Gamma Ising Model

MAJOR PROJECT REPORT SUBMITTED FOR THE DEGREE OF MASTER OF SCIENCE (PHYSICS)

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MAY 2021

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Dr. Amit Kumar Pal, Dr. Kusum Dhochak, Dr. Prithvi Narayan and Dr. Uma Divakaran for their continuous support and guidance throughout the lifecycle of this project.

I would like to thank my project partner and my friend, Pram Milan, for all the long insightful discussions we have had, related/not related to physics.

I would like to thank Neeraj Kumar for sharing this wonderful template with me and in turn saving me from all the time and hassle required to work on it.

Special thanks to Dr. Prithvi Narayan who has been a great source of inspiration for me, under whom I've learnt a lot in these two years, which will hopefully stay with me for a lifetime.

Special thanks to Dr. Kusum Dhochak, for her patience and support during all our discussions and doubt clearing sessions.

I would also like to thank all my friends and batchmates at IIT-Palakkad, for all the late night discussions and basically making my stay here much more enjoyable than I expected.

Last but not the least, I am indebted to my parents for the unconditional support they have shown in whatever decisions I've made in life.

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1 ABSTRACT

The aim of this project is to generalise the Transverse field Ising model (TFIM) by introducing more degrees of freedom at each lattice site in the 1D Ising chain and investigate whether this model exhibits Quantum Phase Transition (QPT). Traditionally, there are two degrees of freedom available at each lattice site (spin up and spin down) in a TFIM. We increase the dimension of hilbert space at each site by replacing pauli matrices with higher dimensional matrices called gamma matrices. Next, we come up with a model Hamiltonian and generalize the standard techniques (used to solve TFIM) to arrive at energy dispersion relation and explore the QPTs.

2 INTRODUCTION

Exactly solvable models are useful in gaining insight into many-particle systems. If the problem to be solved can be related to an exactly solvable one, however vaguely, one can usually gain some insight. One can explore different aspects (often physical quantities) of exactly solvable models in condensed matter physics such as behaviours relating to symmetry breaking, phases transitions, thermodynamic quantities, universality etc. One of the few exactly solvable models where we can actually compute thermodynamic quantities and interpret them is the Transverse field Ising model (TFIM).

It was introduced in 1960s to study an order-disorder transition in hydrogen-bonded ferroelectric systems. TFIM is an exactly solvable model that has been studied extensively. A key feature of this setup is that, in a quantum sense, the spin projection along the direction of interaction term and the spin projection along the field term are not commuting observable quantities. That is, they cannot both be observed simultaneously. This means that classical statistical mechanics cannot describe this model, and a quantum treatment is needed.

Another important attribute of such models is phase transitions. Phase transitions are identified by sudden changes in macroscopic properties of the system as a control parameter is varied. In this report, we will consider only subclass of phase transitions, called Quantum Phase Transitions (QPT). QPTs are identified by an abrupt change in the ground state of a many body system. In classical Ising model, the phase transition is driven by thermal fluctuations with temperature as a control parameter. But QPTs occur as a result of competing ground state phases, and is purely driven by quantum fluctuations with physical control parameters such as pressure, external field etc at absolute zero temperature. TFIM is one of the models which displays QPT. Our aim is to go beyond TFIM by generalising it and studying exotic features of this larger class of exactly solvable model.

The outline of the project is as follows: We start with a brief introduction of the TFIM in Section 3, revisiting the techniques used to exactly solve TFIM. We then inspect the behaviour of energy gap near criticality. In Section 4, we follow similar steps used to exactly solve TFIM, towards generalisation of TFIM. Starting with gamma matrices, we construct the Hamiltonian and follow the standard techniques to solve this model. Later, we explore the behaviour of this model near phase transition point, ending the the discussion with concluding remarks and future scope of the project.

3

QUANTUM MECHANICAL 1D ISING MODEL: A BRIEF OVERVIEW

Transverse field Ising model is essentially a quantum model and a prototypical example of a Quantum Phase Transition (a transition driven by quantum fluctuations instead of thermal fluctations). Since the phase transitions are driven purely due to quantum fluctuations, transitions occurring in this model are only in $T = 0$k limit. Interactions between spin-like degrees of freedom can describe the order-disorder transition in many systems and TFIM successfully describes a class of such systems.

3.1 PAULI MATRICES

Before beginning with analyzing the TFIM, it is a good idea to refresh the basic algebraic properties of the Pauli matrices, since they will come in handy in the next section.

The pauli spin matrices are defined as,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

They are involuntary,

$$(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^y)^2 = \mathbb{1} \tag{2}$$

with each of them having eigenvalues ± 1

They obey the following commutation and anti commutation relations,

$$[\sigma^a, \sigma^b] = 2i\epsilon_{abc}\sigma^c \tag{3}$$

$$\{\sigma^a, \sigma^b\} = 2\delta_{ab}\mathbb{1} \tag{4}$$

We can also define raising and lowering operators using pauli matrices such that,

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y) \qquad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y) \tag{5}$$

3.2 CONSTRUCTION OF HAMILTONIAN

The Hamiltonian of TFIM of spins on a 1D lattice is given by,

$$H = -\mathrm{J}\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - h\sum_i \hat{\sigma}_i^z \qquad \text{where} \quad h = \mathrm{J}g \tag{6}$$

Here J is the interaction strength between the spins and h is the external magnetic field. Before we begin to understand the TFIM, notice that this Hamiltonian shows a discrete global $\mathcal{Z}_2$ symmetry (π rotation about the field direction),

$$\sigma^x \rightarrow -\sigma^x \tag{7}$$
$$\sigma^y \rightarrow -\sigma^y \tag{8}$$
$$\sigma^z \rightarrow \sigma^z \tag{9}$$
$$H \rightarrow H \tag{10}$$

To obtain a fundamental understanding of the physics of the TFIM, we will consider two simple limits, namely the strong field limit and the zero field limit.

Strong field limit

In the strong field limit $h >> J$, the Hamiltonian becomes,

$$H = -h\sum_i \hat{\sigma}_i^z \tag{11}$$

Then the Hamiltonian is block diagonal in every spin subspace and the local Hilbert spaces for every site i decouple. The system describes free spins with a magnetic field along the z axis, therefore the ground state becomes,

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow ...\rangle \tag{12}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ^z

Note that there is a $\mathcal{Z}_2$ symmetry preserved by the ground state in this limit as,

$$M_x = \langle 0|\,\sigma^x\,|0\rangle = 0 \tag{13}$$
$$M'_x = \langle 0| - \sigma^x\,|0\rangle = 0 \implies M_x = M'_x \tag{14}$$

Weak field limit

In the weak field limit $h << J$, the Hamiltonian becomes,

$$H = -J\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \tag{15}$$

which is the classical 1D Ising model. At $T = 0\ k$, the ground state is doubly degenerate,

$$|0\rangle = |\rightarrow\rightarrow\rightarrow\rightarrow ...\rangle \qquad or \qquad |\leftarrow\leftarrow\leftarrow\leftarrow ...\rangle \tag{16}$$

where $|\rightarrow\rangle = \frac{|\uparrow\rangle+|\downarrow\rangle}{2}$ and $|\leftarrow\rangle = \frac{|\uparrow\rangle-|\downarrow\rangle}{2}$

Both the states have same energy eigenvalues. These states break the $\mathcal{Z}_2$ symmetry as,

$$M_x = \langle 0|\,\sigma^x\,|0\rangle = 1 \tag{17}$$
$$M'_x = \langle 0| - \sigma^x\,|0\rangle = -1 \implies M_x \neq M'_x \tag{18}$$

Therefore, the ground state spontaneously breaks $\mathcal{Z}_2$ symmetry. So it is natural to expect that there is a phase transition occurring somewhere between the two limits, that is, when J is of the order h.

The Ising chain has N lattice sites with each site representing a 2-dimensional hilbert space (spin 1/2 particle). Such a spin 1/2 system can be mapped to a system of spinless fermion with spin-up state associated with an unoccupied site and spin-down state associated with an occupied site.

We need to find out transformations such that the following commutation and anti commutations relations are preserved,

$$\{c_i^\dagger, c_j\} = 0 \tag{19}$$

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \tag{20}$$

Following the intuition of Jordan and Wigner, we define spin operators as,

$$\sigma_i^z = (1 - 2c_i^\dagger c_i)$$

$$\begin{aligned} \sigma_i^+ &= \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i \\ &= \prod_{j<i}\sigma_j^z c_i \end{aligned} \tag{21}$$

$$\begin{aligned} \hat{\sigma}_i^- &= \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i^\dagger \\ &= \prod_{j<i}\hat{\sigma}_j^z c_i^\dagger \end{aligned} \tag{22}$$

Analogously, using $\sigma_j^z = (\sigma_j^z)^{-1}$, we can write,

$$c_i = (\prod_{j<i}\sigma_j^z)\sigma_i^+ \tag{23}$$

$$c_i^\dagger = (\prod_{j<i}\sigma_j^z)\sigma_i^- \tag{24}$$

which satisfy the relations given in (19),(20). Also,

$$\sigma_i^x = \prod_{j<i}(1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger) \tag{25}$$

After applying the JW transformations given in (21) to the TFIM Hamiltonian (6), we get

$$H = -\sum_{i=1}^{N}[J(c_{i+1}c_i + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger) - 2hc_i^\dagger c_i + h] \tag{26}$$

3.4 FOURIER TRANSFORMATION

We exploit the translation symmetry present in the problem. So we apply a fourier transform and move to momentum space k. We define,

$$c_i = e^{\frac{i\pi}{4}} \sum_q e^{\frac{i2\pi q}{N}} c_q \tag{27}$$

$$c_q = e^{\frac{-i\pi}{4}} \sum_i^N e^{\frac{-i2\pi q}{N}} c_i \tag{28}$$

where,

$$k = \frac{2\pi q}{N} \tag{29}$$

Using the above definitions, the Hamiltonian we get, is given by,

$$H^{\pm} = \sum_q 2\left[h - \mathrm{J}cos(\frac{2\pi}{N}q)\right] c_q^\dagger c_q + \mathrm{J}\sum_q sin(\frac{2\pi}{N}q)\left[c_q c_{-q} + c_{-q}^\dagger c_q^\dagger\right] - h \tag{30}$$

This Hamiltonian can be written as a sum of 2x2 matrices,

$$H = \sum_q \begin{pmatrix} c_q^\dagger & c_{-q} \end{pmatrix} \underbrace{\begin{pmatrix} h - \mathrm{J}cos(\frac{2\pi q}{N}) & -\mathrm{J}sin(\frac{2\pi q}{N}) \\ -\mathrm{J}sin(\frac{2\pi q}{N}) & \mathrm{J}cos(\frac{2\pi q}{N}) - h \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix} \tag{31}$$

3.5 BOGOLIUBOV TRANSFORMATION

We map this Hamiltonian to a new set of fermionic operators χ_k via Bogoliubov transformation.

$$\begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix} = \begin{bmatrix} cos\theta_q & sin\theta_q \\ -sin\theta_q & cos\theta_q \end{bmatrix} \begin{bmatrix} \chi_q \\ \chi_{-q}^\dagger \end{bmatrix} \tag{32}$$

which is just O(2) rotation in fourier space. Using this in (30) and demanding the off diagonal terms to be zero (matrix to be diagonal), we get

$$tan(2\theta_q) = \frac{sin(\frac{2\pi}{N}q)}{h/\mathrm{J} - cos(\frac{2\pi}{N}q)} \tag{33}$$

Therefore, the final form of Hamiltonian is,

$$H = \sum_q \epsilon\left(\frac{2\pi}{N}q\right)\{\chi_q^\dagger \chi_q - \frac{1}{2}\} \tag{34}$$

where

$$\epsilon\left(\frac{2\pi}{N}q\right) = 2\sqrt{J^2 + h^2 - 2hJ\cos\left(\frac{2\pi}{N}q\right)} \tag{35}$$

This is the energy dispersion relation. Here, $\epsilon(\frac{2\pi}{N}q)$ is the energy gap (or energy difference) between an excited state and the ground state (for single particle excitations). **Note** that energy gaps can be calculated even without performing bogoliubov transformation, as eigenvalues of $\mathcal{M}$ matrix are nothing but the single particle excitation energies. This is an important fact which will be used later in the gamma Ising model.

3.6 Quantum phase transitions

For $q \to 0$ the energy gap becomes,

$$\epsilon(0) = 2|J - h| \qquad \forall \quad J > 0 \tag{36}$$

This gap vanishes for special choice of interaction coupling constant i.e, $J = h$ which is the signature of QPT and marks the phase boundary between ordered and disordered phase.

Critical Exponents

Critical exponents describe behaviour of various physical quantities (such as susceptibility, correlation length etc) near phase transition point.There are two critical exponents which can be extracted from the energy dispersion relation. Let's call them z and ν_z.

Critical exponent νz is computed as,

$$\epsilon(0) \sim (J - h)^{\nu z} \tag{37}$$

Critical exponent z is computed as,

$$\epsilon(k \to 0) \sim k^z \tag{38}$$

For TFIM,

$$\epsilon(0) \sim (J - h) \qquad\qquad \text{with } J = h \text{ as gap closes.} \tag{39}$$

And

$$\epsilon\,(k) = 2\sqrt{J^2 + h^2 - 2hJ\cos{(k)}} \tag{40}$$

$$\epsilon(k \to 0) \approx 2\sqrt{J^2 + J^2 - 2J^2(1 - \frac{k^2}{2})} \tag{41}$$

$$\sim k \tag{42}$$

So the critical exponents in TFIM are $z = 1$ and $\nu z = 1$

4

GAMMA ISING MODEL

4.1 GAMMA MATRICES

We want to define a more generalised exactly solvable model. That is, we wish to the increase the hilbert space (or degrees of freedom) at each site. So the pauli matrices must be replaced by higher dimensional matrices (called gamma matrices) and their algebraic structure as well commutation relations must be modified accordingly. Note that gamma matrices would reduce to pauli matrices in some limit and consequently, gamma ising model would reduce to TFIM. Just as done in TFIM, we start with introducing gamma matrices and their properties, as they form building blocks of our model and will be used in construction of the Hamiltonian. Gamma matrices are d dimensional hermitian matrices which satisfy,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu} \qquad \text{where} \quad \mu, \nu = 1, 2.....2d \tag{43}$$

We define the Gamma matrices in the following manner,

$$\begin{aligned}
\Gamma^1 &= \underbrace{\sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}}_{\text{d times}} \\
\Gamma^2 &= \sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
\Gamma^3 &= \sigma^3 \otimes \sigma^1 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
\Gamma^4 &= \sigma^3 \otimes \sigma^2 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
\Gamma^5 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
\Gamma^6 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
&\vdots \\
\Gamma^{2d-1} &= \sigma^3 \otimes \sigma^3 \otimes ... \otimes \sigma^3 \otimes \sigma^1 \\
\Gamma^{2d} &= \sigma^3 \otimes \sigma^3 \otimes ... \otimes \sigma^3 \otimes \sigma^2
\end{aligned} \tag{44}$$

Each gamma matrix is a tensor product of d two dimensional matrices and there are $2d$ such gamma matrices. We can construct another gamma matrix from the above definitions. That is,

$$\Gamma^{2d+1} \equiv (-i)^d \Gamma^1 \Gamma^2 ... \Gamma^{2d} \tag{45}$$

$$= \sigma^3 \otimes \sigma^3 \otimes ... \otimes \sigma^3 \tag{46}$$

For $d = 1$, this reduces to,

$$\Gamma^3 \equiv (-i)\sigma^1\sigma^2 \tag{47}$$

$$= \sigma^3 \tag{48}$$

It follows that,

$$\{\Gamma^{2d+1}, \Gamma^{\mu}\} = 0 \qquad \text{where} \quad \mu = 1, 2 \ldots 2d \tag{49}$$

And,

$$(\Gamma^{2d+1})^2 = 1 \tag{50}$$

with Γ^{2d+1} having eigenvalues ± 1.

Constructing raising and lower operators in terms of gamma matrices,

$$\Gamma^{1\pm} = \frac{\Gamma^1 \pm i\Gamma^2}{2} \tag{51}$$

$$\Gamma^{2\pm} = \frac{\Gamma^3 \pm i\Gamma^4}{2} \tag{52}$$

$$\vdots \tag{53}$$

$$\Gamma^{d\pm} = \frac{\Gamma^{2d-1} \pm i\Gamma^d}{2} \tag{54}$$

They satisfy the following anti commutation relations $\forall i, j \in 1, 2 \ldots d$

$$\{\Gamma^{i+}, \Gamma^{j+}\} = \{\Gamma^{i-}, \Gamma^{j-}\} = 0 \tag{55}$$

$$\{\Gamma^{i+}, \Gamma^{j-}\} = \delta_{ij} \tag{56}$$

We also define a spin operator S^i which measures spin at ith site and has eigenvalues ± 1

$$S^i \equiv 2\Gamma^{i+}\Gamma^{i-} - \mathbb{1} \tag{57}$$

The explicit representation of matrix S^i is given by,

$$S^i = \mathbb{1} \otimes \ldots \otimes \underbrace{\sigma^3}_{\text{ith position}} \otimes \ldots \otimes \mathbb{1} \tag{58}$$

Then it can be verified that,

$$\Gamma^{2d+1} = \prod_i S^i \tag{59}$$

4.2 CONSTRUCTION OF HAMILTONIAN

A naive way of generalising the Hamiltonian is by replacing the pauli matrices with gamma matrices in the TFIM Hamiltonian where each lattice site on the 1D Ising chain is a 2d dimensional Hilbert space. That is,

$$H_G = \sum_{A,\mu,\nu} \mathrm{J}_{\mu\nu} \Gamma^{\mu}_{A} \Gamma^{\nu}_{A+1} + h \sum_A \Gamma^{2d+1}_A \tag{60}$$

where $\mu, \nu = 1, 2 \ldots 2d$ and $J_{\mu\nu}$ is a set of $4d^2$ coupling constants. Here A refers to the lattice index and μ, ν are the sub lattice indices (indices within each site A). For $d = 1$, this Hamiltonian is,

$$H_G = (J_{11}\sigma^1_A\sigma^1_{A+1} + J_{12}\sigma^1_A\sigma^2_{A+1} + J_{21}\sigma^2_A\sigma^1_{A+1} + J_{22}\sigma^2_A\sigma^2_{A+1}) + h\sum_A \sigma^3_A \tag{61}$$

which can be reduced to the usual TFIM for some special choice of coupling constants i.e, $J_{12} = J_{21} = J_{22} = 0$. Since each Γ^μ is a linear combination of Γ^{i+} and Γ^{i-}, the Hamiltonian given in (54) can be re-written in terms of raising and lowering operators as,

$$H_G = \sum_{A,i,j,s=\pm,s'=\pm} J_{i,j,s,s'}\Gamma^{i,s}_A\Gamma^{j,s'}_{A+1} + h\sum_A \Gamma^{2d+1}_A \tag{62}$$

where $i, j = 1, 2 \ldots d$ are still the sublattice indices and $J_{i,j,s,s'}$ is a new set of $4d^2$ coupling constants

Since we are trying to construct an exactly solvable model, we would like to use the JW formalism used in solving TFIM. It will be seen later that the application of JW transformation to this type of Hamiltonian does not give quadratic terms (in fermionic operators) and therefore, the Hamiltonian must be modified accordingly. The modified Hamiltonian, which we will work with in the rest of the project, is given by,

$$\boxed{H_G = \sum_{A,\mu,\nu} J_{\mu\nu}\Gamma^\mu_A\Gamma^{2d+1}_A\Gamma^\nu_{A+1} + \sum_{A,i} h_i S^i_A} \tag{63}$$

where $\mu, \nu = 1, 2 \ldots 2d$ and $J_{\mu\nu}$ is a set of $4d^2$ coupling constants.

In terms of raising and lowering operators,

$$\boxed{H_G = \sum_{A,i,j,s=\pm,s'=\pm} J_{i,j,s,s'}\Gamma^{i,s}_A\Gamma^{2d+1}_A\Gamma^{j,s'}_{A+1} + \sum_{A,i}\left(2h_i\Gamma^{i+}\Gamma^{i-} - h_i\right)} \tag{64}$$

where $i, j = 1, 2 \ldots d$ are still the sublattice indices and $J_{i,j,s,s'}$ is a new set of $4d^2$ coupling constants related to $J_{\mu\nu}$ (See Appendix A)

4.3 SYMMETRIES IN HAMILTONIAN

Even for the simplest generalization such as $d = 2$, we will have to work with 16 coupling constants which is a huge parameter space. It is always good to look out for symmetries which reduces the number of parameters in the problem. There is a possibility of symmetries arising for some special choice of coupling constants.

Space inversion symmetry

We demand that the hamiltonian must be invariant under space inversion (more to write on this). Without proving it here, we will state that a space inversion symmetry arises if we choose,

$$J_{i,j,s,\tilde{s}} = s\tilde{s}J_{j,i,\tilde{s},s} \tag{65}$$

The usual way of proceeding with the JW transformation is by applying a set of transformations to the Hamiltonian such that it becomes quadratic in fermionic operators. The transformations (in fermionic operators) must satisfy the following fermionic algebra,

$$\begin{aligned} \{c^i_A, c^j_B\} &= 0 \\ \{c^{i\dagger}_A, c^{j\dagger}_B\} &= 0 \\ \{c^i_A, c^{j\dagger}_B\} &= \delta_{ij}\delta_{AB} \end{aligned} \tag{66}$$

Note that there are d fermions at each site, i.e one fermion at each sublattice site.

We define the transformations as follows,

$$c^i_A = \left(\prod_k \prod_{B<A} S^k_B\right) \Gamma^{i-}_A \tag{67}$$

$$= \left(\prod_{B<A} \Gamma^{2d+1}_B\right) \Gamma^{i-}_A \tag{68}$$

$$\tag{69}$$

$$c^{i\dagger}_A = \left(\prod_k \prod_{B<A} S^k_B\right) \Gamma^{i+}_A \tag{70}$$

$$= \left(\prod_{B<A} \Gamma^{2d+1}_B\right) \Gamma^{i+}_A \tag{71}$$

It can be verified that the above transformations satisfy the fermionic algebra given in (66). (See appendix B)

This transformation can be inductively inverted by noting that $(S^i_A)^2 = 1$. Thus, the inverted transformation is,

$$\Gamma^{i-}_A = \left(\prod_k \prod_{B<A} S^k_B\right) c^i_A \tag{72}$$

$$= \left(\prod_{B<A} \Gamma^{2d+1}_B\right) c^i_A \tag{73}$$

$$\tag{74}$$

$$\Gamma^{i+}_A = \left(\prod_k \prod_{B<A} S^k_B\right) c^{i\dagger}_A \tag{75}$$

$$= \left(\prod_{B<A} \Gamma^{2d+1}_B\right) c^{i\dagger}_A \tag{76}$$

Consequently from eq (15),

$$S^i_A = 2c^{i\dagger}_A c^i_A - 1 \tag{77}$$

See appendix (C).

With all the transformations at our disposal, we try to apply it to the Hamiltonian given in (62). The interaction part in the Hamiltonian will present terms such as,

$$\Gamma_A^{i+}\Gamma_{A+1}^{j-} = c_A^{i\dagger}\left(\prod_k S_A^k\right)c_{A+1}^j \tag{78}$$

$$= c_A^{i\dagger} S_A^i \left(\prod_{k\neq i} S_A^k\right)c_{A+1}^j \tag{79}$$

$$= -\left(\prod_{k\neq i} S_A^k\right)c_A^{i\dagger}c_{A+1}^j \tag{80}$$

This term is quadratic for $d = 1$ but for $d > 1$ it is higher than quadratic in fermions. Therefore, this cannot be solved by standard methods. Suppose the Hamiltonian is of the form as given in (63). Then the interaction term becomes,

$$\Gamma_A^{i+}\Gamma_A^{2d+1}\Gamma_{A+1}^{j-} = c_A^{i\dagger}\left(\Gamma_A^{2d+1}\right)^2 c_{A+1}^j \tag{81}$$

$$= c_A^{i\dagger}c_{A+1}^j \tag{82}$$

which is quadratic and can be solved using standard methods.

The field term becomes,

$$\sum_{A,i} h_i S_A^i = \sum_{A,i}\left(2h_i c_A^{i\dagger}c_A^i - h_i\right) \tag{83}$$

$$= 2\sum_{A,i}\left(h_i c_A^{i\dagger}c_A^i\right)\underbrace{-Nh}_{\text{constant}} \qquad \text{where } h = \sum_i h_i \tag{84}$$

To write the Hamiltonian in concise form, we replace,

$$c_A^{i\dagger} = c_A^{i+} \tag{85}$$

$$c_A^i = c_A^{i-} \tag{86}$$

Then, the complete Hamiltonian after JW transformation is given by,

$$H_G = \sum_{A,i,j,s=\pm,s'=\pm} J_{i,j,s,s'} c_A^{is} c_{A+1}^{js'} + 2\sum_{A,i} h_i c_A^{i+}c_A^{i-} + \text{constant} \tag{87}$$

$$\begin{aligned} &= \sum_{A,i,j}\Big(J_{i,j,+,+}c_A^{i+}c_{A+1}^{j+} + J_{i,j,+,-}c_A^{i+}c_{A+1}^{j-} + J_{i,j,-,+}c_A^{i-}c_{A+1}^{j+} \\ &+ J_{i,j,-,-}c_A^{i-}c_{A+1}^{j-}\Big) + 2\sum_{A,i} h_i c_A^{i+}c_A^{i-} + \text{constant} \end{aligned} \tag{88}$$

It can be seen that there is a translational invariance in the Hamiltonian. As done before in TFIM, we exploit the translation symmetry by doing a fourier transformation. The fourier transforms for various terms in the Hamiltonian above are given by,

$$\sum_A c_A^{i,s} c_{A+1}^{j,s'} = \sum_q e^{\frac{-2\pi i q s}{N}} c_q^{i,s} c_{-ss'q}^{j,s'} \tag{89}$$

See appendix (D)

To make the Hamiltonian look cleaner, we do some rearrangements and introduce a few more notations to the terms in the Hamiltonian. We start with the two terms given by,

$$\sum_A \left(J_{i,j,+,-} c_A^{i+} c_{A+1}^{j-} + J_{i,j,-,+} c_A^{i-} c_{A+1}^{j+} \right) \tag{90}$$

$$= \sum_q c_q^{i+} c_q^{j-} \underbrace{(J_{i,j,+,-} e^{-\frac{2\pi i q}{N}} - J_{i,j,-,+} e^{\frac{2\pi i q}{N}})}_{\equiv 2A_{ij}(q)} \tag{91}$$

$$= 2 \sum_q A_{ij}(q) c_q^{i+} c_q^{j-} \tag{92}$$

$$= \sum_q A_{ij}(q) c_q^{i+} c_q^{j-} + \sum_q A_{ij}(q) c_q^{i+} c_q^{j-} \tag{93}$$

Since we are working in thermodynamic limit, we can change q → -q in the second term without being careful about the boundary terms,

$$= \sum_q A_{ij}(q) c_q^{i+} c_q^{j-} + \sum_q A_{ij}(-q) c_{-q}^{i+} c_{-q}^{j-} \tag{94}$$

Interchanging i↔ j, in the second term,

$$= \sum_q A_{ij}(q) c_q^{i+} c_q^{j-} - \sum_q A_{ji}(-q) c_{-q}^{i-} c_{-q}^{j+} \tag{95}$$

$$\tag{96}$$

For next two terms,

$$\sum_A J_{i,j,\pm,\pm} c_A^{i\pm} c_{A+1}^{j\pm} \tag{97}$$

$$= J_{i,j,\pm,\pm} \sum_q c_q^{i\pm} c_{-q}^{j\pm} e^{\mp \frac{2\pi i q}{N}} \tag{98}$$

$$= \frac{J_{i,j,\pm,\pm}}{2} \left(\sum_q c_q^{i\pm} c_{-q}^{j\pm} e^{\mp \frac{2\pi i q}{N}} + \sum_q c_q^{i\pm} c_{-q}^{j\pm} e^{\mp \frac{2\pi i q}{N}} \right) \tag{99}$$

$$\tag{100}$$

Changing q $\rightarrow$ -q in the second term,

$$= \frac{J_{i,j,\pm,\pm}}{2}\left(\sum_q c_q^{i\pm}c_{-q}^{j\pm}e^{\mp\frac{2\pi iq}{N}} + \sum_q c_{-q}^{i\pm}c_q^{j\pm}e^{\pm\frac{2\pi iq}{N}}\right) \tag{101}$$

$$= \frac{J_{i,j,\pm,\pm}}{2}\left(\sum_q c_q^{i\pm}c_{-q}^{j\pm}e^{\mp\frac{2\pi iq}{N}} - \sum_q c_q^{j\pm}c_{-q}^{i\pm}e^{\pm\frac{2\pi iq}{N}}\right) \tag{102}$$

Interchanging i$\leftrightarrow$ j, in the second term,

$$= \sum_q c_q^{i\pm}c_{-q}^{j\pm}\underbrace{\left(\frac{J_{i,j,\pm,\pm}e^{\mp\frac{2\pi iq}{N}} - J_{j,i,\pm,\pm}e^{\pm\frac{2\pi iq}{N}}}{2}\right)}_{B_{ij}^{\pm}(q)} \tag{103}$$

$$= \sum_q B_{ij}^{\pm}(q)c_q^{i\pm}c_{-q}^{j\pm} \tag{104}$$

The magnetic field term is trivial,

$$2\sum_{A,i} h_i c_A^{i+}c_A^{i-} \tag{105}$$

$$= 2\sum_{q,i} h_i c_q^{i+}c_q^{i-} \tag{106}$$

$$= \sum_{q,i} h_i(c_q^{i+}c_q^{i-} - c_{-q}^{i-}c_{-q}^{i+}) \tag{107}$$

The complete Hamiltonian (88) after fourier transformation becomes,

$$H_G = \sum_{q,i,j}\left(B_{ij}^{+}(q)c_q^{i+}c_{-q}^{j+} + A_{ij}(q)c_q^{i+}c_q^{j-} - A_{ji}(-q)c_{-q}^{i-}c_{-q}^{j+} + B_{ij}^{-}c_q^{i-}c_{-q}^{j-}\right) \tag{108}$$

$$+ \sum_{q,i} h_i(c_q^{i+}c_q^{i-} - c_{-q}^{i-}c_{-q}^{i+}) \tag{109}$$

$$H_G = \sum_{q,i,j}\Big[B_{ij}^{+}(q)c_q^{i+}c_{-q}^{j+} + A_{ij}(q)c_q^{i+}c_q^{j-} - A_{ji}(-q)c_{-q}^{i-}c_{-q}^{j+} + B_{ij}^{-}c_q^{i-}c_{-q}^{j-} \tag{110}$$

$$+ h_i\delta_{ij}(c_q^{i+}c_q^{i-} - c_{-q}^{i-}c_{-q}^{i+})\Big] \tag{111}$$

We have dropped the constant term from the Hamiltonian above as it just shifts energies of all the states by a constant and does not change the physics of the system. In matrix form, we have,

$$\boxed{H_G = \sum_{q,i,j}\underbrace{\begin{pmatrix} c_q^{i+} & c_{-q}^{i-}\end{pmatrix}}_{M_1}\underbrace{\begin{pmatrix} A_{ij} + h_i\delta_{ij} & B_{ij}^{+}(q) \\ B_{ij}^{-}(q) & -A_{ji} - h_i\delta_{ij}\end{pmatrix}}_{\text{M}}\underbrace{\begin{pmatrix} c_q^{j-} \\ c_{-q}^{j+}\end{pmatrix}}_{M_2}} \tag{112}$$

with

$$A_{ij}(q) = \left(\frac{\mathrm{J}_{i,j,+,-}e^{-\frac{2\pi iq}{N}} - \mathrm{J}_{i,j,-,+}e^{\frac{2\pi iq}{N}}}{2} \right) \tag{113}$$

$$B^{\pm}_{ij}(q) = \left(\frac{\mathrm{J}_{i,j,\pm,\pm}e^{\mp\frac{2\pi iq}{N}} - \mathrm{J}_{j,i,\pm,\pm}e^{\pm\frac{2\pi iq}{N}}}{2} \right) \tag{114}$$

Here, the dimension of matrix M is $2d \times 2d$, M_1 is $1 \times 2d$ and M_2 is $2d \times 1$ respectively. For the matrix M to be hermitian,

$$B^{\pm}_{ij}(q) = -B^{\pm}_{ji}(-q) \tag{115}$$

$$A^{*}_{ij}(q) = A_{ji}(q) \tag{116}$$

$$B^{\pm}_{ij}(q) = B^{\mp}_{ji}{}^{*}(q) \tag{117}$$

Again, to make the matrix look simpler we do some reparametrizations,

$$A_{ij}(q) = cos\frac{2\pi q}{N}\kappa_{ij} + sin\frac{2\pi q}{N}K_{ij} \tag{118}$$

$$B^{+}_{ij}(q) = cos\frac{2\pi q}{N}L_{ij} + sin\frac{2\pi q}{N}\gamma_{ij} \tag{119}$$

where,

$$\kappa_{ij} = \frac{\mathrm{J}_{i,j,+,-} - \mathrm{J}_{j,i,-,+}}{2} \qquad K_{ij} = -i\left(\frac{\mathrm{J}_{i,j,+,-} + \mathrm{J}_{j,i,-,+}}{2}\right) \tag{120}$$

$$L_{ij} = \frac{\mathrm{J}_{i,j,+,+} - \mathrm{J}_{j,i,+,+}}{2} \qquad \gamma_{ij} = -i\left(\frac{\mathrm{J}_{i,j,+,+} + \mathrm{J}_{j,i,+,+}}{2}\right) \tag{121}$$

We can think of $\boldsymbol{\kappa}$ as a matrix with elements κ_{ij}, $\boldsymbol{h}$ as a matrix with elements $h_i\delta_{ij}$ and so on. Then the matrix M becomes,

$$M = \begin{pmatrix} \boldsymbol{\kappa} cos\frac{2\pi q}{N} + \boldsymbol{K} sin\frac{2\pi q}{N} + \boldsymbol{h} & \boldsymbol{L} cos\frac{2\pi q}{N} + \boldsymbol{\gamma} sin\frac{2\pi q}{N} \\ \boldsymbol{L}^{\dagger} cos\frac{2\pi q}{N} + \boldsymbol{\gamma}^{\dagger} sin\frac{2\pi q}{N} & -\boldsymbol{\kappa}^{T} cos\frac{2\pi q}{N} + \boldsymbol{K}^{T} sin\frac{2\pi q}{N} - \boldsymbol{h} \end{pmatrix}_{2d\times 2d} \tag{122}$$

Imposing the hermiticity conditions to (120) and (121),

$$\boldsymbol{L}^{T} = -\boldsymbol{L}, \qquad \boldsymbol{\gamma}^{T} = \boldsymbol{\gamma}, \qquad \boldsymbol{K}^{\dagger} = \boldsymbol{K}, \qquad \boldsymbol{\kappa}^{\dagger} = \boldsymbol{\kappa} \tag{123}$$

Imposing space inversion symmetry using (65) gives,

$$K_{ij} = 0 \implies \boldsymbol{K} = 0 \tag{124}$$

$$L_{ij} = 0 \implies \boldsymbol{L} = 0 \tag{125}$$

$$\kappa_{ij} = \mathrm{J}_{i,j,+,-} \tag{126}$$

$$\gamma_{ij} = -i\mathrm{J}_{i,j,+,+} \tag{127}$$

The matrix M then becomes,

$$M = \begin{pmatrix} \boldsymbol{\kappa} cos\frac{2\pi q}{N} + \boldsymbol{h} & \boldsymbol{\gamma} sin\frac{2\pi q}{N} \\ \boldsymbol{\gamma}^{\dagger} sin\frac{2\pi q}{N} & -\boldsymbol{\kappa}^{T} cos\frac{2\pi q}{N} - \boldsymbol{h} \end{pmatrix}_{2d \times 2d} \tag{128}$$

Therefore, the Hamiltonian finally becomes,

$$\boxed{H_G = \sum_{q,i,j} \underbrace{\begin{pmatrix} c_q^{i+} c_{-q}^{i-} \end{pmatrix}}_{M_1} \underbrace{\begin{pmatrix} \boldsymbol{\kappa} cos\frac{2\pi q}{N} + \boldsymbol{h} & \boldsymbol{\gamma} sin\frac{2\pi q}{N} \\ \boldsymbol{\gamma}^{\dagger} sin\frac{2\pi q}{N} & -\boldsymbol{\kappa}^{T} cos\frac{2\pi q}{N} - \boldsymbol{h} \end{pmatrix}}_{\text{M}} \underbrace{\begin{pmatrix} c_q^{j-} \\ c_{-q}^{j+} \end{pmatrix}}_{M_2}} \tag{129}$$

For $d = 1$, it reduces to,

$$H = \sum_q \begin{pmatrix} c_q^{1+} & c_{-q}^{1-} \end{pmatrix} \begin{pmatrix} \kappa_{11} cos\frac{2\pi q}{N} + h_1 & \gamma_{11} sin\frac{2\pi q}{N} \\ \gamma_{11}^{\dagger} sin\frac{2\pi q}{N} & -\kappa_{11}^{T} cos\frac{2\pi q}{N} - h_1 \end{pmatrix} \begin{pmatrix} c_q^{1-} \\ c_{-q}^{1+} \end{pmatrix} \tag{130}$$

which is the TFIM Hamiltonian (31) with appropriate choice of coupling constants.

Unlike in TFIM, it is not easy to get a full solution of this Hamiltonian. But working with eigenvalues of matrix M is sufficient to study Quantum Phase Transitions.

4.6 QUANTUM PHASE TRANSITIONS

Having a quadratic fermionic Hamiltonian lets us explore the excitation spectrum and look for QPTs. Our aim is to find out whether the phase transition survives for $d > 1$. As before, we work with energy gap to figure out whether there exists a phase transition. Moreover, we also try to figure out the critical exponents.

In this project, we only work with $d = 2$ case. The Hamiltonian for $d = 2$ is given by,

$$H_G = \sum_q \begin{pmatrix} c_q^{1+} c_q^{2+} c_{-q}^{1-} c_{-q}^{2-} \end{pmatrix} \underbrace{\begin{pmatrix} \kappa_{11} cos\frac{2\pi q}{N} + h_1 & \kappa_{12} cos\frac{2\pi q}{N} & \gamma_{11} sin\frac{2\pi q}{N} & \gamma_{12} sin\frac{2\pi q}{N} \\ \kappa_{21} cos\frac{2\pi q}{N} & \kappa_{22} cos\frac{2\pi q}{N} + h_2 & \gamma_{12} sin\frac{2\pi q}{N} & \gamma_{22} sin\frac{2\pi q}{N} \\ \gamma_{11}^{*} sin\frac{2\pi q}{N} & \gamma_{12}^{*} sin\frac{2\pi q}{N} & -\kappa_{11} cos\frac{2\pi q}{N} - h_1 & -\kappa_{21} cos\frac{2\pi q}{N} \\ \gamma_{12}^{*} sin\frac{2\pi q}{N} & \gamma_{22}^{*} sin\frac{2\pi q}{N} & -\kappa_{12} cos\frac{2\pi q}{N} & -\kappa_{22} cos\frac{2\pi q}{N} - h_2 \end{pmatrix}}_{\text{M}} \begin{pmatrix} c_q^{1-} \\ c_q^{2-} \\ c_{-q}^{1+} \\ c_{-q}^{2+} \end{pmatrix} \tag{131}$$

The eigenvalues of this matrix M gives us the energy dispersion relation. That is, the energy difference between an excited state and the ground state.

We already know that when a QPT occurs, the energy gap vanishes as $q \to 0$ in the TFIM for some special choice of coupling constants. We **assume** that even for $d > 1$, the phase transition occurs at $q \to 0$ and then try to figure out the coupling constants (if they exist) for which the above assumption is valid.

The M matrix in terms of $J_{i,j,s,s'}$ is,

$$\begin{pmatrix} 2\left(h_1+\cos[k]J_{1,1,+,-}\right) & 2\cos(k)J_{1,2,+,-} & -2i\sin(k)J_{1,1,+,+} & -2i\sin(k)J_{1,2,+,+} \\ 2\cos(k)J_{2,1,+,-} & 2\left(h_2+\cos(k)J_{2,2,+,-}\right) & -2i\sin(k)J_{2,1,+,+} & -2i\sin(k)J_{2,2,+,+} \\ -2i\sin(k)J_{1,1,-,-} & -2i\sin(k)J_{1,2,-,-} & -2\left(h_1+\cos(k)J_{1,1,+,-}\right) & -2\cos(k)J_{2,1,+,-} \\ -2i\sin(k)J_{2,1,-,-} & -2i\sin(k)J_{2,2,-,-} & -2\cos(k)J_{1,2,+,-} & -2\left(h_2+\cos(k)J_{2,2,+,-}\right) \end{pmatrix} \tag{132}$$

At the phase transition point, the energy gap must vanish. So we put $q=0$ (or $k=0$)

$$\begin{pmatrix} 2\left(h_1+J_{1,1,+,-}\right) & 2J_{1,2,+,-} & 0 & 0 \\ 2J_{2,1,+,-} & 2\left(h_2+J_{2,2,+,-}\right) & 0 & 0 \\ 0 & 0 & -2\left(h_1+J_{1,1,+,-}\right) & -2J_{2,1,+,-} \\ 0 & 0 & -2J_{1,2,+,-} & -2\left(h_2+J_{2,2,+,-}\right) \end{pmatrix} \tag{133}$$

The eigenvalues of this matrix come in pairs,

$$\epsilon_1^{\pm}(0) = \pm\sqrt{\left(J_{1,1,+,-}-J_{2,2,+,-}+h_1-h_2\right)^2+4J_{1,2,+,-}J_{2,1,+,-}} \mp J_{1,1,+,-} \mp J_{2,2,+,-} \mp h_1 \mp h_2$$
$$\epsilon_2^{\pm}(0) = \pm\sqrt{\left(J_{1,1,+,-}-J_{2,2,+,-}+h_1-h_2\right)^2+4J_{1,2,+,-}J_{2,1,+,-}} \pm J_{1,1,+,-} \pm J_{2,2,+,-} \pm h_1 \pm h_2$$

A few things to note about the above matrix. If $J_{1,2,+,-}=J_{2,1,+,-}=J_{1,2,+,-}=J_{2,1,+,-}=0$, the matrix becomes diagonal. We can perform a change of basis by interchanging rows $R_2 \leftrightarrow R_3$ and column $C_2 \leftrightarrow C_3$. The resulting matrix looks like a matrix of two **decoupled** TFIMs with eigenvalues,

$$\epsilon_1^{\pm}(0) = \pm 2(h_1+J_{1,1,+,-}) \tag{134}$$
$$\epsilon_2^{\pm}(0) = \pm 2(h_2+J_{2,2,+,-}) \tag{135}$$

The eigenvalues above are of the same form as in (36). Moreover, if we make $h_2+J_{2,2,+,-}=0$ along with the above assumptions, then we get the usual TFIM with eigenvalues,

$$\epsilon_1^{\pm}(0) = \pm 2(h_1+J_{1,1,+,-}) \tag{136}$$
$$\epsilon_2^{\pm}(0) = 0 \tag{137}$$

Again, $\epsilon_1^{\pm}(0)$ is of the form (36)

Looking at the eigenvalues, we can say that there exists some choice of coupling constants such that the energy gap becomes zero. Therefore, there exists a QPT. There are infinitely many choice of coupling constants such that the eigenvalues become zero. One (or more) of the eigenvalues above must be zero if the energy gap vanishes.

The determinant of the matrix in (133) is

$$16\left[J_{2,1,+,-}J_{1,2,+,-}-(h_1+J_{1,1,+,-})(h_2+J_{2,2,+,-})\right]^2 \tag{138}$$

which must be zero since one of its eigenvalues is vanishing. This implies,

$$J_{2,1,+,-}J_{1,2,+,-} = (h_1 + J_{1,1,+,-})(h_2 + J_{2,2,+,-}) \tag{139}$$

This is a more general condition which could be used later to find out the critical exponent, which is a work in progress.

5

DISCUSSION AND CONCLUSION

The TFIM has been studied and generalised to the gamma Ising model. At various instances, we have seen that the gamma Ising model reduces to TFIM for $d = 1$. We have also seen that there exists a QPT for $d = 2$ case.

Further, a lot of work has to be done in this project. One can redo the calculations in finite A (lattice sites) limit to work with the boundary conditions and see how they affect the JW transformed Hamiltonian. Since the exact solution of this model has not been computed, one can attempt to do a Bogoliubov transformation to proceed with finding the Hamiltonian in terms of Bogoliubov quasi-particles. One can also look out for additional symmetries in the Hamiltonian which would reduce the number of parameters in the problem. As mentioned before, one could try to figure out quantities such as critical exponents which describe the behaviour of various physical quantities near phase transitions, starting with $d = 2$. We can also make use of numerics as done before in TFIM, to inspect single particle excitations near phase transition point. Finally, attempts can be made to increase d and investigate the nature of QPTs.

A

RELATION BETWEEN $J_{\mu\nu}$ AND $J_{i,j,s,s'}$

$$
\begin{aligned}
\underbrace{J_{i,j,+,+}}_{J_{i,j,s,s'}} &= \underbrace{J_{2i-1,2j-1} - i\, J_{2i-1,2j} - i\, J_{2i,2j-1} - J_{2i,2j}}_{J_{\mu\nu}} \\
J_{i,j,+,-} &= J_{2i-1,2j-1} + i\, J_{2i-1,2j} - i\, J_{2i,2j-1} + J_{2i,2j} \\
J_{i,j,-,+} &= J_{2i-1,2j-1} - i\, J_{2i-1,2j} + i\, J_{2i,2j-1} + J_{2i,2j} \\
J_{i,j,-,-} &= J_{2i-1,2j-1} + i\, J_{2i-1,2j} + i\, J_{2i,2j-1} - J_{2i,2j}
\end{aligned}
\tag{140}
$$

From the above equations, it is clear that each $J_{i,j,s,s'}$ is a linear combination of four $J_{\mu\nu}$'s

B

VERIFICATION OF JW TRANSFORMATION

B.1 $\{c^i_A, c^j_B\} = 0$

$$\begin{aligned}
&\left\{\left(\prod_{C<A}\Gamma^{2d+1}_C\right)\Gamma^{i-}_A, \left(\prod_{D<B}\Gamma^{2d+1}_D\right)\Gamma^{j-}_B\right\}\\
&= \underbrace{\left(\prod_{C<A}\Gamma^{2d+1}_C\right)\Gamma^{i-}_A}_{\text{commutes}}\left(\prod_{D<B}\Gamma^{2d+1}_D\right)\Gamma^{j-}_B + \left(\prod_{D<B}\Gamma^{2d+1}_D\right)\Gamma^{j-}_B\left(\prod_{C<A}\Gamma^{2d+1}_C\right)\Gamma^{i-}_A\\
&= \Gamma^{i-}_A\left(\prod_{C<A}\Gamma^{2d+1}_C\right)\left(\prod_{D<B}\Gamma^{2d+1}_D\right)\Gamma^{j-}_B + \Gamma^{j-}_B\left(\prod_{D<B}\Gamma^{2d+1}_D\right)\left(\prod_{C<A}\Gamma^{2d+1}_C\right)\Gamma^{i-}_A
\end{aligned}$$

With no loss of generality, we can assume $B > A$. Since $(\Gamma^{2d+1})^2 = 1$, we have

$$\begin{aligned}
&= \Gamma^{i-}_A\underbrace{\left(\prod_{A\leq D<B}\Gamma^{2d+1}_D\right)}_{\text{commutes with }\Gamma^{i-}_A\text{ and }\Gamma^{j-}_B}\Gamma^{j-}_B + \Gamma^{j-}_B\left(\prod_{A\leq D<B}\Gamma^{2d+1}_D\right)\Gamma^{i-}_A\\
&= \left(\prod_{A<D<B}\Gamma^{2d+1}_D\right)(\Gamma^{i-}_A\Gamma^{2d+1}_A\Gamma^{j-}_B + \Gamma^{j-}_B\Gamma^{2d+1}_A\Gamma^{i-}_A)
\end{aligned}$$

Since,

$$\begin{aligned}
&\{\Gamma^{2d+1}, \Gamma^{\mu}\} = 0\\
&\Rightarrow \Gamma^{i-}_A\Gamma^{2d+1}_A = -\Gamma^{2d+1}_A\Gamma^{i-}_A
\end{aligned}$$

So,

$$\begin{aligned}
&= \left(\prod_{A<D<B}\Gamma^{2d+1}_D\right)(\Gamma^{i-}_A\Gamma^{2d+1}_A\Gamma^{j-}_B - \underbrace{\Gamma^{j-}_B\Gamma^{i-}_A}_{\text{commutes (different lattice sites)}}\Gamma^{2d+1}_A)\\
&= \left(\prod_{A<D<B}\Gamma^{2d+1}_D\right)(\Gamma^{i-}_A\Gamma^{2d+1}_A\Gamma^{j-}_B - \Gamma^{i-}_A\underbrace{\Gamma^{j-}_B\Gamma^{2d+1}_A}_{\text{commutes (different lattice sites)}})\\
&= \left(\prod_{A<D<B}\Gamma^{2d+1}_D\right)(\Gamma^{i-}_A\Gamma^{2d+1}_A\Gamma^{j-}_B - \Gamma^{i-}_A\Gamma^{2d+1}_A\Gamma^{j-}_B)\\
&= 0
\end{aligned}$$

B.2 $\{c_A^{i\dagger}, c_B^{j\dagger}\} = 0$

Can be proved using similar procedure as done above in B.1

B.3 $\{c_A^{i}, c_B^{j\dagger}\} = \delta_{ij}\delta_{AB}$

$$
\begin{aligned}
&\{c_A^{i}, c_B^{j\dagger}\} \\
&= \{\left(\prod_{C<A}\Gamma_C^{2d+1}\right)\Gamma_A^{i-}, \left(\prod_{D<B}\Gamma_D^{2d+1}\right)\Gamma_B^{j+}\} \\
&= \Gamma_A^{i-}\left(\prod_{C<A}\Gamma_C^{2d+1}\right)\left(\prod_{D<B}\Gamma_D^{2d+1}\right)\Gamma_B^{j+} + \Gamma_B^{j+}\left(\prod_{D<B}\Gamma_D^{2d+1}\right)\left(\prod_{C<A}\Gamma_C^{2d+1}\right)\Gamma_A^{i-} \\
&= \left(\prod_{A<D<B}\Gamma_D^{2d+1}\right)(\Gamma_A^{i-}\Gamma_A^{2d+1}\Gamma_B^{j+} + \Gamma_B^{j+}\Gamma_A^{2d+1}\Gamma_A^{i-})
\end{aligned}
$$

<u>For $A \neq B$</u>

$$
\begin{aligned}
&\left(\prod_{A<D<B}\Gamma_D^{2d+1}\right)(\Gamma_A^{i-}\Gamma_A^{2d+1}\Gamma_B^{j+} + \Gamma_B^{j+}\Gamma_A^{2d+1}\Gamma_A^{i-}) && (141)\\
&= 0 && (142)
\end{aligned}
$$

<u>For $i \neq j$</u>

$$
\begin{aligned}
&\left(\prod_{A<D<B}\Gamma_D^{2d+1}\right)(\Gamma_A^{i-}\Gamma_A^{2d+1}\Gamma_B^{j+} + \Gamma_B^{j+}\Gamma_A^{2d+1}\Gamma_A^{i-}) && (143)\\
&= 0 && (144)
\end{aligned}
$$

<u>For $i = j$ and $A = B$</u>

$$
\begin{aligned}
&\underbrace{\left(\prod_{A<D<B}\Gamma_D^{2d+1}\right)}_{\text{won't exist}}(\Gamma_A^{i-}\underbrace{\Gamma_A^{2d+1}}_{\text{won't exist}}\Gamma_A^{i+} + \Gamma_A^{i+}\underbrace{\Gamma_A^{2d+1}}_{\text{won't exist}}\Gamma_A^{i-}) && (145)\\
&= (\Gamma_A^{i-}\Gamma_A^{i+} + \Gamma_A^{i+}\Gamma_A^{i-}) && (146)\\
&= \{\Gamma_A^{i-}, \Gamma_A^{i+}\} && (147)\\
&= 1 \qquad\qquad \text{(using (56))} && (148)
\end{aligned}
$$

C

JW TRANSFORMATION OF S^i

$$
\begin{aligned}
S^i_A &= \Gamma^{i+}_A \Gamma^{i-}_A - \mathbb{1} \\
&= \left(\prod_{B<A} \Gamma^{2d+1}_B \right) c^{i\dagger}_A \left(\prod_{B<A} \Gamma^{2d+1}_B \right) c^i_A - \mathbb{1} \\
&= 2c^{i\dagger}_A \left(\prod_{B<A} \Gamma^{2d+1}_B \right)^2 c^i_A - \mathbb{1} \\
&= 2c^{i\dagger}_A c^i_A - \mathbb{1}
\end{aligned}
$$

D

FOURIER TRANSFORMATION

We will use the following convention,

$$c_A^{i,s} = \frac{1}{\sqrt{N}} \sum_q e^{i\frac{2\pi q A s}{N}} c_q^{i,s} \qquad \text{where } s = \pm$$

Note that,

$$\frac{1}{N} \sum_A^N e^{i\frac{2\pi(q+q')A}{N}} = \delta_{q,-q'}$$

Then we get,

$$\begin{aligned}
\sum_A c_A^{i,s} c_{A+1}^{j,s'} &= \left(\sum_{q,q'} c_q^{i,s} c_{q'}^{j,s'} e^{\frac{2\pi i q' s'}{N}} \right) \left(\frac{1}{N} \sum_A e^{\frac{2\pi i A(sq+s'q')}{N}} \right) \\
&= \sum_q e^{\frac{-2\pi i q s}{N}} c_q^{i,s} c_{-ss'q}^{j,s'}
\end{aligned}$$

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