Schwinger's Oscillator

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1 Introduction

There exists a connection between angular momentum algebra and algebra of two uncoupled oscillators.
Note: $\hbar = 1$ units everywhere

1.1 Angular momentum algebra

Let us define $\hat{J}_+$ and $\hat{J}_-$ ladder operators such that,

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \tag{1}$$
$$\hat{J}_- = \hat{J}_x - i\hat{J}_y \tag{2}$$

The angular momentum algebra is given by the following relations:-

$$\hat{J}_+|j,m\rangle = \sqrt{(j-m)(j+m+1)} \quad |j,m+1\rangle \tag{3}$$
$$\hat{J}_-|j,m\rangle = \sqrt{(j+m)(j-m+1)} \quad |j,m-1\rangle \tag{4}$$
$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k \tag{5}$$

where |j,m> is the representation of states in this algebra.

1.2 Harmonic oscillator algebra

Let's say that we have two uncoupled oscillators. Operators of 1st oscillator are denoted with subscript 1 and the other by subscript 2. By uncoupled, we mean that all the operators of one oscillator would commute with the other. That is,

If,

1st Oscillator	**2nd Oscillator**
$N_1 = a_1^\dagger a_1$	$N_2 = a_2^\dagger a_2$
$[a_1, a_1^\dagger] = 1$	$[a_2, a_2^\dagger] = 1$
$[N_1, a_1] = -a_1$	$[N_2, a_2] = -a_2$
$[N_1, a_1^\dagger] = a_1^\dagger$	$[N_2, a_2^\dagger] = a_2^\dagger$

Then,

$$[N_1, N_2] = 0, \quad [a_1, a_2^\dagger] = 0, \quad [a_1, a_2] = 0 \quad \text{and so on.} \tag{6}$$

Now, since N_1 and N_2 commute, we can build simultaneous eigenstates with eigenvalues n_1 and n_2. Therefore,

$$N_1|n_1,n_2\rangle = n_1|n_1,n_2\rangle \quad (7)$$
$$N_2|n_1,n_2\rangle = n_2|n_1,n_2\rangle \quad (8)$$
$$(9)$$

Also,

$$a_1|n_1,n_2\rangle = \sqrt{n_1}|n_1,n_2\rangle \quad (10)$$
$$a_1^\dagger|n_1,n_2\rangle = \sqrt{n_1+1}|n_1,n_2\rangle \quad (11)$$
$$a_2|n_1,n_2\rangle = \sqrt{n_2}|n_1,n_2\rangle \quad (12)$$
$$a_2^\dagger|n_1,n_2\rangle = \sqrt{n_2+1}|n_1,n_2\rangle \quad (13)$$

New Definitions

Let , $\hat{J}_+ = \hbar a_1^\dagger a_2$, $\hat{J}_- = \hbar a_2^\dagger a_1$ and $\hat{J}_z = \frac{\hbar}{2}(N_1 - N_2)$

Using the new definitions we operate them on the HO states,

$$\hat{J}_+|n_1,n_2\rangle = \sqrt{n_2(n_1+1)}\hbar|n_1+1,n_2-1\rangle \quad (14) \quad \texttt{\{eq:13\}}$$
$$\hat{J}_-|n_1,n_2\rangle = \sqrt{n_1(n_2+1)}\hbar|n_1-1,n_2+1\rangle \quad (15) \quad \texttt{\{eq:14\}}$$
$$\hat{J}_z|n_1,n_2\rangle = \frac{1}{2}(n_1-n_2)\hbar|n_1,n_2\rangle \quad (16) \quad \texttt{\{eq:15\}}$$

1.3 Connection

We see that if we replace $n_1 \equiv j+m$ and $n_2 \equiv j-m$ in eq (14), (15) and (16), we get back the angular momentum algebra. Therefore, there is a connection between angular momentum algebra and uncoupled oscillators. In subsequent sections, we try to derive the angular momentum algebra working only in the Harmonic oscillator language.

2 N Harmonic Oscillators

Say we have N harmonic oscillators i.e operators $a_i, a_i^\dagger$ with $i = 1,\ldots N$ which satisfy the following commutators

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i^\dagger, a_j^\dagger] = 0 = [a_i, a_j] \quad (17)$$

Construct the following operators

$$J_{ij} \equiv a_i^\dagger a_j, \qquad \forall \quad 1 \le i,j \le N \quad (18)$$

Note that there are N^2 such operators. These operators satisfy the commutation relations

$$\begin{aligned} [J_{ij}, J_{kl}] &= [a_i^\dagger a_j, a_k^\dagger a_l] \\ &= a_i^\dagger [a_j, a_k^\dagger] a_l + a_k^\dagger [a_i^\dagger, a_l] a_j \\ &= \delta_{jk} J_{il} - \delta_{il} J_{kj} \end{aligned} \quad (19) \quad \texttt{\{eq:sunalgebra\}}$$

Also note that the following linear combination of J operators

$$Q \equiv \sum_{i=1}^{N} J_{ii} \implies [Q, J_{ij}] = 0 \qquad \forall \quad 1 \le i, j \le N \tag{20}$$

Exercise: Do an explicit check of eq (19) for N=2 (Schwinger's Oscillators)
Solution : We know that,

$$\begin{aligned} \hat{J}_+ = a_1 \dagger a_2 \quad \hat{J}_- = a_2^\dagger a_1 \\ N_1 = a_1^\dagger a_1 \quad N_2 = a_2^\dagger a_2 \\ N = N_1 + N_2 \\ \hat{J}_z = \frac{\hbar}{2}(N_1 - N_2) \end{aligned}$$

Also,

$$[J_z, J_+] = J_z \tag{21}$$
{eq:20}

$$[J_+, J_-] = 2J_z \tag{22}$$
{eq:21}

From our HO definitions,

$$\begin{aligned} J_{11} \equiv N_1 \qquad J_{12} \equiv J_+ \\ J_{21} \equiv J_- \qquad J_{22} \equiv N_2 \\ Q \equiv N \qquad J_z \equiv \frac{J_{11} - J_{22}}{2} \end{aligned}$$

checking for eq (21) and eq (22) using eq (19),

$$\begin{aligned} [J_z, J_+] &= \frac{1}{2}[J_{11} - J_{22}, J_{21}] \\ &= \frac{1}{2}(\delta_{11} J_{12} - \delta_{12} J_{11} - \delta_{21} J_{22} + \delta_{22} J_1 2) \\ &= \frac{1}{2}(J_{12} + J_{12}) \\ &= J_{12} \\ &\equiv J_+ \end{aligned}$$

And,

$$\begin{aligned} [J_+, J_-] &= [J_{12}, J_{21}] \\ &= \delta_{22} J_{11} - \delta_{11} J_{22} \\ &\equiv J_z \qquad \text{upto some factors} \end{aligned}$$

2.1 States of N Harmonic Oscillators

Let us denote the state which is annihilated by all oscillators to be

$$|\mathbf{0}\rangle \equiv |0,0,\dots 0\rangle \tag{23}$$

The state space of N Harmonic oscillators is given by

$$|n_1,n_2,\dots,n_N\rangle \equiv (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\dots|\mathbf{0}\rangle \qquad\qquad n_i \in \mathbb{Z}, n_i \geq 0 \tag{24}$$ {eq:StateHarmonic}

Note that the action of the operator Q defined above is very simple on such a state :

$$Q|n_1,n_2,\dots,n_N\rangle = (n_1+n_2+\dots n_N)|n_1,n_2,\dots,n_N\rangle \tag{25}$$

2.1.1 Representations of J operators

We are interested in finding representations of the J operators. By this we mean a subspace of states of the form eq (24) on which the action of J operators close. To simply the problem, first note that $[Q,J]=0$, so we might as well focus on fixed Q sectors.

$Q=1$ sector Note that for $Q=1$, states are of the form

$$|1,0,\dots\rangle \ , \ |0,1,0,\dots\rangle \ , \ \dots|0,\dots,0,1\rangle \tag{26}$$ {eq:Q=1states}

Exercise: Check number of states for Q=2,3
Solution:
Q=2

$|2,0,...,0\rangle,|0,2,0,...,0\rangle$ ..... N states
$|1,1,0,...,0\rangle,|1,0,1,...,0\rangle$ ..... ${}^{N}C_2$ states
Total = N + ${}^{N}C_2$

Q=3

$|3,0,...,0\rangle,|0,3,0,...,0\rangle$ ..... N states
$|1,2,0,...,0\rangle,|1,0,2,...,0\rangle$ ..... ${}^{N}C_2$ states
$|2,1,0,...,0\rangle,|2,0,1,...,0\rangle$ ..... ${}^{N}C_2$ states
$|1,1,1,0,...,0\rangle,|1,0,1,1,...,0\rangle$ ..... ${}^{N}C_3$ states
Total = N + $2*{}^{N}C_2+{}^{N}C_3$

For general Q

$a_{i_1}^\dagger a_{i_2}^\dagger\dots a_{i_N}^\dagger|\mathbf{0}\rangle \qquad \forall i_1,i_2,\dots i_Q \in 1,2,\dots,N$

Total = ${}^{N+Q-1}C_Q$ states

Since there are N of such states, they form an N dimensional representation. More precisely, the action of operators J_{ij} on these states can be represented by $N \times N$ matrices which we will denote by $\mathscr{D}(J_{ij})$. For example, for the operator J_{12}

$$J_{12}|0,1,\ldots,0\rangle = |1,0,0,\ldots\rangle \tag{27}$$

whereas for every other state in eq (26) we have that the action of J_{12} vanishes. We can hence represent the action J_{12} on the states eq (26) by a matrix. More explicitly, if we label the $|1,0,\ldots\rangle$ by row 1 and $|0,1,0,\ldots\rangle$ by row 2 and so on, the matrix corresponding to J_{12} is given by

$$\mathscr{D}(J_{12}) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \tag{28}$$

and so on. If one thinks through this carefully the matrix corresponding to operator J_{ij} will be zero everywhere except i'th row j'the coloumn. i.e

$$[\mathscr{D}(J_{ij})]_{\alpha,\beta} = \delta_{i\alpha}\delta_{j\beta} \tag{29}$$ {eq:Repsun}

where we have labelled the row / columns by the indices α,β with $1 \le \alpha,\beta \le N$. We will sometimes use M_{ij} as a shorthand for the matrix $\mathscr{D}(J_{ij})$.

Note that the commutation relations of $\mathscr{D}(J_{ij})$ matrices follows from the commutations rules given in eq(19) (CHECK THIS EXPLICITLY TO MAKE SURE). i.e

$$[\mathscr{D}(J_{ij}),\mathscr{D}(J_{kl})] = \delta_{jk}\mathscr{D}(J_{il}) - \delta_{il}\mathscr{D}(J_{kj}) \tag{30}$$

Exercise: Do an explicit check of eq (19)
Solution: Say we do it for J_{12} and J_{21}.
<u>LHS</u> :-

$$\begin{aligned} &[D(J_{12}),D(J_{21})] \\ =&D(J_{12})D(J_{21}) - D(J_{21})D(J_{12}) \end{aligned}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}
$$

$$
= J_{11} - J_{22}
$$

RHS :-

$$
\begin{aligned} &\delta_{22} J_{11} - \delta_{11} J_{22} \\ &= J_{11} - J_{22} \end{aligned}
$$

Hence proved.

2.2 A different basis

We will write the states given in eq (24) in a different basis. We choose to label the states by the eigenvalues of $J_{ii} - J_{i+1} \quad \forall\ i \in 1, \dots N-1$. More explicitly,

$$
\boxed{n_1 - n_2, n_2 - n_3, ...} \equiv |n_1, n_2, \dots\rangle \tag{31}
$$

For instance, the states $|1,0,\dots\rangle \leftrightarrow \boxed{|1,0,\dots\rangle}$ and $|0,1,0,\dots\rangle \leftrightarrow \boxed{-1,1,0,\dots}$ The basis $\boxed{\lambda_1, \lambda_2, \dots}$ is the Dynkin basis of SU(N) which will be encountered later. The $Q = 1$ representation states for N=4 are given by,

$$
\begin{gathered} \boxed{1,0,0} \\ \boxed{-1,1,0} \\ \boxed{0,-1,1} \\ \boxed{0,0-1} \end{gathered}
$$

For $Q = 2$ sector and N=3, the states are,

$$
\begin{gathered} |2,0,0\rangle \leftrightarrow \boxed{2,0} \\ |1,1,0\rangle \leftrightarrow \boxed{0,1} \\ |0,2,0\rangle \leftrightarrow \boxed{-2,2} \\ |1,0,1\rangle \leftrightarrow \boxed{1,-1} \\ |0,1,1\rangle \leftrightarrow \boxed{-1,0} \\ |0,0,2\rangle \leftrightarrow \boxed{0,-1} \end{gathered}
$$

2.3 Comments on representations of SU(N)

It must be obvious that the representations obtained above is the same as Q symmetric boxes of SU(N) representation. Using Young's tableau, these representations are given by,

$$\underbrace{\square\square\square\cdots\square\square\square}_{\text{Q boxes}}$$

The Dynkin index is,

$$\boxed{Q,0,\ldots}$$

Question : How obtain representations which have vertical boxes in the diagram? That is,

$$\begin{array}{l}\overbrace{\square\square\square\cdots\square\square\square}^{\text{Q-1 boxes}}\\ \square\end{array} \longleftrightarrow \boxed{Q-1,1,0\ldots}$$

One way out is to introduce more oscillators.

3 Multiple N harmonic Oscillators

We introduce more oscillators to address the issue above. Say we have multiple oscillators of the form,

$$a_{i_\alpha,\alpha}, a^\dagger_{i_\alpha,\alpha} \qquad \text{where} \qquad i_\alpha \in \mathbb{Z}_+$$

We will refer to oscillators with different labels of α as *flavours*. The construction of state space is then given by,

$$\prod_{i_\alpha,\alpha} (a^\dagger_{i_\alpha,\alpha})^{n_{i_\alpha,\alpha}} |\mathbf{0}\rangle \tag{32}$$

The definition of Q is as usual,

$$Q \equiv \sum_{i,\alpha_i} a^\dagger_{i_\alpha,\alpha} a_{i_\alpha,\alpha} \tag{33}$$

The SU(N) generators are,

$$J_{ij} = \sum_\alpha a^\dagger_{i,\alpha} a_{j,\alpha}$$

For a fixed Q, let us choose to focus on states with atleast Q different types of flavours. For examples, we have the following sectors,

$Q = 1$ sector

The states in this sector are the same as before except that a_i is now replaced with $a_{i,1}$

$Q = 2$ sector

In this sector we will have the states in the following form,

$$a^\dagger_{i,1} a^\dagger_{j,2} |\mathbf{0}\rangle \qquad \forall\ i,j \in 1,\ldots,N \tag{34}$$

Now the difference is that it is possible to have two irreps of SU(N):

$$\left(a^\dagger_{i,1} a^\dagger_{j,2} \pm a^\dagger_{j,1} a^\dagger_{i,2}\right) |\mathbf{0}\rangle$$

⋮

⋮

And so on.

4 $U(N)$

$U(N)$ is the group of all unitary $N \times N$ matrices. For any such matrix g one can find the hermitian $N \times N$ matrix H such that

$$g \equiv \exp\{iH\}, \qquad\qquad H^\dagger = H \tag{35}$$

Note that hermitian $N \times N$ matrices are parametrized by N^2 real numbers. We can parametrize these matrices via

$$H = \sum_{i,j=1}^{N} \theta_{ij} M_{ij} \tag{36}$$

where we guarantee hermiticity by choosing $\theta_{ij} = \theta^*_{ji}$ and $M_{ij} = M^\dagger_{ji}$. One possible choice of basis matrices M_{ij} is given by $\mathscr{D}(J_{ij})$ given in eq (29). We write below the commutations between M_{ij} matrices which is just eq (29) in the new notation

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{il} M_{kj} \tag{37}$$

By definition this is called the lie algebra of group $U(N)$ and is denoted by $u(N)$.

Connection to $SU(N)$ $SU(N)$ is the group of all $N \times N$ unitary matrices whose determinant is 1. Note that any $U(N)$ matrix g can be written as a product of determinant (which is a phase) times an $SU(N)$ matrix $\tilde{g}$. i.e

$$\tilde{g} = \frac{1}{\det g}\, g \tag{38}$$

The $SU(N)$ matrices can be parametrized by

$$\tilde{g} = \exp(iH), \quad \mathrm{tr}(H) = 0, H^\dagger = H \tag{39}$$

The matrices $\tilde{H}$ can be parametrized by $\tilde{M}_{ij}$ almost same matrices M_{ij} as before, except that the trace is vanishing. More precisely,

$$\tilde{M}_{ij} = M_{ij} - \delta_{ij}\,\frac{\sum_k M_{kk}}{N} \tag{40}$$

which automatically satisfies $\mathrm{tr}(\tilde{M}_{ij}) = 0, \forall i, j$. Note that there are only $N^2 - 1$ independent matrices $\tilde{M}_{ij}$, since $\sum_{ii} \tilde{M}_{ii} = 0$. An explicit expression for the $\tilde{M}_{ij}$ matrices are given below

$$[\tilde{M}_{ij}]_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \frac{\delta_{ij}\delta_{\alpha\beta}}{N} \tag{41}$$

Also note that since $\sum_k M_{kk}$ commutes with all M_{ij}, the new matrices $\tilde{M}_{ij}$ satisfies the same algebra as before

$$[\tilde{M}_{ij}, \tilde{M}_{kl}] = \delta_{jk}\tilde{M}_{il} - \delta_{il}\tilde{M}_{kj} \tag{42}$$

This is called the lie algebra of the group $SU(N)$.

4.1 Representations of $U(N)$

By definition, a D dimensional representation of the lie algebra $u(N)$ is a way of associating a $D \times D$ matrix $\mathscr{D}(M_{ij})$ corresponding to the matrix M_{ij} such that the following commutation relations are satisfied

$$[\mathscr{D}(M_{ij}), \mathscr{D}(M_{kl})] = \delta_{jk}\mathscr{D}(M_{il}) - \delta_{il}\mathscr{D}(M_{kj}) \tag{43}$$

{eq:Def of Rep}

Two representations $\mathscr{D}_1(M_{ij}), \mathscr{D}_2(M_{ij})$ are considered equivalent if there exists a similarity transformation via some matrix S between them. i.e

$$\mathscr{D}_1(M_{ij}) = S^{-1}\mathscr{D}_2(M_{ij})S, \qquad \forall i, j \tag{44}$$

{eq:Eqrep}

Note that it is the same S for all N^2 matrices $\mathscr{D}_1(M_{ij}), \mathscr{D}_2(M_{ij})$.

4.1.1 Fundamental Representation

This representation is the obvious representation where the matrix corresponding to M_{ij} is M_{ij} itself. This is of course N dimensional representation. i.e

$$\mathscr{D}(M_{ij}) = M_{ij} \tag{45}$$

4.1.2 Anti-Fundamental Representation

We will denote this representation by $\bar{\mathscr{D}}$. This is also an M dimensional representation but the difference from the Fundamental representation is that the matrix corresponding to M_{ij} is $-M_{ij}^T$

$$\bar{\mathscr{D}}(M_{ij}) = -M_{ij}^T \tag{46}$$

{eq:40}

CHECK THAT THIS SATISFIES THE RELATIONS eq (43).

Exercise: Check eq(46) satisfies eq(43)
Solution:

$$\begin{aligned}\bar{\mathscr{D}}(M_{ij}) &= -M_{ij}^T \\ &= -M_{ji}\end{aligned}$$

LHS:-

$$\begin{aligned}&[\mathscr{D}(M_{ij}), \mathscr{D}(M_{kl})] \\ &= [M_{ji}, M_{kl}] \\ &= \delta_{il} M_{jk} - \delta_{jk} M_{li}\end{aligned}$$

RHS:-

$$\begin{aligned}&= \delta_{jk}(-M_{li}) - \delta_{li}(-M_{jk}) \\ &= \delta_{il} M_{jk} - \delta_{jk} M_{li}\end{aligned}$$

Hence proved.

References

[1] A J Macfarlane. "On q-analogues of the quantum harmonic oscillator and the quantum group SU(2)q". In: *Journal of Physics A: Mathematical and General* 22.21 (Nov. 1989), pp. 4581–4588. DOI: 10.1088/0305-4470/22/21/020. URL: https://doi.org/10.1088/0305-4470/22/21/020.