rule)

$$\log_a x \times \log_x b = \log_a b$$

Proof. If we can show that $a^{\log_a x \times \log_x b} = b$, we are done.

$$a^{\log_a x \times \log_x b} = \left(a^{\log_a x}\right)^{\log_x b} = x^{\log_x b} = b$$

□

Corollary 2.12

In general,

$$(\log_a b)(\log_b c)(\log_c d)(\log_d e) \cdots (\log_y z) = \log_a z$$

Corollary 2.13

$$\log_a b = \frac{1}{\log_b a}$$

Example 2.14

Compute

$$\log_5(9) \log_4(125) \log_3(16)$$

Solution. Upon simplification,

$$\log_5(3^2) \log_4(5^3) \log_3(4^2) = 2 \cdot 3 \cdot 2 \cdot [\log_5(3) \log_3(4) \log_4(5)] = 12 \log_5 5 = \boxed{12}$$

§3 More Examples

Example 3.1 (2010 AMC 12A Problem 11)

The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

Solution: We can simplify the expression to write

$$7^7 \cdot 7^x = 8^x \implies 7^7 = \left(\frac{8}{7}\right)^x$$

Since we want to bring x down, we take the logarithm with base 8/7. Thus, $x = \log_{8/7} 7^7 \implies \boxed{b = 8/7}$.

Example 3.2 (2003 AMC 12B Problem 17)

If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

Solution: Since both logs are equal to 1,

$$xy^3 = x^2y \implies y^2 = x$$

Thus, $xy = y^3$ and $\log(xy) = 3\log y$. We can write

$$\log(x^2y) = \log[(y^2)^2y] = \log y^5 = 5\log y = 1 \iff \log y = \frac{1}{5}$$

Using this, we can write

$$\boxed{\log(xy) = 3\log y = \frac{3}{5}}$$

Example 3.3 (NYCIML F06A19)
If $\log_b(a)\log_c(a)\log_c(b) = 25$ and $\frac{a^2}{c^2} = c^k$. What is the sum of all possible values of k?

Solution. When we see the same numbers in bases and arguments, we think of the chain rule. Indeed, we can simplify in the following way

$$\log_b(a)\log_c(a)\log_c(b) = \log_c(a)[\log_c(b)\log_b(a)] = [\log_c(a)]^2 = 25 \implies \log_c a = \pm 5$$

Using the second condition, we can write

$$a^2 = c^{k+2}$$

Taking the logarithm with base c gives

$$2\log_c a = k + 2$$
$$k + 2 = \pm 10$$

Thus, $k = 8, -12$. The sum of the possible values of k is $8 - 12 = \boxed{4}$

Example 3.4 (2008 AMC 12A Problem 16)
The numbers $\log(a^3b^7), \log(a^5b^{12}), \log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12th term of the sequence is $\log b^n$. What is n?

Solution. When we're given an arithmetic progression, it's useful to determine the first term and the common difference, because all other quantities can be easily derived from them. Since we already have the first term, we find the common difference by simply taking the difference of two consecutive terms.

$$d = \log(a^5b^{12}) - \log(a^3b^7) = \log(a^2b^5)$$

The 12th term of this A.P is

$$a_{12} = a_1 + (12-1)d = \log(a^3b^7) + 11\log(a^2b^5) = \log(a^{25}b^{62}) = \log b^n$$

Since the final expression only contains b, we need to express a in terms of b. For that, we use the fact that if a, b, c are three consecutive terms of an arithmetic progression, $2b = a + c$. This result is easily derivable [let $(a, b, c) = (k - d, k, k + d)$]. Thus, we can write

$$2\log(a^5b^{12}) = \log(a^3b^7) + \log(a^8b^{15})$$
$$\log(a^{10}) + \log(b^{24}) = \log(a^{11}) + \log(b^{22})$$

Where we used $\log(xy) = \log(x + y)$ in the second step. Using the power rule,

$$10\log a + 24\log b = 11\log a + 22\log b \implies \log a = 2\log b$$

Which gives us $a = b^2$. Progress! Substituting this in the expression for $\log b^n$,

$$\log(a^{25}b^{62}) = \log[(b^2)^{25}b^{62} = \log(b^{50+62}) = \log b^n$$

We can finally write $\boxed{n = 112}$

Example 3.5 (2000 AIME II Problem 1)
The number $\frac{2}{\log_4 2006^6} + \frac{3}{\log_5 2000^6}$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution. Firstly, we use the power rule to write

$$\frac{2}{\log_4 2006^6} + \frac{3}{\log_5 2000^6} = \frac{2}{6\log_4 2006} + \frac{3}{6\log_5 2006} = \frac{1}{3\log_4 2006} + \frac{1}{2\log_5 2006}$$

To change the logs into a common base (which in this case is 2006), we use 2 to write

$$\frac{m}{n} = \frac{\log_{2006} 4}{3} + \frac{\log_{2006} 5}{2} = \frac{2\log_{2006} 4 + 3\log_{2006} 5}{6}$$

Using $a \log b = \log b^a$,

$$\frac{m}{n} = \frac{\log_{2006} 16 + \log_{2006} 125}{6} = \frac{\log_{2006}(125 \times 16)}{6} = \frac{\log_{2006} 2006}{6} = \frac{1}{6}$$

Thus $m + n = 1 + 6 = \boxed{7}$.

Example 3.6
Define $f(x)$ by

$$f(x) = \log_2 3 \log_3 4 \log_4 5 \cdots \log_{x-1} x$$

Compute $\sum_{n=2}^{10} f(2^n)$.

Solution. Using corollary 2.12, we can condense $f(x)$ into

$$f(x) = \log_2 x$$

$f(2^n)$ is

$$f(2^n) = \log_2(2^n) = n \log_2 2 = n$$

So the sum is just

$$\sum_{n=2}^{10} n$$

Which is

$$2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \boxed{54}$$

If you want to add all those numbers, be my guest! A slicker way is to use the result from arithmetic sequences that the sum of the first k natural numbers is $\frac{k(k+1)}{2}$. So the sum we need is

$$\frac{10(11)}{2} - 1 = \boxed{54}$$

§4 Problems

Problem 4.1. Find $\log_a m$ if

$$\log_{300}(\log_a m^{10}) = 0$$

Problem 4.2. Rewrite the following logarithm in terms of $\log_a x, \log_a y, \log_a z$

$$\log_a \left(\frac{x^4 y}{z^5}\right)$$

Problem 4.3. Which of the following isn't generally true?

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x + \log_y = \log_a x \log_a y$
3.

Problem 4.4. Prove that $\log_a b = \frac{1}{\log_b a}$

Problem 4.5. Solve for $x \in \mathbb{R}$ in the equation

$$\log_{2x} 216 = x$$

Problem 4.6. Find the numerical value of

$$\log_5 \left(\frac{(125)(625)}{25}\right)$$

Problem 4.7. If $\log_a x = b$ and $\log_x a = 5$, what is $(1 - 5b)^{100}$?

Problem 4.8. If $a = b^x$, $b = c^y$, $c = d^z$, express xyz in terms of a, b, c.

Problem 4.9. If $x = \log(\log_2(a))$ and x is a solution to $2^{x+2} = 20^x$, find the value of a.

Problem 4.10. (2015 AMC 12A Problem 14) What is the value of a for which

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$$

Problem 4.11. (2008 AMC 12B) A circle has a radius of $\log(a^2)$ and a circumference of $\log_{10}(b^4)$. What is $\frac{\log_a b}{\pi}$?