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Sets

A set is a well defined collection of object or thing.

ex- A = {a, e, i, o, u} set of vowels.

→ elements are written in curly bracket separated by comma

→ Sets are denoted by capital letters like A, B, X, Y, -

Representation of Set :-

1) Roster Form or Tabular Method :-

In this form all the elements of set are listed, the elements being separated by commas and enclosed within curly bracket.

eg. A = {1, 2, 3, 4, 5}

2) Set builder Form :-

In this method elements of set are not listed but these are represented by some common property.

$A = \{x : 1 \leq x \leq 5 ; x \in N\}$

Types of Set :-

1) Empty Set :- A set having no elements is called empty set. It is denoted by ϕ or { }

also known as Null set

2) Singleton Set :- A set having only one element is called singleton set.

Finite Set :- A set having finite no. of element.

A = {1, 2, 3, 4, 5}

no. of elements = 5

Null set is a subset of every set.
Every set is a subset of itself.

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Infinite Set :- A set having infinite no. of elements.
ex A set of natural no.
$A = \{1, 2, 3, 4, 5, 6, \ldots\}$

Subset :- A set A is called subset of set B if for all $x \in A \Rightarrow x \in B$.
It is expressed as $A \subseteq B$
ex- $B = \{a, e, i, o, u\}$
$A = \{a, i, u\}$, then $A \subseteq B$

Power Set :- Family of all subsets of A is called power set of A.
ex $A = \{a, b\}$
$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$
$\Rightarrow n[P(A)] = 2^n$, where n = no. of elements.
Proper Subset :- If $A \subseteq B$ but $A \neq B$ ~~the~~ then $A \subset B$ ie. there is at least one element in B not in A.

Equal Sets :- Two sets A and B are said to be equal if the no. of elements ~~in~~ in A & B are same and corresponding elements are also same.
ex. $A = \{a, b, c\}$, $B = \{b, c, a\}$

Equivalent Sets :- Two sets A and B are said to be equivalent if the no. of elements in A & B is same.
ex $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$
no. of elements = 4 no. of elements = 4.

Universal Set → A set contain all the elements under consideration in a given problem. It is denoted by U. U = {Months of the year}
A = {March, April}

Cardinality of Set :-

The no. of elements in the set is called cardinality of set.
It is denoted by $|A|$ or $n(A)$.
ex.- $A = \{2, 4, 6, 8\}$, $B = \{2, 4, 8, 16\}$
$|A| = 4$ or $n(A) = 4$ $n(B) = 4$ or $|B| = 4$

Operation on Sets :-

1) Union of Sets :- If A and B are two sets then
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
ex. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
$A \cup B = \{1, 2, 3, 4, 5, 6\}$

2) Intersection of Sets :- If A and B are two sets then
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
ex.- $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
$A \cap B = \{3, 4\}$

3) Complement of a Set → If U is universal set and A be any subset of U. then
$A^C = \{x \mid x \in U \text{ \& } x \notin A\}$
ex.- $U = \{1, 2, 3, 4, 5, 6\}$
$A = \{2, 4, 6\}$
then $A^C = \{1, 3, 5\}$

4) Difference of Sets :- If A & B are two sets then
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$
ex.- $A = \{1, 2, 3, 4, 5\}$ $B = \{2, 3\}$
$A - B = \{1, 4, 5\}$

Ordered Set :- The ordered set is defined as ordered collection of distinct objects.

ex {Sun, Mon, Tue, Wed, Thu, Fri, Sat}

Cartesion Product of two sets

Let A & B are two sets then

$$A \times B = \{(x,y) \mid x \in A, y \in B\}$$

ex let $A = \{1,2\}$, $B = \{3,4\}$

then $A \times B = \{(1,3),(1,4),(2,3),(2,4)\}$

Ex= If $A = \{1,4\}$, $B = \{4,5\}$, $C = \{5,7\}$ then determine $(A \times B) \cap (A \times C)$

Soln

Ex= Let A, B, C, D be any ~~three~~ four sets then prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

Soln

Ordered Pair :- An ordered pair of objects is a pair of objects arranged in some order. Thus in the set $\{a,b\}$ a is first meber & b is second.

Ex- If A & B are any two sets then prove that

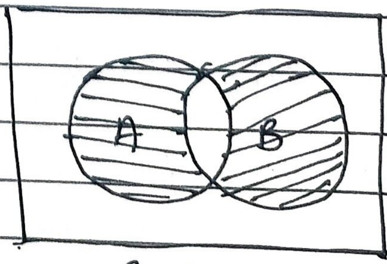
$$A \cap (B - A) = \phi$$

Symmetric Difference :- The symmetric difference of A & B denoted by $A \oplus B$ consists of those elements which belong to A or B but not to both, that is,

$$A \oplus B = (A \cup B) - (A \cap B)$$

also

$$A \oplus B = (A - B) \cup (B - A)$$



$A \oplus B$

Relation

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Relation ⇒ Let A & B be non-empty sets, then any subset R of the A×B is called a relation from A to B.
It is denoted by R
$R = \{(x,y) : x \in A, y \in B \text{ and } x R y\}$

Total no. of Relations from A to B
If set A has m elements and set B has n elements then A×B has mn elements. So, total no. of Relation from A to B is 2^{mn}.

ex. Let $A = \{a,b,c\}$, $B = \{2,4,6,10\}$ &
$R = \{(a,2), (b,4), (c,10)\}$. Is R a relation from A to B?

Domain of R ⇒ Let $R = \{(x,y) : x \in A, y \in B\}$ be a relation from A to B then set of first co-ordinates of every elements of R is called Domain of R. It is denoted by dom(R)

ex. $R = \{(a,1), (a,3), (b,4), (c,3)\}$
$Dom(R) = \{a,b,c\}$

Range of R ⇒ Let $R = \{(x,y) : x \in A, y \in B\}$ be a relation from A to B then set of second co-ordinates of every element of R is called Range of R. It is denoted by Ran(R)

ex. $R = \{(a,1), (a,3), (b,1), (b,2), (b,3)\}$
$Ran(R) = \{1,2,3\}$

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Operations on Relation →

1) Complement of a Relation →

Consider a relation R from A to B then complement of relation R denoted by $\bar{R}$ or R' or R^c is a relation from A to B st

$$R' = \{(a,b) : (a,b) \notin R\}$$

ex- Let R be a relation from X to Y where
$X = \{1,2,3\}$, $Y = \{8,9\}$
$R = \{(1,8), (2,8), (1,9), (3,9)\}$. find R'.

2) Inverse Relation →

Consider a relation R from A to B then ~~comple~~ inverse of relation R is denoted by R^{-1} is a relation from B to A st

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

ex- Let R be a relation from A to B where
$A = \{1,2,4,8\}$, $B = \{2,4\}$
$R = x+y$ is divisible by 2
find inverse relation.

Solⁿ

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Intersection and union of Relation:-

If R and S are two relations then

$R \cup S = \{(x,y) : xRy \text{ or } xSy\}$

$R \cap S = \{(x,y) : xRy \text{ and } xSy\}$

ex: Let $R_1 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$
$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$
then find $R_1 \cup R_2$ & $R_1 \cap R_2$

Solⁿ

Properties of Relation:-

A relation R on a set A satisfies certain properties are defined as -

1) Reflexive Relation:-

A relation R on a set A is reflexive if
$(a,a) \in R \quad \forall\ a \in A$ or $\boxed{aRa\ \forall\ a \in A}$

ex: Let $A = \{a,b\}$
& $R = \{(a,a), (a,b), (b,b)\}$

Solⁿ $(a,a) \in R \Rightarrow aRa$
$(b,b) \in R \Rightarrow \cancel{b}Ra$

So, R is reflexive

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Irreflexive Relation :-

A relation R on set A is irreflexive if,
$(a,a) \notin R \quad \forall\ a \in A$
i.e. $a \not{R} a \quad \forall\ a \in A$

ex. Let $A = \{1,2\}$ & $R = \{(1,2),(2,1)\}$
$(1,1) \notin R$, and $(2,2) \notin R$
$\Rightarrow$ R is irreflexive.

Non-Reflexive Relation :-

A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive

ex. Let $A = \{1,2,3\}$
$R = \{(1,1),(1,2),(2,3)\}$
$\Rightarrow$ R is neither reflexive nor irreflexive

Symmetric Relation :- A relation R on set A is said to be symmetric relation if

$(a,b) \in R \Rightarrow (b,a) \in R \quad \forall\ (a,b) \in A$

Asymmetric Relation :- A relation R on set A is said to be asymmetric relation if

$(a,b) \in R$ but $(b,a) \notin R$

Anti-symmetric Relation :- A relation R on a set A is said to be antisymmetric if

$(a,b) \in R$ and $(b,a) \in R \Rightarrow \boxed{a=b}$

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Transitive Relation :- A relation on set A is said to transitive if
$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
$\forall\ a,b,c \in A$

Ex- Give an example of a relation which is -
i) reflexive and transitive but not symmetric.
ii) symmetric and transitive but not reflexive
iii) reflexive and symmetric but not transitive.
iv) reflexive and transitive but neither symmetric nor antisymmetric.

Equivalence Relation :-

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive
ie R is an equivalence relation on A if it has following three properties.
1) $(a,a) \in R \quad \forall\ a \in A$
2) $(a,b) \in R \Rightarrow (b,a) \in R$
3) $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$

Classification of functions -

1) **One-One function** ⇒ Let $f: A \to B$ be a function and it is called one-one if
$x \neq y \Rightarrow f(x) \neq f(y)$ for $x, y \in A$
or if $f(x) = f(y) \Rightarrow x = y$

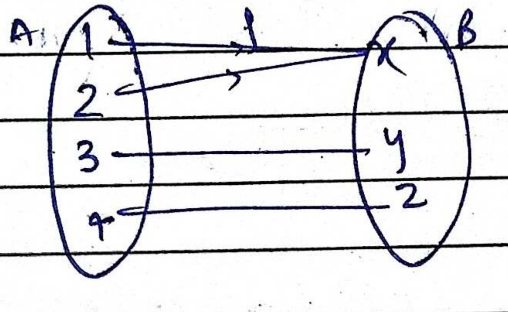
ex- Let $f: R \to R$ s.t $f(x) = ax + b$, $a \neq 0$

$f(x) = f(y)$
$ax + b = ay + b$
$ax = ay$
$\boxed{x = y}$

So, $f: R \to R$ is a one-one function.

2) **Onto Function** ⇒ A function $f: A \to B$ is called onto if Range $f = B$

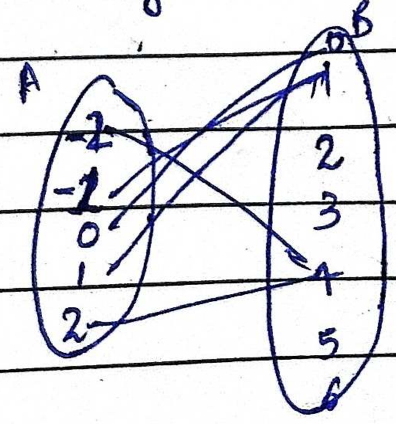
ex- Let $A = \{1, 2, 3, 4\}$ & $B = \{x, y, z\}$



Here Range $f = \{x, y, z\} = B$ which is onto

3) **Many One Function** ⇒ If two or more than two elements of domain X have f-image in Y ie $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

ex- for x^2



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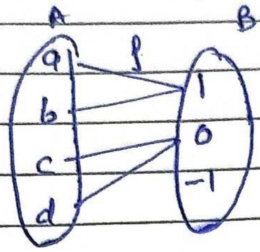
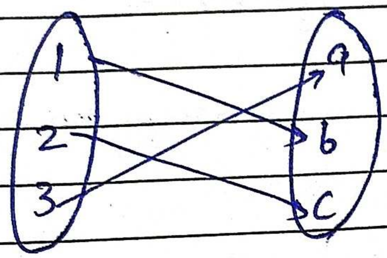
4) <u>Into Mapping</u> :- $f : A \to B$ is said to into mapping if $\{f(x) : x \in A\} \subset B$



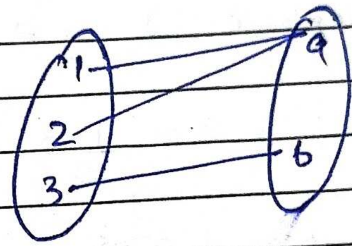
This is into mapping.

~~Bijective Function~~ :-

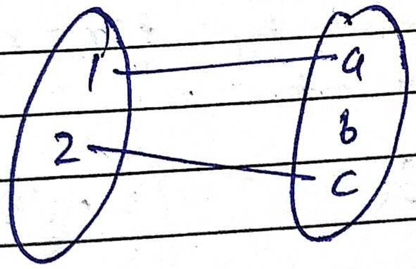
Ex:-



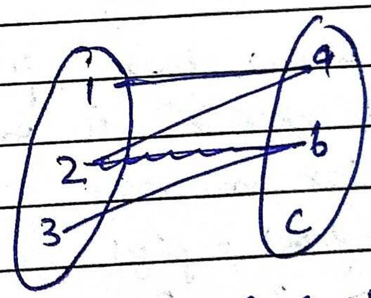
one-one, onto



many-one, onto



one-one, into



many-one, into

\# <u>Bijective Function</u> :- Let $f : A \to B$ be a function & and it is called bijective if f is one-one and onto.

Ex- $f: R \rightarrow R$, $f(x) = 2x-3$. Is f is bijective?

Sol^n For one-one

$f(x) = f(y)$

$\Rightarrow 2x-3 = 2y-3$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

$\Rightarrow f$ is one-one

For onto

suppose $\exists$ $y \in$ Ran f s.t. $y = 2x-3$

$\Rightarrow x = \frac{y+3}{2} \in R$ $\qquad$ $R \rightarrow$ real no.

$\Rightarrow$ Ran $f = R$

$\Rightarrow f$ is onto function

So, f is bijective function.

Q Show that $f: Z^+ \rightarrow Z^+$ defined by $f(x) = x^2$ $\forall x \in Z^+$ is one-one but not onto.

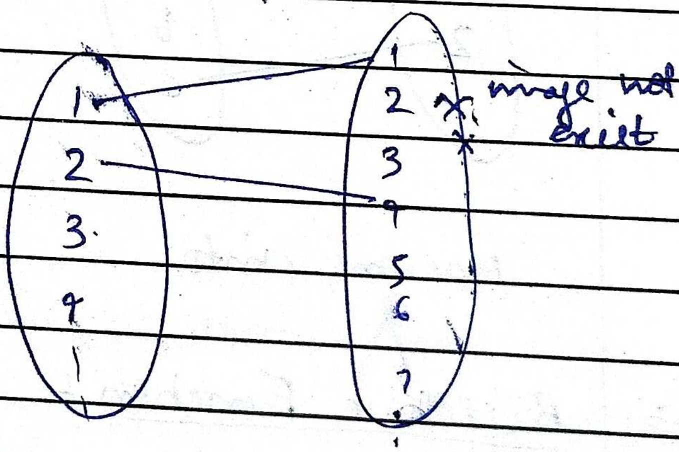
Sol^n For one one

$f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$

f is one-one



for onto

$z = x^2$

$\Rightarrow x = \sqrt{z} \in Z^+$

for $z=2$

$\nexists$ x s.t. $f(x) = 2$

$\Rightarrow f(x)$ is not onto

$\Rightarrow f$ is one-one but not onto.

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Q Find the domain of the function

i) $f(x) = \frac{x}{x^2+1}$ ii) $f(x) = \sqrt{x-4}$

Ch-5- Boolean Algebra

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Unit - 3

Chapter - 4

Posets, Hasse Diagram and Lattices

A relation R on a set A is said to be partial order if R is reflexive, anti-symmetric and transitive. Thus, a relation R on a set A is partial order relation if the following conditions hold.

1) Reflexivity : $aRa \ \forall \ a \in A$
2) Anti-symmetry : If aRb, bRa then $a=b$.
3) Transitivity : If aRb, bRc then aRc.

The set A together with the partial order R is called partially ordered set.

We generally denote a partial order by the symbol $\leq$ in place of R.

ex- The relation $\leq$ of divisibility is partial order on the set N of natural no. Here $a \leq b$ means $a|b$ (a divides b)

Here,
The element 2 and 5 are not comparable. since neither 2 divides 5 nor 5 divides 2. Thus in a poset every pair of poset need to be comparable.

Totally ordered Set ~~or~~ (Chain) :- Let $(A, \leq)$ be a poset $(A, \leq)$ is called chain or totally ordered set if every two element in A are comparable. That is, if $a, b \in A$ then $a \leq b$ or $b \leq a$. They are also called linearly orderly sets.

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Minimal and Maximal elements:-

⇒ Let $(P, \leq)$ be a poset. An element a in P is called a minimal element if there is no other element b in P s.t $b \neq a$ and $b \leq a$.

A minimal element in a poset need not to be unique. All those elements, which appear at the lowest level of a Hasse diagram of a partially ordered set are minimal elements.

Every finite poset has at least one minimal element

⇒ Let $(P, \leq)$ be a poset. An element a in P is called a maximal element if there is no b in P s.t $a \neq b$ and $a \leq b$. ~~In other words~~, .

A maximal element in a poset need not to be unique. All those elements which appear at the highest levels of a Hasse diagram of a poset are maximal elements.

Least & Greatest Elements:-

⇒ Let $(P, \leq)$ be a poset. If there exists an element $a \in P$ s.t $a \leq x$ $\forall$ $x \in P$, then a is called the least element in P.
The least element also called first element or zero element of P.

~~Representati~~

Ex:- let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$

R_1 is P.O.R.

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	(2,3)	(2,4)
3	3,1	3,2	3,3	(3,4)
4	4,1	4,2	4,3	4,4

⇒ Let $(P, \leq)$ be a poset. If there exists an element $a \in P$ s.t. $x \leq a$ for all $x \in P$, then a is called the greatest element in P.

The greatest element is also called last element or unit element of P, and usually denoted by 1.

Upper Bounds and Least Upper Bound

Let $(P, \leq)$ be a poset and let A be a subset of P. An element $x \in P$ is called an upper bound of A if $a \leq x$ $\forall a \in A$

\# Let $(P, \leq)$ be a poset and let $A \subseteq P$. An element $x \in P$ is said to be a least upper bound or supremum of A if x is a upper of A and $x \leq y$ for all upper bounds y of A. Supremum of A is denoted by $\sup(A)$.

Lower Bounds and Greatest Lower Bound

Let $(P, \leq)$ be a poset and A be a subset of P. An element $x \in P$ is said to be a lower bound of A if $x \leq a$ $\forall a \in A$.

\# Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $x \in P$ is said to be a greatest lower bound or infimum of A, if x is a lower bound and $y \leq x$ for all lower bounds y of A. It is denoted by $\inf(A)$.

Lattice

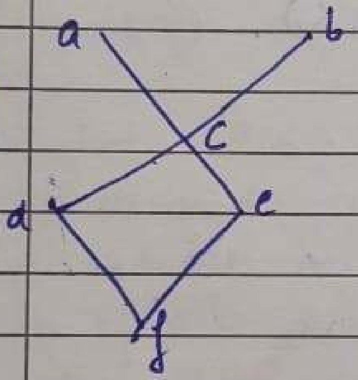
A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ of two element of L has a greatest lower bound and a least upper bound.

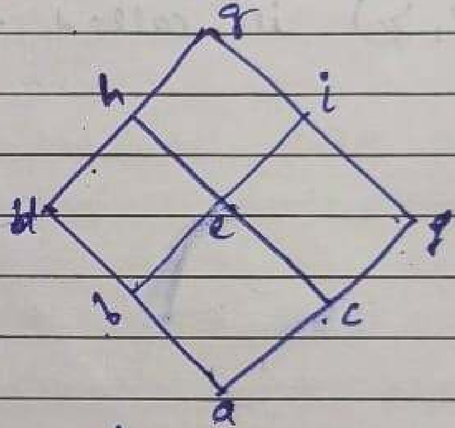
$\Rightarrow$ poset $(L, \leq)$ is a lattice if for every $a, b \in L$ $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L.

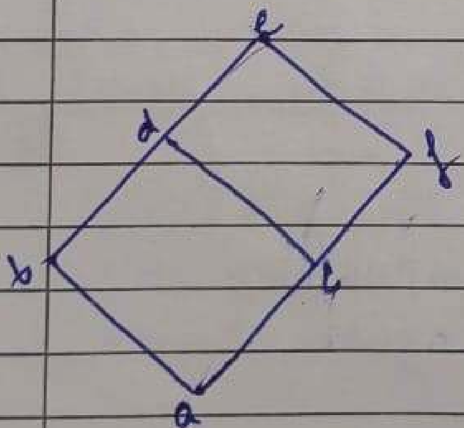
$\vee \rightarrow$ join, $a+b$, lcm $\wedge \rightarrow$ meet, $a \cdot b$, gcd

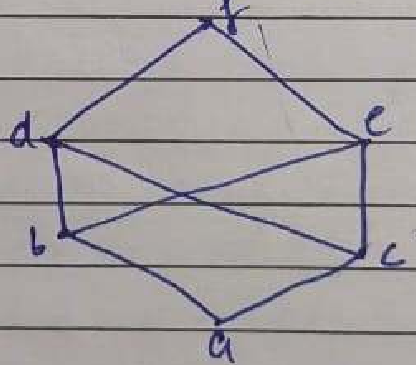
We denote $\sup\{a, b\}$ by $a \vee b$ and call it the join of a and b and $\inf\{a, b\}$ by $a \wedge b$ and call it the meet of a and b.

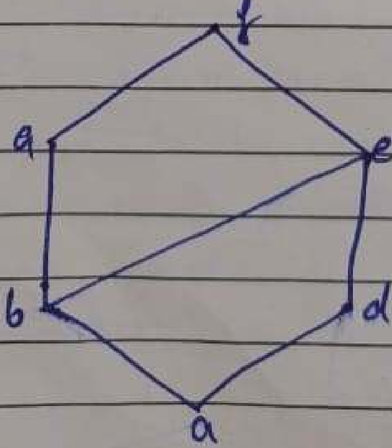
Q











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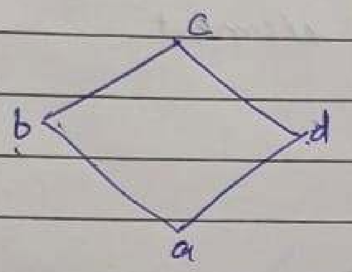
Properties of lattice :

Thm: If $(L, \leq)$ is a lattice then for any $a, b, c \in L$ the following holds :

1) $a \wedge a = a$ and $a \vee a = a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4) $a \wedge (a \vee b) = a$.

Ex.

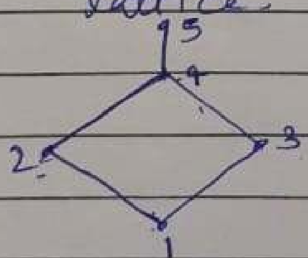


sup $\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

inf $\wedge$	a	b	c	d
a	a	a	a	a.
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since sup and inf exist for each pair So, it is lattice.

Ex.



$\vee$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	4	4	5
3	3	4	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Since sup and inf exist for each pair, So, it is lattice.

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Dual of a Poset :-

Let R be a relation defined on a set X. Then the converse of R, denoted by $\bar{R}$, is a relation on X defined by

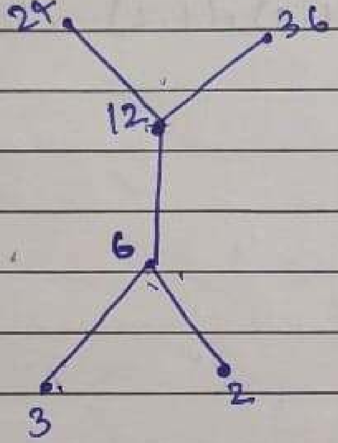
$$a\bar{R}b \text{ if and only if } bRa \text{ , } a, b \in X$$

If R is a partial order relation denoted by $\leq$ on X then $\bar{R}$, the converse of R, is denoted by $\geq$.

$\Rightarrow$ Let $(P, \leq)$ be a poset, then $(P, \geq)$ is also a poset. $(P, \geq)$ is called the dual of $(P, \leq)$.

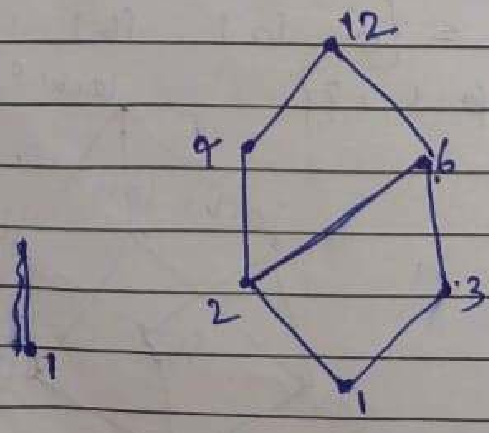
Ex- $X = \{2, 3, 6, 12, 24, 36\}$
$\leq$ be defined as $x|y$
then
$R = (2,6), (2,12), (2,24), (2,36), (3,6), (3,12),$
$(3,24), (3,36), (12,24), (12,36), (6,12),$
$(6,24), (6,36)$

$R = (2,6), (3,6), (3,24), (3,36), (12,24), (12,36)$
$(6,12), (6,36)$



Ex- $X = \{1, 2, 3, 4, 6, 12\}$, $x \leq y \Leftrightarrow x|y$

$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6)$
$(2,12), (3,6), (3,12), (6,12), (4,12)\}$



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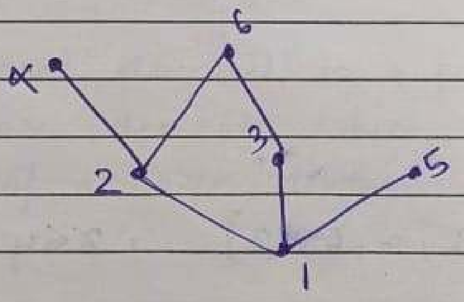
Hasse Diagram :-

A partial order $\leq$ on a set X can be represented by means of a diagram known as Hasse diagram of $(X, \leq)$.

eg. Let $X = \{1, 2, 3, 4, 5, 6\}$ then $/$ is a partial order relation on X. Draw the Hasse diagram of $(X, /)$.

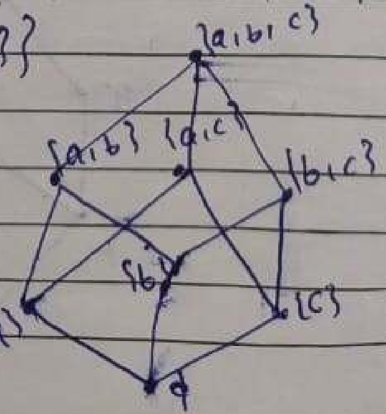
Sol: $R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$



$(X, /)$.

ex: Draw the Hasse diagram for the POSET $(P(S), \subseteq)$ where $P(S)$ is power set on $S = \{a, b, c\}$

Sol: $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$



21

DETERMINANTS

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Every square matrix is associated with a unique number called the determinant of the matrix.

In this lesson, we will learn various properties of determinants and also evaluate determinants by different methods.

OBJECTIVES

After studying this lesson, you will be able to :

- define determinant of a square matrix;
- define the minor and the cofactor of an element of a matrix;
- find the minor and the cofactor of an element of a matrix;
- find the value of a given determinant of order not exceeding 3;
- state the properties of determinants;
- evaluate a given determinant of order not exceeding 3 by using expansion method;

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of solution of equations
- Knowledge of number system including complex number
- Four fundamental operations on numbers and expressions

21.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

On solving this system of equations for x and y, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \text{ provided } a_1b_2 - a_2b_1 \neq 0$$

The number $a_1b_2 - a_2b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

i.e. a_{11} belongs to the 1st row and 1st column
a_{12} belongs to the 1st row and 2nd column
a_{21} belongs to the 2nd row and 1st column
a_{22} belongs to the 2nd row and 2nd column

21.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

write the elements in the following manner :

$$\begin{array}{cc} a_1 & a_{12} \\ a_{21} & a_{22} \end{array}$$

Multiply the elements by the arrow. The sign of the arrow going **downwards** is positive, i.e., a_{11} a_{22} and the sign of the arrow going **upwards** is negative, i.e., $-a_{21}$ a_{12}

Add these two products, i.e., a_{11} a_{22} +($-a_{21}.a_{12}$) or a_{11} a_{22} $-a_{21}$ a_{12} which is the required value of the determinant.

Example 21.1 Evaluate :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)\,(a+b) - (2a)\,(2b)$

$= a^2+2ab+b^2 - 4ab = a^2+b^2 - 2ab = (a-b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2+x+1)\,(x-1) - (x^2-x+1)\,(x+1)$

$= (x^3-1) - (x^3+1) = -2$

Example 21.2 Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution :

(i) Now, $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x = -x - 9$

According to the question,

$-x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now, $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the equation

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x-1) + 1(x-1) = 0$

or, $(2x+1)(x-1) = 0$

or, $x = 1, -\frac{1}{2}$

21.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has $(3)^2 = 9$ elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements

$a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 ,we write a determinant of order3 as follows: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or |A|, |B| etc.

$$\Delta = |a_{ij}| \text{ , where } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

21.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix".
With the 1 $\times$ 1 matrix [a], we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3 $\times$ 3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we associate the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\, a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 21.3 If A = $\begin{pmatrix} 3 & 6 \\ 1 & 5 \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15-6 = 9$

Example 21.4 If A= $\begin{pmatrix} a+b & a \\ b & a-b \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix}$ = (a+b) (a−b) − b × a = $a^2 - b^2 - ab$

Note : 1. The determinant of a unit matrix I is 1.

2. A square matrix whose determinant is zero, is called the singular matrix.

21.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements a_{11}, a_{12} and a_{13} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternatively, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign. Therefore, a_{11} has been assigned positive sign.

Note : We can expand the determinant using any row or column. The value of the determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 21.5 Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1 \text{ x } (20 - 2) - 2 \times (10 - 3) + 3 \times (4 - 12)$

$= 18 - 14 - 24$

$= -20$

Example 21.6 Expand the determinant, by using the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\, 3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -2 \times (3-4) + 1 \times (1-6) - 3 \times (2-9)$

$= 2 - 5 + 21$

$= 18$

CHECK YOUR PROGRESS 21.1

1. Find |A|, if

(a) $A = \begin{bmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{bmatrix}$ (b) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $A = \begin{bmatrix} \sin\alpha + \cos\beta & \cos\beta + \cos\alpha \\ \cos\beta - \cos\alpha & \sin\alpha - \sin\beta \end{bmatrix}$ (d) $\begin{bmatrix} a+bi & c+di \\ c-di & a-bi \end{bmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{bmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ (d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

21.6 MINORS AND COFACTORS

21.6.1 *Minor of a_{ij} in $|A|$*

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in $|A|$ is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of $|A|$ and is denoted by M_{ij}. For example, minor of 3 in the determinant

$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 21.7 Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1^{st} row and 1^{st} column. Thus to find the minor of a_{11}, we delete the 1^{st} row and 1^{st} column of $|A|$.

The minor M_{11} of a_{11} is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \; ; \qquad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \; ; \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

21.6.2 *Cofactors of a_{ij} in |A|*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is usually denoted by C_{ij}, Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Example 21.8 Find the cofactors of the elements a_{11}, a_{12}, and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

The cofactor of any element a_{ij} is $(-1)^{i+j} M_{ij}$, then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$= (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{31}\, a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$$

and $C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32}\, a_{13} - a_{12}\, a_{33})$

Example 21.9 Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution : The elements of the second row are $a_{21}=5$; $a_{22}=2$; $a_{23}=4$.

$$\text{Minor of } a_{21} \text{ (i.e., 5)} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$\text{Minor of } a_{22} \text{ (i.e., 2)} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$$

$$\text{and Minor of } a_{23} \text{ (i.e., 4)} = \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1}M_{21} = -(48) = -48$$

$$C_{22} = (-1)^{2+2}M_{22} = +(-13) = -13$$

and $$C_{23} = (-1)^{2+3}M_{23} = -(-42) = 42$$

CHECK YOUR PROGRESS 21.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinat

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Solve for x, the following equations:

(a) $\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

21.7 PROPERTIES OF DETERMINANTS

We shall now discuss some of the properties of determinants. These properties will help us in expanding the determinants.

Property 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first column, we have

$$\Delta \quad = 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix}$$

$$= 2\,(3-0) - 0\,(1-6) + 4\,(0+9)$$

$$= 6 + 36 = 42$$

Let Δ' be the determinant obtained by interchanging rows and columns of Δ. Then

$$\Delta' = \begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

Expanding the determinant Δ' by second column, we have (Recall that a determinant can be expanded by any of its rows or columns)

$$(-)\,0\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-)\,0\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + (-3)\,(-2-12) + 0$$

$$= 42$$

Thus, we see that $\Delta = \Delta'$

Property 2: If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$= 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$= 2\,(4-3) - 3\,(2-9) + 1\,(1-6)$

$= 2 + 21 - 5 = 18$

Let Δ' be the determinant obtained by interchanging C_1 and C_2

$$\text{Then } \Delta' = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 3\,(2-9) - 2(4-3) + 1(6-1)$

$= -21 - 2 + 5 = -18$

Thus we see that $\Delta' = -\Delta$

Corollary

If any row (or a column) of a determinant is passed over 'n' rows (or columns), then the resulting determinant Δ' is $\Delta = (-1)^n\ \Delta$

For example ,

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 5 & 6 \\ 0 & 4 & 2 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 5 & 6 \\ 0 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix}$$

$= 2\,(10-24) - 3\,(2-0) + 5(4)$

$= -28 - 6 + 20 = -14$

Property 3: If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

$$\textbf{Proof : } \text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

be a determinant with identical columns C_1 and C_2 and let Δ' determinant obtained from Δ by interchanging C_1 and C_2

Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is the same as Δ, but by property 2, the value of the determinant changes in sign, if its any two adjacent rows (or columns) are interchnaged

Therefore $\quad \Delta' = -\Delta$

Thus, we find that

or $\quad 2\Delta = 0 \Rightarrow \Delta = 0$

Hence the value of a determinant is zero, if it has two identical rows (or columns).

Property 4: If each element of a row (or column) of a determinant is multiplied by the same constant say, $k \neq 0$, then the value of the determinat is multiplied by that constant k.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2(3-0) - 1\,(0-0) + (-5)\,(0+12\,)$$

$$= 6 - 60 = -54$$

Let us multiply column 3 of Δ by 4. Then, the new determinant Δ' is :

$$\Delta' = \begin{vmatrix} 2 & 1 & -20 \\ 0 & -3 & 0 \\ 4 & 2 & -4 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$\Delta' = 2\,(12-0) - 1\,(0-0) + (-20)\,(0+12)$$

$$= 24 - 240 = -216$$

$$= 4\,\Delta$$

Corollary :
If any two rows (or columns) of a determinant are proportional, then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & ka_1 \\ a_2 & b_2 & ka_2 \\ a_3 & b_3 & ka_3 \end{vmatrix}$

Note that elements of column 3 are k times the corresponding elements of column 1

$$\text{By Property 4, } \Delta = k\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$$

$$= k \times 0 \qquad \text{(by Property 2)}$$

$$= 0$$

Property 5: If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.

Proof : Let $\Delta = \begin{vmatrix} a_1 + \alpha & b_1 + \beta & c_1 + \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, on expanding the determinant by the first row, we have

$$\Delta = (a_1 + \alpha)(b_2c_3 - b_3c_2) - (b_1 + \beta)(a_2c_3 - a_3c_2) + (c_1 + \gamma)(a_2b_3 - a_3b_2)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) + \alpha(b_2c_3 - b_3c_2)$$

$$-\beta(a_2c_3 - a_3c_2) + \gamma(a_2b_3 - a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, the determinant Δ can be expressed as the sum of the determinants of the same order.

Property 6: The value of a determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding elements of one or more other rows (or columns)

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' be the determinant obtained from Δ by corresponding elements of R_3

i.e. $R_1 \to R_1 + kR_3$

Then, $$\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or, $\Delta' = \Delta + k \times 0$ (Row 1 and Row 3 are identical)

$\Delta' = \Delta$

21.8 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a determinant easily by applying the aforesaid properties. The purpose of simplification of a determinant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the determinant by that row (or column). We denote 1st, 2nd and 3rd row by R_1, R_2, and R_3 respectively and 1st,2nd and 3rd column by C_1, C_2 and C_3 respectively.

Example 21.10 Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

Solution : $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \rightarrow C_1 + (C_2 + C_3)$

$$\therefore \qquad \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \quad \text{(on expanding by } C_1\text{)}$$

(since w is a non-real cube root of unity, therefore, $1+w+w^2 = 0$)

Example 21.11 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution : $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} [R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3]$$

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 0 & 1 & -b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1, we have

$$\Delta = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix} = (a-c)(b-c)(b-a)$$

$$= (a-b)(b-c)(c-a)$$

Example 21.12

Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$

Solution : $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad R_1 \to R_1 - (R_2 + R_3)$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\,[b(a+b) - bc] - 2b[bc - c(c+a)]$$

$$= 2bc\,[a+b-c] - 2bc[b-c-a]$$

$$= 2bc\,[(a+b-c) - (b-c-a)]$$

$$= 2bc\,[a+b-c-b+c+a]$$

$$= 4abc$$

Example 21.13 Evaluate:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3 \;=0,$$

Example 21.14 Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution : Δ

$$= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \qquad C_3 \to C_2 + C_3$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$$= (ab+bc+ca) \times 0 \qquad \text{(by Property 3)}$$

$$= 0$$

Example 21.15 Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : Δ

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$= abc(-a)\begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)(-4bc)$$

$$= 4a^2b^2c^2$$

Example 21.16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2\,(a+3)$$

Solution : Δ

$$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (a+3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a+3) \times (1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3)$$

CHECK YOUR PROGRESS 21.3

1. Show that $\begin{vmatrix} x+3 & x & x \\ x & x+3 & x \\ x & x & x+3 \end{vmatrix} = 27\,(x+1)$

2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc+ca+ab+abc$

4. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for x :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

21.11 Application of Determinants

Determinant is used to find area of a triangle.

21.11.1 Area of a Triangle

We know that area of a triangle ABC, (say) whose vertices are $(x_1 y_1)$, $(x_2 y_2)$ and $(x_3 y_3)$ is given by

$$\text{Area of } (\Delta ABC) = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] \qquad ...(i)$$

Also, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ [expanding along C_1]

$$= x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)$$

$$= x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \qquad ...(ii)$$

from (*i*) and (*ii*)

$$\text{Area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

Thus the area of a triangle having vertices as $(x_1\ y_1)$, $(x_2\ y_2)$ and $(x_3\ y_3)$ is given by

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

21.11.2 Condition of collinearity of three points :

Let A(x_1 y_1), B(x_2 y_2) and C(x_3 y_3) be three points then

A, B, C are collinear if area of ΔABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

21.11.2 Equation of a line passing through the given two points

Let the two points be P(x_1, y_1) and Q(x_2 y_2) and R(x y) be any point on the line joining P and Q since the points P, Q and R are collinear.

$$\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Thus the equation of the line joining points (x_1 y_1) and (x_1 y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Example 21.17 Find the area of the triangle with vertices P(5, 4), Q(–2, 4) and R(2, –6)

Solution : Let A be the area of the triangle PQR, then

$$A = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[5(4-(-6)) - 4(-2-2) + 1(12-8)\right]$$

$$= \frac{1}{2}[50+16+4] = \frac{1}{2}(70) = 35 \text{ sq units.}$$

Example 21.18 Show that points (a, b + c), (b, c + a) and (c, a + b) are collinear.

Solution : We have

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$c_2 \to c_2 + c_1$$

$$= \begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = (a+b+c) \times 0 = 0$$

Hence, the given points are collinear.

Example 21.19 Find equation of the line joining A(1, 3) and B(2, 1) using determinants.

Solution : Let P(x, y) be any point on the line joining A(1, 3) and B(2, 1). Then

$$\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x(3-1) - y(1-2) + 1(1-6) = 0$

$\Rightarrow \quad 2x + y - 5 = 0$

This is the required equation of line AB.

CHECK YOUR PROGRESS 21.4

1. Find area of the ΔABC when A, B and C are (3, 8), (–4, 2) and (5, –1) respectively.
2. Show that points A(5, 5), B(–5, 1) and C(10, 7) are collinear.
3. Using determinants find the equation of the line joining (1, 2) and (3, 6).

LET US SUM UP

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.

- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$
- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a *singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.
- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=det

http://mathworld.wolfram.com/Determinant.html

http://en.wikipedia.org/wiki/Determinant

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Matrices_and_determinants.html

TERMINAL EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the first column.

3. Evaluate $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$ 4. Solve for x, if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

5. Using properties of determinants, show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate: (a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

,w being an imaginary cube-root of unity

7. Find the area of the triangle with vertices at the points :

(i) (2, 7), (1, 1) and (10, 8) (ii) $(-1,-8),(-2,-3)$ and $(3,2)$

(ii) (0, 0) (6, 0) and (4, 3) (iv) (1, 4), (2, 3) and (–5, –3)

8. Using determinants find the value of k so that the following points become collinear

(i) $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$

(ii) $(k,-2),(5,2)$ and $(6,8)$

(iii) $(3,-2),(k,2)$ and $(8,8)$

(iv) $(1,-5)(-4,5)(k,7)$

9. Using determinants, find the equation of the line joining the points

(i) (1, 2) and (3, 6) (ii) (3, 1) and (9, 3)

10. If the points (a, 0), (0, b) and (1, 1) are collinear then using determinants show that $ab = a + b$

ANSWERS

CHECK YOUR PROGRESS 21.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$
2. (a) and (d)
3. (a) 18 (b) −54

   (c) $adf + 2bce - ae^2 - fb^2 - de^2$ (d) $x-1$

CHECK YOUR PROGRESS 21.2

1. $M_{21} = 39;\ C_{21} = -39$

   $M_{22} = 3;\ C_{22} = 3$

   $M_{23} = -11;\ C_{23} = 11$

2. $M_{13} = -5;\ C_{13} = -5$

   $M_{23} = -7;\ C_{23} = 7$

   $M_{33} = 1;\ C_{33} = 1$

3. (a) 19 (b) 0 (c) –131

   (d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) $x = 2$ (b) $x = 2, 3$ (c) $x = 2, -\dfrac{17}{7}$

CHECK YOUR PROGRESS 21.3

7. (a) a^3 (b) $2abc(a+b+c)^3$

8. $x = \dfrac{2}{3}, \dfrac{11}{3}, \dfrac{11}{3}$

CHECK YOUR PROGRESS 21.4

1. $\dfrac{75}{2}$ sq units (3) $y = 2x$

TERMINAL EXERCISE

1. $M_{11} = -2,\ M_{12} = -1,\ M_{13} = 1,\ M_{21} = -7,\ M_{22} = -5,\ M_{23} = -1,$
$M_{31} = -8,\ M_{32} = -7,\ M_{33} = -2$

$C_{11} = -2,\ C_{12} = 1,\ C_{13} = 1,\ C_{21} = 7,\ C_{22} = -5,\ C_{23} = 1,$
$C_{31} = -8,\ C_{32} = 7,\ C_{33} = -2$

2. 0

3. –31

4. $x = 0,\ x = 1$

6. ( ) -8 (b) 0

7. (*i*) $\frac{45}{2}$ sq units (*ii*) 5 sq units

(*iii*) 9 sq units (*iv*) $\frac{15}{2}$ sq units

8. (*i*) $k = -1, \frac{1}{2}$ (*ii*) $k = \frac{13}{3}$

(*iii*) k = 5 (*iv*) $k = -5$

9. (*i*) $y = 2x$ (ii) $x = 3y$

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LIMIT AND CONTINUITY

Consider the function $f(x)=\dfrac{x^2-1}{x-1}$

You can see that the function f(x) is not defined at $x = 1$ as $x-1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x-1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x+1)(x-1)}{(x-1)}=x+1$, because $x-1\neq 0$ and so division by $(x-1)$ is possible.

Table -1

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

Table - 2

x	f(x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.
However, in this case f(x) is not defined at $x = 1$. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function f (x) =2x. Here, we are interested to see its behavior near the point 1 and at $x = 1$. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at $x = 1$ and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near x = 2 and at x = 2 ?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

OBJECTIVES

After studying this lesson, you will be able to :

- define limit of a function
- derive standard limits of a function
- evaluate limit using different methods and standard limits.
- define and interprete geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

25.1 LIMIT OF A FUNCTION

In the introduction, we considered the function $f(x)=\dfrac{x^2-1}{x-1}$. We have seen that as x approaches l, f (x) approaches 2. In general, if a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x)

Symbolically it is written as

$$\lim_{x\to a} f(x) = L$$

Now let us find the limiting value of the function $(5x-3)$ when x approaches 0.

i.e. $$\lim_{x\to 0}(5x-3)$$

For finding this limit, we assign values to x from left and also from right of 0.

x	–0.1	–0.01	–0.001	–0.0001 ....
5x – 3	–3.5	–3.05	–3.005	–3.0005 ....

x	0.1	0.01	0.001	0.0001 ....
5x – 3	–2.5	–2.95	–2.995	–2.9995 ....

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

i.e., $$\lim_{x\to 0}(5x-3) = -3$$

This is illustrated graphically in the Fig. 20.1

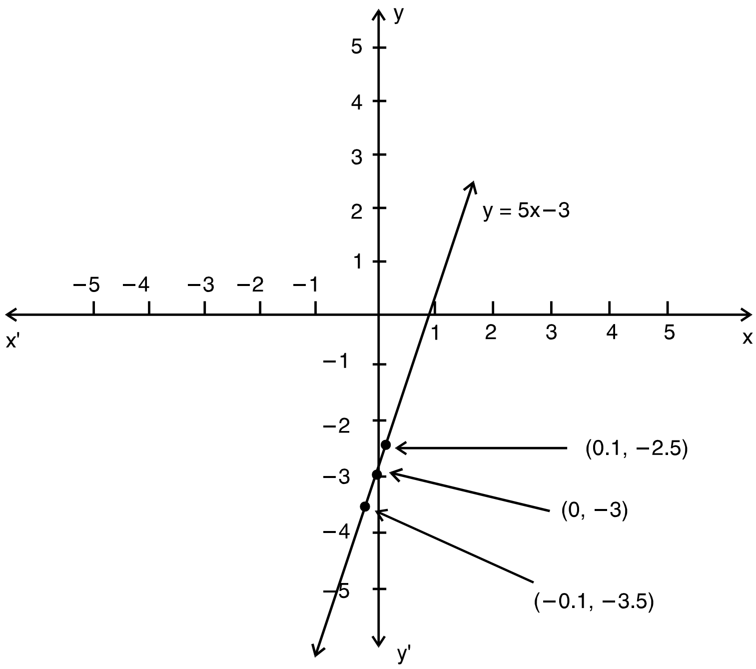


Fig. 25.1

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

Consider an example : Find $\lim_{x\to 3} f(x)$, where $f(x) = \dfrac{x^2-9}{x-3}$

We can solve it by the method of substitution. Steps of which are as follows :

Remarks : It may be noted that f (3) is not defined, however, in this case the limit of the

Step 1: We consider a value of x close to a say $x = a + h$, where h is a very small positive number. Clearly, as $x \to a$, $h \to 0$	For $f(x) = \dfrac{x^2 - 9}{x - 3}$ we write $x = 3 + h$, so that as $x \to 3, h \to 0$
Step 2 : Simplify $f(x) = f(a + h)$	Now $f(x) = f(3 + h)$ $= \dfrac{(3+h)^2 - 9}{3 + h - 3}$ $= \dfrac{h^2 + 6h}{h}$ $= h + 6$
Step 3 : Put $h = 0$ and get the requried result	$\therefore \lim\limits_{x \to 3} f(x) = \lim\limits_{h \to 0} (6 + h)$ As $x \to 0$, $h \to 0$ Thus, $\lim\limits_{x \to 3} f(x) = 6 + 0 = 6$ by putting $h = 0$.

function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

Consider the example :

Find $\lim\limits_{x \to 1} f(x)$, where $\quad f(x) = \begin{cases} \dfrac{x^3 - 1}{x^2 - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

Here, for $x \neq 1$, $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$

It shows that if f (x) is of the form $\dfrac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

Step 1. Factorise g (x) and h (x)	**Sol.** $f(x)=\dfrac{x^3-1}{x^2-1}$ $=\dfrac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}$ $(\because x \neq 1, \therefore x-1 \neq 0$ and as such can be cancelled)
Step 2 : Simplify f (x)	$\therefore \quad f(x)=\dfrac{x^2+x+1}{x+1}$
Step 3 : Putting the value of x, we get the required limit.	$\therefore \lim\limits_{x \to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{1+1+1}{1+1}=\dfrac{3}{2}$ Also $f(1)=1$(given) In this case, $\lim\limits_{x \to 1} f(x) \neq f(1)$

Thus, the limit of a function f (x) as $x \to a$ may be different from the value of the functioin at $x=a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate $\quad \lim\limits_{x \to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps :

Step 1. Rationalise the factor containing square root.

Step 2. Simplify.

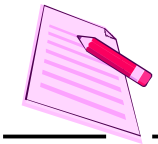
Step 3. Put the value of x and get the required result.

Solution :

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$



$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = 1$$

25.2 LEFT AND RIGHT HAND LIMITS

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^-$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^+$.

Thus, if a function f (x) approaches a limit ℓ_1, as x approaches 'a' from left, we say that the left hand limit of $f(x)$ as $x \to a$ is ℓ_1.

We denote it by writing

$$\lim_{x \to a^-} f(x) = \ell_1 \qquad \text{or} \qquad \lim_{h \to 0} f(a-h) = \ell_1, h > 0$$

Similarly, if f (x) approaches the limit ℓ_2, as x approaches 'a' from right we say, that the right hand limit of $f(x)$ as $x \to a$ is ℓ_2.

We denote it by writing

$$\lim_{x \to a^+} f(x) = \ell_2 \qquad \text{or} \qquad \lim_{h \to 0} f(a+h) = \ell_2, h > 0$$

Working Rules

Finding the right hand limit i.e.,

$$\lim_{x \to a^+} f(x)$$

Put $x = a + h$

Find $\lim_{h \to 0} f(a+h)$

Finding the left hand limit, i.e,

$$\lim_{x \to a^-} f(x)$$

Put $x = a - h$

Find $\lim_{h \to 0} f(a-h)$

Note : In both cases remember that h takes only positive values.

25.3 LIMIT OF FUNCTION y = f(x) AT x = a

Consider an example :

Find $\lim\limits_{x\to 1} f(x)$, where $f(x) = x^2 + 5x + 3$

Here $$\lim_{x\to 1^+} f(x) = \lim_{h\to 0}\left[(1+h)^2 + 5(1+h) + 3\right]$$

$$= \lim_{h\to 0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(i)}$$

and $$\lim_{x\to 1^-} f(x) = \lim_{h\to 0}\left[(1-h)^2 + 5(1-h) + 3\right]$$

$$= \lim_{x\to 0}\left[1 - 2h + h^2 + 5 - 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(ii)}$$

From (i) and (ii), $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$

Now consider another example :

Evaluate : $$\lim_{x\to 3}\frac{|x-3|}{x-3}$$

Here $$\lim_{x\to 3^+}\frac{|x-3|}{x-3} = \lim_{h\to 0}\frac{|(3+h)-3|}{[(3+h)-3]}$$

$$= \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} \text{ (as h>0, so } |h| = h)$$

$$= 1 \qquad \text{.....(iii)}$$

and $$\lim_{x\to 3^-}\frac{|x-3|}{x-3} = \lim_{h\to 0}\frac{|(3-h)-3|}{[(3-h)-3]}$$

$$= \lim_{h\to 0}\frac{|-h|}{-h} = \lim_{h\to 0}\frac{h}{-h} \quad \text{(as } h > 0\text{, so } |-h| = h)$$

$$= -1 \qquad \text{.....(iv)}$$

$\therefore$ From (iii) and (iv), $\lim\limits_{x\to 3^+}\dfrac{|x-3|}{x-3} \neq \lim\limits_{x\to 3^-}\dfrac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit $\neq$ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x\to 1}(x^2+5x+3)$ exists (which is equal to 9) and $\lim_{x\to 3}\frac{|x-3|}{x-3}$ does not exist.

Note :

I $\left.\begin{array}{l}\lim_{x\to a^+} f(x)=\ell \\ \text{and} \quad \lim_{x\to a^-} f(x)=\ell\end{array}\right\} \Rightarrow \lim_{x\to a} f(x)=\ell$

II $\left.\begin{array}{l}\lim_{x\to a^+} f(x)=\ell_1 \\ \text{and} \quad \lim_{x\to a^-} f(x)=\ell_2\end{array}\right\} \Rightarrow \lim_{x\to a} f(x)$ does not exist.

III $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist $\Rightarrow \lim_{x\to a} f(x)$ does not exist.

25.4 BASIC THEOREMS ON LIMITS

1. $\lim_{x\to a} cx = c \lim_{x\to a} x$, c being a constant.

To verify this, consider the function $f(x)=5x$.

We observe that in $\lim_{x\to 2} 5x$, 5 being a constant is not affected by the limit.

$$\therefore \quad \lim_{x\to 2} 5x = 5\lim_{x\to 2} x$$
$$= 5\times 2 = 10$$

2. $$\lim_{x\to a}\left[g(x)+h(x)+p(x)+....\right]=\lim_{x\to a} g(x)+\lim_{x\to a} h(x)+\lim_{x\to a} p(x)+........$$

where $g(x), h(x), p(x), ...$ are any function.

3. $$\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$$

To verify this, consider $f(x)=5x^2+2x+3$

and $g(x) = x + 2$.

Then $$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(5x^2+2x+3\right)$$

$$=5\lim_{x\to 0} x^2+2\lim_{x\to 0} x+3=3$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x+2) = \lim_{x\to 0} x + 2 = 2$$

$\therefore$ $$\lim_{x\to 0}(5x^2+2x+3)\lim_{x\to 0}(x+2) = 6 \qquad \text{.....(i)}$$

Again $$\lim_{x\to 0}[f(x)\cdot g(x)] = \lim_{x\to 0}[(5x^2+2x+3)(x+2)]$$

$$= \lim_{x\to 0}(5x^3+12x^2+7x+6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6 \qquad \text{.....(ii)}$$

From (i) and (ii), $\lim_{x\to 0}[(5x^2+2x+3)(x+2)] = \lim_{x\to 0}(5x^2+2x+3)\lim_{x\to 0}(x+2)$

4. $$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ provided $\lim_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2+5x+6}{x+2}$

we have $$\lim_{x\to -1}(x^2+5x+6) = (-1)^2 + 5(-1) + 6 = 1-5+6 = 2$$

and $$\lim_{x\to -1}(x+2) = -1+2 = 1$$

$$\frac{\lim_{x\to -1}(x^2+5x+6)}{\lim_{x\to -1}(x+2)} = \frac{2}{1} = 2 \qquad \text{.....(i)}$$

Also $$\lim_{x\to -1}\frac{(x^2+5x+6)}{x+2} = \lim_{x\to -1}\frac{(x+3)(x+2)}{x+2} \quad \left[\begin{array}{l}\because x^2+5x+6 \\ = x^2+3x+2x+6 \\ = x(x+3)+2(x+3) \\ = (x+3)(x+2)\end{array}\right]$$

$$= \lim_{x\to -1}(x+3)$$

$$= -1+3 = 2 \qquad \text{.....(ii)}$$

$\therefore$ From (i) and (ii),

$$\lim_{x \to -1} \frac{x^2 + 5x + 6}{x + 2} = \frac{\lim\limits_{x \to -1} \left(x^2 + 5x + 6\right)}{\lim\limits_{x \to -1} (x + 2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x \to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim\limits_{x \to a} f(x) = \ell$ and $\lim\limits_{x \to a} g(x) = m$, then

(i) $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x) = k\ell$ where k is a constant.

(ii) $\lim\limits_{x \to a} \left[f(x) \pm g(x)\right] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = \ell \pm m$

(iii) $\lim\limits_{x \to a} \left[f(x) \cdot g(x)\right] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = \ell \cdot m$

(iv) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{\ell}{m}$, provided $\lim\limits_{x \to a} g(x) \neq 0$

The above results can be easily extended in case of more than two functions.

Example 25.1 Find $\lim\limits_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

Solution : $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x - 1} = (x+1)$ $\quad [\because x \neq 1]$

$\therefore$ $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (x+1) = 1 + 1 = 2$

Note : $\dfrac{x^2 - 1}{x - 1}$ is not defined at x=1. The value of $\lim\limits_{x \to 1} f(x)$ is independent of the value of $f(x)$ at $x = 1$.

Example 25.2 Evaluate : $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$.

Solution : $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x \to 2} \left(x^2+2x+4\right) \qquad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4 = 12$$

Example 25.3 Evaluate : $\lim\limits_{x \to 2} \dfrac{\sqrt{3-x}-1}{2-x}$.

Solution : Rationalizing the numerator, we have

$$\frac{\sqrt{3-x}-1}{2-x} = \frac{\sqrt{3-x}-1}{2-x} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \frac{3-x-1}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$\therefore \qquad \lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \to 2} \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \lim_{x \to 2} \frac{1}{\left(\sqrt{3-x}+1\right)} = \frac{1}{\left(\sqrt{3-2}+1\right)} = \frac{1}{1+1} = \frac{1}{2}$$

Example 25.4 Evaluate : $\lim\limits_{x \to 3} \dfrac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$.

Solution : Rationalizing the numerator as well as the denominator, we get

$$\lim_{x \to 3} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3} = \lim_{x \to 3} \frac{\left(\sqrt{12-x}-x\right)\left(\sqrt{12-x}+x\right)\cdot\left(\sqrt{6+x}+3\right)}{\sqrt{6+x}-3\left(\sqrt{6+x}+3\right)\left(\sqrt{12-x}+x\right)}$$

$$= \lim_{x \to 3} \frac{\left(12-x-x^2\right)}{6+x-9} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

$$= \lim_{x \to 3} \frac{-(x+4)(x-3)}{(x-3)} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x} \qquad [\because x \neq 3]$$

$$= -(3+4) \cdot \frac{6}{6} = -7$$

Note : Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero.

However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :

Example 25.5 Find $\lim_{x \to 0} \frac{1}{x}$, if it exists.

Solution : We choose values of x that approach 0 from both the sides and tabulate the correspondling values of $\frac{1}{x}$.

x	−0.1	−.01	−.001	−.0001
$\frac{1}{x}$	−10	−100	−1000	−10000

x	0.1	.01	.001	.0001
$\frac{1}{x}$	10	100	1000	10000

We see that as $x \to 0$, the corresponding values of $\frac{1}{x}$ are not getting close to any number.

Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist. This is illustrated by the graph in Fig. 20.2

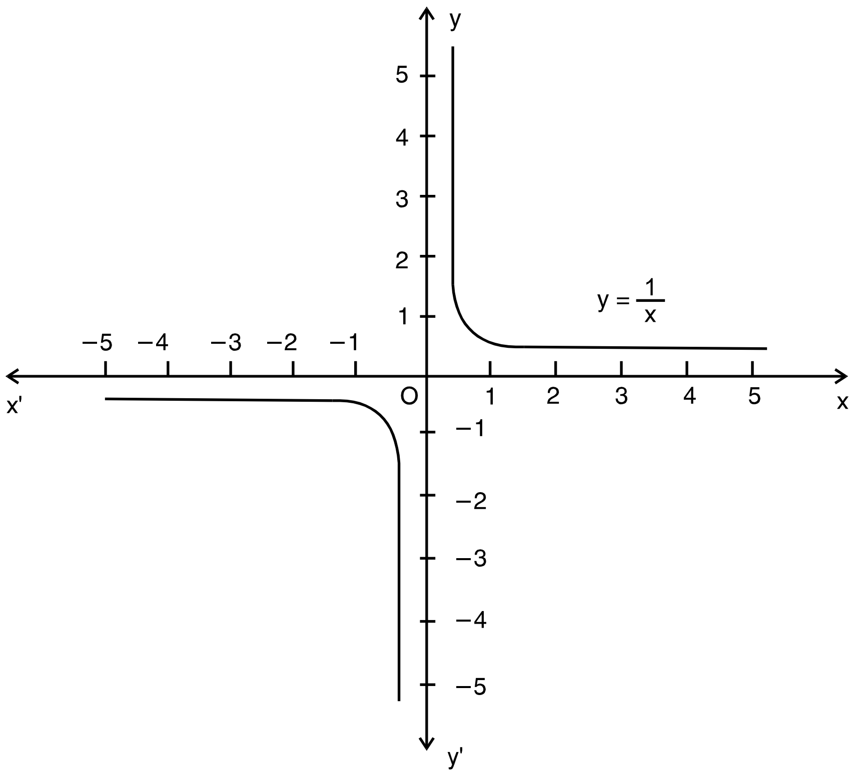


Fig. 25.2

Example 25.6 Evaluate : $\lim_{x \to 0} (|x| + |-x|)$

Solution : Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x \to 0^-} (|x|+|-x|) = \lim_{h \to 0} (|0-h|+|-(0-h)|)$$

$$= \lim_{h \to 0} (|-h|+|-(-h)|)$$

$$= \lim_{h \to 0} h+h = \lim_{h \to 0} 2h = 0 \qquad \text{...(i)}$$

and $$\lim_{x \to 0^+} (|x|+|-x|) = \lim_{h \to 0} (|0+h|+|-(0+h)|)$$

$$= \lim_{x \to 0} h+h = \lim_{h \to 0} 2h = 0 \qquad \text{...(ii)}$$

From (i) and (ii),

$$\lim_{x \to 0^-} (|x|+|-x|) = \lim_{h \to 0^+} [|x|+|-x|]$$

Thus, $$\lim_{h \to 0} [|x|+|-x|] = 0$$

Note : We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

Example 25.7 Find the vlaue of 'a' so that

$\lim_{x \to 1} f(x)$ exist, where $f(x) = \begin{cases} 3x+5, & x \leq 1 \\ 2x+a, & x > 1 \end{cases}$

Solution :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x+5) \qquad [\because f(x) = 3x+5 \text{ for } x \leq 1]$$

$$= \lim_{h \to 0} [3(1-h)+5]$$

$$= 3 + 5 = 8 \qquad \text{.....(i)}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x+a) \qquad [\because f(x) = 2x+a \text{ for } x > 1]$$

$$= \lim_{h \to 0} (2(1+h)+a)$$

$$= 2 + a \qquad \text{.....(ii)}$$

We are given that $\lim_{x \to 1} f(x)$ will exists provided

$$\Rightarrow \qquad \lim_{x \to 1^-} = \lim_{x \to 1^+} f(x)$$

$\therefore$ From (i) and (ii),

$$2 + a = 8$$

$\therefore$ or, $a = 6$

Example 25.8 If a function f (x) is defined as

$$f(x)=\begin{cases} x \quad , & 0 \le x < \frac{1}{2} \\ 0 \quad , & x=\frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\lim\limits_{x \to \frac{1}{2}} f(x)$.

Solution : Here

$$f(x)=\begin{cases} x \quad , & 0 \le x < \frac{1}{2} & \text{.....(i)} \\ 0 \quad , & x=\frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 & \text{.....(ii)} \end{cases}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}-h\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{2}-h\right) \quad \left[\because \frac{1}{2}-h < \frac{1}{2} \text{ and from (i), } f\left(\frac{1}{2}-h\right) = \frac{1}{2}-h\right]$$

$$= \frac{1}{2}-0=\frac{1}{2} \qquad \text{.....(iii)}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}+h\right)$$

$$= \lim_{h \to 0}\left[\left(\frac{1}{2}+h\right)-1\right] \left[\because \frac{1}{2}+h > \frac{1}{2} \text{ and from (ii), } f\left(\frac{1}{2}+h\right) = \left(\frac{1}{2}+h\right)-1\right]$$

$$= \frac{1}{2}+-1$$

$$= -\frac{1}{2} \qquad \text{.....(iv)}$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore \quad \lim\limits_{x \to \frac{1}{2}} f(x)$ does not exist.

CHECK YOUR PROGRESS 25.1

1. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 2}[2(x+3)+7]$ (b) $\lim\limits_{x\to 0}\left(x^2+3x+7\right)$ (c) $\lim\limits_{x\to 1}\left[(x+3)^2-16\right]$

(d) $\lim\limits_{x\to -1}\left[(x+1)^2+2\right]$ (e) $\lim\limits_{x\to 0}\left[(2x+1)^3-5\right]$ (f) $\lim\limits_{x\to 1}(3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim\limits_{x\to 5}\dfrac{x-5}{x+2}$ (b) $\lim\limits_{x\to 1}\dfrac{x+2}{x+1}$ (c) $\lim\limits_{x\to -1}\dfrac{3x+5}{x-10}$

(d) $\lim\limits_{x\to 0}\dfrac{px+q}{ax+b}$ (e) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$ (f) $\lim\limits_{x\to -5}\dfrac{x^2-25}{x+5}$

(g) $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x^2-3x+2}$ (h) $\lim\limits_{x\to \frac{1}{3}}\dfrac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$ (b) $\lim\limits_{x\to 0}\dfrac{x^3+7x}{x^2+2x}$ (c) $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}$

(d) $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim\limits_{x\to 3}\dfrac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$ (e) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim\limits_{x\to 0}\dfrac{2}{x}$, if it exists. (b) Find $\lim\limits_{x\to 2}\dfrac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim\limits_{x\to 0}\dfrac{x}{5-|x|}$ (b) $\lim\limits_{x\to 2}\dfrac{1}{|x+2|}$ (c) $\lim\limits_{x\to 2}\dfrac{1}{|x-2|}$

(d) Show that $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x)=\begin{cases}-2x+3, & x\le 1\\ 3x-5, & x>1\end{cases} \text{ as } x\to 1$$

(b) If $f(x)=\begin{cases}x^2, & x\le 1\\ 1, & x>1\end{cases}$, find $\lim\limits_{x\to 1}f(x)$

(c) Find $\lim\limits_{x \to 4} f(x)$ if it exists, given that $f(x) = \begin{cases} 4x+3, & x<4 \\ 3x+7, & x \geq 4 \end{cases}$

8. Find the value of 'a' such that $\lim\limits_{x \to 2} f(x)$ exists, where $f(x) = \begin{cases} ax+5, & x<2 \\ x-1, & x \geq 2 \end{cases}$

9. Let $f(x) = \begin{cases} x, & x<1 \\ 1, & x=1 \\ x^2, & x>1 \end{cases}$

Establish the existence of $\lim\limits_{x \to 1} f(x)$.

10. Find $\lim\limits_{x \to 2} f(x)$ if it exists, where

$$f(x) = \begin{cases} x-1, & x<2 \\ 1, & x=2 \\ x+1, & x>2 \end{cases}$$

25.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ where n is a positive integer.

Proof : $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x-a} = \lim\limits_{h \to 0} \dfrac{(a+h)^n - a^n}{a+h-a}$

$$= \lim_{h \to 0} \frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots\ldots + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots\ldots + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots\ldots + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + \ldots\ldots + 0$$

$$= n\,a^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

Note : However, the result is true for all n

(ii) Prove that (a) $\lim_{x \to 0} \sin x = 0$ **and (b)** $\lim_{x \to 0} \cos x = 1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0, A \to C$

or when $x \to 0, AC \to 0$

and $BC \to AB$, i.e., $BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

Thus we have

$$\lim_{x \to 0} \sin x = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = 1$$

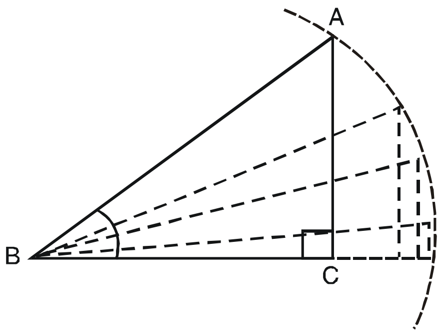


Fig. 25.3

(iii) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC \perp OX$.

Let $\angle AOX = x$ radians, where $0 < x < \frac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of ΔOBD.

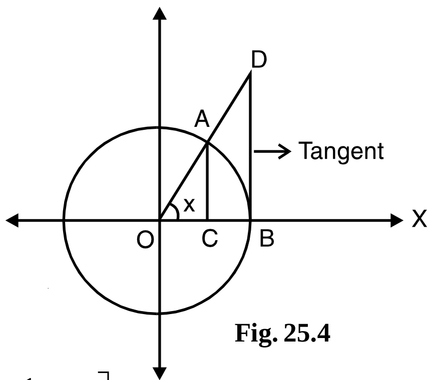


Fig. 25.4

$$\text{or} \ \frac{1}{2} OC \times AC < \frac{1}{2} x (1)^2 < \frac{1}{2} OB \times BD$$

$$\left[\because \text{area of triangle} = \frac{1}{2} \text{base} \times \text{height and area of sector} = \frac{1}{2} \theta \, r^2 \right]$$

$$\therefore \quad \frac{1}{2} \cos x \sin x < \frac{1}{2} x < \frac{1}{2} \cdot 1 \cdot \tan x$$

$$\left[\because \ \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB \right]$$

i.e., $\cos x < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ [Dividing throughout by $\frac{1}{2}$ sin x] or $$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or $$\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$$

i.e., $$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \frac{1}{\cos x}$$

or $$1 < \lim_{x\to 0} \frac{\sin x}{x} < 1 \qquad \left[\because \lim_{x\to 0} \cos x = 1 \text{ and } \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$$

Thus, $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + ...\infty\right]$$

$$= \left[1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + ...\infty\right]$$

$$\therefore \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to 0} \left[1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + ...\infty\right]$$

$$= \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ...\infty\right]$$

$= e$ (By definition)

Thus $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

(v) Prove that

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\log(1+x) = \lim_{x\to 0} \log(1+x)^{1/x}$$

$$= \log e \left(\text{Using } \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \right)$$

$$= 1$$

(vi) Prove that $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Proof : We know that $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)$

$$\therefore \quad e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots - 1\right) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$= \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)}{x} = \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$\therefore \quad \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$= 1 + 0 + 0 + \ldots\ldots = 1$$

Thus, $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Example 25.9 Find the value of $\displaystyle \lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

Solution : We know that

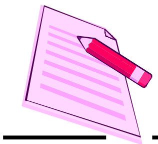
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \qquad \ldots\ldots(i)$$

$\therefore$ Putting $x = -x$ in (i), we get

$$\lim_{x\to 0} \frac{e^{-x} - 1}{-x} = 1 \qquad \ldots\ldots(ii)$$

Given limit can be written as

$$\lim_{x\to 0} \frac{e^x - 1 + 1 - e^{-x}}{x} \qquad \text{[Adding (i) and (ii)]}$$



$$= \lim_{x \to 0}\left[\frac{e^x - 1}{x} + \frac{1 - e^{-x}}{x}\right] = \lim_{x \to 0}\left[\frac{e^x - 1}{x} + \frac{e^{-x} - 1}{-x}\right]$$

$$= \lim_{x \to 0}\frac{e^x - 1}{x} + \lim_{x \to 0}\frac{e^{-x} - 1}{-x} = 1 + 1 = 2 \qquad \text{[Using (i) and (ii)]}$$

Thus $\quad \displaystyle\lim_{x \to 0}\frac{e^x - e^{-x}}{x} = 2$

Example 25.10 Evaluate : $\displaystyle\lim_{x \to 1}\frac{e^x - e}{x - 1}$.

Solution : Put $x = 1 + h$, where $h \to 0$

$$\lim_{x \to 1}\frac{e^x - e}{x - 1} = \lim_{h \to 0}\frac{e^{1+h} - e}{h} = \lim_{h \to 0}\frac{e^1 \cdot e^h - e}{h} = \lim_{h \to 0}\frac{e(e^h - 1)}{h}$$

$$= e\lim_{h \to 0}\frac{e^h - 1}{h} = e \times 1 = e.$$

Thus, $\quad \displaystyle\lim_{x \to 1}\frac{e^x - e}{x - 1} = e$

Example 25.11 Evaluate : $\displaystyle\lim_{x \to 0}\frac{\sin 3x}{x}$.

Solution : $\quad \displaystyle\lim_{x \to 0}\frac{\sin 3x}{x} = \lim_{x \to 0}\frac{\sin 3x}{3x}\cdot 3 \qquad$ [Multiplying and dividing by 3]

$$= 3\lim_{3x \to 0}\frac{\sin 3x}{3x} \qquad [\because \text{ when } x \to 0, 3x \to 0]$$

$$= 3.1 \qquad \left[\because \lim_{x \to 0}\frac{\sin x}{x} = 1\right]$$

$$= 3$$

Thus, $\quad \displaystyle\lim_{x \to 0}\frac{\sin 3x}{x} = 3$

Example 25.12 Evaluate $\displaystyle\lim_{x \to 0}\frac{1 - \cos x}{2x^2}$.

Solution : $\quad \displaystyle\lim_{x \to 0}\frac{1 - \cos x}{2x^2} = \lim_{x \to 0}\frac{2\sin^2\frac{x}{2}}{2x^2} \qquad \left[\begin{array}{l}\because \cos 2x = 1 - 2\sin^2 x, \\ \therefore 1 - \cos 2x = 2\sin^2 x \\ \text{or } 1 - \cos x = 2\sin^2\frac{x}{2}\end{array}\right]$

$$= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{2\times\frac{x}{2}}\right)^2 \quad \text{[Multiplying and dividing the denominator by 2]}$$

$$= \frac{1}{4}\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{4}\times 1 = \frac{1}{4}$$

$$\therefore \quad \lim_{x\to 0}\frac{1-\cos x}{2x^2} = \frac{1}{4}$$

Example 25.13 Find the value of $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$.

Solution : Put $x = \frac{\pi}{2}+h \qquad \because \quad$ when $x \to \frac{\pi}{2}, h \to 0$

$$\therefore \quad 2x = \pi + 2h$$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{[\pi-(\pi+2h)]^2}$$

$$= \lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2} = \lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{4h^2} = \frac{1}{2}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2 = \frac{1}{2}\times 1 = \frac{1}{2}$$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2} = \frac{1}{2}$$

$$= \frac{a}{b}$$

$$\therefore \quad \lim_{x\to 0}\frac{\sin ax}{\tan bx} = \frac{a}{b}$$

CHECK YOUR PROGRESS 25.2

1. Evaluate each of the following :

(a) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x}$ (b) $\lim\limits_{x \to 0} \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

2. Find the value of each of the following :

(a) $\lim\limits_{x \to 1} \dfrac{e^{-x} - e^{-1}}{x - 1}$ (b) $\lim\limits_{x \to 1} \dfrac{e - e^{x}}{x - 1}$

3. Evaluate the following :

(a) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{2x}$ (b) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{5x^2}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x}$

(d) $\lim\limits_{x \to 0} \dfrac{\sin a x}{\sin b x}$

4. Evaluate each of the following :

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (b) $\lim\limits_{x \to 0} \dfrac{1 - \cos 8x}{x}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin 2x(1 - \cos 2x)}{x^3}$

(d) $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{3 \tan^2 x}$

5. Find the values of the following :

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos a x}{1 - \cos b x}$ (b) $\lim\limits_{x \to 0} \dfrac{x^3 \cot x}{1 - \cos x}$ (c) $\lim\limits_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$

6. Evaluate each of the following :

(a) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (b) $\lim\limits_{x \to 1} \dfrac{\cos \frac{\pi}{2} x}{1 - x}$ (c) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

7. Evaluate the following :

(a) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\tan 3x}$ (b) $\lim\limits_{\theta \to 0} \dfrac{\tan 7\theta}{\sin 4\theta}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin 2x + \tan 3x}{4x - \tan 5x}$

25.6 CONTINUITY OF A FUNCTION AT A POINT

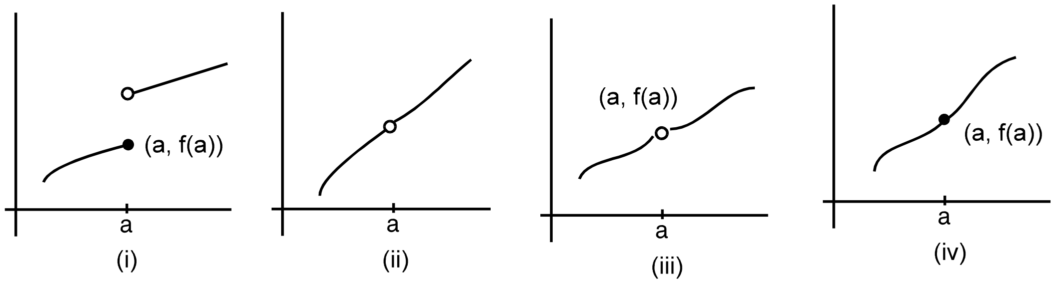


Fig. 25.5

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

Example 25.14 Examine the continuity of the function $f(x) = x - a$ at x = a.

Solution : $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} [(a+h) - a]$$

$$= 0 \qquad .....(i)$$

Also $\qquad f(a) = a - a = 0 \qquad .....(ii)$

From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

Thus $f(x)$ is continuous at x = a.

Example 25. 15 Show that $f(x) = c$ is continuous.

Solution : The domain of constant function c is R.Let 'a' be any arbitrary real number.

$$\therefore \qquad \lim_{x \to a} f(x) = c \text{ and } f(a) = c$$

$$\therefore \qquad \lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f(x)$ is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

Example 25.16 Show that $f(x) = cx + d$ is a continuous function.

Solution : The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to a} [c\,(a+h) + d]$$

$$= ca + d \qquad .....(i)$$

Also $\qquad f(a) = ca + d \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ $\quad f(x)$ is continuous at $x = a$

and $\quad$ since a is any arbitrary , $\quad f(x)$ is a continuous function.

Example 25.17 Prove that $f(x) = \sin\, x$ is a continuous function.

Solution : Let $f(x) = \sin x$

The domain of sin x is R. let 'a' be any arbitrary real number.

$$\therefore \quad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \sin(a+h)$$

$$= \lim_{h \to 0} [\sin a.\cos h + \cos a.\sin h]$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \left[\because \lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos\, x = 1\right]$$

$$= \sin a \qquad .....(i)$$

Also $f(a) = \sin a \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ sin x is continuous at x = a

$\because$ sin x is continuous at x = a and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

Definition :

1. A function f (x) is said to be continuous in an open inteval]a,b[if it is continuous at every point of]a,b[*.

2. A function f (x) is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval]a,b[and is continuous at the point a from the right and continuous at b from the left.

i.e. $\lim_{x \to a^+} f(x) = f(a)$

and $\lim_{x \to b^-} f(x) = f(b)$

* *In the open interval]a,b[we do not consider the end points a and b.*

CHECK YOUR PROGRESS 25.3

1. Examine the continuity of the functions given below :

   (a) $f(x) = x - 5$ at $x = 2$ (b) $f(x) = 2x + 7$ at $x = 0$

   (c) $f(x) = \frac{5}{3}x + 7$ at $x = 3$ (d) $f(x) = px + q$ at $x = -q$

2. Show that f (x)=2a+3b is continuous, where a and b are constants.

3. Show that $5x + 7$ is a continuous function

4. (a) Show that cos x is a continuous function.

   (b) Show that cot x is continuous at all points of its domain.

5. Find the value of the constants in the functions given below :

   (a) $f(x) = px - 5$ and $f(2) = 1$ such that f (x) is continuous at x = 2.

   (b) $f(x) = a + 5x$ and $f(0) = 4$ such that f (x) is continuous at x = 0.

   (c) $f(x) = 2x + 3b$ and $f(-2) = \frac{2}{3}$ such that f(x) is continuous at $x = -2$.

25.7 DISCONTINUITY OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example 25.18 Show that the function $f(x) = e^x$ is a continuous function.

Solution : Domain of e^x is R. Let $a \in R$. where 'a ' is arbitrary.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h) \text{, where h is a very small number.}$$

$$= \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h = e^a \lim_{h \to 0} e^h = e^a \times 1 \qquad \text{.....(i)}$$

$$= e^a \qquad \text{.....(ii)}$$

Also $f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

Example 25.19 By means of graph discuss the continuity of the function $f(x)=\dfrac{x^2-1}{x-1}$.

Solution : The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

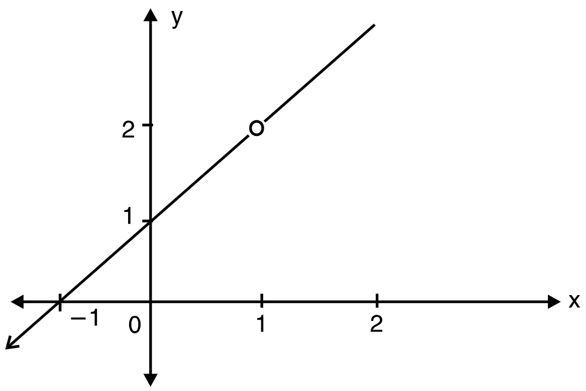


Fig. 25.6

CHECK YOUR PROGRESS 25.4

1. (a) Show that $f(x)=e^{5x}$ is a continuous function.

   (b) Show that $f(x)=e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x)=e^{3x+2}$ is a continuous function.

   (d) Show that $f(x)=e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) f (x) = x+1.

   (b) $f(x)=\dfrac{x+2}{x-2}$

   (c) $f(x)=\dfrac{x^2-9}{x+3}$

   (d) $f(x)=\dfrac{x^2-16}{x-4}$

25.8 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function f (x) = 4. Graph of the function f (x) = 4 is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

(ii) If a function is continuous then the constant multiple of that function is also continuous.

Consider the function $f(x)=\frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$$

$$=\lim_{h\to 0}\frac{7}{2}(a+h)$$

$$=\frac{7}{2}a \qquad .....(i)$$

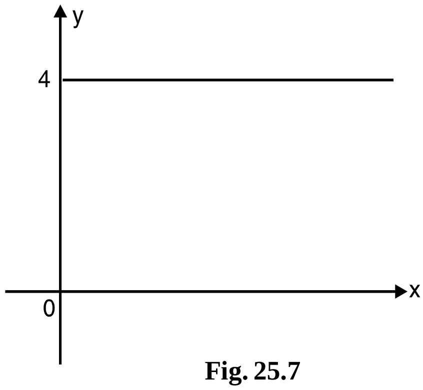


Fig. 25.7

Also $\qquad f(a)=\frac{7}{2}a \qquad .....(ii)$

$\therefore$ From (i) and (ii),

$$\lim_{x\to a} f(x)=f(a)$$

$\therefore$ $f(x)=\frac{7}{2}x$ is continuous at $x = a$.

As $\frac{7}{2}$ is constant, and x is continuous function at $x = a$, $\frac{7}{2}x$ is also a continuous function at $x = a$.

(iii) Consider the function $f(x)=x^2+2x$. We know that the function x^2 and 2x are continuous.

Now $\qquad \lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$

$$=\lim_{h\to 0}\left[(a+h)^2+2(a+h)\right]$$

$$=\lim_{h\to 0}\left[a^2+2ah+h^2+2a+2ah\right]$$

$$=a^2+2a \qquad .....(i)$$

Also $\qquad f(a)=a^2+2a \qquad .....(ii)$

$\therefore$ From (i) and (ii), $\lim_{x\to a} f(x)=f(a)$

$\therefore$ f(x) is continuous at $x = a$.

Thus we can say that if x^2 and 2x are two continuous functions at $x = a$ then $\left(x^2+2x\right)$ is also continuous at $x = a$.

(iv) Consider the function $f(x)=\left(x^2+1\right)(x+2)$. We know that $\left(x^2+1\right)$ and $(x+2)$ are two continuous functions.

Also $\qquad f(x)=\left(x^2+1\right)(x+2)$

$$=x^3+2x^2+x+2$$

Notes

As $x^3, 2x^2, x$ and 2 are continuous functions, therefore.

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $\left(x^2 + 1\right)$ and (x+2) are two continuous functions then $\left(x^2 + 1\right)(x + 2)$ is also a continuous function.

(v) Consider the function $f(x) = \dfrac{x^2 - 4}{x + 2}$ at $x = 2$. We know that $\left(x^2 - 4\right)$ is continuous at $x = 2$. Also $(x + 2)$ is continuous at $x = 2$.

Again
$$\lim_{x \to 2} \frac{x^2 - 4}{x + 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x + 2}$$
$$= \lim_{x \to 2} (x - 2)$$
$$= 2 - 2 = 0$$

Also
$$f(2) = \frac{(2)^2 - 4}{2 + 2}$$
$$= \frac{0}{4} = 0$$

$\therefore$ $\lim_{x \to 2} f(x) = f(2)$. Thus $f(x)$ is continuous at $x = 2$.

$\therefore$ If $x^2 - 4$ and $x + 2$ are two continuous functions at $x = 2$, then $\dfrac{x^2 - 4}{x + 2}$ is also continuous.

(vi) Consider the function $f(x) = |x - 2|$. The function can be written as

$$f(x) = \begin{cases} -(x - 2), & x < 2 \\ (x - 2), & x \geq 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h), \quad h > 0$$
$$= \lim_{h \to 0} \left[(2 - h) - 2\right]$$
$$= 2 - 2 = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h), \quad h > 0 \qquad \text{......(i)}$$
$$= \lim_{x \to 2} \left[(2 + h) - 2\right]$$
$$= 2 - 2 = 0 \qquad \text{.....(ii)}$$

Also
$$f(2) = (2 - 2) = 0 \qquad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim_{x \to 2} f(x) = f(2)$

Thus, $|x - 2|$ is continuous at $x = 2$.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) $f(x) \cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

Note : Every constant function is continuous.

25.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$ (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} (px + q) = pa + q \quad [= \text{value of } p\ x + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + a_n x^n$

where $a_0, a_1, a_2 .....a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1 x, a_2 x^2, ...a_n x^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., (x – 5) is also continuous.

Notes

$\therefore$ Using the property (iv), we can say that the function $\dfrac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at $x = 5$.

In general if $f(x) = \dfrac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and $q(x) \neq 0$, then f (x) is continuous if p (x) and q (x) both are continuous.

Example 25.20 Examine the continuity of the following function at $x = 2$.

$$f(x) = \begin{cases} 3x - 2 & \text{for } x < 2 \\ x + 2 & \text{for } x \geq 2 \end{cases}$$

Solution : Since f (x) is defined as the polynomial function $3x - 2$ on the left hand side of the point $x = 2$ and by another polynomial function $x + 2$ on the right hand side of $x = 2$, we shall find the left hand limit and right hand limit of the function at $x = 2$ separately.

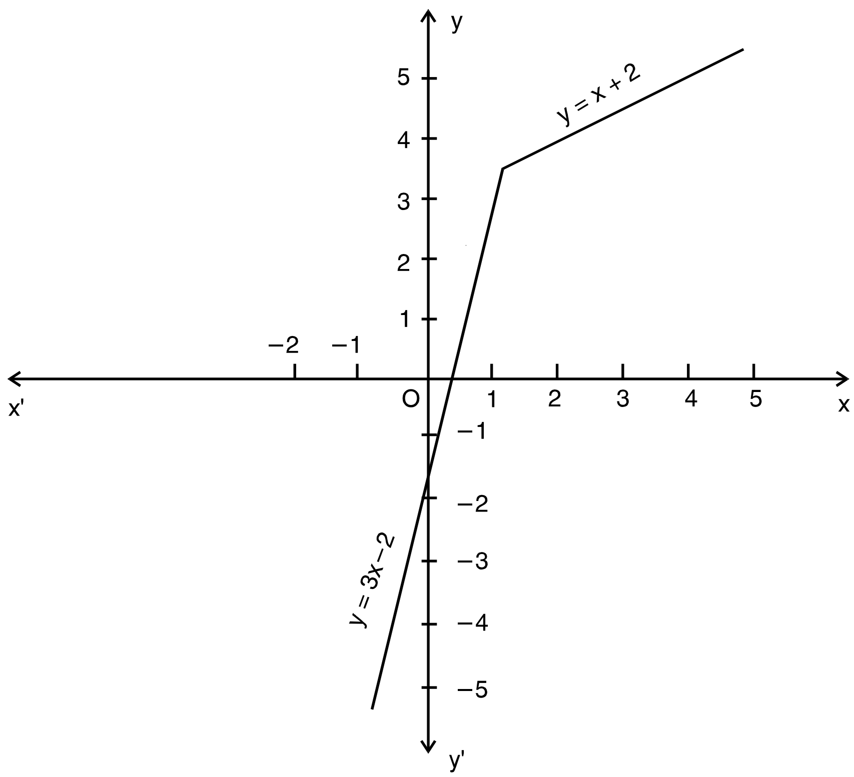


Fig. 25.8

Left hand limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} (3x - 2) = 3 \times 2 - 2 = 4$

Right hand limit at $x = 2$;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x+2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x \to 2} f(x) = 4$.

Also f (x) is defined by (x +2) at x = 2

$\therefore$ $$f(2) = 2 + 2 = 4.$$

Thus, $$\lim_{x \to 2} f(x) = f(2)$$

Hence f (x) is continuous at x = 2.

Example 25.21

(i) Draw the graph of f (x) = |x|.

(ii) Discusss the continuity of f (x) at x = 0.

Solution : We know that for $x \geq 0, |x| = x$ and for $x < 0, |x| = -x$. Hence f (x) can be written as.

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

(i) The graph of the function is given in Fig 20.9

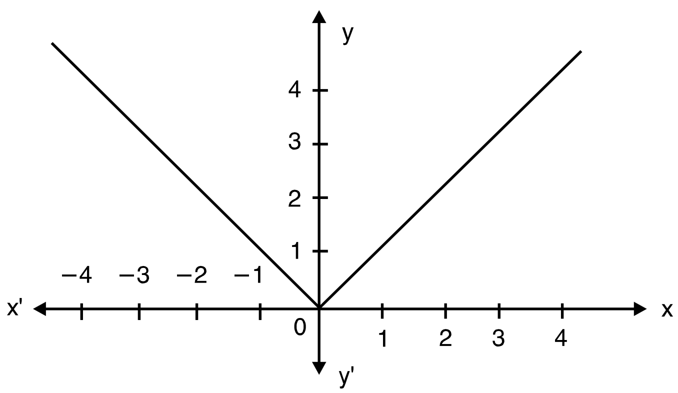


Fig. 25.9

(ii) Left hand limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

Thus, $$\lim_{x \to 0} f(x) = 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0} f(x) = f(0)$$

Hence the function $f(x)$ is continuous at $x = 0$.

Example 25.22 Examine the continuity of $f(x) = |x - b|$ at $x = b$.

Solution : We have $f(x) = |x - b|$. This function can be written as

$$f(x) = \begin{cases} -(x-b), & x < b \\ (x-b), & x \geq b \end{cases}$$

Left hand limit $= \lim\limits_{x \to b^-} f(x) = \lim\limits_{h \to 0} f(b-h)$

$$= \lim_{h \to 0}[-(b-h-b)]$$

$$= \lim_{h \to 0} h = 0 \qquad .....(i)$$

Right hand limit $= \lim\limits_{x \to b^+} f(x) = \lim\limits_{h \to 0} f(b+h)$

$$= \lim_{h \to 0}[(b+h)-b]$$

$$= \lim_{h \to 0} h = 0 \qquad .....(ii)$$

Also, $\quad f(b) = b - b = 0 \qquad .....(iii)$

From (i), (ii) and (iii), $\quad \lim\limits_{x \to b} f(x) = f(b)$

Thus, f(x) is continuous at $x = b$.

Example 25.23 If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

find whether f(x) is continuous at $x = 0$ or not.

Solution : Here $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \lim\limits_{x \to 0^-} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0-h)}{0-h} = \lim\limits_{h \to 0} \dfrac{-\sin 2h}{-h}$

$$= \lim_{h \to 0}\left(\frac{\sin 2h}{2h} \times \frac{2}{1}\right) = 1 \times 2 = 2 \qquad .....(i)$$

Right hand limit $= \lim\limits_{x \to 0^+} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0+h)}{0+h} = \lim\limits_{h \to 0} \dfrac{\sin 2h}{2h} \times \dfrac{2}{1}$

$$= 1 \times 2 = 2 \qquad (ii)$$

Also $\quad f(0) = 2 \quad$ (Given) $\quad$ (iii)

From (i) to (iii),

$$\lim_{x \to 0} f(x) = 2 = f(0)$$

Hence f(x) is continuous at x = 0.

Signum Function : The function f (x)=sgn(x) (read as signum x) is defined as

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Find the left hand limit and right hand limit of the function from its graph given below:

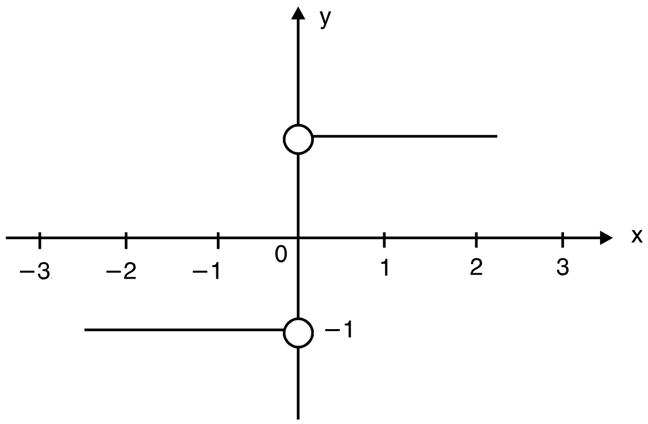


Fig. 25.11

From the graph, we see that as $x \to 0^+$, $f(x) \to 1$ and as $(x) \to 0^-$, $f(x) \to -1$

Hence, $\lim_{x \to 0^+} f(x) = 1$, $\lim_{x \to 0^-} f(x) = -1$

As these limits are not equal, $\lim_{x \to 0} f(x)$ does not exist. Hence f (x) is discontinuous at x =0.

Greatest Integer Function : Let us consider the function f (x)=[x] where [x] denotes the greatest integer less than or equal to x. Find whether f (x) is continuous at

(i) $x = \frac{1}{2}$ (ii) x =1

To solve this, let us take some arbitrary values of x say 1.3, 0.2,–0.2..... By the definition of greatest integer function,

$$[1.3] = 1, [1.99] = 1, [2] = 2, [0.2] = 0, [-0.2] = -1, [-3.1] = -4, \text{ etc.}$$

In general :

for $-3 \le x < -2$, $[x] = -3$

for $-2 \le x < -1$, $[x] = -2$

for $-1 \le x < 0$, $[x] = -1$

for $0 \le x < 1$, $[x] = 0$

for $1 \le x < 2$, $[x]=1$ and so on.

The graph of the function f (x) = [x] is given in Fig. 25.12

(i) From graph

$$\lim_{x \to \frac{1}{2}^-} f(x) = 0, \quad \lim_{x \to \frac{1}{2}^+} f(x) = 0,$$

$\therefore$ $$\lim_{x \to \frac{1}{2}} f(x) = 0$$

Also $$f\left(\frac{1}{2}\right) = [0.5] = 0$$

Thus $$\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at

$$x = \frac{1}{2}$$

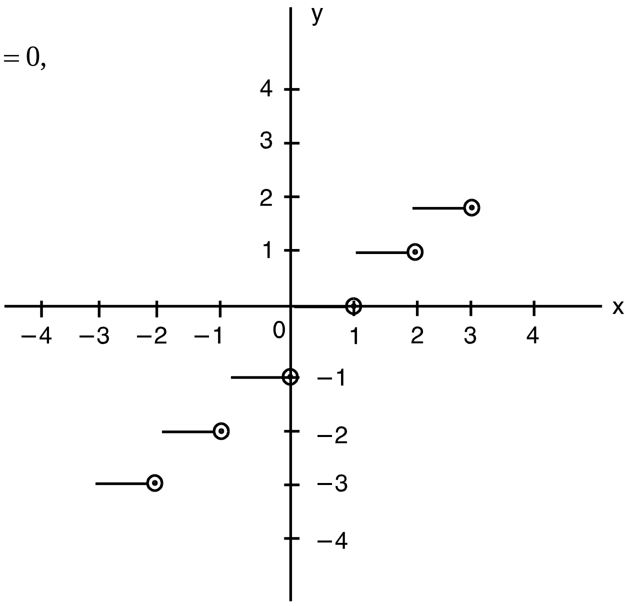


Fig. 25.12

(ii) $$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1$$

Thus $\lim_{x \to 1} f(x)$ does not exist.

Hence, f (x) is discontinuous at x = 1.

Note : The function f (x) = [x] is also known as Step Function.

Example 25.24 At what points is the function $\dfrac{x-1}{(x+4)(x-5)}$ continuous?

Solution : Here $f(x) = \dfrac{x-1}{(x+4)(x-5)}$

The function in the numerator i.e., x – 1 is continuous. The function in the demoninator is (x+4) (x – 5) which is also continuous.

But f (x) is not defined at the points – 4 and 5.

$\therefore$ The function f (x) is continuous at all points except – 4 and 5 at which it is not defined.

In other words, f (x) is continuous at all points of its domain.

CHECK YOUR PROGRESS 25.5

1. (a) If $f(x) = 2x + 1$, when $x \ne 1$ and f(x)=3 when x = 1, show that the function f (x) continuous at x =1.

(b) If $f(x)=\begin{cases}4x+3, & x\neq 2\\ 3x+5, & x=2\end{cases}$, find whether the function f is continuous at x = 2.

(c) Determine whether f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}4x+3, & x\leq 2\\ 8-x, & x>2\end{cases}$$

(d) Examine the continuity of f (x) at x = 1, where

$$f(x)=\begin{cases}x^2, & x\leq 1\\ x+5, & x>1\end{cases}$$

(e) Determine the values of k so that the function

$$f(x)=\begin{cases}kx^2, & x\leq 2\\ 3, & x>2\end{cases}\ \text{is con tinuous at } x=2.$$

2. Examine the continuity of the following functions :

(a) $f(x)=|x-2|$ at $x=2$ (b) $f(x)=|x+5|$ at $x=-5$

(c) $f(x)=|a-x|$ at $x=a$

(d) $f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$ at x = 2

(e) $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x\neq a\\ 1, & x=a\end{cases}$ at x = a

3. (a) If $f(x)=\begin{cases}\sin 4x, & x\neq 0\\ 2, & x=0\end{cases}$, at x = 0

(b) If $f(x)=\begin{cases}\dfrac{\sin 7x}{x}, & x\neq 0\\ 7, & x=0\end{cases}$, at x = 0

(c) For what value of a is the function

$$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & x\neq 0\\ a, & x=0\end{cases}\ \text{continuous at } x=0\ ?$$

4. (a) Show that the function f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, & \text{for } x\neq 2\\ 3, & \text{for } x=2\end{cases}$$

(b) Test the continuity of the function f (x) at x = 1, where

$$f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-1} & \text{for } x\neq 1\\ -2 & \text{for } x=1\end{cases}$$

(c) For what value of k is the following function continuous at x = 1?

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{when } x\neq 1\\ k & \text{when } x=1\end{cases}$$

(d) Discuss the continuity of the function f (x) at x = 2, when

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 7, & x=2\end{cases}$$

5. (a) If $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, find whether f is continuous at x = 0.

(b) Test the continuity of the function f (x) at the origin.

where $$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$$

6. Find whether the function f (x)=[x] is continuous at

(a) $x=\frac{4}{3}$ (b) x = 3 (c) $x=-1$ (d) $x=\frac{2}{3}$

7. At what points is the function f (x) continuous in each of the following cases ?

(a) $f(x)=\dfrac{x+2}{(x-1)(x-4)}$ (b) $f(x)=\dfrac{x-5}{(x+2)(x-3)}$ (c) $f(x)=\dfrac{x-3}{x^2+5x-6}$

(d) $f(x)=\dfrac{x^2+2x+5}{x^2-8x+16}$

LET US SUM UP

- If a function f(x) approaches l when x approches a, we say that l is the limit of f (x). Symbolically, it is written as

$$\lim_{x\to a} f(x)=\ell$$

- If $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$, then

(i) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k\ell$

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m$

(iii) $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = \ell m$

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\ell}{m}$, provided $\lim_{x \to a} g(x) \neq 0$

- **LIMIT OF IMPORTANT FUNCTIONS**

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \sin x = 0$

(iii) $\lim_{x \to 0} \cos x = 1$

(iv) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(v) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

(vi) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

(vii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=HB8CzZEd4xw

http://www.zweigmedia.com/RealWorld/Calcsumm3a.html

http://www.intuitive-calculus.com/limits-and-continuity.html

TERMINAL EXERCISE

Evaluate the following limits :

1. $\lim_{x \to 1} 5$

2. $\lim_{x \to 0} \sqrt{2}$

3. $\lim_{x \to 1} \frac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$

4. $\lim_{x \to -2} \frac{x^2 + 2x}{x^3 + x^2 - 2x}$

5. $\lim_{x \to 0} \frac{(x + k)^4 - x^4}{k(k + 2x)}$

6. $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$

7. $\lim_{x\to -1}\left[\frac{1}{x+1}+\frac{2}{x^2-1}\right]$

8. $\lim_{x\to 1}\frac{(2x-3)\sqrt{x}-1}{(2x+3)(x-1)}$

9. $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

10. $\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{x^2-1}\right]$

11. $\lim_{x\to \pi}\frac{\sin x}{\pi - x}$

12. $\lim_{x\to a}\frac{x^2-(a+1)x+a^2}{x^2-a^2}$

Find the left hand and right hand limits of the following functions :

13. $f(x)=\begin{cases}-2x+3 \text{ if } x\le 1\\ 3x-5 \quad \text{if } x>1\end{cases}$ as $x\to 1$

14. $f(x)=\frac{x^2-1}{|x+1|}$ as $x\to 1$

Evaluate the following limits :

15. $\lim_{x\to 1^-}\frac{|x+1|}{x+1}$

16. $\lim_{x\to 2^+}\frac{|x-2|}{x-2}$

17. $\lim_{x\to 2^-}\frac{x-2}{|x-2|}$

18. If $f(x)=\frac{(x+2)^2-4}{x}$, prove that $\lim_{x\to 0} f(x)=4$ though $f(0)$ is not defined.

19. Find k so that $\lim_{x\to 2} f(x)$ may exist where $f(x)=\begin{cases}5x+2, x\le 2\\ 2x+k, x>2\end{cases}$

20. Evaluate $\lim_{x\to 0}\frac{\sin 7x}{2x}$

21. Evauate $\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2}{x^2}\right]$

22. Evaluate $\lim_{x\to 0}\frac{1-\cos 3x}{x^2}$

23. Find the value of $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\sin 3x}$

24. Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

25. Evaluate $\lim\limits_{\theta \to 0} \dfrac{\sin 5\theta}{\tan 8\theta}$

Examine the continuity of the following :

26. $f(x)\begin{cases} 1+3x \text{ if } x > -1 \\ 2 \text{ if } x \le -1 \end{cases}$

at $x = -1$

27. $f(x) = \begin{cases} \dfrac{1}{x} - x, 0 < x < \dfrac{1}{2} \\ \dfrac{1}{2}, x = \dfrac{1}{2} \\ \dfrac{3}{2} - x, \dfrac{1}{2} < x < 1 \end{cases}$

at $x = \dfrac{1}{2}$

28. For what value of k, will the function

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} \text{ if } x \ne 4 \\ k \text{ if } x = 4 \end{cases}$$

be continuous at x = 4 ?

29. Determine the points of discontinuity, if any, of the following functions :

(a) $\dfrac{x^2 + 3}{x^2 + x + 1}$ (b) $\dfrac{4x^2 + 3x + 5}{x^2 - 2x + 1}$

(b) $\dfrac{x^2 + x + 1}{x^2 - 3x + 1}$ (d) $f(x) = \begin{cases} x^4 - 16, & x \ne 2 \\ 16, & x = 2 \end{cases}$

30. Show that the function $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos, & x \ne 0 \\ 2, & x = 0 \end{cases}$ is continuous at x = 0

31. Determine the value of 'a', so that the function f (x) defined by

$$f(x) = \begin{cases} \dfrac{a \cos x}{\pi - 2x}, x \ne \dfrac{\pi}{2} \\ 5, \ x = \dfrac{\pi}{2} \end{cases}$$

is continuous.

ANSWERS

CHECK YOUR PROGRESS 25.1

1. (a) 17 (b) 7 (c) 0 (d) 2
   (e) - 4 (f) 8

2. (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{2}{11}$ (d) $\frac{q}{b}$ (e) 6
   (f) -10 (g) 3 (h) 2

3. (a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{1}{2}$

4. (a) $\frac{1}{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2\sqrt{6}}$ (d) 2 (e) -1

5. (a) Does not exist (b) Does not exist

6. (a) 0 (b) $\frac{1}{4}$ (c) does not exist

7. (a) $1, -2$ (b) 1 (c) 19

8. $a = -2$

10. limit does not exist

CHECK YOUR PROGRESS 25.2

1. (a) 2 (b) $\frac{e^2 - 1}{e^2 + 1}$

2. (a) $-\frac{1}{e}$ (b) $-e$

3. (a) 2 (b) $\frac{1}{5}$ (c) 0 (d) $\frac{a}{b}$

4. (a) $\frac{1}{2}$ (b) 0 (c) 4 (d) $\frac{2}{3}$

5. (a) $\frac{a^2}{b^2}$ (b) 2 (c) $\frac{1}{2}$

6. (a) 1 (b) $\frac{\pi}{2}$ (c) 0

7. (a) $\frac{5}{3}$ (b) $\frac{7}{4}$ (c) -5

CHECK YOUR PROGRESS 25.3

1. (a) Continuous (b) Continuous
   (c) Continuous (d) Continuous

5. (a) p =3 (b) a = 4
   (c) $b = \frac{14}{9}$

CHECK YOUR PROGRESS 25.4

2. (a) Continuous
   (b) Discontinuous at x = 2
   (c) Discontinuous at $x = -3$
   (d) Discontinuous at x = 4

CHECK YOUR PROGRESS 25.5

1. (b) Continuous (c) Discontinuous
   (d) Discontinuous (e) $k = \frac{3}{4}$

2 (a) Continuous (c) Continuous,
  (d) Discontinuous (e) Discontinuous

3 (a) Discontinuous (b) Continuous (c) $\frac{5}{3}$

4 (b) Continuous (c) k = 2
  (d) Discontinuous

5. (a) Discontinuous (b) Discontinuous

6 (a) Continuous (b) Discontinuous
  (c) Discontinuous (d) Continuous

7. (a) All real number except 1 and 4
   (b) All real numbers except –2 and 3
   (c) All real number except –6 and 1
   (d) All real numbers except 4

TERMINAL EXERCISE

1. 5   2. $\sqrt{2}$ 3. 4   4. $-\frac{1}{3}$

5. $2x^2$ 6. 1   7. $-\frac{1}{2}$ 8. $-\frac{1}{10}$

9.	-8	10.	$\frac{1}{2}$
11.	1	12.	$\frac{a-1}{2a}$
13.	$1, -2$	14.	$-2, 2$
15.	-1	16.	1
17.	-1	19.	$k = 8$
20.	$\frac{7}{2}$	21.	1
22.	$\frac{9}{2}$	23.	1
24.	$\frac{2}{\pi}$	25.	$\frac{5}{8}$
26.	Discontinuous		
27.	Discontinuous		
28.	$k = 8$		
29.	(a) No	(b)	$x = 1$
	(c) $x = 1, x = 2$	(d)	$x = 2$
31.	10		