Logarithms

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September 15, 2022

Today's Plan

Discuss the definition and laws of logarithm via problem-solving. It's going to be dense, so you might want to take notes!

Motivation

The inverse of addition is subtraction. The inverse of multiplication is division. But what is the inverse of exponentiation?
To make the above point clearer, consider the following examples

$$x + 2 = b \implies x = b - 2$$

$$ax = b \implies x = \frac{b}{a}$$

But

$$a^x = b \implies x = ?$$

Definition

The logarithm function is the inverse of the exponential function.

Definition of Logarithm

For $a, x, b \in \mathbb{R}$

$$a^x = b \implies x = \log_b a$$

Basically answers the question "What should the number a be raised to the power of to get b?" So if someone asked you what you should raise 2 to get 3, your answer would be $\log_2 3$.

Couple things to note:

1. $b > 0$
2.

Laws

Identity

$\log_a a = 1$

Proof. Quite obvious.

Laws

Sum Rule

$$\log_a x + \log_a y = \log_a(xy)$$

Proof. Let $\alpha = \log_a x$, $\beta = \log_a y$. This gives us $x = a^\alpha$, $y = a^\beta$ Since we have an xy term, we might as well multiply them both. That results in

$$xy = a^{\alpha+\beta}$$

Taking the logarithm with base a on both sides gives

$$\log_a(xy) = \alpha + \beta$$

Recalling the definition of α, β, we get

$$\log_a x + \log_a y = \log_a(xy)$$

Laws

Difference Rule

$$\log_a x - \log_a y = \log_a(x/y)$$

Proof. Let $\alpha = \log_a x$, $\beta = \log_a y$. This gives us $x = a^\alpha$, $y = a^\beta$. Similar to the previous proof, we compute x/y because that's present in the result.

$$\frac{x}{y} = a^{\alpha-\beta}$$

Taking the logarithm base a on both sides,

$$\log_a\left(\frac{x}{y}\right) = \alpha - \beta$$

Recalling our definitions of α, β, we get

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Laws

Power Rule

$$\log_a x^n = n \log_a x$$

Proof. Define $\alpha \equiv \log_a x$. Then,

$$a^\alpha = a^{\log_a x^n} = x^n = \left(a^{\log_a x}\right)^n$$

Comparing the coefficients,

$$\alpha = \log_a x^n = n \log_a x$$

Corollary: $(\log_a a)^m = m$

Laws

Useful Theorem

$$\log_{a^n} b = \frac{1}{n} \log_a b$$

Proof. Let $\xi = \log_{a^n} b$. Then, by definition

$$(a^n)^\xi = b$$

But $b = a^{\log_a b}$.

$$a^{n\xi} = a^{\log_a b}$$

Comparing exponents,

$$n\xi = \log_a b \implies \log_{a^n} b = \frac{1}{n} \log_a b$$

Corollary: $\log_{a^n} b^n = \log_a b$

Example

Compute

$$\log_{\sqrt{2}} 4$$

Laws

Chain Rule/Base Change Rule

$$\log_a x \times \log_x b = \log_a b$$

Proof. If we can show that $a^{\log_a x \times \log_x b} = b$, we are done.

$$a^{\log_a x \times \log_x b} = \left(a^{\log_a x}\right)^{\log_x b} = x^{\log_x b} = b$$

Corollary 1

In general,

$$(\log_a b)(\log_b c)(\log_c d)(\log_d e)\cdots(\log_y z) = \log_a z$$

Corollary 2

$$\log_a b = \frac{1}{\log_b a}$$

Example 1

(2010 AMC 12A Problem 11)

The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

Solution: We can simplify the expression to write

$$7^7 \cdot 7^x = 8^x \implies 7^7 = \left(\frac{8}{7}\right)^x$$

Since we want to bring x down, we take the logarithm with base $8/7$. Thus, $x = \log_{8/7} 7^7 \implies \boxed{b = 8/7}$.

Example 2

(2003 AMC 12B Problem 17)

If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

Solution: Since both logs are equal to 1,

$$xy^3 = x^2y \implies y^2 = x$$

Thus, $xy = y^3$ and $\log(xy) = 3\log y$. We can write

$$\log(x^2y) = \log[(y^2)^2y] = \log y^5 = 5\log y = 1 \iff \log y = \frac{1}{5}$$

Using this, we can write

$$\boxed{\log(xy) = 3\log y = \frac{3}{5}}$$

Example 3

Define $f(x)$ by

$$f(x) = \log_2 3 \log_3 4 \log_4 5 \cdots \log_{x-1} x$$

Compute $\sum_{n=2}^{10} f(2^n)$.

Solution. Using corollary 2.12, we can condense $f(x)$ into

$$f(x) = \log_2 x$$

$f(2^n)$ is

$$f(2^n) = \log_2(2^n) = n \log_2 2 = n$$

So the sum is just

$$\sum_{n=2}^{10} n$$

Which is

$$2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \boxed{54}$$

If you want to add all those numbers individually, be my guest! A slicker...

... way is to use the result from arithmetic sequences that the sum of the first k natural numbers is $\frac{k(k+1)}{2}$. So the sum we need is

$$\frac{10(11)}{2} - 1 = \boxed{54}$$