How Non linearities influence the Ecosystem

-Aadarsh, Bibaswan,Yashowardhan

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Review of 1-D and 2-D systems

In 1-D systems:

If a system can is given by $dx/dt=f(x)$

- ❖ We calculate fixed points from $dx/dt=0$.
- ❖ A fixed point can be attractor or repeller depending on if $df/dx < 0$ or > 0.

In 2-D systems:

A system is given by $dx/dt=f1(x,y)$ and $dy/dt=f2(x,y)$

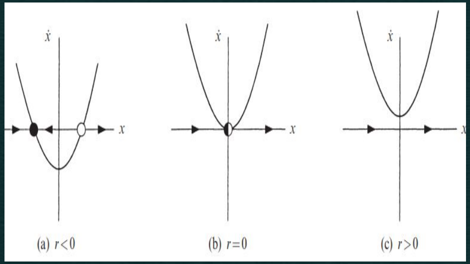
- ❖ We calculate the fixed points from $f1(x,y)=f2(x,y)=0$.
- ❖ We calculate the Jacobian and evaluate $|J-\lambda I|=0$ and find the eigenvalues.
- ❖ Depending upon the signs of the eigenvalues we can say if the fixed point is stable or unstable.

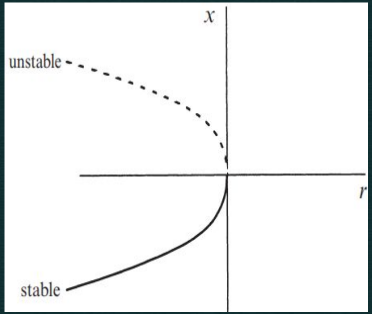
But how does behaviour change when the parameters of $f(x)$ change?

Bifurcations as change of parameter

Saddle node bifurcations:

$$\dot{x} = r + x^2$$





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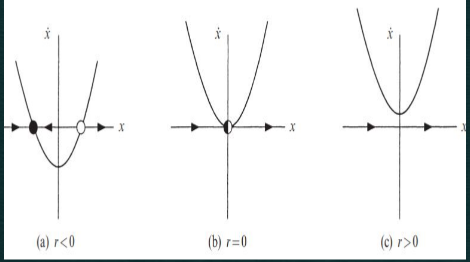
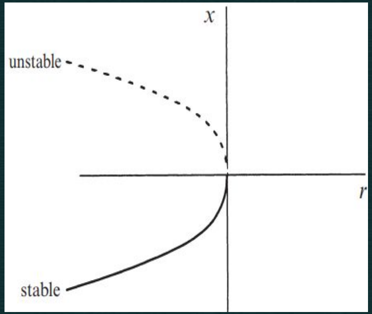
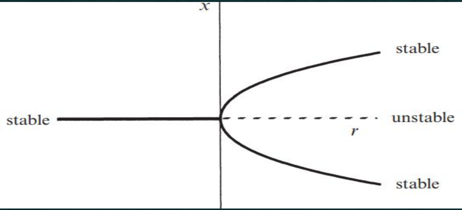




$$\dot{x} = r + x^2$$

Pitchfork Bifurcation:

Supercritical bifurcation: $\dot{x} = rx - x^3$



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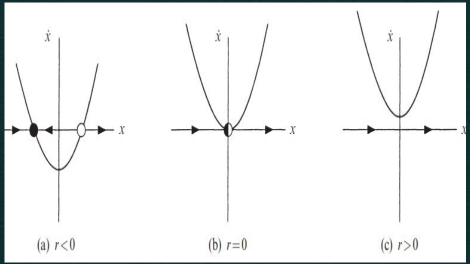

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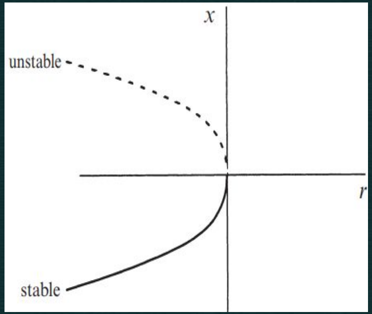

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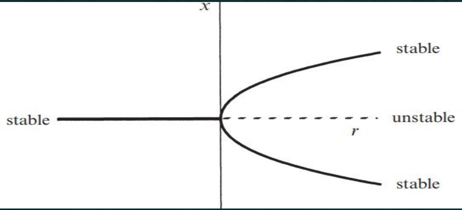
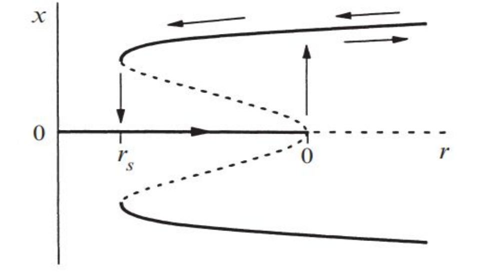


Subcritical bifurcation: $\dot{x} = rx + x^3,$

Here we have a destabilizing term. So we introduce a new term

$$\dot{x} = rx + x^3 - x^5$$



Imperfect Bifurcations and Catastrophe

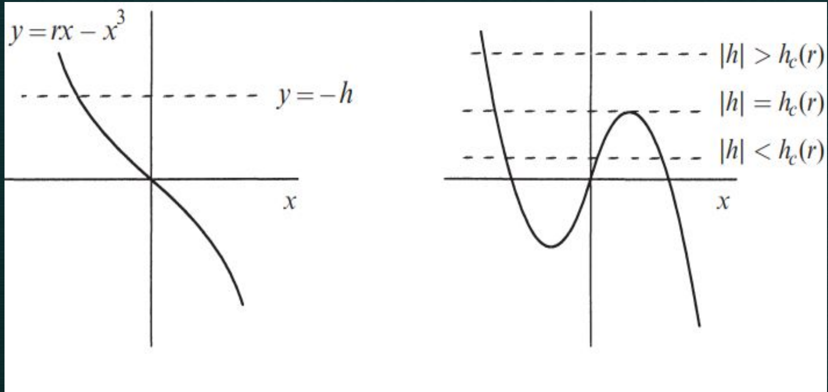
We saw symmetry in Pitchfork and Saddle node bifurcations, but such is not the case in real world. There are imperfections. The graph then is no longer symmetric.

To include this imperfection we add another parameter:

$$\dot{x} = h + rx - x^3$$

Here h is the imperfection parameter

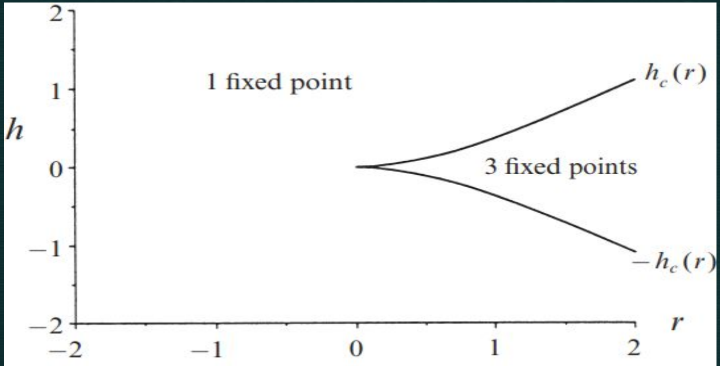
To find the fixed points we equate $rx - x^3$ and -h.



(a) $r \leq 0$ (b) $r > 0$

Imperfect Bifurcation and Catastrophe

Now we plot h as $h=h(r)$
$h_c(r)$ and $-h_c(r)$ are the critical curves and region between them is where 3 fixed points occur.
Here $h_c(r)=2r/3\sqrt{(r/3)}$.
The point where $h_c(r)$ and $-h_c(r)$ meet i.e, (0,0) is called cusp point.



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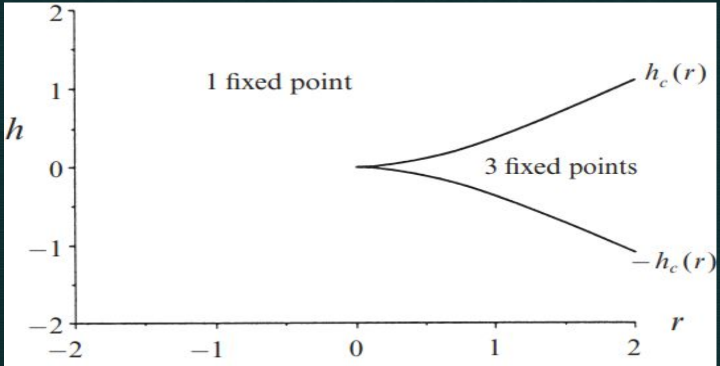
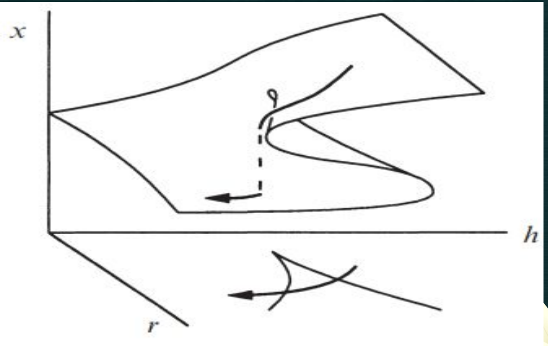


We can plot the 3-d graph for x,h and r. We see that thereis a fold in the graph. This fold when projected to h-r plane we get the above graph. Here we can clearly see that at the particular parameter there is jump in x. This is called catastrophe.



Predator Prey Systems

Lotka-Volterra Model-Description of Parameters

Prey: $$\frac{dx}{dt} = \alpha x - \beta xy$$

Predators: $$\frac{dy}{dt} = \delta xy - \gamma y$$

Lotka-Volterra Model-Description of Parameters

Young being born

Prey: $\dfrac{dx}{dt} = \alpha x - \beta xy$

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Prey being eaten

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Population growth from eating prey

Lotka-Volterra Model-Description of Parameters

Young being born

Prey being eaten

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Predators: $\frac{dy}{dt} = \delta xy - \gamma y$

Population growth from eating prey

Natural death of predators

Fixed Point Stability Analysis

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Fixed Point Stability Analysis

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy \\ \frac{dy}{dt} &= \delta xy - \gamma y\end{aligned}}$$

$$\boxed{\begin{gathered}\frac{dx}{dt} = 0 \\ x = 0 \quad \text{or} \quad y = \frac{\alpha}{\beta}\end{gathered}}$$

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Stationary Points

$$\boxed{\begin{gathered}(0, 0) \\ \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)\end{gathered}}$$

Fixed Point Stability Analysis

$$M = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) & \frac{\partial}{\partial y}\left(\frac{dx}{dt}\right) \\ \frac{\partial}{\partial x}\left(\frac{dy}{dt}\right) & \frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) \end{bmatrix}$$

$$M = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}$$

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$$\boxed{M = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}}$$

$$M = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$$

$$\begin{vmatrix} \alpha - \lambda & 0 \\ 0 & -\gamma - \lambda \end{vmatrix} = 0$$

$$(\lambda - \alpha)(\lambda + \gamma) = 0$$

$$\lambda = \alpha,\ -\gamma$$

Fixed Point Stability Analysis

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So the fixed point is SADDLE

Fixed Point Stability Analysis

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Stationary Point

$$\boxed{\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)}$$

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$$\boxed{M = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}}$$

$$M = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \alpha\gamma = 0$$

$$\lambda = \pm i\sqrt{\alpha\gamma}$$

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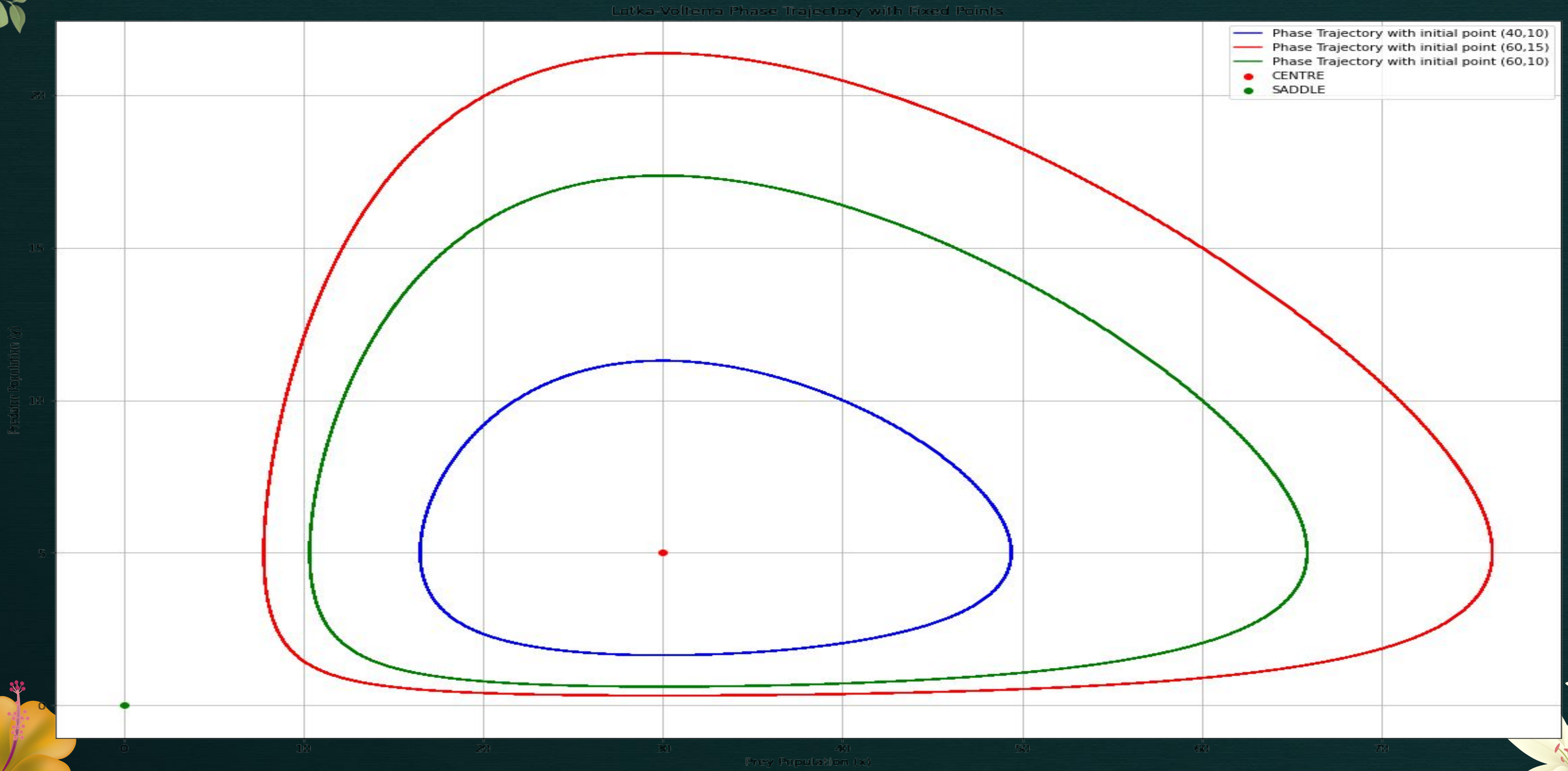
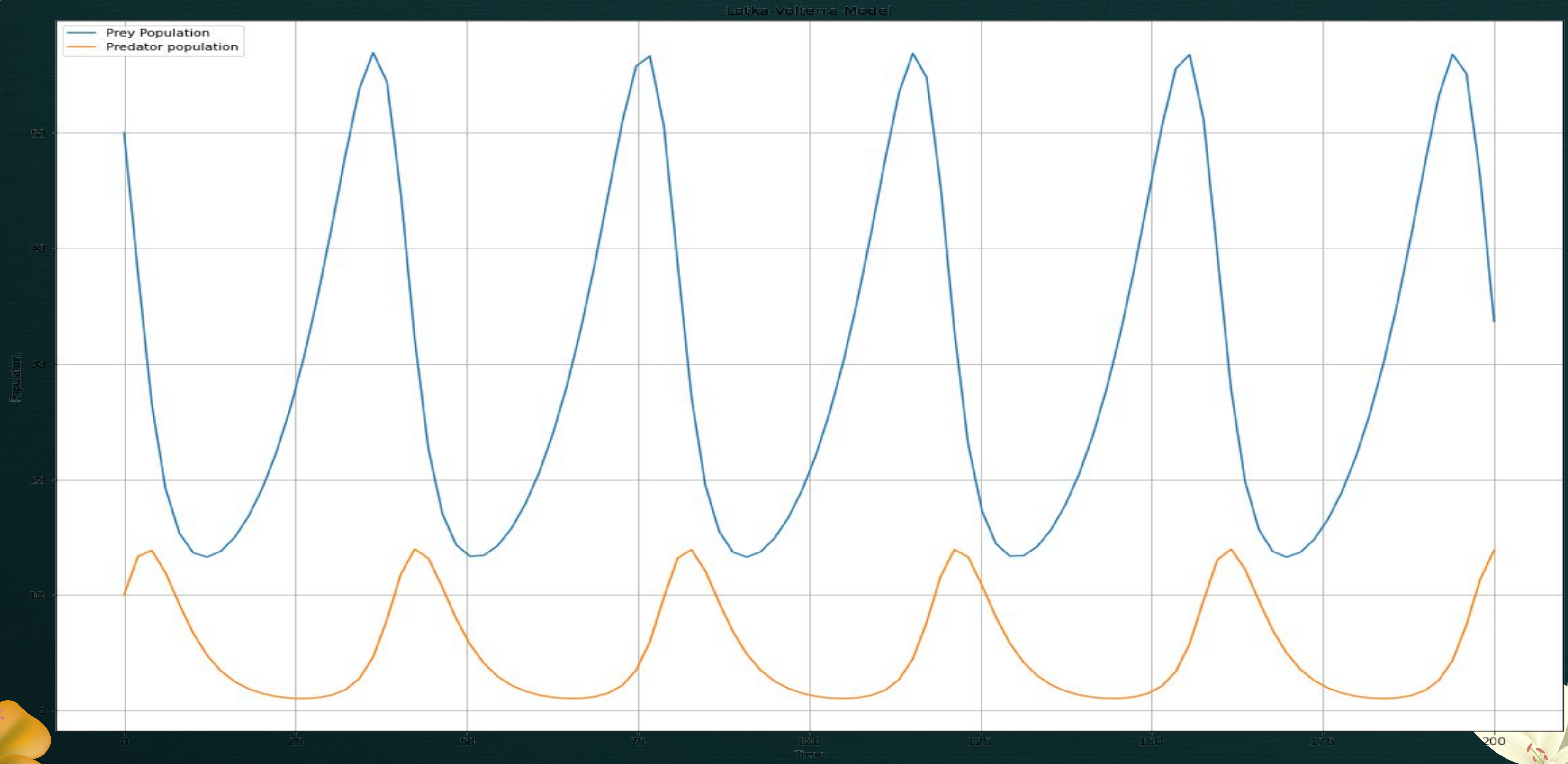
So the fixed point is CENTRE

Phase – Portrait of Lotka Volterra model



Time series of Predator and prey



Motivation to Apply NLD in Ecological Contexts

Non linear dynamics is all about predicting things which are extremely hard to predict, and there's hardly anything more unpredictable than mother nature herself. Like the battle between an immovable object and an unstoppable force, deterministic equations applied to the wildly diverse natural world seems to be an odd combination but one that is useful and ends up matching our intuitions in alot of ways as we shall see.

To just give you a taste of why dealing with ecology is not to be left upto guesswork, here is a historical cautionary tale that speaks to how chaos can ensue in nature:

OPERATION CAT DROP, Borneo:

1) WHO decided to spray DDT to stop the spread of malaria on the island of borneo
2) Due to high levels of DDT, lizards on the island became poisoned as well, killing the cats who ate them
3) Due to a lack of cats, the rat population boomed and they eventually started infesting the island!
4) Therefore they were left with no resort but to stop spraying DDT and airdrop 14,000 Cats to stop the ensuing madness, and now this has become a prime example for unintended consequences in nature.

Budworms in the North Eastern US- The Model

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But how do we make sense of the seemingly sudden outbreaks of budworms that cause the entire forest to die out in the first place? Is it random or is there some underlying mechanism that can be modelled?

Ludwig et. al proposed the following model:

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - p(N).$$

Budworms in the North Eastern US- The Model





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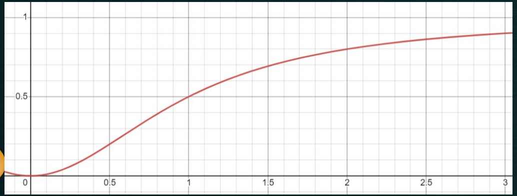
The predation term is propounded to be sigmoidal in shape, and so was chosen to finally be, for simplicity:



$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}.$$

Theory behind the parameters:

1. r_b is the linear birth rate of the budworms, represents the linear growth term in the logistic growth model. Increases with size of the forest
2. K_b is the carrying capacity of the forest, the max number of budworms it could possibly hold, relates to the density of foliage on trees, the more the merrier.
3. B refers to the max possible predation that birds or other prey could possibly do in this time frame, essentially we are holding the population of birds and other predators are roughly constant for the time being.
4. A refers to the point at which predation becomes ‘active’, as in its likely that the budworms will go unnoticed until their population reaches a certain level, predators would rather eat something else than search for the relatively tiny number of budworms. So this manifests as A being the scale at which the population is relevant. Again proportional to size of the forest

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Note on non-dimensionalisation:

$$u = \frac{N}{A}, \quad r = \frac{Ar_B}{B}, \quad q = \frac{K_B}{A}, \quad \tau = \frac{Bt}{A}$$

The equation then becomes:

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u; r, q),$$

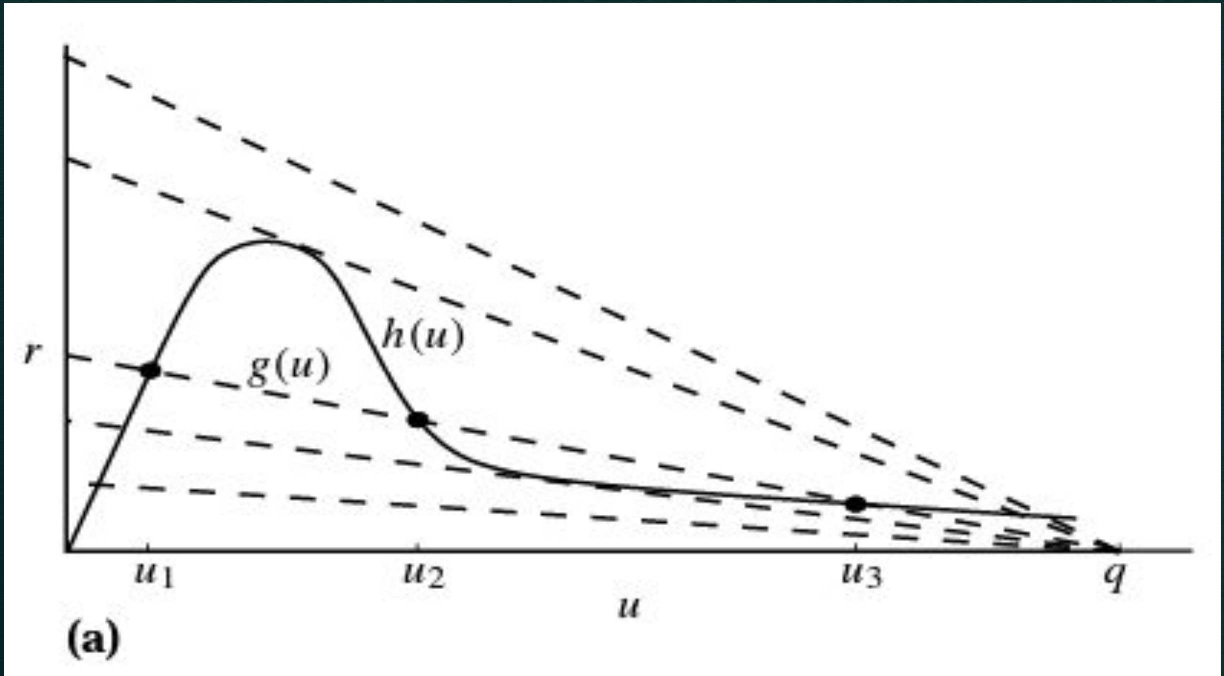
Thus the fixed points are solutions to:

$$f(u; r, q) = 0 \quad \Rightarrow \quad ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = 0.$$

Since u=0 is a trivial solution, we can choose to look at :

$$r\left(1 - \frac{u}{q}\right) = \frac{u}{1+u^2}.$$

Phase Diagram for the insect population : 3 relevant graphs

1. 

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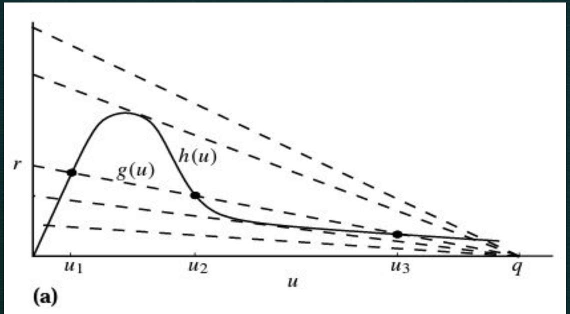

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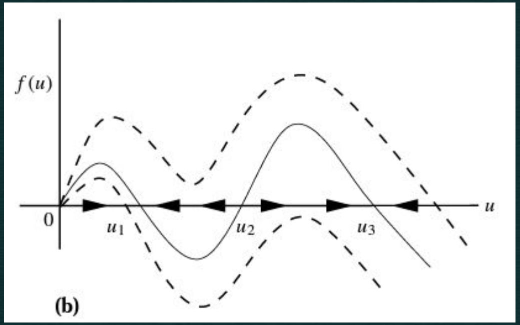

When they exist the fixed points have the following interpretation:
U1– refuge level, budworms are not existential threats.
U2– threshold value, don’t want to let them cross this value ever.
U3–outbreak.

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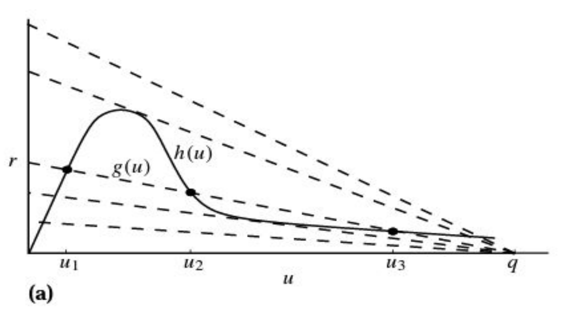

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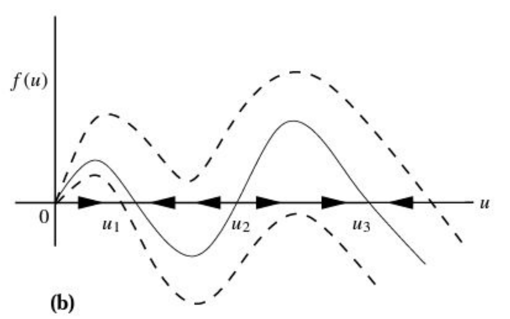

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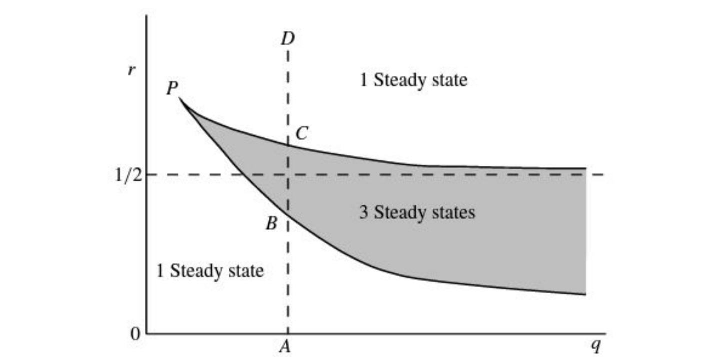

Figure 1.6. Parameter domain for the number of positive steady states for the budworm model (1.8). The boundary curves are given parametrically (see Exercise 1) by $r(a) = 2a^3/(a^2+1)^2$, $q(a) = 2a^3/(a^2-1)$ for $a \geq \sqrt{3}$, the value giving the cusp point P.

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Graphs: murray mathematical biology

Hysteresis In the Jungle Kingdom- Cusp Catastrophe

1. So far we have treated the parameters r_B, K_B, A and B as constants but there is a caveat to that, they are roughly constant on the timescale of months to a year which is the reproduction cycle of the budworm larvae, however they all tend to change as the forest grows or shrinks.

$$u = \frac{N}{A}, \quad r = \frac{Ar_B}{B}, \quad q = \frac{K_B}{A}, \quad \tau = \frac{Bt}{A}$$

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2. We brought down the analysis to two essential parameters: r and q in the previous slides.
   a. q remains constant even on the longer time scale as it is proportional to the ratio of two things which increase with the size of the forest.
   b. R on the other hand changes and increases over time, which is not good for us at all given the r-q diagram! What this means is over time the phase diagram shifts from 3 fixed points to 1 crossing the upper cusp as the forest grows which means the number of budworms suddenly will increase and the forest dies out because they eat too many leaves.

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3. It is often too late to save the forest as well at this point since when the forest size decreases, we return to 3 fixed points yet, C is a stable Node and B is an unstable one, which means the budworms will stay at C until the forest is so small we are on the lower cusp,at which point the forest is pretty much knocked out! Hysteresis! Only when the forest dies out completely and r again becomes low, do we see the budworms reducing to their original level

Cyclical Nature- predator prey systems:

Compare this to the model of the predator and prey that bibaswan discussed earlier, here the rise in the number of budworms is related to the the declining rate of balsam firs, and birch trees are in competition with the balsams for the same resources.

However the balsams, even when low in number are superior in their utilisation of those resources!

Therefore we can think of the balsams and firs as kind of lotka volterra system where the growth of both is coupled to each other.

As you saw earlier there is a rather cyclical nature to predator prey systems and similar is the case here! Over the time scale of decades the birch trees are again replaced and eventually the whole cycle repeats!

Conclusions:

Adarsh Showed how introducing even a single parameter into a non linear model can create so many different types of behaviours in the form of saddle node and pitchfork bifurcations, as well as showing you the hysteresis that arises from this inclusion

Bibaswan Introduced to you the concept of Predator-prey models and their complexity and approach to finding solutions while focussing on the lotka volterra model which is the fundamental model of this kind

Yashowardhan explained the approach to studying the dynamics of one of the most well studied ecological models, which is that of spruce budworms and how sometimes ecological collapse is completely unavoidable. Along the way we saw how non dimensionalisation is essential to developing any model.

Bibliography:

Non-Linear Dynamics & Chaos - Steven Strogatz
Murray Mathematical Biology - James D. Murray