APG3013F - NUMERICAL METHODS

ASSIGNMENT 2 - PARAMETRIC ADJUSTMENTS

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1.1 Chapter 1 - Introduction

1.1.1 About the report

Investigate the method of parametric adjustment of redundant observations in a traverse network. For this adjustment the least squares model is applied in which the sum of the squares of the weighted residuals is minimized. The report also looks at the application of error ellipses and chi- squared tests to assess the quality of the residuals.

1.1.2 Structure of report

The aim of the report, problem and steps to solve the problem will be presented in this chapter. The setting, explanation and analysis of findings will follow up in the next chapters. Conclusions and recommendations will then be drawn based on the findings.

1.1.3 Aim of report

The objective of this investigation is to simulate a traverse network and apply a parametric adjustment to the sequence of points. By identifying the unknowns, observations, fixed and free points the A (design matrix), P (weight matrix) and L (matrix of observations) matrices can all be constructed to compute the solutions vector x as well as residuals vector V. The final adjusted points will thus be formed by the sum of the initial polar points and the computed residuals. The postriori and covariance matrix will also needed to be calculated in order to plot error ellipses as well as assess the quality of residuals.

1.1.4 The problem

The problems of adjustment lie in the amount of data being used. A number of difficulties may stem from mishandling points,identifying the incorrect unknowns, calculating initial points incorrectly, calculating the incorrect A, x, V, L matrices and generally not applying the proper algebra to obtain solutions.

1.1.5 Solution steps

The steps to solving the solutions have been summarized below:

1. Evaluate the numerical data provided and structure input data.

2. Determine how many observations are available and which unknowns are to be estimated.

3. Establish the degrees of freedom of the adjustment.

$$dof = n - u$$

   n: observations
   u: unknowns

4. Formulate one (only) observation equations for each observation.

$$L_i + v_i = F_i(x_a, x_b, x_c...) i = 1, 2...n$$

5. Linearize observation functions if it is non-linear. Formulate a Taylor expansion by differentiating the observation equations with respect to the unknown parameters.

$$L_i + v_i = F_0 + a_i dx_a + b_i dx_b + c_i dx_c...$$

   with

$$a_i = \frac{\partial F}{\partial x_a}$$

$$b_i = \frac{\partial F}{\partial x_b}$$

6. Establish the weight matrix P_{ll} associated with the observations

7. Evaluate provisional values for the unknown parameters. Usually done by polars, intersections, resection or similar calculations for survey adjustment.

8. Form the design matrix A and the l misclosure vector.

9. Evaluate the normal equation matrix $A^T PA$, $A^T Pl$ and invert the normal to find the cofactor matrix Q.

10. Evaluate the unknowns x

$$x = (A^T PA)^{-1} A^T Pl$$

11. Add the unknowns to the provisional values.

$$\bar{x} = x_0 + x$$

12. After updating the unknowns iterate the solution by repeating steps 5 - 11 until the solutions x converges.

13. Evaluate the residuals

$$v = Ax - l$$

14. Evaluate the adjusted observations

$$\bar{L} = L_0 + v$$

15. Complete a global check by comparing the adjusted observations with the function values calculated from the adjusted unknowns.

16. Carry out error analysis by calculating the standard deviation of an observation of unit weight a posteriori

$$\sigma_0 = \sqrt{\frac{V^T PV}{n-u}}$$

and the variance

$$\sigma_0^2 = \frac{V^T PV}{n-u}$$

17. Compute the variance covariance of the unknowns

$$\Sigma_{xx} = \sigma_0^2 Q_{xx}$$

where $Q_{xx} = (A^T PA)^{-1}$ is the co-factor matrix of the unknowns

18. Calculate the error ellipse using the variance covariance matrix

19. Calculate or estimate a provisional value for σ_{0p} to compare with σ_0. Perform a chi-squared test to conclude the hypothesis.

1.2 CHAPTER 2 - INPUT DATA

1.2.1 Input data for the problem

The data provided include a traverse network from points SUR09 to SUR12. The specific structure of nodes and edges is relevant to the task although all points and orientation corrections were not initially given. The data has been divided into 3 files of which one contains point data including provisional points calculated by polars, an observation files to show relationships between points as well as observations and a file containing provisional orientation corrections at each point along the main traverse. The figures below display the traverse network and the input data provided for the investigation.

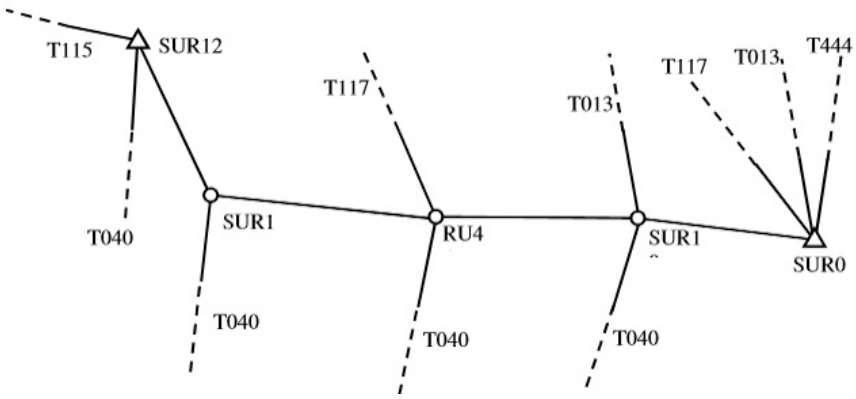

Figure 1.1: Traverse network

From the figure we see that unknown points have been represented by a circle. In the points list below these provisional points are of the Type 0 distinguishing it from the known and fixed points of Type 1.

points	Y	X	Type
DP	51711.35	58779.23	1
KB	50750.11	58299.8	1
Cecilia	54285.72	63409.75	1
DRC Rbsch	48811.38	59862.58	1
Plum	50345.4	58164.13	1
SUR09	49609.09	58738.76	1
SUR12	49925.71	59448.14	1
SUR10	49666.2	58961.0	0
SUR11	49732.0	59294.5	0
RU4A	49686.5	59119.8	0

Figure 1.2: Table of points

The figure 2 above contains all observations for the traverse. From the table the type justifies the type of observation. Type 0 for distance and Type 1 for directions.

Source	Target	Distance	Direction	Sigma Dist	Sigma Dir	Target Type
SUR09	SUR10	0	14.424444	0	0.01	1
SUR09	DP	0	88.888333	0	0.25	1
SUR09	KB	0	111.03388	0	0.25	1
SUR09	Plum	0	127.96111	0	0.25	1
SUR10	SUR09	0	194.43777	0	0.01	1
SUR10	DRC Rbsch	0	316.5225	0	0.25	1
SUR10	RU4A	0	7.2472222	0	0.01	1
SUR10	KB	0	121.39666	0	0.25	1
RU4A	SUR10	0	187.24944	0	0.01	1
RU4A	DRC Rbsch	0	310.33555	0	0.25	1
RU4A	SUR11	0	14.648888	0	0.01	1
RU4A	DP	0	99.554166	0	0.25	1
SUR11	RU4A	0	194.64805	0	0.01	1
SUR11	DRC Rbsch	0	301.68305	0	0.25	1
SUR11	SUR12	0	51.554166	0	0.01	1
SUR12	SUR11	0	231.55611	0	0.01	1
SUR12	DRC Rbsch	0	290.40638	0	0.25	1
SUR12	Cecilia	0	47.746666	0	0.25	1
SUR09	SUR10	229.598	0	40000	0	0
SUR10	RU4A	159.918	0	40000	0	0
RU4A	SUR11	180.52	0	40000	0	0
SUR11	SUR12	247.223	0	40000	0	0

Figure 1.3: Table of observations

points	correction
Zsur09	0.008278
Zsur10	-0.00542
Zru4a	-0.00853
Zsur12	-0.0055
Zsur11	-0.00478

Figure 1.4: Orientation corrections

The orientation corrections and the provisional points have been pre-calculated for convenience in computation of the solutions vector x and the residuals v.

1.3 CHAPTER 3 - CALCULATING PROVISIONAL VALUES

1.3.1 Theory - Provisional points by polar method

The polar formula calculates a second points coordinates from the initial pointâĂŹs known coordinates as well as the observed direction and distance from the known to the unknown point. The formula below describes how a known point 'a' can be used to find point bâĂŹs x and y coordinate:

$$Xb = Xa + Scos(\alpha)$$

$$Yb = Ya + Ssin(\alpha)$$

where S is the distance and alpha is the direction from a to b.
The provisional points have been pre-calculated and are referenced in figure 2 above by type 0.

1.3.2 Theory - Provisional orientation correction

To obtain orientation corrections a join direction must be found between the point of observation and the known points in acceptable range of the point. Note that the point of observation can either be known (fixed) or unknown (free), as long as a provisional point is available the calculation can be made and corrected within the adjustment. The join directions must then be differenced with direction observed to find the orientation correction. It is recommended that more than one join direction to known points is made and the differenced averaged to obtain a more acceptable orientation solution. Refer to figure 4 above for orientation corrections in decimal degrees along the main points. The join direction is calculated as follows: For 2 points a and b the join direction is found by:

$$t_{ab} = tan^{-1}\frac{Yb - Ya}{Xb - Xa}$$

where t_{ab} is the direction from a to b.

1.4 Chapter 4 - Constructing Matrices

1.4.1 Setting up the design matrix A

The observations in this adjustment is non-linear and is thus linearized by the Taylor series. Each observation is partially derived with respect the unknowns giving the form:

$$\partial l_1 = a_1\frac{\partial l_1}{\partial x_a} + b_1\frac{\partial l_1}{\partial x_b} + c_1\frac{\partial l_1}{\partial x_c} + ...$$

For this solution there are 2 forms of partial differentiation; one with respect to unknowns of the direction equation and one with respect to unknowns of the distance equation. Within

the direction equation it partially derived with respect to points x or y and for orientation correction z. The distance equation is only differentiated with respect to points x and y. Partial derivatives are summarized below:
Functional models

$$s_{ab} = \sqrt{(X_b - X_a)^2 + (Y_b - Y_a)^2} \qquad t_{ab} = tan^{-1}\frac{Yb - Ya}{Xb - Xa} - z$$

Partial derivatives

$$\frac{\partial s_{ab}}{\partial Yb} = \frac{(Y_b - Y_a)}{s_{ab}} \qquad \frac{\partial s_{ab}}{\partial Ya} = \frac{-(Y_b - Y_a)}{s_{ab}}$$

$$\frac{\partial s_{ab}}{\partial Xb} = \frac{(X_b - X_a)}{s_{ab}} \qquad \frac{\partial s_{ab}}{\partial Xa} = \frac{-(X_b - X_a)}{s_{ab}}$$

$$\frac{\partial t_{ab}}{\partial Yb} = \frac{-(Y_b - Y_a)}{s_{ab}^2} \qquad \frac{\partial t_{ab}}{\partial Ya} = \frac{-(Y_b - Y_a)}{s_{ab}^2}$$

$$\frac{\partial t_{ab}}{\partial Xb} = \frac{-(X_b - X_a)}{s_{ab}^2} \qquad \frac{\partial t_{ab}}{\partial Xa} = \frac{-(X_b - X_a)}{s_{ab}^2}$$

$$\partial z = -1$$

The A matrix is the matrix of coefficients of the unknowns. It is constructed by deriving the partials of observations with respect to the unknowns associated with a column in the A matrix. The A matrix or Design matrix for this investigation has been solved as follows:

$$
\begin{bmatrix}
\frac{\partial t_1}{\partial Y_{SUR10}} & \frac{\partial t_1}{\partial X_{SUR10}} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\frac{\partial t_5}{\partial Y_{SUR10}} & \frac{\partial t_1}{\partial X_{SUR10}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{\partial t_6}{\partial Y_{SUR10}} & \frac{\partial t_1}{\partial X_{SUR10}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{\partial t_7}{\partial Y_{SUR10}} & \frac{\partial t_7}{\partial X_{SUR10}} & 0 & 0 & \frac{\partial t_7}{\partial Y_{RU4A}} & \frac{\partial t_7}{\partial X_{RU4A}} & -1 & 0 & 0 & 0 & 0 \\
\frac{\partial t_8}{\partial Y_{SUR10}} & \frac{\partial t_1}{\partial X_{SUR10}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{\partial t_9}{\partial Y_{SUR10}} & \frac{\partial t_9}{\partial X_{SUR10}} & 0 & 0 & \frac{\partial t_9}{\partial Y_{RU4A}} & \frac{\partial t_9}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & \frac{\partial t_{10}}{\partial Y_{RU4A}} & \frac{\partial t_{10}}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & \frac{\partial t_{11}}{\partial Y_{SUR11}} & \frac{\partial t_{11}}{\partial X_{SUR11}} & \frac{\partial t_{11}}{\partial Y_{RU4A}} & \frac{\partial t_{11}}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & \frac{\partial t_{12}}{\partial Y_{RU4A}} & \frac{\partial t_{12}}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & \frac{\partial t_{13}}{\partial Y_{SUR11}} & \frac{\partial t_{13}}{\partial X_{SUR11}} & \frac{\partial t_{13}}{\partial Y_{RU4A}} & \frac{\partial t_{13}}{\partial X_{RU4A}} & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\partial t_{13}}{\partial Y_{SUR11}} & \frac{\partial t_{13}}{\partial X_{SUR11}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\partial t_{14}}{\partial Y_{SUR11}} & \frac{\partial t_{14}}{\partial X_{SUR11}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\partial t_{15}}{\partial Y_{SUR11}} & \frac{\partial t_{15}}{\partial X_{SUR11}} & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\frac{\partial s_{19}}{\partial Y_{SUR10}} & \frac{\partial s_{19}}{\partial X_{SUR10}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial s_{20}}{\partial Y_{SUR10}} & \frac{\partial s_{20}}{\partial X_{SUR10}} & 0 & 0 & \frac{\partial s_{20}}{\partial Y_{RU4A}} & \frac{\partial s_{20}}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial s_{21}}{\partial Y_{SUR11}} & \frac{\partial s_{21}}{\partial X_{SUR11}} & \frac{\partial s_{21}}{\partial Y_{RU4A}} & \frac{\partial s_{21}}{\partial X_{RU4A}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial s_{22}}{\partial Y_{SUR11}} & \frac{\partial s_{22}}{\partial X_{SUR11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

1.4.2 The weight matrix

The weight matrix P is populated by the precisions of each observation. For this investigation the observations are uncorrelated and thus will form a principle diagonal matrix.
where P is calculted by :

$$P_i = \frac{\sigma_o^2}{\sigma_i^2}$$

σ_o is the standard deviation of unit weight usually taken as 1 and σ_i is the precision (standard deviation) of the observation.
The weight matrix for this adjustment is 22X22 and summarized as follows:

$$\begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & \frac{1}{\sigma^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & \frac{1}{\sigma^2} & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & \frac{1}{\sigma^2} & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & \frac{1}{\sigma^2} & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{\sigma^2} & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & \frac{1}{\sigma^2} \end{bmatrix}$$

1.4.3 The Misclosure Vector

The vector of misclosures denoted by l contains the difference between he observed quantities and the calculated quantities using provisional values for unknowns. The form is usually $l_i - l_{i0}$ where l_i is the observation and l_{i0} the calculated measurement. The misclosure vector for this investigation is of size 22X1. Hence for each observation there is the difference $l_i - l_{i0}$. Following is a summary of this vector:

$$\begin{bmatrix} t_{sur09sur10} - t_{sur09sur100} \\ .... \\ .... \\ .... \\ .... \\ s_{sur12sur11} - s_{sur12sur110} \end{bmatrix}$$

1.4.4 Calculating the solutions Vector x

The derivative of V^TPV with respect to x, $\frac{\partial V^TPV}{\partial x}$, will obtain a minimum when we set the differential to 0. From this derivation we also find the equation in terms of x and can thus find solutions vector for adjustment. The equation $x = (A^TPA)^{-1}A^TPl$ is therefore the computation of the solutions vector.

$$x = \begin{bmatrix} \delta Y_{SUR10} \\ \delta X_{SUR10} \\ \delta Y_{SUR11} \\ \delta X_{SUR11} \\ \delta Y_{RU4A} \\ \delta X_{RU4A} \\ Z_{SUR10} \\ Z_{SUR09} \\ Z_{SUR11} \\ Z_{SUR12} \\ Z_{RU4A} \end{bmatrix}$$

1.4.5 Variance covariance matrix of unknowns

The variance covariance matrix shows us the correlation between the unknowns. These correlates can used to calculate the error ellipses on each point however only the Y and X correlates for unknowns is needed.

$$\begin{bmatrix} Y_{SUR10} & X_{SUR10} & Y_{SUR11} & X_{SUR11} & Y_{RU4A} & X_{RU4A} & Z_{SUR10} & Z_{SUR09} & Z_{SUR11} & Z_{SUR12} & Z_{RU4A} \end{bmatrix}$$

$$\begin{bmatrix}
\sigma^2_{Ysur10} & \sigma_{XYsur10} & . & . & . & & & & & & & & \\
\sigma_{YXsur10} & \sigma^2_{Xsur10} & & & & & & & & & & & \\
. & & \sigma^2_{Ysur11} & \sigma_{XYsur11} & & & & & & & & & \\
. & & \sigma_{YXsur11} & \sigma^2_{Xsur11} & & & & & & & & & \\
. & & & & \sigma^2_{Yru4a} & \sigma_{XYru4a} & & & & & & & \\
 & & & & \sigma_{YXru4a} & \sigma^2_{Xru4a} & & & & & & & \\
 & & & & & & . & & & & & & \\
 & & & & & & & . & & & & & \\
 & & & & & & & & . & & & & \\
 & & & & & & & & & . & & & \\
 & & & & & & & & & & . & & \\
 & & & & & & & & & & & \sigma_{zsur12ru4a} & \sigma^2_{zru4a}
\end{bmatrix}$$

1.5 Chapter 5 - Error ellipse

1.5.1 Theory

Error ellipse define an area of spatial confidence in two or more dimensions. For a point p in a 2-dimensional graph the ellipse about p will represent the spatial distribution of errors, i.e, it is the area of confidence for a bivariate distribution. To construct an error ellipse we need the size of the semi-major and semi-minor axis. We also require the orientation of the ellipse with respect to the x, y system. The covariance matrix of p is used to determine these quantities. Assuming the covariance matrix of the position x, y is given by:

$$\Sigma xy = \begin{bmatrix} \sigma x^2 & \sigma xy^2 \\ \sigma y^2 & \sigma xy^2 \end{bmatrix}$$

To determine σu^2 and σv^2 , the semi-major and semi- minor axis respectively we use the formula:

$$\sigma u^2 = \frac{\sigma x^2 + \sigma y^2 + \sqrt{(\sigma x^2 + \sigma y^2)^2 + 4\sigma xy2}}{2}$$

$$\sigma v^2 = \frac{\sigma x^2 + \sigma y^2 - \sqrt{(\sigma x^2 + \sigma y^2)^2 + 4\sigma xy2}}{2}$$

And the orientation α can be found by :

$$tan(2\alpha) = \frac{-2\sigma xy}{\sigma x^2 - \sigma y2}$$

1.6 Chapter 6 - Goodness of fit test

1.6.1 Theory - chi-squared

The chi-squared distribution is used to test the quality of the adjustment and to test for outliers. For the purpose of this investigation the chi-squared statistic is used to compare the population variance against the variance of each observation of the traverse.

A chi-squared confidence interval is given by:

$$P(\chi^2_{1-\frac{\alpha}{2}} < \chi < \chi^2_{\frac{\alpha}{2}})$$

where α is the level of confidence

The variance of a population is based on the chi-squared statistic

$$\chi^2 = \frac{vS^2}{\sigma^2}$$

The confidence of the population variance is given by

$$P(\frac{\chi^2_{1-\frac{\alpha}{2}}}{vS^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{\frac{\alpha}{2}}}{vS^2})$$

where v is the degrees of freedom minus 1 $n-1$ and S^2 is the sample variance. On completion of the weighted least squares adjustment where the weight is deteremined by

$$P_i = \frac{\sigma_0^2}{\sigma_i^2}$$

the χ^2 (chi-squared distribution) can be performed on the reference variance.

If the weights of the direction and distances are found by

$$P_{dir} = \frac{1}{\sigma^2_{dir}} \qquad P_{dist} = \frac{1}{\sigma^2_{dist}}$$

then the a priori (before adjustment) value for the reference value is 1 and its a posteriori (after adjustment) computed value is σ_0^2.

1.6.2 Theory - process of hypothesis test

1. Hypothesis:

$$H_0 : S^2 = 1$$

$$H_a : S^2 \neq 1$$

2. Test statistic

$$\chi^2 = \frac{vS^2}{\sigma^2}$$

3. Rejection region

$$\chi^2 > \chi^2_{\frac{\alpha}{2},v}$$

1.7 CHAPTER 7 - ANALYSIS

1.7.1 Visual outcome

The figure below is a visualization of the traverse network. Using a python module matplotlib and network x we were able to connect all the nodes/points and display them visually. For all graphs the y-axis is vertical and x-axis is horizontal.

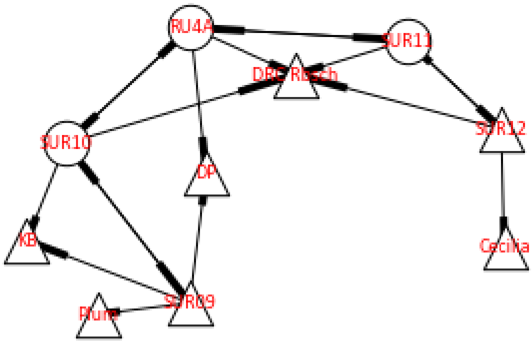

Figure 1.5: Coordinate specified network

Using the modules mentioned above error ellipses were plotted at unknown points and networked to form the figure below. The ellipse at each point has been scaled up significantly for better visualization.

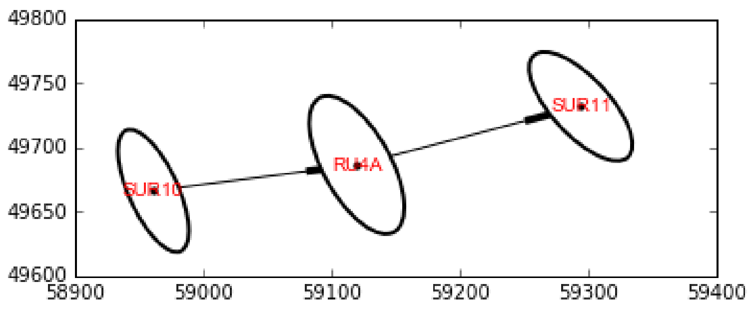

Figure 1.6: Error ellipse diagram

1.7.2 Conclusion

As stipulated before the data structure management was the most cumbersome issue of performing this adjustment. The parametric least squares requires that all observations be evaluated and as a consequence requires a number of checks and corrections when calculating the A, L, P, x or V matrices. A number of issues besides data handling did arise in the computation of adjustment such as the shift between radians and degrees to correct orientations and forming error ellipses. The output of the adjustment has been stored in output1.txt and output2.txt files. Issues aside the adjustment was successfully completed.To reflect on the analysis two points have been made to conclude this investigation.

1. The orientation of the ellipses is suggestive that the major axes of the relative ellipses are perpendicular to the fixed points and minor points are towards the fixed points. This is indicative that a weakness in orientation is present and that more azimuth control is required.

2. A confidence level of 95% was used to calculate this confidence interval. The test statistic (in outputfile2.txt) lies within the confidence region and we thus accept the null hypothesis and conclude that test statistic is not an outlier.