Pg. 1

Aersp 304 Project 1.)

a.)

$\vec{r} = x\hat{b}_1 + y\hat{b}_2 \qquad \dot{\vec{r}} = \dot{x}\hat{b}_1 + x\dot{\hat{b}}_1 + \dot{y}\hat{b}_2 + y\dot{\hat{b}}_2$

$\dot{\hat{b}}_1 = \dot{\theta}\hat{b}_3 \times \hat{b}_1 = \omega\hat{b}_2 \qquad \dot{\hat{b}}_2 = \dot{\theta}\hat{b}_3 \times \hat{b}_2 = -\omega\hat{b}_1$

$\dot{\vec{r}} = (\dot{x} - \omega)\hat{b}_1 + (\dot{y} + \omega)\hat{b}_2 \qquad |\dot{\vec{r}}| = |\vec{v}| = \sqrt{(\dot{x} - y\omega)^2 + (\dot{y} + x\omega)^2}$

$T = \frac{1}{2}mv^2, \quad T = \frac{1}{2}m_3(\dot{x}^2 - 2\omega y\dot{x} + y^2\omega^2 + \dot{y}^2 + 2\omega x\dot{y} + x^2\omega^2)$

$T = \frac{1}{2}m_3\dot{x}^2 - m_3\omega y\dot{x} + \frac{1}{2}m_3y^2\omega^2 + \frac{1}{2}m_3\dot{y}^2 + m_3\omega x\dot{y} + \frac{1}{2}m_3x^2\omega^2$

$V = -G\frac{m_3 m_1}{\rho_1} - G\frac{m_3 m_2}{\rho_2} \quad \rho_1 = \sqrt{(x - r_1)^2 + y^2} \quad \rho_2 = \sqrt{(x - r_2)^2 + y^2}$

$L = T - V, \quad L = \frac{1}{2}m_3\dot{x}^2 - m_3\omega y\dot{x} + \frac{1}{2}m_3y^2\omega^2 + \frac{1}{2}m_3\dot{y}^2 + m_3\omega x\dot{y} + \frac{1}{2}m_3x^2\omega^2 + G\frac{m_3 m_1}{\sqrt{(x-r_1)^2+y^2}} + G\frac{m_3 m_2}{\sqrt{(x-r_2)^2+y^2}}$

E.o.M. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$

x:

$\frac{d}{dt}(m_3\dot{x} - m_3\omega y) = m_3\omega\dot{y} + m_3x\omega^2 - G\frac{m_3m_1(x - r_1)}{\underbrace{((x-r_1)^2 + y^2)^{3/2}}_{\rho_1^3}} - G\frac{m_3m_1(x - r_2)}{\underbrace{((x-r_1)^2 + y^2)^{3/2}}_{\rho_2^3}}$

$r_1 = -\mu, r_2 = 1 - \mu, G = \frac{\omega^2(r_1 + r_2)^3}{(m_1 + m_2)} \quad r_1 + r_2 = 1$

m_3 cancels

$\ddot{x} - 2\omega\dot{y} - \omega^2x = -\frac{\omega^2 m_1(x + \mu)}{(m_1 + m_2)\rho_1^3} - \frac{\omega^2 m_2(x - 1 + \mu)}{(m_1 + m_2)\rho_2^3}, \quad \frac{m_1}{m_1 + m_2} = 1 - \mu, \quad \frac{m_2}{m_1 + m_2} = \mu$

$$\boxed{\ddot{x} - 2\omega\dot{y} - \omega^2x = -\frac{\omega^2(1-\mu)(x+\mu)}{\rho_1^3} - \frac{\omega^2\mu(x - 1 + \mu)}{\rho_2^3}} \qquad (2)$$

y:

$\frac{d}{dt}(m_3\dot{y} + m_3\omega x) = -m_3\omega\dot{x} + m_3\omega^2y - G\frac{m_3m_1y}{\rho_1^3} - G\frac{m_3m_2y}{\rho_2^3}$

m_3 cancels, sub in μ, and ω^2

$$\boxed{\ddot{y} + 2\omega\dot{x} - \omega^2y = -\omega^2\frac{(1-\mu)y}{\rho_1^3} - \omega^2\frac{\mu y}{\rho_2^3}} \qquad (3)$$

Time-scaled $t = \frac{1}{\omega}T$,

$$\boxed{x: x'' - 2y' - x = -\frac{(1-\mu)(x+\mu)}{\rho_1^3} - \frac{\mu(x - 1 + \mu)}{\rho_2^3}}$$

$(\ddot{x}) = \frac{d^2(x)}{dT^2} \quad (\dot{x}) = \frac{d(x)}{dT}$

$(\ddot{y}) = \frac{d^2(y)}{dT^2} \quad (\dot{y}) = \frac{d(y)}{dT} \qquad (4)$

$$\boxed{y: y'' + 2x' - y = -\frac{(1-\mu)y}{\rho_1^3} - \frac{\mu y}{\rho_2^3}} \qquad (5)$$

Aerosp 304 Project 1)

a)

$$x'' - 2y' = \frac{\partial U}{\partial x} = U_x, \quad U = \tfrac{1}{2}(x^2+y^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}, \quad \frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{\rho_1^3} - \frac{\mu(x-1+\mu)}{\rho_2^3}$$

using previous values of ρ_1 and ρ_2

Compared to (4), this derivation checks out.

$$y'' + 2x' = \frac{\partial U}{\partial y} = U_y, \quad \frac{\partial U}{\partial y} = y - \frac{(1-\mu)y}{\rho_1^3} - \frac{\mu y}{\rho_2^3},$$

Compared to (5), this checks out.

b) Lagrange points are assumed to be at rest in the rotating frame,

$$\text{so } U_x = 0,\ U_y = 0,\quad x = \frac{(1-\mu)(x+\mu)}{\rho_1^3} + \frac{\mu(x-1+\mu)}{\rho_2^3}$$

$$y = \frac{(1-\mu)y}{\rho_1^3} + \frac{\mu y}{\rho_2^3} \qquad \rho_1 = \rho_2 = 1$$

$$\rho_1 = 1,\ \sqrt{(x-r_1)^2 + y^2} = 1,\ (x-r_1)^2 + y^2 = 1,\ (x+\mu)^2 + y^2 = 1$$

$$\rho_2 = 1,\ \sqrt{(x-r_2)^2 + y^2} = 1,\ (x-r_2)^2 + y^2 = 1,\ (x-1+\mu)^2 + y^2 = 1$$

$$(x+\mu)^2 = (x-1+\mu)^2 \Rightarrow \cancel{x^2} + 2\mu x + \cancel{\mu^2} = \cancel{x^2} - 2x - 2\mu + \cancel{2x\mu} + \cancel{\mu^2} + 1$$

$$2x = 1 - 2\mu \qquad x = \frac{1}{2} - \mu$$

$$\left(\tfrac{1}{2}\right)^2 + y^2 = 1,$$

$$y^2 = 3/4, \quad y = \pm\sqrt{3}/2$$

System	μ	L_4	L_5
Sun-Earth	3.0039×10^{-6}	$(0.499997, \sqrt{3}/2)$	$(0.499997, -\sqrt{3}/2)$
Earth-Moon	1.2151×10^{-2}	$(0.487849, \sqrt{3}/2)$	$(0.487849, -\sqrt{3}/2)$
Saturn-Titan	2.366×10^{-4}	$(0.499763, \sqrt{3}/2)$	$(0.499763, -\sqrt{3}/2)$

Aersp 304 project 1

(i) $x = x_0 + \delta x \quad y = y_0 + \delta x \quad x'' - 2y' = U_x = x - \frac{(1-\mu)(x+\mu)}{p_1^3} - \frac{\mu(x-1+\mu)}{p_2^3}$

$x' = 0 + \delta x' \quad y' = \delta y' \quad y'' + 2x' = U_y = y - \frac{(1-\mu)y}{p_1^3} - \frac{\mu y}{p_2^3}$

$x'' = \delta x'' \quad y'' = \delta y''$

$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix} \Rightarrow \begin{array}{l} \delta x' = \delta x' \checkmark \\ \delta y' = \delta y' \\ \delta x'' = \delta x U_{xx} + \delta y U_{xy} + 2\delta y' \\ \delta y'' = \delta x U_{xy} + \delta y U_{yy} - 2\delta x' \end{array}$$

$$U_{xx} = 1 - \left(\frac{(1-\mu)p_1^3 - 3(1-\mu)(x+\mu)(x+\mu)p_1}{p_1^6}\right) - \left(\frac{\mu p_2^3 - 3\mu(x-1+\mu)(x-1+\mu)p_2}{p_2^6}\right)$$

$$\boxed{\begin{array}{l} U_{xx} = 1 - \frac{1-\mu}{p_1^3} + \frac{3(1-\mu)(x+\mu)^2}{p_1^5} - \frac{\mu}{p_2^3} + \frac{3\mu(x-1+\mu)^2}{p_2^5} \\ U_{xy} = \frac{3(1-\mu)(x+\mu)y}{p_1^5} + \frac{3\mu(x-1+\mu)y}{p_2^5} \\ U_{yy} = 1 - 3\frac{(1-\mu)y^2}{p^5} - \frac{3\mu y^2}{p_2^5} \end{array}}$$

$p_1 = \sqrt{(x+\mu)^2 + y^2} \quad p_2 = \sqrt{(x-1+\mu)^2 + y^2}$