HOMEWORK 1:
BACKGROUND TEST

Yash Trivedi
ytrivedi@wisc.edu

Minimum Background Test [80 pts]

1 Vectors and Matrices [20 pts]

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 9 \\ 8 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

1. What is the inner product of the vectors $\mathbf{y}$ and $\mathbf{z}$? (this is also sometimes called the *dot product*, and is sometimes written as $\mathbf{y}^T\mathbf{z}$)
   111

2. What is the product $X\mathbf{y}$?

$$\begin{pmatrix} 145 \\ 111 \end{pmatrix}$$

2 Calculus [20 pts]

1. If $y = 4x^3 - x^2 + 7$ then what is the derivative of y with respect to x?
   $12x^2 - 2x$

2. If $y = \tan(z)x^{6z} - \ln(\frac{7x+z}{x^4})$, what is the partial derivative of y with respect to x?
   $(6z)tan(z)x^{6z-1} + (\frac{21x^4+4x^3z}{x^4(7x+z)})$

3 Probability and Statistics [20 pts]

Consider a sample of data $S = \{0, 1, 1, 0, 0, 1, 1\}$ created by flipping a coin x seven times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. What is the sample mean for this data?
   $\frac{4}{7}$

2. What is the sample variance for this data?
   0.244

3. What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p(x = 1) = 0.7, p(x = 0) = 0.3$.
   $(0.7)^4 * (0.3)^3$

4. Note that the probability of this data sample would be greater if the value of $p(x = 1)$ was not 0.7, but instead some other value. What is the value that maximizes the probability of the sample S? Please justify your answer.
   $\frac{4}{7}$ (by the MLE principle)

A	B	$P(A, B)$
0	0	0.1
0	1	0.4
1	0	0.2
1	1	0.3

5. Consider the following joint probability table where both A and B are binary random variables:
   (a) What is $P(A = 0, B = 0)$?
   0.1
   (b) What is $P(A = 1)$?
   0.5
   (c) What is $P(A = 0|B = 1)$?
   $\frac{0.4}{0.7}$
   (d) What is $P(A = 0 \vee B = 0)$?
   0.7

4 Big-O Notation [20 pts]

For each pair (f, g) of functions below, list which of the following are true: $f(n) = O(g(n))$, $g(n) = O(f(n))$, both, or neither. Briefly justify your answers.

1. $f(n) = \frac{n}{2}$, $g(n) = \log_2(n)$.
   $g(n) = O(f(n))$. Since $f(n)$ has a linear rate of growth which is higher than $g(n)$ which is logarithmic.
2. $f(n) = \ln(n)$, $g(n) = \log_2(n)$.
   $Both$. They differ only by a multiplication constant.
3. $f(n) = n^{100}$, $g(n) = 100^n$.
   $f(n) = O(g(n))$. Take the example case of n = 1000.

Medium Background Test [20 pts]

5 Algorithm [5 pts]

Divide and Conquer: Assume that you are given a sorted array with n integers in the range $[-10, +10]$. Note that some integer values may appear multiple times in the array. Additionally, you are told that somewhere in the array the integer 0 appears exactly once. Provide an algorithm to locate the 0 which runs in $O(\log(n))$. Explain your algorithm in words, describe why the algorithm is correct, and justify its running time.

Since the numbers are sorted, we can use the binary search algorithm to search for 0. Binary search maintains two indices - low (initially, zero) and high (initially, n - 1). While $low <= high$ it calculates the mid-point index which is equal to $\frac{low+high}{2}$. If the value at the mid-point index matches the key we are looking for, it returns the value of the mid-point index. If the value at the mid-point index is higher than zero (the key), we decrease the $high$ index to be $mid - 1$, else if the value at the mid-point index is lower than zero (the key), we increase the low index to be $mid + 1$. We continue this procedure till we find the key we are looking for.

The algorithm is correct since it exhaustively searches the relevant portions of the search space until the key is found or returns a failure if they key doesn't exist.

The algorithm has a running time $O(log(n))$ since it always partitions the search space in half on every iteration.

6 Probability and Random Variables [5 pts]

6.1 Probability

State true or false. Here Ω denotes the sample space and A^c denotes the complement of the event A.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(B) = P(B|A)P(A)$.
   True

2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(A|B)$.
   False

3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A \cup B \cup C)}{P(B \cup C)} \geq P(A|B \cup C)P(B \cup C)$.
   True

4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.
   False

5. For any n events $\{A_i\}_{i=1}^n$, if $P(\bigcap_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$, then $\{A_i\}_{i=1}^n$ are mutually independent.
   False

6.2 Discrete and Continuous Distributions

Match the distribution name to its probability density / mass function. Below, $|\boldsymbol{x}| = k$.

(a) Laplace (h)

(b) Multinomial (i)

(c) Poisson (l)

(d) Dirichlet (k)

(e) Gamma (j)

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$

(g) $f(x; n, \alpha) = \binom{n}{x}\alpha^x(1-\alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise

(h) $f(x; b, \mu) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$

(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!}\Pi_{i=1}^k \alpha_i^{x_i}$ for $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise

(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise

(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for $x_i \in (0,1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise

(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in Z^+$; 0 otherwise

6.3 Mean and Variance

Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.

1. What is the mean of the random variable?
   np

2. What is the variance of the random variable?
   $np(1-p)$

7 Linear algebra [5 pts]

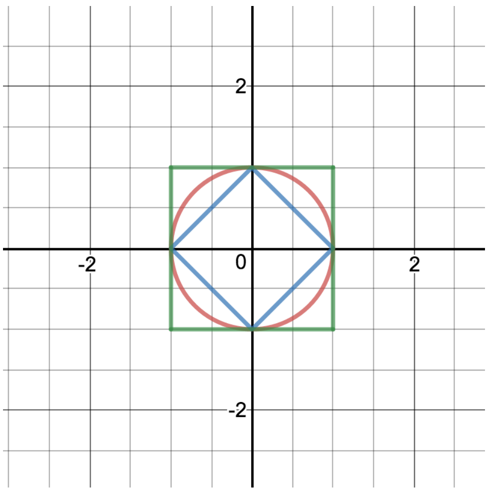
7.1 Norm-enclature

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ with the following norms:

1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)

2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)

3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)
   $||\mathbf{x}||_1 \leq 1$ - blue
   $||\mathbf{x}||_2 \leq 1$ - red
   $||\mathbf{x}||_\infty \leq 1$ - green



7.2 Geometry

Prove that these are true or false. Provide all steps.

1. The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$.
   Let **a** be the closest point to the origin which lies on the given hyperplane. **a** can be written as $\mathbf{a} = min$ $p^T p$ such that $w^T p + b = 0$. Solving the optimization problem using Lagrange multipliers, we get $\mathbf{a} = \frac{\lambda}{2}\mathbf{w}$. Substituting the value in the constraint, we get $\mathbf{a} = \frac{-b}{w^T w} w$. Computing the distance between **a** and the origin, distance = $\sqrt[2]{\mathbf{a}^T\mathbf{a}} = \sqrt[2]{\left(\frac{-b}{w^T w}\right)^2 w^T w} = \frac{b}{\sqrt[2]{w^T w}} = \frac{b}{||w||_2}$.

2. The Euclidean distance between two parallel hyperplane $\mathbf{w}^T\mathbf{x} + b_1 = 0$ and $\mathbf{w}^T\mathbf{x} + b_2 = 0$ is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$ (Hint: you can use the result from the last question to help you prove this one).

   The distance between the hyperplanes is the distance between the two points x_1 and x_2 where the line through origin parallel to the normal vector **a** intersects the hyperplane. The two points of intersection, by the result above, are $x_1 = \left(\frac{b_1}{||\mathbf{a}||_2^2}\right)\mathbf{a}$ and $x_2 = \left(\frac{b_2}{||\mathbf{a}||_2^2}\right)\mathbf{a}$. Hence, the distance between the two points is $||x_1 - x_2||_2 = \frac{|b_1 - b_2|}{||\mathbf{a}||_2}$.

8 Programming Skills - Test Yourself

You do not need to turn this in; it is on the honor system. Please attempt the following two problems on the HankerRank website.

Grading Students (Easy): https://www.hackerrank.com/challenges/grading/problem

Verify BST (Medium): https://www.hackerrank.com/challenges/ctci-is-binary-search-tree/problem