* Bisection method

$f(x)=0$

Find two values a and b such that $f(a)\times f(b)<0$

The roots of $f(x)$ lie between a and b

Find $x_1 = \frac{a+b}{2}$, then check for sign of x_1

→ Find the real root of the equation $x^3-x-4=0$ using bisection method correct upto 3 d.p.

$f(x) = x^3 - x - 4 = 0$

$f(0) = -4$

$f(1) = -4$

$f(2) = 2$

$x_1 = \frac{1+2}{2} = 1.5$

$f(1.5) = -2.125$

$x_2 = \frac{1.5+2}{2} = 1.75$

$f(1.75) = -0.3906$

$x_3 = \frac{1.75+2}{2} = 1.875$

$f(1.875) = 0.7167$

$x_4 = \frac{1.875+1.75}{2}$

$= 1.8125$

$f(1.8125) = 0.1418$

$x_5 = \frac{1.8125+1.75}{2}$

$= 1.78125$

$f(1.78125) = -0.1296$

$x_6 = \frac{1.78125+1.8125}{2}$

$= 1.7968$

$f(1.7968) = 4.8027\times 10^{-3}$

Date __/__/__

$$x_7 = \frac{1.7968 + 1.78125}{2}$$

$$= 1.789025$$

$f(1.789025) = -0.06305$

$$x_8 = \frac{1.789025 + 1.7968}{2}$$

$$= 1.7929$$

$f(1.7929) = -0.0295$

$$x_9 = \frac{1.7929 + 1.7968}{2}$$

$$= 1.79485$$

$f(1.79485) = -0.0127$

$$x_{10} = \frac{1.79485 + 1.7968}{2}$$

$$= 1.7958$$

$f(1.7958) = -4.43119 \times 10^{-3}$

$$x_{11} = \frac{1.7958 + 1.7968}{2}$$

$$= 1.7963$$

$f(1.7963) = -1.901 \times 10^{-4}$

$$x_{12} = \frac{1.7963 + 1.7968}{2}$$

$$= 1.79655$$

Since x_{11} is equal to x_{12} upto 3 d.p.
The approximate value of x is $\underline{\underline{1.796}}$

→ find the real root of $x^3 - 2x - 5 = 0$ upto 3 d.p.

$f(x) = x^3 - 2x - 5 = 0$

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

Root lies between 2 and 3

$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = 5.625$

$x_2 = \frac{2.5+2}{2} = 2.25$

$f(2.25) = 1.8906$

$x_3 = \frac{2.25+2}{2} = 2.125$

$f(2.125) = 0.3457$

$x_4 = \frac{2.125+2}{2} = 2.0625$

$f(2.0625) = -0.3513$

$x_5 = \frac{2.0625 + 2.125}{2}$

$= 2.0937$.

$f(2.0937) = -8.9416 \times 10^{-3}$

$$x_6 = \frac{2.0937 + 2.0625}{2}$$

$$= 2.0781$$

$$f(2.0781) = -0.1819$$

$$x_7 = \frac{2.0781 + 2.125}{2}$$

$$= 2.1015$$

$$f(2.1015) = 0.0784$$

$$x_8 = \frac{2.1015 + 2.0781}{2}$$

$$= 2.0898$$

$$f(2.0898) = -0.5289$$

$$x_9 = \frac{2.0898 + 2.1015}{2}$$

$$= 2.0956$$

$$f(2.0956) = 0.0122$$

$$x_{10} = \frac{2.0956 + 2.0898}{2}$$

$$= 2.0927$$

$$f(2.0927) = -0.0206$$

$$x_{11} = \frac{2.0927 + 2.0956}{2}$$

$$= 2.0941$$

$$f(2.0941) = -ve$$

$$x_{12} = \frac{2.0941 + 2.0956}{2}$$

$$= 2.0948$$

Date ___ /___ /______

Saathi

Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q) Find the root of the equation $x^3+x-1=0$ by newton raphson method correct upto 3 decimal places

$f(x) = x^3+x-1$

$f(0) = -1$

$f(1) = 1$

By newton raphson method

$$x_{n+1} = x_n - \frac{x_n^3+x_n-1}{3x_n^2+1}$$

$$x_{n+1} = \frac{3x_n^3+x_n-x_n^3-x_n+1}{3x_n^2+1}$$

$$x_{n+1} = \frac{2x_n^3+1}{3x_n^2+1}$$

Let $x_0 = 1$ be the initial approximation

$$x_1 = \frac{2(1)^3+1}{3(1)^2+1}$$

$x_1 = 0.75$

$$x_2 = \frac{2(0.75)^3+1}{3(0.75)^2+1}$$

$= 0.5673$

$$x_3 = \frac{2(0.5673)^3+1}{3(0.5673)^2+1}$$

$= 0.6945$

$$x_4 = \frac{2(0.6945)^3+1}{3(0.6945)^2+1}$$

$= 0.6824$

$x_5 = 0.6823$

Q) $f(x) = x^3 - 9x^2 + 18$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(0) = 18$
$f(1) = 10$
$f(2) = -10$

$$x_{n+1} = x_n - \frac{x_n^3 - 9x_n^2 + 18}{3x_n^2 - 18x_n}$$

$$x_{n+1} = \frac{3x_n^3 - 18x_n^2 - x_n^3 + 9x_n^2 - 18}{3x_n^2 - 18x_n}$$

$$x_{n+1} = \frac{2x_n^3 - 9x_n^2 - 18}{3x_n^2 - 18x_n}$$

let $x_0 = 1$ be the initial approximation

$$x_1 = \frac{2(1)^3 - 9(1)^2 - 18}{3(1)^2 - 18(1)}$$

$x_1 = 1.6666$

$x_2 = 1.5572$

$x_3 = 1.5548$

$x_4 = 1.5548$

Date __ / __ / __

Regular falsi method

The regular falsi method is given by

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Q) Find the approximate root of the equation $x^3 - 9x + 1 = 0$ correct upto 3 d.p.

$f(0) = 1$ — a

$f(1) = -7$ — b

let $n = 1$

$$x_1 = \frac{0(-7) - 1(1)}{-7 - 1}$$

$= 0.125$ — b

let $n = 2$

$f(0.125) = -0.1230$ — f(b)

$$x_2 = \frac{0(-0.1230) - 0.125(1)}{-0.1230 - 1}$$

$= 0.1113$ — b

$f(0.1113) = -ve$ — f(b)

$$x_3 = \frac{0(-3.2125 \times 10^{-4}) - 0.1113(1)}{-3.2125 \times 10^{-4} - (1)}$$

$x_3 = 0.1112$

Date ___ / ___ / ______

Iteration method (fixed point iteration)

i) $f(x)=0$ is given
and let x_0 be the initial root of $f(x)=0$

ii) Rewrite the equation $f(x)=0$ as
$$x = \phi(x)$$

iii) If $|\phi'(x)| < 1$

then equation $\phi(x) = x$ can be used as iteration formula
$$x_{n+1} = \phi(x_n) \quad , \quad n = 0, 1, 2, 3, 4 \dots$$

Q) Find a real root of the equation
$x^3 - 9x + 1 = 0$ using fixed point iteration

$f(2) = -9$
$f(3) = 1$

let $x_0 = 2.7$ be the initial root

case I

$x^3 - 9x + 1 = 0$

$9x = x^3 + 1$

$x = \frac{x^3+1}{9}$

$\phi(x) = \frac{x^3+1}{9}$

$\phi'(x) = \frac{1}{3}x^2$

$\phi'(2.7) = 2.43$

case II

$x^3 - 9x + 1 = 0$

$x(x^2 - 9) + 1 = 0$

$x = \frac{-1}{x^2-9} = \frac{1}{9-x^2}$

$\phi(x) = \frac{1}{9-x^2}$

$\phi'(x) = \frac{2x}{(9-x^2)^2}$

$\phi'(2.7) = 1.847$

Date __/__/__

convergence method
Engineering application

case III

$$x^3 - 9x + 1 = 0$$

$$x^3 = 9x - 1$$

$$x = (9x-1)^{1/3}$$

$$\phi(x) = (9x-1)^{1/3}$$

$$\phi'(x) = \frac{1}{3}(9x-1)^{-2/3} \times 9$$

$$\phi'(2.7) = 0.3677$$

Here $\phi(x) = (9x-1)^{1/3}$ satisfies

$$|\phi'(x_0) < 1|$$

let $x_{n+1} = \phi(x_n)$

$$x_{n+1} = (9x_n - 1)^{1/3}$$

Put $n=0$

$$x_1 = (9(2.7)-1)^{1/3}$$

$$x_1 = 2.856$$

Put $n=1$

$$x_2 = (9(2.856)-1)^{1/3}$$

$$x_2 = 2.912$$

Put $n=2$

$$x_3 = 2.932$$

Put $n=3$

$$x_4 = 2.939$$

Put $n=4$

$$x_5 = 2.942$$

Put $n=5$

$$x_6 = 2.942$$

Date ___ / ___ / ___

Saathi

Secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Q) $x^3 - x - 1 = 0$ correct upto 4 d.p.

$f(x) = x^3 - x - 1$

$f(0) = -1$

$f(1) = -1$

$f(2) = 5$

$x_0 = 1 \quad x_1 = 2$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_2 = \frac{1 f(2) - 2 f(1)}{f(2) - f(1)}$$

$$= \frac{5 - 2(-1)}{5 - (-1)}$$

$$= 1.6668$$

$$x_3 = \frac{2 f(1.6666) - 1.6666 f(2)}{f(1.6668) - f(2)}$$

$$x_3 = \frac{3.9249 - 8.333}{1.9624 - 5}$$

$$x_3 = 1.4511$$

Date ___/___/___

System of Linear equations 3

* cramers rule

Q) $x - y + z = 1$

$2x - y = 1$

$3x + 3y - 4z = 2$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 3 & 3 & -4 \end{vmatrix} = 5$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 2 & 3 & -4 \end{vmatrix} = 5$$

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & -4 \end{vmatrix} = 5$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 5$$

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ $z = \frac{D_z}{D}$

$= 1$ $= 1$ $= 1$

$(x, y, z) = (1, 1, 1)$

Date ___/___/_____

* Method of inversion

$2x - y + z = 1$

$x + 2y + 3z = 8$

$3x + y - 4z = 1$

Matrix form $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$

$D(A) = -40 \neq 0$

$\therefore$ inverse exists

$$Ax = B$$
$$A^{-1}(Ax) = A^{-1}B$$
$$(A^{-1}A)x = A^{-1}B$$
$$Ix = A^{-1}B$$
$$x = A^{-1}B$$

$$A^{-1} = \begin{bmatrix} 0.275 & 0.075 & 0.125 \\ -0.325 & 0.275 & 0.125 \\ 0.125 & 0.125 & -0.125 \end{bmatrix}$$

$$x = A^{-1}B$$
$$= \begin{bmatrix} 0.275 & 0.075 & 0.125 \\ -0.325 & 0.275 & 0.125 \\ 0.125 & 0.125 & -0.125 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Saathi

Date ___ /___ /_____

* Gauss Elimination method (Direct method)

$x+y+z=6$
$2x-3y+3z=5$
$3x+2y-z=4$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 1 \\ 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -14 \end{bmatrix}$$

$5R_3 - R_2 = R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -63 \end{bmatrix}$$

$x+y+z=6.$
$-5y+1z=-7$
$-21z=-63$
$z=3$

$-5y+1z=-7$
$-5y+3=-7$
$-5y=-10$
$y=2$

$x+2+3=6$
$x+5=6$
$x=1$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Gauss Jordan Elimination (Direct method)

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + y + 3z = 14$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 2 & 2 & 4 & 18 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -4 & 0 & -8 \\ 0 & -8 & -3 & -25 \end{array}\right]$$

$R_2 = R_2 / -4$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & 1 & 0 & -8 \\ 0 & -8 & -3 & -25 \end{array}\right]$$

$R_1 = R_1 - 3R_2$
$R_3 = R_3 + 8R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 37 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -3 & -89 \end{array}\right]$$

$R_3 = R_3 / -3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 37 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & \end{array}\right]$$

Q) $6x - y - z = 19$

$3x + 4y + z = 26$

$x + 2y + 6z = 22$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 19 \\ 3 & 4 & 1 & 26 \\ 6 & -1 & -1 & 22 \end{array}\right]$$

$R_2 = R_2 - 3R_1$
$R_3 = R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 19 \\ 0 & -2 & -17 & -31 \\ 0 & -13 & -37 & -92 \end{array}\right]$$

$R_2 = R_2 / -2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 19 \\ 0 & 1 & 8.5 & 15.5 \\ 0 & -13 & -37 & -92 \end{array}\right]$$

$R_1 = R_1 - 2R_2$
$R_3 = R_3 + 13R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -11 & -12 \\ 0 & 1 & 8.5 & 15.5 \\ 0 & 0 & 73.5 & 109.5 \end{array}\right]$$

Date __ / __ / ____

Gauss Jacobi's method (Iterative method)

Q) $15x + 2y + z = 18$
$2x + 20y - 3z = 19$
$3x - 6y + 25z = 22$

$$x = \frac{18 - 2y - z}{15}$$

$$y = \frac{19 + 3z - 2x}{20}$$

$$z = \frac{22 - 3x + 6y}{25}$$

First iteration
let $x = 0$, $y = 0$, $z = 0$ be the initial approxima-

$$x_1 = \frac{18}{15}$$

$$y_1 = \frac{19}{20}$$

$$z_1 = \frac{22}{25}$$

Second iteration

$x_2 = 1.0146$

$y_2 = 0.962$

$z_2 = 0.964$

Date ___ /___ /______

Saathi

Third iteration

$x_3 = 1.0074$

$y_3 = 0.9931$

$z_3 = 0.9891$

Fourth iteration

$x_4 = 1.0016$

$y_4 = 0.9976$

$z_4 = 0.9974$

Fifth iteration

$x_5 = 1.0004$

$y_5 = 0.9994$

$z_5 = 0.9992$

Sixth iteration

$x_6 = 1.0001$

$y_6 = 0.9998$

$z_6 = 0.9997$

Gauss seidel method (Iterative method)

Q) $28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

$$x = \frac{32 - 4y + z}{28}$$

$$y = \frac{35 - 4z - 2x}{17}$$

$$z = \frac{24 - x - 3y}{10}$$

First iteration

Put $y = 0$ & $z = 0$

$$x_1 = \frac{32}{28} = 1.1428$$

Put $x = 1.1428$ $z = 0$

$y_1 = 1.9243$

Put $x = 1.1428$ $y = 1.9243$

$z_1 = 1.7084$

Second iteration

$x_2 = 0.9289$

$y_2 = 1.5475$

$z_2 = 1.8428$

Date ___/___/___

Third iteration

$x_3 = 0.9876$

$y_3 = 1.5090$

$z_3 = 1.8485$

Fourth iteration

$x_4 = 0.9933$

$y_4 = 1.5070$

$z_4 = 1.8485$

Fifth iteration

$x_5 = 0.9935$

$y_5 = 1.507$

$z_5 = 1.8485$

Date ___ / ___ / ______

CURVE FITTING

4

Let $y = f(x)$ be a given function, 'h' be any fixed value and let $x = x_0, x_1, x_2, x_3 \ldots x_n$ Such that

$x_1 = x_0 + h, \quad x_2 = x_1 + h \quad \ldots$

$$x_n = x_{n-1} + h$$

h = difference between interval

$x_1 = x_0 + h$
$x_2 = x_0 + 2h$
$x_3 = x_0 + 3h$

$$x_n = x_0 + nh$$

Difference operators

1) Shift operator: denoted by 'E'
$E f(x)$

$E^1 f(x) = f(x+h)$
$E^2 f(x) = f(x+2h)$
$E^{1/2} f(x) = f(x + h/2)$

2) Forward difference operator: denoted by Δ

Date ___/___/___

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$
$$= \Delta(f(x+h) - f(x))$$
$$= f(x+2h) - f(x+h) - f(x+h) - f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x))$$
$$= \Delta(f(x+2h) - 2f(x+h) + f(x))$$

$$= f(x+3h) - 2f(x+2h) + f(x+h) - [f(x+2h) - 2f(x+h) + f(x)]$$

$$= f(x+3h) - 2f(x+2h) + f(x+h) - f(x+2h) + 2f(x+h) - f(x)$$

$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

3) Backward difference operator ∇ - nebla

$$\nabla f(x) = f(x) - f(x-h)$$
$$\nabla^2 f(x) = \nabla(\nabla f(x))$$
$$= \nabla(f(x) - f(x-h))$$
$$= f(x) - f(x-h) - [f(x-h) - f(x-2h)]$$
$$= f(x) - f(x-h) - f(x-h) + f(x-2h)$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

Date ___ / ___ / ____

4) central difference operator. 'δ' - del

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Relation between difference operators

1) Prove that $\Delta = E - 1$

$\Delta f(x) = f(x+h) - f(x)$

$= E^1 f(x) - f(x)$

$= f(x)\,[E^1 - 1]$

$\Delta = E - 1$

2) Prove that $\nabla = 1 - E^{-1}$

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= f(x)\,[1 - E^{-1}]$

$\nabla = 1 - E^{-1}$

3) Prove that $\delta = E^{1/2} - E^{-1/2}$

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$= f(x)\,[E^{1/2} - E^{-1/2}]$

$\delta = E^{1/2} - E^{-1/2}$

Saathi

Date ___/___/___

Interpolation

The method of finding the value of y corresponding to x which lies between x_0, x_n

Newton's forward difference interpolation formula

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

where $p = \frac{x - x_0}{h}$

Q)

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

y at x = 1.2

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		1.33		
			-0.19	
1.4	4.82			
		1.14		-0.41
			-0.6	
1.8	5.96			
		0.54		
2.2	6.5			

$p = \frac{1.2 - 1}{0.4} = 0.5$ $\quad \frac{0.5(0.5-1)}{2!} \times -0.19$

$y = 3.49 + 0.5 \times 1.33 + \frac{0.5(0.5-1)(0.5-2)}{3!}(-0.41)$

$y = 4.153125$

Date __/__/__

Backward difference interpolation formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

a) $f(42)$

	x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	20	354				
			-22			
x_1	25	332		-19		
			-41		29	
x_2	30	291		10		-37
			-31		-8	
x_3	35	260		2		8
			-29		0	
x_4	40	231		2		
			-27			
x_5	45	204				

$$p = \frac{x - x_0}{h} = \frac{42 - 45}{5} = -0.6$$

$$y = 204 + (-0.6 \times -27) + \frac{(-0.6)(-0.6+1)}{2!} \times 2 + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!} \times 0 + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times 8 + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!} \times$$

$$= 218.66$$

Date ___/___/___

* Lagrange's Interpolation formula

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

* Lagrange's Reverse interpolation

x y

Find values of x

Method Same as Interpolation formula

Date __/__/____

Curve fitting

x	y	
x_0	y_0	(x_0, y_0)
x_1	y_1	(x_1, y_1)
x_2	y_2	(x_2, y_2)
x_3	y_3	(x_3, y_3)
x_4	y_4	(x_4, y_4)

Q) Fit a curve $y = a + bx$ to the following data

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

The normal equations

$$\Sigma y = na + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

x	y	Σx	Σxy	Σx^2
1	2.4		2.4	1
2	3		6	4
3	3.6		10.8	9
4	4		16	16
6	5		30	36
8	6		48	64
$\Sigma x = 24$	24		113.2	130

$$24 = 6a + 24b$$
$$113.2 = 24a + 130b$$

$a = 1.9765$

$b = 0.5059$

$$y = 1.9765 + 0.5059x$$

Saathi

Date ___/___/___

Q) Fit a parabola $y = a + bx + cx^2$ to the following data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

$n = 5$

The normal equations

$\Sigma y = na + b\Sigma x + c\Sigma x^2$
$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$
$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

x	y	x^2	xy	x^3	x^2y	x^4
0	1	0	0	0	0	0
1	1.8	1	1.8	1	1.8	1
2	1.3	4	2.6	8	5.2	16
3	2.5	9	7.5	27	22.5	81
4	6.3	16	25.2	64	100.8	256
10	12.9	30	37.1	100	130.3	354

$12.9 = 5a + 10b + 30c$
$37.1 = 10a + 30b + 100c$
$130.3 = 30a + 100b + 354c$

$a = 1.42$ $b = -1.07$ $c = 0.55$

$y = 1.42 + -1.07x + 0.55x^2$

Date ___ /___ /______

Q) Fit a curve $y = ax^b$ to the following data

x	50	450	750	1200	4400	4800	5300
y	28	30	32	36	51	58	69

$y = ax^b$

Taking log to the base 10 on both sides

$$\log y = \log a + b \log x$$
$$Y = A + bX$$

$$\Sigma Y = na + b\Sigma X$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

x	$X = \log x$	$Y = \log y$	XY	X^2
50	1.6989	1.4471	2.4584	2.8862
450	2.6532	1.4771	3.9190	7.039
750	2.8751	1.5051	4.3273	8.2662
1200	3.0791	1.5563	4.7920	9.4801
4400	3.6434	1.7075	6.2211	13.27
4800	3.6812	1.7634	6.4914	13.551
5300	3.7242	1.8388	6.8480	13.868
	21.3551	11.2953	35.0572	68.367

$$11.2953 = 7a + 21.3551b$$
$$35.0572 = 21.3551a + 68.3677b$$

$a = 1.0466$ $\qquad b = 0.1858$

$$y = 1.0466\, x^{0.1858}$$

Date ___ / ___ / ______

* Newton's Divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$$

Q) Find $f'(0)$ & $f''(0)$ from the following table

x	0	1	2	3	4	5
$f(x)$	4	8	15	7	6	2

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		4				
1	8		3			
		7		-18		
2	15		-15		40	
		-8		22		-72
3	7		7		-32	
		-1		-10		
4	6		-3			
		-4				
5	2					

$$\frac{1}{1}\left[4 - \frac{1}{2}(3) + \frac{1}{3}(-18) - \frac{1}{4}(40) + \frac{1}{5}(-72)\right]$$

$$= -27.9$$

Date ___/___/_____

	x	y				
x_0	4	48				
			52			
				15		
x_1	5	100			1	
			97			0
				21		
x_2	7	294			1	0
			202			
				27		0
x_3	10	900			1	
			310			
				33		
x_4	11	1210				
			409			
x_5	13	2028				

$$y = 48 + (8-4)\,52 + (8-4)(8-5)\,15 + (8-4)(8-5)(8-7)\,1$$

$$y = \underline{\underline{448}}$$

$$y = 48 + (15-4)\,52 + (15-4)(15-5)\,15 + (15-4)(15-5)(15-7)(1)$$

$$y = \underline{\underline{3150}}$$

Date ___ /___ /______

Solutions of ordinary differential equation – 5

$y_1 = \frac{dy}{dx} = f(x, y)$

$y(x_0) = y_0 \rightarrow$ Initial solution of differential equation

When $x = x_0$ then $y = y_0$

Taylor series method

$$y = y_0 + (x - x_0)(y_1)_0 + \frac{(x - x_0)^2}{2!}(y_2)_0 + \frac{(x - x_0)^3}{3!}(y_3)_0$$
$$+ \frac{(x - x_0)^4}{4!}(y_4)_0 + \dots$$

Where $(y_1)_0 = \frac{dy}{dx}$ at $x = x_0$

$(y_2)_0 = \frac{d^2y}{dx^2}$ at $x = x_0$

$(y_3)_0 = \frac{d^3y}{dx^3}$ at $x = x_0$

Date __/__/__

Euler's method

consider the O.D.E.

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Euler's formula is given by

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where $h = \frac{x_n - x_0}{n}$

n - take n here as 4 or 5

Greater the value of n, more precise the answer

Q) Find y(2.2) using euler's method from the eq^n

$$\frac{dy}{dx} = -xy^2 \qquad y(2) = 1$$

$$x_0 = 2 \qquad y_0 = 1 \qquad x_n = 2.2$$

$$h = \frac{2.2 - 2}{4} = 0.05$$

By Eulers method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_1 = y_0 + 0.05f(2, 1)$$

$$y_1 = 0.90$$

Date ___ / ___ / ___

$x_0 = 2$ $y_0 = 1.$

$x_1 = 2.05$ $y_1 = 0.90$

$x_2 = 2.1$ $y_2 = 0.8169$

$x_3 = 2.15$ $y_3 = 0.7468$

$x_4 = 2.2$

$y_2 = y_1 + 0.05 f(2.05, 0.90)$

$= \underline{\underline{0.8169}}$

$y_3 = y_2 + 0.05 f(2.1, 0.8169)$

$= 0.7468$

$y_4 = y_3 + 0.05 f(2.15, 0.7468)$

$= 0.6868$

The value of y corresponding to $x = 2.2$ is 0.6868.

Date ___/___/___

R.K. method of 4^{th} order

consider $\frac{dy}{dx} = f(x, y)$.

given $y(x_0) = y_0$

$$y_{n+1} = y_n + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

where

$K_1 = hf(x_n, y_n)$

$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$

$K_4 = hf(x_n + h, y_n + K_3)$

Q) y at $x = 0.1, 0.2$ $\frac{dy}{dx} = 3e^x + 2y$

$y(0) = 0$

$y_0 = 0$ $x_0 = 0$ $h = 0.1$

$K_1 = 0.1f(x_0, y_0)$ $n = 0$
$= 0.1f(0, 0)$
$= 0.3$

$K_2 = 0.1f(0.05, 0.15)$
$= 0.3453$

Date __/__/____

$K_3 = 0.1f(0.05, 0.17265)$
$= 0.3499$

$K_4 = 0.1f(0.1, 0.3499)$
$= 0.4015$

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 0 + \frac{1}{6}[0.3 + 2(0.3453) + 2(0.3499) + 0.4015]$$

$= \underline{\underline{0.34865}}$ at $x = 0.1$

$K_1 = 0.1f(0.1, 0.34865)$
$= 0.4012$

$K_2 = 0.1f(0.15, 0.54925)$
$= 0.4584$

$K_3 = 0.1f(0.15, 0.5778)$
$= 0.4641$

$K_4 = 0.1f(0.2, 0.8127)$
$= 0.5289$

$$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 0.34865 + \frac{1}{6}[0.4012 + 2(0.4584) + 2(0.4641) + 0.5289]$$

$= \underline{\underline{0.8111}}$ at $x = 0.2$

Date ___/___/_____

R.K. method of second order

consider

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$$

where

$K_1 = h \cdot f(x_n, y_n)$

$K_2 = h \cdot f(x_n + h, y_n + K_1)$

Q) $y(0.1)$ & $y(0.2)$

$\frac{dy}{dx} = y - x$ $\quad y(0) = 2$

$x_0 = 0$ $\quad y_0 = 2$ $\quad h = 0.1$

$f(x,y) = y - x$

$K_1 = 0.1 f(0, 2)$
$= 0.2$

$K_2 = 0.1 f(0.1, 2.2)$
$= 0.21$

$y_1 = 2 + \frac{1}{2}[0.2 + 0.21]$

$= 2.205$ at $x = 0.1$

Date ___ /___ /___

$$K_1 = 0.1f(0.1, 2.205)$$
$$= 0.2105$$

$$K_2 = 0.1f(0.2, 2.4155)$$

$$= 0.2215$$

$$y_2 = y_1 + \frac{1}{2}[K_1 + K_2]$$

$$= 2.205 + \frac{1}{2}[0.2105 + 0.2215]$$

$$= \underline{\underline{2.421}} \quad \text{at} \quad x = 0.2$$

Date ___ / ___ / ___

Modified Eulers method

consider $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$

Eulers method

$y_{n+1} = y_n + hf(x_n, y_n)$

Modified Eulers method

$y_{n+1}^{(m)} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-$

Q) $y(0.2)$ & $y(0.4)$ given

$\frac{dy}{dx} = e^x + y$

$y(0) = 0$ $h = 0.1$

$x_0 = 0$ $y_0 = 0$

By Eulers method

$y_1 = 0 + 0.1f(0,0)$

$= 0.1$

$x_1 = 0.1$ $y_1 = 0.1$

Date ___/___/___

Saathi

By modified Eulers

$y_1^{(1)} = y_0 + \frac{0.1}{2}[f(0,0) + f(0.1, 0.1)]$

$y_1^{(1)} = 0.1102$

$y_1^{(2)} = 0.1107$

$y_1^{(3)} = 0.1107$

At $x = 0.1$ $y = 0.1107$

By Eulers method

$y_2 = y_1 + 0.1 f(0.1, 0.1107)$

$= 0.2322$

$x_2 = 0.2$ $y_2 = 0.2322$

By modified Eulers

$y_2^{(1)} = 0.\cancel{2322}^{1107} + \frac{0.1}{2}[f(\cancel{0.2}^{0.1}, 0.\cancel{2322}^{1107}) + f(0.2, 0.2322)]$

$= 0.\cancel{3656}$ 0.2441

$y_2^{(2)} = 0.\cancel{3723}$ 0.2447

$y_2^{(3)} = \cancel{0.3726}$ 0.2447

$y_2^{(4)} = 0.3726$

Date ___/___/___

Milne's Predictor and corrector method

$\frac{dy}{dx} = f(x, y)$

x_0	y_0	$f(x_0, y_0) = f_0$
x_1	y_1	$f(x_1, y_1) = f_1$
x_2	y_2	$f(x_2, y_2) = f_2$
x_3	y_3	$f(x_3, y_3) = f_3$

$$y_4^p = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$y_4^p =$ ____ corresponding to x^4

$x_4 \quad y_4^p \rightarrow f(x_4, y_4^p) \quad [f_4]$

$$y_4^c = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$y_4^c =$ ____ final answer

Date __/__/__

Q) $\frac{dy}{dx} = x^2 + \frac{y}{2}$ $y(1) = 2$, $y(1.1) = 2.2156$

$y(1.2) = 2.4549$ $y(1.3) = 2.7514$

find y at $x = 1.4$

x	y	f	
1	2	2	$- f_0$
1.1	2.2156	2.3178	$- f_1$
1.2	2.4549	2.6674	$- f_2$
1.3	2.7514	3.0657	$- f_3$

$y_4^P = 2 + \frac{4(0.1)}{3} [2(2.3178) - 2.6674 + 2(3.0657)]$

$y_4^P = 3.0799$

$x = 1.4$ $y_4^P = 3.0799$ $3.4999 - f_4$

$y_4^C = 2.4549 + \frac{0.1}{3} [2.6674 + 4(3.0657) + 3.499$

$y_4^C = \underline{\underline{3.0692}}$

Numerical differentiation and integration

* Derivatives using forward difference formula

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \dots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\right]$$

* Derivatives using backward difference formula

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{132}{180}\nabla^6 y_n + \dots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\right]$$

* Simpsons $\frac{3}{8}$ th rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)\right.$$
$$\left.+\,2(y_3+y_6+y_9+\dots)\right]$$
$$= \frac{3h}{8}\left[X+3T+2R\right]$$

X = Sum of the extreme ordinates

T = Sum of the remaining ordinates

R: Sum of the multiples of 3

n = multiples of 3
= 6, 9, 12, 15

Q) $\int_0^2 e^{x^2}\,dx$ using trapezoidal rule $n=10$

$$h = \frac{2-0}{10} = 0.2$$

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
y	1	1.0408	1.1735	1.4333	1.8964	2.7182	4.2206	7.0993	12.9358	25.5337	54.5981

$$\frac{0.2}{2}\left[(1+54.5981)+2(1.0408+1.1735+1.4333+1.8964+2.7182+4.2206+7.0993+12.9358+25.5337)\right]$$

$$= \frac{0.2}{2}\left[(1+54.5981)+2(58.0516)\right]$$

$$= \underline{\underline{17.17013}}$$

Date __/__/____

Boole's rule

$$\int_a^b f(x)\,dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8\right.$$

In boole's rule $n =$ multiple of 4

Weddle's rule.

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6\right.$$

In weddle's rule $n =$ multiple of 6.

→ Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ using weddle's rule

$$h = \frac{6-0}{6} = 1$$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.03846	0.027[illegible]
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\frac{3(1)}{10}\left[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.03846) + 2(0.02702)\right]$$

$= 1.~~0815~~$ 1.38154

Date __/__/____

* Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[x + 2R\right]$$

Where x = Sum of extreme ordinates
R = Sum of remaining ordinates

* Simpson's $\frac{1}{3}$rd rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_{n-2}) + 4(y_1 + y_3 + y_5 + \dots\ y_{n-1})\right]$$

$$= \frac{h}{3}\left[x + 2E + 4O\right]$$

Where x = Sum of the extreme ordinates
E = Sum of the even ordinates
O = Sum of odd ordinates

n = even number
4, 6, 8, 10

Date ___ / ___ / ___

Saathi

* Newton's Divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$$

Q) Find $f'(0)$ & $f''(0)$ from the following table

x	0	1	2	3	4	5
$f(x)$	4	8	15	7	6	2

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		4				
1	8		3			
		7		-18		
2	15		-15		40	
		-8		22		-72
3	7		7		-32	
		-1		-10		
4	6		-3			
		-4				
5	2					

$$\frac{1}{1}\left[4 - \frac{1}{2}(3) + \frac{1}{3}(-18) - \frac{1}{4}(40) + \frac{1}{5}(-72)\right]$$

$= -27.9$