Ch-5- Boolean Algebra

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Unit - 3

Chapter - 4

Posets, Hasse Diagram and Lattices

A relation R on a set A is said to be partial order if R is reflexive, anti-symmetric and transitive. Thus, a relation R on a set A is partial order relation if the following conditions hold.

1) Reflexivity : $aRa \ \forall \ a \in A$
2) Anti-symmetry : If aRb, bRa then $a=b$.
3) Transitivity : If aRb, bRc then aRc.

The set A together with the partial order R is called partially ordered set

We generally denote a partial order by the symbol $\leq$ in place of R.

ex- The relation $\leq$ of divisibility is partial order on the set N of natural no. Here $a \leq b$ means $a|b$ (a divides b)

Here,
The element 2 and 5 are not comparable since neither 2 divides 5 nor 5 divides 2. Thus in a poset every pair of poset need to be comparable.

Totally ordered Set ~~or~~ (Chain) :- Let $(A, \leq)$ be a poset $(A, \leq)$ is called chain or totally ordered set if every two element in A are comparable. That is, if $a, b \in A$ then $a \leq b$ or $b \leq a$. They are also called linearly orderly sets.

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Minimal and Maximal elements :-

⇒ Let $(P, \leq)$ be a poset. An element a in P is called a minimal element if there is no other element b in P s.t $b \neq a$ and $b \leq a$.

A minimal element in a poset need not to be unique. All those elements, which appear at the lowest level of a Hasse diagram of a partially ordered set are minimal elements.

Every finite poset has at least one minimal element

⇒ Let $(P, \leq)$ be a poset. An element a in P is called a maximal element if there is no b in P s.t $a \neq b$ and $a \leq b$. ~~In other words~~,

A maximal element in a poset need not to be unique. All those elements which appear at the highest levels of a Hasse diagram of a poset are maximal elements.

Least & Greatest Elements :-

⇒ Let $(P, \leq)$ be a poset. If there exists an element $a \in P$ s.t $a \leq x$ $\forall$ $x \in P$, then a is called the least element in P.
The least element also called first element or zero element of P.

~~Representati~~

Ex:- let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$

R_1 is P.O.R

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	(2,3)	(2,4)
3	3,1	3,2	3,3	(3,4)
4	4,1	4,2	4,3	4,4

⇒ Let $(P, \leq)$ be a poset. If there exists an elements $a \in P$ s.t. $x \leq a$ for all $x \in P$, then a is called the greatest element in P.

The greatest element is also called last element or unit element of P, and usually denoted by 1.

Upper Bounds and Least Upper Bound

Let $(P, \leq)$ be a poset and let A be a subset of P. An element $x \in P$ is called an upper bound of A if $a \leq x$ $\forall$ $a \in A$

Let $(P, \leq)$ be a poset and let $A \subseteq P$. An element $x \in P$ is said to be a least upper bound or supremum of A if x is a upper of A and $x \leq y$ for all upper bounds y of A. Supremum of A is denoted by $\sup(A)$.

Lower Bounds and Greatest Lower Bound

Let $(P, \leq)$ be a poset and A be a subset of P. An element $x \in P$ is said to be a lower bound of A if $x \leq a$ $\forall$ $a \in A$.

Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $x \in P$ is said to be a greatest lower bound or infimum of A, if x is a lower bound and $y \leq x$ for all lower bounds y of A. It is denoted by $\inf(A)$.

Lattice

A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ of two element of L has a greatest lower bound and a least upper bound.

⇛ poset $(L, \leq)$ is a lattice if for every $a, b \in L$ sup $\{a, b\}$ and inf $\{a, b\}$ exist in L.

$\vee \rightarrow$ join, $a+b$, lcm $\qquad \wedge \rightarrow$ meet, $a \cdot b$, gcd

We denote sup $\{a, b\}$ by $a \vee b$ and call it the join of a and b and inf $\{a, b\}$ by $a \wedge b$ and call it the meet of a and b.

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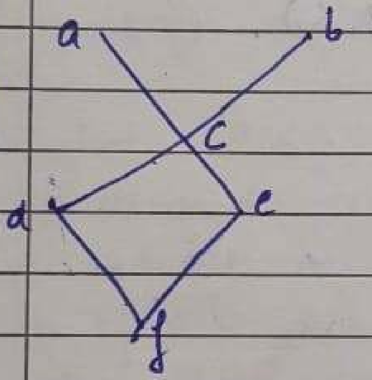
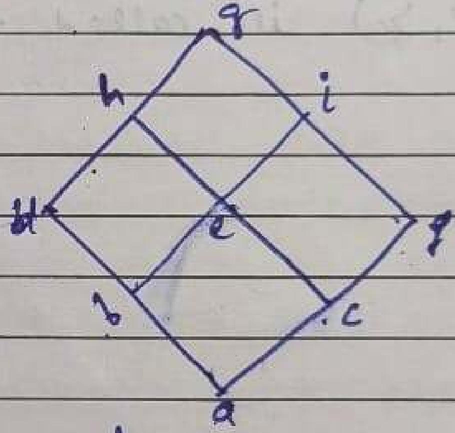
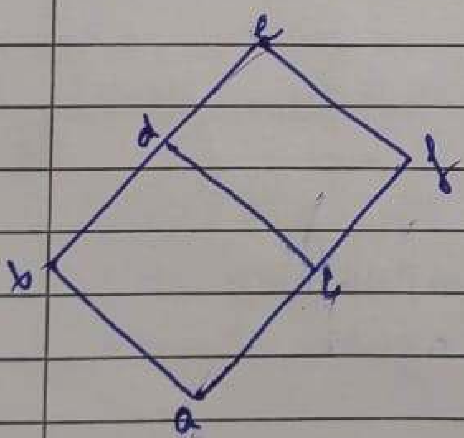
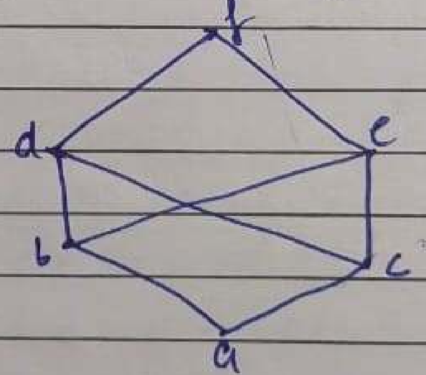
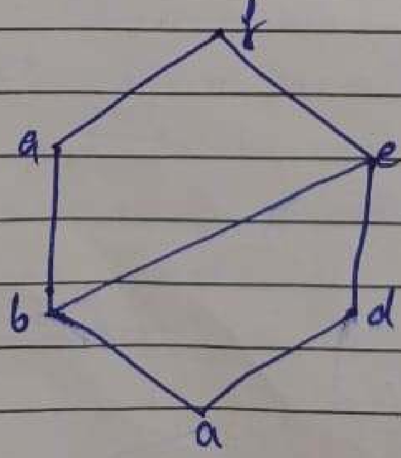








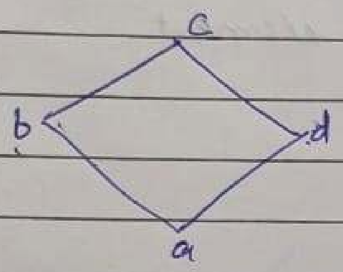


Properties of lattice :

Thm: If $(L, \leq)$ is a lattice then for any $a, b, c \in L$ the following holds:

1) $a \wedge a = a$ and $a \vee a = a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4) $a \wedge (a \vee b) = a$

Ex.

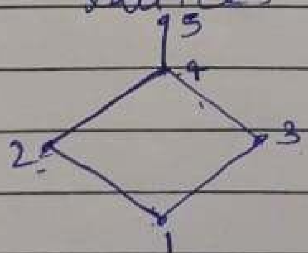


sup $\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

inf $\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since sup and Inf exist for each pair So, it is lattice.

Ex.



$\vee$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	4	4	5
3	3	4	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Since sup and inf exist for each pair, So, it is lattice.

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Dual of a Poset :-

Let R be a relation defined on a set X. Then the converse of R, denoted by $\bar{R}$, is a relation on X defined by

$$a\bar{R}b \text{ if and only if } bRa \text{ , } a,b \in X$$

If R is a partial order relation denoted by $\leq$ on X then $\bar{R}$, the converse of R, is denoted by $\geq$.

⇒ Let $(P, \leq)$ be a poset, then $(P, \geq)$ is also a poset. $(P, \geq)$ is called the dual of $(P, \leq)$.

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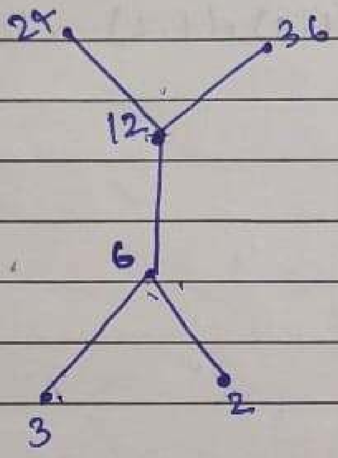
Ex- $X = \{2, 3, 6, 12, 24, 36\}$

$\leq$ be defined as $x|y$

then

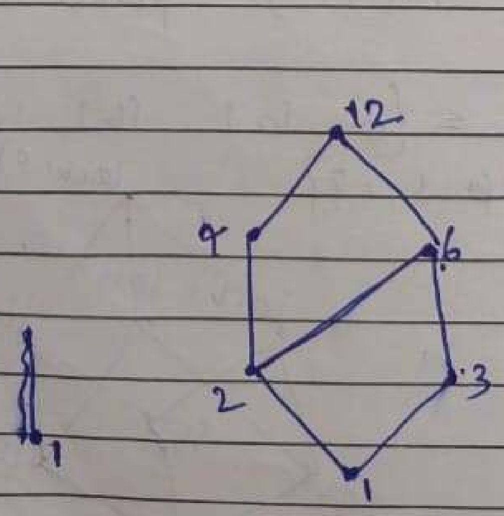
$R = (2,6), (2,12), (2,24), (2,36), (3,6), (3,12)$
$(3,24), (3,36), (12,24), (12,36), (6,12)$
$(6,24), (6,36)$

$R = (2,6), (3,6), (3,24), (3,36), (12,24), (12,36)$
$(6,12), (6,36)$



Ex- $X = \{1, 2, 3, 4, 6, 12\}$, $x \leq y \Leftrightarrow x|y$

$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6)$
$(2,12), (3,6), (3,12), (6,12), (4,12)\}$



Hasse
↑
Poset
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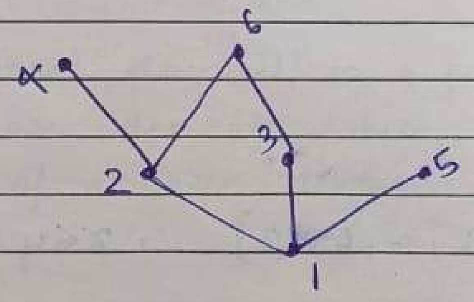
Hasse Diagram :-

A partial order $\leq$ on a set X can be represented by means of a diagram known as Hasse diagram of $(X, \leq)$.

eg. let $X = \{1, 2, 3, 4, 5, 6\}$ then / is a partial order relation on X. Draw the Hasse diagram of $(X, /)$.

sol: $R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$



$(X, /)$.

ex= Draw the Hasse diagram for the POSET $(P(S), \subseteq)$ where $P(S)$ is power set on $S = \{a, b, c\}$

sol: $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$