The prompts and responses below have been edited only for formatting compatibility.

User Prompt

Is it true that if $a_1 < a_2 < \cdots$ is a sequence of integers with

$$\liminf a_n^{1/2^n} > 1$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational?

Model Response (*Aletheia*, ablation run)

Introduction. In this paper, we examine the arithmetic nature of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}},$$

where $(a_n)_{n\geq 1}$ is a strictly increasing sequence of integers. We assume the sequence exhibits double exponential growth, quantified by the condition:

$$L := \liminf_{n\to\infty} a_n^{1/2^n} > 1. \tag{1}$$

This problem relates to classical results by Erdős, Graham, and Badea concerning the irrationality of series with rapidly growing denominators [1, 2]. We prove that condition (1) is sufficient to guarantee the irrationality of S.

1 Main Result

Theorem 1. *Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of integers satisfying (1). Then the sum S is an irrational number.*

Proof. We proceed by contradiction. Assume that S is a rational number.

Reduction to a Tail Sequence. Since $L > 1$, the terms a_n tend to infinity. Let c be a constant such that $1 < c < L$. By the definition of the limit inferior, there exists an index N_0 such that for all $n \geq N_0$, $a_n^{1/2^n} > c$. Consequently, $a_n \geq 2$ for all $n \geq N_0$. The original series S can be decomposed into a finite sum and an infinite tail:

$$S = \sum_{n=1}^{N_0-1} \frac{1}{a_n a_{n+1}} + \sum_{n=N_0}^{\infty} \frac{1}{a_n a_{n+1}}.$$

Since the finite sum is rational, S is rational if and only if the tail sum is rational. We define a shifted sequence $(b_n)_{n\geq 1}$ by $b_n = a_{N_0+n-1}$. The sequence (b_n) satisfies:

1. $b_n \in \mathbb{Z}$, $b_n \geq 2$, and $b_{n+1} > b_n$ for all $n \geq 1$.
2. The sum $S' = \sum_{n=1}^{\infty} \frac{1}{b_n b_{n+1}}$ is rational. Let $S' = p/q$ for positive integers p, q.
3. The growth rate is preserved in the limit. Let $\mathcal{L} = \liminf_{n\to\infty} b_n^{1/2^n}$. We have:

$$\mathcal{L} = \liminf_{n\to\infty} \left(a_{N_0+n-1}^{1/2^{N_0+n-1}} \right)^{2^{N_0-1}} = L^{2^{N_0-1}}.$$

Since $L > 1$ and $N_0 \geq 1$, we have $\mathcal{L} > 1$.

Integer Constraints from Rationality. Let $P_n = \prod_{k=1}^{n} b_k$. Define the partial sum $S'_n = \sum_{k=1}^{n-1} \frac{1}{b_k b_{k+1}}$ and the remainder $R_n = S' - S'_n$. Substituting $S' = p/q$:

$$R_n = \frac{p}{q} - S'_n \implies qP_nR_n = pP_n - q(P_nS'_n).$$

The term $P_nS'_n = \sum_{k=1}^{n-1} \frac{P_n}{b_k b_{k+1}}$ is an integer because for every $k < n$, the distinct factors b_k and b_{k+1} divide P_n. Thus, the quantity $K_n := qP_nR_n$ is an integer. Since the terms of the series are strictly positive, $R_n > 0$, implying $K_n \geq 1$. This yields a lower bound for the remainder:

$$R_n \geq \frac{1}{qP_n}. \tag{2}$$

Recurrence Bound on Sequence Growth. We establish an upper bound for b_{n+1} derived from the properties of R_n. Using the telescoping inequality $\frac{1}{b_k b_{k+1}} < \frac{1}{b_k} - \frac{1}{b_{k+1}}$, we sum from $k = n$ to infinity:

$$R_n = \sum_{k=n}^{\infty} \frac{1}{b_k b_{k+1}} < \frac{1}{b_n}.$$

From $K_n = qP_nR_n$, we have:

$$K_n < \frac{qP_n}{b_n} = qP_{n-1} \quad \text{(with } P_0 = 1\text{)}.$$

Applying this to index $n+1$, we obtain $K_{n+1} < qP_n$. Next, we use the recurrence relation for the remainder:

$$R_n = \frac{1}{b_n b_{n+1}} + R_{n+1}.$$

Multiplying by $qP_{n+1} = qP_nb_{n+1}$:

$$qP_{n+1}R_n = \frac{qP_nb_{n+1}}{b_nb_{n+1}} + qP_{n+1}R_{n+1}.$$

Simplifying the first term on the right-hand side using $P_n/b_n = P_{n-1}$:

$$b_{n+1}(qP_nR_n) = qP_{n-1} + (qP_{n+1}R_{n+1}).$$

Substituting K_n and K_{n+1}:

$$b_{n+1}K_n = qP_{n-1} + K_{n+1}.$$

Since $K_n \geq 1$, we have $b_{n+1} \leq b_{n+1}K_n$. Therefore:

$$b_{n+1} \leq qP_{n-1} + K_{n+1}.$$

Using the bound $K_{n+1} < qP_n$:

$$b_{n+1} < qP_{n-1} + qP_n = qP_{n-1}(1 + b_n).$$

Since $b_n \geq 2$, $1 + b_n \leq 2b_n$. Thus:

$$b_{n+1} < 2qP_{n-1}b_n = 2qP_n.$$

Substituting this into $P_{n+1} = P_n b_{n+1}$, we obtain the growth constraint:

$$P_{n+1} < 2qP_n^2. \tag{3}$$

Convergence of Sequence Limits. We analyze the asymptotic behavior of P_n. Taking the natural logarithm of (3):

$$\ln P_{n+1} < 2\ln P_n + \ln(2q).$$

Dividing by 2^{n+1}:

$$\frac{\ln P_{n+1}}{2^{n+1}} < \frac{\ln P_n}{2^n} + \frac{\ln(2q)}{2^{n+1}}.$$

Let $y_n = 2^{-n}\ln P_n$. Then $y_{n+1} < y_n + 2^{-(n+1)}\ln(2q)$. Consider the auxiliary sequence $z_n = y_n + 2^{-n}\ln(2q)$. We observe that z_n is strictly decreasing. To prove convergence, we show it is bounded below. Since $\mathcal{L} > 1$, for sufficiently large n, $b_n > c^{2^n}$ for some $c > 1$. This implies P_n grows at least double exponentially, so y_n is bounded away from 0. Thus z_n converges, implying y_n converges. Let $Y = \lim_{n\to\infty} y_n$ and define:

$$\Pi = \lim_{n\to\infty} P_n^{1/2^n} = e^Y.$$

We determine the limit of the normalized sequence terms. Since $b_n = P_n/P_{n-1}$:

$$b_n^{1/2^n} = \frac{P_n^{1/2^n}}{(P_{n-1}^{1/2^{n-1}})^{1/2}}.$$

Taking limits as $n \to \infty$:

$$\lim_{n\to\infty} b_n^{1/2^n} = \frac{\Pi}{\sqrt{\Pi}} = \sqrt{\Pi}.$$

Since the limit exists, it must equal the limit inferior $\mathcal{L}$. Therefore:

$$\Pi = \mathcal{L}^2. \tag{4}$$

Contradiction via Remainder Estimation

We now derive a conflicting lower bound for Π. Fix any constant C such that $1 < C < \mathcal{L}$. For sufficiently large k, $b_k > C^{2^k}$. The term $b_k b_{k+1}$ grows as $C^{2^k} C^{2^{k+1}} = C^{3 \cdot 2^k}$. The remainder R_n is bounded by:

$$R_n = \sum_{k=n}^{\infty} \frac{1}{b_k b_{k+1}} < \sum_{k=n}^{\infty} C^{-3 \cdot 2^k}.$$

For $C > 1$, this series is dominated by its first term (bounded by a geometric series with ratio $\leq 1/2$ for large n):

$$R_n < 2C^{-3 \cdot 2^n}.$$

Recall the rationality lower bound (2), $R_n \geq \frac{1}{qP_n}$. Combining these:

$$\frac{1}{qP_n} < 2C^{-3 \cdot 2^n} \implies P_n > \frac{1}{2q} C^{3 \cdot 2^n}.$$

Raising to the power $1/2^n$ and taking the limit $n \to \infty$:

$$\Pi \geq \lim_{n \to \infty} (2q)^{-1/2^n} C^3 = C^3.$$

Since this holds for all $C < \mathcal{L}$, we conclude:

$$\Pi \geq \mathcal{L}^3. \tag{5}$$

Conclusion

We have derived two relations: $\Pi = \mathcal{L}^2$ and $\Pi \geq \mathcal{L}^3$. Combining them yields $\mathcal{L}^2 \geq \mathcal{L}^3$. Since $\mathcal{L} > 1$, we divide by $\mathcal{L}^2$ to obtain:

$$1 \geq \mathcal{L}.$$

This contradicts the condition $\mathcal{L} > 1$. Therefore, the assumption that S is rational implies a contradiction. $\square$

References

[1] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monographies de L'Enseignement Mathématique, No. 28, Université de Genève, Geneva, 1980.

[2] C. Badea, "The irrationality of certain infinite series," *Glasgow Mathematical Journal*, vol. 29, no. 2, pp. 221–228, 1987.

Model Response (*Aletheia*, initial run)

Theorem 2. *Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of integers such that*

$$\liminf_{n\to\infty} a_n^{1/2^n} > 1.$$

Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational.

Proof. **1. Growth Estimates Based on Hypothesis**
Let $\lambda = \liminf_{n\to\infty} a_n^{1/2^n}$. By hypothesis, $\lambda > 1$. Select a constant c such that $1 < c < \lambda$. By the definition of limit inferior, there exists an integer N_0 such that for all $n \geq N_0$:

$$a_n^{1/2^n} > c \implies a_n > c^{2^n}.$$

Let $u_n = \ln a_n$. Taking the logarithm, we have $u_n > 2^n \ln c$. Let $T_n = \sum_{i=1}^{n} u_i$. Bounding the sum from below:

$$T_n > \sum_{i=N_0}^{n} 2^i \ln c = \ln c(2^{n+1} - 2^{N_0}).$$

Let $\alpha = \ln c > 0$. For sufficiently large n, we have:

$$T_n > \alpha 2^n. \tag{6}$$

Since a_n is a strictly increasing sequence of integers, $a_n \geq 1$ for all sufficiently large n. We assume $n \geq N_0$ throughout the remaining argument.

2. Assumption of Rationality and Lower Bound
Assume, for the sake of contradiction, that S is rational. Let $S = \frac{p}{q}$ for positive integers p, q. Define the n-th partial sum S_n and the remainder r_n:

$$S_n = \sum_{k=1}^{n} \frac{1}{a_k a_{k+1}}, \qquad r_n = S - S_n = \sum_{k=n+1}^{\infty} \frac{1}{a_k a_{k+1}}.$$

Since the terms are positive, $S_n < S$, so $r_n > 0$. Consider the denominators in S_n. Each term is of the form $\frac{1}{a_k a_{k+1}}$. Notice that $a_k a_{k+1}$ divides the product $P_{n+1} = \prod_{i=1}^{n+1} a_i$. Let $L_n = \operatorname{lcm}(a_1 a_2, \ldots, a_n a_{n+1})$. Then $L_n \leq P_{n+1}$. Writing $S_n = \frac{M_n}{L_n}$ for some integer M_n, we have:

$$r_n = \frac{p}{q} - \frac{M_n}{L_n} = \frac{pL_n - qM_n}{qL_n}.$$

Since $r_n > 0$, the numerator is a positive integer, so $pL_n - qM_n \geq 1$. Thus:

$$r_n \geq \frac{1}{qL_n} \geq \frac{1}{qP_{n+1}}. \tag{7}$$

Using the relation $P_{n+1} = a_{n+1}P_n$, we write this as:

$$r_n \geq \frac{1}{qa_{n+1}P_n}.$$

3. Universal Upper Bound for Remainder
We now bound $r_{n+1} = \sum_{k=n+2}^{\infty} \frac{1}{a_k a_{k+1}}$. Since (a_k) is strictly increasing, $a_{k+1} > a_k$, so $\frac{1}{a_k a_{k+1}} < \frac{1}{a_k^2}$. Thus, $r_{n+1} < \sum_{k=n+2}^{\infty} \frac{1}{a_k^2}$.
We partition the indices $k \geq n+2$ into sets I_j:

$$I_j = \{k \geq n+2 \mid 2^j a_{n+2} \leq a_k < 2^{j+1}a_{n+2}\}, \quad j = 0, 1, 2, \ldots$$

For any $k \in I_j$, we have $\frac{1}{a_k^2} \leq \frac{1}{(2^j a_{n+2})^2}$. For any $k \in I_j$, $c^{2^k} < a_k < 2^{j+1}a_{n+2}$. Taking logs:

$$2^k \ln c < (j+1)\ln 2 + \ln a_{n+2} \implies k < \log_2\left(\frac{(j+1)\ln 2 + \ln a_{n+2}}{\ln c}\right).$$

Let $L = \ln a_{n+2}$. There exists a constant $C > 0$ such that $|I_j| \leq C(\ln L + j)$. Substituting this into the sum:

$$r_{n+1} < \sum_{j=0}^{\infty} \frac{C(\ln L + j)}{(2^j a_{n+2})^2} = \frac{C}{a_{n+2}^2}\left(\ln L \sum_{j=0}^{\infty} \frac{1}{4^j} + \sum_{j=0}^{\infty} \frac{j}{4^j}\right).$$

Thus, for some constant K:

$$r_{n+1} < \frac{K \ln\ln a_{n+2}}{a_{n+2}^2}. \tag{8}$$

4. Deriving the Recurrence Relation
Recall $r_n = \frac{1}{a_{n+1}a_{n+2}} + r_{n+1}$. Combining (2) and (3):

$$\frac{1}{qa_{n+1}P_n} \leq \frac{1}{a_{n+1}a_{n+2}} + \frac{K \ln\ln a_{n+2}}{a_{n+2}^2}.$$

Multiplying by $a_{n+1}a_{n+2}$ and using $a_{n+1}/a_{n+2} < 1$:

$$\frac{a_{n+2}}{qP_n} < 1 + K \ln\ln a_{n+2}.$$

Taking logarithms ($u_n = \ln a_n$, $T_n = \ln P_n$):

$$u_{n+2} < \ln q + T_n + \ln(1 + K \ln u_{n+2}).$$

For any $\epsilon > 0$ and sufficiently large n, $\ln(1 + K \ln u_{n+2}) < \epsilon u_{n+2}$. Thus:

$$(1-\epsilon)u_{n+2} < T_n + \ln q.$$

Substituting $u_{n+2} = T_{n+2} - T_{n+1}$:

$$T_{n+2} < T_{n+1} + \frac{1}{1-\epsilon} T_n + C_\epsilon.$$

5. Contradiction
The characteristic equation $x^2 - x - \frac{1}{1-\epsilon} = 0$ has a positive root:

$$\rho(\epsilon) = \frac{1 + \sqrt{1 + \frac{4}{1-\epsilon}}}{2}.$$

For $\epsilon = 0.05$, $\rho(0.05) \approx 1.64 < 2$. Thus $T_n = O(1.7^n)$. However, from Step 1, $T_n > \alpha 2^n$. This implies:

$$\alpha 2^n < C(1.7)^n \implies \alpha < C \left(\frac{1.7}{2}\right)^n.$$

As $n \to \infty$, this implies $\alpha \leq 0$, contradicting $\alpha > 0$. The assumption that S is rational must be false. $\square$

Conclusion: The sum $S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$ is irrational.

User Prompt

Find a meaningful, interesting generalization of this theorem or its proof.
Theorem Fix $d \geq 2$, and let c_d be the unique positive real root of $x^d - x^{d-1} - 1 = 0$. Suppose a strictly increasing sequence $\{a_n\}$ of integers satisfy

$$\liminf_{n\to\infty} a_n^{1/b^n} > 1$$

for some $b > c_d$. Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1} \cdots a_{n+d-1}}$$

is irrational.
Proof For two functions f and g, we write $f \precsim g$ when there exists some constant $C > 0$ satisfying $f \leq Cg$. Assume for contradiction that $S = p/q$. We may assume there is some constant $c > 1$ such that $a_n > c^{b^n}$ for all n. We set

$$P_n := \prod_1^n a_i, \quad S_n = \sum_{m=1}^{n} \frac{1}{a_m a_{m+1} \cdots a_{m+d-1}}.$$

Since $S > S_n$ for all n, we have

$$r_n := S - S_n = \frac{p}{q} - \frac{*}{P_{n+d-1}} \geq \frac{1}{qP_{n+d-1}}.$$

Let us define

$$I_j(n) := \{k \geq n+1 \mid 2^j a_{n+1}^d \leq a_k^d < 2^{j+1} a_{n+1}^d\}, \quad j = 0, 1, 2, \ldots$$

so that each $k \in I_j(n)$ satisfies

$$c^{d \cdot b^k} < a_k^d < 2^{j+1} a_{n+1}^d.$$

The cardinality of $I_j(n)$ can be bounded from above as

$$|I_j(n)| \leq \max_{k \in I_j(n)} k \leq \log_b \left(\frac{\log_c(2^{j+1} a_{n+1}^d)}{d} \right) \precsim \log j + \log\log a_{n+1}.$$

It follows that

$$r_n < \sum_{k=n+1}^{\infty} \frac{1}{a_k^d} = \sum_{j=0}^{\infty} \sum_{k \in I_j(n)} \frac{1}{a_k^d} \leq \sum_{j=0}^{\infty} \frac{|I_j(n)|}{2^j a_{n+1}^d} \precsim \sum_{j=0}^{\infty} \frac{\log j + \log\log a_{n+1}}{2^j a_{n+1}^d} \precsim \frac{\log\log a_{n+1}}{a_{n+1}^d}.$$

Combining the lower and the upper bounds for r_n and r_{n+1}, we have

$$\begin{aligned} \frac{1}{qP_{n+d-1}} \leq r_n &= \sum_{j=1}^{d-1} \frac{1}{a_{n+j} \cdots a_{j+n+d-1}} + r_{n+d-1} \\ &\precsim \frac{d-1}{a_{n+1} \cdots a_{n+d}} + \frac{\log\log a_{n+d}}{a_{n+d}^d} \leq \frac{\log\log a_{n+d}}{a_{n+1} \cdots a_{n+d}}. \end{aligned}$$

Fix an arbitrarily small $\epsilon > 0$. Multiplying both sides by P_{n+d}, we obtain that

$$a_{n+d} = P_{n+d}/P_{n+d-1} \precsim P_n \log\log a_{n+d} \precsim P_n a_{n+d}^{\epsilon} = P_n (P_{n+d}/P_{n+d-1})^{\epsilon}.$$

It follows that we have $P_{n+d} \precsim P_{n+d-1} P_n^{1/(1-\epsilon)}$. The sequence $u_n := \log_c P_n$ satisfies the recursive inequality

$$u_{n+d} \leq u_{n+d-1} + \frac{1}{1-\epsilon} u_n + K$$

for some constant $K = K(\epsilon) > 0$. For a sufficiently small ϵ, the equation

$$x^d = x^{d-1} + \frac{1}{1-\epsilon}$$

has exactly one positive real root $s = s(d, \epsilon)$, slightly larger than c_d, and one or zero negative real root, depending on the parity of d; moreover, the negative real root, if exists, belong to $(-1, 0)$. By choosing a sufficiently small ϵ satisfying $s = s(d, \epsilon) < b$, we have that $u_n \precsim s^n$. This contradicts the estimate $\log_c P_n \geq \log_c a_n \geq b^n$. **QED**

Model Response (Gemini Deep Think)

A meaningful and natural generalization of this theorem extends the result to sums of reciprocals of products with **arbitrary integer weights**.
The original theorem relies on a product of d consecutive terms, effectively assigning a weight of 1 to every integer in a "sliding window." By generalizing these exponents, we can quantify how the density or sparsity of the denominator affects the growth rate required for the series to be irrational.

Theorem (Generalized)

Fix an integer $d \geq 1$ and a tuple of non-negative integers $\mathbf{w} = (w_0, w_1, \ldots, w_{d-1})$ with $w_{d-1} \geq 1$. Let $W = \max(w_0, \ldots, w_{d-1})$.
Let $c_{\mathbf{w}}$ be the unique real root greater than 1 of the polynomial:

$$P(x) = (x-1)\sum_{j=0}^{d-1} w_j x^j - W x^{d-1}$$

Suppose a strictly increasing sequence of integers $\{a_n\}$ satisfies

$$\liminf_{n\to\infty} a_n^{1/b^n} > 1$$

for some $b > c_{\mathbf{w}}$. Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n^{w_0} a_{n+1}^{w_1} \cdots a_{n+d-1}^{w_{d-1}}}$$

is irrational.
—

Proof Sketch

The generalization follows the "gap principle" strategy used in the original proof, balancing the decay of the tail against the growth of the common denominator.
1. **The Common Denominator (LCM):**
Assume for contradiction that $S = p/q$. Let S_n be the n-th partial sum. The denominators of the terms in S_n involve integers up to a_{n+d-1}.
Because the window shifts by 1 at each step, any specific integer a_k appears in d consecutive terms with exponents $w_{d-1}, \ldots, w_0$. When taking the Least Common Multiple (LCM) to form the common denominator Q_n, the exponent of a_k is bounded by the maximum weight W.

$$Q_n \lesssim \prod_{k=1}^{n+d-1} a_k^W \approx P_{n+d-1}^W$$

2. **The Tail Estimate:**
The remainder $r_n = S - S_n$ is dominated by the first term of the tail (index $n+1$).

$$r_n \approx \frac{1}{a_{n+1}^{w_0} a_{n+2}^{w_1} \cdots a_{n+d}^{w_{d-1}}}$$

3. **The Inequality:**
For S to be rational, the error r_n cannot be "too small" relative to the denominator Q_n. Specifically, $r_n \geq 1/(qQ_n)$. This implies $Q_n r_n \gtrsim 1$. Substituting the bounds:

$$P_{n+d-1}^W \cdot \frac{1}{\prod_{j=0}^{d-1} a_{n+1+j}^{w_j}} \gtrsim 1 \implies \prod_{j=0}^{d-1} a_{n+1+j}^{w_j} \lesssim P_{n+d-1}^W$$

Taking logarithms (assuming $\log a_k \sim b^k$) and dividing by b^{n+1}:

$$\sum_{j=0}^{d-1} w_j b^j \lesssim W \frac{b^{d-1}}{b-1}$$

Rearranging this inequality yields the polynomial $P(b) \leq 0$. If $b > c_{\mathbf{w}}$ (the root), this inequality is violated for large n, proving irrationality.
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Why this is Meaningful
1. **It Unifies "Dense" and "Sparse" Problems** **The Original Case:** If $\mathbf{w} = (1, \dots, 1)$, then $W = 1$. The sum is $\frac{x^d - 1}{x-1}$.

$$(x-1)\frac{x^d - 1}{x-1} - 1 \cdot x^{d-1} = x^d - x^{d-1} - 1 = 0$$

This recovers the original constant c_d.
The "Gap" Case: Consider $S = \sum \frac{1}{a_n a_{n+2}}$. Here $\mathbf{w} = (1, 0, 1)$ and $W = 1$. The polynomial becomes:

$$(x-1)(1+x^2) - x^2 = x^3 - 2x^2 + x - 1 = 0$$

The root is $c \approx 1.75$, which is larger than the $c \approx 1.46$ required for the consecutive block $a_n a_{n+1} a_{n+2}$. This quantifies the intuition that **"missing terms" weaken the convergence**, forcing the sequence a_n to grow faster to guarantee irrationality.
2. **It Handles Heavy Weights**
Sylvester-Type Series: If $d = 1$ and $w_0 = 1$, the polynomial is $(x-1)(1) - 1 = x - 2$, recovering the classic result that $\sum 1/a_n$ is irrational if $a_{n+1} \sim a_n^2$ (limit $b = 2$).
If we increase the weight of the *last* term (e.g., $1/(a_n a_{n+1}^2)$), the root increases, making irrationality harder to prove. This is because the "cost" of the LCM ($W = 2$) rises, while the benefit to the tail decay is limited to the future terms.
3. **Universal Irrationality for Double-Exponential Sequences**
Since $c_d \to 1$ as $d \to \infty$ for the uniform case, this generalization implies a powerful corollary:
For **any** sequence $\{a_n\}$ with double-exponential growth (any $b > 1$), there exists a sufficiently large block size d such that $\sum \frac{1}{a_n \cdots a_{n+d-1}}$ is irrational.