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User Prompt

Let C be a positive integer. For X a finite set of real numbers, denote with $A(X, C)$ the set of real numbers obtained by taking integer linear combinations of elements of X , with coefficients bounded by C . Find an upper bound, which is polynomial in $|X|$, for the number of intervals of the form $[2^n, 2^{n+1})$, with n nonnegative integer, that intersect an element of $A(X, C)$. If possible find an asymptotically sharp upper bound.

Model Response:

Let C be a positive integer and $X = \{x_1, \dots, x_k\}$ be a finite set of $k \geq 1$ real numbers. The set $A(X, C)$ is defined as

$$A(X, C) = \left\{ \sum_{i=1}^k a_i x_i : a_i \in \mathbb{Z}, |a_i| \leq C \right\}.$$

We denote by $N(X, C)$ the number of integers $n \geq 0$ such that the dyadic interval $[2^n, 2^{n+1})$ intersects $A(X, C)$. Let $Y = A(X, C) \cap [1, \infty)$. $N(X, C)$ is the number of such intervals intersecting Y .

Theorem 1. *Let $k = |X|$ and $C \geq 1$. The number of dyadic intervals $N(X, C)$ is bounded by*

$$N(X, C) \leq (2k - 1)(\lfloor \log_2(2kC + 1) \rfloor + 2).$$

Furthermore, this bound is asymptotically sharp, i.e.,

$$\sup_{|X|=k} N(X, C) = \Theta(k(\log k + \log C)).$$

We utilize Siegel's Lemma to establish a Gap Principle.

Lemma 1 (Siegel's Lemma). *Let B be an $M \times N$ matrix with integer entries, with $N > M > 0$. Let $H \geq 1$ be an upper bound for the absolute values of the entries of B . Then there exists a non-zero integer vector $c \in \mathbb{Z}^N \setminus \{0\}$ such that $Bc = 0$ and*

$$\|c\|_\infty \leq (NH)^{M/(N-M)}.$$

Proof. See, for example, M. Hindry and J. H. Silverman, Diophantine Geometry: An Introduction, Springer, 2000, Part D, Lemma 4.1. \square

A sequence of positive real numbers (z_i) is called R -separated if $z_{i+1}/z_i \geq R$ for all i .

Lemma 2 (Gap Principle). *Let $k \geq 1, C \geq 1$. Let $R = 2kC + 1$. Any R -separated sequence of distinct positive elements in $A(X, C)$ has length $m \leq 2k - 1$.*

Proof. Let $C' = 2kC$. Note that $R = C' + 1$. Suppose for contradiction that there exists an R -separated sequence $0 < z_1 < z_2 < \dots < z_m$ in $A(X, C)$ with $m = 2k$. For each $i = 1, \dots, m$, $z_i \in A(X, C)$, so $z_i = \sum_{j=1}^k a_{ij}x_j$ with $a_{ij} \in \mathbb{Z}$ and $|a_{ij}| \leq C$. Let $v_i = (a_{i1}, \dots, a_{ik})$.

Let B be the $k \times m$ matrix whose columns are the vectors v_i . The entries of B are bounded by $H = C$. We apply Lemma 1 with $M = k$ and $N = m = 2k$. Since $N > M$, there exists a non-zero integer vector $c = (c_1, \dots, c_m) \in \mathbb{Z}^m$ such that $Bc = 0$. The bound on the coefficients is

$$\|c\|_\infty \leq (NH)^{M/(N-M)} = (2kC)^{k/(2k-k)} = 2kC = C'.$$

The condition $Bc = 0$ implies a linear relation among the elements z_i :

$$\sum_{i=1}^m c_i z_i = \sum_{i=1}^m c_i \sum_{j=1}^k a_{ij} x_j = \sum_{j=1}^k x_j \left(\sum_{i=1}^m c_i a_{ij} \right) = 0.$$

Let J be the largest index such that $c_J \neq 0$. Then $c_J z_J = -\sum_{i=1}^{J-1} c_i z_i$. Taking absolute values (and noting $z_i > 0$):

$$|c_J| z_J = \left| \sum_{i=1}^{J-1} c_i z_i \right| \leq \sum_{i=1}^{J-1} |c_i| z_i \leq C' \sum_{i=1}^{J-1} z_i.$$

Since the sequence (z_i) is R -separated and increasing, $z_i \leq R^{-(J-1-i)} z_{J-1}$ for $i \leq J-1$.

$$\sum_{i=1}^{J-1} z_i \leq z_{J-1} \sum_{l=0}^{J-2} R^{-l} < z_{J-1} \sum_{l=0}^{\infty} R^{-l} = z_{J-1} \frac{R}{R-1}.$$

Also, by the separation property, $z_J \geq R z_{J-1}$. Combining these inequalities:

$$|c_J| R z_{J-1} \leq |c_J| z_J < C' z_{J-1} \frac{R}{R-1}.$$

Since $z_{J-1} > 0$ and $R > 0$, we can divide by $R z_{J-1}$:

$$|c_J| < \frac{C'}{R-1}.$$

We chose $R = C' + 1$, so $R - 1 = C'$.

$$|c_J| < \frac{C'}{C'} = 1.$$

This contradicts the fact that c_J is a non-zero integer. Therefore, the length of the sequence must be $m \leq 2k - 1$. \square

Lemma 3. *Let $I = [a, b)$ be an interval with $1 \leq a < b$. The number of dyadic intervals $[2^n, 2^{n+1})$ with $n \geq 0$ that intersect I , denoted by $D(I)$, is bounded by*

$$D(I) \leq \lfloor \log_2(b/a) \rfloor + 2.$$

Proof. Since $a \geq 1$, the exponents n must be non-negative. The smallest exponent n_{\min} is $\lfloor \log_2 a \rfloor$. The largest exponent n_{\max} must satisfy $2^{n_{\max}} < b$, so $n_{\max} = \lceil \log_2 b \rceil - 1$. The count is $D(I) = n_{\max} - n_{\min} + 1 = \lceil \log_2 b \rceil - \lfloor \log_2 a \rfloor$. Let $R = b/a$. Let $r = \log_2 R$ and $x = \log_2 a$. $D(I) = \lceil r + x \rceil - \lfloor x \rfloor$. Let $I_r = \lfloor r \rfloor$, $f_r = \{r\}$, and $I_x = \lfloor x \rfloor$, $f_x = \{x\}$. $D(I) = \lceil I_r + f_r + I_x + f_x \rceil - I_x = I_r + \lceil f_r + f_x \rceil$. Since $0 \leq f_r, f_x < 1$, we have $0 \leq f_r + f_x < 2$. Thus $\lceil f_r + f_x \rceil \in \{0, 1, 2\}$. Therefore, $D(I) \leq I_r + 2 = \lfloor \log_2 R \rfloor + 2$. \square

We now prove the upper bound in Theorem 1.

Proof of the Upper Bound. Let $Y = A(X, C) \cap [1, \infty)$. If Y is empty, $N(X, C) = 0$. Assume Y is non-empty. Y is finite. Let $R = 2kC + 1$. We construct a maximal R -separated subsequence of Y greedily. Let $s_1 = \min Y$. Since $Y \subset [1, \infty)$, $s_1 \geq 1$. For $j \geq 1$, if s_j is defined, let $Y_j = \{y \in Y : y \geq Rs_j\}$. If Y_j is non-empty, define $s_{j+1} = \min Y_j$. Otherwise, the sequence terminates. Let the sequence be $S = \{s_1, \dots, s_m\}$. By construction, $s_{j+1} \geq Rs_j$. By Lemma 2, $m \leq 2k - 1$. We show that Y is covered by the union of the intervals $J_j = [s_j, Rs_j)$ for $j = 1, \dots, m$. Let $y \in Y$. Since $s_1 = \min Y$, $y \geq s_1$. Let J be the largest index such that $s_J \leq y$. If $J = m$. The sequence terminated because Y_m is empty. Thus, any $y' \in Y$ with $y' \geq s_m$ must satisfy $y' < Rs_m$. So $y \in J_m$. If $J < m$. Then s_{J+1} exists, and $y < s_{J+1}$ by the maximality of J . If we had $y \geq Rs_J$, then $y \in Y_J$. By definition, $s_{J+1} = \min Y_J \leq y$, a contradiction. Thus $y < Rs_J$. So $y \in J_J$. Therefore, $Y \subset \bigcup_{j=1}^m J_j$.

The total number of dyadic intervals intersecting Y is bounded by the sum of the counts for each J_j . By Lemma 3, since $s_j \geq 1$ and the ratio for J_j is R , the number of dyadic intervals intersecting J_j is at most $\lfloor \log_2 R \rfloor + 2$.

$$N(X, C) \leq \sum_{j=1}^m D(J_j) \leq m(\lfloor \log_2 R \rfloor + 2) \leq (2k - 1)(\lfloor \log_2(2kC + 1) \rfloor + 2).$$

\square

This upper bound is $O(k \log(kC)) = O(k(\log k + \log C))$.