Power Dissipated in a Series LCR Circuit

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1 Mathematical Derivation

Average of a quantity $f(t)$ in a domain $[a, b]$ is

$$\langle f \rangle = \frac{\int_a^b f(t)\mathrm{d}t}{\int_a^b \mathrm{d}t} = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \tag{1}$$

We apply the same principles to power dissipated in an LCR circuit. Instantaneous power $p(t)$ is given as

$$p(t) = i(t)v(t)$$

Since $v(t) = v_0 \sin(\omega t)$ and $i(t) = i_0 \cos(\omega t + \phi)$ $\left(\text{where}\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)$, we can write

$$p(t) = i_0 v_0 \sin(\omega t)\cos(\omega t + \phi) \tag{2}$$

This is the instantaneous power. If you want to find the power dissipated at the instant $t = 2$ seconds, you plug in $t = 2$ and that's your answer. We're more interested in the average power. So since we'll be doing some integration (in the form of eq. 1), we'll do some cleaning up. Turns out we can write

$$\sin A \cos B = \frac{\cos(A-B) - \cos(A+B)}{2} \tag{3}$$

Proof here. Comparing (2) and (3), we see that $A = \omega t$ and $B = \omega t + \phi$. Since $A + B = 2\omega t + \phi$, $A - B = -\phi$, $\cos(-x) = +\cos(x)$ (since the cosine function is even), we can write

$$p(t) = \frac{i_0 v_0}{2}\left[\cos(\phi) - \cos(2\omega t + \phi)\right] \tag{4}$$

We're ready to find the average using (1). We want to find the average power dissipated in one cycle of alternation of voltage. That means $a = 0, b = T$. So we write

$$\langle p \rangle = \frac{1}{T}\frac{i_0 v_0}{2}\int_0^T [\cos\phi - \cos(2\omega t + \phi)]\mathrm{d}t = \frac{1}{T}\frac{i_0 v_0}{2}\left[\int_0^T \cos\phi \mathrm{d}t - \int_0^T \cos(2\omega t + \phi)\mathrm{d}t\right] \tag{5}$$

The first integrand is constant w.r.t the variable t, so we can pull it out of the integral. So

$$\int_0^T \cos\phi \mathrm{d}t = \cos\phi \int_0^T \mathrm{d}t = T\cos\phi$$

For the second one, we use the fact that $\int \cos(ax+b)\mathrm{d}x = \frac{\sin(ax+b)}{a} + C$. We'll work the second integral out later and move on for now, noting that it is 0. Then, (5) condenses nicely to

$$\langle p \rangle = \frac{1}{T}\frac{i_0 v_0}{2}(T\cos\phi) = \frac{i_0}{\sqrt{2}}\frac{v_0}{\sqrt{2}}\cos\phi = I_{rms}V_{rms}\cos\phi \tag{6}$$

Since $V_{rms} = I_{rms}Z$,

$$\boxed{\langle p \rangle = I_{rms}^2 Z \cos\phi = \frac{V_{rms}^2}{Z}\cos\phi} \tag{7}$$

$\cos\phi$ is referred to as the **power factor** and is given by

$$\cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

2 Integral

We want to compute

$$\int_0^T \cos(2\omega t + \phi)\mathrm{d}t$$

First we note that $\omega T = 2\pi$. We solve this integral using substitution. Let $\alpha = 2\omega t + \phi$, then $\mathrm{d}\alpha = 2\omega \mathrm{d}t \implies \mathrm{d}t = \frac{\mathrm{d}\alpha}{2\omega}$. Also, when $t = 0, \alpha = \phi$ and $t = T \implies \alpha = 4\pi + \phi$. So our integral is

$$\frac{1}{2\omega}\int_{\phi}^{4\pi+\phi} \cos\alpha \mathrm{d}\alpha = \frac{1}{2\omega}[\sin(4\pi + \phi) - \sin\phi] = 0$$

Where we used $\sin(4\pi + \xi) = \sin(\xi)$. How to do this for the exam? Just write that the average sum of the values of a cosine function in its period is 0.