Reinforcement Learning: Basics and DQN

Changyou Chen

Department of Computer Science and Engineering
Universitpy at Buffalo, SUNY
changyou@buffalo.edu

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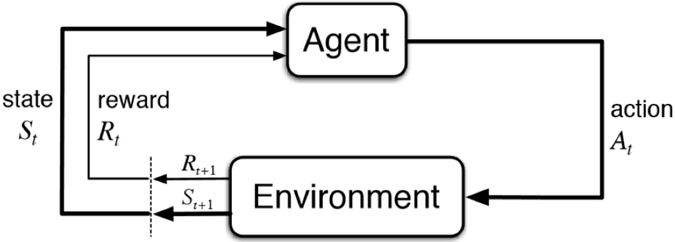
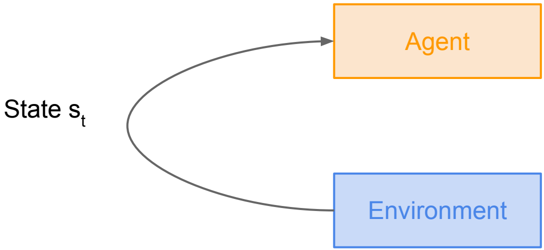
Reinforcement Learning

- Problems involving an agent interacting with an environment, which provides numeric reward signals.
- **Goal**: Learn how to take actions in order to maximize reward.

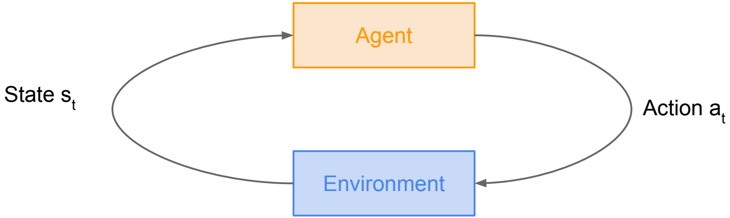


Reinforcement Learning

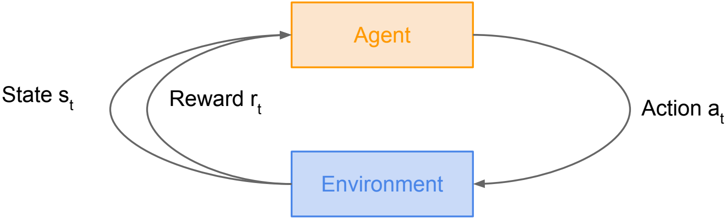
Reinforcement Learning



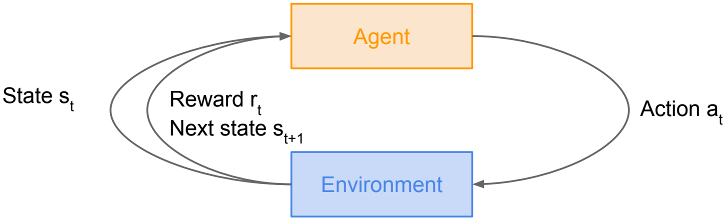
Reinforcement Learning



Reinforcement Learning

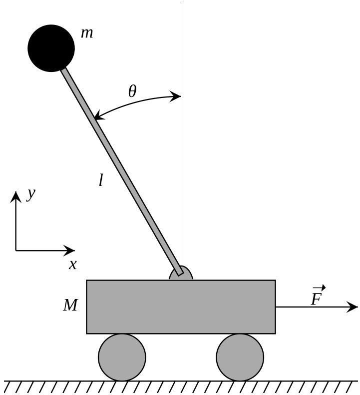


Reinforcement Learning



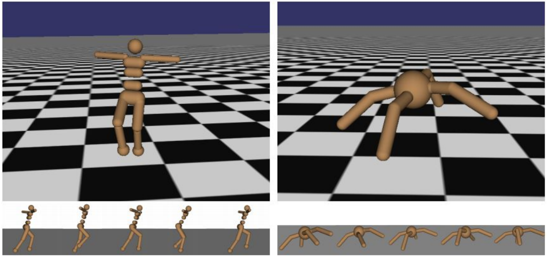
Cart-Pole Problem

- **Objective**: Balance a pole on top of a movable cart.
- **State**: angle, angular speed, position, horizontal velocity.
- **Action**: horizontal force applied on the cart.
- **Reward**: 1 at each time step if the pole is upright.



Robot Locomotion

- **Objective**: Make the robot move forward.
- **State**: Angle and position of the joints.
- **Action**: Torques applied on joints.
- **Reward**: 1 at each time step upright + forward movement.

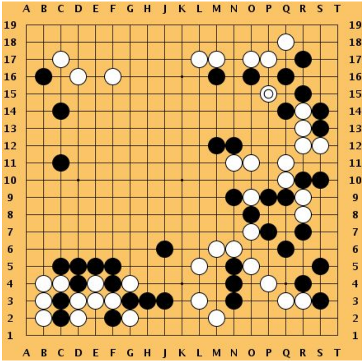


Atari Games

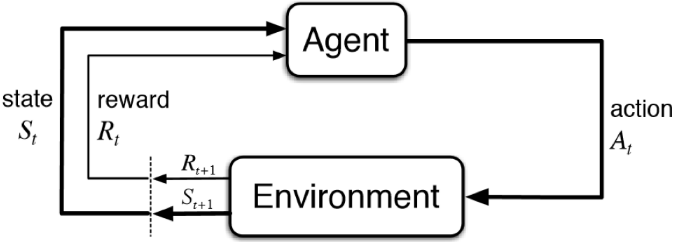
- **Objective**: Complete the game with the highest score.
- **State**: Raw pixel inputs of the game state.
- **Action**: Game controls e.g. Left, Right, Up, Down.
- **Reward**: Score increase/decrease at each time step.

Go

- **Objective**: Win the game.
- **State**: Position of all pieces.
- **Action**: Where to put the next piece down.
- **Reward**: 1 if win at the end of the game, 0 otherwise.



How to Mathematically Formalize the RL Problem?



Markov Decision Process

- Mathematical formulation of the RL problem.
- **Markov property**: Current state completely characterizes the state of the world.

$\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{P}, \gamma)$

- $\mathcal{S}$: set of possible states.
- $\mathcal{A}$: set of possible actions.
- $\mathcal{R}$: distribution of reward given (state, action) pairs.
- $\mathbb{P}$: transition probability, *i.e.*, distribution over next state given a (state, action) pair.
- γ: discount factor.

Markov Decision Process

1. At time step $t = 0$, environment samples initial state $s_0 \sim p(s_0)$.
2. Then, for $t = 0$ until done:
   - Agent selects an action a_t.
   - Environment samples reward $r_t \sim \mathcal{R}(\cdot|s_t, a_t)$.
   - Environment samples next state $s_{t+1} \sim \mathbb{P}(\cdot|s_t, a_t)$.
   - Agent receives reward r_t and next state s_{t+1}.
3. A policy π is a function from $\mathcal{S}$ to $\mathcal{A}$ that specifies what action to take in each state:
   - usually it is modeled as a conditional distribution of action given state.
4. **Objective**: find policy π^* that maximizes cumulative discounted reward: $\sum_{t \geq 0} \gamma^t r_t$.

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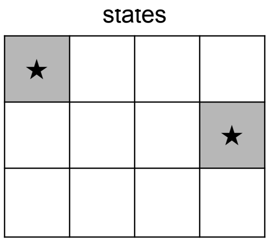
We need to learn the policy π^* and sometimes (parts of) the MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{P}, \gamma)$.

A Simple MDP: Policy for Grid World

actions = {

1. right
2. left
3. up
4. down

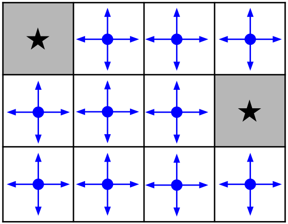
}



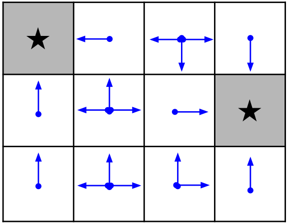
Set a negative "reward" for each transition (e.g. r = -1)

Objective: reach one of terminal states (greyed out) in least number of actions

A Simple MDP: Policy for Grid World



Random Policy



Optimal Policy

The Optimal Policy

1. We want to find optimal policy π^* that maximizes the sum of rewards.
2. Directly summing over rewards endows randomness, *e.g.*, initial state, transition probability, reward probability.
3. We should maximize the *expected total reward*:

$$\pi^* = \arg\max_{\pi} \mathbb{E}\left[\sum_{t \geq 0} \gamma^t r_t\right],$$
$$\text{with } s_0 \sim p(s_0), a_t \sim \pi(\cdot|s_t), s_{t+1} \sim p(\cdot|s_t, a_t)$$

Exact Methods and Monte Carlo Approximation[1]

[1] Adapted from: https://drive.google.com/file/d/0BxXI_RttTZAhVXBlMUVkQ1BVVDQ/view, https://drive.google.com/file/d/0BxXI_RttTZAhREJKRGhDT25OOTA/view

Exact Methods

Optimal control

- Given an MDP: $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{P}, \gamma)$, find the optimal policy π^*.

Exact methods

- Value iteration.
- Policy iteration.

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- Given an MDP: $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{P}, \gamma)$, find the optimal policy π^*.

Exact methods

- Value iteration.
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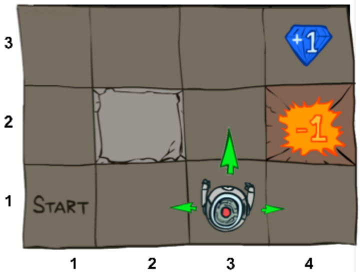
Optimal Value Function

- Define the optimal value function $V^*(s)$ as the sum of discounted rewards when starting from state s and acting optimally:

$$V^*(s) \triangleq \max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t, s_{t+1}) | \pi, s_0 = s\right]$$
$$= \max_{\pi} \mathbb{E}\left[\sum_{t=0}^{H} \gamma^t R(s_t, a_t, s_{t+1}) | \pi, s_0 = s, H \to \infty\right]$$

- What are the values $V^*(1,1)$, $V^*(1,2)$, $\cdots$, $V^*(3,4)$?



Value Iteration: Dynamic Programming

- $V_0^*(s)$: optimal value for state s when $H = 0$:

$$V_0^*(s) = 0, \forall s$$

- $V_1^*(s)$: optimal value for state s when $H = 1$:

$$V_1^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_0^*(s'))$$

- $V_2^*(s)$: optimal value for state s when $H = 2$:

$$V_2^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1^*(s'))$$

- $V_k^*(s)$: optimal value for state s when $H = k$:

$$V_k^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k-1}^*(s'))$$

Value Iteration

Algorithm:

Start with $V_0^*(s) = 0$ for all s.

For k = 1, … , H:

For all states s in S:

$$V_k^*(s) \leftarrow \max_a \sum_{s'} P(s'|s,a) \left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

$$\pi_k^*(s) \leftarrow \arg\max_a \sum_{s'} P(s'|s,a) \left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

This is called a value update or Bellman update/back-up

Value Iteration

Theorem

Value iteration converges. At convergence, we have found the optimal value function V^* for the discounted infinite horizon problem, which satisfies the Bellman equations:

$$\forall s, V^*(s) = \max_a \sum_{s'} P(s'|s) \left[R(s, a, s') + \gamma V^*(s')\right]$$

Now we know how to act for infinite horizon with discounted rewards:

- Run value iteration until convergence.
- This produces V^*, which tells us the optimal policy:

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s) \left[R(s, a, s') + \gamma V^*(s')\right]$$

Q-Values

- $Q^*(s,a)$: expected reward starting in s, taking action a, and (thereafter) acting optimally.
- Bellman equation:

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma \max_{a'} Q^*(s',a')\right)$$

- Q-value iteration:

$$Q_{k+1}(s,a) = \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma \max_{a'} Q_k(s',a')\right)$$

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma \max_{a'} Q_k(s',a')\right)$$

$$= \arg\max_a Q_{k+1}(s,a)$$

Exact Methods

Optimal control

- Given an MDP: $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathbb{P}, \gamma)$, find the optimal policy π^*.

Exact methods

- Value iteration.
- Policy iteration.

Policy Evaluation

- Recall value iteration:

$$V_k^*(s) = \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

- Policy evaluation for a given π:

Deterministic π: $V_i^\pi(s) = \sum_{s'} P(s'|s,\pi(s))\left(R(s,\pi(s),s') + \gamma V_{i-1}^*(s')\right)$

Stochastic π: $V_i^\pi(s) = \mathbb{E}_{a\sim\pi(\cdot|s)} \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_{i-1}^*(s')\right)$

- At convergence:

$$\forall s,\, V^\pi(s) = \sum_{s'} P(s'|s,\pi(s))\left(R(s,\pi(s),s') + \gamma V^*(s')\right)$$

Policy Iteration

One iteration of policy iteration:

- Policy evaluation for current policy π_k :
  - Iterate until convergence

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} P(s'|s, \pi_k(s)) \left[R(s, \pi(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: find the best action according to one-step look-ahead

$$\pi_{k+1}(s) \leftarrow \arg\max_a \sum_{s'} P(s'|s, a) \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

Policy Iteration

Theorem

Policy iteration is guaranteed to converge. At convergence, the current policy and its value function are the optimal policy and the optimal value function.

Limitations:

- Iteration over / storage for all states and actions: requires small, discrete state-action space:
  - sampling-based approximations
- Update equations require access to dynamics mode.

Policy Iteration

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Sampling-Based Approximation

- Q-value iteration?
- Value iteration?
- Policy iteration:
  - policy evaluation?
  - policy improvement?

Sampling-Based Approximation

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Recap: Q-Values

- $Q^*(s, a)$: expected reward starting in s, taking action a, and (thereafter) acting optimally.
- Bellman equation:

$$Q^*(s, a) = \sum_{s'} P(s'|s, a) \left(R(s, a, s') + \gamma \max_{a'} Q^*(s', a') \right)$$

- Q-value iteration:

$$Q_{k+1}(s, a) = \sum_{s'} P(s'|s, a) \left(R(s, a, s') + \gamma \max_{a'} Q_k(s', a') \right)$$

(Tabular) Q-Learning

$$Q_{k+1}(s,a) = \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma \max_{a'} Q_k(s',a')\right)$$

$$\Rightarrow Q_{k+1}(s,a) = \mathbb{E}_{s' \sim P(s'|s,a)}\left[R(s,a,s') + \gamma \max_{a'} Q_k(s',a')\right]$$

(Tabular) Q-Learning: replace expectation by samples:

- For an state-action pair (s,a), sample $s' \sim P(s'|s,a)$.
- Calculate the new sample estimate based on the old estimate $Q_k(s,a)$:

$$\text{target}(s') = R(s,a,s') + \gamma \max_{a'} Q_k(s',a')$$

- Update the new sample estimate by a running average:

$$Q_{k+1}(s,a) = (1-\alpha)Q_k(s,a) + \alpha \times \text{target}(s')$$

(Tabular) Q-Learning

Algorithm:

Start with $Q_0(s, a)$ for all s, a.

Get initial state s

For k = 1, 2, … till convergence

Sample action a, get next state s'

If s' is terminal:

$$\text{target} = R(s, a, s')$$

Sample new initial state s'

else:

$$\text{target} = R(s, a, s') + \gamma \max_{a'} Q_k(s', a')$$

$$Q_{k+1}(s, a) \leftarrow (1 - \alpha) Q_k(s, a) + \alpha \, [\text{target}]$$

$$s \leftarrow s'$$

How to Sample Actions?

- Random actions.
- Action that maximizes $Q_k(s, a)$ (greedy).
- ϵ-Greedy: choose random actions with prob.ϵ; otherwise choose actions greedily.
- Caveats:
  - You have to explore enough
  - You have to eventually make the learning rate small enough, but not decrease it too quickly

Sampling-Based Approximation

- Q-value iteration?
- Value iteration?
- Policy iteration:
  - policy evaluation?
  - policy improvement?

Value Iteration

$$V_{k+1}^*(s) = \max_a \mathbb{E}_{s' \sim P(s'|s,a)} \left[R(s, a, s') + \gamma V_k^*(s') \right]$$

- V^* does not depend on actions, have to integrate it out.
- Unclear how to draw samples through max.

Sampling-Based Approximation

- Q-value iteration?
- Value iteration?
- Policy iteration:
  - policy evaluation?
  - policy improvement?

Policy Iteration

One iteration of policy iteration:

- Policy evaluation for current policy π_k :
  - Iterate until convergence

$$V_{i+1}^{\pi_k}(s) \leftarrow \mathbb{E}_{s' \sim P(s'|s,\pi_k(s))}[R(s,\pi_k(s),s') + \gamma V_i^{\pi_k}(s')]$$

Can be approximated by samples
This is called Temporal Difference (TD) Learning

- Policy improvement: find the best action according to one-step look-ahead

$$\pi_{k+1}(s) \leftarrow \arg\max_a \mathbb{E}_{s' \sim P(s'|s,a)}[R(s,a,s') + \gamma V^{\pi_k}(s')]$$

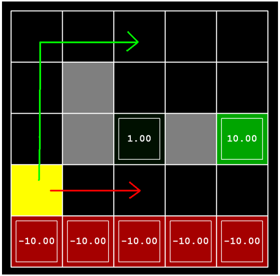
Unclear what to do with the max (for now)

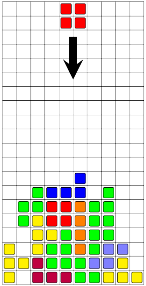
Limitation

Limitations:

- Iteration over / storage for all states and actions: requires small, discrete state-action space:
  - sampling-based approximations
- Update equations require access to dynamics mode:
  - Q/V function fitting.

Can tabular methods scale?





Gridworld
10^1

Tetris
10^60

Atari
10^308 (ram) 10^16992 (pixels)

Approximate Q-Learning

- Instead of a table, we have a parametrized Q-function $Q_\theta(s, a)$:
  - Can be a linear function in features:

$$Q_\theta(s, a) = \theta_0 f_0(s, a) + \theta_1 f_1(s, a) + \cdots + \theta_n f_n(s, a)$$

  - Or a complicated neural network.
- Learning rule:

$$\theta_{k+1} = \theta_k - \alpha \nabla_\theta \left[\frac{1}{2} \left(Q_\theta(s, a) - \text{target}(s') \right)^2 \right] |_{\theta=\theta_k}$$

$$\text{target}(s') \triangleq R(s, a, s') + \gamma \max_{a'} Q_{\theta_k}(s', a')$$

Connection to Tabular Q-Learning

- Suppose $\theta \in \mathbb{R}^{|S|\times|A|}, \quad Q_\theta(s,a) \equiv \theta_{sa}$

$$\begin{aligned}
&\nabla_{\theta_{sa}}\left[\frac{1}{2}(Q_\theta(s,a) - \text{target}(s'))^2\right] \\
&= \nabla_{\theta_{sa}}\left[\frac{1}{2}(\theta_{sa} - \text{target}(s'))^2\right] \\
&= \theta_{sa} - \text{target}(s')
\end{aligned}$$

- Plug into update: $\theta_{sa} \leftarrow \theta_{sa} - \alpha(\theta_{sa} - \text{target}(s'))$

$$= (1-\alpha)\theta_{sa} + \alpha[\text{target}(s')]$$

- Compare with Tabular Q-Learning update:

$$Q_{k+1}(s,a) \leftarrow (1-\alpha)Q_k(s,a) + \alpha\left[\text{target}(s')\right]$$

Convergence of Approximate Q-Learning

- It is not guaranteed to converge, even if the function approximation is expressive enough to represent the true Q-function:
  - The approximation is sequential, so if each time there induces a large enough error, the aggregate error might be exploded.