UNIT I :- Matrix

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Matrix :-

A matrix A is a rectangular array in which elements are arranged in square bracket and column.

Ex :-

$$A = \begin{array}{c} \\ \end{array}\begin{array}{l} \begin{bmatrix} 2 & 3 & 4 & 2 \\ 5 & 6 & 7 & 1 \\ -1 & -2 & 3 & 4 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \\ \;\downarrow \quad\; \downarrow \quad\; \downarrow \quad\; \downarrow \quad 3\times 4 \\ \;C_1 \quad C_2 \quad C_3 \quad C_4 \end{array}$$

* Elements of matrix are represented by a_{11}, a_{12}, a_{13} .... and a_{21}, a_{22}, a_{23} .... in short elements ar represented by a_{ij} where $i = 1, 2, 3$ ...... and $j = 1, 2, 3$ ....

* Matrix is denoted by $A = [a_{ij}]_{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}_{m\times n}$$

$$A = [a_{ij}]_{3\times 4}$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times4}$$

* In matrix A Horizontal lines represent = Row
  vertical lines represent = column

Types of Matrices :-

(1) **Row Matrix :-** A matrix which contain only one row and any column is called row matrix

Ex :-

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times4}$

(ii) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{1\times6}$ (iii) $\begin{bmatrix} 7 \end{bmatrix}_{1\times1}$

(ii) **Column Matrix :-** A matrix which contain only one column and any column is called column matrix

Ex :-

(i) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times1}$ (ii) $\begin{bmatrix} 8 \end{bmatrix}_{1\times1}$ (iii) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{5\times1}$

(iii) Null Matrix or zero matrix :-

A matrix in which all elements are zero is called null matrix and is denoted by O

Ex:-

$$A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \quad C = \begin{bmatrix} 0 \end{bmatrix}_{1\times1}$$

(iv) Square matrix :-

A matrix is called square matrix if no of rows = no of column.

Ex:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 5 \end{bmatrix}_{3\times3} \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}_{2\times2} \quad C = \begin{bmatrix} 5 \end{bmatrix}_{1\times1}$$

(v) Diagonal Martix :-

A square Martix is said to be diagonal Matrix if its all non-diagonal entries are zero

Ex:-

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

Principal diagonal

$a_{11}, a_{22}, a_{33} \ldots a_{nn}$ = diagonal element

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(vi) Scalar Matrix : A square matrix M is said to be scalar matrix if

(i) All non diagonal elements are zero

(ii) All diagonal elements are same

Ex :-

$$M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3} \qquad M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

(vii) Identity Matrix or Unit Matrix :-

A square matrix M is said to be unit matrix if

(i) All non diagonal elements are zero

(ii) All diagonal elements are equal to 1.

Ex :-

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \qquad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

* unit matrix is denoted by I or I_n

(viii) Transpose of Matrix :

— * — * — * — * — * — Let M be any given matrix A matrix obtained by inter-changing corresponding row to corresponding column is known as transpose of matrix M and is denoted by M^T or M'

Ex :- $M = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 2 \\ -1 & -2 & 3 \end{bmatrix}_{3\times3}$ $M^T = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 7 & -2 \\ 6 & 2 & 3 \end{bmatrix}_{3\times3}$

$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}$

$B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & -3 \end{bmatrix}_{2\times3}$ $B^T = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & -3 \end{bmatrix}_{3\times2}$

(ix) Symmetric Matrix :

— * — * — * — * — A square Matrix is said to be symmetric Matrix if $\boxed{M = M^T}$

* In this Matrix element $a_{ij} = a_{ji}$

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Ex:

$$M = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \quad M^T = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

$$M = M^T$$

$\Rightarrow$ M is a symmetric matrix

$$A = \begin{bmatrix} a & b & c \\ b & d & f \\ c & f & g \end{bmatrix}_{3\times3} \quad A^T = \begin{bmatrix} a & b & c \\ b & d & f \\ c & f & g \end{bmatrix}_{3\times3}$$

$$A = A^T$$

$\Rightarrow$ A is a symmetric Matrix

(x) Skew Symmetric Matrix :

—*—*—*—*—* A square Matrix is said to be Skew Symmetrix matrix if $\boxed{M = -M^T}$

* In this matrix element $a_{ij} = -a_{ji}$
* Diagonal element must be zero

$$M = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \quad M^T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$M = -M^T$$

$\Rightarrow$ M is skew symmetric matrix.

(xi) Upper triangular Matrix (U.T.M)

A Square Matrix whose all element below the Principal diagonal are zero is called U.T.M

Ex:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(xii) Lower triangular Matrix (L.T.M) :

A Square Matrix whose all elements above the Principal diagonal are zero is called L.T.M

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 2 & 5 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(xiii) Orthogonal of a matrix :

A Square matrix M is said to be orthogonal if $MM^T = I$ (Identity Matrix)

Ex:-

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta.\sin\theta + \sin\theta.\cos\theta \\ -\sin\theta.\cos\theta + \cos\theta.\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad AA^T = I$$

A is orthogonal matrix

(xiv) conjugate of a Matrix :

—*—*—*—*—*—

Ex :-

$$A = \begin{bmatrix} 4i-1 & 3 & 2i \\ 2i & -3 & -2+i \\ 4 & 2 & -3 \end{bmatrix}$$

$$\bar{A} = \bar{A} = \begin{bmatrix} -4i-1 & 3 & -2i \\ -2i & -3 & -2-i \\ 4 & 2 & -3 \end{bmatrix}$$

(xv) Matrix A^θ :- $(\bar{A})^T = \overline{(A^T)}$

—*—*— Transpose of a conjugate matrix

Ex :-

$$A = \begin{bmatrix} 2+i & 3 & 4 \\ 3i & 4 & 5i \\ 0 & 3 & -4i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2-i & 3 & 4 \\ -3i & 4 & -5i \\ 0 & 3 & 4i \end{bmatrix}$$

$$(\bar{A})^T = (A^\theta) = \begin{bmatrix} 2-i & -3i & 0 \\ 3 & 4 & 3 \\ 4 & -5i & 4i \end{bmatrix}$$

(xvi) Hermitian Matrix :

—*—*—*—* $\boxed{A = A^\theta}$

Ex :-

$$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 2-3i & 3-i \\ 2+3i & 2 & 1+2i \\ 3+i & 1-2i & 5 \end{bmatrix}$$

$$(\bar{A})^T = A^\theta = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

since $A = A^\theta \Rightarrow A$ is ~~sym~~ Hermitian matrix

(xvii) Skew Hermitian Matrix :

—*—*—*—*—*— $\boxed{A = -A^\theta \text{ or } A^\theta = -A}$

Ex :-

$$A = \begin{bmatrix} 0 & 2-3i & 4+5i \\ -2-3i & 0 & 2i \\ -4+5i & 2i & 0 \end{bmatrix}$$

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$$\bar{A} = \begin{bmatrix} 0 & 2+3i & 4-5i \\ -2+3i & 0 & -2i \\ -4-5i & -2i & 0 \end{bmatrix}$$

$$\underset{A^{\theta}}{(\bar{A})^{T}} = \begin{bmatrix} 0 & -2+3i & -4-5i \\ 2+3i & 0 & -2i \\ 4-5i & -2i & 0 \end{bmatrix}$$

$$(\bar{A})^{\theta} = -\begin{bmatrix} 0 & 2-3i & 4+5i \\ -2-3i & 0 & 2i \\ -4+5i & 2i & 0 \end{bmatrix}$$

$\boxed{A^{\theta} = -A} \Rightarrow$ A is skew Hermitian matrix

(xviii) Matrix multiplication :

— * — * — * — * —

let $[A]_{m\times n}$ and $[B]_{n\times p}$

* Multiplications AB is possible only when no of columns of matrix A = No of row of matrix B

Ex :-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3} \qquad B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times 2}$$

$$AB = \begin{bmatrix} 0+(-1)+4 & 0+0+(-2) \\ 1+(-2)+6 & -2+0+(-3) \\ 2+(-3)+8 & -4+0+(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}_{3\times 2}$$

(xix) Idempotent matrix :- $\boxed{A^2 = A}$

—*—*—*—*

Ex :-

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+2+(-4) & -4+(-6)+8 & -8+(-8)+12 \\ -2+3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \qquad A^2 = A$$

$\Rightarrow$ A is an idempotent matrix

(xx) Involuntary matrix :- $\boxed{A^2 = I}$ (identity matrix)

—*—*—*—*

Ex :-

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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$$A^2 = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(xxi) # Identity matrix is always an involuntary matrix.

— x — x — x — x — x — x —

Ex:-

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

→ A is an involuntary matrix.

(xxi) Nilpotent Matrix :

— * — * — * — A matrix M is said to be nilpotent if $\boxed{M^k = 0}$, where k is a positive integer. The least positive integer k for which $\boxed{M^k = 0}$ is called index.

Ex :-

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{a^2b^2} - \cancel{a^2b^2} & \cancel{ab^3} - \cancel{ab^3} \\ -\cancel{a^3b} + \cancel{a^3b} & -\cancel{a^2b^2} + \cancel{a^2b^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boxed{\text{Index} = 2}$$

(xxii) Unitary Matrix :-

— * — * — * — * — $\boxed{AA^\theta = I}$ or $\boxed{A^\theta A = I}$

Ex :-

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}, \quad \bar{A} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$(\bar{A})^T = A^\theta = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AA^\theta = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

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$$= \frac{1}{3}\begin{bmatrix} 1+(1+i)(1-i) & 1+i+(-1)(1+i) \\ 1-i+(-1)(1-i) & (1-i)(1+i)+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+1-i^2 & 1+i-1-i^2 \\ 1-i-1+i & 1-i^2+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2-(-1) & 0 \\ 0 & 2-(-1) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\boxed{AA^{\theta} = I}$$

→ A is a unitary matrix.

(xxiii) **Equal matrix or Equivalent matrix:**

—*—*—*—*—*—*—*—

Two matrix A and B are said to be equal if

(i) Both matrix have same size.
(ii) corresponding elements of matrix A and B are equal.

Ex:- $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(xxiv) **Elementary Transformations.**

—*—*—*—*—*—*—*—

(1) We can interchange any two rows and column $(R_i \leftrightarrow R_j)$ or $(C_i \leftrightarrow C_j)$

Ex:-

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

$C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 5 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(ii) We can multiply any row or column by a non-zero number k (scalar)

Ex:-

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix}$$

$R_1 \rightarrow 2R_1$

$$\sim \begin{bmatrix} 2 & 6 \\ 2 & 4 \\ 2 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \rightarrow 2R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 2 \end{bmatrix}$$

(iii) We can add any two rows or column $(R_i \rightarrow R_i + R_j)$ or $(C_j \rightarrow C_j + C_k)$

Ex:-

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$\sim \begin{bmatrix} 3 & 7 \\ 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 4 & -3 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

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{2 Lecture}

Rank by Echelon Form :

Rank (A) = $\rho(A)$ = No. of non-zero row in echelon form.

Echelon Form :

A matrix A is said to be in echelon form if A satisfy following conditions.

(i) The non-zero element in a row is unity (1)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) All zero rows if any are at the bottom of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) The no. of zero preceding the first non-zero element in a row is less than the no. of such zero in the succeeding row

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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(iv) Matrix should be in U.T.M.

* Find Rank

Q. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

Solⁿ..

$R_2 \to R_2 - 3R_1 \;,\; R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$$

$R_2 \to R_2/(-5)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7/5 \\ 0 & -1 & -5 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7/5 \\ 0 & 0 & -18/5 \end{bmatrix}$$

$R_3 \to R_3 / {-18/5}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) =$ No. of Non-zero row $= 3$

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Q. $A = \begin{bmatrix} 4 & 1 & 3 & 8 \\ 6 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Solⁿ:-

$R_3 \to R_3 - R_2$,

$$\sim \begin{bmatrix} 4 & 1 & 3 & 8 \\ 6 & 2 & 6 & -1 \\ 4 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 4 & 1 & 3 & 8 \\ 6 & 2 & 6 & -1 \\ 16 & 4 & 12 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 4 & 1 & 3 & 8 \\ 6 & 2 & 6 & -1 \\ 0 & 0 & 0 & -17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{4} R_1$, $R_3 \to \frac{-1}{17} R_3$

$$\sim \begin{bmatrix} 1 & 1/4 & 3/4 & 2 \\ 6 & 2 & 6 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$R_2 \to R_2 - 6R_1$

$$\sim \begin{bmatrix} 1 & 1/4 & 3/4 & 2 \\ 0 & 1/2 & 3/2 & -13 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to 2R_2$

$$\sim \begin{bmatrix} 1 & 1/4 & 3/4 & 2 \\ 0 & 1 & 3 & -26 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$.

Q. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

Solⁿ.

$R_3 \leftrightarrow R_1$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 7 & 8 \\ 0 & 7 & 8 \end{bmatrix}$$

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$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{7} R_2$

$R_1 \rightarrow -1 R_1$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 8/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

Normal Form of a matrix

—*—*—*—*—*—*—*—* A matrix A can be reduced by elementary row and column transformation into one of the following equivalent matrix.

(i) $\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ (iii) $\begin{bmatrix} I_n & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_n \end{bmatrix}$

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Q. Reduce the following matrix to the normal form and hence find its rank:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solⁿ:-

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$R_4 \rightarrow R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$R_4 = R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - C_1 + C_2$

$C_3 \rightarrow C_3 - C_1 + 3C_2$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

UNIT - II

Mathematical Logic

Statement or proposition:
A sentence which is either true or false is called statement.

Eg:-
1. Ram is good boy.
2. Hari is rich man.
3. It is raining.
4. what is your name?
5. Oh my God!
6. Shut up
7. Blood is green.

4,5,6 are not a statements

Compound statement:
A statement obtain from the combination of two or more statement is called compound statement.

Eg:- P: Blood is green
q: I am hungry

$P \vee q$ = Blood is green or I am hungry

P: Ram is good boy
q: shyam is good boy

$P \wedge q$ = Ram and shyam are good boys.

Atomic or simple statement:

A statement which is not a combination of other statement is called simple statement.

Eg :- P: Blood is green
q: $2+6=11$

Note

* Generally statements are denoted by P, q, r, etc.

Connective:

Two statements can be combine by "and", "or", "not" etc. called connectives. There are following connectives with their symbols.

English word	connectives Name	symbols
Not	Negation	$\sim$, $\neg$
AND	conjunction	$\wedge$
OR	Disjunction	$\vee$
one way implication	conditional	$\rightarrow$, $\Rightarrow$
Iff or If and only if	Biconditional	$\leftrightarrow$, $\Leftrightarrow$

1. Negation of statement :- Let P be statement them negation of P difined as:

(i) If P is true then negation of P is false.
(ii) If P is false then negation of P is True

* we can use "not" or "It is false that" to negate the statement.

P : Riya is poor
~P : Riya is not poor
~P : It is false that Riya is poor

P	~P
T	F
F	T

2. Conjunction of statement -

The conjunction of two statements p and q is defind as $p \wedge q$ (read as p and q) statement is true when both p and q are true and false when either one of them of both are False.

p : Ansh is a bad boy
q : shubham is a bad boy

$p \wedge q$: Ansh and shubham are bad boys

Truth Table of $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction of statement -

Let p and q be two statements, then the disjunction of p and q is defined as $p \vee q$ (read as p or q) is True when one or both statements are True and False it both statements are False.

Eg :- p : I am hungry
q : I Like to eat pizza.

$p \vee q$: I am hungry or I like to eat pizza.

Truth Table of $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. Conditional statement : Let p and q be two statement then the statement "If p then q" is called conditional statement.

* p is called antecedent or hypothesis
* q is called consequent or conclusion
* denoted by $p \rightarrow q$ or $p \Rightarrow q$ (implication)

Truth Table of $p \rightarrow q$ or $p \Rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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eg :- p : I am hungry
q : I will eat

5. Biconditional statement :
Let p and q be two statements then the statement "p if and only if q" is called Biconditional statement.

eg :- p : I am 18 year old.
q : I can vote

Truth Table For $p \leftrightarrow q$ or $p \Leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Toutology :
A statement which is always True is called Toutology.

Eg :- $p \rightarrow p$

p	p	$p \rightarrow p$
T	T	T
F	F	T

Contradiction:

A statement which is always False is called contradiction.

Eg :- $P \wedge \sim P$

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

Some important formulas

(1). Commutative Law

(a) $P \wedge q = q \wedge P$

(b) $P \vee q = q \vee P$

(2) Associative Law

(a) $P \vee (q \vee r) = (P \vee q) \vee r$

(b) $P \wedge (q \wedge r) = (P \wedge q) \wedge r$

* (3) Distributive Law

(a) $P \wedge (q \vee r) = (P \wedge q) \vee (P \wedge r)$

(b) $P \vee (q \wedge r) = (P \vee q) \wedge (P \vee r)$

* (4) Idempotent Law

(a) $P \wedge P = P$

(b) $P \vee P = P$

(5) Law of absouption

(a) $p \wedge (p \vee q) = p$

(b) $p \vee (p \wedge q) = p$

* (6) Involution Law

$\sim(\sim p) = p$

(7) Complement Law

(a) $p \vee \sim p = T$

(b) $p \wedge \sim p = F$

(8) Operation with T

(a) $p \vee T = T$

(b) $p \wedge T = p$

(9) Operation with F

(a) $p \vee F = p$

(b) $p \wedge F = F$

* (10) De-Morgan's Law

(a) $\sim(p \wedge q) = \sim p \vee \sim q$

(b) $\sim(p \vee q) = \sim p \wedge \sim q$

* (11) $p \rightarrow q = \sim p \vee q$

* (12) $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

Normal Form

(1) Disjunctive Normal Form (DNF):-

A Formula which has a sum of elementary product and is equivalent to the given formula is called DNF.

Eg:- (i) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$
(ii) $p \vee q \vee r$
(iii) $(p \wedge q) \vee (p \wedge r)$

(2.) Conjunction Normal Form (CNF):-

A Formula which has a product of elementary sum and is equivalent to the given formula is called CNF.

Eg:- (i) $(p \vee q) \wedge (q \vee r) \wedge (r \vee p)$
(ii) $p \wedge q \wedge r$
(iii) $(p \vee q) \wedge (p \wedge r)$

Rules to obtain DNF / CNF

1. Remove $\rightarrow, \leftrightarrow$ connectives by using proper Formu
2. Eliminate $\sim$ before sum and product by using De-morgan's Law.
3. Apply Distribution Law untill the form DNF or CNF obtained.

Argument:

An argument is a statement which is formed by a given set of statement called premises and gives conclusion.

$$\left.\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array}\right] \rightarrow \text{Premises}$$

$q \rightarrow$ conclusion

$$[P_1 \wedge P_2 \wedge P_3 \ldots\ldots \wedge P_n] \rightarrow q$$

P_1 - Today is sunday

P_2 - Market is closed today

q - I am not going to market

$P_1 \wedge P_2 \rightarrow q$

Fallacy (Invalid):

An argument which is not valid is called fallcy.

valid Argument:

An argument is said to be valid if the conclusion is toutology.

check the argument is valid or not?

$$\frac{\begin{matrix} p \\ p \to q \end{matrix}}{q}$$

$[p \land (p \to q)] \to q$

p	q	$p \to q$	$p \land (p \to q)$	$[p \land (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Law of Detachment (Modus ponens)

$$\frac{\begin{matrix} p \\ p \to q \end{matrix}}{q}$$

$[p \land (p \to q)] \to q$

Law of syllogism

$$\frac{\begin{matrix} p \to q \\ q \to r \end{matrix}}{p \to r}$$

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$$[(p \to q) \land (q \to r)] \to p \to r$$

Let $[(p \to q) \land (q \to r)] = x$

p	q	r	$p \to q$	$q \to r$	x	$p \to r$	$x \to p \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

∵ since conclunsion is toutology
∴ statement is valid

The End

UNIT-3

GRAPH THEORY

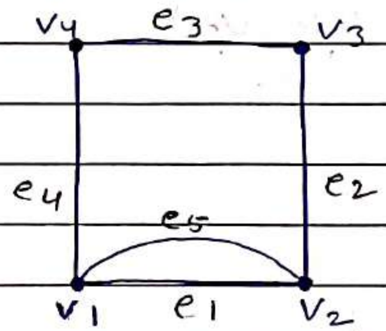
Graph:

A Graph is a pair of (V, E) where $V = \{v_1, v_2, v_3 \ldots\}$ is called the set of vertex or nodes and $E = \{e_1, e_2, e_3 \ldots\}$ is a set such that each element e_K of E is an unordered pair of (v_i, v_j) of vertices.

$E = \{e_1, e_2, e_3 \ldots\}$ is called set of edges

Parallel Edges in the graph-

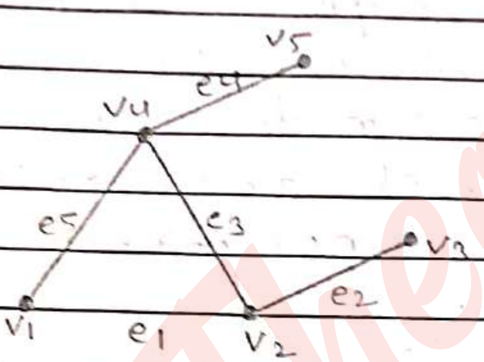
An Edges having the same pair of end vertex are called parallel edges.



e_1 & e_5 are parallel edges

Self-loop in the Graph:

A Graph is called simple graph if graph is free from parallel edges and self-Loop.



Finite Graph:

A graph is said to be finite if the set v and E are finite.

$V = \{v_1, v_2, v_3\}$

$E = \{\underbrace{(v_1, v_2)}_{e_1}, \underbrace{(v_3, v_2)}_{e_2}\}$

$\therefore E = \{e_1, e_2\}$

Infinite Graph:

A graph is said to be infinite if the set v and E are infinite.

$V = \{v_1, v_2, v_3 \text{------}\}$

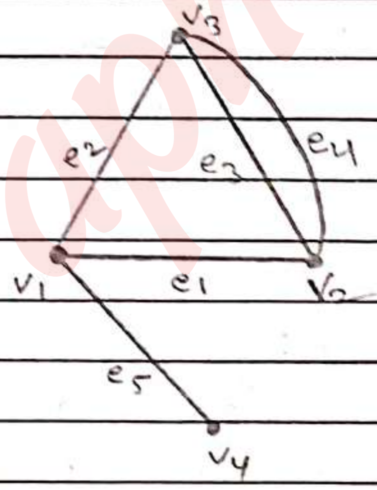
$E = \{e_1, e_2, e_3 \text{------}\}$

Incident and degree:

Let e_k be any edge joining two vertices v_i and v_j in the graph G. Then the edge e_k is said to be incident on v_i and v_j.

Adjacent vertex:

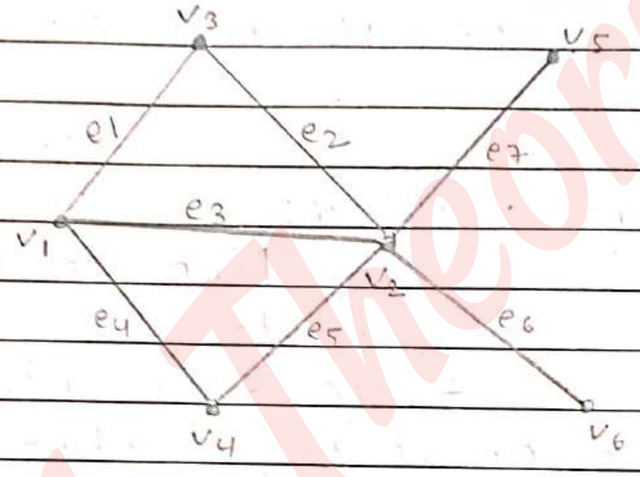
Two vertices in the graph is said to be adjacent if there exist an edge which are incident on both vertices.



$\{(v_1, v_4), (v_1, v_3), (v_1, v_2), (v_2, v_3)\}$ are adjacent.
but,
$\{(v_2, v_4), (v_3, v_4)\}$ are not adjacent.

Adjacent Edges:

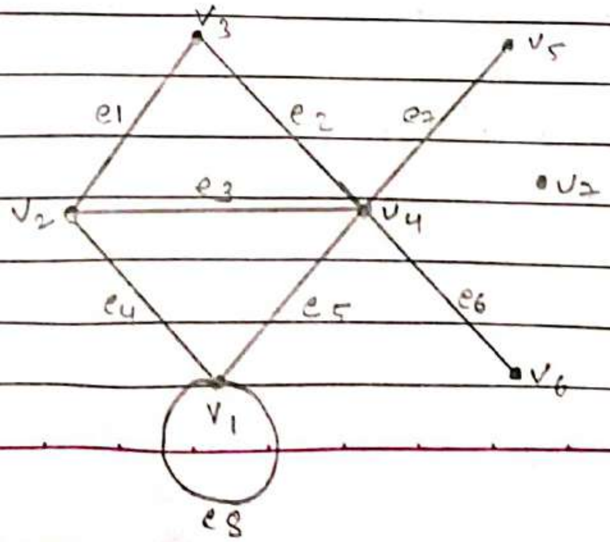
Two non parallel edges which incident on same vertex is called adjacent edges.



$(e_1, e_2), (e_3, e_1), (e_3, e_2)$ .... are adjacent edges but (e_2, e_4) are not adjacent.

* Degree of vertex:

The degree of vertex v is equal to the number of edges which are incident on v



- Denoted by $d(v)$
- Degree of self-loop is counted as two

$d(v_1) = 4$
$d(v_2) = 3$
$d(v_3) = 2$
$d(v_4) = 5$
$d(v_5) = 1$
$d(v_6) = 1$
$d(v_7) = 0$

Isolated vertex:

A vertex which has degree zero is called isolated vertex

Pendant vertex:

A vertex of degree one(1) is called pendent vertex.

Null Graph:

A graph is said to be null graph if the set of vertex is non-empty but the set of edges are empty is called null graph.

Eg:-

$V = \{v_1, v_2, v_3, v_4\}$

$\cdot v_1 \quad \cdot v_3$
$\cdot v_2 \quad \cdot v_4$] Null Graph

* Sum of degrees of all vertices will be twice the no. of edges

Proof

Let G be any graph and $v_1, v_2, v_3 \ldots v_n$ n vertices since each edge incident on two vertex, so each edge contribute 2 times when we find the degree of edges. Thus the form of all vertex = 2 x Number of degree.

Handshaking Lemma Mathematically

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) = 2e$$

Even vertex:

A vertex v is said to be even if degree of vertex v is even.

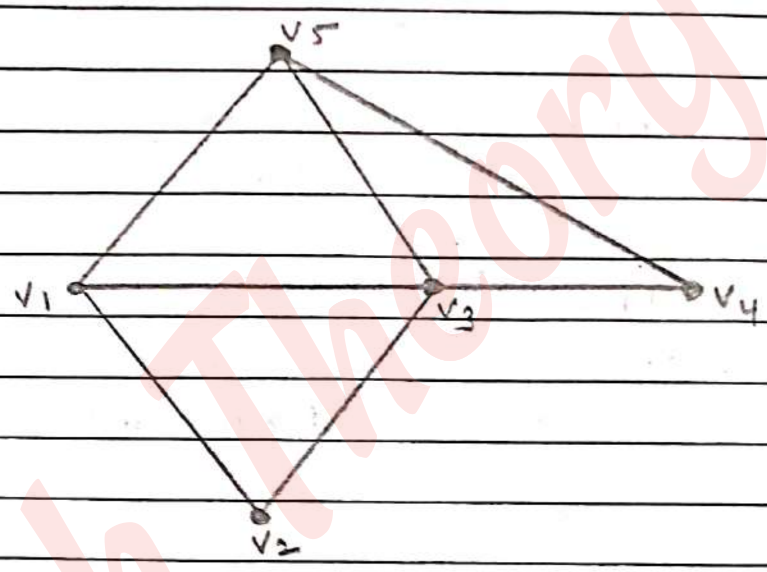
Odd vertex:

A vertex v is said to be odd if degree of vertex v is odd.

VVI

Th-2

The number of odd vertex in a graph is always even



Let a be any graph.

Let $v_1, v_2, v_3, \ldots\ldots v_n$ are n vertex
Let k vertices is of odd degree. Then number of even vertex is equal to n-k.

Now,

$$\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{n} d(v_i) + \sum_{i=n-k}^{n} d(v_i)$$

$\downarrow$ Even No. $\qquad \downarrow$ I even no. $\qquad \downarrow$ II even no.

Since the sum of L.H.S term is an even number and the sum of R.H.S 2nd term is also even

$\Rightarrow$ The sum $\sum_{i=1}^{k} d(v_i)$ is also even

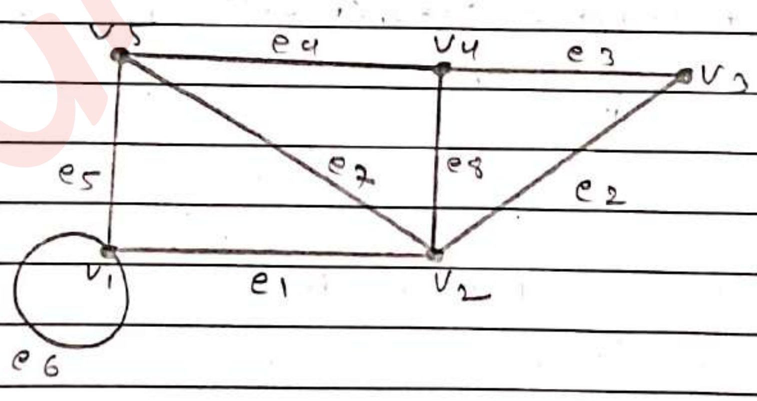
since sum $\sum_{i=1}^{n} d(v_i)$ is even

$\Rightarrow$ The number of odd vertices is also even

Proved

walk:

A walk is an alternating sequence of vertex and edges in which repetition of edges and vertex are allowed



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_8 v_2 e_1 v_1$ is walk

Find walk start and end at v_5 and cover all edges and vertex at least one time

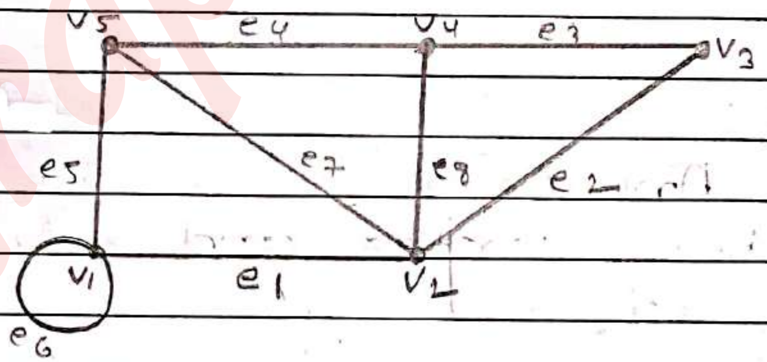
$v_5 e_5 v_1 e_6 v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_8 v_2 e_7 v_5 e_4 v_4 e_8 v_2 e_7 v_5$

open walk:

A walk in which end point are distinct is called open walk.

Note:

In open walk vertices and edges may be respected. except end vertices.

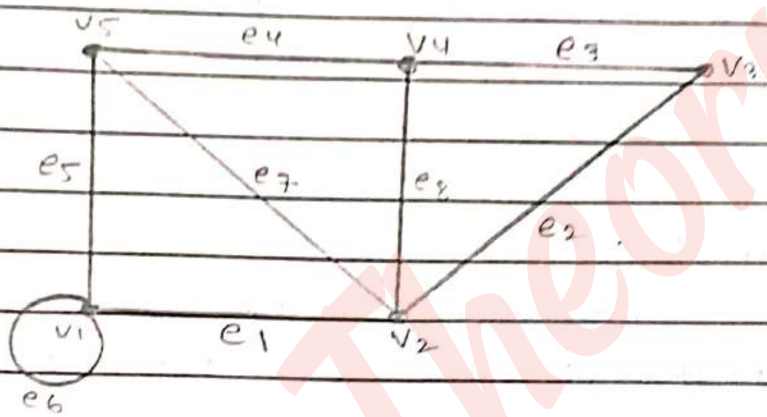


(i) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ is an open walk.

(ii) $v_5 e_5 v_1 e_1 v_2 e_2 v_3$ is an open walk.

Closed walk:

A walk is said to be closed if the end vertices of the walk are same.

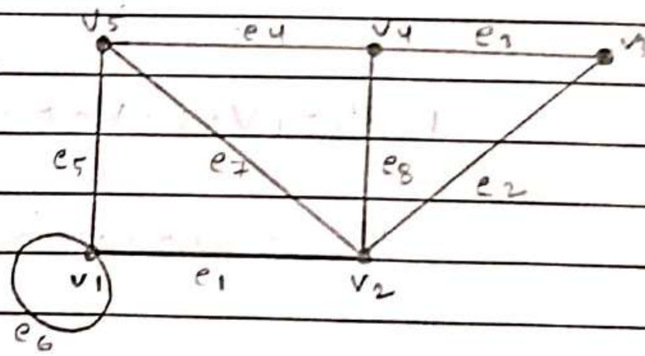


(i) $v_1 e_6 v_1$ is a closed walk

(ii) $v_5 e_5 v_1 e_1 v_2 e_8 v_4 e_4 v_5$ is a closed walk

Path:

An open walk in which number vertex and edges are reapeted is called path.



(i) $v_1 e_1 v_2 e_7 v_5$ is path

(ii) $v_1 e_6 v_1 e_5 v_5$ is not path

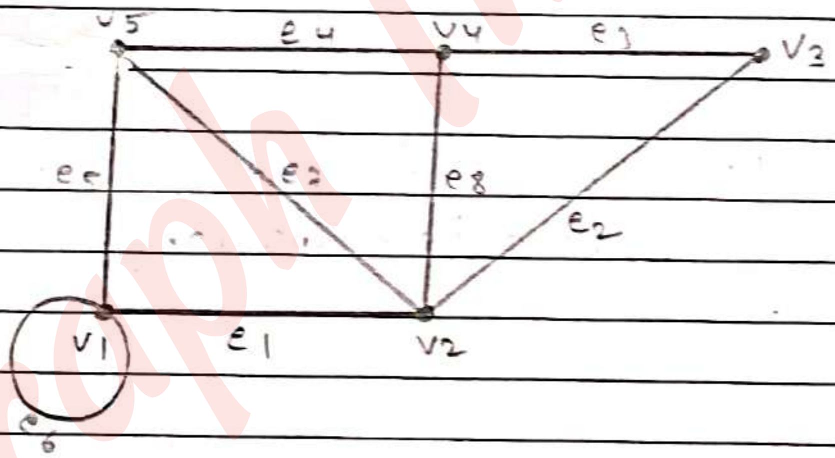
(iii) $v_4 e_4 v_5 e_5 v_1$ is a path.

Length of the Path:

The number of edges in the path is called length of the path.

circuit (cycle):

A closed walk in which number vertex and edges are repeated except end vertices is called circuit.



$v_1 e_6 v_1$ is a circuit.

$v_1 e_1 v_2 e_8 v_4 e_4 v_5 e_5 v_1$ is a circuit

Note:

In circuit the degree of each vertex is equal to 2.

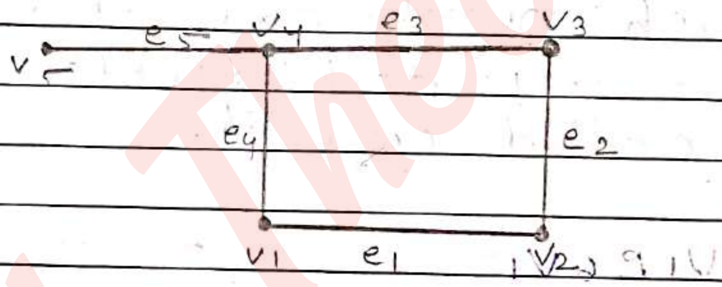
Note:

Every self-loop is a circuit but every circuit is not a self-loop

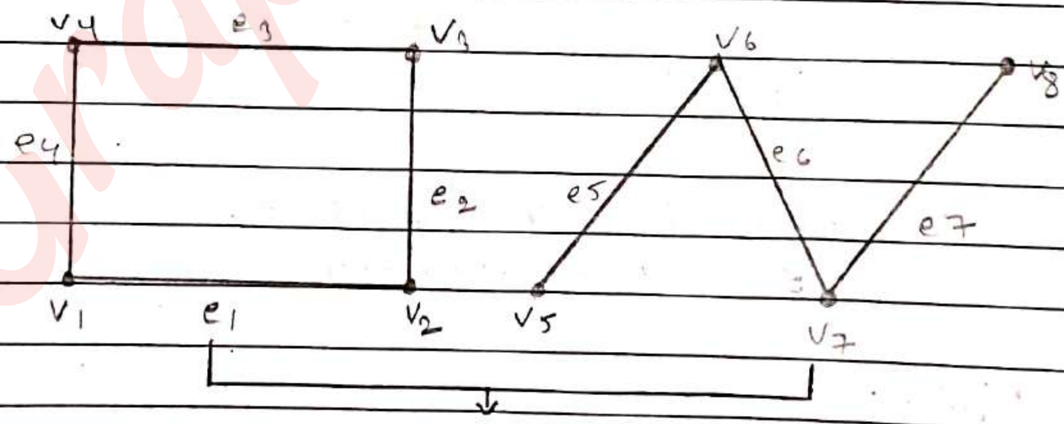
Connected & Disconnected Graph:

A graph G is said to be connected if there exist a path between each pair of vertices. other graph is said to be disconnected.

Eg:-



$(v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_5 v_5)$
Connected Graph



Disconnected Graph

* In disconnected graph, Every component of disconnected graph is connected.

Component:

A component of a graph G is maximal connected graph G.

V.V.I.

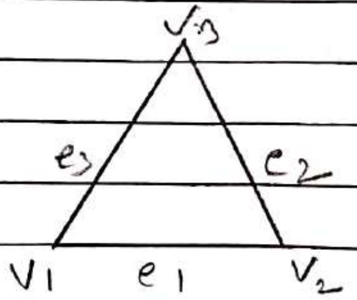
Complete Graph:

A simple graph is said to be complete if each pair of vertices are adjacent in graph.

- If there are n vertices in a complete graph their it is denoted by K_n
- The number of edge in complete graph is $\frac{n(n-1)}{2}$
- The degree of each vertex is equal to $(n-1)$

Eg:-

K_3



* no. of edge $= \frac{3(3-1)}{2}$

$= 3$

* degree of vertex $= K_3$

$= (3-1)$

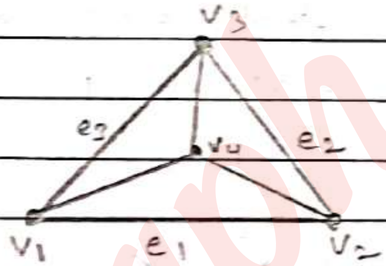
$= 2$

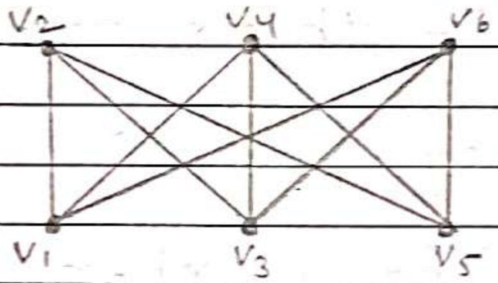
Regular Graph:

A graph G is said to be regular if the degree of each vertex is equal.

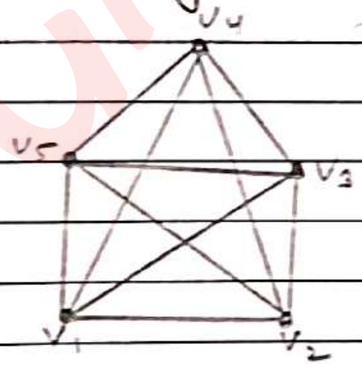
Let the degree of each vertex in a regular graph is equal to r then graph is called r- regular graph.

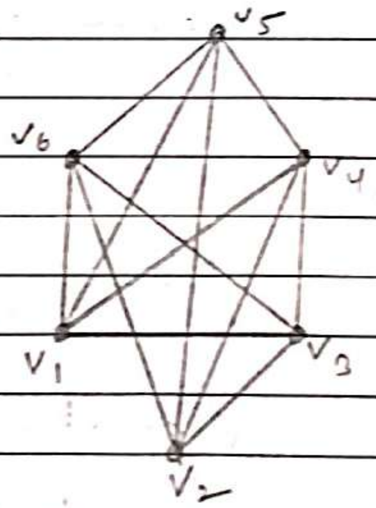
3 - regular graph





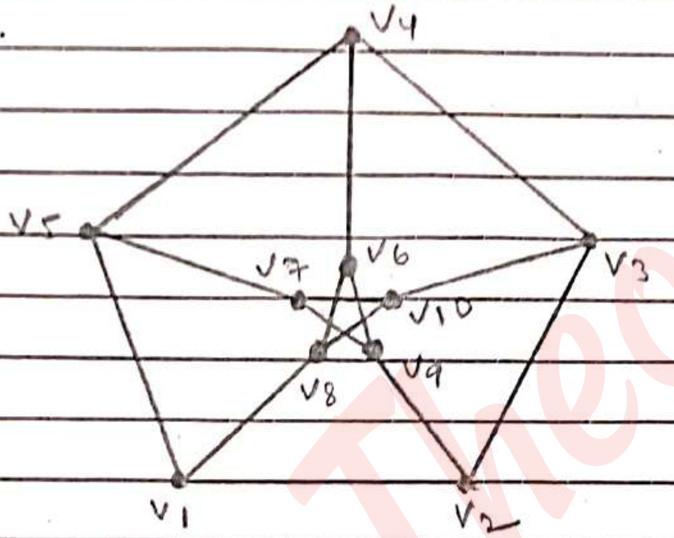
4- regular graph





Peterson Graph:

A 3-regular graph which is as follows is called peterson graph.



subgraph:

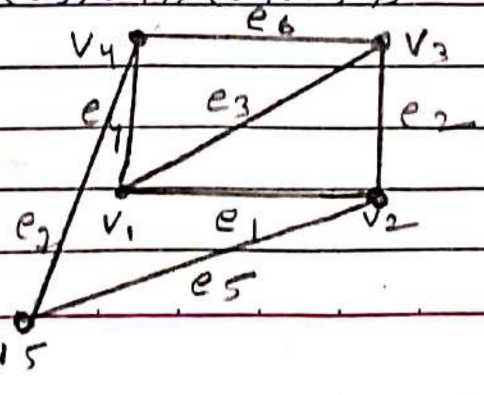
Let $G=(V,E)$ be a graph then $H=(V',E')$ is said to be subgraph if V' is a subset of V and E' is a subset of E.

<u>Graph</u>

$G=(V,E)$

$V=\{v_1,v_2,v_3,v_4,v_5\}$

$E=\{(v_1,v_2),(v_2,v_3),(v_3,v_1),(v_4,v_1),(v_5,v_2),(v_3,v_4),(v_4,v_5)\}$

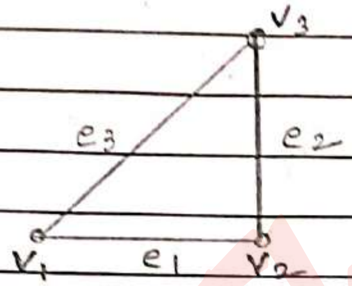


Subgraph

$H = (V', E')$

$V' = \{V_1, V_2, V_3\}$

$E' = \{(V_1, V_2), (V_2, V_3), (V_3, V_1)\}$



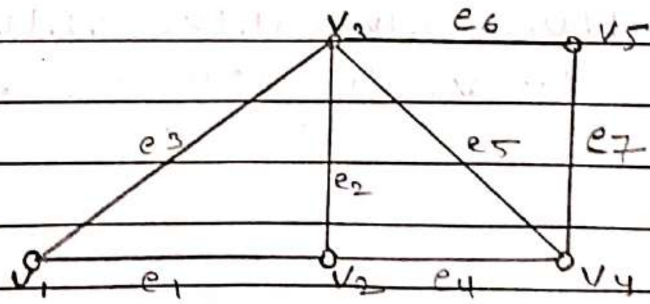
Edge - Disjoint Subgraph:

Two subgraphs H_1 and H_2 of Graph G is said to be edge disjoint subgraph if number edge is common in both subgraph.

$G = (V, E)$

$V = \{V_1, V_2, V_3, V_4, V_5\}$

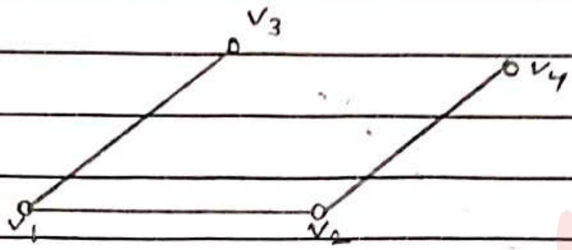
$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_1), (V_2, V_4), (V_4, V_3), (V_3, V_5), (V_5, V_4)\}$



$H_1 = (V_1, E_1)$

$V_1 = \{v_1, v_2, v_3, v_4\}$

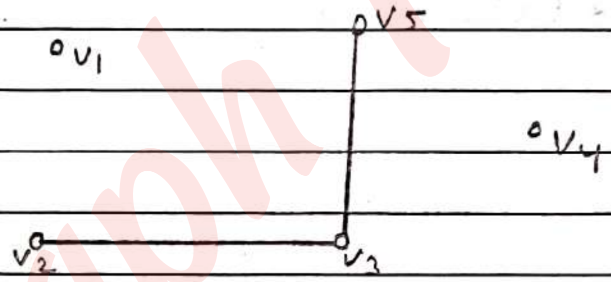
$E_1 = \{(v_1, v_2), (v_3, v_1), (v_2, v_4)\}$



$H_2 = (V_2, E_2)$

$V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$E_2 = \{(v_2, v_3), (v_3, v_5)\}$



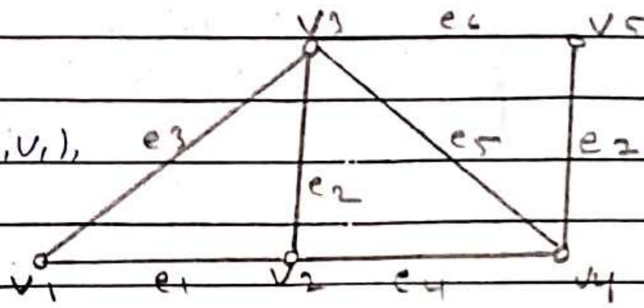
Vertex - Disjoint subgraph:

Two subgraph H_1 and H_2 of graph G is said to be vertex-Disjoint subgraph if number vertex is common in both subgraph.

$G = (V, E)$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

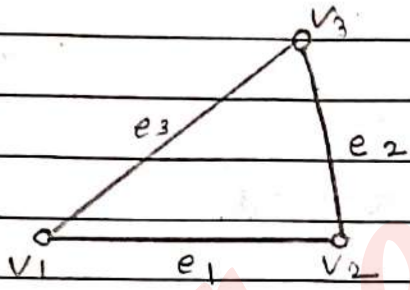
$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_2, v_4), (v_4, v_3), (v_3, v_5), (v_5, v_4)\}$



$H_1 = (V_1, E_1)$

$V_1 = \{v_1, v_2, v_3\}$

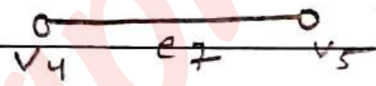
$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$



$H_2 = (V_2, E_2)$

$V_2 = \{v_4, v_5\}$

$E_2 = \{(v_4, v_5)\}$



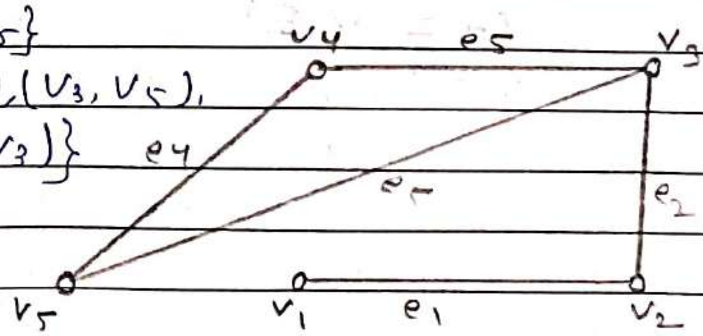
Spanning Subgraph:

A subgraph H of graph G is called spanning subgraph if all the vertices are present in subgraph.

$H = (V_1, E_1)$

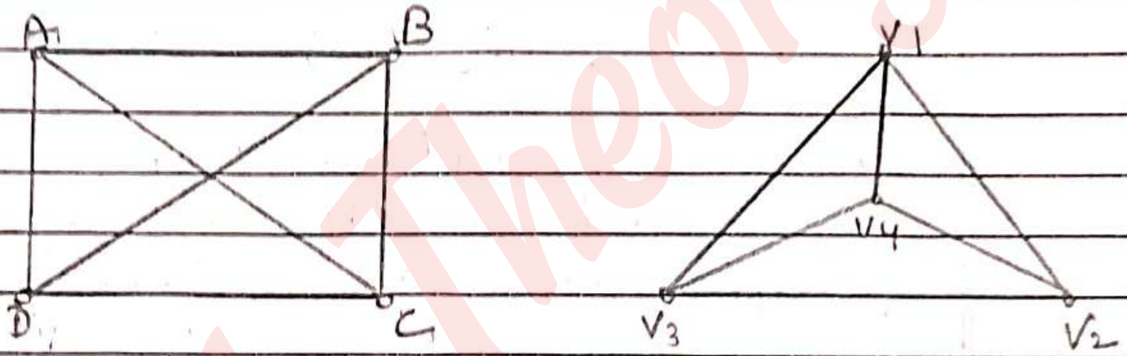
$V_1 = \{v_1, v_2, v_3, v_4, v_5\}$

$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_5), (v_5, v_4), (v_4, v_3)\}$



Isomorphic Graph:

Two graphs G_1 and G_2 are said to be isomorphic if there exist one to one correspondence between their vertices and their edges which are incident on vertices.

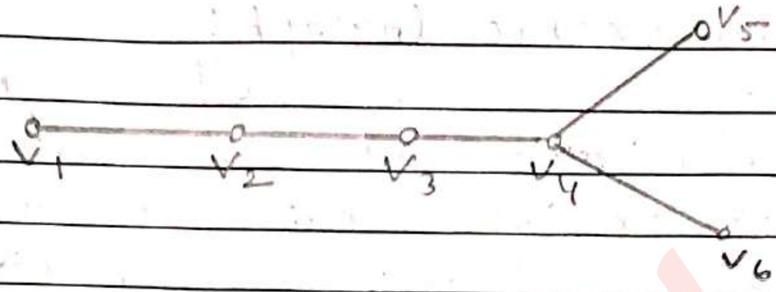


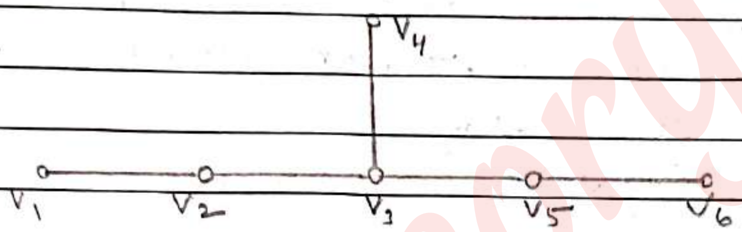
G_1 G_2

Number of vertex = 4	Number of vertex = 4
Number of Edge = 6	Number of Edge = 6

Vertex correspondence	Edge correspondence
$A \longleftrightarrow V_1$	$(A, B) \longleftrightarrow (V_1, V_2)$
$B \longleftrightarrow V_2$	$(B, C) \longleftrightarrow (V_2, V_3)$
$C \longleftrightarrow V_3$	$(C, D) \longleftrightarrow (V_3, V_1)$
$D \longleftrightarrow V_4$	$(D, A) \longleftrightarrow (V_1, V_4)$
	$(A, C) \longleftrightarrow (V_2, V_4)$
	$(B, D) \longleftrightarrow (V_3, V_4)$

Both graph are isomorphic

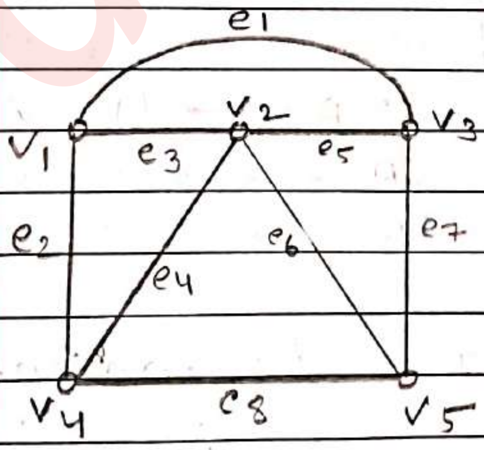




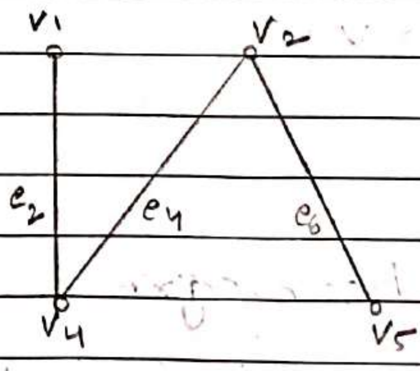
There does not exist one to one correspondence between edges so graphs are not isomorphic.

Complement of a subgraph:

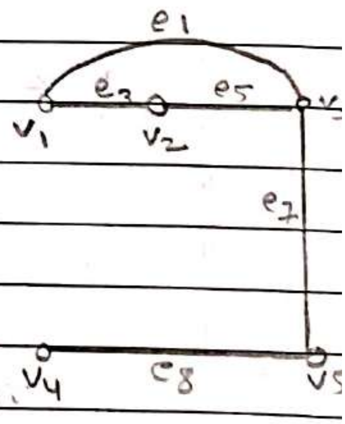
Let $H = (V', E')$ be a subgraph of a graph $G = (V, E)$.
Then complement of a subgraph H is $\bar{H} = (V, E - E')$.



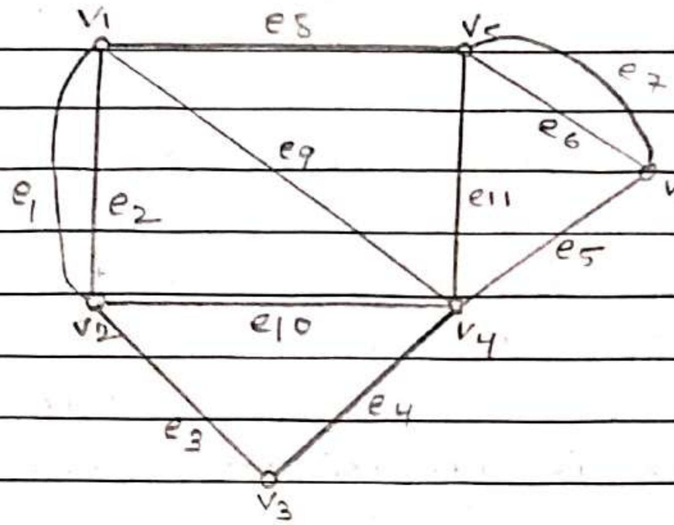
Graph-G



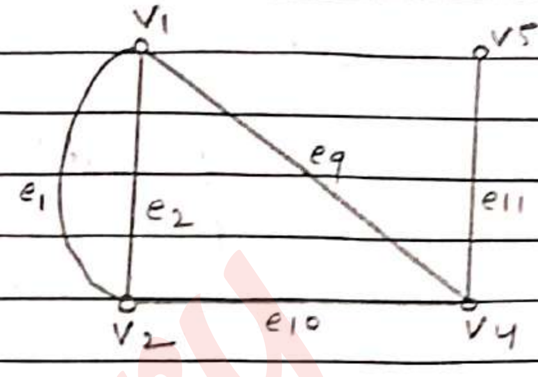
subgraph-H



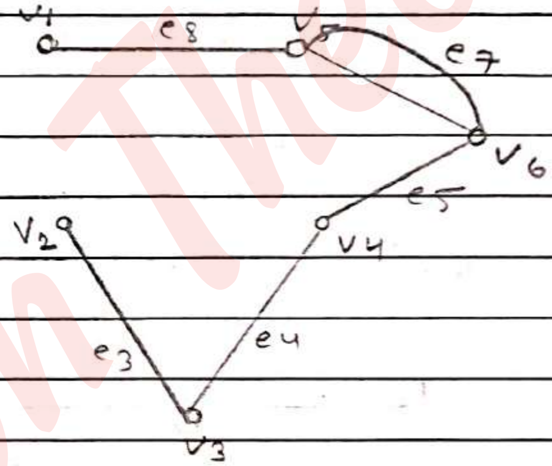
component of subgraph



Graph - G



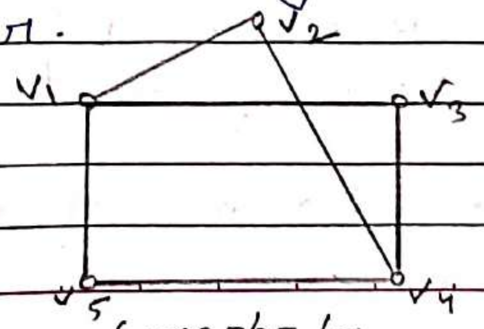
subgraph - H



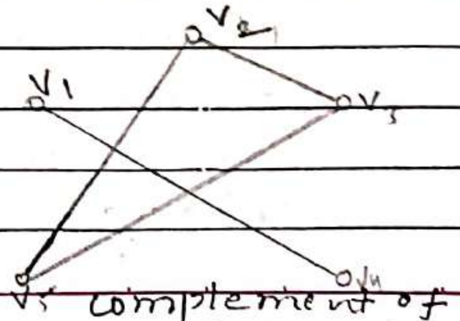
complement of subgraph

complement of simple graph:

Let $G = (V, E)$ be a simple Graph. Then complement of simple graph G is equal to the graph in which number Edge is present which is in G.



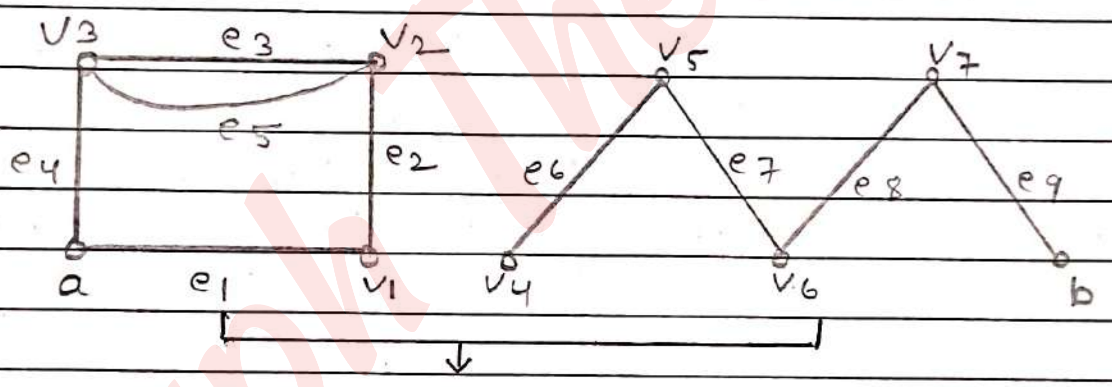
Graph - G



complement of simple graph

Th - 1

A graph G is disconnected if and only if the vertex V of G can be partitioned into two non empty disjoint subsets V_1 and V_2 such that there exist number edge in G whose one end vertex is in V_1 and other in V_2



Disconnected

$V = \{a, v_1, v_2, v_3, v_4, v_5, v_6, v_7, b\}$
$V_1 = \{a, v_1, v_2, v_3\}$
$V_2 = \{v_4, v_5, v_6, v_7, b\}$

Proof

Let G be a disconnected graph
Let a be any vertex in G.
Let V_1 be the set of all those vertices which are join to "a" by any path.
since graph is disconnected.

$\Rightarrow V_1$ does not contains all the vertices of V Let V_2 be the set of all those vertices which are not in V_1

$\Rightarrow V_1$ and V_2 are required partition

Coversely,

Let V_1 and V_2 be two partition of V which are disjoint Let $a \in V_1$ and $b \in V_2$

Since there does not exist any path from a to b

$\Rightarrow G$ is disconnected.

V.V.I.

Note-1

The maximum number of edge in simple Graph with n vertices is equal to $\frac{n(n-1)}{2}$

Th-1

A simple graph with n vertices and k components have at most $\frac{(n-k)(n-k+1)}{2}$ edges

Proof

Let G be a simple graph with n vertices and k components. Let $n_1, n_2, n_3 \dots n_k$ be the number of vertices of k components.

$$\Rightarrow n_1 + n_2 + n_3 + \dots\dots + n_k = n \quad \text{--- (1)}$$

Now,

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$\sum_{i=1}^{k} (n_i - 1) = (n_1 + n_2 + n_3 + \dots + n_k) - (1+1+1 \dots + k)$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k \quad \text{--- (2)}$$

squaring both side of eqn (2)

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$\sum_{i=1}^{k} (n_i - 1)^2 + 2(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} (n_i^2 + 1 - 2i) \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2 \sum_{i=1}^{k} \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + (1 + 1 + 1 + \cdots + k) - 2(n_1 + n_2 + n_3 + \cdots + n_k) \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k(k - 2n) - 1(k - 2n)$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + (k - 2n)(k - 1) \quad \text{——} \quad (3)$$

Now,

Maximum number of edge in simple graph with n vertex $\frac{n(n-1)}{2}$

Now,

Let maximum of edge in i^{th} component is $\frac{n_i(n_i-1)}{2}$

Now,

Again we have total k-component

$$\sum_{i=1}^{k} n_i(n_i-1) = \sum_{i=1}^{k} \frac{(n_i^2 - n_i)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[n^2 + (k-2n)(k-1) - (n_1+n_2+n_3+\dots+n_k)\right]$$

$$= \frac{1}{2}\left[n^2 + (k-2n)(k-1) - n\right]$$

$$= \frac{1}{2}\left[n^2 + k^2 - k - 2nk + 2n - n\right]$$

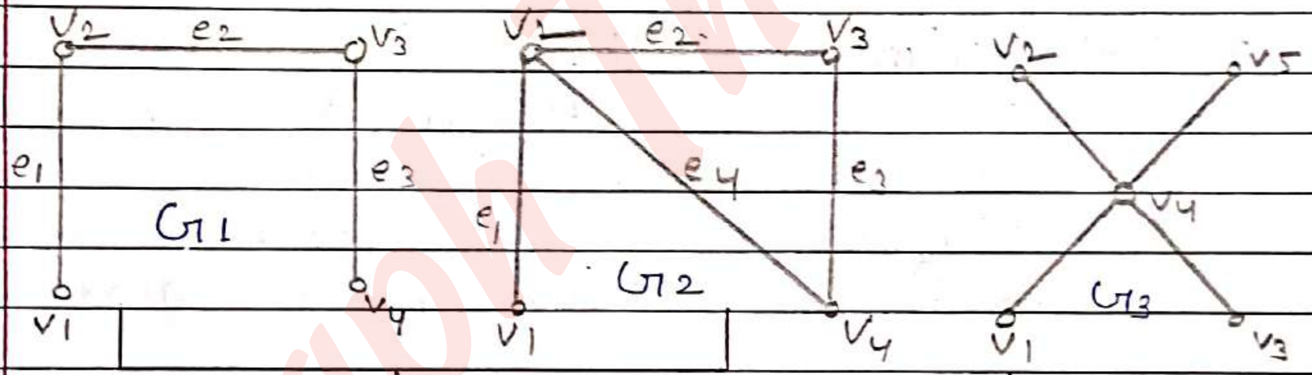
$$= \frac{1}{2}\left[n^2 + k^2 - k - 2nk + n\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right.$$

Proved

Euler Path:

An open walk in a graph which contains all edges of the graph called Euler path.



Euler Path

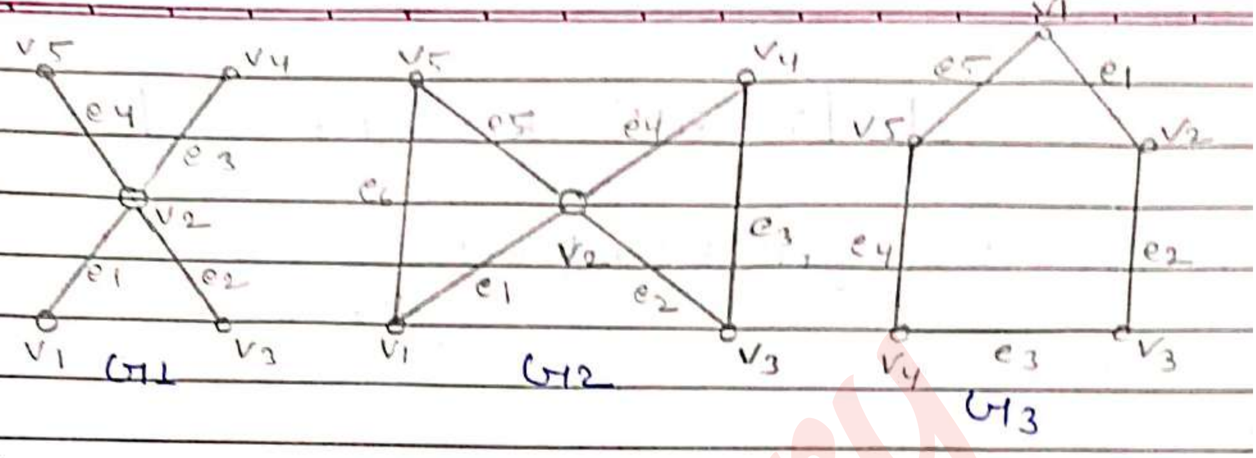
Not contain Euler Path

$G_1 = v_1 e_1 v_2 e_2 v_3 e_3 v_4$

$G_2 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_2$

Euler Graph:

A closed walk in a graph which contains all edges of the graph called Euler graph or Euler circuit or Euler line.



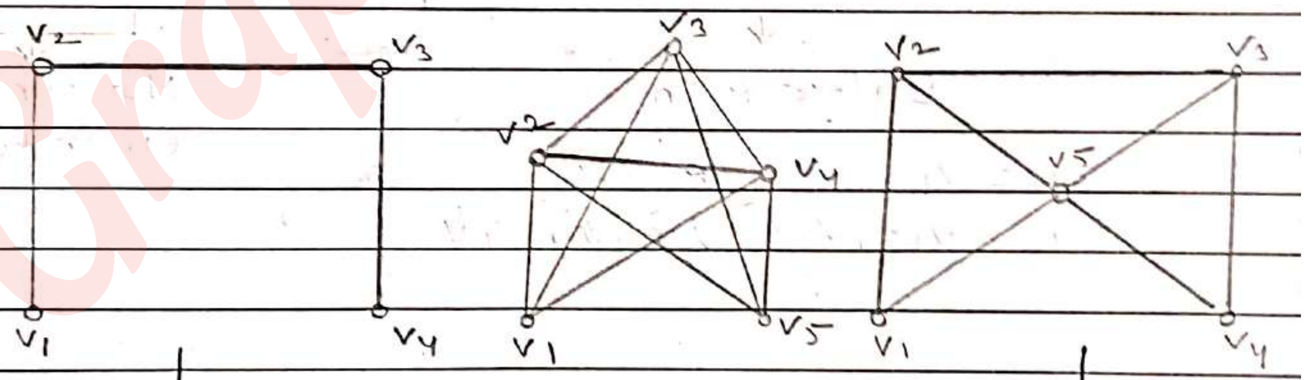
G_1 = Not an Euler graph

$G_2 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_2 e_5 v_5 e_6 v_1$] → Euler graph

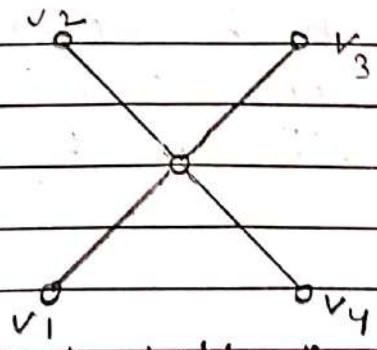
$G_3 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_1$

Hamiltonian Path:

An walk in a graph which covers all the vertices exactly once without repeating end vertex called Hamiltonian path.



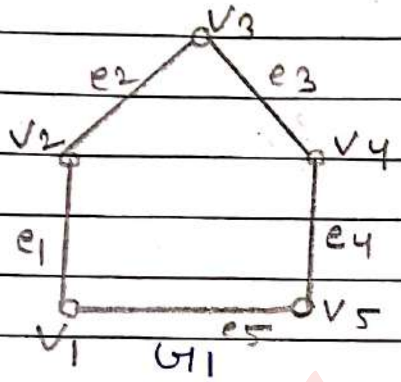
All are Hamiltonian Path



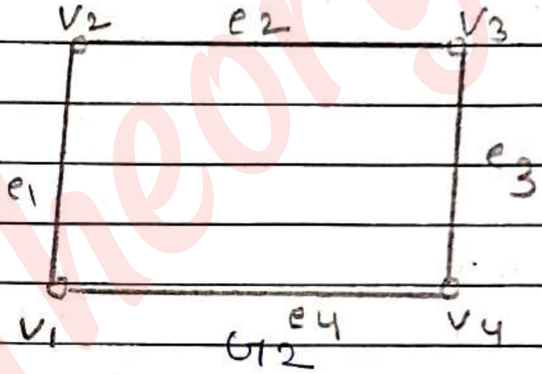
Not Hamiltonian Path

Hamiltonian Graph:

A closed walk in a graph which covers all the vertices exactly once except end points is called Hamiltonian graph or Hamiltonian circuit.



G_1

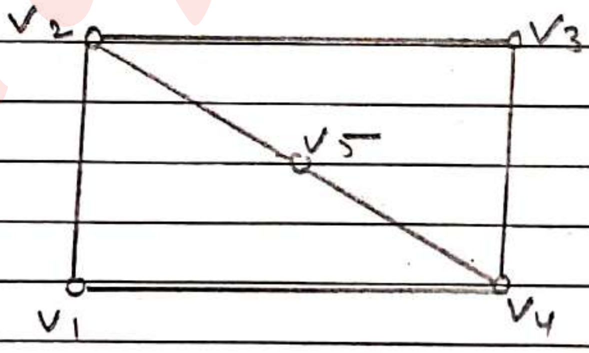


G_2

Hamiltonian Graph

$G_1 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_1$

$G_2 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$



Not a Hamiltonian Graph

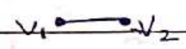
Unit :- 04

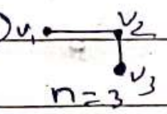
Sr. No.________ Date:________

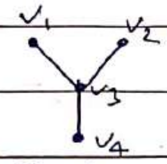
Trees and cut-sets.

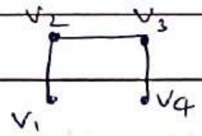
⇒ ~~Tree~~ Tree :- A ~~graph~~ connected graph without any circuit is called tree.

Ex :- (i) v_1•
$n=1$
(Trivial Tree)

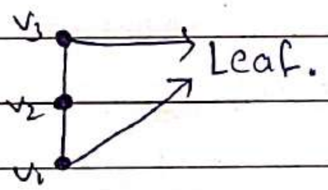
(ii) 

(iii) 

(iv) 

(v) 

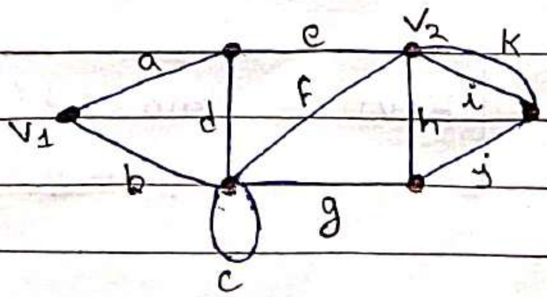
* The edge of tree is called branches.
* The vertex of degree 1 is called leaf



Distance and centre in a ~~tree~~ Graph:

⇒ Let G be a connected graph and v_i, v_j are two vertices in a graph G. then, The distance $d(v_i, v_j)$ between the vertices v_i and v_j is the length of the shortest path between v_i and v_j

$d(v_1, v_2) = 2$



$(b, f) = 2$
$(a, e) = 2$
$(b, c, f) = 3$
$(b, g, h) = 3$

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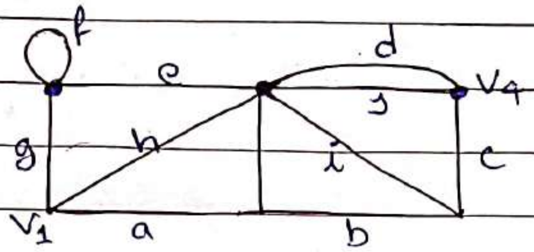
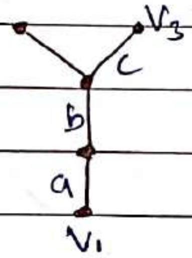
* Distance in connected graph may not be unique.

Ex:-

$d(v_1, v_4) = 2$



* In a tree distance between two vertex is unique.



$d(v_1 v_3) = 3$

Very - very Important.

Ecentricity of a vertex :-

This ~~even~~ Ecentricity $E(v)$ of a vertex v in a graph G is the distance between v and the vertex v_i farthest from v in G.

$$\boxed{E(v) = \max_{v_i \in G} d(v, v_i)}$$

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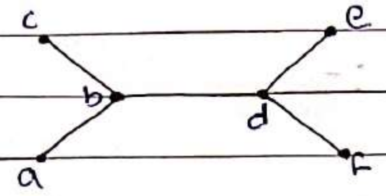
Ex:-



$E(c) = 3$

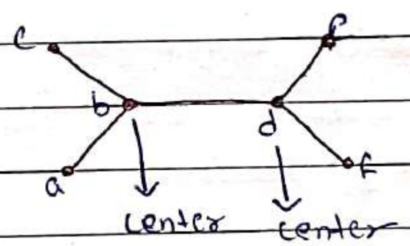
$E(d) = 2$

$E(e) = 3$

$E(f) = 3$

Center of a graph :-

⇒ A center of a graph G is a vertex whose Ecentricity in G is minimum.



$E(a) = 3$

$\boxed{E(b) = 2}$

$E(c) = 3$

$\boxed{E(d) = 2}$

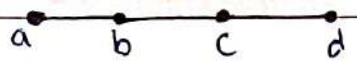
$E(e) = 3$

$E(f) = 3$

Redius of a tree :-

⇒ The ecentricity of a center in a tree is called Redius of a tree.

Ex:-



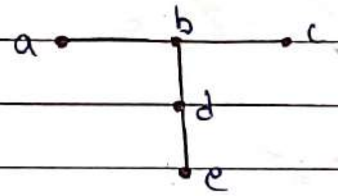
$E(a) = 3$

$E(b) = 2$ $\qquad C(T_1) = b, c$

$E(c) = 2$

$E(d) = 3$ $\qquad \boxed{R(T_1) = 2}$ Ans.

Ex:-



$E(a) = 3$

$E(b) = 2$ $\qquad C(T_2) = b, d$

$E(c) = 3$

$E(d) = 2$ $\qquad \boxed{R(T_2) = 2}$ Ans.

$E(e) = 3$

Diameter of a tree :-

⇒ The diameter of a tree is defined as the length of longest path in a tree.

$E(a) = 3$

$E(b) = 2$

$E(c) = 2$

$E(d) = 3$

$C(T_1) = a, d$

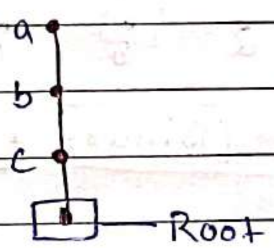
$R(T_1) = 2$

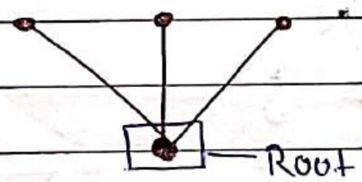
$\therefore \boxed{d(T_1) = 3}$ Ans.

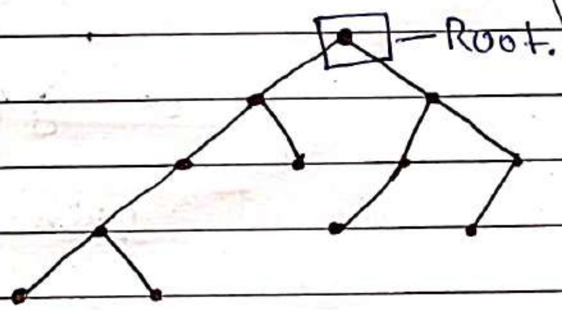
Rooted Tree:-

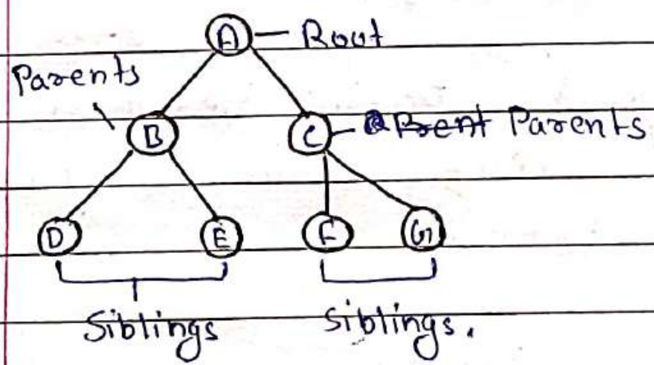
⇒ A ~~tree~~ rooted tree is a tree with a distinguisted vertex called the root(Node).

Ex:-









Leaf Node :- A vertex which do not have any child is called leaf Node.

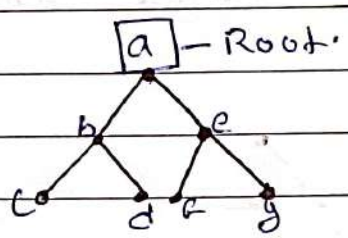
Binary tree:-

⇒ A tree in which tree is exactly one vertex of degree two and each of remaning verties is of degree one or three is called binary tree.

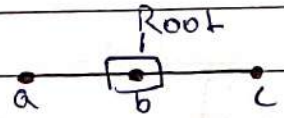
Note:- In binary tree, vertex of degree two is called root.

Ex:-

(i)



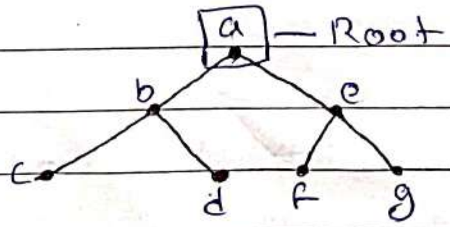
(Binary tree)



Binary tree

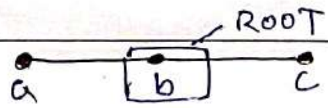
Height of Binary tree:-

→ The length of largest path between root and the leaf node is called hight of Binary tree.



(Binary tree)

$\boxed{H(B.T) = 2}$ Ans



(Binary tree)

$\boxed{H(B.T) = 1}$

Sr. No. ______ (1) Date: ______

Th-1. There is one and only one path between every pair of vertices in a tree T.

Proof. Let T be a tree.

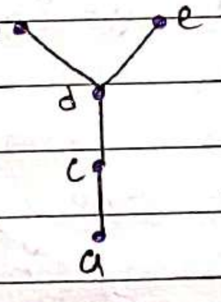
⇒ T is connected. Since T is connected there must exist at least one Path between every Pair of vertices in T Now suppose, That between two vertices a and b of T, there are two distinct Paths, The union of these two Paths will contain a circuit.

⇒ T is not a tree

So we get contradiction.

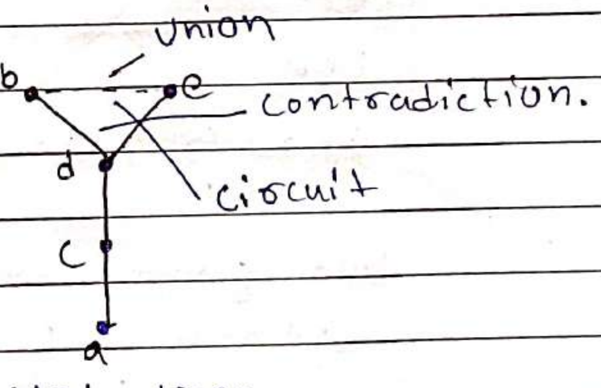
⇒ There is one and only Path between every Pair of verties in a tree T.

ex!- (I)



Tree

(II)



Not tree.

Evergreen

Th-2 If in a graph G, there is one and only one path between every pair of vertices, then G is a tree.

Proof:- Let G be a connected graph. A circuit in a graph G implies that there is at least one pair of vertices a, b such that there are two distinct path between a and b.

Since G has one and only path between every pair of vertices.

⇒ G has no circuit.

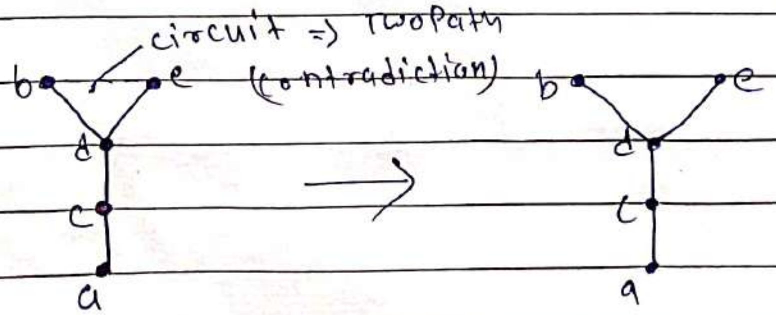
⇒ G is connected without any circuit.

⇒ G is a tree.

$\boxed{\text{Tree} = \text{connected} + \text{No Circuit}}$

Let G be a connected graph and G contain circuit, then there is two path from a to b but given that there is one and only Graph-G path b/w a and b ⇒ contraction.

⇒ G has no circuit ⇒ G is a tree.



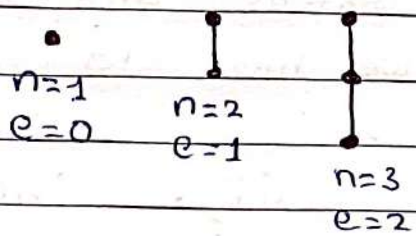
Graph-G.

Connected graph with no circuit. G is tree.

Th-3. A tree of n vertices has n-1 edges.

Proof. We Prove this theorem by Induction theorem is true for $n = 1, 2, 3$.

Let us assume that the theorem is true for k vertices and $k < n$.



Now,

Let us suppose T is tree with n vertices. Let e_k be an edge with end vertices v_i and v_j. There is no other Path between v_i and v_j except e_k. Therefore detection of e_k from T will disconnect the graph. Further more $T - e_k$ consist of exantly two components and since we have no circuit in T to begin with each of these components is a tree. Both these trees t_1 and t_2 have fewer vertices each.

There fore by induction each contain one less edge than the no. of vertices in it.

These $T - e_k$ have $n-2$ edges.

$\Rightarrow$ T has exactly $n-1$ edges.

Ex:-

Let the theorem is tree for k vertex.

$k < n$

< 10

$k - 1$.

$n = k + l$

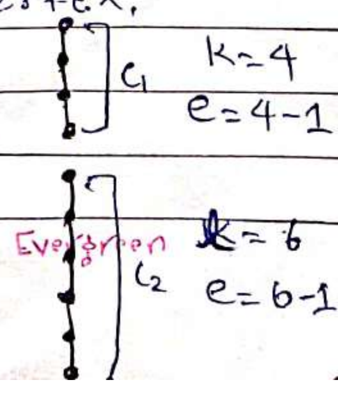
$10 = 4 + 6$

$e = 8$

$= 10 - 2 = n - 2$

$= n - 2 + 1$.

$\boxed{e = n - 1}$



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Th-4 Every connected graph with n vertices and (n-1) edges is a tree.

Proof. Let G be a connected graph with n vertices and (n-1) edges.
we have to show that G has no circuit.
Let us consider G has at least one circuit, since removing an edge from a circuit does not disconnecte a graph so we may remove edges, but no verties from circuits in G unitil the resulting graph G^* is circuit free.
Now G^* is a connected graph with n vertices and contains no ~~vertices~~ circuit. This G^* has (n-1) edge, But now the graph G has more than (n-1) edges.
⇒ contraditition.
⇒ G has no circuit
⇒ G is a tree.

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Th-5. A graph G with n vertices, (n-1) edges and no circuit is tree.

Proof. Let G be a graph with n vertices, (n-1) edges and has no circuit. we have to show that G is connected, If Possible, suppose that G is disconnected Then G will consists of two or more circuit-less components.

Let G consists of two components G_1 and G_2 we add an edge e between a vertex v_1 in G_1 and v_2 in G_2.
since v_1 and v_2 are in different components of G.
⇒ There is no path between v_1 and v_2 in G.
⇒ Addition of edge e will not create a circuit,
This G is a circuitless, connected graph of n vertices and n edges which is not possible because a tree with n vertices has (n-1) edges.

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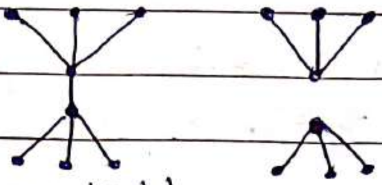
Ex:-



(Connected) (Disconnect)

minimally connected Graph:-

⇒ A connected graph G is said to be minimally connected if removal of any edge from G disconnects the graph.

V.V.I

Th:- 06 Every tree has either one or two centres.

Proof [Note:- In any tree, there are at least two Pendant vertices.]

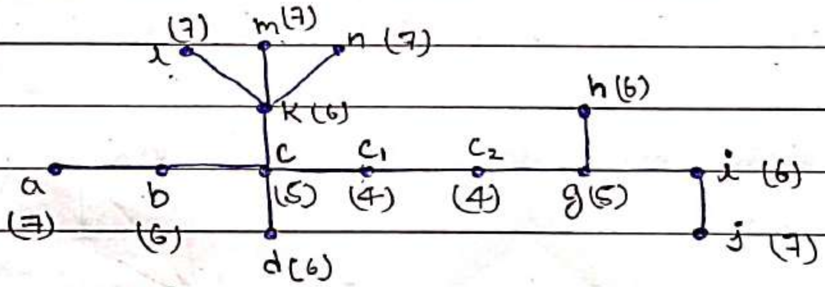


figure - 1 (Graph-T)

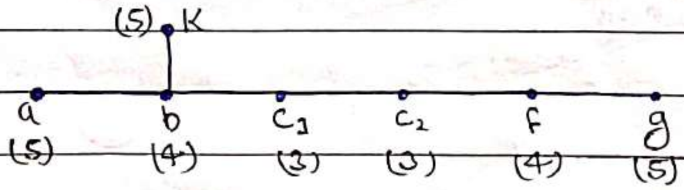


figure -2 (Graph - T')

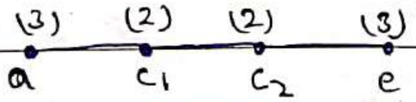


figure -3 (Graph - T'')

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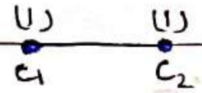

figure - 4 (graph T''')

Let T be a tree having more than two vertices.

⇒ T Must have at least two Pendant vertices (By Note)

Remove all the pendent vertices from T to obtain another graph T'. Then T' is a tree we observe that the ecentricities of the vertices in T' get reduced by one from T.

⇒ All vertices that are centree in T will still remain centres in T' Remove again all Pendent vertices from T' to obtain another tree T'' contices until we are left with either a vertex or an edge whose end vertices are the centre of T.

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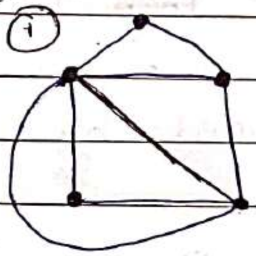
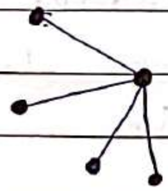
Spanning tree:- / skeleton :-

⇒ A subgraph T of a connected graph G is said to be spanning tree of G if the sub graph T is a Tree. and contains all the vertics of G.

Ex:- ①

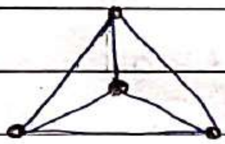


(Connected graph)

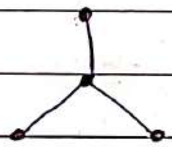


(Spanning tree)

②



(Connected graph)



(Spanning tree)

Branches and Chords :-

⇒ Let G be a connected graph with n vertices and e edges let T be a spanning tree of G.
" The edgs of spanning tree are called branches of T "

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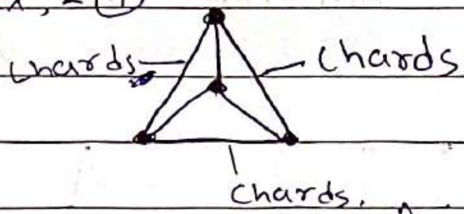
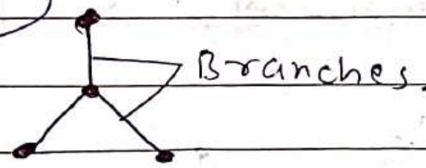
and "The edge of G which are not in the given spanning tree is called chord or link ~~on~~ or tie".

EX'; - (I)



(Connected chords graph)

(II)

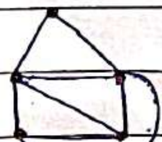


(Spanning Tree)

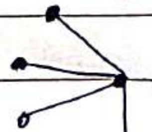
Th:- 1:- Every connected graph has at least one spanning tree.

Proof:- L G be a connected graph. if G has no circuit then it is its own spanning tree. If G has a circuit then delete an edge from the circuit The graph obtained by removing an edge from a circuit in G will remain connected. If there are more circuits, repeat the process until we get a connected, circuit - free graph that contains all the vertices of G. This graph will then be a spanning tree of G.

(1)

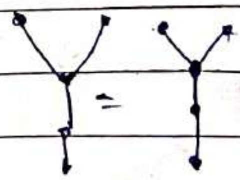


=>



Connected graph

Circuit   non circuit (Tree)



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Let G be Connected Graph

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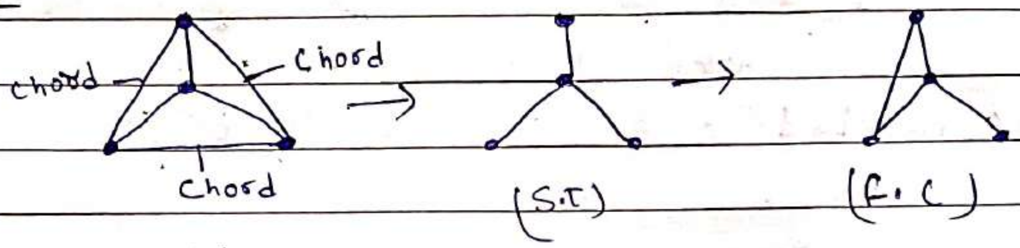
fundamental circuits :-

⇛ A circuit fromed by adding a ~~code~~ chord in to spanning trere of a connected graph G, is called fundamental circuits.

Ex:-



* No. of fundamental circuit in G = e-n+1

Where, e = no. of edge.

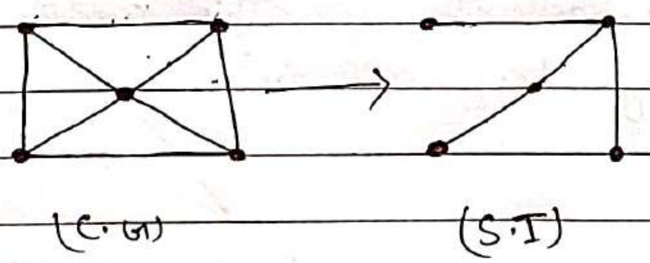
n = no. of vertices.

Solu,

= 6-4+1

= 3 —— fundamental circuits दर्शाता।

Q. Find all fundamental circuits.

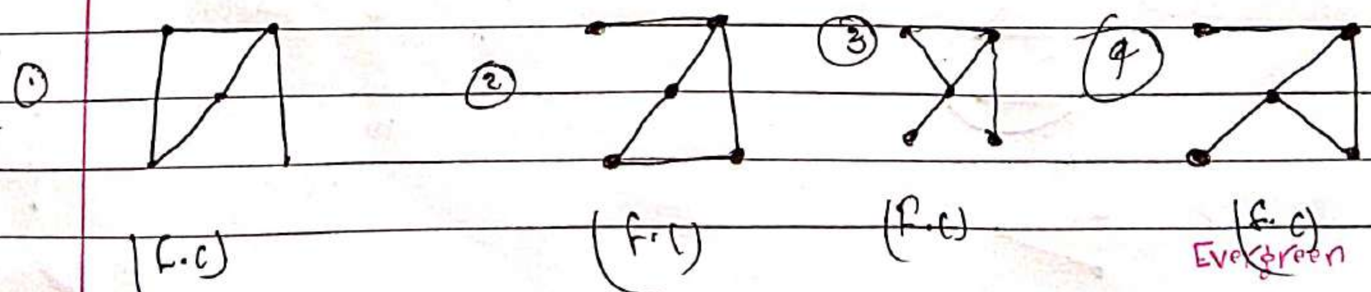


= e-n+1

= 8-5+1

= 3+1

= 4



Rank and nullity:-

⇒ let G be a graph with n vertices, e edge and k - Components, then Rank of graph G.

$\boxed{r = n-k}$

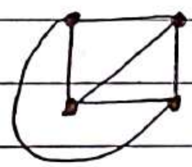
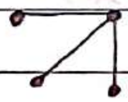
and nullity (cyclomalik no. or betti no) = $\boxed{e-(n-k)}$

$\mu = \boxed{e-r}$

* for connected graph $\boxed{\text{rank } (r) = \text{no. of branches of graph } G.}$

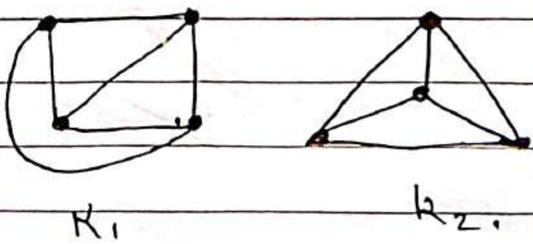
* $\boxed{\text{Nullity } (\mu) = \text{No. of Chords of Graph } G.}$

Ex:-

⇒  ⇒ 

$r = 3$
$\mu = 3$
} Ans

Ex:- disconnected graph.



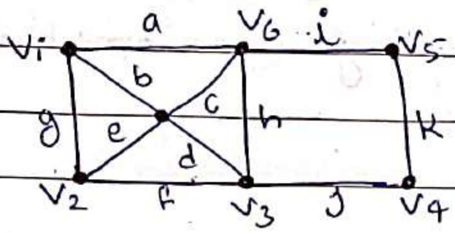
$n = 8$
$e = 12$
$K = 2$

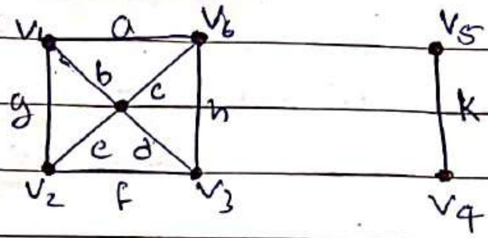
$r = n-k = 8-2 = 6$
and nullity $= e-r = 12-6 = 6$
} Ans.

Cut-set :-

⇒ A cut set S of a connected graph G is a minimal set of edges of G whose removeal from G disconnect the graph G.

(1)

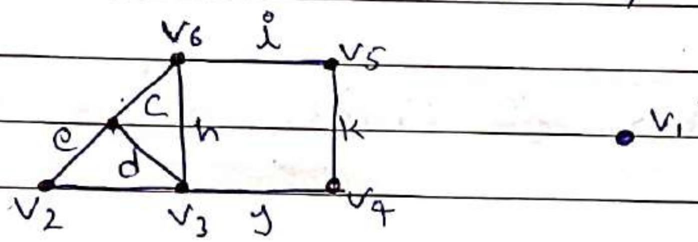




Cut-set

$S_1 = \{i, j\}$

(2)



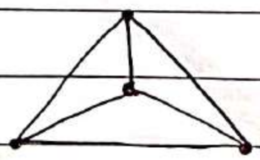
$S_2 = \{a, b, g\}$

cut set.

$S_3 = \{a, c, d, f\}$

cut set.

H.W.



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Properties of Cut-set:-

⇒ ① Every Cut-set in a connected graph G Must contain at least one branch of every spanning tree of G.

② In a connected graph G any minimal set of edges containing at least one ~~brain~~ branch of every spanning tree of G is a cut-set.

③ Every circuit has an even no. of edges in common with any cut set.

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Matrices and Graph :-

① Incidence Matrix:-

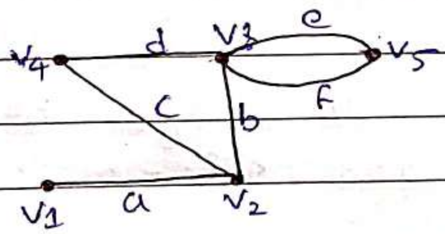
⇒ ∃ let G be a graph with n vertices v_1, v_2, v_3 --- v_n and m edges e_1, e_2, e_3 --- e_n and no Self-loop then incidence matrix defined as follows.

⇒ $a_{ij} = 1$. if j^{th} edge in incidence on i^{th} vertex v_i and $a_{ij} = 0$

* Incidance matrix is denoted by A (G).
* Incidance Matrix is also Known as "bit matrix"
* n rows represent n verties m columns represent m edges.

Ex:- write Incidence matrix.

①



	a	b	c	d	e	f
v_1	1	0	0	0	0	0
v_2	1	1	1	0	0	0
v_3	0	1	0	1	1	1
v_4	0	0	1	1	0	0
v_5	0	0	0	0	1	1

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Properties of Incidence.

1.) Each column of I.M of graph G without any self loops has exactly two.

2) The sum of each row gives the degree of the corresponding vertex.

3) A row with all zero represent an isolated vertex.

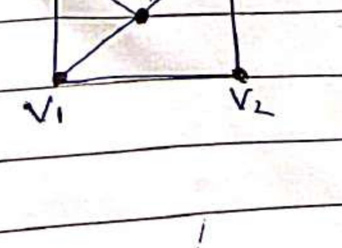
② Adjacency matrix :-

⇒ Adjancency Matrix. of a graph with n vertices and no Parallel edges (self-loops are allowed) is $n \times n$ matrix which is defined as.

⇒ $x_{ij} = 1$ If there is an edge between i^{th} and j^{th} vertices and $x_{ij} = 0$ otherwise.

* A.M is denoted by X.
(Adjacency Matrix)

Ex:- ① Write A.M of this graph.



$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 1 & 1 & 0 \end{array}$$

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Properties of Adjacency Matrix.

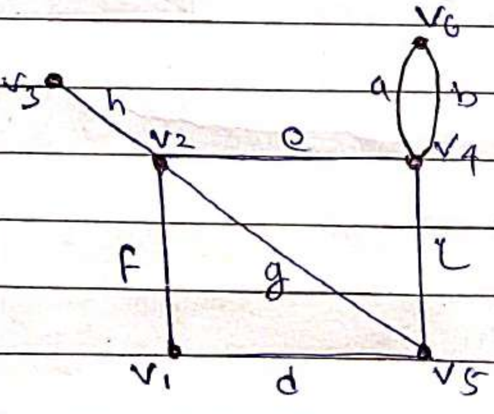
1. A.M of graph G is symmetric
2. If graph has no self-loop then the diagonal entrices of A.M are zero and vice versa.
3. If the graph has no self-loops then sum of i^{th} row (or column) is the degree of v_i.

3) Path - Matrix.

⇒ A Path Matrix is defined for a given pair of vertices in a graph, say (u,v) and is written as $P(u,v)$ and defined as.

$P_{ij} = 1$ if j^{th} edge lies in i^{th} path and $P_{ij} = 0$.

Ex:- find Path matrix between v_3 and v_4



$P_1 = \{H, e\}$

$P_2 = \{H, G, C\}$

$P_3 = \{H, F, D, C\}$

~~$v_3 h v_2 e v_4 = \{h, e\}$~~

$$P(v_3 v_4) = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

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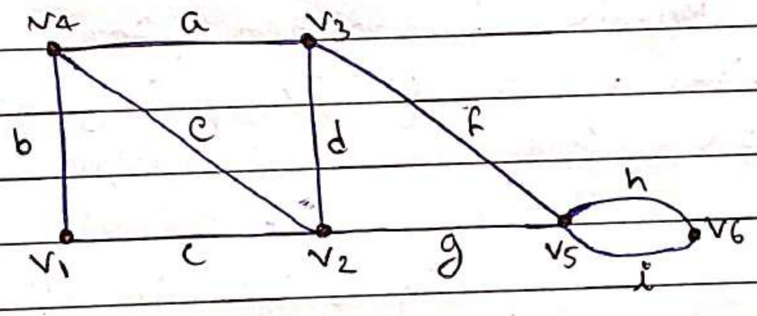
④ # Circuit Matrix :-

⇒ Let G be a graph with e edge end q different circuits then circuit Matrix B is defiened as.

b_{ij} = 1 if i^{th} circuit include j^{th} edgs.
$b_{ij} = 0$

ex:- find circuit matrix.



c_1 = {b, c, e}
c_2 = {a, d, e}
c_3 = {d, g, f}
c_4 = {h, i}
c_5 = {a, b, c, d}
c_6 = {a, b, c, g, f}
c_7 = {a, e, g, f}

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$$C(G) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

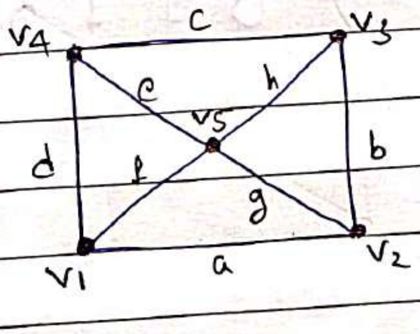
(5) Cut Set Matrix :-

⇒ A Matrix c is set to be cut-set Matrix if.

$c_{ij} = 1$ it ith cut-set jth edges.

$c_{ij} = 0$ other wise.

* Row corrospond to cut-set and columans use correspond to edge.



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$S_1 = \{d, e, c\}$

$S_2 = \{e, h, g, f\}$

$S_3 = \{a, f, d\}$

$S_4 = \{a, g, b\}$

$S_5 = \{b, h, c\}$

$S_6 = \{a, f, e, c\}$

$S_7 = \{a, g, h, c\}$

$S_8 = \{b, h, e, d\}$

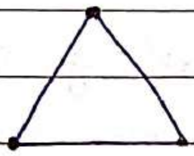
$S_9 = \{b, g, f, d\}$

$$
\begin{array}{c|cccccccc|}
 & a & b & c & d & e & f & g & h \\
\hline
S_1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
S_2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
S_3 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
S_4 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
S_5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
S_6 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
S_7 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
S_8 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
S_9 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
\hline
\end{array}
$$

Planar graph :-

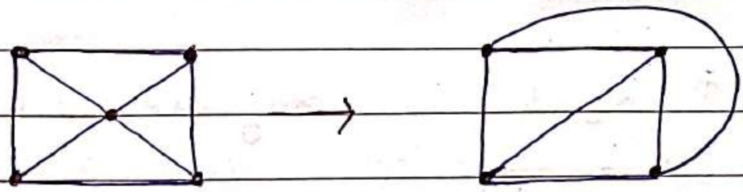
⇒ A graph G is said to be planar if The graph can be drown in the plane so that no edges crosses to each other. ~~at~~ axcept at the vertices.

Ex:- ①



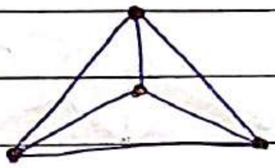
Planar graph

②



graph-G. → Planar graph.

③

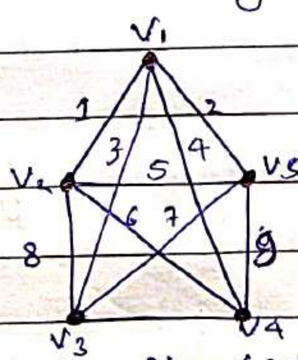


Planar graph.

non-planar Graph :-

⇒ A graph G that can not be drawn on a Plane without Crossing of its edges is called non-Planar graph.

ex:- ①



Graph-(G)

(non-Planar Graph)

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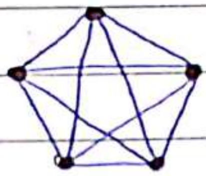
Kuratowski's two graphs
(कुरातोस्की)

⇒ A complete graph with 5 vertices. is known as first Kuratowski's graph and the second graph of Kuratowski's is $K_{3,3}$

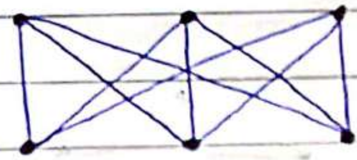
* $K_{3,3}$ is regular connected graph with 6 vertices and 9 edges.

Ex:- ①



first graph.

②



$K_{3,3}$ (second graph)

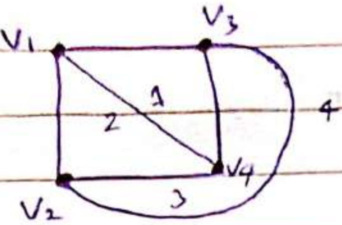
* Both graph are non-Planar. $K_{3,3}$ graph is known as utility graph.

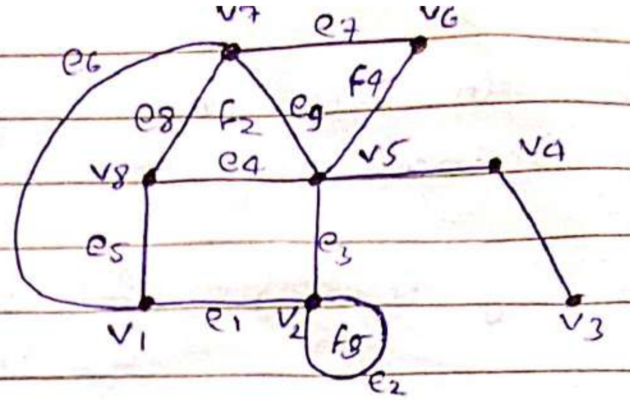
Region:-

A Region of a Planar graph is an area of the Plane that is bainded by edges.

A region is finite if the area it enclose is finite, otherwise it is called infinite.

Evergreen





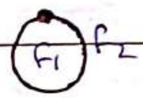
V.Imp (15)

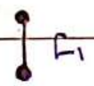
Euler's formula :-

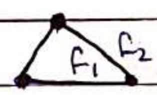
Th-1 :- A connected planar graph with n vertices and e edges has e-n+2 regions.

Proof :- Let G be a connected ~~Graph~~ planar graph with n vertices and e edge. This theorem is proof by Induction method.

for n = 1 | n = 2 | n = 3

(1) 

(2) 

(3) 

$f = e-n+2$
$= \cancel{1} - \cancel{1} + 2$
$= 2$ Ans

$f = e-n+2$
$= 1 - \cancel{2} + \cancel{2}$
$= 1$ Ans

$f = e-n+2$
$= \cancel{3} - \cancel{3} + 2$
$= 2$ Ans

This imply the theorm is true for $n=1, e=1, n=2, e=1, n=3, e=3$. Now let us consider the theorm is true for $e-1$ edges.

Now Let G be a connected graph with n vertices, e edges and f- regions.

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Case -I When G has no circuit $\Rightarrow$ G is a tree.
$\Rightarrow$ G has only one region.

now by formula $f = e - n + 2$

$f = (\cancel{n}-1) - \cancel{n} + 2$ $\{ e = n-1$ G is a tree,

$= -1 + 2$

$\boxed{f = 1}$

Case:-II:- when G has a circuit. Then let circuit has an edge 'a'

now removed of an edge 'a' will merge 2 regions into 1 region,

now in this case no. of edges $= e - 1$

and no of regions $= f - 1$

By using formula $\Rightarrow f = e - n + 2$

$f - 1 = e - 1 - n + 2$

$f = e - \cancel{1} - n + 2 + \cancel{1}$

$\boxed{f = e - n + 2}$

proved

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V. Imp.

NOTE:- if a Connected simple Planar graph G has $n \geq 3$ vertices e-edes and f - regions then.

$$\boxed{3f \leq 2e} \text{ — (i)}$$

$$3(e-n+2) \leq 2e$$

$$3e-3n+6 \leq 2e$$

$$\boxed{e \leq 3n-6} \text{ — (ii)}$$

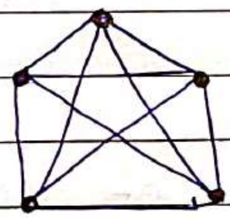
Imp.

(1) ~~K_5 is non-Planar~~

Q. Proved that K_5 is non-Planar.

Soln

$n=5$, $e=10$



Apply

$3f < 2e$

$3(e-n+2) \leq 2e$

$3(10-5+2) \leq 2\times10$

$21 \leq 20$ (condition is failing)

Hence:- K_5 is non-Planar.

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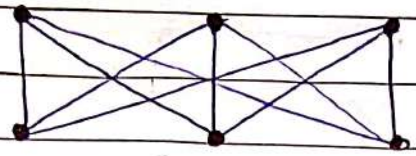
Q. Prove that $K_{3,3}$ is non Planar.

Sol



$K_{3,3}$

$4f \le 2e$ $n=6$
$e=9$.

$4(e-n+2) \le 2e$

$4(9-6+2) \le 2\times 9$

$20 \le 18$ (condition failing)

$\Rightarrow K_{3,3}$ is non Planar.

Detection of Planarity :-

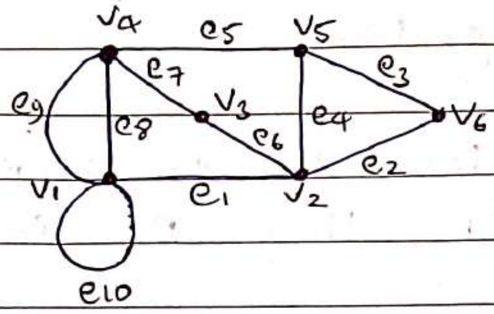
Rule :- ① Remove all self-loop

② Remove Parallel edges.

③ Removal of a vertex of degree 2 by merging 2 edge in series.

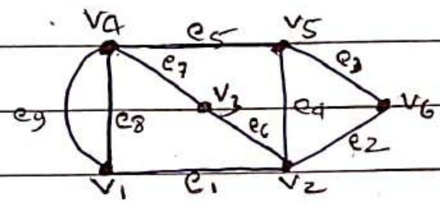
④ Repeat step 2nd and 3rd again & again untill graph become simply.

Q1. Reduce or (simplify) the graph without affacting its Planarity.

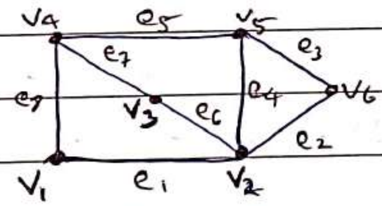


Solv.

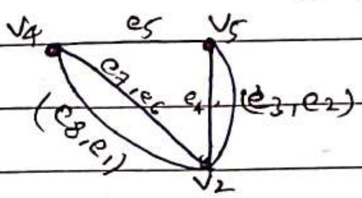
Step :- 1 Remove Self - loop.



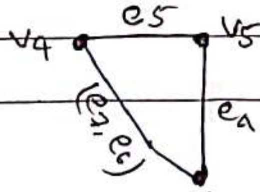
Step :- 2 Remove Parallel edge.



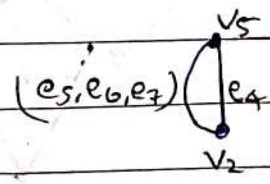
Step :- 3 Merging of an edge')



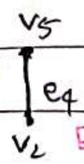
Step :- 4 repeat step - 2



Step :- 5 repeat. Step - 3



Step :- 6 :- Repeat step - 2



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Dual Graph :-

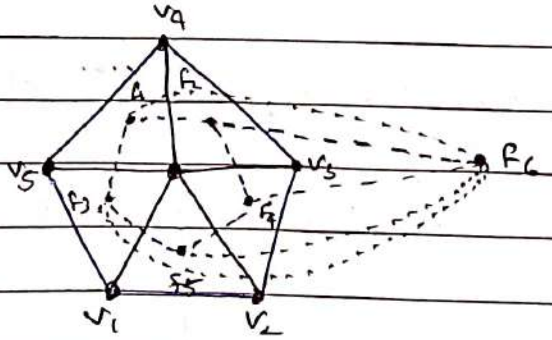
=) Given a Plane reprusentatin of Plannar graph we constrnct another graph G^* is called dual graph.

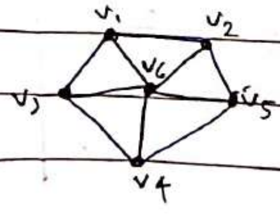
Rules:- ① Inside each region f_i of grap G, we take a Point p_i these p_i is vertices of G^*

② if two regions f_i and f_j are adjacent we draw a line to join the point p_i and p_j of f_i and f_j.

③ For an edge e lying entirely in one region f_k, draw a self loop at Point p_k of f_k.

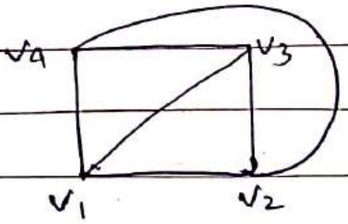
Q. Find the dual of the graph G.

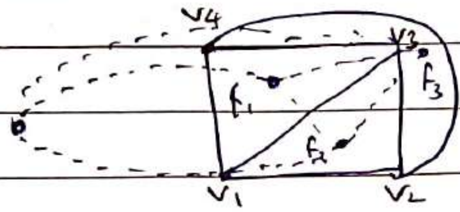


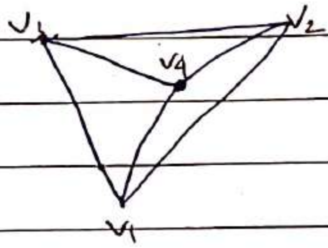


Q. find the dual of graph.

Sol

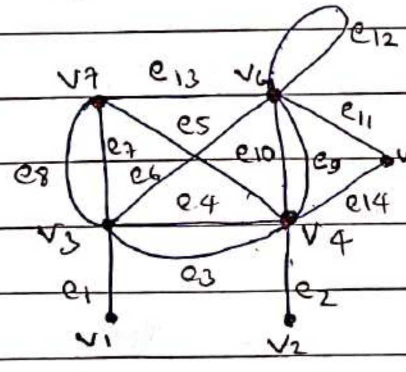






Extra

Th-6. Every tree has either one or two centre.



1. Degeree each vertex.
2. even vertex.
3. odd vertex.
4. Pendent vertex.

①	② even vertex.	odd vertex:-	③ Pendent vertex
d(v1) = 1			
d(v2) = 1	d(v3) = 6	d(v1) = 1	
d(v3) = 6	d(v5) = 2	d(v2) = 1	d(v1), d(v2)
d(v4) = 7	d(v7) = 4	d(v4) = 7	
d(v5) = 2		d(v6) = 7	
d(v6) = 7			
d(v7) = 4			

Unit — 05

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Digraph

Directed graph or Digraph:-

=) A directed graph G consists of two sets.

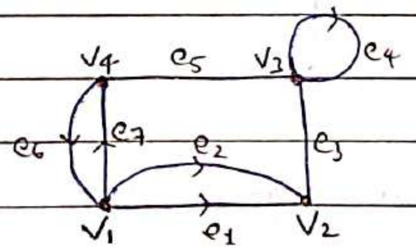
1.) A finite set V of vertex.
2.) A finite set E of edges, contains ordered pair of vertices.

(v_1, v_2) $v_1 \longrightarrow v_2$

$(v_1, v_2) \neq v_2 \longrightarrow v_1$

* if an edge $e_k = (v_i, v_j)$, then this edges is represented ~~betⁿ vi and vj~~ in the digraph by a line segement - betⁿ vi and vj with arrow directed from vi to vj

$v_i \xrightarrow[e_k]{} v_j$



Note:- Let $e_k = (v_i, v_j)$ is an edge $v_i \xrightarrow[e_k]{} v_j$ then vi is called initial vertex and vi is called forminal vertex.

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Parallel edge (Digraph) :-

⇒ Two edge are said to be Parallel edge in digraph if their initial and terminal vertex are same.

ex:- In digraph G e_1 and e_2 are parallel edges but e_c and G are not Parallel edges.

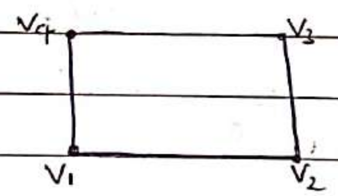
Self Loop (Digraph) :-

⇒ A edges for which the inital and the terminal vertex are same, is called self loop.

ex:- In digraph G edge e_4 from a self-loop.

NOTE:- Let $e_k = (v_i, v_j)$ be can edge in digraph. Then the edges e_k is said to be incident out of its intitial vertex v_i and incident into its terminal vertex v_j.

ex:- In digraph G edge e_5 is incident out of v_4 ~~on~~ and incident into v_3.



In degree and out-degree of a vertex.

⇒ The number of edges incident into a vertex v_i is called the in degree of vertex v_i and is denoted ~~dy~~ by $d^-(v_i)$ And the number of edges incident out of a vertex v_i is called out-degree of a vertex v_i and is denoted by $d^+(v_i)$.

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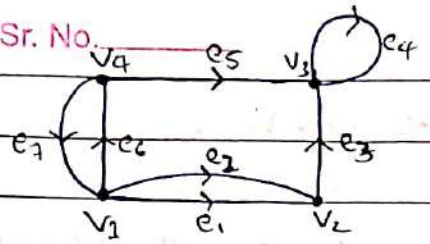
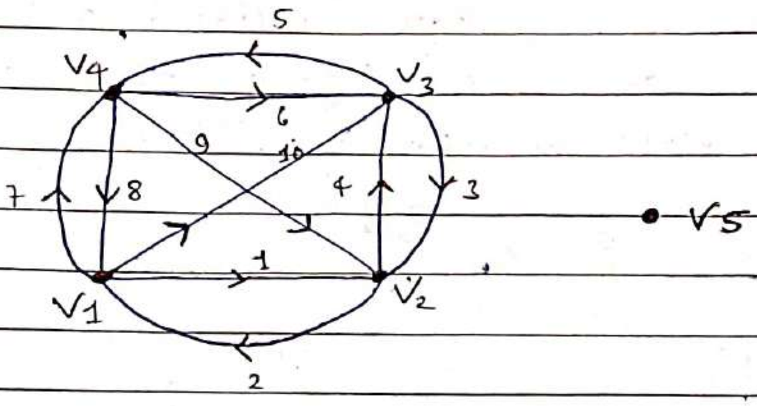
①



$d^-(v_1) = 1$	$d^-(v_3) = 3$
$d^+(v_1) = 3$	$d^+(v_3) = 1$
$d^-(v_2) = 2$	$d^-(v_4) = 1$
$d^+(v_2) = 1$	$d^+(v_4) = 2$

②



$d^-(v_1) = 2$

$d^+(v_1) = 3$

$d^-(v_2) = 3$

$d^+(v_2) = 2$

$d^-(v_3) = 3$

$d^+(v_3) = 2$

$d^-(v_4) = 2$

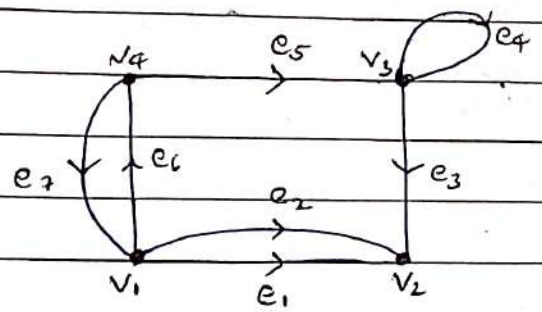
$d^+(v_4) = 3$

$d^-(v_5) = 0$

In degree and out-degree of vertex.

⇒ The no. of edge incident into a vertex v_i is called the in degree of vertex v_i and is denoted by $d^-(v_i)$.
And the no. of edges incident out of a vertex v_i is called out degree of a vertex v_i and is ~~verte~~ denoted by $d^+(v_i)$.

Ex:-

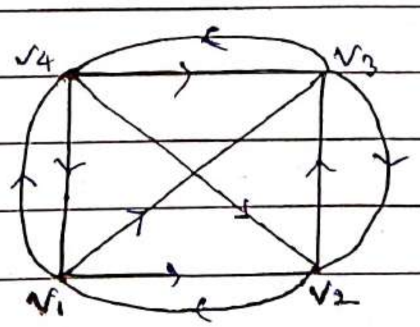


$d^-(v_1) = 1$ $\quad$ $d^-(v_3) = 2$

$d^+(v_1) = 3$ $\quad$ $d^+(v_3) = 2$

$d^-(v_2) = 3$ $\quad$ $d^-(v_4) = 1$

$d^+(v_2) = 0$ $\quad$ $d^+(v_4) = 2$



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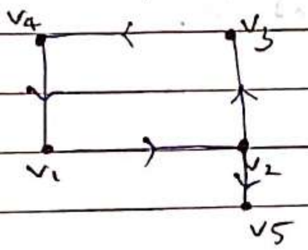
Isolated vertex:-

⇒ A vertex is said to be isolated vertex if $d^-(v_i) = d^+(v_i) = 0$

Pendent vertex:-

⇒ A vertex v is said to be pendent vertex if $d^-(v) + d^+(v) = 1$.

Ex:-



$d^-(v_5) = 1$

$d^+(v_5) = 0$

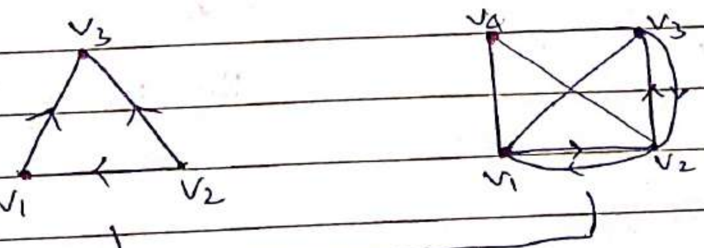
$d^-(v_5) + d^+(v_5) = 1 + 0 = 1$

⇒ v_5 is a pendent vertex.

Some type of Digraph:-

1. Simple digraph:- A digraph that has no self-loop and no parallel edges is called simple digraph.

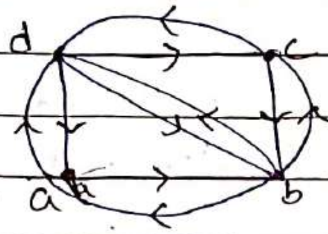
Ex:-



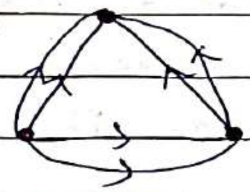
Simple digraph.

(2) Symmetric Digraph :-

⇒ A digraph in which for every edge from a to be there is an edge from b to a is called Symmetric digraph.



symmetric.

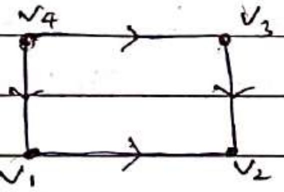


non-symmetric

Asymmetric Graph :-

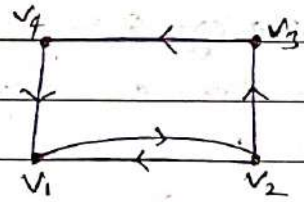
⇒ A digraph which has at most one edge between a pair of vertices is called asymmetric graph.

Ex :- (1)



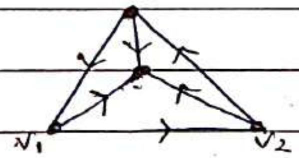
Asymmetric graph

(2)



Not Asymmetric.

(3)



Asymmetric graph.

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Complete Digraph :-

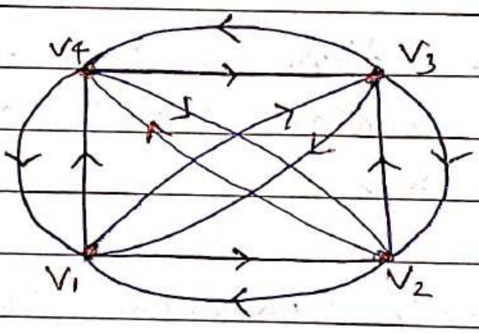
⇒ There are two types of complete graph.

1) Complete symmetric graph.
2) Complete Asymmetric graph.

(1) Complete symmetric graph.

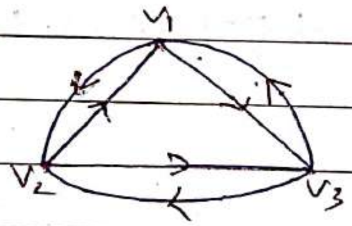
⇒ A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.

Ex:- (1)



complete symmetric digraph

(2)



(2) Complete Asymmetric digraph.

⇒ A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between $\boxed{\text{every}}$ pair of vertices.

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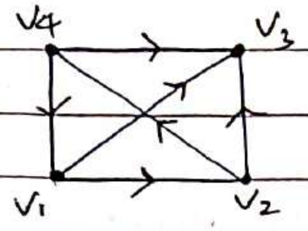
Ex:-



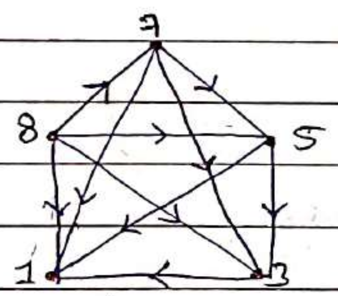
Complete Asymmetric digraph.

Digraphs and Binary Relation.

⇒ A binary relation on a set $X = \{x_1, x_2, x_3 --- x_n\}$ is a collection of ordered pair of elements of X. Ordered pair (x_i, x_j) means x_i is related to x_j.

Q. Draw a digraph that represent the relation "is greater then" on a set = {1, 3, 5, 7, 8}.

Solⁿ: Binary relation :- {(8,7) (8,5) (8,3) (8,1) (7,5) (7,3) (7,1) (5,3) (5,1) (3,1)}



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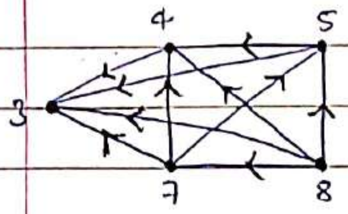
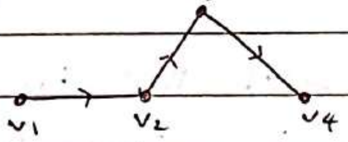
Q. Draw a digraph that ~~that~~ represent the relation "is greater than" on a set $X = \{3, 4, 7, 5, 8\}$

Sol:- Binary relation :- {(8,7) (8,5) (8,4) (8,3), (7,5) (7,4) (7,3) ~~(4,3)~~ (5,4) (5,3) (4,3) }

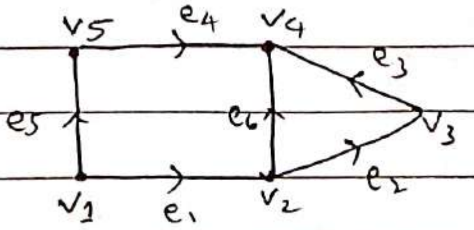


Directed Walk :-

⇒ A directed walk from a vertex v_i to v_j is an alternating sequence of vertex and edges, begining with v_i and ending with v_j such that each edge is directed from the vertex Preceding it to that vertex following it.



Note:- Repetation of edge is not allowed.



$v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4$ is a directed walk

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$v_1\, e_5\, v_5\, e_4\, v_4$ is a directed walk.

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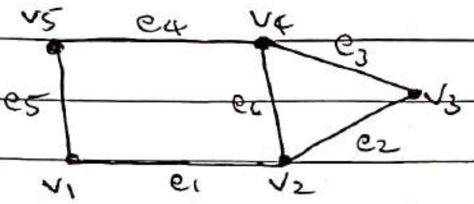
Semi walk :-

⇒ A semi-walk in a direted graph is a walk in the corresponding undirected graph but not in directed graph.

Ex:- $v_1 e_1 v_2 e_6 v_4 e_3 v_3$ is a semi walk.



Directed Path :-

⇒ An open directed walk in which no edge appears more than ~~ones~~ once is called directed path.

Ex:- ~~$v_1 v_5$~~ $v_1 e_5 v_5$ is a directed path.
$v_1 e_1 v_2 e_6 v_4$ is a directed Path.

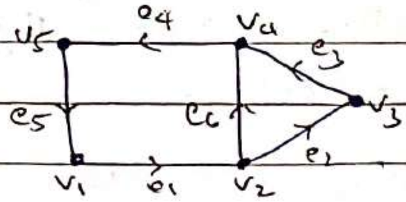
Semi-directed Path.

⇒ A semi-directed Path in a directed graph is a Path in corresponding undirected graph but not in directed Graph.

Ex:- $v_1 e_1 v_2 e_6 v_4 e_4 v_5$ is a semi-directed Path.

Directed circuit :-

⇒ A closed directed walk in which no vertex apears more than once ~~at~~ except and vertices is called directed circuit.



Ex:- $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_1$ is directed circuit.

Semi-directed circuit :-

⇒ A semi-directed circuit in a directed graph is a circuit in corresponding undirected graph but not in directed graph.

Ex:- $v_2 e_6 v_4 e_3 v_3 e_2 v_2$ is a semi-directed circuit.

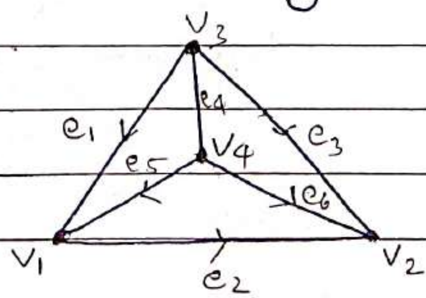
Imp.

Connected Directed Graphs.

⇒ Connected graph is of two types.

1. Strongly connected :-

⇒ A digraph G is said to be strongly connected if there exist at least one directed path from every vertex to every other vertex.



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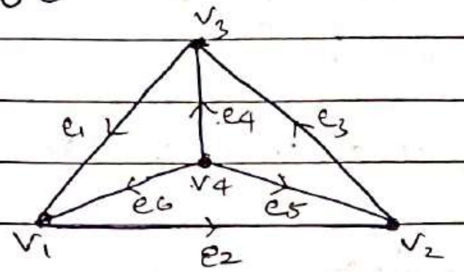
2) weakly connected:

⇒ A digraph G is said to be strongly connected if the corresponding undirected graph is connected but digraph is not strongly connected.



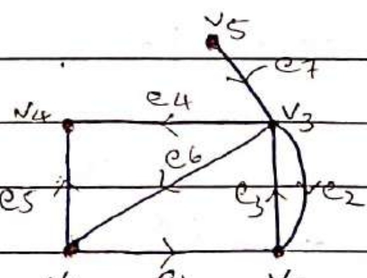
Matrices In digraph.

1) ⇒ Incidence matrix:-

⇒ Let G be a digraph with n vertices and e ~~edges~~ edges and having no self-loop, then incidence matrix is defined as $a_{ij} = 1$ if the j^{th} edge is incident out of i^{th} vertex.

$a_{ij} = -1$ if the j^{th} edge is incident into the i^{th} vertex.

$a_{ij} = 0$ if j^{th} edge is not incident on i^{th} vertex.



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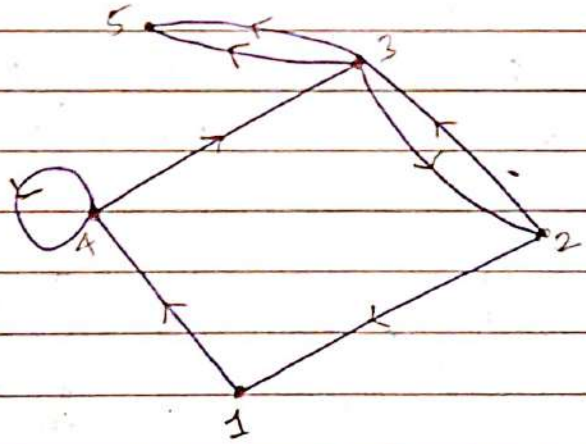
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(2) Adjacency matrix.

⇒ Let G be a digraph with in vertex then adjacency matrix is defined as,

$x_{ij} = \{ k =$ if there are k edges directed from i^{th} vertex to j^{th} vertex. $\}$

$x_{ij} = 0$ otherwise.

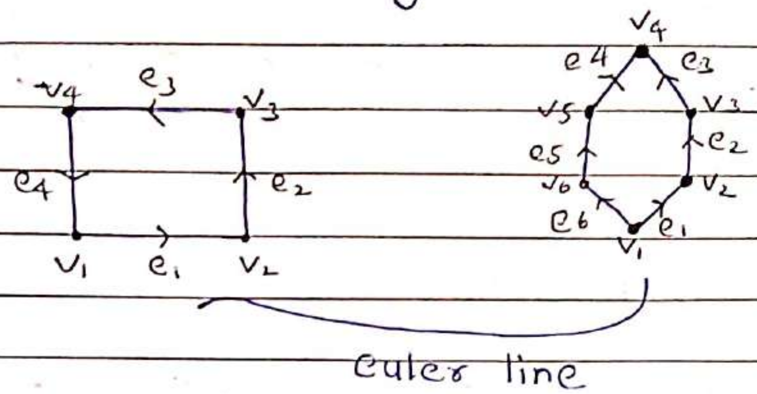


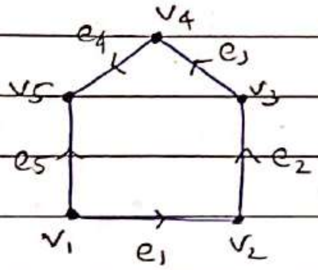
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Evergreen

Directed Euler line:

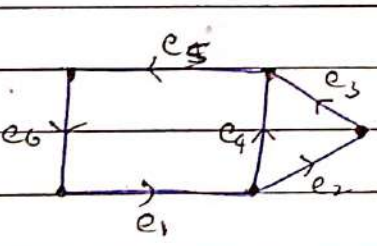
⇒ A directed euler line in a digraph is a close directed walk that traverse every edge of G exactly once.



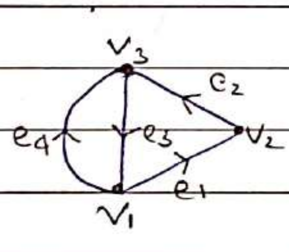


Euler's Digraph :-

⇒ A digraph is called Euler's digraph. if it contains a directed Euler line.



Not euler Digraph.



$v_1 e_1 v_2 e_2 v_3 e_3 v_1 e_4 v_3$

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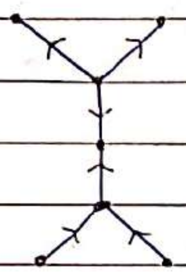
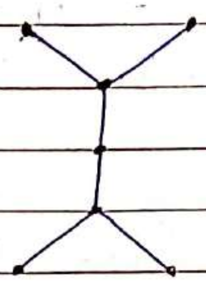
Directed tree :-

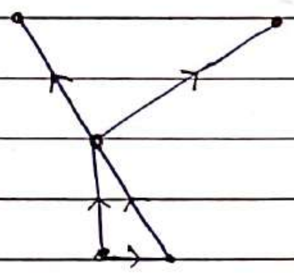
⇒ A directed graph is said to be directed tree it the corresponding undirected graph is a tree.



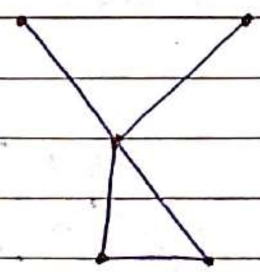
Directed Tree



Tree



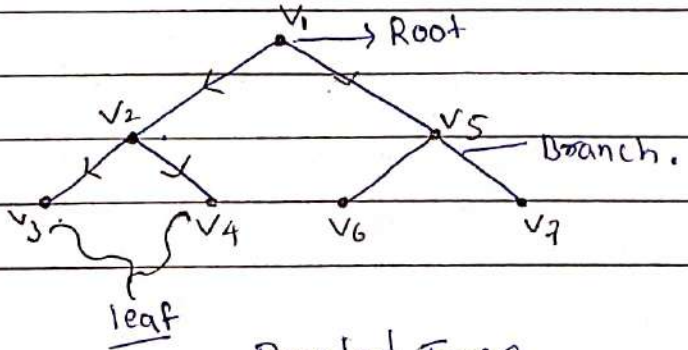
not a directed Tree



Not Tree

Rooted tree:

→ A directed tree is called rooted tree if there is only one vertex, whose in-degree is zero and the degree of all other vertices is equal to 1.



Rooted Tree

Evergreen

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* The vertex whose in-degree is '0' is called "root" of tree.

* A vertex whose out degree zero is called "leaf" of the tree and the vertex whose out-degree is non-zero is called branches or internal node.

Binary tree.

=> A rooted tree in which out degree of the root is 2 and the remaining interned vertices at out degree at most 2 is called binary tree.